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Ambitwistor integrands from tensionless chiral superstring integrands

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ABSTRACT: It is established that in the tensionless limit the chiral superstring integrand is reduced to the chiral integrand of the ambitwistor string.

KEYWORDS: Scattering Amplitudes, String theory and cosmic strings

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Contents

1	Introduction	1
2	Holomorphic factorization of superstring scattering amplitudes	2
3	Finding the chiral correlator on the supermoduli space	3
4	Comparison to the ambitwistor string	4
5	Conclusion	6

1 Introduction

Since the twistor string formulation of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills [1–3], much effort has been expended in developing worldsheet approaches to scattering amplitudes in more general quantum field theories. An important advance was made by the scattering equation formalism of Cachazo, He and Yuan [4–8], which realised scattering amplitudes in a large class of theories at tree level as integrals over the moduli space $\mathcal{M}_{0,n}$ of marked Riemann spheres of the form

$$\int_{\mathcal{M}_{0,n}} \frac{d^n \sigma}{\text{SL}(2, \mathbb{C})} \mathcal{I}_L \mathcal{I}_R \prod_{i=1}^{n-3} \delta(E_i) \quad (1.1)$$

where the E_i are conditions known as the scattering equations that serve to fully localise the integrand at discrete points on the moduli space. The half integrands $\mathcal{I}_{L,R}$ encode kinematical or colour degrees of freedom. The freedom in choosing these objects leads to integral formulae for a large class of quantum field theories.

Soon after the development of the CHY formalism, it was observed that the half integrands $\mathcal{I}_{L,R}$ used in evaluating amplitudes in gauge theory and gravity¹ can be equivalently computed as correlation functions of a two-dimensional theory known as the *ambitwistor string* [13]. Since this model provided an explicit worldsheet interpretation for the CHY approach, the restriction to tree level amplitudes could be overcome. The genus expansion of the ambitwistor string has since been used to derive moduli space formulae at one-loop [14–16] and two-loop [17] orders.

Although the ambitwistor string provides a consistent scheme through which the half integrands in the CHY framework may be determined, its relationship with the more conventional Ramond-Neveu-Schwarz (RNS) string remains to be fully understood. While

¹It is a fact that *any* quantum field theory admits such a representation [9–11]. However, the integrands giving gauge and gravity amplitudes at tree level are known to enjoy properties such as manifest color-kinematic duality [4, 12] that make them especially interesting.

it may be formally obtained as a low-energy ($\alpha' \rightarrow 0$) limit of the RNS string [13], a more careful analysis seems to indicate that the two are better related by the *tensionless* ($\alpha' \rightarrow \infty$) limit [18, 19].

In this letter, considering the latter viewpoint, by direct computation of the chiral integrand [20] of superstring NS states we demonstrate that it reduces to the corresponding half integrand in the ambitwistor string as the tensionless limit is approached. More concretely, we show that the superstring integrand (we denote the spin structure by δ) takes the form

$$\mathcal{A}_{g,n}[\delta] = \exp(\text{KN}) \times \mathcal{I}_{g,n}^{\alpha'}[\delta] \quad (1.2)$$

(where KN generalises the Koba-Nielsen factor to higher loops) such that in the limit of infinite α' , $\mathcal{I}_{g,n}^{\alpha'}[\delta]$ equals the chiral half integrand computed by the ambitwistor string.

2 Holomorphic factorization of superstring scattering amplitudes

The scattering of n NS states in superstring perturbation theory is defined by a formal integral over the supermoduli space $\mathfrak{M}_{g,n}$ (with measure $d\mu_{g,n}$) of super Riemann surfaces with n NS punctures

$$\int_{\mathfrak{M}_{g,n}} d\mu_{g,n} \langle |\delta(H_A|B)|^2 \rangle_{B,C} \times \mathcal{O}_n \quad (2.1)$$

where H_A is a basis of Beltrami superdifferentials and B and C are ghost superfields encoding the bc and $\beta\gamma$ systems such that

$$B(z, \theta) = \beta(z) + \theta b(z), \quad (2.2)$$

and

$$C(z, \theta) = c(z) + \theta \gamma(z). \quad (2.3)$$

We compute the quantity $\mathcal{O}_n$ by making use of the chiral splitting theorem due to d'Hoker and Phong [20], which says

$$\mathcal{O}_n = \int_{\mathbb{R}^{10g}} d^{10}p_I \left| \left\langle \exp \left(\frac{i}{\alpha'} \int \chi(z) \psi^\mu \partial x^\mu(z) \right) \times \prod_i \mathcal{V}(z_i, \theta_i, k_i, \epsilon_i) \right\rangle \right|^2 \quad (2.4)$$

where $\chi(z)$ is the gravitino, parametrising odd moduli on the supermoduli space and

$$\mathcal{V}(z_i, \theta_i, k_i, \epsilon_i) = \int d\tilde{\theta}_i \exp \left(i k_i^\mu x_+^\mu(z_i) + \frac{2i}{\alpha'} \tilde{\theta}_i \epsilon_i^\mu \partial x_+^\mu(z_i) + \theta_i k_i^\mu \psi^\mu(z_i) + \tilde{\theta}_i \epsilon_i^\mu \psi^\mu(z_i) \right). \quad (2.5)$$

The chiral fields $x_+(z)$ and $\psi(z)$ are purely holomorphic and obey (given a spin structure δ) the operator product expansions

$$x_+^\mu(z) x_+^\nu(z') \sim -\eta^{\mu\nu} \alpha' \ln(E(z, z')) \quad (2.6)$$

and

$$\psi^\mu(z) \psi^\nu(z') \sim \eta^{\mu\nu} S_\delta(z, z'). \quad (2.7)$$

where $E(z_i, z_j)$ is the prime form on a genus g Riemann surface and $S_\delta(z_i, z_j)$ is the genus g Szego kernel for spin structure δ . The loop momenta p_I are defined as monodromies of the chiral ∂x_+ fields around the $\mathfrak{A}_I$ cycles of the Riemann surface

$$\oint_{\mathfrak{A}_I} \partial x_+^\mu(z) dz = -i\alpha' p_I^\mu. \quad (2.8)$$

This condition is satisfied by performing the effective replacement

$$\partial x^\mu(z) \rightarrow \partial x^\mu(z) - i\alpha' p_I^\mu \omega_I(z). \quad (2.9)$$

With these definitions, we define the *chiral correlation function*

$$\mathcal{A}_{g,n}[\delta] = \left\langle \prod_A \delta(\langle H_A | B \rangle) \exp \left(\frac{i}{\alpha'} \int \chi(z) \psi^\mu \partial x^\mu(z) \right) \times \prod_i \mathcal{V}(z_i, \theta_i, k_i, \epsilon_i) \right\rangle_{B, C, x_+, \psi}. \quad (2.10)$$

The full superstring integrand is then expressed as a sum over two complex conjugate copies thereof

$$\mathcal{I}_{g,n} = \sum_{\delta, \delta'} \eta_\delta \tilde{\eta}_{\delta'} \mathcal{A}_{g,n}[\delta] \overline{\mathcal{A}_{g,n}[\delta]}. \quad (2.11)$$

where the complex conjugate chiral integrand is be evaluated at loop momenta $-p_I$. The constants η_δ and $\tilde{\eta}_{\delta'}$ take values ± 1 and perform the GSO projection based on the combination chosen.

3 Finding the chiral correlator on the supermoduli space

We start with the evaluation of (2.10) on $\mathfrak{M}_{g,n}$. In doing this, we note that we will implicitly absorb the integrals over the auxiliary Grassmann parameters $\tilde{\theta}_i$ into the measure of $\mathfrak{M}_{g,n}$. This is only done to make the resulting expressions more compact.

We begin by computing the matter part of the chiral correlator, defined intrinsically on the supermoduli space $\mathfrak{M}_{g,n}$. To do this we integrate over the chiral $x_+(z)$ and $\psi(z)$ in turn. The $x_+(z)$ integration introduces additional terms quadratic in $\psi(z)$. The $\psi(z)$ integral is then performed using the effective propagator

$$\hat{S}_\delta(z, z') = S_\delta(z, z') + \frac{1}{\alpha'} \int S_\delta(z, w) K(w, v) S_\delta(v, z') dw dv \quad (3.1)$$

where

$$K(w, v) = \chi(w) \partial_w \partial_v \ln(E(w, v)) \chi(v). \quad (3.2)$$

This modified contraction is effected by factoring out

$$\mathcal{X}_{\text{PCO}} = \left\langle \exp \left(\frac{i}{\alpha'} \int \chi(z) \partial x_+(z) \cdot \psi(z) dz \right) \right\rangle. \quad (3.3)$$

When this is done, the chiral correlator is evaluated as a conventional Gaussian integral. The result is given by

$$\mathcal{X}_{\text{PCO}} \times \exp \left(\text{KN} + H_{0,0} + H_{0,1} + \mathcal{O} \left(\frac{1}{\alpha'} \right) \right) \quad (3.4)$$

where

$$\text{KN} = \alpha' \sum_{i,I} k_i \cdot p_I \int_P^{z_i} \omega_I(z) dz + \alpha' \sum_{i \neq j} \frac{1}{2} k_i \cdot k_j \ln(E(z_i, z_j)) \quad (3.5)$$

and

$$\begin{aligned} H_{0,0} = & \int \chi(z) \chi(z') P(z) \cdot P(z') S_\delta(z, z') dz dz' \\ & + 2 \sum_i \left(\int [\chi(z) \theta_i P(z) \cdot k_i S_\delta(z, z_i) + \chi(z) \tilde{\theta}_i P(z) \cdot \epsilon_i S_\delta(z, z_i)] dz \right) + \\ & \sum_{i \neq j} \left[\theta_i \theta_j k_i \cdot k_j S_\delta(z_i, z_j) + \tilde{\theta}_i \tilde{\theta}_j \epsilon_i \cdot \epsilon_j S_\delta(z_i, z_j) - 2 \theta_i \tilde{\theta}_j \epsilon_j \cdot k_i S_\delta(z_j, z_i) \right] + \sum_i 2 \theta_i \tilde{\theta}_i \epsilon_i \cdot P(z_i). \end{aligned} \quad (3.6)$$

The quantity $H_{0,1}$ vanishes for even spin structure. For odd spin structure it is given by

$$H_{0,1} = \int \chi(z) \psi^{(0)} \cdot P(z) dz + \sum_i \left[\theta_i k_i \cdot \psi^{(0)} + \tilde{\theta}_i \epsilon_i \cdot \psi^{(0)} \right] \quad (3.7)$$

where $\psi^{(0)}$ is the (ten dimensional) zero mode of the worldsheet fermion. In these expressions

$$P^\mu(z) = \sum_i k_i^\mu \ln(E(z, z_i)) + \sum_I p_I^\mu \omega_I(z). \quad (3.8)$$

Note that ω_I are Abelian differentials of the first kind.

We now make contact with the representation (1.2) by pointing out that the factor KN is not dependent on any Grassmann valued variables. Accordingly, we factor it out, leaving behind $H_{0,0}$, $H_{0,1}$ and terms subleading in $\frac{1}{\alpha'}$ contributing to $\mathcal{I}_{g,n}^{\alpha'}[\delta]$

4 Comparison to the ambitwistor string

To compare our result to the corresponding chiral integrand in the ambitwistor string, we need to reduce our result to an expression on the ordinary moduli space $\mathcal{M}_{g,n}$. However, projecting down to the ordinary moduli space from the supermoduli space isn't always possible.² Due to the general difficulties involved in projecting down to $\mathcal{M}_{g,n}$, it is best to work in small enough neighbourhoods $\mathcal{U} \in \mathcal{M}_{g,n}$ onto which holomorphic projections always exist.³

Let us then take some (arbitrary) point $\Sigma \in \mathcal{M}_{g,n}$. For a small enough neighbourhood $\mathcal{U}_\Sigma$, the moduli of $\mathfrak{M}_{g,n}$ can be parametrised by the ordinary bosonic moduli and a gravitino $\chi(z)$ that can be expanded as [23]

$$\chi(z) = \sum_i^{N_{\text{odd}}} \chi_{\alpha_i} \delta(z - z_{\alpha_i}) \quad (4.1)$$

where the insertion points z_{α_i} are holomorphic functions of the bosonic moduli.⁴ The number N_{odd} of Grassmann directions spanned by the χ_{α_i} is 0 for genus zero and genus

²It is known that $\mathfrak{M}_g$ is not projected [21, 22] while the question remains unanswered in the presence of punctures.

³I am grateful to Seyed Farough Moosavian and Edward Witten for discussions on this point.

⁴The requirement that the PCOs depend on the moduli is due to the presence of spurious singularities. See [24–26] for discussions of this issue.

one with even spin structure. For genus one with odd spin structure $N_{\text{odd}} = 1$ and for genus $g \geq 2$ we have $N_{\text{odd}} = 2g - 2$.

In this local patch, the basis for the Beltrami superdifferentials is

$$H_A = (\mu_a | \delta_{z, z_{\alpha_i}}), \quad (4.2)$$

where the μ_a are the N_{even} ,⁵ ordinary Beltrami differentials labelling deformations of the bosonic moduli and $\delta_{z, z_{\alpha_i}}$ are N_{odd} delta functions evaluating at z_{α_i} .

We have shown that in this set up, upon integrating away the fermionic degrees of freedom, the matter part of the chiral correlation function and the ghost partition function precisely match the corresponding results of the ambitwistor string [17]. Putting these together, we have proved that the chiral correlator of the RNS superstring reduces to the corresponding chiral half integrand of the ambitwistor string in the $\alpha' \rightarrow \infty$ limit. Concretely we have (specialising to the case of even spin structure)

$$\mathcal{I}_{g,n}^{\alpha'}[\delta] = \mathcal{Z}_{gh}[\delta] \left(\text{Pf} \Psi_{g,n}[\delta] + \mathcal{O}\left(\frac{1}{\alpha'}\right) \right). \quad (4.3)$$

Here, $\mathcal{Z}_{gh}[\delta]$ is the ghost partition function

$$\mathcal{Z}_{gh}[\delta] = \frac{1}{Z^5} \left\langle \prod_{i=1}^{N_{\text{even}}} \langle \mu_i | b \rangle \prod_{i=1}^{N_{\text{odd}}} \delta(\beta(z_{\alpha_i})) \prod_{i=1}^{N_B} c(z_{a_i}) \prod_{i=1}^{N_F} \delta(\gamma(z_{b_i})) \right\rangle_{\delta} \quad (4.4)$$

where $\langle \mu_i | b \rangle$ is used to denote the inner product between the Beltrami differential μ_i and the field b and it is to be understood that the expectation value is to be taken with respect to the bc and $\beta\gamma$ systems. Z is the chiral scalar partition function (see for example [17, 27] for an explicit representation in terms of the prime form and theta functions on $\mathcal{M}_{g,n}$). N_B and N_F are only nonzero at genus zero and one,⁶ corresponding to insertions of c and γ fields to account for superconformal Killing vectors. The z_{a_i} and z_{b_i} are arbitrarily chosen external punctures, not to be confused with PCO insertions. $\mathcal{Z}_{gh}[\delta]$ is computed by bosonisation [28, 29].

The quantity $\Psi_{g,n}$ is a matrix, given by⁷

$$\Psi_{g,n} = \begin{pmatrix} \mathbf{A} & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B} \end{pmatrix} \quad (4.5)$$

⁵For genus $g = 0$ $N_{\text{even}} = 0$, for $g = 1$ $N_{\text{even}} = 1$ and for $g \geq 2$ $N_{\text{even}} = 3g - 3$.

⁶At $g = 0$, $N_B = 3$ and $N_F = 2$, at $g = 1$ for even spin structure $N_B = 1$ and $N_F = 0$ and for odd spin structure $N_B = N_F = 1$.

⁷In evaluating the Pfaffian in (4.3), it is defined after removing a pair of rows and columns and one row and column corresponding to external states respectively at genus zero and genus one with odd spin structure. This is due to the fact that N_F states with marked points z_{b_i} are in the -1 picture.

where

$$\begin{aligned}
\mathbf{A}_{\alpha_i \alpha_j} &= P(z_{\alpha_i}) \cdot P(z_{\alpha_j}) S_{\delta}(z_{\alpha_i}, z_{\alpha_j}), \\
\mathbf{A}_{\alpha_i i} &= P(z_{\alpha_i}) \cdot k_i S_{\delta}(z_{\alpha_i}, z_i), \\
\mathbf{C}_{\alpha_i j} &= P(z_{\alpha_i}) \cdot \epsilon_i S_{\delta}(z_i, z_{\alpha_i}), \\
\mathbf{A}_{ij} &= k_i \cdot k_j S_{\delta}(z_i, z_j), & \mathbf{B}_{ij} &= \epsilon_i \cdot \epsilon_j S_{\delta}(z_i, z_j), \\
\mathbf{C}_{ij} &= \epsilon_i \cdot k_j S_{\delta}(z_i, z_j), & \mathbf{C}_{ii} &= -k_i \cdot P(z_i).
\end{aligned} \tag{4.6}$$

Indeed, the result (4.3) is in agreement with the genus g chiral correlator defined in [17] after taking the $\alpha' \rightarrow 0$ limit is taken. We remark that the result is analogous for the case of odd spin structure; the chiral correlator is resolved into the ambitwistor correlator followed by corrections of order $\frac{1}{\alpha'}$ [30].

Regarding this choice of local projection, there is one subtlety that to our knowledge has not been explicitly considered in previous analyses of the ambitwistor string. The choice of PCO insertions must be made in a manner that avoids spurious singularities. Indeed, this can always be done locally by picking PCOs that depend holomorphically on the bosonic moduli. In piecing together such local descriptions, it is not known if a globally consistent prescription to perform localisation in the ambitwistor string can be found. In practice however, it turns out that such a selection of PCOs exists for the ambitwistor string up to two loop order [17, 30] as a result of the fact that the integral ultimately localises on a discrete set of points in $\mathcal{M}_{2,n}$. Proving this at higher genus requires further study.

For readers familiar with the picture changing operator formalism, we point out that what we have done basically is to derive the ansatz due to Friedan, Martinec and Shenker [31] using the language of supermoduli spaces. Historically, it was observed that this ansatz was poorly defined and led to amplitudes dependent on the choice of PCO insertions [29]. This is due to the fact that the projection was done globally, which as we have noted is generically not possible. It is in accordance with this that we have chosen to work in small neighbourhoods. In the context of the full superstring, local descriptions must be carefully glued together [25, 32]. Accordingly, it should be kept in mind that we have proved that the integrand $\mathcal{I}_{g,n}^{\alpha'}[\delta]$ matches the half integrand in the ambitwistor string in the tensionless limit given a specific local configuration of picture changing operators. The subtleties involved with gluing might however pose a problem if we actually want to perform the integration over the moduli space for the ambitwistor string at higher genus.

5 Conclusion

In this letter, we have proved the claim that the chiral superstring integrand in the RNS formalism is resolved into a Koba-Nielsen term and an α' dependent integrand that reduces to the half integrand of the ambitwistor string in the tensionless limit. We made use of the chiral splitting theorem due to d'Hoker and Phong to define a chiral correlation function for n NS states, which we evaluated on the supermoduli space $\mathfrak{M}_{g,n}$. Due to potential issues involving non-projectedness at higher genus, we had to work on small neighbourhoods in $\mathcal{M}_{g,n}$, onto which a projection is possible. Evaluating the chiral correlation function on

the ordinary moduli space in this fashion, we see that in the limit of vanishing tension the chiral correlator reproduced the chiral correlation function of the ambitwistor string.

A full understanding of the tensionless limit of RNS superstrings and its relation to the ambitwistor string requires clarifying the behaviour of the higher genus Koba-Nielsen factor in the Gross-Mende limit [33, 34]. Due to an infinite number of saddle points on the universal covering space of $\mathcal{M}_{g,n}$, the naïve sum over solutions of the scattering equations cannot be taken (see [35] for the complications arising even at genus zero). Understanding this limit on the non-separating divisors of $\mathcal{M}_{g,n}$ would be especially relevant in understanding the $\alpha' \rightarrow 0$ limit (see [36] for a discussion of the genus zero case.). The interplay (and potential equivalence or inequivalence) of the two limits would have important implications for the duality between colour and kinematics [37, 38].

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