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A proof of loop equations in 2d topological gravity

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Abstract: We study multi-boundary correlators in 2d Witten-Kontsevich topological gravity. We present a proof of the loop equations obeyed by the correlators. While the loop equations were derived a long time ago, our proof is fully explicit in the presence of general couplings t_k. We clarify all the details, in particular the treatment of the genus zero part of the one-boundary correlator. The loop equations are verified by several new examples, including the correlators of Jackiw-Teitelboim gravity in the genus expansion and the exact correlators in the Airy case. We also discuss the free boson/fermion representation of the correlators and compare it with the formulation of Marolf and Maxfield and the string field theory of Ishibashi and Kawai. We find similarities but also some differences.

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1 Introduction

Two dimensional gravity coupled to a conformal matter is a useful toy model for the study of quantum gravity. When coupled to a particular conformal matter, it is exactly solvable via a double-scaled matrix model [4–6]. For instance, 2d gravity coupled to the $(2, 2k-1)$ minimal model is described by a hermitian one-matrix model at a multi-critical point, and

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it corresponds to a particular background of Witten-Kontsevich topological gravity [7, 8] where the first k couplings t_n $(n \leq k)$ are turned on in a certain manner.

Recently, Saad, Shenker and Stanford showed that Jackiw-Teitelboim (JT) gravity [9, 10] is also described by a certain double-scaled matrix model [11], which enables us to study the holographic duality in a solvable model of 2d gravity. It is shown in [12–14] that the matrix model of JT gravity is a special case of Witten-Kontsevich topological gravity where the infinitely many couplings t_n are turned on in a specific way (see (3.28)).[1] This connection between JT gravity and matrix model puts the old story of 2d gravity into the modern perspective of holography. In the matrix model of JT gravity [11], the Hamiltonian H of the boundary theory becomes the random matrix and the path integral of JT gravity on the asymptotically AdS spacetime with boundary lengths β_i $(i = 1, \ldots, n)$ corresponds to the ensemble average of the partition function $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ over the random Hamiltonian H

$$\langle Z(\beta_1) \cdots Z(\beta_n) \rangle. \tag{1.1}$$

In the 2d gravity literature, such quantity (1.1) is known as the correlator of macroscopic loop operators $Z(\beta)$. These correlators satisfy a set of equations called the loop equation. As we review in appendix C, the loop equation of finite N matrix model simply follows from the Schwinger-Dyson equation for the matrix integral. After taking the double-scaling limit of the loop equation, one can derive the Virasoro constraint obeyed by the partition function of the hermitian one-matrix model [16, 17]. As shown in [18], the same Virasoro constraint can be derived from the Kontsevich matrix model [8] as well.

The loop equation in the double-scaled matrix model is customarily written in terms of the resolvent, which is related to the macroscopic loop operator $Z(\beta)$ by the Laplace transformation. The genus expansion of the resolvent can be computed systematically by using the topological recursion, which basically follows from the loop equation [19].

In this paper, we will write down the loop equation for the correlator of macroscopic loop operators $Z(\beta)$ in Witten-Kontsevich topological gravity for the arbitrary background $\{t_k\}$. In particular, we will elaborate on the treatment of the genus-zero part of the one-point function $\langle Z(\beta) \rangle$, i.e. the disk amplitude, and prove the loop equation in the case with non-zero genus-zero part u_0 of the specific heat $u = g_\mathrm{s}^2 \partial_0^2 F$ of topological gravity. We emphasize that although the loop equation of topological gravity was already written in the original paper [17], the details of the treatment of the disk amplitude and the $u_0 \neq 0$ case for the general background have not been worked out in the literature before, as far as we know. Starting from the Virasoro constraints (2.3), we will prove the loop equations (3.1) and (3.2) for Witten-Kontsevich topological gravity with arbitrary couplings $\{t_k\}$.

Our loop equation involves the effective potential $V_\mathrm{eff}(\xi)$ whose explicit form was recently obtained for the general background with $u_0 \neq 0$ [20]. The effective potential $V_\mathrm{eff}(\xi)$ is defined by the leading term of the genus expansion of the Baker-Akhiezer function $\psi(\xi) = e^{-\frac{1}{2g_\mathrm{s}} V_\mathrm{eff}(\xi) + \mathcal{O}(g_\mathrm{s}^0)}$ and it is related to the genus-zero part of the eigenvalue density by

$$\rho_0(E) = \frac{1}{2\pi g_\mathrm{s}} \mathrm{Im} V_\mathrm{eff}'(-E + \mathrm{i}0). \tag{1.2}$$

[1] JT gravity can also be viewed as the $p \to \infty$ limit of the $(2, p)$ minimal model coupled to 2d gravity [11, 15].

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The $u_0 \neq 0$ case is of particular interest since the edge of the spectrum of $\rho_0(E)$ is shifted from $E = 0$ to a non-zero threshold energy $E_0 = -u_0$. Near $E = E_0$, $\rho_0(E)$ generically behaves as

$$\rho_0(E) \sim \sqrt{E - E_0}. \tag{1.3}$$

This shift of threshold energy played an important role in the recent study of JT gravity with conical defects [21, 22].

We revisit the loop equation in topological gravity partly because we are motivated by the recent discussion of the null state originating from the diffeomorphism invariance of the gravitational path integral [1]. It has long been speculated that the Virasoro constraint of double-scaled matrix model represents the diffeomorphism invariance of 2d gravity, and as a consequence the loop equation defines a null state. We give a concrete expression to this argument using the free boson/fermion representation of the correlators which we developed in [20]. We find similarities and differences between our expression and that advocated in [1]. We will give a cautionary remark on the naive application of the sewing operation in the gravitational path integral, which might be related to the differences between the two expressions. Our expression of loop equation has a close connection to the closed string field theory of non-critical strings developed by Ishibashi and Kawai [2, 3] but the details are slightly different.

This paper is organized as follows. In section 2, we summarize the definitions and basic properties of the multi-boundary correlators in Witten-Kontsevich topological gravity. We also introduce a continuum analog of the Virasoro operators and recall the explicit form of the effective potential. In section 3, we give a precise description and a proof of the general loop equations. We also make an interpretation of the absence of the disk amplitude. The loop equations are then verified in the JT gravity and the Airy cases as well as in the most general case. In section 4, we first recall that the Virasoro constraint and the loop equation can be nicely expressed in the free boson/fermion representation of the Witten-Kontsevich τ-function. Then we comment on the difference between our result and the approach of Marolf and Maxfield in [1]. We also comment on the similarity and difference between our result and the closed string field theory of Ishibashi and Kawai [2, 3]. Finally we conclude in section 5. Some of the details of the proof of (3.1) and (3.2) in the main text are relegated to the appendices A and B. In appendices C and D, we review the loop equation at finite N and the cut-and-join representation of the Witten-Kontsevich τ-function, respectively.

2 Multi-boundary correlators in Witten-Kontsevich gravity

2.1 Generating function for intersection numbers

In Witten-Kontsevich topological gravity [7, 8] (see e.g. [13] for a recent review) observables are made up of the intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \qquad d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

They are associated with a closed Riemann surface Σ of genus g with n marked points $p_1, \ldots, p_n$. We let $\mathcal{M}_{g,n}$ denote the moduli space of Σ and $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford

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compactification of $\mathcal{M}_{g,n}$. Here $\tau_{d_i} = \psi_i^{d_i}$ and ψ_i is the first Chern class of the complex line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber is the cotangent space to p_i. The generating function for the intersection numbers is defined as

$$F(\{t_k\}) := \sum_{g=0}^{\infty} g_{\rm s}^{2g-2} F_g(\{t_k\}), \qquad F_g(\{t_k\}) := \left\langle e^{\sum_{d=0}^{\infty} t_d \tau_d} \right\rangle_g . \tag{2.2}$$

Here $g_{\rm s}$ is the genus counting parameter. F is uniquely determined either by the KdV equations with the string equation [7, 8] or by the Virasoro constraints [16, 17]. In our previous papers [20, 23] we formulated a systematic method of computing multi-boundary correlators based on the former conditions. In this paper we instead investigate the implication of the latter conditions for the multi-boundary correlators.

The Virasoro constraints are written as the highest weight conditions

$$L_m e^F = 0 \qquad m \geq -1. \tag{2.3}$$

The Virasoro generators are given by

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{k \geq 0} \frac{(2k+2m+1)!!}{(2k-1)!!} \tilde{t}_k \partial_{k+m} + \frac{g_{\rm s}^2}{4} \sum_{\substack{k,l \geq 0 \\ k+l=m-1}} (2k+1)!!(2l+1)!! \partial_k \partial_l \quad (m \geq 1), \\ L_0 &= \frac{1}{16} + \frac{1}{2} \sum_{k \geq 0} (2k+1) \tilde{t}_k \partial_k, \\ L_{-1} &= \frac{t_0^2}{4 g_{\rm s}^2} + \frac{1}{2} \sum_{k \geq 0} \tilde{t}_{k+1} \partial_k, \end{aligned} \tag{2.4}$$

where

$$\partial_k := \frac{\partial}{\partial t_k}, \qquad \tilde{t}_k := t_k - \delta_{k,1}. \tag{2.5}$$

L_m satisfy

$$[L_m, L_n] = (m-n) L_{m+n} \qquad m, n \geq -1. \tag{2.6}$$

For later convenience, let us introduce the Itzykson-Zuber variables [18]

$$I_n(v) \equiv I_n(v, \{t_k\}) = \sum_{m=0}^{\infty} t_{n+m} \frac{v^m}{m!} \quad (n \geq 0). \tag{2.7}$$

They satisfy

$$\partial_v I_n(v) = I_{n+1}(v). \tag{2.8}$$

Throughout this paper I_n without specifying its argument should always be understood as

$$I_n = I_n(u_0) \tag{2.9}$$

with

$$u_0 := \partial_0^2 F_0. \tag{2.10}$$

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F and multi-boundary correlators can be expressed in terms of either $\{t_k\}$ or $\{I_n\}$. The relation between t_k and I_n can also be expressed as

$$t_k = \sum_{m=0}^{\infty} \frac{(-u_0)^m}{m!} I_{k+m}. \tag{2.11}$$

The variables I_n are useful because F_g $(g \geq 2)$ are polynomials in I_n $(n \geq 2)$ and $(1-I_1)^{-1}$. The (genus zero) string equation also takes the simple form

$$I_0 = u_0. \tag{2.12}$$

2.2 Connected n-boundary correlator

In this paper we are interested in the n-boundary connected correlators

$$Z_n(\beta_1, \ldots, \beta_n) = \langle Z(\beta_1) \cdots Z(\beta_n) \rangle_{\text{conn}}. \tag{2.13}$$

They are given by the gravitational path integrals over all possible connected Riemann surfaces with n boundaries (or more specifically, n macroscopic loops in the matrix model language) of length $\beta_1, \ldots, \beta_n$. They are generated from F as [24]

$$Z_n(\beta_1, \ldots, \beta_n) \simeq B(\beta_1) \cdots B(\beta_n) F, \tag{2.14}$$

where

$$B(\beta) = \frac{g_{\text{s}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \beta^{k+\frac{1}{2}} \partial_k \tag{2.15}$$

is the boundary creation operator.[2] The symbol "$\simeq$" in (2.14) means that the equality holds up to an additional non-universal part [24]. More specifically, we call

$$Z_n^{\text{u}}(\beta_1, \ldots, \beta_n) := B(\beta_1) \cdots B(\beta_n) F \tag{2.17}$$

the universal parts of Z_n and decompose Z_n as

$$Z_n(\beta_1, \ldots, \beta_n) = Z_n^{\text{u}}(\beta_1, \ldots, \beta_n) + Z_n^{\text{nu}}(\beta_1, \ldots, \beta_n). \tag{2.18}$$

[2] As we saw in (2.1), insertion of τ_d adds a marked point, i.e. a puncture on the Riemann surface, which corresponds to a microscopic loop in the dual matrix model. The macroscopic loop operator $Z(\beta)$ in the limit $\beta \to 0$ is expanded in terms of the microscopic loop operator τ_d as

$$Z(\beta) \simeq \frac{g_{\text{s}}}{\sqrt{2\pi}} \sum_{d=0}^{\infty} \beta^{d+\frac{1}{2}} \tau_d. \tag{2.16}$$

The insertion of τ_d is represented by the derivative ∂_d when acting on the free energy F. Therefore $B(\beta)$ of the form (2.15) is viewed as the operator that creates a macroscopic loop, i.e. a boundary on the Riemann surface.

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The deviations $Z_n^{\rm nu}$, which we call the non-universal parts, appear only for $n = 1, 2$ and are given by [20]

$$\begin{aligned} Z_1^{\rm nu}(\beta) &= \frac{1}{g_{\rm s}}\sqrt{\frac{\beta}{2\pi}}\int_{-\infty}^{0} dv\,(I_0(v) - v)\,e^{\beta v} \\ &= \frac{1}{\sqrt{2\pi}g_{\rm s}}\sum_{k=0}^{\infty}(-1)^k\beta^{-k-\frac{1}{2}}\tilde{t}_k, \\ Z_2^{\rm nu}(\beta_1,\beta_2) &= \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2}. \end{aligned} \tag{2.19}$$

It is easy to check that

$$\begin{aligned} B(\beta_2)Z_1^{\rm nu}(\beta_1) &= Z_2^{\rm nu}(\beta_1,\beta_2), \\ B(\beta_3)Z_2^{\rm nu}(\beta_1,\beta_2) &= 0. \end{aligned} \tag{2.20}$$

Therefore, Z_n for general n can also be expressed as

$$\begin{aligned} Z_n(\beta_1,\ldots,\beta_n) &= B(\beta_1)\cdots B(\beta_{n-1})Z_1(\beta_n) \qquad n \geq 2, \\ Z_1(\beta) &= B(\beta)F + Z_1^{\rm nu}(\beta). \end{aligned} \tag{2.21}$$

2.3 Full n-boundary correlator

For our purposes it is convenient to consider the full n-boundary correlators as well. We let them be denoted by

$$\mathcal{Z}_n(\beta_1,\ldots,\beta_n) = \langle Z(\beta_1)\cdots Z(\beta_n)\rangle. \tag{2.22}$$

They are given by the gravitational path integrals over all possible Riemann surfaces, including disconnected ones, with n boundaries of length $\beta_1,\ldots,\beta_n$. They are expressed in terms of the connected correlators as

$$\begin{aligned} \mathcal{Z}_1(\beta) &= Z_1(\beta), \\ \mathcal{Z}_2(\beta_1,\beta_2) &= Z_2(\beta_1,\beta_2) + Z_1(\beta_1)Z_1(\beta_2), \\ \mathcal{Z}_3(\beta_1,\beta_2,\beta_3) &= Z_3(\beta_1,\beta_2,\beta_3) + Z_2(\beta_1,\beta_2)Z_1(\beta_3) + Z_2(\beta_2,\beta_3)Z_1(\beta_1) \\ &\quad + Z_2(\beta_3,\beta_1)Z_1(\beta_2) + Z_1(\beta_1)Z_1(\beta_2)Z_1(\beta_3). \end{aligned} \tag{2.23}$$

In general, the relation between the full and the connected correlators is expressed by means of the generating functionals

$$\mathcal{Z}[J] = e^{\mathcal{F}[J]}, \tag{2.24}$$

where

$$\begin{aligned} \mathcal{Z}[J] &:= 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int d\beta_1\cdots d\beta_n J(\beta_1)\cdots J(\beta_n)\mathcal{Z}_n(\beta_1,\ldots,\beta_n), \\ \mathcal{F}[J] &:= \sum_{n=1}^{\infty}\frac{1}{n!}\int d\beta_1\cdots d\beta_n J(\beta_1)\cdots J(\beta_n)Z_n(\beta_1,\ldots,\beta_n). \end{aligned} \tag{2.25}$$

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Equivalently, in terms of $\mathcal{F}[J]$ the full and the connected n-boundary correlators are expressed respectively as

$$\begin{aligned}\mathcal{Z}_n(\beta_1,\dots,\beta_n) &= \frac{\delta}{\delta J(\beta_1)}\cdots\frac{\delta}{\delta J(\beta_n)}e^{\mathcal{F}}\Big|_{J=0},\\ Z_n(\beta_1,\dots,\beta_n) &= \frac{\delta}{\delta J(\beta_1)}\cdots\frac{\delta}{\delta J(\beta_n)}\mathcal{F}\Big|_{J=0}.\end{aligned} \tag{2.26}$$

Let us now introduce the operator

$$\mathcal{B}(\beta) := B(\beta) + Z_1^{\mathrm{nu}}(\beta) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\left[g_{\mathrm{s}}\beta^{k+\frac{1}{2}}\partial_k + g_{\mathrm{s}}^{-1}(-1)^k\beta^{-k-\frac{1}{2}}\tilde{t}_k\right], \tag{2.27}$$

where $B(\beta)$ and $Z_1^{\mathrm{nu}}(\beta)$ are given in (2.15) and (2.19) respectively. The meaning of $\mathcal{B}(\beta)$ is understood as follows. Recall that for any operators X, Y satisfying $[X,[X,Y]] = [Y,[X,Y]] = 0$ the Baker-Campbell-Hausdorff formula is written as

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{2.28}$$

By setting $X = \int d\beta J(\beta)\mathcal{B}(\beta)$ and $Y = -\int d\beta J(\beta)B(\beta)$ one obtains

$$\begin{aligned}e^{\int d\beta J(\beta)\mathcal{B}(\beta)} &= e^{\int d\beta J(\beta)Z_1^{\mathrm{nu}}(\beta)+\frac{1}{2}\left[\int d\beta_1 J(\beta_1)\mathcal{B}(\beta_1), -\int d\beta_2 J(\beta_2)B(\beta_2)\right]}e^{\int d\beta J(\beta)B(\beta)}\\ &= e^{\int d\beta J(\beta)Z_1^{\mathrm{nu}}(\beta)+\frac{1}{2}\int d\beta_1 d\beta_2 J(\beta_1)J(\beta_2)Z_2^{\mathrm{nu}}(\beta_1,\beta_2)}e^{\int d\beta J(\beta)B(\beta)}.\end{aligned} \tag{2.29}$$

From this one sees that

$$\begin{aligned}&e^{\int d\beta J(\beta)\mathcal{B}(\beta)}e^F\\ &= e^{\int d\beta J(\beta)Z_1^{\mathrm{nu}}(\beta)+\frac{1}{2}\int d\beta_1 d\beta_2 J(\beta_1)J(\beta_2)Z_2^{\mathrm{nu}}(\beta_1,\beta_2)}e^{\int d\beta J(\beta)B(\beta)}e^F\\ &= e^{\int d\beta J(\beta)Z_1^{\mathrm{nu}}(\beta)+\frac{1}{2}\int d\beta_1 d\beta_2 J(\beta_1)J(\beta_2)Z_2^{\mathrm{nu}}(\beta_1,\beta_2)}e^{\sum_{n=0}^{\infty}\frac{1}{n!}\int d\beta_1\cdots d\beta_n J(\beta_1)\cdots J(\beta_n)B(\beta_1)\cdots B(\beta_n)F}\\ &= e^F e^{\mathcal{F}[J]}.\end{aligned} \tag{2.30}$$

Using (2.26) one obtains

$$\mathcal{Z}_n(\beta_1,\dots,\beta_n) = e^{-F}\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_n)e^F. \tag{2.31}$$

Therefore, $\mathcal{B}(\beta)$ is interpreted as the boundary creation operator that generates the full correlators.

2.4 Continuous Virasoro operator

Let us introduce

$$L(\beta) := \sum_{m=-1}^{\infty}\frac{\beta^{m+1}}{(m+1)!2^m}L_m. \tag{2.32}$$

$L(\beta)$ is viewed as the continuum analog of the Virasoro generators L_m. Indeed, it follows from (2.6) that $L(\beta)$ satisfies

$$[L(\beta_1), L(\beta_2)] = (\beta_1-\beta_2)L(\beta_1+\beta_2), \tag{2.33}$$

which is viewed as a continuum limit of (2.6). Moreover, L and $\mathcal{B}$ satisfy

$$[L(\beta),\mathcal{B}(\beta')] = -\beta'\mathcal{B}(\beta+\beta'). \tag{2.34}$$

We give a proof of this relation in appendix A.

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2.5 Effective potential

The correlators Z_n as well as F are uniquely characterized by the KdV equations. The KdV equations are obtained as the compatibility conditions of the Schrödinger equation

$$\left(\frac{g_{\rm s}^2}{2}\partial_0^2 + u\right)\psi = \xi\psi \tag{2.35}$$

with $u = g_{\rm s}^2\partial_0^2 F$ and the KdV flow equations (see [25] for their explicit forms in our convention). The Baker-Akhiezer function $\psi(\xi;\{t_k\})$ is a solution to these auxiliary linear differential equations. It plays an important role in topological gravity. In our previous papers we systematically investigated the multi-boundary correlators Z_n [20, 23] and the open free energy [25] using the fact that they are expressed in terms of $\psi(\xi)$. For instance, $Z_1(\beta)$ is related to $\psi(\xi)$ as

$$\partial_0 Z_1(\beta) = \frac{\sqrt{2}}{g_{\rm s}}\int_{-\infty}^{\infty} d\xi e^{\beta\xi}\psi(\xi)^2. \tag{2.36}$$

In terms of $\psi(\xi)$ the effective potential $V_{\rm eff}(\xi)$ is introduced as the leading order exponent in the small $g_{\rm s}$ expansion

$$\psi(\xi) = e^{-\frac{1}{2g_{\rm s}}V_{\rm eff}(\xi)+\mathcal{O}(g_{\rm s}^0)}. \tag{2.37}$$

The explicit form of $V_{\rm eff}(\xi)$ for general Witten-Kontsevich gravity was first obtained in [20]. It is given by[3]

$$V_{\rm eff}(\xi) = -2\sum_{n=1}^{\infty}\frac{I_n-\delta_{n,1}}{(2n+1)!!}2^{n+\frac{1}{2}}(\xi-u_0)^{n+\frac{1}{2}}. \tag{2.38}$$

In this paper we will deal with its first derivative

$$V'_{\rm eff}(\xi) = -2\sum_{n=1}^{\infty}\frac{I_n-\delta_{n,1}}{(2n-1)!!}2^{n-\frac{1}{2}}(\xi-u_0)^{n-\frac{1}{2}}. \tag{2.39}$$

In fact, $V'_{\rm eff}$ will appear in the form of the differential operator $V'_{\rm eff}(\partial_\beta)$, which consists of half-integer powers of $(\partial_\beta - u_0)$. The half-integer power of the differential operator ∂_β is defined as (see e.g. [17])

$$\partial_\beta^{k-\frac{1}{2}}\beta^{n-\frac{1}{2}} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-k+1)}\beta^{n-k} \qquad (k,n\in\mathbb{Z}). \tag{2.40}$$

It follows that

$$(\partial_\beta - u_0)^{k-\frac{1}{2}}e^{\beta u_0}\beta^{n-\frac{1}{2}} = e^{\beta u_0}\partial_\beta^{k-\frac{1}{2}}\beta^{n-\frac{1}{2}} = e^{\beta u_0}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-k+1)}\beta^{n-k}. \tag{2.41}$$

[3] The effective potential in this paper is related to that in [20] by $V_{\rm eff}^{\rm here} = \sqrt{2}V_{\rm eff}^{\rm there}$ with the identification $I_n = (-1)^n B_{n-1}$ $(n\geq 2)$. As explained in [26], generalization from the JT gravity case to the Witten-Kontsevich case is straightforward.

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2.6 One-boundary correlator and disk amplitude

In this paper we will use the following decomposition of the one-boundary correlator

$$Z_1(\beta) = Z_1^{g\geq 1}(\beta) + Z_1^{g=0}(\beta). \tag{2.42}$$

The higher genus part is given by

$$\begin{aligned} Z_1^{g\geq 1}(\beta) &= B(\beta) F^{g\geq 1} \\ &= B(\beta) \sum_{g=1}^{\infty} g_{\mathrm{s}}^{2g-2} F_g. \end{aligned} \tag{2.43}$$

The genus zero part, i.e. the disk amplitude is given by [20]

$$Z_1^{g=0}(\beta) = \frac{1}{g_{\mathrm{s}}} \sqrt{\frac{\beta}{2\pi}} \int_{-\infty}^{u_0} dv \, (I_0(v) - v) \, e^{\beta v}. \tag{2.44}$$

We further decompose it as

$$Z_1^{g=0}(\beta) = Z_1^{\mathrm{nu}}(\beta) + Z_1^{\mathrm{u},g=0}(\beta), \tag{2.45}$$

where $Z_1^{\mathrm{nu}}(\beta)$ is the non-universal part given in (2.19) and

$$Z_1^{\mathrm{u},g=0}(\beta) = \frac{1}{g_{\mathrm{s}}} \sqrt{\frac{\beta}{2\pi}} \int_{0}^{u_0} dv \, (I_0(v) - v) \, e^{\beta v} = B(\beta) \frac{F_0}{g_{\mathrm{s}}^2} \tag{2.46}$$

is the genus-zero universal part.

3 Proof of loop equations

3.1 Loop equations

The main purpose of this paper is to prove the loop equations for general Witten-Kontsevich gravity. The fundamental loop equation is written as

$$\int_0^{\beta} ds \left[Z_2(s, \beta - s) + Z_1^{g\geq 1}(s) Z_1^{g\geq 1}(\beta - s) \right] = \frac{1}{g_{\mathrm{s}}} V'_{\mathrm{eff}}(\partial_\beta) Z_1^{g\geq 1}(\beta). \tag{3.1}$$

More generally, connected multi-boundary correlators satisfy

$$\begin{aligned} &\frac{1}{g_{\mathrm{s}}} V'_{\mathrm{eff}}(\partial_\beta) \widetilde{Z}_{n+1}(\beta, \beta_1, \ldots, \beta_n) \\ &= \int_0^{\beta} ds \left[Z_{n+2}(s, \beta - s, \beta_1, \ldots, \beta_n) + \sum_{I \subset S} \widetilde{Z}_{|I|+1}(s; \beta_I) \widetilde{Z}_{|S-I|+1}(\beta - s; \beta_{S-I}) \right] \\ &\quad + \sum_{j=1}^{n} \beta_j Z_n(\{\beta_i + \delta_{ij}\beta\}_{i=1}^n) \qquad n \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{3.2}$$

Here

$$\widetilde{Z}_{n+1} = \begin{cases} Z_1^{g\geq 1} & n = 0 \\ Z_{n+1} & n \geq 1 \end{cases}, \qquad \widetilde{Z}_{|I|+1}(s; \beta_I) = \begin{cases} Z_1^{g\geq 1}(s) & |I| = 0 \\ Z_{|I|+1}(s, \beta_{i_1}, \ldots, \beta_{i_{|I|}}) & |I| \geq 1 \end{cases} \tag{3.3}$$

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with $I = \{i_1, i_2, \ldots, i_{|I|}\}$, $S = \{1, 2, \ldots, n\}$ and the sum is taken for all possible subsets I of S including the empty set. (3.2) includes (3.1) as the $n = 0$ case.

As we reviewed in appendix C, the above loop equations have appearances naturally expected from those for the finite-size matrix model. Moreover, apart from the treatment of the genus-zero contribution, (3.1) was derived in [17] for double-scaled matrix models. To the best of our knowledge, however, the precise treatment of the genus-zero contribution, i.e. the absence of the disk amplitude as presented in (3.1) and (3.2) has never been clearly stated in the literature. Also, in [17] the general form of the loop equation was extrapolated from that of the multi-critical models, but it is not so clear how their derivation is generalized in the case of $u_0 \neq 0$. Having the explicit form (2.39) of $V'_{\text{eff}}(\xi)$ obtained recently, we think it is meaningful to revisit the derivation in a more specific manner. In what follows we will present a rigorous, concrete proof of (3.1) and (3.2). In section 3.4 we will also remark on what the absence of the disk amplitude implies.

3.2 Proof of fundamental loop equation

Let us first prove (3.1). By plugging (2.4) into (2.32) the operator $L(\beta)$ is explicitly written as

$$\begin{aligned} L(\beta) &= \frac{g_{\text{s}}^2}{2} \sum_{k,l\geq 0} \frac{(2k+1)!!(2l+1)!!\beta^{k+l+2}}{(k+l+2)!2^{k+l+2}} \partial_k \partial_l \\ &\quad + \sum_{\substack{m\geq -1,\ k\geq 0 \\ (m,k)\neq(-1,0)}} \frac{(2k+2m+1)!!\beta^{m+1}}{(2k-1)!!(m+1)!2^{m+1}} \tilde{t}_k \partial_{k+m} + \frac{\beta}{16} + \frac{t_0^2}{2g_{\text{s}}^2} \\ &=: L(\beta)\Big|_{\partial\partial} + L(\beta)\Big|_{\tilde{t}\partial} + \frac{\beta}{16} + \frac{t_0^2}{2g_{\text{s}}^2}. \end{aligned} \tag{3.4}$$

Let us rewrite the above expression in terms of the boundary creation operator $B(\beta)$ given in (2.15) instead of the derivative ∂_k. The first term is immediately rewritten as

$$\begin{aligned} L(\beta)\Big|_{\partial\partial} &= \frac{g_{\text{s}}^2}{2\pi} \sum_{k,l\geq 0} \frac{\Gamma(k+\frac{3}{2})\Gamma(l+\frac{3}{2})}{\Gamma(k+l+3)} \beta^{k+l+2} \partial_k \partial_l \\ &= \int_0^\beta ds B(s) B(\beta - s). \end{aligned} \tag{3.5}$$

As we prove in appendix B, the second term can be expressed as

$$L(\beta)\Big|_{\tilde{t}\partial} = -2 \int_0^\beta ds Z_1^{\text{u},g=0}(\beta - s) B(s) - \frac{1}{g_{\text{s}}} V'_{\text{eff}}(\partial_\beta) B(\beta), \tag{3.6}$$

where $Z_1^{\text{u},g=0}(\beta)$ and $V'_{\text{eff}}(\xi)$ are given in (2.46) and (2.39) respectively. Thus (3.4) is rewritten as

$$L(\beta) = \int_0^\beta ds B(s) B(\beta - s) - 2 \int_0^\beta ds Z_1^{\text{u},g=0}(\beta - s) B(s) - \frac{1}{g_{\text{s}}} V'_{\text{eff}}(\partial_\beta) B(\beta) + \frac{\beta}{16} + \frac{t_0^2}{2g_{\text{s}}^2}. \tag{3.7}$$

The Virasoro constraints (2.3) imply that

$$L(\beta) e^F = 0. \tag{3.8}$$

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By using (3.7) this equation is written as

$$
\begin{aligned}
0 = e^{-F}L(\beta)e^{F} &= \int_0^\beta ds[B(s)B(\beta-s)F + (B(s)F)(B(\beta-s)F)] \\
&\quad - 2\int_0^\beta ds Z_1^{\mathrm{u},g=0}(\beta-s)B(s)F - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)B(\beta)F + \frac{\beta}{16} + \frac{t_0^2}{2g_{\mathrm{s}}^2} \\
&= \int_0^\beta ds\,[Z_2^{\mathrm{u}}(s,\beta-s) + Z_1^{\mathrm{u}}(s)Z_1^{\mathrm{u}}(\beta-s)] \\
&\quad - 2\int_0^\beta ds Z_1^{\mathrm{u},g=0}(\beta-s)Z_1^{\mathrm{u}}(s) - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)Z_1^{\mathrm{u}}(\beta) + \frac{\beta}{16} + \frac{t_0^2}{2g_{\mathrm{s}}^2} \\
&= \int_0^\beta ds\left[Z_2(s,\beta-s) + Z_1^{g\geq 1}(s)Z_1^{g\geq 1}(\beta-s)\right] - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)Z_1^{g\geq 1}(\beta) \\
&\quad - \int_0^\beta ds Z_1^{\mathrm{u},g=0}(\beta-s)Z_1^{\mathrm{u},g=0}(s) - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)Z_1^{\mathrm{u},g=0}(\beta) + \frac{t_0^2}{2g_{\mathrm{s}}^2}.
\end{aligned}
\tag{3.9}
$$

In the last step we have used

$$
Z_2 = Z_2^{\mathrm{u}} + Z_2^{\mathrm{nu}}, \qquad Z_1^{\mathrm{u}} = Z_1^{g\geq 1} + Z_1^{\mathrm{u},g=0} \tag{3.10}
$$

and

$$
\int_0^\beta ds Z_2^{\mathrm{nu}}(s,\beta-s) = \int_0^\beta ds \frac{\sqrt{s(\beta-s)}}{2\pi\beta} = \frac{\beta}{16}. \tag{3.11}
$$

Expanding the last expression of (3.9) in g_{s}, one finds that the first two terms give contributions that are non-negative powers of g_{s} while the last three terms are of the order of g_{s}^{-2}. They both vanish independently in order for (3.9) to hold. Thus we have proved (3.1).

3.3 Proof of general loop equation

Let us next prove (3.2). For any function $f(\{t_k\})$ it follows from (3.7) that

$$
\begin{aligned}
e^{-F}L(\beta)e^{F}f &= fe^{-F}L(\beta)e^{F} \\
&\quad + \int_0^\beta ds\left[B(s)B(\beta-s) + 2(B(\beta-s)F)B(s) - 2Z_1^{\mathrm{u},g=0}(\beta-s)B(s)\right]f \\
&\quad - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)B(\beta)f \\
&= \left[\int_0^\beta ds B(s)B(\beta-s) + 2\int_0^\beta ds Z_1^{g\geq 1}(\beta-s)B(s) - \frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)B(\beta)\right]f.
\end{aligned}
\tag{3.12}
$$

In the last step we have used (3.8) and $BF = Z_1^{\mathrm{u}} = Z_1^{g\geq 1} + Z_1^{\mathrm{u},g=0}$.

On the other hand, starting from (3.8) and using (2.31) and (2.34) we obtain

$$
\begin{aligned}
0 &= e^{-F}\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_n)L(\beta)e^{F} \\
&= e^{-F}L(\beta)\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_n)e^{F} - e^{-F}[L(\beta),\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_n)]e^{F} \\
&= e^{-F}L(\beta)e^{F}\mathcal{Z}_n(\beta_1,\ldots,\beta_n) + e^{-F}\sum_{k=1}^{n}\beta_k\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_k+\beta)\cdots\mathcal{B}(\beta_n)e^{F} \\
&= \left[e^{-F}L(\beta)e^{F} + \mathcal{T}(\beta)\right]\mathcal{Z}_n(\beta_1,\ldots,\beta_n).
\end{aligned}
\tag{3.13}
$$

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Here we have introduced a formal shift operator $\mathcal{T}(\beta)$ which acts on any function f of n-variables $\beta_1, \ldots, \beta_n$ as

$$\mathcal{T}(\beta) f(\beta_1, \ldots, \beta_n) := \sum_{k=1}^{n} \beta_k f(\beta_1, \ldots, \beta_k + \beta, \ldots, \beta_n). \tag{3.14}$$

By using (3.12) and (2.26), the equation (3.13) is expressed as

$$\begin{aligned} 0 = & \left[\int_0^\beta ds B(s) B(\beta - s) \right. \\ & \left. + 2 \int_0^\beta ds Z_1^{g \geq 1}(\beta - s) B(s) - \frac{1}{g_{\mathrm{s}}} V'_{\mathrm{eff}}(\partial_\beta) B(\beta) + \mathcal{T}(\beta) \right] \frac{\delta}{\delta J(\beta_1)} \cdots \frac{\delta}{\delta J(\beta_n)} e^{\mathcal{F}} \Big|_{J=0} \\ = & \frac{\delta}{\delta J(\beta_1)} \cdots \frac{\delta}{\delta J(\beta_n)} \mathcal{A} e^{\mathcal{F}} \Big|_{J=0} \end{aligned} \tag{3.15}$$

with

$$\begin{aligned} \mathcal{A} = & \int_0^\beta ds B(s) B(\beta - s) \mathcal{F} + \int_0^\beta ds (B(s)\mathcal{F})(B(\beta - s)\mathcal{F}) \\ & + 2 \int_0^\beta ds Z_1^{g \geq 1}(\beta - s) B(s) \mathcal{F} - \frac{1}{g_{\mathrm{s}}} V'_{\mathrm{eff}}(\partial_\beta) B(\beta) \mathcal{F} + \mathcal{T}(\beta) \mathcal{F}. \end{aligned} \tag{3.16}$$

Note that B and $\mathcal{T}$ in $\mathcal{A}$ do not act on $e^{\mathcal{F}}$ in the last line of (3.15). We can rewrite (3.15) as

$$\begin{aligned} 0 = & \left[\frac{\delta}{\delta J(\beta_1)} \cdots \frac{\delta}{\delta J(\beta_n)} \mathcal{A} \right] e^{\mathcal{F}} \Big|_{J=0} \\ & + \sum_{I \subsetneq S} \prod_{i_k \in I} \left[\frac{\delta}{\delta J(\beta_{i_1})} \cdots \frac{\delta}{\delta J(\beta_{i_{|I|}})} \mathcal{A} \right] \prod_{j_k \in S - I} \left[\frac{\delta}{\delta J(\beta_{j_1})} \cdots \frac{\delta}{\delta J(\beta_{j_{|S-I|}})} e^{\mathcal{F}} \right] \Bigg|_{J=0}, \end{aligned} \tag{3.17}$$

where $I = \{i_1, i_2, \ldots, i_{|I|}\}$, $S = \{1, 2, \ldots, n\}$ and the sum is taken for all possible proper subsets I of S including the empty set. Based on this expression one can show by induction (with respect to n) that

$$\frac{\delta}{\delta J(\beta_1)} \cdots \frac{\delta}{\delta J(\beta_n)} \mathcal{A} \Big|_{J=0} = 0. \tag{3.18}$$

This gives (3.2).

3.4 Remark on disk amplitude contribution

For a better understanding of the structure of the loop equation let us elaborate on the absence of the disk amplitude in (3.2).

To do this, let us first rewrite the genus-zero part (2.44) as follows. Consider the Taylor series expansion of $(I_0(v) - v)$ about the point $v = u_0$. By using the property (2.8) and the string equation (2.12) it is expressed as

$$I_0(v) - v = \sum_{n=1}^{\infty} (I_n(u_0) - \delta_{n,1}) \frac{(v - u_0)^n}{n!}. \tag{3.19}$$

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Plugging this into (2.44) and evaluating the integral one obtains

$$Z_1^{g=0}(\beta) = \frac{e^{\beta u_0}}{\sqrt{2\pi} g_{\rm s}} \sum_{n=1}^{\infty} (-1)^n \beta^{-n-\frac{1}{2}} (I_n - \delta_{n,1}). \tag{3.20}$$

From this expression one observes that the disk amplitude contains negative powers of β. This is in contrast to the higher genus amplitudes, which contain only nonnegative powers of β [14, 20] (see (3.25) in the next subsection).

Comparing the expression (3.20) with (2.39) one can regard $V'_{\rm eff}(\xi)$ as a formal Laplace transform of the disk amplitude

$$-\frac{1}{2g_{\rm s}} V'_{\rm eff}(\xi) \sim \int_0^{\infty} d\beta Z_1^{g=0}(\beta) e^{-\beta\xi}. \tag{3.21}$$

Of course, this should not be viewed as a mathematically rigorous relation, because the Laplace transform converges only for $e^{\beta u_0}\beta^{-n-\frac{1}{2}}$ with $n < \frac{1}{2}$, but actually n is summed over positive integers. Nevertheless, at the price of mathematical rigor this formal relation provides us with an intuitive understanding of the structure of the loop equations, as we see below.

Let us now come back to the loop equation (3.2). If we naively place $Z_1(\beta)$ instead of $Z_1^{g\geq 1}(\beta)$, the r.h.s. of (3.2) gets an extra contribution

$$2\int_0^{\beta} ds Z_1^{g=0}(s) \widetilde{Z}_{n+1}(\beta - s, \{\beta_k\}_{k=1}^n). \tag{3.22}$$

This gives rise to divergence due to the negative powers of β as seen in (3.20) and thus should not be included in the loop equation. However, let us be tolerant for a while and attempt to evaluate it using (3.21). Let $\widetilde{Z}^*_{n+1}$ denote the Laplace transform of $\widetilde{Z}_{n+1}$:

$$\widetilde{Z}^*_{n+1}(\xi) = \int_0^{\infty} d\beta e^{-\beta\xi} \widetilde{Z}_{n+1}(\beta), \qquad \widetilde{Z}_{n+1}(\beta) = \frac{1}{2\pi {\rm i}} \int_{\mathcal{C}} d\xi e^{\beta\xi} \widetilde{Z}^*_{n+1}(\xi). \tag{3.23}$$

Here we have introduced the abbreviated notation $\widetilde{Z}^*_{n+1}(\xi) \equiv \widetilde{Z}^*_{n+1}(\xi; \{\beta_k\}_{k=1}^n)$, $\widetilde{Z}_{n+1}(\beta) \equiv \widetilde{Z}_{n+1}(\beta, \{\beta_k\}_{k=1}^n)$ and the contour $\mathcal{C}$ is chosen accordingly so that the inverse Laplace transform makes sense. Recall that the Laplace transform maps a convolution product to an ordinary product. Rewriting the convolution (3.22) using (3.21) and (3.23) we see that

$$\begin{aligned} 2\int_0^{\beta} ds Z_1^{g=0}(s) \widetilde{Z}_{n+1}(\beta - s) &\sim 2\frac{1}{2\pi {\rm i}} \int_{\mathcal{C}} d\xi e^{\beta\xi} \left(-\frac{1}{2g_{\rm s}} V'_{\rm eff}(\xi) \right) \widetilde{Z}^*_{n+1}(\xi) \\ &= -\frac{1}{g_{\rm s}} V'_{\rm eff}(\partial_\beta) \frac{1}{2\pi {\rm i}} \int_{\mathcal{C}} d\xi e^{\beta\xi} \widetilde{Z}^*_{n+1}(\xi) \\ &= -\frac{1}{g_{\rm s}} V'_{\rm eff}(\partial_\beta) \widetilde{Z}_{n+1}(\beta). \end{aligned} \tag{3.24}$$

Being transposed to the other side of the loop equation, this becomes precisely what we have already had on the l.h.s. of (3.2)! Therefore, we can think that the disk amplitude contribution (3.22) is not removed by hand from the loop equation, but rather it turns into the term involving $V'_{\rm eff}(\partial_\beta)$ as a mathematically well-defined contribution.

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3.5 Examples: JT gravity and Airy case

One can check the loop equation (3.2) order by order in the genus expansion, using the formalism of the genus expansion of the multi-boundary correlators developed in our previous paper [20]. Although we have applied this formalism to the JT gravity case in [20], our formalism can be trivially generalized to 2d topological gravity with arbitrary background couplings $\{t_k\}$, as explained in [26]. Indeed, we have checked that the loop equation (3.2) is satisfied up to the first few orders in the genus expansion for the general background $\{t_k\}$. For instance, the one- and two-boundary correlators in the genus expansion are obtained as [14, 20]

$$\begin{aligned}
Z_1^{g\geq 1}(\beta) = \frac{e^{\beta u_0}}{\sqrt{2\pi\beta^3}} \Bigg[& \left(\frac{I_2\beta^2}{24t^2} + \frac{\beta^3}{24t} \right) g_{\mathrm{s}} \\
& + \Bigg[\left(\frac{I_5}{1152t^4} + \frac{29I_3^2}{5760t^5} + \frac{11I_2I_4}{1440t^5} + \frac{5I_2^2I_3}{144t^6} + \frac{7I_2^4}{288t^7} \right) \beta^2 \\
& \quad + \left(\frac{7I_2^3}{288t^6} + \frac{29I_2I_3}{1440t^5} + \frac{I_4}{384t^4} \right) \beta^3 + \left(\frac{7I_2^2}{480t^5} + \frac{29I_3}{5760t^4} \right) \beta^4 \\
& \quad + \frac{29I_2\beta^5}{5760t^4} + \frac{\beta^6}{1152t^3} \Bigg] g_{\mathrm{s}}^3 + \mathcal{O}(g_{\mathrm{s}}^5) \Bigg], \\
Z_2(\beta_1,\beta_2) = \frac{\sqrt{\beta_1\beta_2}}{2\pi} e^{(\beta_1+\beta_2)u_0} \Bigg[& \frac{1}{\beta_1+\beta_2} \\
& + \left(\frac{I_3}{24t^3} + \frac{I_2^2}{12t^4} + \frac{I_2(\beta_1+\beta_2)}{12t^3} + \frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{24t^2} \right) g_{\mathrm{s}}^2 + \mathcal{O}(g_{\mathrm{s}}^4) \Bigg],
\end{aligned} \tag{3.25}$$

where

$$t = 1 - I_1. \tag{3.26}$$

Substituting these into (3.1) and using the formula (2.41) one can check that the both sides of the equation give

$$\frac{\beta}{16} + \left[\left(\frac{I_3}{384t^3} + \frac{49I_2^2}{9216t^4} \right) \beta^2 + \frac{49I_2}{9216t^3}\beta^3 + \frac{35}{16384t^2}\beta^4 \right] g_{\mathrm{s}}^2 + \mathcal{O}(g_{\mathrm{s}}^4). \tag{3.27}$$

This confirms (3.1), which is (3.2) for $n = 0$, up to this order.

JT gravity is a special case of topological gravity with infinitely many couplings turned on in a specific way [12–14]

$$t_0 = t_1 = 0, \qquad t_k = \frac{(-1)^k}{(k-1)!} \quad (k \geq 2). \tag{3.28}$$

In this case

$$u_0 = I_1 = 0, \qquad I_k = \frac{(-1)^k}{(k-1)!} \quad (k \geq 2) \tag{3.29}$$

and $V'_{\mathrm{eff}}(\xi)$ in (2.39) is given by

$$V'_{\mathrm{eff}}(\xi) = \sqrt{2}\sin(2\sqrt{\xi}). \tag{3.30}$$

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In JT gravity the one- and two-boundary correlators at the first few orders in the genus expansion are given by [11, 14, 20][4]

$$
\begin{aligned}
Z_1^{g\geq 0}(\beta) &= \frac{1}{\sqrt{2\pi\beta^3}}\left[\frac{\beta^2+\beta^3}{24}g_{\rm s}\right.\\
&\qquad\left.+\left(\frac{29\beta^2}{3072}+\frac{169\beta^3}{11520}+\frac{139\beta^4}{11520}+\frac{29\beta^5}{5760}+\frac{\beta^6}{1152}\right)g_{\rm s}^3+\mathcal{O}(g_{\rm s}^5)\right], \qquad (3.31)\\
Z_2(\beta_1,\beta_2) &= \frac{\sqrt{\beta_1\beta_2}}{2\pi}\left[\frac{1}{\beta_1+\beta_2}+\left(\frac{1}{16}+\frac{\beta_1+\beta_2}{12}+\frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{24}\right)g_{\rm s}^2+\mathcal{O}(g_{\rm s}^4)\right].
\end{aligned}
$$

These expressions are nothing but (3.25) evaluated at the special values (3.29). One can check that the loop equation (3.2) for $n=0$ is satisfied with (3.30) and (3.31), though this is evident from the previous example with general t_k.

Another interesting example is what is called the Airy case corresponding to the trivial background $t_k=0$ $(k\geq 0)$. In this case $u_0=I_n=0$ and $V'_{\rm eff}(\xi)$ in (2.39) becomes

$$
V'_{\rm eff}(\xi)=2\sqrt{2\xi}. \qquad (3.32)
$$

The n-boundary correlators in the Airy case for $n=1,2,3$ are known in a closed form [23, 27, 28]

$$
\begin{aligned}
Z_1(\beta) &= \frac{1}{\sqrt{2\pi\beta^3}g_{\rm s}}e^{\frac{g_{\rm s}^2\beta^3}{24}},\\
Z_2(\beta_1,\beta_2) &= Z_1(\beta_1+\beta_2)\mathrm{Erf}\left(\frac{g_{\rm s}}{2\sqrt{2}}\sqrt{\beta_1\beta_2(\beta_1+\beta_2)}\right),\\
Z_3(\beta_1,\beta_2,\beta_3) &= Z_1\left(\sum_{i=1}^{3}\beta_i\right)\\
&\quad\times\left[1-4T\left(\frac{g_{\rm s}}{2}\sqrt{\beta_1(\beta_2+\beta_3)(\beta_1+\beta_2+\beta_3)},\sqrt{\frac{\beta_2\beta_3}{\beta_1(\beta_1+\beta_2+\beta_3)}}\right)\right.\\
&\quad-4T\left(\frac{g_{\rm s}}{2}\sqrt{\beta_2(\beta_3+\beta_1)(\beta_1+\beta_2+\beta_3)},\sqrt{\frac{\beta_3\beta_1}{\beta_2(\beta_1+\beta_2+\beta_3)}}\right)\\
&\quad\left.-4T\left(\frac{g_{\rm s}}{2}\sqrt{\beta_3(\beta_1+\beta_2)(\beta_1+\beta_2+\beta_3)},\sqrt{\frac{\beta_1\beta_2}{\beta_3(\beta_1+\beta_2+\beta_3)}}\right)\right],
\end{aligned} \qquad (3.33)
$$

where $\mathrm{Erf}(z)$ and $T(z,a)$ denote the error function and the Owen's T-function respectively

$$
\mathrm{Erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z dt\, e^{-t^2}, \quad T(z,a)=\frac{1}{2\pi}\int_0^a dt\frac{e^{-\frac{1}{2}z^2(1+t^2)}}{1+t^2}. \qquad (3.34)
$$

We have checked that the loop equations (3.2) for $n=0,1$ are indeed satisfied by the correlators in the Airy case (3.33) with $V'_{\rm eff}(\xi)$ in (3.32).

[4] In (3.31) we have set the asymptotic value γ of the dilaton as $\gamma=\frac{1}{2\pi^2}$. S_0 in [11] is related to $g_{\rm s}$ by $g_{\rm s}=(2\pi^2)^{\frac{3}{2}}e^{-S_0}$ [14, 20].

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4 Relation to other approaches

4.1 Free boson/fermion representation

It is well-known that $\tau = e^F$ of Witten-Kontsevich topological gravity is the tau-function of the KdV hierarchy and it has a free boson/fermion representation (see e.g. [29–32] and references therein)

$$e^F = \langle t|V\rangle, \tag{4.1}$$

where the state $\langle t|$ is given by the coherent state of free boson

$$\langle t| = \langle\Omega| \exp\left(\sum_{k=0}^{\infty} \frac{t_k \alpha_{2k+1}}{g_{\rm s}(2k+1)!!}\right) \tag{4.2}$$

with α_n obeying the usual commutation relation of the free boson

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}, \qquad \langle\Omega|\alpha_n = 0 \quad (n<0). \tag{4.3}$$

Note that only the odd modes α_{2k+1} of α_n appear in (4.2) since the KdV hierarchy is a mod-2 reduction of the KP hierarchy.

The state $|V\rangle$ in (4.1) is written in terms of the free fermions ψ_r, ψ_r^* ($r \in \mathbb{Z}+\frac{1}{2}$) obeying the anti-commutation relation

$$\{\psi_r, \psi_s^*\} = \delta_{r+s,0}, \qquad \psi_r|\Omega\rangle = \psi_r^*|\Omega\rangle = 0 \quad (r>0). \tag{4.4}$$

They are related to α_n by the usual bosonization

$$\alpha_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} :\psi_r \psi_{n-r}^*: . \tag{4.5}$$

Then $|V\rangle$ is written as

$$|V\rangle = \exp\left(\sum_{m,n=0}^{\infty} A_{m,n}\psi_{-m-\frac{1}{2}}\psi^*_{-n-\frac{1}{2}}\right)|\Omega\rangle. \tag{4.6}$$

The generating function of $A_{m,n}$ for the Witten-Kontsevich τ-function is obtained in [33–35]:

$$\sum_{m,n=0}^{\infty} A_{m,n} z^{-m-1} w^{-n-1} = \frac{1}{z-w} + \frac{a(w)b(-z) - a(-z)b(w)}{z^2-w^2}, \tag{4.7}$$

where $a(z)$ and $b(z)$ are given by

$$\begin{aligned} a(z) &= \sum_{m=0}^{\infty}\left(\frac{-g_{\rm s}}{288}\right)^m \frac{(6m)!}{(2m)!(3m)!} z^{-3m}, \\ b(z) &= -\sum_{m=0}^{\infty}\left(\frac{-g_{\rm s}}{288}\right)^m \frac{(6m)!}{(2m)!(3m)!}\frac{6m+1}{6m-1} z^{-3m+1}. \end{aligned} \tag{4.8}$$

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Note that the derivative ∂_k with respect to the coupling t_k is mapped to the operator α_{2k+1} when acting on the state $\langle t|$ in (4.2)

$$\partial_k \langle t| = \frac{1}{g_{\rm s}(2k+1)!!}\langle t|\alpha_{2k+1}, \quad (k \geq 0). \tag{4.9}$$

Using the commutation relation (4.3) one can also show that

$$\begin{aligned} t_k \langle t| &= g_{\rm s}(2k-1)!!\langle t|\alpha_{-2k-1} \\ &= g_{\rm s}\frac{(-1)^k}{(-2k-1)!!}\langle t|\alpha_{-2k-1}, \quad (k \geq 0), \end{aligned} \tag{4.10}$$

where we have used $(2k-1)!!(-2k-1)!! = (-1)^k$. Then the Virasoro constraint (2.3) with L_m in (2.4) can be translated to the free boson/fermion language via the dictionary (4.9), (4.10)

$$\widehat{L}_n|V\rangle = 0 \quad (n \geq -1), \tag{4.11}$$

where the Virasoro generator $\widehat{L}_n$ is given by

$$\widehat{L}_n = \frac{1}{4}\sum_{k\in\mathbb{Z}} : \alpha_{2k+1}\alpha_{2n-2k-1} : + \frac{1}{16}\delta_{n,0} - \frac{1}{2g_{\rm s}}\alpha_{2n+3}. \tag{4.12}$$

Note that the linear term $-\frac{1}{2g_{\rm s}}\alpha_{2n+3}$ in (4.12) arises from the shift of $\tilde{t}_1 = t_1 - 1$ [18, 36]. Another useful expression of $|V\rangle$ is the cut-and-join representation found in [37]

$$\begin{aligned} |V\rangle &= e^W|\Omega\rangle, \\ W &= \frac{2g_{\rm s}}{3}\sum_{m=-1}^{\infty}\alpha_{-2m-3}\widehat{L}'_m, \end{aligned} \tag{4.13}$$

where $\widehat{L}'_m = \widehat{L}_m + \frac{1}{2g_{\rm s}}\alpha_{2m+3}$. See appendix D for a derivation of this expression.

The boundary creation operator $\mathcal{B}(\beta)$ in (2.27) can also be translated to the free boson/fermion language as

$$\mathcal{B}(\beta)\langle t| = \langle t|\widehat{Z}(\beta), \tag{4.14}$$

where $\widehat{Z}(\beta)$ is given by

$$\widehat{Z}(\beta) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\frac{\beta^{k+\frac{1}{2}}}{(2k+1)!!}\widetilde{\alpha}_{2k+1}. \tag{4.15}$$

Here $\widetilde{\alpha}_n$ is defined by

$$\widetilde{\alpha}_n = \alpha_n - g_{\rm s}^{-1}\delta_{n,-3} = e^{-\frac{1}{3g_{\rm s}}\alpha_3}\alpha_n e^{\frac{1}{3g_{\rm s}}\alpha_3}, \tag{4.16}$$

which is related to the shift of $\tilde{t}_k = t_k - \delta_{k,1}$. One can show that the operators $\widehat{Z}(\beta)$ mutually commute [20]

$$[\widehat{Z}(\beta), \widehat{Z}(\beta')] = 0, \tag{4.17}$$

and the full correlator (2.31) is written as

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = \frac{\langle t|\widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)|V\rangle}{\langle t|V\rangle}. \tag{4.18}$$

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In our previous paper [20], we identified the state $|V\rangle$ as the Hartle-Hawking state $|\mathrm{HH}\rangle$ [38]

$$|\mathrm{HH}\rangle = |V\rangle. \tag{4.19}$$

This is based on the argument in [39] that the Hartle-Hawking state is "the most symmetric state." Indeed, $|V\rangle$ is invariant under the Virasoro generators (4.12) and $|V\rangle$ can be thought of as the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum. In particular, the constraint $\widehat{L}_0|V\rangle = 0$ corresponds to the Wheeler-DeWitt equation. This indicates that the state $|V\rangle$ is a natural candidate for the Hartle-Hawking state $|\mathrm{HH}\rangle$.

Our (4.19) is consistent with the identification of the one-point function $\langle Z(\beta)\rangle$ as the wavefunction of the Hartle-Hawking state, which is commonly adopted in 2d gravity literature (see e.g. [40] for a review)

$$\langle Z(\beta)\rangle = \Psi_{\mathrm{HH}}(\beta) = \langle Z(\beta)|\mathrm{HH}\rangle, \tag{4.20}$$

where $\langle Z(\beta)|$ is given by

$$\langle Z(\beta)| = \frac{\langle t|\widehat{Z}(\beta)}{\langle t|\mathrm{HH}\rangle}. \tag{4.21}$$

More generally, the multi-point correlator (4.18) is written as

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = \langle Z(\beta_1)\cdots Z(\beta_n)|\mathrm{HH}\rangle, \tag{4.22}$$

where $\langle Z(\beta_1)\cdots Z(\beta_n)|$ is given by

$$\langle Z(\beta_1)\cdots Z(\beta_n)| = \frac{\langle t|\widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)}{\langle t|\mathrm{HH}\rangle}. \tag{4.23}$$

In [41, 42], the Virasoro constraint of matrix model is interpreted as the gauge symmetry of closed string field theory in a minimal model background. This suggests that the Virasoro constraint is the analogue of the bulk diffeomorphism invariance. Since the loop equation (3.2) is equivalent to the Virasoro constraint, one can regard the loop equation (3.2) as a manifestation of the bulk diffeomorphism invariance. The loop equation (3.2) relates the amplitudes with different number of boundaries. This can be thought of as a gauge redundancy due to the "large" diffeomorphism relating different topologies of spacetime [43]. In the language of [1], the Virasoro constraint defines a null state

$$|\mathcal{N}\rangle = \widehat{L}(\beta)|\mathrm{HH}\rangle = 0, \tag{4.24}$$

where $\widehat{L}(\beta)$ is obtained from (2.32) as

$$\widehat{L}(\beta) = \sum_{m=-1}^{\infty} \frac{\beta^{m+1}}{(m+1)!2^m}\widehat{L}_m. \tag{4.25}$$

More generally, acting $\widehat{Z}(\beta)$'s on $|\mathcal{N}\rangle$ also gives rise to a null state

$$\widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)|\mathcal{N}\rangle = \widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)\widehat{L}(\beta)|\mathrm{HH}\rangle = 0. \tag{4.26}$$

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As we have seen in the previous section, after rewriting (4.26) as

$$\widehat{L}(\beta)\widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)|\mathrm{HH}\rangle - \sum_{i=1}^{n}\widehat{Z}(\beta_1)\cdots[\widehat{L}(\beta),\widehat{Z}(\beta_i)]\cdots\widehat{Z}(\beta_n)|\mathrm{HH}\rangle = 0, \tag{4.27}$$

and using $[\widehat{L}(\beta),\widehat{Z}(\beta')] = -\beta'\widehat{Z}(\beta+\beta')$, (4.26) becomes equivalent to the loop equation (3.2). Thus we can regard the loop equation (3.2) as the equation for the null state due to the large diffeomorphism invariance.

4.2 Relation to Marolf-Maxfield [1]

Let us discuss the relation between our expression (4.22) and the one proposed by Marolf and Maxfield in [1]

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = \frac{\langle\mathbb{HH}|\widehat{Z}(\beta_1)\cdots\widehat{Z}(\beta_n)|\mathbb{HH}\rangle}{\langle\mathbb{HH}|\mathbb{HH}\rangle}. \tag{4.28}$$

This is different from our (4.22). In their formulation the Hartle-Hawking state is represented by both the bra $\langle\mathbb{HH}|$ and ket $|\mathbb{HH}\rangle$ as in (4.28), while in our formulation the bra and ket are treated asymmetrically and the Hartle-Hawking state is represented by the ket $|\mathrm{HH}\rangle$ only. In other words, our expression (4.22) corresponds to a special (Euclidean) time-slicing of the spacetime where the initial state has no boundary and all the boundaries are on the final state. On the other hand, the proposal (4.28) in [1] is based on a certain assumption of the cutting and sewing of the gravitational path integral and the existence of the CPT conjugation.

However, as emphasized in [44], the sewing of path integral in quantum gravity is quite different from the ordinary quantum field theories without gravity. Let us recall the argument in [44]. When the manifold M is cut into two pieces M_1 and M_2, the path integral of quantum fields over M is obtained by gluing M_1 and M_2 along the common boundary $\Sigma = \partial M_1 = \partial M_2$

$$\int_M \mathcal{D}\phi\, e^{-S(\phi)} = \int_\Sigma \mathcal{D}\phi_b \psi_{M_1}(\phi_b)\psi_{M_2}(\phi_b) = \langle\psi_{M_1}|\psi_{M_2}\rangle, \tag{4.29}$$

where $\psi_{M_i}(\phi_b)$ $(i=1,2)$ is the wavefunction defined by the path integral over M_i with the fixed boundary value $\phi|_\Sigma = \phi_b$

$$\psi_{M_i}(\phi_b) = \int_{M_i;\phi|_\Sigma=\phi_b} \mathcal{D}\phi\, e^{-S(\phi)}. \tag{4.30}$$

In the case of quantum gravity we have to perform the path integral over the metrics, which in particular includes the integral over the moduli space of metrics. Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the moduli spaces of M, M_1 and M_2 respectively. In calculating ψ_{M_1} and ψ_{M_2} we integrate over $\mathcal{M}_1$ and $\mathcal{M}_2$, and as a consequence the inner product $\langle\psi_{M_1}|\psi_{M_2}\rangle$ is given by the integral over $\mathcal{M}_1\times\mathcal{M}_2$. However, $\mathcal{M}$ is not equal to the product of $\mathcal{M}_1$ and $\mathcal{M}_2$ in general

$$\mathcal{M} \neq \mathcal{M}_1 \times \mathcal{M}_2. \tag{4.31}$$

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Therefore, the inner product $\langle\psi_{M_1}|\psi_{M_2}\rangle$ does not correspond to the integral over M. In other words, the sewing operation does not commute with the integration over the moduli [44]. As mentioned in [44], the sewing operation in the gravitational path integral is valid only for a given point in moduli space and it breaks down when we integrate over the moduli space. The sewing operation works for local fields on spacetime, but the moduli space is defined from the global property of spacetime.

Of course, one can also consider the cutting and sewing of the moduli space integral. For instance, the Weil-Petersson volume of the moduli space of Riemann surfaces satisfies the recursion relation found by Mirzakhani [45],[5] which comes from the pant decomposition of the underlying Riemann surfaces. This recursion relation essentially says that the higher genus Weil-Petersson volume is obtained by summing over all possible pant decompositions. In particular, we have to include the contribution of a pair of pants connecting two punctures on either connected or disconnected Riemann surfaces with lower genera, which can be thought of as a contribution of wormhole connecting either the same universe or disjoint universes.[6] In other words, in 2d gravity we have to sum over all possible cutting/sewing of the spacetime and include the contribution of wormholes. We expect that this is a general feature of the gravitational path integral.

The above argument urges us to reconsider the derivation of the expression (4.28) in [1] more carefully.[7] In fact, the correlator in 2d quantum gravity, which is exactly solved in terms of the double-scaled matrix model, takes the form (4.22), not (4.28). The Hilbert space based on the conventional free boson/fermion representation that we studied in section 4.1 does not seem to be identical with the one proposed in [1] and we do not know how to relate our (4.22) with (4.28) proposed in [1]. We leave this as an important future problem.

4.3 Relation to Ishibashi-Kawai [2, 3]

Next we consider the relation to the closed string field theory (SFT) of non-critical strings developed in a series of papers by Ishibashi and Kawai [2, 3]. Their SFT naturally arises in the quantization of 2d gravity in the temporal gauge [50]. Let us briefly recall the formalism of [2, 3]. First they introduce the creation and annihilation operators $\Psi^\dagger(\beta), \Psi(\beta)$ of the macroscopic loops obeying the commutation relation

$$[\Psi(\beta), \Psi^\dagger(\beta')] = \delta(\beta - \beta'), \tag{4.32}$$

and define the "vacuum state" $|0\rangle$ as

$$\Psi(\beta)|0\rangle = \langle 0|\Psi^\dagger(\beta) = 0. \tag{4.33}$$

[5] As shown by Eynard and Orantin [46], this recursion relation is equivalent to the topological recursion of the double-scaled matrix model.

[6] Here, by "wormhole" we mean a wormhole with a puncture, which is topologically equivalent to a pair of pants.

[7] The naive applications of sewing operation in the third quantization of universes [47] or "universe field theories" (see [48, 49] and references therein) suffer from the same problem.

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Then the partition function e^F is written as

$$e^F = \langle 0|v\rangle, \tag{4.34}$$

where $|v\rangle$ is given by

$$|v\rangle = \lim_{\tau\to\infty} e^{-\tau\mathcal{H}}|0\rangle. \tag{4.35}$$

Here $\mathcal{H}$ can be thought of as the Fokker-Planck Hamiltonian for the stochastic quantization [50]. $\mathcal{H}$ describes the splitting and joining of loops and its explicit form is given by

$$\mathcal{H} = \int_0^\infty d\beta\beta\Psi^\dagger(\beta)\left[T(\beta) + \beta^{-1}\rho(\beta)\right], \tag{4.36}$$

with

$$T(\beta) = \int_0^\beta ds\Psi(s)\Psi(\beta - s) + g_{\rm s}^2\int_0^\infty ds\, s\Psi^\dagger(s)\Psi(\beta + s). \tag{4.37}$$

This satisfies the continuum Virasoro algebra

$$[T(\beta), T(\beta')] = g_{\rm s}^2(\beta - \beta')T(\beta + \beta'). \tag{4.38}$$

$\rho(\beta)$ in (4.36) represents the tadpole term. In this formalism, the correlator of macroscopic loops is written as

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = g_{\rm s}^{-n}\frac{\langle 0|\Psi(\beta_1)\cdots\Psi(\beta_n)|v\rangle}{\langle 0|v\rangle}. \tag{4.39}$$

It is argued in [2, 3] that the state $|v\rangle$ in (4.35) satisfies the constraint

$$\left[T(\beta) + \beta^{-1}\rho(\beta)\right]|v\rangle = 0. \tag{4.40}$$

This condition fixes the form of $\rho(\beta)$ in terms of the disk amplitude $Z_1^{g=0}(\beta)$

$$\beta^{-1}\rho(\beta) = -g_{\rm s}^2\int_0^\beta ds Z_1^{g=0}(s)Z_1^{g=0}(\beta - s). \tag{4.41}$$

Note that this is just a formal expression since this integral is divergent due to the negative powers of s in $Z_1^{g=0}(s)$ (see (3.20)). From (4.40) the following form of the loop equation is obtained in [2, 3]

$$\int_0^\beta ds\langle\Psi(s)\Psi(\beta - s)\rangle_J + g_{\rm s}^2\int_0^\infty dssJ(s)\langle\Psi(\beta + s)\rangle_J + \beta^{-1}\rho(\beta) = 0, \tag{4.42}$$

where $\langle\cdots\rangle_J$ is defined by

$$\langle\cdots\rangle_J = \frac{\langle 0|(\cdots)\exp\left[\int_0^\infty dsJ(s)\Psi(s)\right]|v\rangle}{\langle 0|\exp\left[\int_0^\infty dsJ(s)\Psi(s)\right]|v\rangle}. \tag{4.43}$$

The loop equation (4.42) is almost identical to our result, but there are some subtle differences. Let us compare (4.42) and the free boson/fermion formalism. One can formally introduce the "boundary annihilation operator"

$$\widehat{Z}^\dagger(\beta) = \lim_{\varepsilon\to+0}\frac{\rm i}{\sqrt{2\pi}}\sum_{k=0}^\infty(-1)^k\left[(\beta + {\rm i}\varepsilon)^{k+\frac{1}{2}}\frac{\alpha_{2k+1}}{(2k+1)!!} - (\beta - {\rm i}\varepsilon)^{-k-\frac{1}{2}}\frac{\alpha_{-2k-1}}{(-2k-1)!!}\right], \tag{4.44}$$

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which satisfies

$$[\widehat{Z}(\beta), \widehat{Z}^\dagger(\beta')] = \beta\delta(\beta - \beta'). \tag{4.45}$$

Then it is tempting to identify

$$\begin{aligned} \Psi(\beta) &\leftrightarrow g_{\rm s}\widehat{Z}(\beta), \\ \Psi^\dagger(\beta) &\leftrightarrow \frac{1}{g_{\rm s}\beta}\widehat{Z}^\dagger(\beta). \end{aligned} \tag{4.46}$$

This identification works at the level of commutation relation, but the Hilbert spaces on which these operators act are different. In particular, there is no such state $|0\rangle$ annihilated by $\widehat{Z}(\beta)$ for all $\beta \geq 0$ at least in the Fock space of free boson/fermion.

Also, it is argued in [3] that the algebra of $\widetilde{T}(\beta) = T(\beta) + \beta^{-1}\rho(\beta)$ does not close due to the presence of the tadpole term $\rho(\beta)$

$$[\widetilde{T}(\beta_1), \widetilde{T}(\beta_2)] = g_{\rm s}^2(\beta_1 - \beta_2)\widetilde{T}(\beta_1 + \beta_2) - g_{\rm s}^2\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}\rho(\beta_1 + \beta_2), \tag{4.47}$$

and the consistency of the constraint $\widetilde{T}(\beta)|v\rangle = 0$ in (4.40) is a subtle issue. On the other hand, in our case the constraint algebra is closed

$$[\widehat{L}(\beta_1), \widehat{L}(\beta_2)]|V\rangle = (\beta_1 - \beta_2)\widehat{L}(\beta_1 + \beta_2)|V\rangle = 0 \tag{4.48}$$

and there is no problem associated with the tadpole term. This difference can be traced back to the fact that only the positive powers of β appear in the definition of $L(\beta)$ in (2.32) and the convolution of the disk amplitudes in (4.41) is already subtracted from the beginning (see (3.7))

$$L(\beta) = \int_0^\beta ds B(s)B(\beta - s) + \cdots \neq \int_0^\beta ds \mathcal{B}(s)\mathcal{B}(\beta - s) + \cdots . \tag{4.49}$$

Note that $B(\beta)$ denotes the universal part which contains only the positive powers of β.

It is interesting to observe that the cut-and-join operator W in (4.13) has a similar form with the SFT Hamiltonian $\mathcal{H}$ in (4.36). W in (4.13) can be thought of as the SFT Hamiltonian written in terms of the microscopic loop operators α_n.

5 Conclusions and outlook

In this paper we have presented a detailed proof of the loop equations (3.1) and (3.2) obeyed by the multi-boundary correlators in Witten-Kontsevich topological gravity with arbitrary background $\{t_k\}$. Since the Virasoro operator $L(\beta)$ in (2.32) contains only nonnegative powers of β, the disk amplitude $Z_1^{g=0}(\beta)$ does not appear in the convolution part (i.e. the s-integral) of (3.1) and (3.2). We emphasize that our loop equations (3.1) and (3.2) are valid for the general background $\{t_k\}$ including the $u_0 \neq 0$ case, which has not been worked out in the literature before. As a concrete example, we have demonstrated that our loop equations are indeed satisfied for JT gravity and the Airy case.

One of the motivations of our study of the loop equation is to understand the relation to the discussion of the null state by Marolf and Maxfield [1]. In section 4, we have argued that

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our loop equation has an interpretation as the null state in the free boson/fermion language of the Witten-Kontsevich τ-function. It is interesting that the loop equation relates the multi-boundary correlators with different number of boundaries and it can be thought of as a consequence of the "large" diffeomorphism relating different topologies of spacetime [43].

As we mentioned in section 4, our expression of multi-boundary correlator (4.22) is different from (4.28) of Marolf and Maxfield [1]. Since 2d gravity is completely solved by the double-scaled matrix model and (4.22) is what we get from the general formula of Witten-Kontsevich topological gravity, we have to take the result (4.22) very seriously. The integrable structure of Witten-Kontsevich topological gravity leading to the result (4.22) is tightly constrained and there is no natural way to rewrite (4.22) into the form of (4.28). We suspect that one of the possible pitfalls of the discussion in [1] is the naive application of the sewing operation in the gravitational path integral. As discussed in [44], the sewing operation breaks down when we integrate over the moduli space of metrics. It would be desirable to reconsider the argument in [1] in view of the remark in [44]. We leave this as an important future problem.

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A Proof of commutation relation (2.34)

In this section we prove the commutation relation (2.34). The l.h.s. of (2.34) is written as

$$[L(\beta), \mathcal{B}(\beta')] = \sum_{m=-1}^{\infty} \frac{\beta^{m+1}}{(m+1)!2^m} \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \left(g_{\mathrm{s}}^{-1}(-1)^k \beta'^{-k-\frac{1}{2}} [L_m, \tilde{t}_k] + g_{\mathrm{s}} \beta'^{k+\frac{1}{2}} [L_m, \partial_k] \right). \tag{A.1}$$

One can show that

$$\begin{aligned} [L_m, \tilde{t}_k] &= \begin{cases} \dfrac{1}{2} \dfrac{(2k+1)!!}{(2k-2m-1)!!} \tilde{t}_{k-m} & (m \leq k), \\ \dfrac{g_{\mathrm{s}}^2}{2} (2k+1)!!(2m-2k-1)!! \partial_{m-k-1} & (m \geq k+1), \end{cases} \\ [L_m, \partial_k] &= \begin{cases} -\dfrac{t_0}{2g_{\mathrm{s}}^2} & (m,k) = (-1,0), \\ -\dfrac{1}{2} \dfrac{(2k+2m+1)!!}{(2k-1)!!} \partial_{k+m} & \text{otherwise.} \end{cases} \end{aligned} \tag{A.2}$$

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Using these relations we obtain

$$
\begin{aligned}
&[L(\beta), \mathcal{B}(\beta')] \\
&= \frac{1}{\sqrt{2\pi} g_{\rm s}} \sum_{m=-1}^{\infty} \sum_{k=m}^{\infty} \frac{\beta^{m+1}(-1)^k \beta'^{-k-\frac{1}{2}}}{(m+1)! 2^{m+1}} \frac{(2k+1)!!}{(2k-2m-1)!!} \tilde{t}_{k-m} \\
&\quad + \frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{\beta^{m+1}(-1)^k \beta'^{-k-\frac{1}{2}}}{(m+1)! 2^{m+1}} (2k+1)!!(2m-2k-1)!! \partial_{m-k-1} \\
&\quad - \frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\substack{m \geq -1,\ k \geq 0 \\ (m,k) \neq (-1,0)}} \frac{\beta^{m+1} \beta'^{k+\frac{1}{2}}}{(m+1)! 2^{m+1}} \frac{(2k+2m+1)!!}{(2k-1)!!} \partial_{k+m}.
\end{aligned}
\tag{A.3}
$$

Note that the contribution of the $(m,k) = (-1,0)$ case of $[L_m, \partial_k]$ is included as the $(m,k) = (-1,-1)$ case of the first term.

By setting $\widetilde{m} = m+1$ and $\ell = k-m$ the first term of (2.34) is rewritten as

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi} g_{\rm s}} \sum_{m=-1}^{\infty} \sum_{k=m}^{\infty} \frac{\beta^{m+1}(-1)^k \beta'^{-k-\frac{1}{2}}}{(m+1)! 2^{m+1}} \frac{(2k+1)!!}{(2k-2m-1)!!} \tilde{t}_{k-m} \\
&= -\beta' \frac{1}{\sqrt{2\pi} g_{\rm s}} \sum_{\ell=0}^{\infty} (-1)^\ell \beta'^{-\ell-\frac{1}{2}} \tilde{t}_\ell \sum_{\widetilde{m}=0}^{\infty} \frac{1}{\widetilde{m}} \left(\frac{\beta}{\beta'}\right)^{\widetilde{m}} \frac{(-1)^{\widetilde{m}}(2\ell+\widetilde{m}-1)!!}{2^{\widetilde{m}}(2\ell-1)!!} \\
&= -\beta' \frac{1}{\sqrt{2\pi} g_{\rm s}} \sum_{\ell=0}^{\infty} (-1)^\ell \beta'^{-\ell-\frac{1}{2}} \tilde{t}_\ell \left(1 + \frac{\beta}{\beta'}\right)^{-\ell-\frac{1}{2}} = -\beta' \mathcal{B}(\beta+\beta')\Big|_{\tilde{t}}.
\end{aligned}
\tag{A.4}
$$

This reproduces the polynomial part of the r.h.s. of (2.34).

Next, by setting $\ell = m-k-1$ the second term of (A.3) becomes

$$
\begin{aligned}
&\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{\beta^{m+1}(-1)^k \beta'^{-k-\frac{1}{2}}}{(m+1)! 2^{m+1}} (2k+1)!!(2m-2k-1)!! \partial_{m-k-1} \\
&= \frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \sum_{\ell=0}^{m-1} \frac{\beta^{m+1}(-1)^{m-\ell-1} \beta'^{\ell-m+\frac{1}{2}}}{(m+1)! 2^{m+1}} (2m-2\ell-1)!!(2\ell+1)!! \partial_\ell \\
&= -\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty} \sum_{m=\ell+1}^{\infty} \frac{\beta^{m+1}(-1)^{m-\ell} \beta'^{\ell-m+\frac{1}{2}}}{(m+1)! 2^{m+1}} (2m-2\ell-1)!!(2\ell+1)!! \partial_\ell \\
&= -\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty} \sum_{m=\ell+1}^{\infty} \frac{\beta^{m+1} \beta'^{\ell-m+\frac{1}{2}}}{(m+1)!} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell-m+\frac{1}{2})} \partial_\ell.
\end{aligned}
\tag{A.5}
$$

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On the other hand, by setting $\ell = k+m$ the third term of (A.3) becomes

$$
\begin{aligned}
&-\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\substack{m\geq -1,\ k\geq 0\\ (m,k)\neq(-1,0)}} \frac{\beta^{m+1}\beta'^{k+\frac{1}{2}}}{(m+1)!2^{m+1}} \frac{(2k+2m+1)!!}{(2k-1)!!}\partial_{k+m} \\
&= -\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty}\sum_{m=-1}^{\ell} \frac{\beta^{m+1}\beta'^{\ell-m+\frac{1}{2}}}{(m+1)!2^{m+1}} \frac{(2\ell+1)!!}{(2\ell-2m-1)!!}\partial_{\ell} \\
&= -\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty}\sum_{m=-1}^{\ell} \frac{\beta^{m+1}\beta'^{\ell-m+\frac{1}{2}}}{(m+1)!} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell-m+\frac{1}{2})}\partial_{\ell}.
\end{aligned}
\tag{A.6}
$$

From (A.5) and (A.6) we see that the sum of the second and the third terms in (A.3) becomes (we set $\widetilde{m} = m+1$)

$$
\begin{aligned}
&-\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty}\sum_{m=-1}^{\infty} \frac{\beta^{m+1}\beta'^{\ell-m+\frac{1}{2}}}{(m+1)!} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell-m+\frac{1}{2})}\partial_{\ell} \\
&= -\beta'\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty}\beta'^{\ell+\frac{1}{2}} \sum_{\widetilde{m}=0}^{\infty} \frac{1}{\widetilde{m}!}\left(\frac{\beta}{\beta'}\right)^{\widetilde{m}} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell-\widetilde{m}+\frac{3}{2})}\partial_{\ell} \\
&= -\beta'\frac{g_{\rm s}}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty}\beta'^{\ell+\frac{1}{2}} \left(1+\frac{\beta}{\beta'}\right)^{\ell+\frac{1}{2}}\partial_{\ell} \\
&= -\beta'\mathcal{B}(\beta+\beta')\Big|_{\partial}.
\end{aligned}
\tag{A.7}
$$

This reproduces the derivative part of the r.h.s. of (2.34). Thus we have proved (2.34).

B Proof of relation (3.6)

In this section we prove the relation (3.6). Let us start from the r.h.s. of (3.6). To evaluate the first term, we first substitute (3.19) and $e^{\beta v} = \sum_{m=0}^{\infty}\frac{\beta^m v^m}{m!}$ into (2.46). This gives

$$
\begin{aligned}
Z_1^{{\rm u},g=0}(\beta) &= \frac{1}{g_{\rm s}}\sqrt{\frac{\beta}{2\pi}}\sum_{l=1}^{\infty}\frac{I_l-\delta_{l,1}}{l!}\sum_{m=0}^{\infty}\frac{\beta^m}{m!}\int_0^{u_0} dv(v-u_0)^l v^m \\
&= \frac{1}{\sqrt{2\pi}g_{\rm s}}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^l(I_l-\delta_{l,1})}{(l+m+1)!}u_0^{l+m+1}\beta^{m+\frac{1}{2}}.
\end{aligned}
\tag{B.1}
$$

Using this expression and the definition (2.15) of $B(\beta)$ we obtain

$$
\begin{aligned}
&2\int_0^{\beta} ds Z_1^{{\rm u},g=0}(\beta-s)B(s) \\
&= \frac{1}{\pi}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(I_l-\delta_{l,1})}{(l+m+1)!}u_0^{l+m+1}\int_0^{\beta} ds(\beta-s)^{m+\frac{1}{2}}s^{n+\frac{1}{2}}\partial_n.
\end{aligned}
\tag{B.2}
$$

The last integral is evaluated as

$$
\begin{aligned}
\int_0^{\beta} ds(\beta-s)^{m+\frac{1}{2}}s^{n+\frac{1}{2}} &= \frac{\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})}{\Gamma(m+n+3)}\beta^{m+n+2} \\
&= \frac{(-1)^{m+1}\pi\Gamma(n+\frac{3}{2})}{\Gamma(-m-\frac{1}{2})(m+n+2)!}\beta^{m+n+2}.
\end{aligned}
\tag{B.3}
$$

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Thus we have

$$
\begin{aligned}
&2\int_0^\beta ds Z_1^{\mathrm{u},g=0}(\beta-s)B(s)\\
&=\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-u_0)^{l+m+1}}{(l+m+1)!}\frac{I_l-\delta_{l,1}}{\Gamma(-m-\frac{1}{2})}\frac{\Gamma(n+\frac{3}{2})}{(m+n+2)!}\beta^{m+n+2}\partial_n\\
&=\sum_{n=0}^{\infty}\sum_{\widetilde{m}=2}^{\infty}\sum_{l=1}^{\widetilde{m}-1}\frac{(-u_0)^{\widetilde{m}}}{\widetilde{m}!}\frac{I_l-\delta_{l,1}}{\Gamma(l-\widetilde{m}+\frac{1}{2})}\frac{\Gamma(n+\frac{3}{2})}{(n-l+\widetilde{m}+1)!}\beta^{n-l+\widetilde{m}+1}\partial_n.
\end{aligned}
\tag{B.4}
$$

In the last step we have set $\widetilde{m}=l+m+1$.

On the other hand, the second term on the r.h.s. of (3.6) is evaluated as

$$
\begin{aligned}
&-\frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)B(\beta)\\
&=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\frac{I_l-\delta_{l,1}}{\Gamma(l+\frac{1}{2})}(\partial_\beta-u_0)^{l-\frac{1}{2}}\beta^{n+\frac{1}{2}}\partial_n\\
&=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\frac{I_l-\delta_{l,1}}{\Gamma(l+\frac{1}{2})}\sum_{m=0}^{\infty}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l-m+\frac{1}{2})}\frac{(-u_0)^m}{m!}\partial_\beta^{l-\frac{1}{2}-m}\beta^{n+\frac{1}{2}}\partial_n\\
&=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{I_l-\delta_{l,1}}{\Gamma(l-m+\frac{1}{2})}\frac{(-u_0)^m}{m!}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n-l+m+2)}\beta^{n-l+m+1}\partial_n.
\end{aligned}
\tag{B.5}
$$

In the last step we have used the formula (2.40).

Now, observe that the summand of (B.4) is identical with that of (B.5). Therefore, subtracting (B.4) from (B.5) we see that the r.h.s. of (3.6) becomes

$$
\begin{aligned}
&-2\int_0^\beta ds Z_1^{\mathrm{u},g=0}(\beta-s)B(s)-\frac{1}{g_{\mathrm{s}}}V'_{\mathrm{eff}}(\partial_\beta)B(\beta)\\
&=\sum_{n=0}^{\infty}\sum_{\substack{m\geq 0,\ l\geq m\\(m,l)\neq(0,0)}}\frac{I_l-\delta_{l,1}}{\Gamma(l-m+\frac{1}{2})}\frac{(-u_0)^m}{m!}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n-l+m+2)}\beta^{n-l+m+1}\partial_n.
\end{aligned}
\tag{B.6}
$$

By adding and subtracting the $(m,l)=(0,0)$ contribution and setting $k=l-m$, (B.6) is rewritten as

$$
\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{k=0}^{n+1}\frac{I_{k+m}-\delta_{k+m,1}}{\Gamma(k+\frac{1}{2})}\frac{(-u_0)^m}{m!}\frac{\beta^{n-k+1}}{(n-k+1)!}-\frac{I_0\beta^{n+1}}{\Gamma(\frac{1}{2})(n+1)!}\right)\Gamma\left(n+\frac{3}{2}\right)\partial_n. \tag{B.7}
$$

By using (2.11) this is rewritten as

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n+1}\frac{t_k-\delta_{k,1}+u_0\delta_{k,0}}{\Gamma(k+\frac{1}{2})}\frac{\beta^{n-k+1}}{(n-k+1)!}-\frac{I_0\beta^{n+1}}{\Gamma(\frac{1}{2})(n+1)!}\right)\Gamma\left(n+\frac{3}{2}\right)\partial_n\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\frac{\tilde{t}_k}{\Gamma(k+\frac{1}{2})}\frac{\beta^{n-k+1}}{(n-k+1)!}\Gamma\left(n+\frac{3}{2}\right)\partial_n.
\end{aligned}
\tag{B.8}
$$

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In the last step we have used the string equation (2.12). By setting $m = n - k$ this is further rewritten as

$$\begin{aligned}
&\sum_{\substack{m\geq -1,\ k\geq 0\\ (m,k)\neq(-1,0)}} \frac{\tilde{t}_k}{\Gamma(k+\frac{1}{2})}\frac{\beta^{m+1}}{(m+1)!}\Gamma\left(k+m+\frac{3}{2}\right)\partial_{k+m} \\
&= \sum_{\substack{m\geq -1,\ k\geq 0\\ (m,k)\neq(-1,0)}} \frac{(2k+2m+1)!!\beta^{m+1}}{(2k-1)!!(m+1)!2^{m+1}}\tilde{t}_k\partial_{k+m}.
\end{aligned} \tag{B.9}$$

This is precisely the second term on the r.h.s. of (3.4) and thus equal to the l.h.s. of (3.6).

C Loop equation at finite N

In this appendix we review the loop equation of matrix model at finite N. See e.g. [40] for a review on this subject.

The loop equation of matrix model follows from the Schwinger-Dyson equation

$$\sum_{i,j=1}^{N}\int dM \frac{\partial}{\partial M_{ij}}\left[\left(e^{\beta M}\right)_{ij} e^{-\frac{1}{g_s}\operatorname{Tr}V(M)+\int_0^\infty d\beta J(\beta)Z(\beta)}\right] = 0, \tag{C.1}$$

where M is the $N\times N$ hermitian matrix and $Z(\beta)$ is defined by

$$Z(\beta) = \operatorname{Tr} e^{\beta M}. \tag{C.2}$$

$J(\beta)$ in (C.1) is the source for $Z(\beta)$. Note that $Z(\beta)$ becomes the usual partition function $\operatorname{Tr} e^{-\beta H}$ under the identification $M = -H$.

Using the relations

$$\begin{aligned}
\sum_{i,j=1}^{N}\frac{\partial}{\partial M_{ij}}\left(e^{\beta M}\right)_{ij} &= \int_0^\beta ds \operatorname{Tr} e^{sM}\operatorname{Tr} e^{(\beta-s)M}, \\
\sum_{i,j=1}^{N}\left(e^{\beta M}\right)_{ij}\frac{\partial}{\partial M_{ij}}\operatorname{Tr} f(M) &= \operatorname{Tr}\left[f'(M)e^{\beta M}\right],
\end{aligned} \tag{C.3}$$

(C.1) is written as

$$\frac{1}{g_s}\left\langle \operatorname{Tr}\left[V'(M)e^{\beta M}\right]\right\rangle_J = \int_0^\beta ds\left\langle \operatorname{Tr} e^{sM}\operatorname{Tr} e^{(\beta-s)M}\right\rangle_J + \int_0^\infty d\beta'\beta' J(\beta')\left\langle \operatorname{Tr} e^{(\beta+\beta')M}\right\rangle_J, \tag{C.4}$$

where $\langle\cdots\rangle_J$ is defined by

$$\langle\cdots\rangle_J = \frac{\int dM(\cdots)e^{-\frac{1}{g_s}\operatorname{Tr}V(M)+\int_0^\infty d\beta J(\beta)Z(\beta)}}{\int dM e^{-\frac{1}{g_s}\operatorname{Tr}V(M)}}. \tag{C.5}$$

Using the identity $\operatorname{Tr} V'(M)e^{\beta M} = V'(\partial_\beta)\operatorname{Tr} e^{\beta M}$, (C.4) is rewritten in terms of $Z(\beta) = \operatorname{Tr} e^{\beta M}$ as

$$\frac{1}{g_s}V'(\partial_\beta)\langle Z(\beta)\rangle_J = \int_0^\beta ds\langle Z(s)Z(\beta-s)\rangle_J + \int_0^\infty d\beta'\beta' J(\beta')\langle Z(\beta+\beta')\rangle_J. \tag{C.6}$$

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By taking the derivative of the both sides of (C.6) with respect to J and setting $J = 0$, we find the loop equation for the multi-point functions of $Z(\beta)$. For instance, by simply setting $J = 0$ in (C.6) we find

$$\frac{1}{g_s}V'(\partial_\beta)\langle Z(\beta)\rangle = \int_0^\beta ds\langle Z(s)Z(\beta - s)\rangle. \tag{C.7}$$

By induction, one can show that the following loop equation is obtained from (C.6)

$$\begin{aligned}
&\frac{1}{g_s}V'(\partial_\beta)Z_{n+1}(\beta, \beta_1, \cdots, \beta_n) \\
&= \int_0^\beta ds \left[Z_{n+2}(s, \beta - s, \beta_1, \cdots, \beta_n) + \sum_{I\subset S} Z_{|I|+1}(s; \beta_I) Z_{|S-I|+1}(\beta - s; \beta_{S-I}) \right] \\
&\quad + \sum_{j=1}^{n} \beta_j Z_n(\{\beta_i + \delta_{ij}\beta\}_{i=1}^n),
\end{aligned} \tag{C.8}$$

where $Z_n(\beta_1, \ldots, \beta_n) = \langle Z(\beta_1)\cdots Z(\beta_n)\rangle_{\text{conn}}$ denotes the connected correlator and the definition of I and S is the same as in (3.2). (C.8) is the finite N version of the loop equation; essentially it has the same form as the loop equation (3.2) in the double-scaled matrix model but the potential V should be replaced by the effective potential V_{eff} in the double-scaled version of the loop equation (3.2). Note that the integral over s in (C.8) is finite and we do not have to subtract the genus-zero part of one-point function at finite N.

D Cut-and-join representation of $|V\rangle$

In this appendix we review the derivation of $|V\rangle$ in (4.13), [37]. The Virasoro constraint $\widehat{L}_m|V\rangle = 0$ is written as

$$\widehat{L}'_m|V\rangle = \frac{1}{2g_s}\alpha_{2m+3}|V\rangle, \tag{D.1}$$

where $\widehat{L}'_m = \widehat{L}_m + \frac{1}{2g_s}\alpha_{2m+3}$. Following [37] we introduce the operator D by

$$D = \frac{1}{2}\sum_{k=0}^{\infty} \alpha_{-2k-1}\alpha_{2k+1}. \tag{D.2}$$

One can show that

$$[D, \widehat{L}'_m] = -m\widehat{L}'_m, \quad [D, \alpha_{2k+1}] = -\left(k + \frac{1}{2}\right)\alpha_{2k+1}. \tag{D.3}$$

Then we expand $|V\rangle$ as

$$|V\rangle = \sum_{n=0}^{\infty} |V_n\rangle, \quad D|V_n\rangle = \Delta_n|V_n\rangle, \tag{D.4}$$

where $|V_n\rangle$ has the increasing weight of D

$$\Delta_0 < \Delta_1 < \Delta_2 < \cdots. \tag{D.5}$$

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We assume that $|V_0\rangle = |\Omega\rangle$ and $\Delta_0 = 0$. Plugging (D.4) into (D.1) we find

$$\widehat{L}'_m|V_n\rangle = \frac{1}{2g_{\rm s}}\alpha_{2m+3}|V_{n+1}\rangle. \tag{D.6}$$

From this we find the condition of the weight Δ_n

$$-m + \Delta_n = -m - \frac{3}{2} + \Delta_{n+1}. \tag{D.7}$$

This is solved as

$$\Delta_n = \frac{3n}{2}. \tag{D.8}$$

Applying α_{-2m-3} to both sides of (D.1) and summing over $m \geq -1$ we find the recursion relation for $|V_n\rangle$

$$\sum_{m=-1}^{\infty} \alpha_{-2m-3}\widehat{L}'_m|V_n\rangle = \frac{1}{g_{\rm s}}D|V_{n+1}\rangle = \frac{3(n+1)}{2g_{\rm s}}|V_{n+1}\rangle. \tag{D.9}$$

This is solved as

$$|V_n\rangle = \frac{1}{n!}\left(\frac{2g_{\rm s}}{3}\sum_{m=-1}^{\infty}\alpha_{-2m-3}\widehat{L}'_m\right)^n|\Omega\rangle. \tag{D.10}$$

Finally, plugging (D.10) into (D.4) we find our desired result (4.13).

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