A Young diagram expansion of the hexagonal Wilson loop (amplitude) in $\mathcal{N} = 4$ SYM

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ABSTRACT: We shall interpret the null hexagonal Wilson loop (or, equivalently, six gluon scattering amplitude) in 4D $\mathcal{N} = 4$ Super Yang-Mills, or, precisely, an integral representation of its matrix part, via an ADHM-like instanton construction. In this way, we can apply localisation techniques to obtain combinatorial expressions in terms of Young diagrams. Then, we use our general formula to obtain explicit expressions in several explicit cases. In particular, we discuss those already available in the literature and find exact agreement. Moreover, we are capable to determine explicitly the denominator (poles) of the matrix part, and find some interesting recursion properties for the residues, as well.

KEYWORDS: Wilson, 't Hooft and Polyakov loops, Supersymmetric Gauge Theory, AdS-CFT Correspondence, Solitons Monopoles and Instantons

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Contents

1	Elements of Wilson loops/amplitudes in $\mathcal{N} = 4$ SYM	1
2	Scalars	4
2.1	Connection to the $\mathcal{N} = 2$ (Ω background) SYM partition function	4
2.2	Matrix equations for the hexagonal Wilson loop	6
2.3	The set of all admissible diagrams	7
2.4	Tangent space decomposition	9
2.5	Matrix part as a sum over diagrams	10
2.5.1	Example $n = 1$	11
2.5.2	Example $n = 2, 3, \dots$	13
2.6	Asymptotic factorisation and recursion property of the residue	14
3	Fermions	16
3.1	Matrix equations and their consequent combinatorial expression	16
3.1.1	The cases $n = 1, n = 2, n = 3, \dots$	19
3.1.2	Asymptotic factorisation and recursion property of the residue	20
4	Young diagram representation for MHV, NMHV and N^2MHV amplitudes	21
4.1	Matrix equations	22
4.2	The fixed points of the moduli space	23
4.3	Representation of $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$ as a sum over diagrams	26
5	Explicit results for $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$	27
5.1	Cases with $r_b = 1$	28
5.2	Cases with $r_b = 2$	29
5.3	Cases with $r_b = 3$	30
5.4	Cases with $r_b = 0$ or $r_b = 4$	30
6	Conclusions and perspectives	31

1 Elements of Wilson loops/amplitudes in $\mathcal{N} = 4$ SYM

Among the different supersymmetric gauge theories a special rôle is played by the maximal one in 4D, $\mathcal{N} = 4$ Super Yang-Mills (SYM), with gauge group $SU(N_c)$ and dimensionless coupling constant g_{YM} . Needless to say that we are interested in gauge theories for phenomenological reasons and in this particular one because it is conformal, albeit there are at least two other connected motivations. One is that this theory forms one side of the

most known example of AdS/CFT correspondence [1–3], as it lives on the $4d$ Minkowski boundary of its gravitational *dual*, the type IIB superstring theory on $AdS_5 \times S^5$. The second reason is that the latter appears to be classically integrable in the sense that it can be written in Lax form [4], and moreover quantum integrable at least in the planar limit, $N_c \rightarrow \infty$ with $g_{YM} \rightarrow 0$ so that the 't Hooft coupling $\lambda \equiv N_c g_{YM}^2 = 16\pi^2 g^2$ stays fixed, in that $\mathcal{N} = 4$ SYM shows remarkable connections with $1+1$ dimensional integrable models [5–17]. The presence of integrability has opened the way to a better comprehension and partial proof of the aforementioned correspondence.

More precisely, quantum integrability was discovered for the spectral problem of $\mathcal{N} = 4$, i.e. the computation of the anomalous dimensions of local (single trace) gauge invariant operators. But, more recently, it played an important rôle also in the evaluation of null polygonal Wilson loops (Wls), which are dual to and the same as gluon scattering amplitudes [18–20]; but in a different, though connected, guise.

In a nutshell, $\mathcal{N} = 4$ SYM, as any conformal quantum field theory, enjoys an Operator Product Expansion (OPE) of the (renormalised) null polygonal Wilson loops [21], the simplest ones of non-local nature. This method has an intrinsic non-perturbative nature, given by the collinearity expansion of the null edges which gives rise to an infinite series over the number of particles (cf. *infra*). This OPE series was developed for the Wls in $\mathcal{N} = 4$ SYM by employing the underlying integrability of the theory, which manifests itself in the flux-tube, dual to the Gubser, Klebanov and Polyakov (GKP) [22] string. The connection with the spin chain picture was initiated in [23, 24], but the OPE series for the Wilson loop was developed in a series of papers [25–27], through the so-called *pentagon approach*. Briefly, the proposal was to write the expectation value of the Wl as an infinite sum over intermediate excitations on the GKP string vacuum: gluons and their bound states, fermions, antifermions and, finally, scalars. In this way, it is reminiscent of the Form Factor (FF) spectral (large distance) expansion of the correlation functions in integrable $2d$ quantum field theories, and the pentagonal operator has been identified [25, 28–30] with a specific *conical twist* field [31, 32]. Therefore, this methodology needs to know the dispersion relations of the GKP string excitations [33] and the $2d$ scattering factors between them [34–36]: these quantities can be gained by filling up (with infinite Bethe roots) the original BPS vacuum of the single trace operators (as ferromagnetic vacuum) to construct the GKP (or anti-ferromagnetic) vacuum and the excitations thereof.

The validity of the OPE series has been checked in the weak coupling (e.g. [26, 37–39]) and the strong coupling regimes (e.g. [27, 40–45]), with comparisons against gauge and string theory, respectively. In the latter regime, string theory has so far given only the leading order as minimization of the world-sheet area living in the AdS_5 space and insisting on the polygon in $4d$ Minkowski [21, 46–48]. This contribution is given by gluons and (bound states of) fermions in the OPE series [42, 45, 49]. Although scalars elude the minimal area argument, nevertheless they are even more important to understand as they provide a comparable non-perturbative contribution, heuristically derivable from the $O(6)$ non-linear sigma model and corrections as emerging from the low energy action of the string on S^5 : for the hexagon this has been proven in the FF set-up by [40, 41, 50–52]. In general, their OPE contribution to the N -sided polygon is a form factors series of a $(N-4)$ -

point correlation function of a string model, which reduces to the $O(6)$ non-linear sigma-model in the strong coupling [40, 41, 50, 51]. In this regime the dynamically generated mass is exponentially suppressed $m \sim e^{-\frac{\sqrt{\lambda}}{4}}$ and entails the short-distance regime for the correlation function and thus a power-law behavior [40]. Thanks to the specific form of this contribution at all couplings, it has been possible to expand it with exact computations in this regime by using integrability (only, so far) derived ideas and methods, so proving it to be of the same order as the one coming from the classical string computation [41, 50, 51]. In this respect, the all coupling form of the contribution of scalars assumes a particular relevance and it will be given a new interpretation in this paper.

In general, if u_i are the rapidities of the GKP string (flux tube), the hexagonal Wilson loops can be represented as the OPE

$$W = \sum_{\mathbf{n}} \frac{1}{S_{\mathbf{n}}} \int \prod_{i=1}^n \frac{du_i}{2\pi} \Pi_{\text{dyn}}(\{u_i\}) \Pi_{FF}(\{u_i\}) \Pi_{\text{mat}}(\{u_i\}). \quad (1.1)$$

over all admissible $\mathbf{n}$ particle states. The explicit form of Π_{dyn} and Π_{FF} can be found e.g. in [54] and $S_{\mathbf{n}}$ is the symmetry factor: given $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2 + \dots + \mathbf{n}_k$, where $\mathbf{n}_j$ are the number of identical particles of a given kind j , then $S_{\mathbf{n}} = \mathbf{n}_1! \mathbf{n}_2! \dots \mathbf{n}_k!$. In this paper, we will be interested only in Π_{mat} , which admits an integral representation as multi-integrals over the nested Bethe (or isotopic) roots [36, 45] (see below (4.1) and notice that it depends only on the rapidities of scalars, fermions and antifermions $\{u_i\}$, $\{v_i\}$, $\{\bar{v}_i\}$). For large number of particles the evaluation of these integrals soon becomes formidable. In fact, our aim here is to represent this integral as a combinatorial sum, which instead allows for explicitly calculation also for a large number of particles.

In papers [50, 51] one of the authors (D.F.) and collaborators have focused on the matrix part with only scalars, cf. (2.1), or only fermions respectively, cf. (3.1), namely as multi-integrals over the nested Bethe roots of $SO(6)$ or $SU(4)$, respectively. They systematically evaluated by residues the result and encoded it in some Young diagram combinatorics [41, 50, 51]. This method gives rather explicit final formulæ in terms of simple rational functions and is reminiscent of the pole contributions to the instanton partition function of $\mathcal{N} = 2$ SYM [55]. But still the diagrams are rather different and comparison is not so strict, neither physically motivated. This is why in the present paper, we want to draw a more refined correspondence between the two methods by making explicit and adequate reference to instantons, namely the equivariant localization technique and its related Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [56]. Importantly, we can in this way treat the most general situation with any number of scalars *and* fermions and reach the simple form of sum over multiple Young diagrams of certain shapes, where each term is the inverse of a nice factorized polynomial expression.

In this perspective, the paper is organized as follows: in section 2, we consider the matrix part of MHV amplitudes where one has only $2n$ scalars. Upon taking into account the structure of the integral representation we construct an appropriate complex manifold defined by a system of matrix equations. Then we apply equivariant localization to get a combinatorial representation of the matrix part of Wl. Several applications and consequences of our formula are given as well. In section 3, we extend the above analysis to the

case with an equal number of fermions and antifermions, without scalars. Then, in section 4, we consider the matrix part of the Wl with N_ϕ scalars, N_ψ fermions and $N_{\bar{\psi}}$ antifermions. In this general case too, we suggest for the integral representation (4.1) the matrix equations (4.8)–(4.10), which, via localization, lead to the combinatorial formula presented in subsection 4.3. Eventually, in section 5 we use our general formula (4.29) to establish the result (5.2) for the denominator of $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$. We also have found two interesting residue formulae (5.3) and (5.4). At the end of this section for several specific values of scalars, fermion, antifermions and the bottom R -charge r_b we derive the matrix parts explicitly.

2 Scalars

In this section we rewrite the integral representation of the matrix part of the hexagonal bosonic Wl in such a way to make the connection with the integral representation of instanton partition function in Ω background more obvious. The structure of integral representation suggests how to construct an ADHM-like moduli space which reproduces the same integral via equivariant localization. As usual the denominator encodes the required set of algebraic data (matrices) and the numerator reflects the symmetry properties of equations to be satisfied by these data. Thus, we introduce the appropriate ADHM-like moduli space defined in terms of six matrices subject to three quadratic equations. It is shown that this moduli space admits a $U(1)^{2n+1}$ symmetry which is used to apply equivariant localization technique. Subsection 2.3 is devoted to the classification of fixed points of moduli space under aforementioned $U(1)^{2n+1}$ action. It is shown that each fixed point can be described by an array of $2n$ Young diagrams. Only six types of diagrams (an empty one, a 1-box, two 2-box, a 3-box and a 4-box diagrams shown in figure 2) are allowed. We investigate the tangent space of an arbitrary fixed point and find a closed character formula which encodes the pattern of how a tangent space decomposes into one (complex) dimensional invariant subspaces of the $U(1)^{2n+1}$ action. Finally, we apply localization technique to present the matrix part as a sum over fixed points, i.e. $2n$ -tuples of above young diagrams constructed in subsection 2.3. We use our formula to derive explicitly the first 2 and 4 particles expressions, $\Pi_{\text{mat}}^{(2)}$ and $\Pi_{\text{mat}}^{(4)}$ respectively and find complete agreement with all previous results available in literature.

2.1 Connection to the $\mathcal{N} = 2$ (Ω background) SYM partition function

As for the hexagonal bosonic Wl in $\mathcal{N} = 4$ SYM, the matrix factor accounting for the 2D scattering of $2n$ scalars (of GKP string or flux tube) does not depend on the 't Hooft coupling constant λ and can be written as a multi-integral over the three kinds of nested Bethe Ansatz roots of a $SO(6)$ spin-chain [36, 45], a_k , b_j and c_k [40, 41]:

$$\begin{aligned} \Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n}) = & \frac{1}{(2n)!(n!)^2} \int_{-\infty}^{\infty} \prod_{k=1}^n \frac{da_k}{2\pi} \prod_{j=1}^{2n} \frac{db_j}{2\pi} \prod_{k=1}^n \frac{dc_k}{2\pi} \times \\ & \times \frac{\prod_{i < j}^n g(a_{ij}) \prod_{i < j}^{2n} g(b_{ij}) \prod_{i < j}^n g(c_{ij})}{\prod_{j=1}^{2n} \left(\prod_{i=1}^n f(a_i - b_j) \prod_{k=1}^n f(c_k - b_j) \prod_{l=1}^{2n} f(u_l - b_j) \right)}, \end{aligned} \quad (2.1)$$

where the functions $f(x)$ and $g(x)$ are defined as

$$f(x) = x^2 + 1/4 = (x - i/2)(x + i/2), \quad g(x) = x^2(x^2 + 1) = x^2(x - i)(x + i). \quad (2.2)$$

Inserting this in (2.1) we get

$$\begin{aligned} \Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n}) &= \frac{1}{(2n)!(n!)^2} \int_{-\infty}^{\infty} \prod_{k=1}^n \frac{da_k dc_k}{(2\pi i)^2} \prod_{k=1}^{2n} \frac{db_k}{2\pi i} \times \\ &\times \frac{\prod'_{i,j=1}^n a_{ij} c_{ij} (a_{ij} + i)(c_{ij} + i) \prod'_{\alpha,\beta=1}^{2n} b_{\alpha\beta} (b_{\alpha\beta} + i)}{\prod_{\alpha=1}^{2n} \prod_{i=1}^n (a_i - b_{\alpha} + \frac{i}{2})(b_{\alpha} - a_i + \frac{i}{2})(c_i - b_{\alpha} + \frac{i}{2})(b_{\alpha} - c_i + \frac{i}{2}) \prod_{\ell=1}^{2n} \prod_{\alpha=1}^{2n} (u_{\ell} - b_{\alpha} + \frac{i}{2})(b_{\alpha} - u_{\ell} + \frac{i}{2})}, \end{aligned} \quad (2.3)$$

where the prime on the product symbol indicates that all the vanishing factors (i.e. the factors a_{ii} , c_{ii} , $b_{\alpha,\alpha}$) are omitted. After generalizing the last expression by the substitution $\frac{i}{2} \rightarrow \epsilon$ and shifting the variables of integration by this amount we get

$$\begin{aligned} \Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n}) &= \frac{1}{(2n)!(n!)^2} \int_{-\infty}^{\infty} \prod_{k=1}^n \frac{da_k dc_k}{(2\pi i)^2} \prod_{k=1}^{2n} \frac{db_k}{2\pi i} \times \\ &\times \frac{\prod'_{i,j=1}^n a_{ij} c_{ij} (a_{ij} + 2\epsilon)(c_{ij} + 2\epsilon) \prod'_{\alpha,\beta=1}^{2n} b_{\alpha\beta} (b_{\alpha\beta} + 2\epsilon)}{\prod_{\alpha=1}^{2n} \prod_{i=1}^n (a_i - b_{\alpha} + \epsilon)(b_{\alpha} - a_i + \epsilon)(c_i - b_{\alpha} + \epsilon)(b_{\alpha} - c_i + \epsilon) \prod_{\ell=1}^{2n} \prod_{\alpha=1}^{2n} (u_{\ell} - b_{\alpha})(b_{\alpha} - u_{\ell} + 2\epsilon)}. \end{aligned} \quad (2.4)$$

A similar integral arises in a completely different context, the $N = 2$ SYM theory with gauge group $U(n)$ in Ω -background parameterized by ϵ_1 and ϵ_2 . In the latter case, the k -instanton contribution to the partition function can be written in the form [55, 57]

$$Z_k = \frac{1}{k!} \oint \prod_{l=1}^k \frac{d\phi_l}{2\pi i} \frac{\prod'_{i,j=1}^k \phi_{ij} (\phi_{ij} + \epsilon_1 + \epsilon_2)}{\prod_{i,j=1}^k (\phi_{ij} + \epsilon_1)(\phi_{ij} + \epsilon_2) \prod_{i=1}^k \prod_{\ell=1}^n (\phi_i - a_{\ell} + \epsilon_1 + \epsilon_2)(\phi_i - a_{\ell})}, \quad (2.5)$$

where a_{ℓ} are the gauge expectation values and the prime on the product symbol again means that the vanishing factors should be suppressed. In the case $\epsilon_1 = \epsilon_2 = \epsilon$ this expression is suspiciously similar to (2.4). On the other hand, upon applying localization techniques for the moduli space of instantons, according to the ADHM construction, the full instanton partition function

$$Z_{\text{inst}} = \sum_k Z_k q^k, \quad (2.6)$$

can be arranged as a sum over Young diagrams [58] in the following way

$$Z_{\text{inst}} = \sum_{\vec{Y}} f_{\vec{Y}} q^{|\vec{Y}|}, \quad (2.7)$$

where $\vec{Y}$ is an array of n Young diagrams $\vec{Y} = \{Y_1, Y_2, \dots, Y_n\}$ and $|\vec{Y}|$ is the total number of boxes. q is the instanton counting parameter, related to the gauge coupling constant in the standard manner: $q = \exp 2\pi i \tau$, with $\tau = \frac{i}{g^2} + \frac{\theta}{2\pi}$ the complexified coupling. The coefficients $f_{\vec{Y}}$ are factorized as

$$f_{\vec{Y}} = \prod_{u=1}^n \prod_{v=1}^n \frac{1}{Z_{bf}(a_u, Y_u \mid a_v, Y_v)}. \quad (2.8)$$

This formula can be obtained applying the so called localization technique for moduli space of instantons. Due to the similarities of the contour integral representation of instanton

Figure 1. Represents a quiver diagram where the arrows indicate the linear maps M_{ab} , M_{ba} , M_{bc} , M_{cb} , M_{bu} , M_{ub} and dots the spaces u , b , a , c on which they act.

partition function (2.5) and the matrix part of the hexagonal bosonic Wilson loop (2.4) it is reasonable to expect that there exists a combinatorial expression like (2.7) also for Π_{mat} . In the next two subsections we will find an ADHM-like construction such that the corresponding localization formula (see (2.34), (2.35) and (2.37)) gives results compatible with (2.4).

2.2 Matrix equations for the hexagonal Wilson loop

A nice exposition on how ADHM construction leads to the integral representation can be found in [59]. Here we consider the opposite problem to obtain an ADHM like construction corresponding to (2.4). The ADHM analogy suggests that we have to introduce four vector spaces

- u : a $2n$ complex dimensional space related to the parameters $u_1, u_2, \dots, u_{2n}$
- b : a $2n$ complex dimensional space related to the parameters $b_1, b_2, \dots, b_{2n}$
- a and c : each a n complex dimensional space related to the parameters $a_1, a_2, \dots, a_n$ and $c_1, c_2, \dots, c_n$ respectively

and six linear maps M_{ab} , M_{ba} , M_{bc} , M_{cb} , M_{bu} , M_{ub} acting among these spaces as shown in quiver diagram figure 1. The choice of quiver diagram is dictated by the factors in denominator of figure 1. Notice that M_{bu} and M_{ub} are $2n \times 2n$ matrices, M_{ab} and M_{cb} are $n \times 2n$ matrices and finally M_{ba} and M_{bc} are $2n \times n$ matrices.

The factors $a_{ij}+2\epsilon$, $c_{ij}+2\epsilon$ and $b_{\alpha\beta}+2\epsilon$ in the numerator of integral representation (2.4) suggests that there must be three matrix equations (analogs of complex ADHM), having the structure of operators acting as $a \rightarrow a$, $c \rightarrow c$, $b \rightarrow b$ respectively. A simple choice which eventually leads to results consistent with the integral representation is

$$M_{ab}M_{ba} = 0, \quad (2.9)$$

$$M_{cb}M_{bc} = 0, \quad (2.10)$$

$$M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub} = 0. \quad (2.11)$$

As usual in order to guaranty smoothness of moduli space these equations are supplemented by stability condition [60].

We found that the suitable *Stability condition* reads: there is no proper subspace of b which contains the image M_{bu} and is invariant under two operators $M_{ba}M_{ab}$ and $M_{bc}M_{cb}$. In addition it is required that $M_{ab}M_{bu}$ and $M_{cb}M_{bu}$ are *onto* maps on a , c respectively. The first condition means that if one acts in all possible ways on the image of M_{bu} the entire space b will be recovered.

Let us introduce also transformations T_a, T_b, T_c, T_u acting in respective spaces (e.g. $T_a : a \rightarrow a$ is a $n \times n$ matrix). While T_u is a genuine symmetry (which is the analog of global gauge transformations in $N = 2$ SYM), the remaining T_a, T_b, T_c are auxiliary transformations (the moduli space is found by factorizing space of solutions over these auxiliary transformations). In addition we introduce also transformation $T_\epsilon \in C^*$ (C^* is the group of multiplication by nonzero complex numbers). Let us quote below the transformation laws of matrices M

$$M_{ab} \rightarrow T_\epsilon T_a M_{ab} T_b^{-1}, \quad M_{ba} \rightarrow T_\epsilon T_b M_{ba} T_a^{-1}, \quad (2.12)$$

$$M_{cb} \rightarrow T_\epsilon T_c M_{cb} T_b^{-1}, \quad M_{bc} \rightarrow T_\epsilon T_b M_{bc} T_c^{-1}, \quad (2.13)$$

$$M_{bu} \rightarrow T_\epsilon^2 T_b M_{bu} T_u^{-1}, \quad M_{ub} \rightarrow T_u M_{ub} T_b^{-1}. \quad (2.14)$$

Notice that the above rules follow the pattern of factors in denominator of (2.4) (e.g. the first transformation rule matches with the factor $(a_i - b_\alpha + \epsilon)$).

Moduli space: by definition a point in moduli space $\mathcal{M}_n$ is a set of matrices

$\{M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}\}$ satisfying the equations (2.9)–(2.11) and the stability condition together with equivalence relation

$$\begin{aligned} & \{M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}\} \\ & \sim \{T_a M_{ab} T_b^{-1}, T_b M_{ba} T_a^{-1}, T_b M_{bc} T_c^{-1}, T_c M_{cb} T_b^{-1}, T_b M_{bu}, M_{ub} T_b^{-1}\}. \end{aligned} \quad (2.15)$$

It is straightforward to check that the matrix equations (2.9)–(2.11) respect these transformations. The left hand sides of matrix equations get transform as

$$M_{ab}M_{ba} \rightarrow T_\epsilon^2 T_a M_{ab} M_{ba} T_a^{-1}, \quad (2.16)$$

$$M_{cb}M_{bc} \rightarrow T_\epsilon^2 T_c M_{cb} M_{bc} T_c^{-1}, \quad (2.17)$$

$$M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub} \rightarrow T_\epsilon^2 T_b (M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub}) T_b^{-1} \quad (2.18)$$

The equations (2.9)–(2.11) are designed so that their transformation laws exactly match with corresponding factors in the numerators of (2.4). For example the equation (2.9) transforms as (2.16) which obviously agrees with the factor $a_{ij} + 2\epsilon$ in (2.4).

2.3 The set of all admissible diagrams

To apply localisation we need to find all fixed points under transformations T_ϵ, T_u . Thus a fixed point is a set of matrices $\{M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}\}$ satisfying the equations (2.9)–(2.11) and stability condition, invariant under T_ϵ, T_u up to auxiliary transformations T_a, T_b, T_c . It is possible to show that at fixed points the matrix $M_{ub} = 0$, so

Figure 2. The list of allowed diagrams. The diagrams are labeled by $0, 1, 2, \tilde{2}, 3, 4$ which indicate the number of boxes. Three dotted lines correspond to the spaces a, b, c such that a box lying on one of these lines represents a basis vector of the respective space.

that due to (2.11)

$$M_{ba}M_{ab} = M_{bc}M_{cb}. \quad (2.19)$$

Let us choose the basis vectors $e_1, e_2, \dots, e_{2n}$ in space u to be eigenvectors of transformation T_u :

$$T_u e_i = T_{u_i} e_i. \quad (2.20)$$

We introduce a useful graphical interpretation as follows:

- To each e_i such that $M_{bu}e_i = 0$ we associate a dot (empty diagram).
- To each e_i such that $M_{bu}e_i \neq 0 \equiv v$ we associate a box . Thus a box represents a non zero basis vector in space b .
- If $M_{ab}v_i \neq 0$ then we can add a extra box (NE direction) , notice this new box represents a vector in the space a .

- If $M_{ba}M_{ab}v_i \neq 0$ we can add a third box (in SE direction) , since we have (2.19) $M_{ba}M_{ab}v = M_{bc}M_{cb}v$ we must add also a box representing $M_{cb}v$ this

results

- If $M_{ab}v_i \neq 0$ and $M_{cb}v_i \neq 0$ then it gives rise to ,
- Because the spaces a and c enter our construction in a symmetric way so obviously we have in addition the diagram .

Due to the matrix equations (2.9) and (2.10) it is not possible to enlarge these diagrams further on. So we conclude that the diagrams listed in figure 2 are the only possible ones.

Since the dimension of u is $2n$ we should have $2n$ diagrams chosen from the list depicted in figure 2. Let n_i be the number of diagrams of type i , $i = 0, 1, 2, \tilde{2}, 3, 4$, then

$$n_4 + n_3 + n_2 + n_{\tilde{2}} + n_1 + n_0 = 2n, \quad (2.21)$$

The dimensions of b , a and c are $2n$, n and n respectively, so

$$2n_4 + n_3 + n_2 + n_{\tilde{2}} + n_1 = 2n, \quad n_4 + n_3 + n_2 = n, \quad n_4 + n_3 + n_{\tilde{2}} = n. \quad (2.22)$$

The last four conditions lead to

$$n_4 = n_0, \quad n_1 = n_3, \quad n_2 = n_{\tilde{2}}, \quad n_0 + n_1 + n_2 = n. \quad (2.23)$$

2.4 Tangent space decomposition

To apply localization formula we have to describe the tangent spaces of $\mathcal{M}_n$ at the fixed points. The procedure is standard: from the total space where the unconstrained variation δM lives, one “subtracts” subspaces corresponding to equations (2.9)–(2.11) and auxiliary transformations. Notice that (cf. transformation laws (2.12))

$$\delta M_{ab} \in T_\epsilon a \otimes b^*, \quad \delta M_{ba} \in T_\epsilon b \otimes a^* \quad (2.24)$$

$$\delta M_{cb} \in T_\epsilon c \otimes b^*, \quad \delta M_{bc} \in T_\epsilon b \otimes c^* \quad (2.25)$$

$$\delta M_{bu} \in T_\epsilon^2 b \otimes u^*, \quad \delta M_{ub} \in u \otimes b^*. \quad (2.26)$$

Obviously the subspace corresponding to the three equations (2.9)–(2.11) is the direct sum of three terms

$$T_\epsilon^2 a \otimes a^* \oplus T_\epsilon^2 c \otimes c^* \oplus T_\epsilon^2 b \otimes b^*. \quad (2.27)$$

Finally the remaining part corresponding to auxiliary transformations $T_a \in a \otimes a^*$, $T_b \in b \otimes b^*$, $T_c \in c \otimes c^*$ simply is

$$a \otimes a^* \oplus b \otimes b^* \oplus c \otimes c^*. \quad (2.28)$$

Combining all together for the tangent space character we get

$$\begin{aligned} \chi_{\tilde{Y}} &= T_\epsilon ((a + c)b^* + b(a^* + c^*)) + T_\epsilon^2 bu^* + ub^* \\ &\quad - (T_\epsilon^2 + 1)(aa^* + bb^* + cc^*), \end{aligned} \quad (2.29)$$

where (without changing notation) we have replaced the spaces by their character (with respect to T_ϵ , T_u transformations) as follows: the character of space u can be written as (see (2.20))

$$u = T_{u_1} + T_{u_2} + T_{u_3} + \dots + T_{u_{2n}}, \quad u^* = T_{u_1}^{-1} + T_{u_2}^{-1} + T_{u_3}^{-1} + \dots + T_{u_{2n}}^{-1} \quad (2.30)$$

as for the spaces a , b and c we will have

$$a = \sum_{i=1}^{2n} a_i, \quad c = \sum_{i=1}^{2n} c_i, \quad b = \sum_{i=1}^{2n} b_i, \quad (2.31)$$

where

$$\begin{aligned} a_k &= \begin{cases} T_\epsilon T_{u_k} & \text{if } Y = 2, 3, 4 \\ 0 & \text{if } Y = 0, 1, \tilde{2} \end{cases}, \quad c_k = \begin{cases} T_\epsilon T_{u_k} & \text{if } Y = \tilde{2}, 3, 4 \\ 0 & \text{if } Y = 0, 1, 2 \end{cases} \\ b_k &= \begin{cases} T_{u_k} & \text{if } Y = 1, 2, \tilde{2}, 3 \\ (T_\epsilon^2 + 1)T_{u_k} & \text{if } Y = 4 \\ 0 & \text{if } Y = 0 \end{cases} \end{aligned} \quad (2.32)$$

	0	1	2	$\tilde{2}$	3	4
0	0	1	1	1	1	$1 + T_\epsilon^{-2}$
1	T_ϵ^2	0	1	1	2	1
2	T_ϵ^2	T_ϵ^2	0	$T_\epsilon^2 + 1$	1	1
$\tilde{2}$	T_ϵ^2	T_ϵ^2	$T_\epsilon^2 + 1$	0	1	1
3	T_ϵ^2	$2T_\epsilon^2$	T_ϵ^2	T_ϵ^2	0	1
4	$T_\epsilon^4 + T_\epsilon^2$	T_ϵ^2	T_ϵ^2	T_ϵ^2	T_ϵ^2	0

Table 1. This table gives $X(T_{u_i}, Y_i | T_{u_j}, Y_j)$ the factor $T_{u_i} T_{u_j}^{-1}$ is suppressed for example $X(u_i, 4 | u_j, 4) = 0$, $X(u_i, 4 | u_j, 3) = T_\epsilon^2 T_{u_i} T_{u_j}^{-1}$ and $X(u_i, 4 | u_j, 0) = (T_\epsilon^2 + T_\epsilon^4) T_{u_i} T_{u_j}^{-1}$.

Notice that since the spaces a and c are n dimensional exactly n terms in each (2.31) must be zero. The characters a^* , b^* , c^* can be found by replacing $a_i \rightarrow a_i^{-1}$, $b_i \rightarrow b_i^{-1}$, $c_i \rightarrow c_i^{-1}$. For a given fixed point specified by diagrams $\vec{Y} = \{Y_1, Y_2, \dots, Y_{2n}\}$ the character has the following structure

$$\chi_{\vec{Y}} = \sum_{i,j=1}^{2n} X(T_{u_i}, Y_i | T_{u_j}, Y_j). \quad (2.33)$$

The summands $X(u_i, Y_i | u_j, Y_j)$ can be easily read off from (2.29):

$$X(T_{u_i}, Y_i | T_{u_j}, Y_j) = T_\epsilon \left(\frac{a_i}{b_j} + \frac{c_i}{b_j} + \frac{b_i}{a_j} + \frac{b_i}{c_j} \right) + T_\epsilon^2 T_{u_j}^{-1} b_i + \frac{T_{u_i}}{b_j} - (T_\epsilon^2 + 1) \left(\frac{a_i}{a_j} + \frac{b_i}{b_j} + \frac{c_i}{c_j} \right).$$

For convenience all $X(T_{u_i}, Y_i | T_{u_j}, Y_j)$'s are listed in table 1. It is essential that though (2.29) besides positive terms includes also negative ones, nevertheless in final formula (2.33) and table 1 all terms are positive, which is required by self consistency (the initial space contains all the sub spaces to be factored out due to matrix equations and equivalence relation (2.15)). Though we did not provide a mathematically rigorous proof, the above property strongly suggests that the moduli space $\mathcal{M}_n$ is a *smooth* algebraic manifold with dimension $4n^2$.

2.5 Matrix part as a sum over diagrams

For given n the matrix part of the hexagonal WI can be represented as

$$\Pi_{\text{mat}}^{(2n)} = \sum_{\vec{Y}} F_{\vec{Y}}, \quad (2.34)$$

where the sum is over all collections of $2n$ diagrams $\vec{Y} \equiv \{Y_1, Y_2, Y_3, \dots, Y_{2n}\}$ from the list given in figure 2 subject to constraints:

- The total number of boxes in middle line (related to the space b) is equal to $2n$.
- Both the upper and lower lines (related to spaces a and c) contain n boxes each.

	0	1	2	$\tilde{2}$	3	4
0	1	u_{ij}	u_{ij}	u_{ij}	u_{ij}	$u_{ij}(u_{ij} - i)$
1	$u_{ij} + i$	1	u_{ij}	u_{ij}	u_{ij}^2	u_{ij}
2	$u_{ij} + i$	$u_{ij} + i$	1	$u_{ij}(u_{ij} + i)$	u_{ij}	u_{ij}
$\tilde{2}$	$u_{ij} + i$	$u_{ij} + i$	$u_{ij}(u_{ij} + i)$	1	u_{ij}	u_{ij}
3	$u_{ij} + i$	$(u_{ij} + i)^2$	$u_{ij} + i$	$u_{ij} + i$	1	u_{ij}
4	$(u_{ij} + i)(u_{ij} + 2i)$	$u_{ij} + i$	$u_{ij} + i$	$u_{ij} + i$	$u_{ij} + i$	1

Table 2. This table gives $P(u_i, Y_i | u_j, Y_j)$ for all diagrams i.e. $Y_i, Y_j \in \{0, 1, 2, \tilde{2}, 3, 4\}$. For example $P(u_i, 2 | u_j, 1) = u_{ij} + i$.

Figure 3. The list of allowed diagram pairs for the case when $n = 1$ is $\{0, 4\}$, $\{1, 3\}$, $\{\tilde{2}, 2\}$, $\{4, 0\}$, $\{3, 1\}$, $\{2, \tilde{2}\}$.

A little consideration ensures that above conditions are equivalent to the requirements (2.23) where n_i ($i = 0, 1, 2, \tilde{2}, 3, 4$) is the number of diagrams of type i (see figure 2) in $\vec{Y}$. We can also deduce that if a diagram of type 0 (1 and 2) enters in $\vec{Y}$ then 4 (3 and $\tilde{2}$ respectively) must be present too.

The summands $F_{\vec{Y}}$ are given by

$$F_{\vec{Y}} = \prod_{i,j=1}^{2n} \frac{1}{P(u_i, Y_i | u_j, Y_j)}. \quad (2.35)$$

Here $P(a, \lambda | b, \mu)$ is a product of factors obtained from the terms of $X(T_{u_i}, Y_i | T_{u_j}, Y_j)$ by usual substitution

$$n T_{\epsilon}^m T_{u_i} T_{u_j}^{-1} \rightarrow (m\epsilon + u_{ij})^n, \quad (2.36)$$

where m and n are integers. The final result (alternatively represented in table 2) is

$$P(a, \lambda | b, \mu) = \begin{cases} 1 & \text{if } \lambda = \mu \\ (a - b + 2\epsilon) (1 + \delta_{4, |\lambda| + |\mu|} (-1 + a - b + \epsilon(|\lambda| - |\mu|))) & \text{if } |\lambda| \geq |\mu| \\ (a - b) (1 + \delta_{4, |\lambda| + |\mu|} (-1 + a - b + 2\epsilon + \epsilon(|\lambda| - |\mu|))) & \text{if } |\lambda| \leq |\mu| \end{cases} \quad (2.37)$$

To derive $P(a, 2 | b, \tilde{2})$ we can use both the second and third lines of (2.37) because they give the same result.

2.5.1 Example $n = 1$

When $n = 1$, as explained in subsection 2.5, the dimensions of the spaces u , b , a , c are 2, 2, 1, 1 respectively. So the sum in (2.34) is over the following six pairs $\vec{Y} = \{Y_1, Y_2\}$:

$\{0, 4\}$, $\{4, 0\}$, $\{1, 3\}$, $\{3, 1\}$, $\{2, \tilde{2}\}$, $\{\tilde{2}, 2\}$ (see figure 3). From (2.35) and the fact that $P(u_i, Y_i|u_i, Y_i) = 1$, $i = 1, 2$ (this can be seen from (2.37) directly) we will get

$$F_{\vec{Y}} = \frac{1}{P(u_1, Y_1|u_2, Y_2)P(u_2, Y_2|u_1, Y_1)}. \quad (2.38)$$

Below we derive $F_{\vec{Y}}$ for all six $\vec{Y}$ s.

1. When $\vec{Y} = \{0, 4\}$ that is $Y_1 = 0$ and $Y_2 = 4$ then from (2.37) we will obtain $P(u_1, Y_1|u_2, Y_2) = u_{12}(u_{12} - 2\epsilon)$ and $P(u_2, Y_2|u_1, Y_1) = (u_{21} + 2\epsilon)(u_{21} + 4\epsilon)$. Inserting these in (2.38) we will obtain

$$F_{\{0,4\}} = \frac{1}{u_{12}(u_{12} - 2\epsilon)^2(u_{12} - 4\epsilon)}. \quad (2.39)$$

2. When $\vec{Y} = \{4, 0\}$ that is $Y_1 = 4$ and $Y_2 = 0$ then from (2.37) we will obtain $P(u_1, Y_1|u_2, Y_2) = (u_{12} + 2\epsilon)(u_{12} + 4\epsilon)$ and $P(u_2, Y_2|u_1, Y_1) = u_{21}(u_{21} - 2\epsilon)$, hence from (2.38) we obtain

$$F_{\{4,0\}} = \frac{1}{u_{12}(u_{12} + 2\epsilon)^2(u_{12} + 4\epsilon)}. \quad (2.40)$$

3. When $\vec{Y} = \{2, \tilde{2}\}$ that is $Y_1 = 2$ and $Y_2 = \tilde{2}$ then $P(u_1, Y_1|u_2, Y_2) = (u_{12} + 2\epsilon)u_{12}$ and $P(u_2, Y_2|u_1, Y_1) = u_{21}(u_{21} + 2\epsilon)$, so that

$$F_{\{2,\tilde{2}\}} = \frac{1}{(u_{12} - 2\epsilon)u_{12}^2(u_{12} + 2\epsilon)}. \quad (2.41)$$

4. When $\vec{Y} = \{\tilde{2}, 2\}$ that is $Y_1 = \tilde{2}$ and $Y_2 = 2$ then $P(u_1, Y_1|u_2, Y_2) = u_{12}(u_{12} + 2\epsilon)$ and $P(u_2, Y_2|u_1, Y_1) = (u_{21} + 2\epsilon)u_{21}$ and

$$F_{\{\tilde{2},2\}} = \frac{1}{(u_{12} - 2\epsilon)u_{12}^2(u_{12} + 2\epsilon)}. \quad (2.42)$$

5. $\vec{Y} = \{1, 3\}$ then $P(u_1, Y_1|u_2, Y_2) = u_{12}^2$ and $P(u_2, Y_2|u_1, Y_1) = (u_{21} + 2\epsilon)^2$ thus

$$F_{\{1,3\}} = \frac{1}{(u_{12} - 2\epsilon)^2 u_{12}^2}. \quad (2.43)$$

6. $\vec{Y} = \{3, 1\}$ so $P(u_1, Y_1|u_2, Y_2) = (u_{12} + 2\epsilon)^2$, $P(u_2, Y_2|u_1, Y_1) = u_{21}^2$ and we get

$$F_{\{3,1\}} = \frac{1}{u_{12}^2(u_{12} + 2\epsilon)^2}. \quad (2.44)$$

Inserting these results in (2.34) we find

$$\Pi_{\text{mat}}^{(2)} = F_{\{0,4\}} + F_{\{4,0\}} + F_{\{2,\tilde{2}\}} + F_{\{\tilde{2},2\}} + F_{\{0,3\}} + F_{\{3,0\}} = \frac{6}{(u_{12}^2 + 1)(u_{12}^2 + 4)},$$

where the value $\epsilon = i/2$ is restored. This result is in agreement with the ones obtained with other approaches.

2.5.2 Example $n = 2, 3, \dots$

Now we consider the case with $n = 2$, so our spaces u , b , a and c have dimensions 4, 4, 2, 2 correspondingly. From the constraints explained in section 2.5 one can see that there are 90 possibilities for $\vec{Y} = \{Y_1, Y_2, Y_3, Y_4\}$. Here are some of them: $\{2, 2, \tilde{2}, \tilde{2}\}$, $\{2, \tilde{2}, 2, \tilde{2}\}$, $\{2, \tilde{2}, \tilde{2}, 2\}$, $\{\tilde{2}, 2, 2, \tilde{2}\}$, $\{\tilde{2}, 2, \tilde{2}, 2\}$, $\{\tilde{2}, \tilde{2}, 2, 2\}$, $\{3, 2, \tilde{2}, 1\}$, $\{3, 2, 1, \tilde{2}\}$, $\{3, \tilde{2}, 2, 1\}$, $\{3, \tilde{2}, 1, 2\}$, $\{3, 1, 2, \tilde{2}\}$, $\dots$. For a given $\vec{Y}$ from (2.35) and (2.37) we can derive the corresponding $F_{\vec{Y}}$. As an example let us derive $F_{\{2,2,\tilde{2},\tilde{2}\}}$ i.e. $Y_1 = 2, Y_2 = 2, Y_3 = \tilde{2}, Y_4 = \tilde{2}$. From (2.37) one can see straightforwardly that

$$\begin{aligned} P(u_1, Y_1|u_2, Y_2) &= 1; \quad P(u_1, Y_1|u_3, Y_3) = u_{13}(u_{13} + 2\epsilon); \quad P(u_1, Y_1|u_4, Y_4) = u_{14}(u_{14} + 2\epsilon); \\ P(u_2, Y_2|u_1, Y_1) &= 1; \quad P(u_2, Y_2|u_3, Y_3) = u_{23}(u_{23} + 2\epsilon); \quad P(u_2, Y_2|u_4, Y_4) = u_{24}(u_{24} + 2\epsilon); \\ P(u_3, Y_3|u_1, Y_1) &= u_{31}(u_{31} + 2\epsilon); \quad P(u_3, Y_3|u_2, Y_2) = u_{32}(u_{32} + 2\epsilon); \quad P(u_3, Y_3|u_4, Y_4) = 1; \\ P(u_4, Y_4|u_1, Y_1) &= u_{41}(u_{41} + 2\epsilon); \quad P(u_4, Y_4|u_2, Y_2) = u_{42}(u_{42} + 2\epsilon); \quad P(u_4, Y_4|u_3, Y_3) = 1. \end{aligned}$$

After inserting these in (2.35) we find

$$F_{\{2,2,\tilde{2},\tilde{2}\}} = \left(u_{13}^2 u_{14}^2 u_{23}^2 u_{24}^2 (u_{13}^2 - 4\epsilon^2)(u_{14}^2 - 4\epsilon^2)(u_{23}^2 - 4\epsilon^2)(u_{24}^2 - 4\epsilon^2) \right)^{-1}. \quad (2.45)$$

The other $F_{\vec{Y}}$ are derived similarly. From (2.34) one obtains

$$\Pi_{\text{mat}}^{(4)} = \frac{N^{(4)}}{\prod_{i=1}^4 \prod_{j=1}^{i-1} (u_{ij}^2 + 1) (u_{ij}^2 + 4)}. \quad (2.46)$$

where

$$\begin{aligned} N^{(4)} &= 36 \left(84e_1^4 - 57e_3e_1^3 + 19e_2^2e_1^2 + 9e_3^2e_1^2 - 448e_2e_1^2 + 183e_4e_1^2 + 1068e_1^2 - \right. \\ &\quad \left. - 6e_2^2e_3e_1 + 167e_2e_3e_1 - 287e_3e_1 - 72e_3e_4e_1 + e_2^4 - 54e_2^3 + 693e_2^2 - \right. \\ &\quad \left. - 45e_3^2 + 144e_4^2 - 2848e_2 + 24e_2^2e_4 - 488e_2e_4 + 1148e_4 + 3504 \right). \end{aligned}$$

Here the e_i ($i = 1, 2, 3, 4$) are elementary symmetric polynomials in four variables

$$e_1 = \sum_{i=1}^4 u_i; \quad e_2 = \sum_{1 \leq i < j \leq 3} u_i u_j; \quad e_3 = \sum_{1 \leq i < j < k \leq 3} u_i u_j u_k; \quad e_4 = u_1 u_2 u_3 u_4. \quad (2.47)$$

Next let us consider the case when $n = 3$ then $\Pi_{\text{mat}}^{(6)}$ (2.34) is given in terms of 1860 summands. To generate all of them via (2.35) is a matter of few seconds by using for example mathematica. Bringing the 1860 terms together and canceling out the fake poles is a much more difficult task. The resulting denominator is a very complicated polynomial. Below we present the result for the case when all u_i are very large so the ϵ in (2.37) can be neglected. We have found

$$\Pi_{\text{mat}}^{(6)} = \frac{N^{(6)}}{\prod_{i=1}^6 \prod_{j=1}^{i-1} u_{ij}^4} + O(\epsilon) \quad (2.48)$$

	number of summands of [50]	number of summands of (2.34)
$n = 1$	6	6
$n = 2$	648	90
$n = 3$	537 840	1 860
$n = 4$	1 655 821 440	44 730

Table 3. Comparison of the number of summands entering in the combinatorial expression given in [50] versus our formula.

where

$$N^{(6)} = 216 \left(12e_6e_2^3 + e_4^2e_2^2 - 3e_3e_5e_2^2 + 75e_5^2e_2 + e_1e_4e_5e_2 - 45e_1e_3e_6e_2 - \right. \\ \left. - 126e_4e_6e_2 + 12e_4^3 - 3e_1e_3e_4^2 - 20e_1^2e_5^2 + 405e_6^2 + 9e_1e_3^2e_5 - \right. \\ \left. - 45e_3e_4e_5 + 81e_3^2e_6 + 75e_1^2e_4e_6 - 135e_1e_5e_6 \right)^2. \quad (2.49)$$

Here e_i , $i = 1, 2, \dots, 6$ are elementary symmetric polynomials from six variables. In [50] it was shown that the denominator of $\Pi_{\text{mat}}^{(2n)}$ has the following structure

$$\Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n}) = \frac{N^{(2n)}(u_1, \dots, u_{2n})}{\prod_{i=1}^{2n} \prod_{j=1}^{i-1} (u_{ij}^2 + 1) (u_{ij}^2 + 4)}. \quad (2.50)$$

As one can see from (2.37) or table 2, poles of the form u_{ij} cancel out.

In order to demonstrate the advantage of our approach let us compare it with the method invented in [50]. If we take into consideration the constraints on the diagrams then for fixed n the number of summands (possible $\vec{Y}$) in (2.34) is equal to

$$\sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \frac{(2n)!}{(k_1!k_2!(n-k_1-k_2)!)^2}$$

whereas it can be shown that the number of summands when the method of [50] is applied equals to

$$\sum_{k=0}^n (2n)!(2n-2k)!$$

For the first few numbers of scalars these gives table 3. So we have a good reason to think that our formula is substantially more efficient for both symbolic and numerical computations.

2.6 Asymptotic factorisation and recursion property of the residue

Using integral representation (2.1) it was shown in [50] that if one shifts $2k$ of $2n$ rapidities in $\Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n})$ by $\Lambda \gg 0$, then the following factorization formula holds

$$\Pi_{\text{mat}}^{(2n)}(u_1 + \Lambda, \dots, u_{2k} + \Lambda, u_{2k+1}, \dots, u_{2n}) = \quad (2.51) \\ = \Lambda^{-8k(n-k)} \left(1 + \frac{4k}{\Lambda} \sum_{m=2k+1}^{2n} u_m - 4(n-k) \sum_{i=1}^{2k} \frac{u_i}{\Lambda} + \dots \right) \times \\ \times \Pi_{\text{mat}}^{(2k)}(u_1, \dots, u_{2k}) \Pi_{\text{mat}}^{2(n-k)}(u_{2k+1}, \dots, u_{2n}).$$

Here we will demonstrate that this property follows from (2.34) in a straightforward way. For a diagram Y we denote by $\bar{Y}$ the unique diagram $Y \neq \bar{Y}$ which has $4 - |Y|$ boxes. It is not difficult to see that the arrays of the form

$$\vec{Y}_\Lambda = \{\underbrace{Y_1, Y_2, \dots, Y_k, \bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_k}_{2k}, \underbrace{Y_{2k+1}, Y_{2k+2}, \dots, Y_{n+k}, \bar{Y}_{2k+1}, \bar{Y}_{2k+2}, \dots, \bar{Y}_{n+k}}_{2n-2k}\} \quad (2.52)$$

and those which can be obtained by permutations not mixing the first $2k$ diagrams with the last $2n-2k$ (i.e. those arrays which contain any diagram together with its partner separately in both groups) give the most relevant contributions in above described clustering limit $\Lambda \rightarrow \infty$ as seen from table 2. We observe from (2.37) that the both cases $\lambda \neq \mu$ and $\lambda = \mu$ though seem different, in fact give the same outcome:

$$\begin{aligned} & (P(u_i + \Lambda, \lambda|u_n, \mu)P(u_i + \Lambda, \lambda|u_m, \bar{\mu})P(u_n, \mu|u_i + \Lambda, \lambda)P(u_m, \bar{\mu}|u_i + \Lambda, \lambda))^{-1} \\ & \times (P(u_j + \Lambda, \bar{\lambda}|u_n, \mu)P(u_j + \Lambda, \bar{\lambda}|u_m, \bar{\mu})P(u_n, \mu|u_j + \Lambda, \bar{\lambda})P(u_m, \bar{\mu}|u_j + \Lambda, \bar{\lambda}))^{-1} \\ & = \frac{1}{\Lambda^8} \left(1 - \frac{4}{\Lambda} (u_{im} + u_{jn})\right) + O(\Lambda^{-10}). \end{aligned} \quad (2.53)$$

Hence it is quite straightforward to observe that contributions of arrays of type (2.52) are given by

$$\begin{aligned} F_{\vec{Y}_\Lambda} & \sim \Lambda^{-8k(n-k)} \left(1 + \frac{4k}{\Lambda} \sum_{m=2k+1}^{2n} u_m - \frac{4(n-k)}{\Lambda} \sum_{i=1}^{2k} u_i + \dots\right) \\ & \times F_{\{Y_1, Y_2, \dots, Y_k, \bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_k\}} F_{\{Y_{2k+1}, Y_{2k+2}, \dots, Y_{n+k}, \bar{Y}_{2k+1}, \bar{Y}_{2k+2}, \dots, \bar{Y}_{n+k}\}}. \end{aligned} \quad (2.54)$$

Due to such a nice factorization, summing over all allowed choices of diagrams we arrive at the desired formula (2.51).

In [50] it was demonstrated that

$$Res_{u_2=u_1+2i} \Pi_{\text{mat}}^{(2n)}(u_1, \dots, u_{2n}) = - \frac{\Pi_{\text{mat}}^{(2n-2)}(u_3, \dots, u_{2n})}{2i \prod_{j=3}^{2n} u_{1j} (u_{1j} + i)^2 (u_{1j} + 2i)}. \quad (2.55)$$

This result too can be easily obtained by using our formula for $\Pi_{\text{mat}}^{(2n)}$. Indeed the arrays $\vec{Y}$ with nonzero residues $Res_{u_2=u_1+2i} F_{\vec{Y}} \neq 0$ must have the structure $\vec{Y} = \{4, 0, Y_3, Y_4, \dots, Y_{2n}\}$, as can be seen from table 2. From (2.35)

$$\begin{aligned} & F_{\{4, 0, Y_3, Y_4, \dots, Y_{2n}\}} \\ & = \frac{F_{\{Y_3, Y_4, \dots, Y_{2n}\}}}{P(u_1, 4|u_2, 0)P(u_2, 0|u_1, 4) \prod_{j=3}^{2n} P(u_1, 4|u_j, Y_j)P(u_2, 0|u_j, Y_j)P(u_j, Y_j|u_1, 4)P(u_j, Y_j|u_2, 0)}. \end{aligned}$$

Using table 2 for the residue we get

$$Res_{u_2=u_1+2i} F_{\{4, 0, Y_3, Y_4, \dots, Y_{2n}\}} = - \frac{F_{\{Y_3, Y_4, \dots, Y_{2n}\}}}{2i \prod_{j=3}^{2n} u_{1j} (u_{1j} + i)^2 (u_{1j} + 2i)}. \quad (2.56)$$

Summing over diagrams (see (2.34)) we recover (2.55).

3 Fermions

This section is analogous to the one before, with the only difference that we consider fermions instead of scalars. More precisely, we write down the analogues of ADHM equations that upon localization lead to (3.1). As a result, we recover also a combinatorial representation for $\Pi_{\text{mat}}^{(n)}$ (see (3.2) together with (3.12)). By using our formula, we can perform explicit computations and study different properties of $\Pi_{\text{mat}}^{(n)}$.

3.1 Matrix equations and their consequent combinatorial expression

In this section, we will consider the MHV case with no scalars, $N_\phi = 0$, but equal number of fermions and antifermions $N_\psi = N_{\bar{\psi}} = n$:

$$\Pi_{\text{mat}}^{(n)} = \frac{1}{(n!)^3} \int_{-\infty}^{\infty} \prod_{k=1}^n \left(\frac{da_k db_k dc_k}{(2\pi)^3} \right) \frac{\prod_{i < j}^n g(a_{ij}) g(b_{ij}) g(c_{ij})}{\prod_{i,j}^n f(a_i - b_j) f(c_i - b_j) \prod_{i,\alpha}^n f(v_\alpha - a_i) f(\bar{v}_\alpha - c_i)}, \quad (3.1)$$

where, like for scalars, we have a multi-integral over the three kinds of nested Bethe Ansatz roots of a $\text{SU}(4)$ spin-chain [36, 45], a_k , b_j and c_k and the same functions $g(x)$ and $f(x)$ (2.2) [27, 51]. In this case, too, we want to apply localization in order to obtain a combinatorial representation. It turns out that from the point of view of localization it is more natural to consider a slightly different quantity

$$\Pi_f^{(n)} = (-)^n \Pi_{\text{mat}}^{(n)}. \quad (3.2)$$

With simple manipulations one gets

$$\begin{aligned} \Pi_f^{(n)} &= \frac{(-1)^n}{(n!)^3} \int_{-\infty}^{\infty} \prod_{k=1}^n \left(\frac{da_k db_k dc_k}{(2\pi i)^3} \right) \times \\ &\times \frac{\prod_{i,j=1}^n a_{ij}(a_{ij}+2\epsilon) b_{ij}(b_{ij}+2\epsilon) c_{ij}(c_{ij}+2\epsilon)}{\prod_{i,j}^n (a_i - b_j + \epsilon)(b_j - a_i + \epsilon)(c_i - b_j + \epsilon)(b_j - c_i + \epsilon) \prod_{i,\alpha}^n (v_\alpha - a_i)(a_i - v_\alpha + 2\epsilon)(\bar{v}_\alpha - c_i)(c_i - \bar{v}_\alpha + 2\epsilon)}. \end{aligned} \quad (3.3)$$

As earlier $\epsilon \equiv i/2$ and the vanishing factors should be suppressed in the product with a prime. For this case the ADHM analogy suggests that we have to introduce five vector spaces:

- v and $\bar{v}$: each one is a n complex dimensional space associated with the parameters $v_1, v_2, \dots, v_n$ and $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ respectively
- a , b , and c : each one is a n complex dimensional space associated with the parameters $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$

and eight linear maps M_{va} , M_{va} , $M_{\bar{v}c}$, $M_{c\bar{v}}$, M_{ab} , M_{ba} , M_{cb} and M_{cb} acting among these spaces as demonstrated in quiver diagram figure 4. The numerator of (3.3) is compatible with the following three matrix equations

$$M_{ab} M_{ba} + M_{a\psi} M_{\psi a} = 0, \quad (3.4)$$

$$M_{cb} M_{bc} + M_{c\bar{\psi}} M_{\bar{\psi} c} = 0, \quad (3.5)$$

$$M_{ba} M_{ab} - M_{bc} M_{cb} = 0, \quad (3.6)$$

Figure 4. Represents a quiver diagram where the arrows indicate the linear maps M_{av} , M_{va} , M_{ab} , M_{ba} , M_{bc} , M_{cb} , $M_{c\bar{v}}$, $M_{\bar{v}c}$ and dots the spaces v , a , b , c , $\bar{v}$ on which they act.

that lead to consistent final results. A suitable *stability condition* is: there is no proper subspace of a which contains the image M_{av} and is invariant under $M_{ab}M_{ba}$ and $M_{ab}M_{bc}M_{cb}M_{ba}$ (i.e. one acts in all possible ways on image M_{av} the entire space a will be recovered). Similarly there is no proper subspace of c which contains the image $M_{c\bar{v}}$ and is invariant under $M_{cb}M_{bc}$ and $M_{cb}M_{ba}M_{ab}M_{bc}$. Finally the direct sum of images of the maps $M_{ba}M_{av}$ and $M_{bc}M_{c\bar{v}}$ covers the space b .

We introduce five transformations, three of which T_a , T_b , T_c are auxiliary while the remaining two T_v , $T_{\bar{v}}$ are diagonal matrices corresponding to genuine symmetries. As before the additional transformation $T_\epsilon \in C^*$ related to the parameter ϵ is introduced. The denominator of $\Pi_f^{(n)}$ (3.3) dictates the following rule of transformations

$$M_{ab} \rightarrow T_\epsilon T_a M_{ab} T_b^{-1}, \quad M_{ba} \rightarrow T_\epsilon T_b M_{ba} T_a^{-1} \quad (3.7)$$

$$M_{cb} \rightarrow T_\epsilon T_c M_{cb} T_b^{-1}, \quad M_{bc} \rightarrow T_\epsilon T_b M_{bc} T_c^{-1} \quad (3.8)$$

$$M_{av} \rightarrow T_\epsilon^2 T_a M_{av} T_v^{-1}, \quad M_{va} \rightarrow T_v M_{va} T_a^{-1} \quad (3.9)$$

$$M_{c\bar{v}} \rightarrow T_\epsilon^2 T_c M_{c\bar{v}} T_{\bar{v}}^{-1}, \quad M_{\bar{v}c} \rightarrow T_{\bar{v}} M_{\bar{v}c} T_c^{-1}. \quad (3.10)$$

It is easy to see that the matrix equations (3.4)–(3.6) transform in accordance with the numerator of (3.3).

The fixed points are given by the set of matrices M_{ab} , M_{ba} , M_{bc} , M_{cb} , M_{av} , M_{va} , $M_{c\bar{v}}$ and $M_{\bar{v}c}$ which are subject to equations (3.4)–(3.6) and are invariant under T_v , $T_{\bar{v}}$ and T_ϵ up to auxiliary transformations T_a , T_b and T_c . One can prove that at fixed points

$$M_{va} = 0, \quad M_{\bar{v}c} = 0. \quad (3.11)$$

Proceeding as in the case of scalars we find that the set of admissible diagrams are those depicted in figure 5.

Figure 5. The list of diagrams for the cases with fermions and antifermions only. The indices v and $\bar{v}$ indicate whether a diagram starts from the spaces a or c respectively. As before three dotted lines correspond to the spaces a, b, c .

	0_v	1_v	2_v	3_v	$0_{\bar{v}}$	$1_{\bar{v}}$	$2_{\bar{v}}$	$3_{\bar{v}}$
0_v	1	v_{ij}	v_{ij}	v_{ij}	1	1	1	$v_i - \bar{v}_j - i$
1_v	$v_{ij} + i$	1	v_{ij}	v_{ij}	1	1	$v_i - \bar{v}_j$	1
2_v	$v_{ij} + i$	$v_{ij} + i$	1	v_{ij}	1	$v_i - \bar{v}_j + i$	1	1
3_v	$v_{ij} + i$	$v_{ij} + i$	$v_{ij} + i$	1	$v_i - \bar{v}_j + 2i$	1	1	1
$0_{\bar{v}}$	1	1	1	$\bar{v}_i - v_j - i$	1	$\bar{v}_{ij}$	$\bar{v}_{ij}$	$\bar{v}_{ij}$
$1_{\bar{v}}$	1	1	$\bar{v}_i - v_j$	1	$\bar{v}_{ij} + i$	1	$\bar{v}_{ij}$	$\bar{v}_{ij}$
$2_{\bar{v}}$	1	$\bar{v}_i - v_j + i$	1	1	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	1	$\bar{v}_{ij}$
$3_{\bar{v}}$	$\bar{v}_i - v_j + 2i$	1	1	1	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	1

Table 4. This table lists the factors $P(v_i, Y_i^v | v_j, Y_j^v)$, $P(v_i, Y_i^v | \bar{v}_j, Y_j^{\bar{v}})$, $P(\bar{v}_i, Y_i^{\bar{v}} | v_j, Y_j^v)$ and $P(\bar{v}_i, Y_i^{\bar{v}} | \bar{v}_j, Y_j^{\bar{v}})$ for $Y_i^v \in \{0, 1, 2, 3\}$ and $Y_i^{\bar{v}} \in \{0_{\bar{v}}, 1_{\bar{v}}, 2_{\bar{v}}, 3_{\bar{v}}\}$. For example $P(\bar{v}_i, 2_{\bar{v}} | \bar{v}_j, 1_{\bar{v}}) = \bar{v}_{ij} + i$.

Our final result for fermionic case can be formulated as¹

$$\Pi_f^{(n)} = \sum_{\vec{Y}} F_{\vec{Y}}, \quad (3.12)$$

where the sum is over collections of $2n$ diagrams $\vec{Y} \equiv \{Y_1^v, Y_2^v, \dots, Y_n^v, Y_1^{\bar{v}}, Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}$. Y^v ($Y^{\bar{v}}$) are chosen from the first (second) row of diagrams listed in figure 5. In addition it is required that the total number of boxes in upper, middle and lower lines (related to the spaces a, b and c) contain precisely n boxes each. In terms of Young diagrams the summands in (3.12) can be represented as

$$F_{\vec{Y}} = \prod_{i,j=1}^n \frac{1}{P(v_i, Y_i^v | v_j, Y_j^v) P(v_i, Y_i^v | \bar{v}_j, Y_j^{\bar{v}}) P(\bar{v}_i, Y_i^{\bar{v}} | v_j, Y_j^v) P(\bar{v}_i, Y_i^{\bar{v}} | \bar{v}_j, Y_j^{\bar{v}})}. \quad (3.13)$$

¹In the next section a more general case, a subcase of which is the present one, is considered. The reader can find the expressions for the characters and other details there.

The factors $P(a, \lambda|b, \mu)$ are listed in table 4, which also can be represented by the formula

$$P(a, \lambda^m|b, \mu^n) = \begin{cases} \text{when } m = n & \begin{cases} 1 & \text{if } \lambda = \mu \\ a - b + i & \text{if } \lambda > \mu \\ a - b & \text{if } \lambda < \mu \end{cases} \\ \text{when } m \neq n & \begin{cases} 1 & \text{if } \lambda + \mu \neq 3 \\ a - b + \frac{i}{2}(1 + \lambda - \mu) & \text{if } \lambda + \mu = 3 \end{cases} \end{cases} \quad (3.14)$$

where the upper indices m and n take the values v and $\bar{v}$.

3.1.1 The cases $n = 1$, $n = 2$, $n = 3, \dots$

If $n = 1$ we have four fixed points given by pairs of diagrams $\{0_v, 3_{\bar{v}}\}$, $\{3_v, 0_{\bar{v}}\}$, $\{1_v, 2_{\bar{v}}\}$ and $\{2_v, 1_{\bar{v}}\}$. For the first fixed point $\vec{Y} = \{0_v, 3_{\bar{v}}\}$, that is $Y_1^v = 0_v$, $Y_1^{\bar{v}} = 3_{\bar{v}}$, (3.13) gives

$$F_{\{0_v, 3_{\bar{v}}\}} = \frac{1}{P(v_1, 0_v|v_1, 0_v)P(v_1, 0_v|\bar{v}_1, 3_{\bar{v}})P(\bar{v}_1, 3_{\bar{v}}|v_1, 0_v)P(\bar{v}_1, 3_{\bar{v}}|\bar{v}_1, 3_{\bar{v}})} . \quad (3.15)$$

From table 4

$$P(v_1, 0_v|v_1, 0_v) = 1, \quad P(\bar{v}_1, 3_{\bar{v}}|\bar{v}_1, 3_{\bar{v}}) = 1, \quad (3.16)$$

$$P(v_1, 0_v|\bar{v}_1, 3_{\bar{v}}) = v_1 - \bar{v}_1 - i, \quad P(\bar{v}_1, 3_{\bar{v}}|v_1, 0_v) = \bar{v}_1 - v_1 + 2i. \quad (3.17)$$

Inserting these expressions into (3.15) we will find $F_{\{0_v, 3_{\bar{v}}\}}$, the contributions of remaining three fixed points are found similarly:

$$F_{\{0_v, 3_{\bar{v}}\}} = \frac{1}{(v_1 - \bar{v}_1 - i)(\bar{v}_1 - v_1 + 2i)}, \quad F_{\{3_v, 0_{\bar{v}}\}} = \frac{1}{(v_1 - \bar{v}_1 + 2i)(\bar{v}_1 - v_1 - i)}, \quad (3.18)$$

$$F_{\{1_v, 2_{\bar{v}}\}} = \frac{1}{(v_1 - \bar{v}_1)(\bar{v}_1 - v_1 + i)}, \quad F_{\{2_v, 1_{\bar{v}}\}} = \frac{1}{(v_1 - \bar{v}_1 + i)(\bar{v}_1 - v_1)}. \quad (3.19)$$

Inserting these into (3.12) and recalling (3.2) we obtain

$$\Pi_{\text{mat}}^{(1)} = -(F_{\{0_v, 3_{\bar{v}}\}} + F_{\{3_v, 0_{\bar{v}}\}} + F_{\{1_v, 2_{\bar{v}}\}} + F_{\{2_v, 1_{\bar{v}}\}}) = \frac{4}{(v_1 - \bar{v}_1)^2 + 4}, \quad (3.20)$$

which coincides with the result in [51].

In [51] it was shown that $\Pi_{\text{mat}}^{(n)}$ can be represented as

$$\Pi_{\text{mat}}^{(n)} = \frac{N^{(n)}}{\prod_{i < j}^n (v_{ij}^2 + 1)(\bar{v}_{ij}^2 + 1) \prod_{i,j=1}^n ((v_i - \bar{v}_j)^2 + 4)}, \quad (3.21)$$

where the numerator is a polynomial in v_i and $\bar{v}_i$.

For $n = 2$ we have 28 fixed points: $\{3_v, 3_v, 0_{\bar{v}}, 0_{\bar{v}}\}$, $\{2_v, 3_v, 0_{\bar{v}}, 1_{\bar{v}}\}$, $\{2_v, 2_v, 1_{\bar{v}}, 1_{\bar{v}}\}$, $\{1_v, 3_v, 0_{\bar{v}}, 2_{\bar{v}}\}$, $\{1_v, 2_v, 1_{\bar{v}}, 2_{\bar{v}}\}$, $\{1_v, 1_v, 2_{\bar{v}}, 2_{\bar{v}}\}$, $\{0_v, 3_v, 0_{\bar{v}}, 3_{\bar{v}}\}$, $\{0_v, 2_v, 1_{\bar{v}}, 3_{\bar{v}}\}$, $\{0_v, 1_v, 2_{\bar{v}}, 3_{\bar{v}}\}$, $\{0_v, 0_v, 3_{\bar{v}}, 3_{\bar{v}}\}$ and those obtained by the permutations of their first two and last two entries. We used (3.12) with (3.13) and the table 4 to derive $\Pi_{\text{mat}}^{(2)}$. The result is

$$\begin{aligned} N^{(2)} = & 8 \left(128 - 18\bar{s}_1 s_1 + 26\bar{s}_2 s_2 + 2\bar{s}_1^2 s_1^2 - 44(\bar{s}_2 + s_2) + 20(\bar{s}_1^2 + s_1^2) - \right. \\ & \left. - 5(\bar{s}_1^2 s_2 + \bar{s}_2 s_1^2) - 3(\bar{s}_1 \bar{s}_2 s_1 + \bar{s}_1 s_1 s_2 - \bar{s}_2^2 - s_2^2) \right), \end{aligned} \quad (3.22)$$

where s_1 and s_2 ($\bar{s}_1$ and $\bar{s}_2$) are the elementary symmetric polynomials in v_1 and v_2 ($\bar{v}_1$ and $\bar{v}_2$). The result is in full agreement with [51].

For $n = 3$ there are 256 fixed points. Typical examples are $\{3_v, 3_v, 3_v, 0_{\bar{v}}, 0_{\bar{v}}, 0_{\bar{v}}\}$, $\{2_v, 3_v, 3_v, 0_{\bar{v}}, 0_{\bar{v}}, 1_{\bar{v}}\}$. Using mathematica we were able to compute the polynomial $N^{(3)}$ explicitly. Unfortunately the result is too lengthy to be presented here.² For $n = 4$ and $n = 5$ we have 2716 and 31504 fixed points and mathematica gives the expression as a huge sum over the fixed points.

3.1.2 Asymptotic factorisation and recursion property of the residue

Let λ be any of the diagrams $Y_1^v, \dots, Y_k^v$ and $\bar{\lambda}$ a diagram from $Y_1^{\bar{v}}, \dots, Y_k^{\bar{v}}$ having $3 - |\lambda|$ boxes. Similarly let $\mu \in \{Y_{k+1}^v, \dots, Y_n^v\}$ and $\bar{\mu} \in \{Y_{k+1}^{\bar{v}}, \dots, Y_n^{\bar{v}}\}$ is such that $3 - |\mu|$. Using table 4 one can check that

$$\begin{aligned} & \left\{ P(v_i + \Lambda, \lambda|v_a, \mu) P(v_a, \mu|v_i + \Lambda, \lambda) P(v_i + \Lambda, \lambda|\bar{v}_b, \bar{\mu}) P(\bar{v}_b, \bar{\mu}|v_i + \Lambda, \lambda) \right. \\ & \times P(\bar{v}_j + \Lambda, \bar{\lambda}|v_a, \mu) P(v_a, \mu|\bar{v}_j + \Lambda, \bar{\lambda}) P(\bar{v}_j + \Lambda, \bar{\lambda}|\bar{v}_b, \bar{\mu}) P(\bar{v}_b, \bar{\mu}|\bar{v}_j + \Lambda, \bar{\lambda}) \left. \right\}^{-1} = \\ & = \frac{1}{\Lambda^4} \left(1 - \frac{2}{\Lambda} (v_{ia} + \bar{v}_{jb}) \right) + O(\Lambda^{-6}), \end{aligned} \quad (3.23)$$

where $i, j \in \{1, 2, \dots, k\}$ and $a, b \in \{k+1, k+2, \dots, n\}$. We will denote by $\vec{Y}_\Lambda$ a particular choice of $\vec{Y}$ such that for any $i = 1, \dots, k$, $|\vec{Y}_i^v| = 3 - |\vec{Y}_{p(i)}^{\bar{v}}|$ with $p(i)$ being some permutation of $1, \dots, k$. It is easy to see that for such a choice of $\vec{Y}_\Lambda$ an analogous property for $k+1, \dots, n$ holds automatically. So from (3.23) and (3.13) we get

$$\begin{aligned} & F_{\vec{Y}_\Lambda}(v_1 + \Lambda, \dots, v_k + \Lambda, v_{k+1}, \dots, v_n; \bar{v}_1 + \Lambda, \dots, \bar{v}_k + \Lambda, \bar{v}_{k+1}) \sim \Lambda^{-4k(n-k)} \\ & \times \left(1 - \frac{2}{\Lambda} \sum_{i,j=1}^k \sum_{a,b=k+1}^n (v_{ia} + \bar{v}_{jb}) \right) F_{\{Y_1^v, \dots, Y_k^v, Y_1^{\bar{v}}, \dots, Y_k^{\bar{v}}\}} F_{\{Y_{k+1}^v, \dots, Y_n^v, Y_{k+1}^{\bar{v}}, \dots, Y_n^{\bar{v}}\}}. \end{aligned} \quad (3.24)$$

One can see from (3.12) that

$$\begin{aligned} & \Pi_{\text{mat}}^{(n)}(v_1 + \Lambda, \dots, v_k + \Lambda, v_{k+1}, \dots, v_n; \bar{v}_1 + \Lambda, \dots, \bar{v}_k + \Lambda, \bar{v}_{k+1}, \dots, \bar{v}_n) \sim \Lambda^{-4k(n-k)} \\ & \times \left(1 - \frac{2}{\Lambda} \sum_{i,j=1}^k \sum_{a,b=k+1}^n (v_{ia} + \bar{v}_{jb}) \right) \Pi_{\text{mat}}^{(k)}(v_1, \dots, v_k; \bar{v}_1, \dots, \bar{v}_k) \Pi_{\text{mat}}^{(n-k)}(v_{k+1}, \dots, v_n; \bar{v}_{k+1}, \dots, \bar{v}_n). \end{aligned} \quad (3.25)$$

In [51] it was demonstrated that

$$i\text{Res}_{\bar{v}_1=v_1+2i} \Pi_{\text{mat}}^{(n)}(v_1, \dots, v_n; \bar{v}_1, \dots, \bar{v}_n) = - \frac{\Pi_{\text{mat}}^{(n-1)}(v_2, \dots, v_n; \bar{v}_2, \dots, \bar{v}_n)}{\prod_{j=2}^n (v_{1j} + i) v_{1j} (v_1 - \bar{v}_j + 2i) (v_1 - \bar{v}_j + i)}. \quad (3.26)$$

²The result is available on request.

This result too can be easily obtained by using our formula for $\Pi_{\text{mat}}^{(n)}$. Indeed, as can be seen from table 4, the arrays $\vec{Y}$ with nonzero residues $\text{Res}_{\bar{v}_1=v_1+2i} F_{\vec{Y}} \neq 0$ have the structure $\vec{Y} = \{3_v, Y_2^v, \dots, Y_n^v; 0_{\bar{v}}, Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}$. From (3.13) we obtain

$$\begin{aligned} F_{\{3_v, Y_2^v, \dots, Y_n^v; 0_{\bar{v}}, Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}} &= \{P(v_1, 3_v | \bar{v}_1, 0_{\bar{v}}) P(\bar{v}_1, 0_{\bar{v}} | v_1, 3_v) \\ &\times \prod_{j=2}^n P(v_1, 3_v | v_j, Y_j^v) P(v_1, 3_v | \bar{v}_j, Y_j^{\bar{v}}) P(\bar{v}_1, 0_{\bar{v}} | v_j, Y_j^v) P(\bar{v}_1, 0_{\bar{v}} | \bar{v}_j, Y_j^{\bar{v}}) \\ &\times P(v_j Y_j^v | v_1, 3_v) P(v_j Y_j^v | \bar{v}_1, 0_{\bar{v}}) P(\bar{v}_j, Y_j^{\bar{v}} | v_1, 3_v) P(\bar{v}_j, Y_j^{\bar{v}} | \bar{v}_1, 0_{\bar{v}})\}^{-1} F_{\{Y_2^v, \dots, Y_n^v; Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}}. \end{aligned} \quad (3.27)$$

Using table 4 for the residue we get

$$i \text{Res}_{\bar{v}_1=v_1+2i} F_{\{3_v, Y_2^v, \dots, Y_n^v; 0_{\bar{v}}, Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}} = - \frac{F_{\{Y_2^v, \dots, Y_n^v; Y_2^{\bar{v}}, \dots, Y_n^{\bar{v}}\}}}{\prod_{j=2}^n (v_{1j} + i) v_{1j} (v_1 - \bar{v}_j + 2i) (v_1 - \bar{v}_j + i)}. \quad (3.28)$$

Summing over the diagrams (see (3.12)) we recover (3.26).

4 Young diagram representation for MHV, NMHV and N²MHV amplitudes

According to [53], inside the hexagonal Wilson loop in $\mathcal{N} = 4$ SYM the factor accounting for the matrix structure can be again written as a multi-integral over the three kinds of nested Bethe Ansatz roots of a spin-chain a_i , b_m , c_j , where $i = 1, \dots, K_1$, $m = 1, \dots, K_2$, $j = 1, \dots, K_3$ [36, 45]:

$$\begin{aligned} \Pi_{\text{mat}}^{(N_\psi, N_\phi, N_{\bar{\psi}})} &= \frac{1}{K_1! K_2! K_3!} \int_{-\infty}^{\infty} \prod_{i=1}^{K_1} \frac{da_i}{2\pi} \prod_{m=1}^{K_2} \frac{db_m}{2\pi} \prod_{j=1}^{K_3} \frac{dc_j}{2\pi} \\ &\times \frac{\prod_{i < k}^{K_1} g(a_{ik}) \prod_{m < n}^{K_2} g(b_{mn}) \prod_{j < k}^{K_3} g(c_{jk})}{\prod_{i=1}^{K_1} \prod_{m=1}^{K_2} f(a_i - b_m) \prod_{j=1}^{K_3} \prod_{m=1}^{K_2} f(b_m - c_j) \prod_{m=1}^{K_2} \prod_{\ell=1}^{N_\phi} f(b_m - u_\ell) \prod_{i=1}^{K_1} \prod_{\alpha=1}^{N_\psi} f(a_i - v_\alpha) \prod_{j=1}^{K_3} \prod_{\beta=1}^{N_{\bar{\psi}}} f(c_j - \bar{v}_\beta)}, \end{aligned} \quad (4.1)$$

with the same functions $f(x)$ and $g(x)$ (2.2). After the usual specification $\epsilon \equiv \frac{i}{2}$ and shifting the integration variables a_i , b_m and c_j by ϵ the last expression becomes

$$\begin{aligned} \Pi_{\text{mat}}^{(N_\psi, N_\phi, N_{\bar{\psi}})} &= \frac{(-1)^C}{K_1! K_2! K_3!} \int_{-\infty}^{\infty} \prod_{i=1}^{K_1} \frac{da_i}{2\pi i} \prod_{m=1}^{K_2} \frac{db_m}{2\pi i} \prod_{j=1}^{K_3} \frac{dc_j}{2\pi i} \\ &\times \frac{\prod'_{i,k=1}^{K_1} a_{ik} (a_{ik} + 2\epsilon) \prod'_{m,n=1}^{K_2} b_{mn} (b_{mn} + 2\epsilon) \prod'_{j,k=1}^{K_3} c_{jk} (c_{jk} + 2\epsilon)}{\prod_{m=1}^{K_2} \prod_{i=1}^{K_1} (a_i - b_m + \epsilon) (b_m - a_i + \epsilon) \prod_{m=1}^{K_2} \prod_{j=1}^{K_3} (c_j - b_m + \epsilon) (b_m - c_j + \epsilon)} \\ &\times \frac{1}{\prod_{m=1}^{K_2} \prod_{\ell=1}^{N_\phi} (u_\ell - b_m) (b_m - u_\ell + 2\epsilon) \prod_{i=1}^{K_1} \prod_{\alpha=1}^{N_\psi} (v_\alpha - a_i) (a_i - v_\alpha + 2\epsilon) \prod_{j=1}^{K_3} \prod_{\beta=1}^{N_{\bar{\psi}}} (\bar{v}_\beta - c_j) (c_j - \bar{v}_\beta + 2\epsilon)}, \end{aligned} \quad (4.2)$$

where $C = K_1 K_2 + K_2 K_3 + K_2 N_\phi + K_1 N_\psi + K_3 N_{\bar{\psi}}$, the prime on the product symbol again indicates that all the vanishing factors must be ignored. Here u_ℓ are the rapidities of the scalars and v_α and $\bar{v}_\beta$ are the rapidities of the fermions and anti-fermions, respectively. N_ϕ , N_ψ and $N_{\bar{\psi}}$ are the numbers of scalars, fermions and anti-fermions. K_j $j = 1, 2, 3$

Figure 6. Represents a quiver diagram where the arrows indicate the linear maps M_{av} , M_{va} , M_{ab} , M_{ba} , M_{bu} , M_{ub} , M_{bc} , M_{cb} , $M_{c\bar{v}}$, $M_{\bar{v}c}$ and dots the spaces v , a , u , b , c , $\bar{v}$ on which they act.

satisfy the following conditions (see [53])

$$N_\psi - 2K_1 + K_2 = \delta_{r_b,3}, \quad (4.3)$$

$$N_{\bar{\psi}} - 2K_3 + K_2 = \delta_{r_b,1}, \quad (4.4)$$

$$N_\phi + K_1 - 2K_2 + K_3 = \delta_{r_b,2}, \quad (4.5)$$

where $0 \leq r_b \leq 4$ is the R charge carried by the bottom pentagon. For MHV and N^2 MHV the parameter r_b is fixed and takes the values $r_b = 0$ and $r_b = 4$ respectively. In NMHV case r_b may assume any of the five allowed values.

The MHV case with $2K_1 = 2K_3 = K_2 = 2n$, $N_\psi = N_{\bar{\psi}} = 0$, $N_\phi = 2n$ is considered in grate details in section 2 while the MHV amplitudes with $K_1 = K_2 = K_3 = n$, $N_\psi = N_{\bar{\psi}} = n$, $N_\phi = 0$ are considered in section 3. Here we treat the general case.

4.1 Matrix equations

Let us start by constructing a more general system of matrix equations corresponding to the integral representation of $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$ (4.2). As a result, by means of localization, we will get an efficient combinatorial procedure for the evaluation of these integrals. Similar to the case considered in section 3 here too it is more natural to consider a slightly different quantity defined as

$$\Pi_f^{(N_\psi N_\phi N_{\bar{\psi}})} = (-)^m \Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}, \quad (4.6)$$

where

$$m = K_1 K_2 + K_2 K_3 + K_2 N_\phi + K_1 N_\psi + K_3 N_{\bar{\psi}} + K_1 + K_2 + K_3. \quad (4.7)$$

As the integrand of (4.2) suggests, one needs to introduce six spaces:

- u , v , $\bar{v}_1$ that are N_ϕ , N_ψ and $N_{\bar{\psi}}$ dimensional complex vector spaces respectively. They are connected to parameters $u_1, u_2, \dots, u_{N_\phi}$, $v_1, v_2, \dots, v_{N_\psi}$ and $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{N_{\bar{\psi}}}$

- a , b and c which are K_1 , K_2 and K_3 dimensional complex vector spaces related to $a_1, a_2, \dots, a_{K_1}$, $b_1, b_2, \dots, b_{K_2}$ and $c_1, c_2, \dots, c_{K_3}$

and ten linear maps (matrices) $M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}, M_{av}, M_{va}, M_{c\bar{v}}, M_{\bar{v}c}$ acting among these spaces as indicated by the quiver diagram given in figure 6.

Admissible matrix equations compatible with the form of the numerator in (4.2) can be chosen as:

$$M_{ab}M_{ba} + M_{av}M_{va} = 0, \quad (4.8)$$

$$M_{cb}M_{bc} + M_{c\bar{v}}M_{\bar{v}c} = 0, \quad (4.9)$$

$$M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub} = 0. \quad (4.10)$$

Similar to previous cases we introduce auxiliary transformations T_a, T_b, T_c in addition with three diagonal matrices $T_u, T_v, T_{\bar{v}}$ corresponding to genuine symmetries. The role of $T_\epsilon \in C^*$ is the same as in previous cases. The denominator of (4.2) dictates the transformation rules of the matrices M :

$$M_{ab} \rightarrow T_\epsilon T_a M_{ab} T_b^{-1}, \quad M_{ba} \rightarrow T_\epsilon T_b M_{ba} T_a^{-1} \quad (4.11)$$

$$M_{cb} \rightarrow T_\epsilon T_c M_{cb} T_b^{-1}, \quad M_{bc} \rightarrow T_\epsilon T_b M_{bc} T_c^{-1} \quad (4.12)$$

$$M_{bu} \rightarrow T_\epsilon^2 T_b M_{bu} T_u^{-1}, \quad M_{ub} \rightarrow T_u M_{ub} T_b^{-1} \quad (4.13)$$

$$M_{av} \rightarrow T_\epsilon^2 T_a M_{av} T_v^{-1}, \quad M_{va} \rightarrow T_v M_{va} T_a^{-1} \quad (4.14)$$

$$M_{c\bar{v}} \rightarrow T_\epsilon^2 T_c M_{c\bar{v}} T_{\bar{v}}^{-1}, \quad M_{\bar{v}c} \rightarrow T_{\bar{v}} M_{\bar{v}c} T_c^{-1}. \quad (4.15)$$

Under these transformations the left hand sides of the matrix equations (4.8)–(4.10) transform respectively as:

$$M_{ab}M_{ba} + M_{av}M_{va} \rightarrow T_\epsilon^2 T_a (M_{ab}M_{ba} + M_{av}M_{va}) T_a^{-1}, \quad (4.16)$$

$$M_{cb}M_{bc} + M_{c\bar{v}}M_{\bar{v}c} \rightarrow T_\epsilon^2 T_c (M_{cb}M_{bc} + M_{c\bar{v}}M_{\bar{v}c}) T_c^{-1}, \quad (4.17)$$

$$M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub} \rightarrow T_\epsilon^2 T_b (M_{ba}M_{ab} - M_{bc}M_{cb} + M_{bu}M_{ub}) T_b^{-1} \quad (4.18)$$

in accordance with the form of numerator of (4.2).

4.2 The fixed points of the moduli space

Moduli space: by definition a point in moduli space $\mathcal{M}_n$ is a set of matrices

$\{M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}, M_{av}, M_{va}, M_{c\bar{v}}, M_{\bar{v}c}\}$ satisfying the stability condition and the equations (4.8)–(4.10) modulo the equivalence relation

$$\{M_{ab}, M_{ba}, M_{bc}, M_{cb}, M_{bu}, M_{ub}, M_{av}, M_{va}, M_{c\bar{v}}, M_{\bar{v}c}\} \sim \{T_a M_{ab} T_b^{-1}, T_b M_{ba} T_a^{-1}, T_b M_{bc} T_c^{-1}, T_c M_{cb} T_b^{-1}, T_b M_{bu}, M_{ub} T_b^{-1}, T_a M_{av}, M_{va} T_a^{-1}, T_c M_{c\bar{v}}, M_{\bar{v}c} T_c^{-1}\}. \quad (4.19)$$

In addition we supplement the matrix equations with a stability condition. Roughly speaking this condition states that starting with the images of $v, u, \bar{v}$ in a, b, c respectively and acting by matrices $M_{ba}, M_{ab}, M_{cb}, M_{bc}$ in various consistent ways one completely covers each of spaces a, b and c .

Figure 7. The list of diagrams if one has fermions antifermions and scalars. The diagrams are labeled by $0_v, \dots, 3_v, 0_u, \dots, 4_u, 0_{\bar{v}}, \dots, 3_{\bar{v}}$, where the indicate coincides with the number of boxes in a diagram and the indices v, u and $\bar{v}$ tell weather a diagram is connected to fermions scalars or antifermions. As before three dotted lines correspond to the spaces a, b, c .

Our next step is to find the points of the moduli space fixed under transformations $T_\epsilon, T_v, T_u, T_{\bar{v}}$. It is possible to show that at fixed points

$$M_{ub} = 0, \quad M_{va} = 0, \quad M_{\bar{v}c} = 0. \quad (4.20)$$

Construction of fixed points goes parallel to the cases discussed in previous cases (see section 2.3). In fact the general case under consideration is a simple combination of the purely bosonic and the case with several fermion antifermion pairs. The complete set of allowed diagrams is depicted in figure 7. Remind that a box of a diagram located on the dotted lines a, b or c represents a basis vector in the respective space.

A fixed point is represented by $N_\psi + N_\phi + N_{\bar{\psi}}$ diagrams from figure 7. Since the dimension of u is N_ϕ we must have N_ϕ diagrams from the second row. The dimension of v and $\bar{v}$ are N_ψ and $N_{\bar{\psi}}$ respectively hence we need N_ψ diagrams from the first and $N_{\bar{\psi}}$ diagrams from the third row. In addition to match the dimensions of spaces a, b and c we should have in total K_1 boxes on first dotted line, K_2 boxes on the second and K_3 on the third lines.

To apply localisation we need the tangent space of $\mathcal{M}$ at the fixed points. As usual we start with the total unconstrained space $\delta\mathcal{M}$ then “subtract” subspaces corresponding to equations (4.8)–(4.10) and auxiliary transformations.

Transformation laws (4.11)–(4.15) dictate the following structure of the variations of matrices M

$$\begin{aligned} \delta M_{ab} &\in T_\epsilon a \otimes b^*, & \delta M_{ba} &\in T_\epsilon b \otimes a^*, & \delta M_{cb} &\in T_\epsilon c \otimes b^*, & \delta M_{bc} &\in T_\epsilon b \otimes c^*, \\ \delta M_{av} &\in T_\epsilon^2 a \otimes v^*, & \delta M_{va} &\in v \otimes a^*, & \delta M_{c\bar{v}} &\in T_\epsilon^2 c \otimes \bar{v}^*, & \delta M_{\bar{v}c} &\in \bar{v} \otimes c^*, \\ & & & & \delta M_{bu} &\in T_\epsilon^2 b \otimes u^*, & \delta M_{ub} &\in u \otimes b^*. \end{aligned} \quad (4.21)$$

Subtracting from the total space of unconstrained deformations the spaces (2.27) and (2.28), corresponding to equations and auxiliary transformations, for the tangent space we get

$$\chi_{\tilde{Y}} = T_\epsilon ((a+c)b^* + b(a^* + c^*)) + T_\epsilon^2 bu^* + ub^* + T_\epsilon^2 av^* + va^* + T_\epsilon^2 c\bar{v}^* + \bar{v}c^* - (T_\epsilon^2 + 1)(aa^* + bb^* + cc^*) . \quad (4.22)$$

The characters of the spaces v , u and $\bar{v}$ are (the same latter is used for both the space and its character)

$$v = \sum_{i=1}^{N_\psi} T_{v_i} , \quad u = \sum_{i=1}^{N_\phi} T_{u_i} , \quad \bar{v} = \sum_{i=1}^{N_{\bar{\psi}}} T_{\bar{v}_i} . \quad (4.23)$$

The dual characters u^* , v^* , $\bar{v}^*$ are obtained by substitution $T_{u_i} \rightarrow T_{u_i}^{-1}$, $T_{v_i} \rightarrow T_{v_i}^{-1}$ and $T_{\bar{v}_i} \rightarrow T_{\bar{v}_i}^{-1}$. Similarly for the spaces a , b and c we have

$$a = \sum_k a_k , \quad b = \sum_k b_k , \quad c = \sum_k c_k \quad (4.24)$$

and the conjugates are obtained by replacing the summands by their inverses. Examining the structure of the diagrams we get convinced that the summands in equations (4.24) explicitly are given by

$$a_k = \begin{cases} T_\epsilon T_{u_k} & \text{if } Y = 2_u, 3_u, 4_u \\ 0 & \text{if } Y = 0_u, 1_u, \tilde{2}_u, 0_v, 0_{\bar{v}}, 1_{\bar{v}}, 2_{\bar{v}} , \\ T_{v_k} & \text{if } Y = 1_v, 2_v, 3_v \\ T_\epsilon^2 T_{\bar{v}_k} & \text{if } Y = 3_{\bar{v}} \end{cases} , \quad (4.25)$$

$$c_k = \begin{cases} T_\epsilon T_{u_k} & \text{if } Y = \tilde{2}_u, 3_u, 4_u \\ 0 & \text{if } Y = 0_u, 1_u, 2_u, 0_v, 1_v, 2_v, 0_{\bar{v}} , \\ T_\epsilon^2 T_{v_k} & \text{if } Y = 3_v \\ T_{\bar{v}_k} & \text{if } Y = 1_{\bar{v}}, 2_{\bar{v}}, 3_{\bar{v}} \end{cases} , \quad (4.26)$$

$$b_k = \begin{cases} T_{u_k} & \text{if } Y = 1_u, 2_u, \tilde{2}_u, 3_u \\ (T_\epsilon^2 + 1)T_{u_k} & \text{if } Y = 4_u \\ 0 & \text{if } Y = 0_u, 0_v, 1_v, 0_{\bar{v}}, 1_{\bar{v}} \\ T_\epsilon T_{v_k} & \text{if } Y = 2_v, 3_v \\ T_\epsilon T_{\bar{v}_k} & \text{if } Y = 2_{\bar{v}}, 3_{\bar{v}} \end{cases} \quad (4.27)$$

In view of above decompositions the detailed structure of the character (4.22) takes the form

$$\begin{aligned} \chi_{\tilde{Y}} = & \sum_{i,j=1}^{N_\psi} \sum_{m,n=1}^{N_\phi} \sum_{k,l=1}^{N_{\bar{\psi}}} \left(X(T_{v_i}, Y_i^v | T_{v_j}, Y_j^v) + X(T_{v_i}, Y_i^v | T_{u_n}, Y_n^u) + X(T_{v_i}, Y_i^v | T_{\bar{v}_k}, Y_k^{\bar{v}}) + \right. \\ & + X(T_{u_n}, Y_n^u | T_{v_i}, Y_i^v) + X(T_{u_n}, Y_n^u | T_{u_m}, Y_m^u) + X(T_{u_n}, Y_n^u | T_{\bar{v}_k}, Y_k^{\bar{v}}) + \\ & \left. + X(T_{\bar{v}_k}, Y_k^{\bar{v}} | T_{v_i}, Y_i^v) + X(T_{\bar{v}_k}, Y_k^{\bar{v}} | T_{u_n}, Y_n^u) + X(T_{\bar{v}_k}, Y_k^{\bar{v}} | T_{\bar{v}_l}, Y_l^{\bar{v}}) \right), \quad (4.28) \end{aligned}$$

where the summands, derived from (4.22) and (4.25)–(4.27), are listed in table 5.

	0_v	1_v	2_v	3_v	0_u	1_u	2_u	$\tilde{2}_u$	3_u	4_u	$0_{\bar{v}}$	$1_{\bar{v}}$	$2_{\bar{v}}$	$3_{\bar{v}}$
0_v	0	1	1	1	0	0	T_ϵ^{-1}	0	T_ϵ^{-1}	T_ϵ^{-1}	0	0	0	T_ϵ^{-2}
1_v	T_ϵ^2	0	1	1	0	T_ϵ	0	T_ϵ	0	T_ϵ^{-1}	0	0	1	0
2_v	T_ϵ^2	T_ϵ^2	0	1	T_ϵ^3	0	0	T_ϵ	T_ϵ	0	0	T_ϵ^2	0	0
3_v	T_ϵ^2	T_ϵ^2	T_ϵ^2	0	T_ϵ^3	T_ϵ^3	T_ϵ^3	0	0	0	T_ϵ^4	0	0	0
0_u	0	0	T_ϵ^{-1}	T_ϵ^{-1}	0	1	1	1	1	$1 + T_\epsilon^{-2}$	0	0	T_ϵ^{-1}	T_ϵ^{-1}
1_u	0	T_ϵ	0	T_ϵ^{-1}	T_ϵ^2	0	1	1	2	1	0	T_ϵ	0	T_ϵ^{-1}
2_u	T_ϵ^3	0	0	T_ϵ^{-1}	T_ϵ^2	T_ϵ^2	0	$T_\epsilon^2 + 1$	1	1	0	T_ϵ	T_ϵ	0
$\tilde{2}_u$	0	T_ϵ	T_ϵ	0	T_ϵ^2	T_ϵ^2	$T_\epsilon^2 + 1$	0	1	1	T_ϵ^3	0	0	T_ϵ^{-1}
3_u	T_ϵ^3	0	T_ϵ	0	T_ϵ^2	$2 T_\epsilon^2$	T_ϵ^2	T_ϵ^2	0	1	T_ϵ^3	0	T_ϵ	0
4_u	T_ϵ^3	T_ϵ^3	0	0	$T_\epsilon^4 + T_\epsilon^2$	T_ϵ^2	T_ϵ^2	T_ϵ^2	T_ϵ^2	0	T_ϵ^3	T_ϵ^3	0	0
$0_{\bar{v}}$	0	0	0	T_ϵ^{-2}	0	0	0	T_ϵ^{-1}	T_ϵ^{-1}	T_ϵ^{-1}	0	1	1	1
$1_{\bar{v}}$	0	0	1	0	0	T_ϵ	T_ϵ	0	0	T_ϵ^{-1}	T_ϵ^2	0	1	1
$2_{\bar{v}}$	0	T_ϵ^2	0	0	T_ϵ^3	0	T_ϵ	0	T_ϵ	0	T_ϵ^2	T_ϵ^2	0	1
$3_{\bar{v}}$	T_ϵ^4	0	0	0	T_ϵ^3	T_ϵ^3	0	T_ϵ^3	0	0	T_ϵ^2	T_ϵ^2	T_ϵ^2	0

Table 5. Gives $X(T_a, \lambda|T_b, \mu)$ with $T_a T_b^{-1}$ suppressed for example $X(T_{v_i}, 2_v|T_{u_j}, 0_u) = T_{v_i} T_{u_j}^{-1} T_\epsilon^3$.

4.3 Representation of $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$ as a sum over diagrams

Due to localization for given N_ϕ , N_ψ and $N_{\bar{\psi}}$ the matrix part of the hexagonal WI is a sum over fixed points:

$$\Pi_f^{(N_\psi N_\phi N_{\bar{\psi}})} = \sum_{\vec{Y}} F_{\vec{Y}}. \quad (4.29)$$

Specifically the sum is over all collections of $N_\phi + N_\psi + N_{\bar{\psi}}$ diagrams of the form $\vec{Y} \equiv \{Y_1^v, Y_2^v, \dots, Y_{N_\psi}^v; Y_1^u, Y_2^u, \dots, Y_{N_\phi}^u; Y_1^{\bar{v}}, Y_2^{\bar{v}}, \dots, Y_{N_{\bar{\psi}}}^{\bar{v}}\}$ with entries taken from the list given in figure 7. As the notation indicates Y_i^v , Y_i^u and $Y_i^{\bar{v}}$ are diagrams taken from the first second or third row in figure 7 respectively. In $\vec{Y}$ the total numbers of boxes on the upper, middle and lower lines are K_1 , K_2 and K_3 respectively.

The summands $F_{\vec{Y}}$ are given by

$$1/F_{\vec{Y}} = \prod_{i,j=1}^{N_\psi} \prod_{m,n=1}^{N_\phi} \prod_{k,l=1}^{N_{\bar{\psi}}} P(v_i, Y_i^v|v_j, Y_j^v) P(v_i, Y_i^v|u_n, Y_n^u) P(v_i, Y_i^v|\bar{v}_k, Y_k^{\bar{v}}) \quad (4.30)$$

$$\times P(u_n, Y_n^u|v_i, Y_i^v) P(u_n, Y_n^u|u_m, Y_m^u) P(u_n, Y_n^u|\bar{v}_k, Y_k^{\bar{v}})$$

$$\times P(\bar{v}_k, Y_k^{\bar{v}}|v_i, Y_i^v) P(\bar{v}_k, Y_k^{\bar{v}}|u_n, Y_n^u) P(\bar{v}_k, Y_k^{\bar{v}}|\bar{v}_l, Y_l^{\bar{v}})$$

where $P(a, \lambda|b, \mu)$ are displayed in table 6.

	0_v	1_v	2_v	3_v	0_u	1_u	2_u	$\tilde{2}_u$	3_u	4_u	$0_{\bar{v}}$	$1_{\bar{v}}$	$2_{\bar{v}}$	$3_{\bar{v}}$
0_v	1	v_{ij}	v_{ij}	v_{ij}	1	1	$v_i - u_j - \frac{i}{2}$	1	$v_i - u_j - \frac{i}{2}$	$v_i - u_j - \frac{i}{2}$	1	1	1	$v_i - \bar{v}_j - i$
1_v	$v_{ij} + i$	1	v_{ij}	v_{ij}	1	$v_i - u_j + \frac{i}{2}$	1	$v_i - u_j + \frac{i}{2}$	1	$v_i - u_j - \frac{i}{2}$	1	1	$v_i - \bar{v}_j$	1
2_v	$v_{ij} + i$	$v_{ij} + i$	1	v_{ij}	$v_i - u_j + \frac{3i}{2}$	1	1	$v_i - u_j + \frac{i}{2}$	$v_i - u_j + \frac{i}{2}$	1	1	$v_i - \bar{v}_j + i$	1	1
3_v	$v_{ij} + i$	$v_{ij} + i$	$v_{ij} + i$	1	$v_i - u_j + \frac{3i}{2}$	$v_i - u_j + \frac{3i}{2}$	1	1	1	$v_i - \bar{v}_j + 2i$	1	1	1	1
0_u	1	1	$u_i - v_j - \frac{i}{2}$	$u_i - v_j - \frac{i}{2}$	1	u_{ij}	u_{ij}	u_{ij}	u_{ij}	$u_{ij}(u_{ij} - i)$	1	1	$u_i - \bar{v}_j - \frac{i}{2}$	$u_i - \bar{v}_j - \frac{i}{2}$
1_u	1	$u_i - v_j + \frac{i}{2}$	1	$u_i - v_j - \frac{i}{2}$	$u_{ij} + i$	1	u_{ij}	u_{ij}	u_{ij}^2	u_{ij}	1	$u_i - \bar{v}_j + \frac{i}{2}$	1	$u_i - \bar{v}_j - \frac{i}{2}$
2_u	$u_i - v_j + \frac{3i}{2}$	1	1	$u_i - v_j - \frac{i}{2}$	$u_{ij} + i$	$u_{ij} + i$	1	$u_{ij}(u_{ij} + i)$	u_{ij}	u_{ij}	1	$u_i - \bar{v}_j + \frac{i}{2}$	$u_i - \bar{v}_j + \frac{i}{2}$	1
$\tilde{2}_u$	1	$u_i - v_j + \frac{i}{2}$	$u_i - v_j + \frac{i}{2}$	1	$u_{ij} + i$	$u_{ij} + i$	$u_{ij}(u_{ij} + i)$	1	u_{ij}	u_{ij}	$u_i - \bar{v}_j + \frac{3i}{2}$	1	1	$u_i - \bar{v}_j - \frac{i}{2}$
3_u	$u_i - v_j + \frac{3i}{2}$	1	$u_i - v_j + \frac{i}{2}$	1	$u_{ij} + i$	$(u_{ij} + i)^2$	$u_{ij} + i$	$u_{ij} + i$	1	u_{ij}	$u_i - \bar{v}_j + \frac{3i}{2}$	1	$u_i - \bar{v}_j + \frac{i}{2}$	1
4_u	$u_i - v_j + \frac{3i}{2}$	$u_i - v_j + \frac{3i}{2}$	1	1	$(u_{ij} + i)(u_{ij} + 2i)$	$u_{ij} + i$	$u_{ij} + i$	$u_{ij} + i$	$u_{ij} + i$	1	$u_i - \bar{v}_j + \frac{3i}{2}$	$u_i - \bar{v}_j + \frac{3i}{2}$	1	1
$0_{\bar{v}}$	1	1	1	$\bar{v}_i - v_j - i$	1	1	1	$\bar{v}_i - u_j - \frac{i}{2}$	$\bar{v}_i - u_j - \frac{i}{2}$	$\bar{v}_i - u_j - \frac{i}{2}$	1	$\bar{v}_{ij}$	$\bar{v}_{ij}$	$\bar{v}_{ij}$
$1_{\bar{v}}$	1	1	$\bar{v}_i - v_j$	1	1	$\bar{v}_i - u_j + \frac{i}{2}$	$\bar{v}_i - u_j + \frac{i}{2}$	1	1	$\bar{v}_i - u_j - \frac{i}{2}$	$\bar{v}_{ij} + i$	1	$\bar{v}_{ij}$	$\bar{v}_{ij}$
$2_{\bar{v}}$	1	$\bar{v}_i - v_j + i$	1	1	$\bar{v}_i - u_j + \frac{3i}{2}$	1	$\bar{v}_i - u_j + \frac{i}{2}$	1	$\bar{v}_i - u_j + \frac{i}{2}$	1	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	1	$\bar{v}_{ij}$
$3_{\bar{v}}$	$\bar{v}_i - v_j + 2i$	1	1	1	$\bar{v}_i - u_j + \frac{3i}{2}$	$\bar{v}_i - u_j + \frac{3i}{2}$	1	$\bar{v}_i - u_j + \frac{3i}{2}$	1	1	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	$\bar{v}_{ij} + i$	1

Table 6. This table gives $P(v_i, Y_i^v | v_j, Y_j^v)$, $P(v_i, Y_i^v | \bar{v}_j, Y_j^{\bar{v}})$, $P(v_i, Y_i^v | u_j, Y_j^u)$, $P(\bar{v}_i, Y_i^{\bar{v}} | v_j, Y_j^v)$, $P(\bar{v}_i, Y_i^{\bar{v}} | \bar{v}_j, Y_j^{\bar{v}})$... for all diagrams that is $Y_i^v \in \{0_v, 1_v, 2_v, 3_v\}$, $Y_i^{\bar{v}} \in \{0_{\bar{v}}, 1_{\bar{v}}, 2_{\bar{v}}, 3_{\bar{v}}\}$ and $Y_i^u \in \{0_u, 1_u, 2_u, \tilde{2}_u, 3_u, 4_u\}$.

5 Explicit results for $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$

Examining numerous cases (some of them are presented in subsections 5.1–5.4) we got convinced that the denominator of

$$\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})} = \frac{N^{(N_\psi N_\phi N_{\bar{\psi}})}}{D^{(N_\psi N_\phi N_{\bar{\psi}})}} \quad (5.1)$$

has the form

$$\begin{aligned} D^{(N_\psi N_\phi N_{\bar{\psi}})} &= \prod_{i=1}^{N_\psi} \prod_{k=1}^{N_\phi} \left(4(u_k - v_i)^2 + 9 \right) \prod_{k=1}^{N_\phi} \prod_{m=1}^{N_{\bar{\psi}}} \left(4(u_k - \bar{v}_m)^2 + 9 \right) \\ &\times \prod_{i=1}^{N_\psi} \prod_{m=1}^{N_{\bar{\psi}}} \left((v_i - \bar{v}_m)^2 + 4 \right) \prod_{i>j}^{N_\psi} (v_{ij}^2 + 1) \prod_{m>n}^{N_{\bar{\psi}}} (\bar{v}_{mn}^2 + 1) \prod_{k>l}^{N_\phi} (u_{kl}^2 + 1) (u_{kl}^2 + 4). \end{aligned} \quad (5.2)$$

We have succeeded to generalize the recursion properties (2.55) and (3.26):

$$\begin{aligned} \text{Res}_{u_2=u_1+2i} \Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}(u_1, \dots, u_{N_\phi}) &= (-)^{m_1+m_2+1} \left(2i \prod_{j=3}^{N_\phi} u_{1j} (u_{1j} + i)^2 (u_{1j} + 2i) \right. \\ &\times \prod_{m=1}^{N_\psi} \left(v_m - u_1 - \frac{i}{2} \right) \left(u_1 - v_m + \frac{3i}{2} \right) \prod_{n=1}^{N_{\bar{\psi}}} \left(\bar{v}_n - u_1 - \frac{i}{2} \right) \left(u_1 - \bar{v}_n + \frac{3i}{2} \right) \Big)^{-1} \\ &\times \Pi_{\text{mat}}^{(N_\psi, N_\phi-2, N_{\bar{\psi}})}(u_3, \dots, u_{N_\phi}) \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} i \text{Res}_{\bar{v}_1=v_1+2i} \Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}(v_1, \dots, v_{N_\psi}; \bar{v}_1, \dots, \bar{v}_{N_{\bar{\psi}}}) \\ = \frac{(-)^{m_1+m_2+1} \Pi_{\text{mat}}^{(N_\psi-1, N_\phi, N_{\bar{\psi}}-1)}(v_2, \dots, v_{N_\psi}; \bar{v}_2, \dots, \bar{v}_{N_{\bar{\psi}}})}{\prod_{k=2}^{N_\psi} (v_{1k} + i) v_{1k} \prod_{j=2}^{N_{\bar{\psi}}} (v_1 - \bar{v}_j + 2i) (v_1 - \bar{v}_j + i) \prod_{n=1}^{N_\phi} \left(v_1 - u_n + \frac{3i}{2} \right) \left(u_n - v_1 - \frac{i}{2} \right)}, \end{aligned} \quad (5.4)$$

where $m_{1,2}$ are defined in (4.7) for pairs Π_{mat} related by above recursions. We do not present proofs of these recursion relations, since structurally they are similar to the proofs of simpler cases considered in sections 2 and 3.

In the upcoming subsections we will use our combinatorial formula to compute $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$ explicitly for some fixed values of r_b , N_ψ , N_ϕ , $N_{\bar{\psi}}$. For some cases³ we have checked our results against (4.1) by performing numerical integrations.

5.1 Cases with $r_b = 1$

- If $N_\psi = 3$, $N_\phi = 0$ and $N_{\bar{\psi}} = 0$, according to (4.3)–(4.5) we have $K_1 = 2$, $K_2 = 1$ and $K_3 = 0$. There are 6 fixed points $\{0_v, 1_v, 2_v\}$, $\{0_v, 2_v, 1_v\}$, $\{1_v, 0_v, 2_v\}$, $\{1_v, 2_v, 0_v\}$, $\{2_v, 0_v, 1_v\}$ and $\{2_v, 1_v, 0_v\}$, thus using our formula (4.29) and (4.6) we get

$$\Pi_{\text{mat}}^{(300)} = \frac{6}{(v_{12}^2 + 1)(v_{13}^2 + 1)(v_{23}^2 + 1)}. \quad (5.5)$$

- When $N_\psi = 0$, $N_\phi = 2$ and $N_{\bar{\psi}} = 1$ we have $K_1 = 1$, $K_2 = 2$ and $K_3 = 1$. Here there are 12 fixed points $\{0_u, 1_u, 3_{\bar{v}}\}$, $\{0_u, 2_u, 2_{\bar{v}}\}$, $\{0_u, 4_u, 0_{\bar{v}}\}$, $\{1_u, 0_u, 3_{\bar{v}}\}$, $\{1_u, 2_u, 1_{\bar{v}}\}$, $\{1_u, 3_u, 0_{\bar{v}}\}$, $\{2_u, 0_u, 2_{\bar{v}}\}$, $\{2_u, 1_u, 1_{\bar{v}}\}$, $\{2_u, 2_u, 0_{\bar{v}}\}$, $\{2_u, 2_u, 0_{\bar{v}}\}$, $\{3_u, 1_u, 0_{\bar{v}}\}$, $\{4_u, 0_u, 0_{\bar{v}}\}$

$$\Pi_{\text{mat}}^{(021)} = \frac{24(-4e_1\bar{v} + 6e_1^2 - 20e_2 + 4\bar{v}^2 + 45)}{(u_{12}^2 + 1)(u_{12}^2 + 4)(4(\bar{v} - u_1)^2 + 9)(4(\bar{v} - u_2)^2 + 9)}, \quad (5.6)$$

where e_1 and e_2 are the elementary symmetric polynomials in u_1 and u_2 .

- The case with $N_\psi = 2$, $N_\phi = 1$ and $N_{\bar{\psi}} = 1$ thus $K_1 = K_2 = 2$ and $K_3 = 1$. We have 27 fixed points and the final result is

$$\begin{aligned} N^{(211)} = 96 \big(& -48s_1u + 12u^2\bar{v}^2 + 80u^2 - 8u\bar{v}^3 - 64u\bar{v} + 8\bar{v}^4 + 143\bar{v}^2 + 612 \\ & -12s_1u^2\bar{v} + 8s_1^2u^2 - 8s_1^2u\bar{v} - 12s_1\bar{v}^3 + 12s_1^2\bar{v}^2 - 111s_1\bar{v} + 90s_1^2 \\ & -20s_2u^2 + 56s_2u\bar{v} - 8s_1s_2u - 12s_2\bar{v}^2 - 16s_1s_2\bar{v} + 12s_2^2 - 201s_2 \big), \end{aligned} \quad (5.7)$$

where s_1 and s_2 are the elementary symmetric polynomials in v_1 and v_2 . The denominator is given by (5.2).

- The case with $N_\psi = 4$, $N_\phi = 0$ and $N_{\bar{\psi}} = 1$ so $K_1 = 3$, $K_2 = 2$ and $K_3 = 1$ this case has 60 fixed points. Our formula gives

$$\begin{aligned} N^{(401)} = 12 \big(& -3s_1\bar{v}^5 - 49s_1\bar{v}^3 - 196s_1\bar{v} + 2\bar{v}^6 + 49\bar{v}^4 + 392\bar{v}^2 + 1040 \\ & + 3s_1^2\bar{v}^4 - 3s_2\bar{v}^4 + 42s_1^2\bar{v}^2 - 63s_2\bar{v}^2 - 28s_1s_2\bar{v} + 152s_1^2 - 340s_2 \\ & - 4s_1s_2\bar{v}^3 + 14s_3\bar{v}^3 + 3s_2^2\bar{v}^2 - 3s_1s_3\bar{v}^2 + 119s_3\bar{v} + 20s_2^2 - 46s_1s_3 \\ & - 30s_4\bar{v}^2 - 3s_2s_3\bar{v} + 15s_1s_4\bar{v} + 2s_3^2 - 5s_2s_4 + 65s_4 \big). \end{aligned} \quad (5.8)$$

Here the s_i are elementary symmetric polynomials v_i , $i = 1, \dots, 4$. For the denominator see (5.2).

³Those with $K_1 + K_2 + K_3 \leq 4$.

- The case with $N_\psi = 0$, $N_\phi = 0$ and $N_{\bar{\psi}} = 5$ thus $K_1 = 1$, $K_2 = 2$ and $K_3 = 3$. We have 60 fixed points and the final result is

$$\Pi_{\text{mat}}^{(005)} = \frac{12(2\bar{s}_1^2 - 5\bar{s}_2 + 25)}{\prod_{i>j}^5 (\bar{v}_{ij}^2 + 1)}, \quad (5.9)$$

where $\bar{s}_i$ are the elementary symmetric polynomials $\bar{v}_i$, $i = 1, \dots, 5$.

5.2 Cases with $r_b = 2$

- The case $N_\psi = 2$, $N_\phi = 0$ and $N_{\bar{\psi}} = 0$ thus $K_1 = 1$, $K_2 = K_3 = 0$. We have only 2 fixed points $\{0_v, 1_v\}$ and $\{1_v, 0_v\}$ using our formula we get

$$\Pi_{\text{mat}}^{(200)} = \frac{2}{v_{12}^2 + 1}. \quad (5.10)$$

- The case $N_\phi = 3$, $N_\psi = N_{\bar{\psi}} = 0$ so $K_1 = K_3 = 1$ and $K_2 = 2$. Here we have 15 fixed points which are: $\{0_u, 0_u, 4_u\}$, $\{0_u, 1_u, 3_u\}$, $\{0_u, 2_u, \tilde{2}_u\}$, $\{0_u, \tilde{2}_u, 2_u\}$, $\{0_u, 3_u, 1_u\}$, $\{0_u, 4_u, 0_u\}$, $\{1_u, 0_u, 3_u\}$, $\{1_u, 3_u, 0_u\}$, $\{2_u, 0_u, \tilde{2}_u\}$, $\{2_u, \tilde{2}_u, 0_u\}$, $\{\tilde{2}_u, 0_u, 2_u\}$, $\{\tilde{2}_u, 2_u, 0_u\}$, $\{3_u, 0_u, 1_u\}$, $\{3_u, 1_u, 0_u\}$, $\{4_u, 0_u, 0_u\}$. Using (4.29), (4.6), (4.30) and table 6 we get

$$\Pi^{(030)} = \frac{6(e_1^2 - 3e_2 + 7)(e_1^2 - 3e_2 + 12)}{\prod_{i<j} (u_{ij}^2 + 1)(u_{ij}^2 + 4)}, \quad (5.11)$$

where e_1 and e_2 are the elementary symmetric polynomials in u_1 , u_2 and u_3 . This coincides with [53].

- Case $N_\psi = N_\phi = N_{\bar{\psi}} = 1$ so $K_1 = K_2 = K_3 = 1$ has 8 fixed points: $\{0_v, 0_u, 3_{\bar{v}}\}$, $\{0_v, 2_u, 1_{\bar{v}}\}$, $\{0_v, 3_u, 0_{\bar{v}}\}$, $\{1_v, 0_u, 2_{\bar{v}}\}$, $\{1_v, 1_u, 1_{\bar{v}}\}$, $\{1_v, \tilde{2}_u, 0_{\bar{v}}\}$, $\{2_v, 0_u, 1_{\bar{v}}\}$, $\{3_v, 0_u, 0_{\bar{v}}\}$. Using our formula we get

$$\Pi_{\text{mat}}^{(111)} = \frac{16(4u^2 - 4uv - 4u\bar{v} + 6v^2 - 8v\bar{v} + 6\bar{v}^2 + 45)}{(4(u-v)^2 + 9)(4(u-\bar{v})^2 + 9)((v-\bar{v})^2 + 4)}. \quad (5.12)$$

- Case $N_\psi = N_\phi = 2$, $N_{\bar{\psi}} = 0$ thus $K_1 = K_2 = 2$, $K_3 = 1$ has 34 fixed points, we get

$$N^{(220)} = 192(-16e_1^3s_1 - 148e_1s_1 + 16e_1^4 - 96e_2e_1^2 + 316e_1^2 + 144e_2^2 - 968e_2 + 1449 + 24e_1^2s_1^2 - 48e_1^2s_2 + 48e_2e_1s_1 - 16e_1s_1s_2 - 80e_2s_1^2 + 224e_2s_2 + 180s_1^2 + 16s_2^2 - 424s_2). \quad (5.13)$$

Here e_1 and e_2 (s_1 and s_2) are the elementary symmetric polynomials in u_1 and u_2 (v_1 and v_2). The denominator is given by (5.2).

- Case $N_\psi = 6$, $N_\phi = 0$ and $N_{\bar{\psi}} = 0$ so $K_1 = 4$, $K_2 = 2$, $K_3 = 1$ has 180 fixed points, we get

$$\Pi_{\text{mat}}^{(600)} = \frac{24(25s_1^2 - 5s_3s_1 + 2s_2^2 - 60s_2 + 10s_4 + 210)}{\prod_{i>j}^6 (v_{ij}^2 + 1)}, \quad (5.14)$$

where s_i are elementary symmetric polynomials in v_i , $i = 1, \dots, 6$.

5.3 Cases with $r_b = 3$

- The case with $N_\psi = N_\phi = 0$ and $N_{\bar{\psi}} = 3$ thus $K_1 = 0$, $K_2 = 1$ and $K_3 = 2$. Here we have 6 fixed points $\{0_{\bar{v}}, 1_{\bar{v}}, 2_{\bar{v}}\}$, $\{0_{\bar{v}}, 2_{\bar{v}}, 1_{\bar{v}}\}$, $\{1_{\bar{v}}, 0_{\bar{v}}, 2_{\bar{v}}\}$, $\{1_{\bar{v}}, 2_{\bar{v}}, 0_{\bar{v}}\}$, $\{2_{\bar{v}}, 0_{\bar{v}}, 1_{\bar{v}}\}$, $\{2_{\bar{v}}, 1_{\bar{v}}, 0_{\bar{v}}\}$, using (4.29), (4.30) and table 6 we get

$$\Pi_{\text{mat}}^{(003)} = \frac{6}{(\bar{v}_{12}^2 + 1)(\bar{v}_{13}^2 + 1)(\bar{v}_{23}^2 + 1)}. \quad (5.15)$$

- Case with $N_\psi = 1$, $N_\phi = 2$ and $N_{\bar{\psi}} = 0$ thus $k_1 = 1$, $k_2 = 2$ and $k_3 = 1$ has 12 fixed points. Our result is

$$\Pi_{\text{mat}}^{(120)} = \frac{24(-4e_1v + 6e_1^2 - 20e_2 + 4v^2 + 45)}{(u_{12}^2 + 1)(u_{12}^2 + 4)(4(u_1 - v)^2 + 9)(4(u_2 - v)^2 + 9)}, \quad (5.16)$$

where e_1 and e_2 are the elementary symmetric polynomials in u_1, u_2 .

- Case with $N_\psi = 1$, $N_\phi = 1$ and $N_{\bar{\psi}} = 2$ therefor $K_1 = 1$ and $K_2 = K_3 = 2$ has 27 fixed points. We get

$$\begin{aligned} N^{(112)} = & -96(12\bar{s}_1u^2v - 8\bar{s}_1^2u^2 + 20\bar{s}_2u^2 + 48\bar{s}_1u - 90\bar{s}_1^2 + 201\bar{s}_2 - 80u^2 - 612 \\ & + 8\bar{s}_1^2uv - 56\bar{s}_2uv + 8\bar{s}_1\bar{s}_2u + 111\bar{s}_1v + 16\bar{s}_1\bar{s}_2v - 12\bar{s}_2^2 + 64uv \\ & + 12\bar{s}_1v^3 - 12\bar{s}_1^2v^2 + 12\bar{s}_2v^2 - 12u^2v^2 + 8uv^3 - 8v^4 - 143v^2), \end{aligned} \quad (5.17)$$

$\bar{s}_1$ and $\bar{s}_2$ are the elementary symmetric polynomials in $\bar{v}_1, \bar{v}_2$. For the denominator see (5.2).

- Case with $N_\psi = 5$, $N_\phi = 0$ and $N_{\bar{\psi}} = 0$ so $K_1 = 3$, $K_2 = 2$ and $K_3 = 1$ has 60 fixed points. The result is

$$\Pi_{\text{mat}}^{(500)} = \frac{24s_1^2 - 60s_2 + 300}{\prod_{i>j}^5 (v_{ij}^2 + 1)}, \quad (5.18)$$

where s_i are elementary symmetric polynomials in v_i , $i = 1, \dots, 5$.

5.4 Cases with $r_b = 0$ or $r_b = 4$

As one can see the constraints (4.3)–(4.5) on K_1 , K_2 and K_3 are the same when $r_b = 0$ and $r_b = 4$ so this two cases give rise to the same $\Pi_{\text{mat}}^{(N_\psi N_\phi N_{\bar{\psi}})}$.

- Case with $N_\psi = 2$, $N_\phi = 1$ and $N_{\bar{\psi}} = 0$ so $k_1 = 2$, $k_2 = 2$ and $k_3 = 1$ has 12 fixed points. The result is

$$\Pi_{\text{mat}}^{(210)} = \frac{192}{(v_{12}^2 + 1)(4(u - v_1)^2 + 9)(4(u - v_2)^2 + 9)}. \quad (5.19)$$

- Case with $N_\psi = 1$, $N_\phi = 2$ and $N_{\bar{\psi}} = 1$ so $K_1 = 2$, $K_2 = 3$ and $K_3 = 2$ has 48 fixed points. The result is

$$\begin{aligned} N^{(121)} = & 384(-84e_1v - 84e_1\bar{v} + 16v^2\bar{v}^2 + 180v^2 - 192v\bar{v} + 180\bar{v}^2 + 1161 \\ & - 16e_1v^2\bar{v} + 24e_1^2v^2 - 16e_1v\bar{v}^2 - 32e_1^2v\bar{v} + 24e_1^2\bar{v}^2 + 180e_1^2 - 80e_2v^2 \\ & + 192e_2v\bar{v} - 16e_1e_2v - 80e_2\bar{v}^2 - 16e_1e_2\bar{v} + 16e_2^2 - 552e_2), \end{aligned} \quad (5.20)$$

where e_1 and e_2 are the elementary symmetric polynomials u_1 and u_2 . For the denominator see (5.2).

- Case with $N_\psi = 4$, $N_\phi = 0$ and $N_{\bar{\psi}} = 0$ thus $K_1 = 3$, $K_2 = 2$ and $K_3 = 1$ has 24 fixed points. The result is

$$\Pi_{\text{mat}}^{(400)} = \frac{24}{\prod_{i>j}^4 (v_{ij}^2 + 1)}. \quad (5.21)$$

6 Conclusions and perspectives

It would be interesting to find a physical interpretation of the ADHM-like moduli space we have constructed. Perhaps the identification of the real ADHM equation counterpart of the matrix equations (2.9)–(2.11) would have some significance as well. Another achievement of this paper may be consider that of a general approach for passing from an integral representation with some group-theoretical structure to a combinatorial sum over Young diagrams.

Thanks to the peculiar two $SU(4)$ matrix structure of the fermions and anti-fermions contributions, the papers [36, 42, 51] have re-summed the leading contributions to those from gluons (and their bound states) at strong coupling so that to give the same Thermodynamic Bethe Ansatz results as (classical) string theory ([51] furnishes by the same method also subleading corrections, waiting for one-loop confirmation). This computation resembles the poles contributions of the instanton partition function of $\mathcal{N} = 2$ SYM [55] in the so-called Nekrasov-Shatashvili (NS) limit [61] (and similarly for the subleading correction to the NS limit, computed in [62, 63]). If this represents a second way to TBA (with respect to the ordinary one [64]), which surprisingly stems from FFs, a third one can be counted as the massive Ordinary Differential Equation/ Integrable Model (ODE/IM) correspondence which [65] has recently shown to ‘solve’ the dual (classical) string theory thanks to the full-fledged quantum integrability structures: not only T -, Y -systems and TBA [46–48], but also the more fundamental Q -functions with relative functional and integral equations. These structures have been derived from the discrete (Ω - and Λ -) symmetries acting on the gauge (or differential equation) moduli space (with the extension of a twist or angular momentum for the string solution). In this perspective, the FF series is incorporated in a full integrability structure in its strong coupling and hence the (exact) all coupling expressions, we are dealing with here, will acquire even more importance as a possible (second) quantization of the massive ODE/IM correspondence. In a nutshell, the extension (at all couplings) and rôle of the Q -functions shall be punctually scrutinized as these are the most fundamental objects on the integrability side and the closest to the ODE wave function according to [66].

Yet, the OPE series is much more complicated than the NS one and, in particular, its strong coupling seems insensible to many details of the weak or all coupling regime (the presence of the scalars, for instance). Maybe this is a positive point in favor of the quantization of massive ODE/IM correspondence as could be argued by looking at the simple and elegant structure of the next to leading expression in the NS regime [62, 63]. In fact, how to quantize the TBA is a long-standing question, but gauge or string theory may know the answer.

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