AdS (super)projectors in three dimensions and partial masslessness

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ABSTRACT: We derive the transverse projection operators for fields with arbitrary integer and half-integer spin on three-dimensional anti-de Sitter space, AdS_3 . The projectors are constructed in terms of the quadratic Casimir operators of the isometry group $\text{SO}(2,2)$ of AdS_3 . Their poles are demonstrated to correspond to (partially) massless fields. As an application, we make use of the projectors to recast the conformal and topologically massive higher-spin actions in AdS_3 into a manifestly gauge-invariant and factorised form. We also propose operators which isolate the component of a field that is transverse and carries a definite helicity. Such fields correspond to irreducible representations of $\text{SO}(2,2)$. Our results are then extended to the case of $\mathcal{N} = 1$ AdS_3 supersymmetry.

KEYWORDS: Supergravity Models, Superspaces

ARXIV EPRINT: [2107.12201](https://arxiv.org/abs/2107.12201)

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1 Introduction

The spin projection operators, or transverse and traceless (TT) spin- s projectors, were first derived in four-dimensional (4d) Minkowski space $\mathbb{M}^4$ by Behrends and Fronsdal [1, 2]. Given a symmetric tensor field on $\mathbb{M}^4$ that obeys the Klein-Gordon equation, it decomposes into a sum of constrained fields describing irreducible representations of the Poincaré group with varying spin. The Behrends-Fronsdal projectors allow one to extract the component of this decomposition corresponding to the representation with the highest spin. Many applications for the TT projectors have been found within the landscape of high energy physics. For example, they played a crucial role in the original formulation of conformal higher-spin gauge actions [3].

Since the work of [1, 2], the spin projection operators have been generalised to diverse dimensions and symmetry groups. In the case of $\mathbb{M}^d$, the TT projectors were first derived by Segal [4] (see also [5–8]) in the bosonic case and later by Isaev and Podoinitsyn [8]

for half-integer spins. In four dimensions, the projection operators in $\mathcal{N} = 1$ Minkowski superspace, $\mathbb{M}^{4|4}$, were introduced by Salam and Strathdee [9] in the case of a scalar superfield, and by Sokatchev [10] for superfields of arbitrary rank. The superprojectors derived in [10] were formulated in terms of Casimir operators. A few years later Rittenberg and Sokatchev [11] made use of a similar method to construct the superprojectors in $\mathcal{N}$ -extended Minkowski superspace $\mathbb{M}^{4|4\mathcal{N}}$ (see also [12]). An alternative powerful construction of the superprojectors in $\mathbb{M}^{4|4\mathcal{N}}$ was given in [13, 14].¹ Recently, the superprojectors in three-dimensional $\mathcal{N}$ -extended Minkowski superspace, $\mathbb{M}^{3|2\mathcal{N}}$, were derived in ref. [17], which built upon the earlier work of [18].

It is of interest to construct spin projection operators for fields on (anti-)de Sitter space, (A)dS. In particular, in order to describe irreducible representations of the AdS_d isometry algebra, $\mathfrak{so}(d-1, 2)$, fields on AdS_d must satisfy certain differential constraints involving the Lorentz-covariant derivative $\mathcal{D}_a$ for AdS_d . Since both dS and AdS spaces have non-vanishing curvature, the construction of the TT projectors proves to be technically challenging. However, recent progress has been made in [19, 20], where the (super)projectors in AdS_4 were derived. The next logical step is to derive the TT (super)projectors in AdS_d . In this work we consider the case $d = 3$, which serves as a starting point for this program.

This paper is organised as follows. In section 2.1, we begin by reviewing on-shell fields in AdS_3 and the corresponding irreducible representations of $\mathfrak{so}(2, 2)$ which they furnish. In section 2.2, we derive the spin projection operators for fields of arbitrary rank. More specifically, let us denote by $\mathcal{V}_{(n)}$ the space of totally symmetric rank- n spinor fields $\phi_{\alpha(n)} := \phi_{\alpha_1 \dots \alpha_n} = \phi_{(\alpha_1 \dots \alpha_n)}$ on AdS_3 . For any integer $n \geq 2$, we derive the rank- n spin projection operator, $\Pi_{[n]}^\perp$, which is defined by its action on $\mathcal{V}_{(n)}$ according to the rule:

$$\Pi_{[n]}^\perp : \mathcal{V}_{(n)} \longrightarrow \mathcal{V}_{(n)}, \quad \phi_{\alpha(n)} \longmapsto \Pi_{[n]}^\perp \phi_{\alpha(n)} =: \phi_{\alpha(n)}^\perp. \quad (1.1)$$

For fixed n , this operator is defined by the following properties:

1. **Idempotence.** $\Pi_{[n]}^\perp$ is a projector in the sense that it squares to itself,

$$\Pi_{[n]}^\perp \Pi_{[n]}^\perp = \Pi_{[n]}^\perp. \quad (1.2a)$$

2. **Transversality.** $\Pi_{[n]}^\perp$ maps $\phi_{\alpha(n)}$ to a transverse field,

$$\mathcal{D}^{\beta(2)} \phi_{\beta(2)\alpha(n-2)}^\perp = 0. \quad (1.2b)$$

3. **Surjectivity.** Every transverse field belongs to the image of $\Pi_{[n]}^\perp$,

$$\mathcal{D}^{\beta(2)} \psi_{\beta(2)\alpha(n-2)} = 0 \quad \implies \quad \Pi_{[n]}^\perp \psi_{\alpha(n)} = \psi_{\alpha(n)}. \quad (1.2c)$$

In other words, $\Pi_{[n]}^\perp$ acts as the identity operator on the space of transverse fields.

¹This approach has found numerous applications, e.g. the derivation of gauge-invariant actions [15, 16].

Any operator satisfying all three of these properties may be considered to be an AdS_3 analogue of the Behrends-Fronsdal projector.² However, the field $\phi_{\alpha(n)}^\perp$ will correspond to a reducible representation of $\mathfrak{so}(2,2)$. In order to isolate the component describing an irreducible representation, it is necessary to bisect the projectors according to $\Pi_{[n]}^\perp = \mathbb{P}_{[n]}^{(+)} + \mathbb{P}_{[n]}^{(-)}$. The operator $\mathbb{P}_{[n]}^{(\pm)}$ is a helicity projector since it satisfies the properties³ (1.2a) and (1.2b) and selects the component of $\phi_{\alpha(n)}$ carrying the definite value $\pm \frac{n}{2}$ of helicity. They are constructed in section 2.3. In section 2.4 we make use of the orthogonal complement of $\Pi_{[n]}^\perp$ to decompose an unconstrained field $\phi_{\alpha(n)}$ into a sum of transverse fields $\phi_{\alpha(n-2j)}^\perp$ where $0 \leq j \leq \lfloor n/2 \rfloor$. We then provide an operator $\mathbb{S}_{\alpha(n-2j)}^\perp$ which extracts the field $\phi_{\alpha(n-2j)}^\perp$ from this decomposition.

Making use of these projection operators, we derive a number of interesting and non-trivial results. In particular, in section 2 we show that all information about (partially) massless fields is encoded in the poles of the transverse projectors. The novelty of our approach is that all projectors are derived in terms of the quadratic Casimir operators of $\mathfrak{so}(2,2)$. This allows us to recast the AdS_3 higher-spin Cotton tensors and their corresponding conformal actions into a manifestly gauge-invariant and factorised form. Similar results are provided for new topologically massive (NTM) spin- s gauge models, which are of order $2s$ in derivatives, where s is a positive (half-)integer. In the case when s is an integer, it is possible to construct NTM models of order $2s - 1$. In $\mathbb{M}^3$ such models were recently proposed in [21], here we extend them to AdS_3 . The above results are detailed in section 2.5. Finally, in section 2.6 we study the flat limit of these results, and obtain new realisations for the spin projection operators, the helicity projectors and the conformal higher-spin actions in $\mathbb{M}^3$.

In section 3, we extend some of these results to the case of $\mathcal{N} = 1$ AdS_3 supersymmetry. Alongside concluding comments, new realisations of the Behrends-Fronsdal projectors in $\mathbb{M}^4$, expressed in terms of the Casimir operators of the 4d Poincaré algebra, are given in section 4. The main body is accompanied by two technical appendices. Appendix A summarises our conventions and notation. We review the generating function formalism in appendix B, which is a convenient framework used in deriving the non-supersymmetric results of section 2.

Our findings in this paper can be viewed as generalisations of the earlier results in AdS_4 [19, 20] and AdS_3 [22], which in turn were based on the structure of (super)projectors in Minkowski (super)space [17, 18]. Throughout this work we make use of the convention

$$U_{\alpha(n)} V_{\alpha(m)} = U_{(\alpha_1 \dots \alpha_n} V_{\alpha_{n+1} \dots \alpha_{n+m})} . \quad (1.3)$$

²We refer to any operator satisfying properties (1.2a), (1.2b) and (1.2c) as a spin projection operator. In section 2.2 we show that, under an additional assumption, such an operator is unique. In general, operators satisfying properties (1.2a) and (1.2b) will be called transverse projectors. The latter are sometimes referred to as TT projectors, which is a slight abuse of terminology, since in vector notation the field $\phi_{\alpha(n)}$ is already traceless.

³Whilst $\mathbb{P}_{[n]}^{(\pm)}$ satisfies the properties (1.2a) and (1.2b), it does not satisfy (1.2c).

2 Transverse projectors in AdS₃

The geometry of AdS₃ is described by the Lorentz covariant derivative,

$$\mathcal{D}_a = e_a^m \partial_m + \frac{1}{2} \omega_a^{bc} M_{bc} = e_a^m \partial_m + \frac{1}{2} \omega_a^{\beta\gamma} M_{\beta\gamma}, \quad (2.1)$$

which satisfies the commutation relation

$$[\mathcal{D}_a, \mathcal{D}_b] = -4\mathcal{S}^2 M_{ab} \iff [\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma\delta}] = 4\mathcal{S}^2 (\varepsilon_{\gamma(\alpha} M_{\beta)\delta} + \varepsilon_{\delta(\alpha} M_{\beta)\gamma}). \quad (2.2)$$

Here e_a^m is the inverse vielbein, ω_a^{bc} is the Lorentz connection and the parameter $\mathcal{S}$ is related to the scalar curvature R via $R = -24\mathcal{S}^2$. The Lorentz generators with vector ($M_{ab} = -M_{ba}$) and spinor ($M_{\alpha\beta} = M_{\beta\alpha}$) indices are defined in appendix A. In our subsequent analysis, we will make use of the quadratic Casimir operators of the AdS₃ isometry algebra $\mathfrak{so}(2, 2) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, for which we choose (see, e.g., [23])

$$\mathcal{F} := \mathcal{D}^{\alpha\beta} M_{\alpha\beta}, \quad [\mathcal{F}, \mathcal{D}_{\alpha\beta}] = 0, \quad (2.3a)$$

$$\mathcal{Q} := \square - 2\mathcal{S}^2 M^{\alpha\beta} M_{\alpha\beta}, \quad [\mathcal{Q}, \mathcal{D}_{\alpha\beta}] = 0. \quad (2.3b)$$

Here $\square := \mathcal{D}^a \mathcal{D}_a = -\frac{1}{2} \mathcal{D}^{\alpha\beta} \mathcal{D}_{\alpha\beta}$ is the d'Alembert operator in AdS₃. The operators $\mathcal{F}$ and $\mathcal{Q}$ are related to each other as follows

$$\mathcal{F}^2 \phi_{\alpha(n)} = n^2 [\mathcal{Q} - (n-2)(n+2)\mathcal{S}^2] \phi_{\alpha(n)} + n(n-1) \mathcal{D}_{\alpha(2)} \mathcal{D}^{\beta(2)} \phi_{\beta(2)\alpha(n-2)}, \quad (2.4)$$

for an arbitrary symmetric rank- n spinor field $\phi_{\alpha(n)}$. The structure $\mathcal{D}_{\alpha(2)} \mathcal{D}^{\beta(2)} \phi_{\beta(2)\alpha(n-2)}$ in (2.4) is not defined for the cases $n = 0$ and $n = 1$. However, it is multiplied by $n(n-1)$ which vanishes for these cases.

2.1 On-shell fields

In any irreducible representation of the AdS₃ isometry group SO(2, 2), the Casimir operators $\mathcal{F}$ and $\mathcal{Q}$ must be multiples of the identity operator. Therefore, in accordance with (2.4), one is led to consider on-shell fields of the type

$$\mathcal{D}^{\beta(2)} \phi_{\beta(2)\alpha(n-2)} = 0, \quad (2.5a)$$

$$(\mathcal{F} - \mu) \phi_{\alpha(n)} = 0, \quad (2.5b)$$

for some real mass parameter μ .

Unitary representations of the Lie algebra $\mathfrak{so}(2, 2)$ may be realised in terms of the on-shell fields (2.5) for certain values of μ . As is well known (see, e.g., [24, 25] and references therein), the irreducible unitary representations of $\mathfrak{so}(2, 2)$ are denoted $D(E_0, s)$, where E_0 is the minimal energy (in units of $\mathcal{S}$), s the helicity and $|s|$ is the spin. In this paper we are interested in only those representations carrying integer or half-integer spin with $|s| \geq 1$ and, consequently, the allowed values of s are $s = \pm 1, \pm \frac{3}{2}, \pm 2, \dots$. In order for the representation $D(E_0, s)$ to be unitary, the inequality $E_0 \geq |s|$, known as the unitarity bound, must be satisfied.

The representation $D(E_0, s) \equiv D(E_0, \sigma|s|)$, with $\sigma := \pm 1$, may be realised on the space of symmetric rank- n spinor fields $\phi_{\alpha(n)}$ satisfying the following differential constraints:

$$\mathcal{D}^{\beta(2)}\phi_{\beta(2)\alpha(n-2)} = 0, \quad (2.6a)$$

$$\mathcal{D}_{(\alpha_1}{}^{\beta}\phi_{\alpha_2\dots\alpha_n)\beta} = \sigma\frac{\rho}{n}\phi_{\alpha(n)}. \quad (2.6b)$$

Here the integer $n \geq 2$ is related to s via $n = 2|s|$. The real parameter $\rho \geq 0$, which carries mass dimension one, is called the pseudo-mass and is related to E_0 through

$$E_0 = 1 + \frac{\rho}{2n\mathcal{S}}. \quad (2.7)$$

In terms of ρ and n , the unitarity bound reads $\rho \geq n(n-2)\mathcal{S}$. With this in mind, we will label the representations using ρ in place of E_0 , and use the notation $\mathfrak{D}(\rho, \sigma\frac{n}{2})$.

The equations (2.6) were introduced in [25]. In the flat-space limit, these equations reduce to those proposed in [26, 27].

The first-order equation (2.6b) is equivalent to (2.5b) with $\mu = \sigma\rho$. Any field $\phi_{\alpha(n)}$ satisfying both constraints (2.6a) and (2.6b), is an eigenvector of the Casimir operator $\mathcal{Q}$,

$$(\mathcal{Q} - m^2)\phi_{\alpha(n)} = 0, \quad m^2 := (\rho/n)^2 + (n-2)(n+2)\mathcal{S}^2. \quad (2.8)$$

In place of (2.6a) and (2.6b), one may instead consider tensor fields $\phi_{\alpha(n)}$ constrained by the equations (2.6a) and (2.8),

$$\mathcal{D}^{\beta(2)}\phi_{\beta(2)\alpha(n-2)} = 0, \quad (2.9a)$$

$$(\mathcal{Q} - m^2)\phi_{\alpha(n)} = 0. \quad (2.9b)$$

In this case, the equation (2.4) becomes

$$(\mathcal{F} - \rho)(\mathcal{F} + \rho)\phi_{\alpha(n)} = 0. \quad (2.10)$$

It follows that such a $\phi_{\alpha(n)}$ furnishes the reducible representation

$$\mathfrak{D}\left(\rho, -\frac{n}{2}\right) \oplus \mathfrak{D}\left(\rho, \frac{n}{2}\right). \quad (2.11)$$

It may be shown that when the pseudo-mass takes on any of the special values

$$\rho \equiv \rho_{(t,n)} = n(n-2t)\mathcal{S}, \quad 1 \leq t \leq \lfloor n/2 \rfloor, \quad (2.12)$$

then the representation $\mathfrak{D}(\rho, \sigma\frac{n}{2})$, with either sign for σ , shortens. At the field-theoretic level, this is manifested by the appearance of a depth- t gauge symmetry

$$\delta_\zeta \phi_{\alpha(n)}^{(t)} = (\mathcal{D}_{\alpha(2)})^t \zeta_{\alpha(n-2t)}, \quad (2.13)$$

under which the system of equations (2.6), with ρ given by (2.12) and σ arbitrary, is invariant.⁴ A field which satisfies the constraints (2.9a) and (2.8), and has pseudo-mass (2.12),

⁴This is true when the gauge parameter satisfies conditions analogous to (2.6), see [22] for the details.

will be said to be partially-massless with depth t and denoted by $\phi_{\alpha(n)}^{(t)}$.⁵ For the field $\phi_{\alpha(n)}^{(t)}$ the second order equation (2.8) takes the form

$$(\mathcal{Q} - \tau_{(t,n)} \mathcal{S}^2) \phi_{\alpha(n)}^{(t)} = 0, \quad \tau_{(t,n)} = [2n(n-2t) + 4(t-1)(t+1)], \quad (2.14)$$

where the parameters $\tau_{(t,n)}$ are known as the partially massless values. For $t > 1$, the pseudo-mass $\rho_{(t,n)}$, eq. (2.12), violates the unitarity bound and hence the partially massless representations are non-unitary.

2.2 Spin projection operators

Given a tensor field $\phi_{\alpha(n)}$ on AdS_3 , the spin projection operator $\Pi_{[n]}^\perp$ with the defining properties (1.2), selects the component $\phi_{\alpha(n)}^\perp$ of $\phi_{\alpha(n)}$ which is transverse. If, in addition, $\phi_{\alpha(n)}$ satisfies the second order mass-shell equation (2.8), then $\Pi_{[n]}^\perp \phi_{\alpha(n)}$ furnishes the reducible representation $\mathfrak{D}(\rho, -\frac{n}{2}) \oplus \mathfrak{D}(\rho, \frac{n}{2})$ of $\mathfrak{so}(2, 2)$.

In this section we derive the spin projection operators $\Pi_{[n]}^\perp$. For this purpose it is convenient to make use of the generating function formalism, which is described in appendix B. In this framework, the properties (1.2a) and (1.2b) take the following form:

$$\Pi_{[n]}^\perp \Pi_{[n]}^\perp \phi_{(n)} = \Pi_{[n]}^\perp \phi_{(n)}, \quad \mathcal{D}_{(-2)} \Pi_{[n]}^\perp \phi_{(n)} = 0. \quad (2.15)$$

It is necessary to separately analyse the cases with n even and n odd.

2.2.1 Bosonic case

We will begin by studying the bosonic case, $n = 2s$, for integer $s \geq 1$. Let us introduce the differential operator $\mathbb{T}_{[2s]}$ of order $2s$ in derivatives⁶

$$\mathbb{T}_{[2s]} = \sum_{j=0}^s 2^{2j} s \frac{(s+j-1)!}{(s-j)!} \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t+1, 2s)} \mathcal{S}^2) \mathcal{D}_{(2)}^{s-j} \mathcal{D}_{(-2)}^{s-j}. \quad (2.16)$$

Here $\tau_{(t,n)}$ denotes the partially massless values (2.14). We refer the reader to appendix B for an explanation of the other notation. Given an arbitrary field $\phi_{(2s)} \in \mathcal{V}_{(2s)}$, using (B.3b) one may show that this operator maps it to a transverse field

$$\mathcal{D}_{(-2)} \mathbb{T}_{[2s]} \phi_{(2s)} = 0. \quad (2.17)$$

However, it is not a projector on $\mathcal{V}_{(2s)}$ since it does not square to itself,

$$\mathbb{T}_{[2s]} \mathbb{T}_{[2s]} \phi_{(2s)} = 2^{2s-1} (2s)! \prod_{t=1}^s (\mathcal{Q} - \tau_{(t, 2s)} \mathcal{S}^2) \mathbb{T}_{[2s]} \phi_{(2s)}. \quad (2.18)$$

To prove this identity, we observe that only the $j = s$ term of the sum in (2.16) survives when $\mathbb{T}_{[2s]}$ acts on a transverse field such as $\mathbb{T}_{[2s]} \phi_{(2s)}$.

⁵Partially massless fields have been studied in diverse dimensions for over 35 years, see e.g. [28–32] for some of the earlier works.

⁶When the upper bound in a product is less than the lower bound, we define the result to be unity.

To obtain a projector, we define the following dimensionless operator

$$\hat{\Pi}_{[2s]}^\perp := \left[2^{2s-1}(2s)! \prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s)} \mathcal{S}^2) \right]^{-1} \mathbb{T}_{[2s]}. \quad (2.19)$$

On $\mathcal{V}_{(2s)}$ it inherits its transversality from $\mathbb{T}_{[2s]}$, and is idempotent by virtue of (2.18). In a fashion similar to the proof of (2.18), it may also be shown that $\hat{\Pi}_{[2s]}^\perp$ acts as the identity on the space of rank-(2s) transverse fields. Thus, $\hat{\Pi}_{[2s]}^\perp$ satisfies the properties (1.2) and is hence the spin projection operator on $\mathcal{V}_{(2s)}$. Making the indices explicit, the latter reads

$$\begin{aligned} \hat{\Pi}_{[2s]}^\perp \phi_{\alpha(2s)} &= \left[\prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s)} \mathcal{S}^2) \right]^{-1} \sum_{j=0}^s 2^{2j-2s} \frac{2s}{s+j} \binom{s+j}{2j} \\ &\times \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t+1,2s)} \mathcal{S}^2) \mathcal{D}_{\alpha(2)}^{s-j} (\mathcal{D}^{\beta(2)})^{s-j} \phi_{\alpha(2j)\beta(2s-2j)}. \end{aligned} \quad (2.20)$$

It is possible to construct a spin projection operator solely in terms of the two quadratic Casimir operators (2.3). To this end, we introduce the operator

$$\Pi_{[2s]}^\perp = \frac{1}{2^{2s-1}(2s)!} \prod_{j=1}^s \frac{(\mathcal{F}^2 - 4(j-1)^2(\mathcal{Q} - 4j(j-2)\mathcal{S}^2))}{(\mathcal{Q} - \tau_{(j,2s)} \mathcal{S}^2)}. \quad (2.21)$$

Let us show that (2.21) satisfies the three defining properties (1.2) on $\mathcal{V}_{(2s)}$. Given an arbitrary transverse field $\psi_{\alpha(2s)}$, $\mathcal{D}_{(-2)}\psi_{(2s)} = 0$, using (2.4) one may show that

$$\prod_{j=1}^s (\mathcal{F}^2 - 4(j-1)^2(\mathcal{Q} - 4j(j-2)\mathcal{S}^2)) \psi_{(2s)} = 2^{2s-1}(2s)! \prod_{j=1}^s (\mathcal{Q} - \tau_{(j,2s)} \mathcal{S}^2) \psi_{(2s)}. \quad (2.22)$$

It follows that $\Pi_{[2s]}^\perp$ acts as the identity on the space of transverse fields,

$$\mathcal{D}_{(-2)}\psi_{(2s)} = 0 \quad \implies \quad \Pi_{[2s]}^\perp \psi_{(2s)} = \psi_{(2s)}. \quad (2.23)$$

Next, the image of any unconstrained field $\phi_{(2s)}$ under $\Pi_{[2s]}^\perp$ is transverse, which follows elegantly from (B.3c)

$$\mathcal{D}_{(-2)} \Pi_{[2s]}^\perp \phi_{(2s)} = \Pi_{[2s]}^\perp \mathcal{D}_{(-2)} \phi_{(2s)} \propto \mathcal{D}_{(2)}^s \mathcal{D}_{(-2)}^{s+1} \phi_{(2s)} = 0. \quad (2.24)$$

Finally, using (2.23) and (2.24) one can show that $\Pi_{[2s]}^\perp$ squares to itself

$$\Pi_{[2s]}^\perp \Pi_{[2s]}^\perp \phi_{(2s)} = \Pi_{[2s]}^\perp \phi_{(2s)}. \quad (2.25)$$

Thus $\Pi_{[2s]}^\perp$ satisfies (1.2a), (1.2b) and (1.2c) and can also be identified as a spin projector.

Although it is not immediately apparent, the two projectors $\hat{\Pi}_{[2s]}^\perp$ and $\Pi_{[2s]}^\perp$ actually coincide. Indeed, an operator satisfying the three properties (1.2), and which commutes with $\mathcal{D}_a$, must be unique. Let us explain why this is so. Take an arbitrary $\phi_{(2s)}$ and act on

it first with $\hat{\Pi}_{[2s]}^\perp$ and then with $\Pi_{[2s]}^\perp$. Since $\hat{\Pi}_{[2s]}^\perp \phi_{(2s)}$ is transverse, and $\Pi_{[2s]}^\perp$ acts as the identity on this space, we have

$$\Pi_{[2s]}^\perp \hat{\Pi}_{[2s]}^\perp \phi_{(2s)} = \hat{\Pi}_{[2s]}^\perp \phi_{(2s)}. \quad (2.26)$$

Next, we perform the same operation but in the opposite order,

$$\hat{\Pi}_{[2s]}^\perp \Pi_{[2s]}^\perp \phi_{(2s)} = \Pi_{[2s]}^\perp \phi_{(2s)}, \quad (2.27)$$

and subtract (2.26) from (2.27). Using the fact that $\Pi_{[2s]}^\perp$ is composed solely from Casimir operators, and hence commutes with $\hat{\Pi}_{[2s]}^\perp$, it follows that on $\mathcal{V}_{(2s)}$ the two are equal to one another,

$$\hat{\Pi}_{[2s]}^\perp \phi_{(2s)} = \Pi_{[2s]}^\perp \phi_{(2s)}. \quad (2.28)$$

So far our analysis of the spin projection operators $\hat{\Pi}_{[2s]}^\perp$ and $\Pi_{[2s]}^\perp$ has been restricted to the linear space $\mathcal{V}_{(2s)}$. However, for fixed s , the operator $\Pi_{[2s]}^\perp$ given by eq. (2.21) is also defined to act on the linear spaces $\mathcal{V}_{(2s')}$ with $s' < s$. In fact, making use of (2.4) and (B.3c), it is possible to show that the following holds true

$$\Pi_{[2s]}^\perp \phi_{(2s')} = 0, \quad 1 \leq s' \leq s-1. \quad (2.29)$$

This important identity states that $\Pi_{[2s]}^\perp$ annihilates any lower-rank field $\phi_{\alpha(2s')} \in \mathcal{V}_{(2s')}$. It should be mentioned that $\Pi_{[2s]}^\perp$ does not annihilate lower-rank fermionic fields $\phi_{\alpha(2s'+1)}$. When acting on $\mathcal{V}_{(2s')}$, the two operators $\hat{\Pi}_{[2s]}^\perp$ and $\Pi_{[2s]}^\perp$ are no longer equal to each other, and in particular $\hat{\Pi}_{[2s]}^\perp \phi_{(2s')} \neq 0$. It is for this reason that we will continue to use different notation for the two operators.

It follows from (2.21) that the poles of $\Pi_{[2s]}^\perp$ correspond to the partially massless values $\tau_{(j,2s)}$ defined by (2.14).

2.2.2 Fermionic case

We now turn our attention to the fermionic case, $n = 2s + 1$, for integers $s \geq 1$. Let us introduce the differential operator $\mathbb{T}_{[2s+1]}$ of order $2s$ in derivatives

$$\mathbb{T}_{[2s+1]} = \sum_{j=0}^s 2^{2j} \frac{(s+j)!}{(s-j)!} \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t+1,2s+1)} \mathcal{S}^2) \mathcal{D}_{(2)}^{s-j} \mathcal{D}_{(-2)}^{s-j}. \quad (2.30)$$

Here $\tau_{(t,n)}$ are the partially massless values (2.14). The operator $\mathbb{T}_{[2s+1]}$ maps $\phi_{(2s+1)}$ to a transverse field

$$\mathcal{D}_{(-2)} \mathbb{T}_{[2s+1]} \phi_{(2s+1)} = 0. \quad (2.31)$$

However, this operator does not square to itself on $\mathcal{V}_{(2s+1)}$

$$\mathbb{T}_{[2s+1]} \mathbb{T}_{[2s+1]} \phi_{(2s+1)} = 2^{2s} (2s)! \prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s+1)} \mathcal{S}^2) \mathbb{T}_{[2s+1]} \phi_{(2s+1)}. \quad (2.32)$$

As a result, one can immediately define the dimensionless operator

$$\hat{\Pi}_{[2s+1]}^\perp := \left[2^{2s}(2s)! \prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s+1)} \mathcal{S}^2) \right]^{-1} \mathbb{T}_{[2s+1]}, \quad (2.33)$$

which is a transverse projector by construction. Following a derivation similar to that of (2.32), it can be shown that the operator $\hat{\Pi}_{[2s+1]}^\perp$ acts like the identity on the space of transverse fields. Hence, the operator $\hat{\Pi}_{[2s+1]}^\perp$ satisfies properties (1.2), and is thus a spin projection operator on $\mathcal{V}_{(2s+1)}$. Converting (2.33) to spinor notation yields

$$\begin{aligned} \hat{\Pi}_{[2s+1]}^\perp \phi_{\alpha(2s+1)} &= \left[\prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s+1)} \mathcal{S}^2) \right]^{-1} \sum_{j=0}^s 2^{2j-2s} \frac{2s+1}{2j+1} \binom{s+j}{2j} \\ &\quad \times \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t+1,2s+1)} \mathcal{S}^2) \mathcal{D}_{\alpha(2)}^{s-j} (\mathcal{D}^{\beta(2)})^{s-j} \phi_{\alpha(2j+1)\beta(2s-2j)}. \end{aligned} \quad (2.34)$$

As in the bosonic case, one can construct a fermionic projector purely in terms of the quadratic Casimir operators (2.3). Let us introduce the operator

$$\Pi_{[2s+1]}^\perp = \frac{1}{2^{2s}(2s)!} \prod_{j=1}^s \frac{(\mathcal{F}^2 - (2j-1)^2 (\mathcal{Q} - (2j-3)(2j+1) \mathcal{S}^2))}{(\mathcal{Q} - \tau_{(j,2s+1)} \mathcal{S}^2)}. \quad (2.35)$$

We wish to show that (2.35) indeed satisfies the properties (1.2) on $\mathcal{V}_{(2s+1)}$. Given an arbitrary transverse field $\psi_{(2s+1)}$, using (2.4) one can derive the identity

$$\begin{aligned} \prod_{j=1}^s \left(\mathcal{F}^2 - (2j-1)^2 (\mathcal{Q} - (2j-3)(2j+1) \mathcal{S}^2) \right) \psi_{(2s+1)} \\ = 2^{2s}(2s)! \prod_{j=1}^s \left(\mathcal{Q} - \tau_{(j,2s+1)} \mathcal{S}^2 \right) \psi_{(2s+1)}. \end{aligned} \quad (2.36)$$

It follows that $\Pi_{[2s+1]}^\perp$ acts like the identity on the space of transverse fields

$$\mathcal{D}_{(-2)} \psi_{(2s+1)} = 0 \implies \Pi_{[2s+1]}^\perp \psi_{(2s+1)} = \psi_{(2s+1)}. \quad (2.37)$$

By making use of (B.3c), one can show that the operator $\Pi_{[2s+1]}^\perp$ maps $\phi_{(2s+1)}$ to a transverse field

$$\mathcal{D}_{(-2)} \Pi_{[2s+1]}^\perp \phi_{(2s+1)} = \Pi_{[2s+1]}^\perp \mathcal{D}_{(-2)} \phi_{(2s+1)} \propto \mathcal{D}_{(2)}^s \mathcal{D}_{(-2)}^{s+1} \phi_{(2s+1)} = 0. \quad (2.38)$$

Finally, using (2.37) in conjunction with (2.38), one can show that $\Pi_{[2s+1]}^\perp$ is idempotent

$$\Pi_{[2s+1]}^\perp \Pi_{[2s+1]}^\perp \phi_{(2s+1)} = \Pi_{[2s+1]}^\perp \phi_{(2s+1)}. \quad (2.39a)$$

Hence, $\Pi_{[2s+1]}^\perp$ satisfies (1.2), and can thus be classified as a spin projector on AdS_3 .

In a similar fashion to the bosonic case, it may be shown that $\hat{\Pi}_{[2s+1]}^\perp$ and $\Pi_{[2s+1]}^\perp$ are equivalent on $\mathcal{V}_{(2s+1)}$,

$$\hat{\Pi}_{[2s+1]}^\perp \phi_{(2s+1)} = \Pi_{[2s+1]}^\perp \phi_{(2s+1)}. \quad (2.40)$$

Stepping away from $\mathcal{V}_{(2s+1)}$, one can show that for fixed s , the projector $\Pi_{[2s+1]}^\perp$ annihilates any lower-rank field $\phi_{(2s'+1)} \in \mathcal{V}_{(2s'+1)}$

$$\Pi_{[2s+1]}^\perp \phi_{(2s'+1)} = 0, \quad 1 \leq s' \leq s-1. \quad (2.41)$$

The two operators $\hat{\Pi}_{[2s+1]}^\perp$ and $\Pi_{[2s+1]}^\perp$ are not equivalent on $\mathcal{V}_{(2s'+1)}$. We remark that $\Pi_{[2s+1]}^\perp$ does not annihilate lower-rank bosonic fields $\phi_{\alpha(2s'+2)}$.

It follows from (2.35) that the poles of $\Pi_{[2s+1]}^\perp$ correspond to the partially massless values $\tau_{(j,2s+1)}$ defined by (2.14).

An important property of the projectors (2.21) and (2.35) is that they are symmetric operators, that is

$$\int d^3x e \psi^{\alpha(n)} \Pi_{[n]}^\perp \phi_{\alpha(n)} = \int d^3x e \phi^{\alpha(n)} \Pi_{[n]}^\perp \psi_{\alpha(n)}, \quad e^{-1} := \det(e_a{}^m), \quad (2.42)$$

for arbitrary well-behaved fields $\psi_{\alpha(n)}$ and $\phi_{\alpha(n)}$.

2.3 Helicity projectors

As previously mentioned, given a rank- n field $\phi_{\alpha(n)}$ satisfying the mass-shell equation (2.8), its projection $\Pi_{[n]}^\perp \phi_{\alpha(n)}$ furnishes the reducible representation $\mathfrak{D}(\rho, -\frac{n}{2}) \oplus \mathfrak{D}(\rho, \frac{n}{2})$. In particular, representations with both signs of helicity $\pm \frac{n}{2}$ appear in this decomposition.

In order to isolate the component of $\phi_{\alpha(n)}$ describing an irreducible representation of $\mathfrak{so}(2,2)$, it is necessary to split the spin projection operators $\Pi_{[n]}^\perp$ according to

$$\Pi_{[n]}^\perp = \mathbb{P}_{[n]}^{(+)} + \mathbb{P}_{[n]}^{(-)}. \quad (2.43)$$

Each of the helicity projectors $\mathbb{P}_{[n]}^{(\pm)}$ should satisfy (1.2a) and (1.2b). In addition, they should project out the component of $\phi_{\alpha(n)}$ carrying a single value of helicity. The last two requirements are equivalent to the equations

$$\mathcal{D}^{\beta(2)} \phi_{\beta(2)\alpha(n-2)}^{(\pm)} = 0, \quad (2.44a)$$

$$(\mathcal{F} \mp \rho) \phi_{\alpha(n)}^{(\pm)} = 0, \quad (2.44b)$$

where we have denoted $\phi_{\alpha(n)}^{(\pm)} := \mathbb{P}_{[n]}^{(\pm)} \phi_{\alpha(n)}$. It follows that $\phi_{\alpha(n)}^{(\pm)}$ furnishes the irreducible representation $\mathfrak{D}(\rho, \pm \frac{n}{2})$.

It is not difficult to show that the following operators satisfy these requirements

$$\mathbb{P}_{[n]}^{(\pm)} := \frac{1}{2} \left(\mathbb{1} \pm \frac{\mathcal{F}}{n\sqrt{\mathcal{Q} - (n+2)(n-2)\mathcal{S}^2}} \right) \Pi_{[n]}^\perp. \quad (2.45)$$

Here $\Pi_{[n]}^\perp$ are the spin projectors written in terms of Casimir operators, and are given by (2.21) and (2.35) in the bosonic and fermionic cases respectively. Of course, on $\mathcal{V}_{(n)}$, one could instead choose to represent the latter in their alternate form (2.19) and (2.33).

Using the defining features of $\Pi_{[n]}^\perp$, it can be shown that the operators $\mathbb{P}_{[n]}^{(+)}$ and $\mathbb{P}_{[n]}^{(-)}$ are orthogonal projectors when restricted to $\mathcal{V}_{(n)}$:

$$\mathbb{P}_{[n]}^{(\pm)} \mathbb{P}_{[n]}^{(\pm)} = \mathbb{P}_{[n]}^{(\pm)}, \quad \mathbb{P}_{[n]}^{(\pm)} \mathbb{P}_{[n]}^{(\mp)} = 0. \quad (2.46)$$

It is also clear that (2.45) projects onto the transverse subspace of $\mathcal{V}_{(n)}$ — it inherits this property from $\Pi_{[n]}$. Moreover, the off-shell field $\phi_{\alpha(n)}^{(\pm)}$ satisfies the constraint

$$\left(\mathcal{F} \mp n\sqrt{\mathcal{Q} - (n-2)(n+2)\mathcal{S}^2}\right)\phi_{\alpha(n)}^{(\pm)} = 0. \quad (2.47)$$

If $\phi_{\alpha(n)}^{(\pm)}$ is on the mass-shell, eq. (2.8), then (2.47) reduces to (2.44b).

2.4 Longitudinal projectors and lower-spin extractors

In this section we study the operator $\Pi_{[n]}^{\parallel}$ which is the compliment of $\Pi_{[n]}^{\perp}$,

$$\Pi_{[n]}^{\parallel} := \mathbb{1} - \Pi_{[n]}^{\perp}. \quad (2.48)$$

By construction, the two operators $\Pi_{[n]}^{\perp}$ and $\Pi_{[n]}^{\parallel}$ resolve the identity, $\mathbb{1} = \Pi_{[n]}^{\parallel} + \Pi_{[n]}^{\perp}$, and form an orthogonal set of projectors

$$\Pi_{[n]}^{\perp}\Pi_{[n]}^{\perp} = \Pi_{[n]}^{\perp}, \quad \Pi_{[n]}^{\parallel}\Pi_{[n]}^{\parallel} = \Pi_{[n]}^{\parallel}, \quad (2.49a)$$

$$\Pi_{[n]}^{\parallel}\Pi_{[n]}^{\perp} = 0, \quad \Pi_{[n]}^{\perp}\Pi_{[n]}^{\parallel} = 0. \quad (2.49b)$$

Moreover, it can be shown that $\Pi_{[n]}^{\parallel}$ projects a field $\phi_{\alpha(n)}$ onto its longitudinal component. A rank- n field $\psi_{\alpha(n)}$ is said to be longitudinal if there exists a rank- $(n-2)$ field $\psi_{\alpha(n-2)}$ such that $\psi_{\alpha(n)}$ may be expressed as $\psi_{\alpha(n)} = \mathcal{D}_{\alpha(2)}\psi_{\alpha(n-2)}$. Such fields are also sometimes referred to as being pure gauge. Therefore, we find that

$$\phi_{\alpha(n)}^{\parallel} := \Pi_{[n]}^{\parallel}\phi_{\alpha(n)} = \mathcal{D}_{\alpha(2)}\phi_{\alpha(n-2)}, \quad (2.50)$$

for some unconstrained field $\phi_{\alpha(n-2)}$. For $\phi_{\alpha(n)}$ off-shell, $\phi_{\alpha(n-2)}$ will be non-local in general. For example, in the case of a vector field ϕ_a , we have $\phi_a^{\parallel} = \mathcal{D}_a\phi$ where $\phi = \frac{1}{\mathcal{Q}}\mathcal{D}^a\phi_a$.

Using the fact that $\Pi_{[n]}^{\perp}$ and $\Pi_{[n]}^{\parallel}$ resolve the identity, one can decompose an arbitrary field $\phi_{\alpha(n)}$ as follows

$$\phi_{\alpha(n)} = \phi_{\alpha(n)}^{\perp} + \mathcal{D}_{\alpha(2)}\phi_{\alpha(n-2)}. \quad (2.51)$$

Here $\phi_{\alpha(n)}^{\perp}$ is transverse and $\phi_{\alpha(n-2)}$ is unconstrained. Repeating this process iteratively, we obtain the following decomposition

$$\phi_{\alpha(n)} = \sum_{j=0}^{\lfloor n/2 \rfloor} (\mathcal{D}_{\alpha(2)})^j \phi_{\alpha(n-2j)}^{\perp}. \quad (2.52)$$

Here each of the fields $\phi_{\alpha(n-2j)}^{\perp}$ are transverse, except of course $\phi^{\perp}$ and $\phi_{\alpha}^{\perp}$. We note that, using (2.43), one may take the decomposition (2.52) a step further and bisect each term into irreducible components which are transverse and have positive or negative helicity,

$$\phi_{\alpha(n)} = \sum_{j=0}^{\lfloor n/2 \rfloor} (\mathcal{D}_{\alpha(2)})^j \left(\phi_{\alpha(n-2j)}^{(+)} + \phi_{\alpha(n-2j)}^{(-)} \right). \quad (2.53)$$

Making use of the projectors (2.21) and (2.35) and their corresponding properties, one can construct operators which extract the component $\phi_{\alpha(n-2j)}^\perp$ from the decomposition (2.52), where $1 \leq j \leq \lfloor n/2 \rfloor$. In particular, we find that the spin $\frac{1}{2}(n-2j)$ component may be extracted via

$$\phi_{\alpha(n)} \mapsto \phi_{\alpha(n-2j)}^\perp = (\mathbb{S}_{[n-2j]}^\perp \phi)_{\alpha(n-2j)} \equiv \mathbb{S}_{\alpha(n-2j)}^\perp(\phi), \quad (2.54)$$

where we have defined

$$\mathbb{S}_{\alpha(n-2j)}^\perp(\phi) = \frac{(-1)^j}{2^{2j}} \binom{n}{j} \prod_{k=1}^j (\mathcal{Q} - \tau_{(k, n-2j+2k)} \mathcal{S}^2)^{-1} \Pi_{[n-2j]}^\perp (\mathcal{D}^{\beta(2)})^j \phi_{\alpha(n-2j)\beta(2j)}. \quad (2.55)$$

From this expression, it is clear that $\mathbb{S}_{\alpha(n-2j)}^\perp(\phi)$ is transverse,

$$0 = \mathcal{D}^{\beta(2)} \mathbb{S}_{\beta(2)\alpha(n-2j-2)}^\perp(\phi). \quad (2.56)$$

Therefore it is appropriate to call $\mathbb{S}_{[n-2j]}^\perp$ the transverse spin $\frac{1}{2}(n-2j)$ extractor. It is not a projector, since it is dimensionful and reduces the rank of the field on which it acts.

Let $\psi_{\alpha(n)}$ be some longitudinal field, $\psi_{\alpha(n)} = \mathcal{D}_{\alpha(2)} \zeta_{\alpha(n-2)}$. We do not assume it to be in the image of $\Pi_{[n]}^\parallel$. However, since $\Pi_{[n]}^\perp$ commutes with $\mathcal{D}_{\alpha(2)}$ and annihilates all lower-rank fields, eq. (2.29), it follows that it also annihilates any rank- n longitudinal field⁷

$$\psi_{\alpha(n)} = \mathcal{D}_{\alpha(2)} \zeta_{\alpha(n-2)} \implies \Pi_{[n]}^\perp \psi_{\alpha(n)} = 0. \quad (2.57)$$

As a consequence, given two integers m, n satisfying $2 \leq m \leq n$, it immediately follows that $\Pi_{[n]}^\parallel$ acts as the identity operator on the space of rank- m longitudinal fields $\psi_{\alpha(m)}$,

$$\psi_{\alpha(m)} = \mathcal{D}_{\alpha(2)} \psi_{\alpha(m-2)} \implies \Pi_{[m+2s]}^\parallel \psi_{\alpha(m)} = \psi_{\alpha(m)}, \quad (2.58)$$

with s a non-negative integer. These properties will be useful in section 2.5.

Decompositions similar to (2.51) are well-known in the literature (usually they are stated without a derivation) and are used in the framework of path-integral quantisation, see e.g. [33]. Making use of the projectors allows one to reconstruct $\phi_{\alpha(n)}^\perp$ and $\phi_{\alpha(n-2)}$ from $\phi_{\alpha(n)}$. Quite often such decompositions are given in vector notation in terms of a symmetric field $\varphi_{a_1 \dots a_s} = \varphi_{(a_1 \dots a_s)}$ subject to the double traceless constraint $\varphi_{a_1 \dots a_{s-4} b c}{}^{bc} = 0$ (Fronsdal's field [34]). The decomposition in AdS₃ reads [33]

$$\varphi_{a_1 \dots a_s} = \varphi_{a_1 \dots a_s}^{\text{TT}} + \eta_{(a_1 a_2} \tilde{\varphi}_{a_3 \dots a_s)} + \mathcal{D}_{(a_1} \zeta_{a_2 \dots a_s)}, \quad \mathcal{D}^b \varphi_{b a_1 \dots a_{s-1}}^{\text{TT}} = 0, \quad (2.59)$$

where $\varphi_{a_1 \dots a_s}^{\text{TT}}$, $\tilde{\varphi}_{a_1 \dots a_{s-2}}$ and $\zeta_{a_1 \dots a_{s-1}}$ are symmetric and traceless. This decomposition for a symmetric second-rank tensor field, $\varphi_{ab} = \varphi_{ba}$, in a curved four-dimensional space was introduced long ago [35–38]. In this paper we consider only symmetric traceless fields $\varphi_{a_1 \dots a_s}$ satisfying the constraint $\varphi_{a_1 \dots a_{s-2} b}{}^b = 0$. In this case, $\tilde{\varphi}_{a_1 \dots a_{s-2}}$ in the decomposition (2.59) is given by

$$\tilde{\varphi}_{a_1 \dots a_{s-2}} = -\frac{s-1}{2s-1} \mathcal{D}^b \zeta_{a_1 \dots a_{s-2} b}. \quad (2.60)$$

⁷This also implies that $\hat{\Pi}_{[n]}^\perp \psi_{\alpha(n)} = 0$, since both $\hat{\Pi}_{[n]}^\perp$ and $\Pi_{[n]}^\perp$ are equal on $\mathcal{V}_{(n)}$.

2.5 Linearised higher-spin Cotton tensors

Further applications of spin projection operators can be found in modern conformal higher-spin theories. In particular, we will show that the spin projectors can be used to obtain new realisations of the linearised higher-spin Cotton tensors, which were recently derived in [22]. For integer $n \geq 2$, the higher-spin bosonic and fermionic Cotton tensors $\mathfrak{C}_{\alpha(n)}(h)$ take the respective closed forms

$$\begin{aligned} \mathfrak{C}_{\alpha(2s)}(h) &= \frac{1}{2^{2s-1}} \sum_{j=0}^{s-1} 2^{2j+1} \binom{s+j}{2j+1} \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t, 2s)} \mathcal{S}^2) \\ &\quad \times \mathcal{D}_{\alpha(2)}^{s-j-1} \mathcal{D}_{\alpha}^{\beta} (\mathcal{D}^{\beta(2)})^{s-j-1} h_{\alpha(2j+1)\beta(2s-2j-1)}, \end{aligned} \quad (2.61a)$$

$$\begin{aligned} \mathfrak{C}_{\alpha(2s+1)}(h) &= \frac{1}{2^{2s}} \sum_{j=0}^s 2^{2j} \binom{s+j}{2j} \frac{(2s+1)}{(2j+1)} \prod_{t=1}^j (\mathcal{Q} - \tau_{(s-t+1, 2s+1)} \mathcal{S}^2) \\ &\quad \times \mathcal{D}_{\alpha(2)}^{s-j} (\mathcal{D}^{\beta(2)})^{s-j} h_{\alpha(2j+1)\beta(2s-2j)}. \end{aligned} \quad (2.61b)$$

The Cotton tensors are primary descendents of the conformal gauge field $h_{\alpha(n)}$, which is a real field defined modulo gauge transformations of the form

$$\delta_{\zeta} h_{\alpha(n)} = \mathcal{D}_{\alpha(2)} \zeta_{\alpha(n-2)}, \quad (2.62)$$

for some real unconstrained gauge parameter $\zeta_{\alpha(n-2)}$. The Cotton tensors (2.61) are characterised by the properties:

1. $\mathfrak{C}_{\alpha(n)}(h)$ is transverse

$$\mathcal{D}^{\beta\gamma} \mathfrak{C}_{\beta\gamma\alpha(n-2)}(h) = 0. \quad (2.63a)$$

2. $\mathfrak{C}_{\alpha(n)}(h)$ is gauge-invariant

$$\mathfrak{C}_{\alpha(n)}(\delta_{\zeta} h) = 0. \quad (2.63b)$$

Making use of the bosonic (2.19) and fermionic (2.33) spin projectors $\hat{\Pi}_{[n]}^{\perp}$, we see that the higher-spin Cotton tensors (2.61) can be recast into the simple form:

$$\mathfrak{C}_{\alpha(2s)}(h) = \frac{1}{2^s} \prod_{t=1}^{s-1} (\mathcal{Q} - \tau_{(t, 2s)} \mathcal{S}^2) \mathcal{F} \hat{\Pi}_{[2s]}^{\perp} h_{\alpha(2s)}, \quad (2.64a)$$

$$\mathfrak{C}_{\alpha(2s+1)}(h) = \prod_{t=1}^s (\mathcal{Q} - \tau_{(t, 2s+1)} \mathcal{S}^2) \hat{\Pi}_{[2s+1]}^{\perp} h_{\alpha(2s+1)}. \quad (2.64b)$$

The identity $\mathcal{F} \mathcal{D}_{(-2)}^s \phi_{\alpha(2s)} = 0$ proves useful in deriving (2.64a). In the flat space limit, $\mathcal{S} \rightarrow 0$, (2.64) coincides with the closed form expressions of $\mathfrak{C}_{\alpha(n)}(h)$ given in [39, 40].⁸ Moreover, we can make use of the equivalent family of projectors $\Pi_{[n]}^{\perp}$ to recast $\mathfrak{C}_{\alpha(n)}(h)$ purely in terms of the quadratic Casimir operators (2.3). Explicitly, they read

$$\mathfrak{C}_{\alpha(2s)}(h) = \frac{\mathcal{F}}{2^{2s-1} (2s-1)!} \prod_{j=1}^{s-1} \left(\mathcal{F}^2 - 4j^2 (\mathcal{Q} - 4(j-1)(j+1) \mathcal{S}^2) \right) h_{\alpha(2s)}, \quad (2.65a)$$

$$\mathfrak{C}_{\alpha(2s+1)}(h) = \frac{1}{2^{2s} (2s)!} \prod_{j=0}^{s-1} \left(\mathcal{F}^2 - (2j+1)^2 (\mathcal{Q} - (2j-1)(2j+3) \mathcal{S}^2) \right) h_{\alpha(2s+1)}. \quad (2.65b)$$

⁸It can be shown that the Cotton tensors are equivalent to those derived in [41, 42].

There are many advantages to expressing the Cotton tensors in terms of spin projection operators. Firstly, in both (2.64) and (2.65), the properties of (i) transversality (2.63a) and (ii) gauge invariance (2.63b) are manifest, as a consequence of the projector properties (1.2b) and (2.57) respectively. Using this gauge freedom, one may impose the transverse gauge condition on $h_{\alpha(n)}$,

$$h_{\alpha(n)} \equiv h_{\alpha(n)}^T, \quad 0 = \mathcal{D}^{\beta(2)} h_{\beta(2)\alpha(n-2)}^T. \quad (2.66)$$

On account of (1.2c), in this gauge the Cotton tensors become manifestly factorised into products of second order differential operators involving all partial masses,

$$\mathfrak{C}_{\alpha(2s)}(h^T) = \frac{1}{2^s} \prod_{t=1}^{s-1} (\mathcal{Q} - \tau_{(t,2s)} \mathcal{S}^2) \mathcal{F} h_{\alpha(2s)}^T, \quad (2.67a)$$

$$\mathfrak{C}_{\alpha(2s+1)}(h^T) = \prod_{t=1}^s (\mathcal{Q} - \tau_{(t,2s+1)} \mathcal{S}^2) h_{\alpha(2s+1)}^T. \quad (2.67b)$$

This property was observed in [22] without the use of projectors. An interesting feature of the new realisation (2.65), which was not observed in [22], is that the Cotton tensors are manifestly factorised in terms of second-order differential operators without having to enter the transverse gauge.

By virtue of the above observations, it follows that the conformal higher-spin action [43, 44]

$$S_{\text{CHS}}^{(n)}[h] = \frac{i^n}{2^{[n/2]+1}} \int d^3x e h^{\alpha(n)} \mathfrak{C}_{\alpha(n)}(h) \quad (2.68)$$

is manifestly gauge invariant and factorised when $\mathfrak{C}_{\alpha(n)}(h)$ is expressed as in (2.65).

Analogous factorised expressions can be given for the so-called new topologically massive (NTM) models. For bosonic fields they were first introduced in [45] in Minkowski space. Extensions of these models to fields with half-integer spin were proposed in [43], where their generalisations to an AdS background were also given. These models are formulated solely in terms of the gauge prepotentials $h_{\alpha(n)}$ and the associated Cotton tensors $\mathfrak{C}_{\alpha(n)}(h)$. Given an integer $n \geq 2$, the gauge-invariant NTM action for the field $h_{\alpha(n)}$ given in [43] is

$$S_{\text{NTM}}^{(n)}[h] = \frac{i^n}{2^{[n/2]+1}} \frac{1}{\rho} \int d^3x e h^{\alpha(n)} (\mathcal{F} - \sigma \rho) \mathfrak{C}_{\alpha(n)}(h), \quad (2.69)$$

where ρ is some positive mass parameter and $\sigma := \pm 1$. Making use of the representation (2.65) leads to a manifestly gauge invariant and factorised form for the action (2.69). The equation of motion obtained by varying (2.69) with respect to the field $h^{\alpha(n)}$ is

$$0 = (\mathcal{F} - \sigma \rho) \mathfrak{C}_{\alpha(n)}(h). \quad (2.70)$$

By analysing (2.70), it can be shown that on-shell, the action (2.69) describes a propagating mode with pseudo-mass ρ , spin $n/2$ and helicity $\sigma n/2$ given $\rho \neq \rho_{(t,2s)}$. For the case $\rho = \rho_{(t,2s)}$, the model describes only pure gauge degrees of freedom.

Recently, a new variant of the NTM model for bosonic fields in $\mathbb{M}^3$ was proposed in [21]. This model also does not require auxilliary fields, but is of order $2s - 1$ in derivatives, whereas those given in [45] are of order $2s$. Given an integer $s \geq 1$, the actions of [21] may be readily extended to AdS_3 as follows

$$\tilde{S}_{\text{NTM}}^{(2s)}[h] = \int d^3x e h^{\alpha(2s)} (\mathcal{F} - \sigma\rho) \mathfrak{W}_{\alpha(2s)}(h), \quad (2.71)$$

where ρ is a positive mass parameter, $\sigma := \pm 1$, and $\mathfrak{W}_{\alpha(2s)}(h)$ is the field strength,

$$\mathfrak{W}_{\alpha(2s)}(h) := \prod_{t=1}^{s-1} (\mathcal{Q} - \tau_{(t,2s)} \mathcal{S}^2) \Pi_{[2s]}^\perp h_{\alpha(2s)}. \quad (2.72)$$

Due to the properties of $\Pi_{[2s]}^\perp$, the action (2.71) is manifestly gauge invariant and factorised. The descendent $\mathfrak{W}_{\alpha(2s)}(h)$ may be obtained from $\mathfrak{C}_{\alpha(2s)}(h)$ by stripping off $\mathcal{F}$:

$$\mathfrak{C}_{\alpha(2s)}(h) = \frac{1}{2s} \mathcal{F} \mathfrak{W}_{\alpha(2s)}(h). \quad (2.73)$$

A similar construction does not appear to be possible in the fermionic case.

The equation of motion obtained by varying (2.71) with respect to the field $h^{\alpha(2s)}$ is

$$0 = (\mathcal{F} - \sigma\rho) \mathfrak{W}_{\alpha(2s)}(h). \quad (2.74)$$

By analysing (2.74), it can be shown that on-shell, the model (2.71) has the same particle content as the NTM model (2.69).

2.6 Results in Minkowski space

In this section we study the flat-space limit of various results derived in section 2. Of particular interest are the transverse projectors which are constructed in terms of the Casimir operators of $\mathfrak{so}(2,2)$. In this limit we obtain novel realisations for the transverse projectors on $\mathbb{M}^3$ which did not appear in [8, 18]. They are expressed in terms of the quadratic Casimir operators of the three dimensional Poincaré algebra $\mathfrak{iso}(2,1)$,

$$\square := \partial^a \partial_a = -\frac{1}{2} \partial^{\alpha\beta} \partial_{\alpha\beta}, \quad (2.75a)$$

$$\mathcal{W} := \partial^{\alpha\beta} M_{\alpha\beta}, \quad [\mathcal{W}, \partial_{\alpha\beta}] = 0. \quad (2.75b)$$

Here $\partial_{\alpha\beta}$ are the partial derivatives of $\mathbb{M}^3$ and $\mathcal{W}$ is the Pauli-Lubanski pseudo-scalar. We recall that an irreducible representation of $\mathfrak{iso}(2,1)$ with mass ρ and helicity $\sigma n/2$ may be realised on the space of totally symmetric rank- n spinor fields $\phi_{\alpha(n)}$ satisfying the differential equations

$$\partial^{\beta(2)} \phi_{\beta(2)\alpha(n-2)} = 0, \quad (2.76a)$$

$$(\mathcal{W} - \sigma n \rho) \phi_{\alpha(n)} = 0, \quad (2.76b)$$

where $\sigma = \pm 1$. These equations are equivalent to those given in [26, 27]. We are concerned only with representations carrying (half-)integer spin.

By taking the limit $\mathcal{S} \rightarrow 0$ of the corresponding AdS_3 expressions given above, one may obtain the following results in Minkowski space:

- The bosonic (2.21) and fermionic (2.35) spin projection operators reduce to

$$\mathcal{P}_{[2s]}^\perp = \frac{1}{2^{2s-1}(2s)!\square^s} \prod_{j=0}^{s-1} \left(\mathcal{W}^2 - (2j)^2 \square \right), \quad (2.77a)$$

$$\mathcal{P}_{[2s+1]}^\perp = \frac{1}{2^{2s}(2s)!\square^s} \prod_{j=0}^{s-1} \left(\mathcal{W}^2 - (2j+1)^2 \square \right). \quad (2.77b)$$

- The orthogonal helicity projectors (2.45) reduce to

$$\mathbb{P}_{[n]}^{(\pm)} = \frac{1}{2} \left(\mathbb{1} \pm \frac{\mathcal{W}}{n\sqrt{\square}} \right) \mathcal{P}_{[n]}^\perp. \quad (2.78)$$

From (2.47) it follows that the field $\phi_{\alpha(n)}^{(\pm)} := \mathbb{P}_{[n]}^{(\pm)} \phi_{\alpha(n)}$ satisfies

$$(\mathcal{W} \mp n\sqrt{\square}) \phi_{\alpha(n)}^{(\pm)} = 0. \quad (2.79)$$

For a $\phi_{\alpha(n)}$ lying on the mass shell, $(\square - \rho^2) \phi_{\alpha(n)} = 0$, this reduces to (2.76b).

- The transverse spin $\frac{1}{2}(n-2j)$ extractors (2.55), where $1 \leq j \leq \lfloor n/2 \rfloor$, are given by

$$\mathbb{S}_{\alpha(n-2j)}^\perp(\phi) = \frac{(-1)^j}{2^{2j}} \binom{n}{j} \frac{1}{\square^j} \mathcal{P}_{[n-2j]}^\perp (\partial^{\beta(2)})^j \phi_{\alpha(n-2j)\beta(2j)}. \quad (2.80)$$

- The new realisations for the higher-spin Cotton tensors (2.65) become

$$\mathcal{C}_{\alpha(2s)}(h) = \frac{\mathcal{W}}{2^{2s-1}(2s-1)!} \prod_{j=1}^{s-1} \left(\mathcal{W}^2 - (2j)^2 \square \right) h_{\alpha(2s)}, \quad (2.81a)$$

$$\mathcal{C}_{\alpha(2s+1)}(h) = \frac{1}{2^{2s}(2s)!} \prod_{j=0}^{s-1} \left(\mathcal{W}^2 - (2j+1)^2 \square \right) h_{\alpha(2s+1)}. \quad (2.81b)$$

It may be shown that each of these expressions are equivalent to the corresponding ones given in [18], except for the lower-spin extractors, which were not discussed in [18].

3 Transverse superprojectors in $\text{AdS}^{3|2}$

In this section, we derive the superprojectors in $\mathcal{N} = 1$ AdS superspace, $\text{AdS}^{3|2}$, and explore several of their applications. We remind the reader that $\text{AdS}^{3|2}$ is the maximally supersymmetric solution of three-dimensional $\mathcal{N} = 1$ AdS supergravity [14].

We begin by reviewing the geometric structure of $\text{AdS}^{3|2}$, as presented in [46], which is described in terms of its covariant derivatives⁹

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} M_{bc}. \quad (3.1)$$

⁹In the hope that no confusion arises, we use the same notation for the vector covariant derivative in AdS_3 and in $\text{AdS}^{3|2}$.

Here E_A^M is the inverse supervielbein and Ω_A^{bc} the Lorentz connection. The covariant derivatives obey the following (anti-)commutation relations¹⁰

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\mathcal{D}_{\alpha\beta} - 4i\mathcal{S}M_{\alpha\beta}, \quad (3.2a)$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma] = -2\mathcal{S}\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}, \quad [\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma\delta}] = 4\mathcal{S}^2(\varepsilon_{\gamma(\alpha}M_{\beta)\delta} + \varepsilon_{\delta(\alpha}M_{\beta)\gamma}), \quad (3.2b)$$

where $\mathcal{S} \neq 0$ is a real constant parameter which determines the curvature of $\text{AdS}^{3|2}$.

We list several identities which prove indispensable for calculations:

$$\mathcal{D}_\alpha \mathcal{D}_\beta = i\mathcal{D}_{\alpha\beta} - 2i\mathcal{S}M_{\alpha\beta} + \frac{1}{2}\varepsilon_{\alpha\beta}\mathcal{D}^2, \quad (3.3a)$$

$$\mathcal{D}^\beta \mathcal{D}_\alpha \mathcal{D}_\beta = 4i\mathcal{S}\mathcal{D}_\alpha, \quad \{\mathcal{D}^2, \mathcal{D}_\alpha\} = 4i\mathcal{S}\mathcal{D}_\alpha, \quad (3.3b)$$

$$\mathcal{D}^2 \mathcal{D}_\alpha = 2i\mathcal{S}\mathcal{D}_\alpha + 2i\mathcal{D}_{\alpha\beta}\mathcal{D}^\beta - 4i\mathcal{S}\mathcal{D}^\beta M_{\alpha\beta}, \quad (3.3c)$$

$$[\mathcal{D}_\alpha \mathcal{D}_\beta, \mathcal{D}^2] = 0 \implies [\mathcal{D}_{\alpha\beta}, \mathcal{D}^2] = 0, \quad (3.3d)$$

where we have denoted $\mathcal{D}^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha$. These relations can be derived from the algebra of covariant derivatives (3.2).

Crucial to our analysis are two independent Casimir operators of the $\mathcal{N} = 1$ AdS_3 isometry supergroup $\text{OSp}(1|2; \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. They are [22, 43]

$$\mathbb{Q} := -\frac{1}{4}\mathcal{D}^2 \mathcal{D}^2 + i\mathcal{S}\mathcal{D}^2, \quad [\mathbb{Q}, \mathcal{D}_A] = 0, \quad (3.4a)$$

$$\mathbb{F} := -\frac{i}{2}\mathcal{D}^2 + 2\mathcal{D}^{\alpha\beta}M_{\alpha\beta}, \quad [\mathbb{F}, \mathcal{D}_A] = 0. \quad (3.4b)$$

Making use of the identity

$$-\frac{1}{4}\mathcal{D}^2 \mathcal{D}^2 = \square - 2i\mathcal{S}\mathcal{D}^2 + 2\mathcal{S}\mathcal{D}^{\alpha\beta}M_{\alpha\beta} - 2\mathcal{S}^2 M^{\alpha\beta}M_{\alpha\beta}, \quad (3.5)$$

allows us to express $\mathbb{Q}$ in terms of the d'Alembert operator $\square = \mathcal{D}^a \mathcal{D}_a$. The operators $\mathbb{Q}$ and $\mathbb{F}$ are related to each other as follows

$$\begin{aligned} \mathbb{F}^2 \Phi_{\alpha(n)} &= \left((2n+1)^2 \mathbb{Q} + (2n+1)(2n^2 + 2n - 1)i\mathcal{S}\mathcal{D}^2 + 4n^2(n+2)^2\mathcal{S}^2 \right) \Phi_{\alpha(n)} \\ &\quad + 4(2n^2 + n - 2)i\mathcal{S}\mathcal{D}_\alpha \mathcal{D}^\beta \Phi_{\beta\alpha(n-1)} - 4in\mathcal{D}_{\alpha\beta}\mathcal{D}^\beta \mathcal{D}^\gamma \Phi_{\gamma\alpha(n-1)} \\ &\quad + 4n(n-1)\mathcal{D}_{\alpha(2)}\mathcal{D}^{\beta(2)}\Phi_{\beta(2)\alpha(n-2)}, \end{aligned} \quad (3.6)$$

for an arbitrary symmetric rank- n spinor superfield $\Phi_{\alpha(n)}$.

3.1 On-shell superfields

We begin by reviewing aspects of on-shell superfields in $\text{AdS}^{3|2}$, as presented in [22]. Given an integer $n \geq 1$, the real symmetric superfield $\Phi_{\alpha(n)}$ is said to be on-shell if it satisfies the two constraints

$$0 = \mathcal{D}^\beta \Phi_{\beta\alpha(n-1)}, \quad (3.7a)$$

$$0 = (\mathbb{F} - \sigma M)\Phi_{\alpha(n)}, \quad (3.7b)$$

¹⁰In vector notation, the commutation relations (3.2b) take the form $[\mathcal{D}_a, \mathcal{D}_\beta] = \mathcal{S}(\gamma_a)_\beta{}^\gamma \mathcal{D}_\gamma$ and $[\mathcal{D}_a, \mathcal{D}_b] = -4\mathcal{S}^2 M_{ab}$.

where $\sigma := \pm 1$ and $M \geq 0$ is a real parameter of unit mass dimension. Such a field furnishes an irreducible representation of the $\mathcal{N} = 1$ AdS₃ superalgebra $\mathfrak{osp}(1|2; \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, which we denote as $\mathfrak{S}(M, \sigma \frac{n}{2})$. It can be shown that the representation $\mathfrak{S}(M, \sigma \frac{n}{2})$ decomposes into two irreducible representations of $\mathfrak{so}(2, 2)$,

$$\mathfrak{S}\left(M, \sigma \frac{n}{2}\right) = \mathfrak{D}\left(\rho_A, \sigma_A \frac{n}{2}\right) \oplus \mathfrak{D}\left(\rho_B, \sigma_B \frac{n+1}{2}\right). \quad (3.8)$$

Here, the pseudo-masses are given by

$$\rho_A = \frac{n}{2n+1} \left| \sigma M - (n+2)\mathcal{S} \right|, \quad \rho_B = \frac{n+1}{2n+1} \left| \sigma M + (n-1)\mathcal{S} \right|, \quad (3.9)$$

and the corresponding signs of the superhelicities are

$$\sigma_A = \frac{\sigma M - (n+2)\mathcal{S}}{\left| \sigma M - (n+2)\mathcal{S} \right|}, \quad \sigma_B = \frac{\sigma M + (n-1)\mathcal{S}}{\left| \sigma M + (n-1)\mathcal{S} \right|}. \quad (3.10)$$

The representation $\mathfrak{S}(M, \sigma \frac{n}{2})$ is unitary if the parameter M obeys the unitarity bound $M \geq 2(n-1)(n+1)\mathcal{S}$. This bound ensures that both representations appearing in the decomposition (3.8) are unitary.

A superfield satisfying the first condition (3.7a) is said to be transverse. Any transverse superfield may be shown to satisfy the following relation

$$-\frac{i}{2} \mathcal{D}^2 \Phi_{\alpha(n)} = \mathcal{D}_{(\alpha_1}{}^{\beta} \Phi_{\alpha_2 \dots \alpha_n) \beta} + (n+2) \mathcal{S} \Phi_{\alpha(n)}. \quad (3.11)$$

If a transverse superfield also satisfies (3.7b), we say that it carries pseudo-mass M , super-spin $n/2$ and superhelicity $\frac{1}{2}(n+\frac{1}{2})\sigma$. From (3.11) it follows that an on-shell superfield (3.7) satisfies

$$-\frac{i}{2} \mathcal{D}^2 \Phi_{\alpha(n)} = \frac{1}{2n+1} (\sigma M + 2n(n+2)\mathcal{S}) \Phi_{\alpha(n)}, \quad (3.12)$$

and hence the second-order mass-shell equation

$$0 = (\mathbb{Q} - \lambda^2) \Phi_{\alpha(n)}, \quad (3.13a)$$

$$\lambda^2 := \frac{1}{(2n+1)^2} [\sigma M + 2n(n+2)\mathcal{S}] [\sigma M + 2(n-1)(n+1)\mathcal{S}]. \quad (3.13b)$$

The equations (3.7a) and (3.12) were introduced in [47]. On the other hand, one may instead consider a superfield $\Phi_{\alpha(n)}$ satisfying (3.7a) and (3.13a). In this case, using the identity (3.6), one can show that (3.13a) becomes

$$0 = \left(\mathbb{F} - \sigma_{(-)} |M_{(-)}| \right) \left(\mathbb{F} - \sigma_{(+)} |M_{(+)}| \right), \quad (3.14)$$

where we have defined $\sigma_{(\pm)} = \text{sgn}(M_{(\pm)})$ and

$$M_{(\pm)} := -(2n^2 + 2n - 1)\mathcal{S} \pm (2n+1)\sqrt{\lambda^2 + \mathcal{S}^2}. \quad (3.15)$$

It follows that such a field furnishes the reducible representation

$$\mathfrak{S}\left(|M_{(-)}|, \sigma_{(-)} \frac{n}{2}\right) \oplus \mathfrak{S}\left(|M_{(+)}|, \sigma_{(+)} \frac{n}{2}\right). \quad (3.16)$$

In $\text{AdS}^{3|2}$ there exist two distinct types of on-shell partially massless superfields [22], which are distinguished by the sign σ of their superhelicity. More specifically, they are described by an on-shell superfield (3.7) whose pseudo-mass and parameter σ assume the special combinations

$$\sigma = +1, \quad M \equiv M_{(t,n)}^{(+)} = 2[n(n-2t+1) - (t-1)]\mathcal{S}, \quad 1 \leq t \leq \lfloor n/2 \rfloor, \quad (3.17a)$$

$$\sigma = -1, \quad M \equiv M_{(t,n)}^{(-)} = 2[n(n-2t) - (t+1)]\mathcal{S}, \quad 0 \leq t \leq \lceil n/2 \rceil - 1. \quad (3.17b)$$

The integer t is called the (super)depth and the corresponding supermultiplets are denoted by $\Phi_{\alpha(n)}^{(t,+)}$ and $\Phi_{\alpha(n)}^{(t,-)}$ respectively. Their second order equations (3.13) take the form

$$0 = (\mathbb{Q} - \lambda_{(t,n)}^{(+)}\mathcal{S}^2)\Phi_{\alpha(n)}^{(t,+)}, \quad 0 = (\mathbb{Q} - \lambda_{(t,n)}^{(-)}\mathcal{S}^2)\Phi_{\alpha(n)}^{(t,-)}, \quad (3.18)$$

where we have introduced the partially massless values

$$\lambda_{(t,n)}^{(+)} = 4(n-t)(n-t+1), \quad \lambda_{(t,n)}^{(-)} = 4t(t+1). \quad (3.19)$$

The gauge symmetry associated with positive and negative superhelicity partially massless superfields of depth- t is

$$\delta_\Lambda \Phi_{\alpha(n)}^{(t,+)} = (\mathcal{D}_{\alpha(2)})^t \Lambda_{\alpha(n-2t)}, \quad 1 \leq t \leq \lfloor n/2 \rfloor, \quad (3.20a)$$

$$\delta_\Lambda \Phi_{\alpha(n)}^{(t,-)} = i^n (\mathcal{D}_{\alpha(2)})^t \mathcal{D}_\alpha \Lambda_{\alpha(n-2t-1)}, \quad 0 \leq t \leq \lceil n/2 \rceil - 1. \quad (3.20b)$$

In particular, the system of equations (3.7) and (3.17) is invariant under these transformations for an on-shell real gauge parameter.

3.2 Superspin projection operators

We wish to find supersymmetric generalisations of the spin projection operators in AdS_3 which were computed in section 2. More precisely, let us denote by $\mathbb{V}_{(n)}$ the space of totally symmetric rank- n superfields $\Phi_{\alpha(n)}$ on $\text{AdS}^{3|2}$. For any integer $n \geq 1$, we define the rank- n superspin projection operator¹¹ $\Pi_{[n]}^\perp$ to act on $\mathbb{V}_{(n)}$ by the rule

$$\Pi_{[n]}^\perp : \mathbb{V}_{(n)} \longrightarrow \mathbb{V}_{(n)}, \quad \Phi_{\alpha(n)} \longmapsto \Pi_{[n]}^\perp \Phi_{\alpha(n)} =: \Phi_{\alpha(n)}^\perp, \quad (3.21)$$

which satisfies the following properties:

1. $\Pi_{[n]}^\perp$ is idempotent,

$$\Pi_{[n]}^\perp \Pi_{[n]}^\perp = \Pi_{[n]}^\perp. \quad (3.22a)$$

2. $\Pi_{[n]}^\perp$ maps $\Phi_{\alpha(n)}$ to a transverse superfield,

$$\mathcal{D}^\beta \Phi_{\beta\alpha(n-1)}^\perp = 0. \quad (3.22b)$$

3. Every transverse superfield $\Psi_{\alpha(n)}$ belongs to the image of $\Pi_{[n]}^\perp$,

$$\mathcal{D}^\beta \Psi_{\beta\alpha(n-1)} = 0 \quad \implies \quad \Pi_{[n]}^\perp \Psi_{\alpha(n)} = \Psi_{\alpha(n)}. \quad (3.22c)$$

¹¹The four-dimensional analogue was recently given in [19].

In other words, the superprojector $\Pi_{[n]}^\perp$ maps $\Phi_{\alpha(n)}$ to a supermultiplet with the properties of a conserved supercurrent.

To obtain a superprojector, we introduce the operator Δ^α_β [43]

$$\Delta^\alpha_\beta := -\frac{i}{2}\mathcal{D}^\alpha\mathcal{D}_\beta - 2\mathcal{S}\delta^\alpha_\beta, \quad \mathcal{D}^\beta\Delta^\alpha_\beta = \Delta^\alpha_\beta\mathcal{D}_\alpha = 0, \quad (3.23)$$

and its corresponding extensions [22]

$$\Delta^\alpha_{[j]\beta} := -\frac{i}{2}\mathcal{D}^\alpha\mathcal{D}_\beta - 2j\mathcal{S}\delta^\alpha_\beta. \quad (3.24)$$

Note that for the case $j = 1$, (3.24) coincides with (3.23). It can be shown that the operator (3.24) has the following properties

$$[\Delta^\alpha_{[j]\beta_1}, \Delta^\alpha_{[k]\beta_2}] = \varepsilon_{\beta_1\beta_2}\mathcal{S}(\mathcal{D}^{\alpha(2)} - \mathcal{S}M^{\alpha(2)}) - \varepsilon^{\alpha_1\alpha_2}\mathcal{S}(\mathcal{D}_{\beta(2)} - \mathcal{S}M_{\beta(2)}), \quad (3.25a)$$

$$\varepsilon^{\beta_1\beta_2}\Delta^\alpha_{[j]\beta_1}\Delta^\alpha_{[j+1]\beta_2} = -j\varepsilon^{\alpha_1\alpha_2}\mathcal{S}(i\mathcal{D}^2 + 4(j+1)\mathcal{S}^2), \quad (3.25b)$$

$$\varepsilon_{\alpha_1\alpha_2}\Delta^\alpha_{[j+1]\beta_1}\Delta^\alpha_{[j]\beta_2} = j\varepsilon_{\beta_1\beta_2}\mathcal{S}(i\mathcal{D}^2 + 4(j+1)\mathcal{S}^2), \quad (3.25c)$$

$$\Delta^\beta_{[j]\alpha}\Delta^\gamma_{[k]\beta} = -\frac{i}{2}\mathcal{D}^2\Delta^\gamma_{[1]\alpha} + (j+k-1)i\mathcal{S}\mathcal{D}^\gamma\mathcal{D}_\alpha + 4jk\mathcal{S}^2\delta_\alpha^\gamma, \quad (3.25d)$$

$$[\Delta^\alpha_{[j]\beta}, \mathcal{D}^2] = 0, \quad (3.25e)$$

for arbitrary integers j and k .

Let us define the operator $\mathbb{T}_{[n]}$, which acts on $\mathbb{V}_{(n)}$ by the rule

$$\mathbb{T}_{[n]}\Phi_{\alpha(n)} \equiv \mathbb{T}_{\alpha(n)}(\Phi) = \Delta^\beta_{[1]\alpha_1}\Delta^\beta_{[2]\alpha_2} \cdots \Delta^\beta_{[n]\alpha_n}\Phi_{\beta(n)}. \quad (3.26)$$

This operator maps $\Phi_{\alpha(n)}$ to a transverse superfield

$$\mathcal{D}^\beta\mathbb{T}_{\beta\alpha(n-1)}(\Phi) = 0. \quad (3.27)$$

To see this, one needs to open the symmetrisation in (3.26)

$$\begin{aligned} \mathcal{D}^\beta\mathbb{T}_{\beta\alpha(n-1)}(\Phi) &= \mathcal{D}^\gamma\Delta^\beta_{[1]\gamma}\Delta^\beta_{[2]\alpha_1} \cdots \Delta^\beta_{[n]\alpha_{n-1}}\Phi_{\beta(n)} \\ &\propto \mathcal{D}^\gamma(\Delta^\beta_{[1]\gamma}\Delta^\beta_{[2]\alpha_1} \cdots \Delta^\beta_{[n]\alpha_{n-1}} + (n! - 1) \text{ permutations})\Phi_{\beta(n)}. \end{aligned} \quad (3.28)$$

By making use of (3.25b), it can be shown that the remaining $(n! - 1)$ terms can be expressed in the same form as the first. Then transversality follows immediately as a consequence of property (3.23). However, $\mathbb{T}_{[n]}$ does not square to itself on $\mathbb{V}_{(n)}$

$$\mathbb{T}_{[n]}\mathbb{T}_{[n]}\Phi_{\alpha(n)} = \frac{1}{(2n+1)^n} \prod_{t=0}^{[n/2]-1} (\mathbb{F} + M_{(t,n)}^{(-)}) \prod_{t=1}^{[n/2]} (\mathbb{F} - M_{(t,n)}^{(+)}) \mathbb{T}_{[n]}\Phi_{\alpha(n)}, \quad (3.29)$$

where $M_{(t,n)}^{(\pm)}$ denotes the pseudo-masses associated with a partially massless superfield (3.17). We can immediately introduce the dimensionless operator

$$\Pi_{[n]}^\perp\Phi_{\alpha(n)} := (2n+1)^n \left[\prod_{t=0}^{[n/2]-1} (\mathbb{F} + M_{(t,n)}^{(-)}) \prod_{t=1}^{[n/2]} (\mathbb{F} - M_{(t,n)}^{(+)}) \right]^{-1} \mathbb{T}_{[n]}\Phi_{\beta(n)}, \quad (3.30)$$

which is idempotent and transverse by construction. In addition, it can be shown that the operator $\Pi_{[n]}^\perp$ acts as the identity on the space of transverse superfields (3.22c). Hence, $\Pi_{[n]}^\perp$ satisfies properties (3.22) and can be identified as a rank- n superprojector on $\text{AdS}^{3|2}$.

An alternative form of the superprojector $\Pi_{[n]}^\perp$ can be derived, which instead makes contact with the Casimir operator $\mathbb{Q}$. Let us introduce the dimensionless operator

$$\hat{\Pi}_{[n]}^\perp \Phi_{\alpha(n)} = \left[\prod_{t=0}^{n-1} (\mathbb{Q} + it\mathcal{SD}^2) \right]^{-1} \hat{\Delta}_{[1]}^{\beta_1}(\alpha_1) \hat{\Delta}_{[2]}^{\beta_2}(\alpha_2) \dots \hat{\Delta}_{[n]}^{\beta_n}(\alpha_n) \Phi_{\beta(n)}, \quad (3.31)$$

where we denote $\hat{\Delta}_{[j]}^\beta$ as

$$\hat{\Delta}_{[j]}^\beta := -\frac{i}{2} \mathcal{D}^2 \Delta_{[j]}^\beta. \quad (3.32)$$

In the flat superspace limit, $\hat{\Pi}_{[n]}^\perp$ coincides with the superprojector derived in [17]. Making use of the properties of $\Pi_{[n]}^\perp$ and the identity

$$-\frac{i}{2} \mathcal{D}^2 \Psi_{\alpha(n)} = \frac{1}{2n+1} (\mathbb{F} + 2n(n+2)\mathcal{S}) \Psi_{\alpha(n)}, \quad (3.33)$$

where $\Psi_{\alpha(n)}$ is an arbitrary transverse superfield, it can be shown that $\hat{\Pi}_{[n]}^\perp \Phi_{\alpha(n)}$ satisfies properties (3.22) and is also a superprojector on $\text{AdS}^{3|2}$. Using an analogous proof employed to show the coincidence of the two bosonic projectors in section 2.2, it can be shown that $\Pi_{[n]}^\perp$ and $\hat{\Pi}_{[n]}^\perp$ are indeed equivalent. So far, we have been unable to obtain an expression for $\Pi_{[n]}^\perp$ which is purely in terms of the Casimir operators $\mathbb{F}$ and $\mathbb{Q}$.

We recall that in the non-supersymmetric case, one starts with a field $\phi_{\alpha(n)}$ lying on the mass-shell (2.9b) and its projection $\Pi_{[n]}^\perp \phi_{\alpha(n)}$ furnishes the reducible representation (2.11). A single irreducible representation from the decomposition (2.11) can be singled out via application of the helicity projectors (2.45). The significance of the condition (2.9b) is that it allows one to resolve the poles in both types of projectors.

In the supersymmetric case, the equation analogous to (2.9b) which $\Phi_{\alpha(n)}$ should satisfy is (3.13a). Upon application of $\Pi_{[n]}^\perp$ on such a $\Phi_{\alpha(n)}$, one obtains the reducible representation (3.16). However, it appears that the imposition of (3.13a) does not allow one to resolve the poles of the superprojector in either of the forms (3.30) or (3.31). Therefore, rather than imposing (3.13a), one must start with a superfield $\Phi_{\alpha(n)}$ obeying the first-order constraint (3.7b), which does allow for resolution of the poles. In this case, after application of $\Pi_{[n]}^\perp$, the superfield $\Phi_{\alpha(n)}$ already corresponds to an irreducible representation with fixed superhelicity, relinquishing the need for superhelicity projectors. Thus, it suffices to provide only the superspin projection operators $\Pi_{[n]}^\perp$.

3.3 Longitudinal projectors

For $n \geq 1$, let us define the orthogonal complement of $\Pi_{[n]}^\perp$ acting on $\Phi_{\alpha(n)}$ by the rule

$$\Pi_{[n]}^\parallel \Phi_{\alpha(n)} = (\mathbb{1} - \Pi_{[n]}^\perp) \Phi_{\alpha(n)}. \quad (3.34)$$

By construction, the operators $\Pi_{[n]}^\perp$ and $\Pi_{[n]}^\parallel$ resolve the identity, $\mathbb{1} = \Pi_{[n]}^\parallel + \Pi_{[n]}^\perp$, and are orthogonal projectors

$$\Pi_{[n]}^\perp \Pi_{[n]}^\perp = \Pi_{[n]}^\perp, \quad \Pi_{[n]}^\parallel \Pi_{[n]}^\parallel = \Pi_{[n]}^\parallel, \quad \Pi_{[n]}^\parallel \Pi_{[n]}^\perp = \Pi_{[n]}^\perp \Pi_{[n]}^\parallel = 0. \quad (3.35)$$

It can be shown that $\Pi_{[n]}^\parallel$ extracts the longitudinal component of a superfield $\Phi_{\alpha(n)}$. A rank- n superfield $\Psi_{\alpha(n)}$ is said to be longitudinal if there exists a rank- $(n-1)$ superfield $\Psi_{\alpha(n-1)}$ such that $\Psi_{\alpha(n)}$ can be expressed as $\Psi_{\alpha(n)} = i^n \mathcal{D}_\alpha \Psi_{\alpha(n-1)}$. Thus, we find

$$\Pi_{[n]}^\parallel \Phi_{\alpha(n)} = i^n \mathcal{D}_\alpha \Phi_{\alpha(n-1)}, \quad (3.36)$$

for some unconstrained real superfield $\Phi_{\alpha(n-1)}$. In order to see this, it proves beneficial to make use of the superprojector $\hat{\Pi}_{[n]}^\perp$, and express the operator $\hat{\Delta}_{[j]\alpha}^\beta$ in the form

$$\hat{\Delta}_{[j]\alpha}^\beta := -\frac{1}{4} \mathcal{D}_\alpha \mathcal{D}^\beta \mathcal{D}^2 + (\mathbb{Q} + i(j-1) \mathcal{S} \mathcal{D}^2) \delta_\alpha^\beta. \quad (3.37)$$

Using the fact that the $\Pi_{[n]}^\parallel$ and $\Pi_{[n]}^\perp$ resolve the identity, it follows that one can decompose any superfield $\Phi_{\alpha(n)}$ in the following manner

$$\Phi_{\alpha(n)} = \Phi_{\alpha(n)}^\perp + i^n \mathcal{D}_\alpha \Phi_{\alpha(n-1)}. \quad (3.38)$$

Here, $\Phi_{\alpha(n)}^\perp$ is transverse and $\Phi_{\alpha(n-1)}$ is unconstrained. Repeating this prescription iteratively yields the decomposition

$$\Phi_{\alpha(n)} = \sum_{j=0}^{[n/2]} (\mathcal{D}_{\alpha(2)})^j \Phi_{\alpha(n-2j)}^\perp + i^n \sum_{j=0}^{[n/2]-1} (\mathcal{D}_{\alpha(2)})^j \mathcal{D}_\alpha \Phi_{\alpha(n-2j-1)}^\perp. \quad (3.39a)$$

Here, the real superfields $\Phi_{\alpha(n-2j)}^\perp$ and $\Phi_{\alpha(n-2j-1)}^\perp$ are transverse, except for $\Phi^\perp$.

It can be shown that the superprojector $\Pi_{[n]}^\perp$ annihilates any longitudinal superfield. Indeed, let us consider the action of $\Pi_{[n]}^\perp$ on a superfield $\Psi_{\alpha(n)} = i^n \mathcal{D}_\alpha \Lambda_{\alpha(n-1)}$. Opening the symmetrisation present in $\Pi_{[n]}^\perp$ gives

$$\begin{aligned} \Pi_{[n]}^\perp \Psi_{\alpha(n)} &= i^n \Delta_{[1]}^{\beta_1}{}_{(\alpha_1} \Delta_{[2]}^{\beta_2}{}_{\alpha_2} \dots \Delta_{[n]}^{\beta_n}{}_{\alpha_n)} \mathcal{D}_{(\beta_1} \Lambda_{\beta_2 \dots \beta_n)} \\ &= \frac{i^n}{n!} \Delta_{[n]}^{\beta_1}{}_{(\alpha_1} \Delta_{[n-1]}^{\beta_2}{}_{\alpha_2} \dots \Delta_{[1]}^{\beta_n}{}_{\alpha_n)} (\mathcal{D}_{\beta_n} \Lambda_{\beta_1 \dots \beta_{n-1}} + (n! - 1) \text{ permutations}). \end{aligned} \quad (3.40)$$

Note that we have made use of the identity (3.25a) to rearrange the operators $\Delta_{[j]\alpha}^\beta$. Making use of the relation (3.25c) allows us to express the other $(n! - 1)$ permutations in the same form as the first. Then due to the property (3.23), it follows that

$$\Psi_{\alpha(n)} = i^n \mathcal{D}_\alpha \Lambda_{\alpha(n-1)} \quad \implies \quad \Pi_{[n]}^\perp \Psi_{\alpha(n)} = 0. \quad (3.41)$$

Consequently, the operator $\Pi_{[n]}^\parallel$ acts as unity on the space of rank- n longitudinal superfields $\Psi_{\alpha(n)}$

$$\Psi_{\alpha(n)} = i^n \mathcal{D}_\alpha \Lambda_{\alpha(n-1)} \quad \implies \quad \Pi_{[n]}^\parallel \Psi_{\alpha(n)} = \Psi_{\alpha(n)}. \quad (3.42)$$

3.4 Linearised higher-spin super-Cotton tensors

In this section, we make use of the rank- n superprojector to study the properties of superconformal higher-spin (SCHS) theories. In particular, we will make use of $\mathbf{\Pi}_{[n]}^\perp$ to construct the higher-spin super-Cotton tensors in $\text{AdS}^{3|2}$, which were recently derived in [22]. The super-Cotton tensors $\mathfrak{W}_{\alpha(n)}(H)$ were shown to take the explicit form

$$\mathfrak{W}_{\alpha(n)}(H) = \Delta_{[1]}^{\beta_1}(\alpha_1) \Delta_{[2]}^{\beta_2}(\alpha_2) \cdots \Delta_{[n]}^{\beta_n}(\alpha_n) H_{\beta(n)}, \quad (3.43)$$

which is a real primary descendent of the SCHS superfield $H_{\alpha(n)}$. The latter is defined modulo gauge transformations of the form

$$\delta_\Lambda H_{\alpha(n)} = i^n \mathcal{D}_\alpha \Lambda_{\alpha(n-1)}, \quad (3.44)$$

where the gauge parameter $\Lambda_{\alpha(n-1)}$ is a real unconstrained superfield. The super-Cotton tensor (3.43) satisfies the defining properties: (i) it is transverse

$$\mathcal{D}^\beta \mathfrak{W}_{\beta\alpha(n-1)}(H) = 0; \quad (3.45a)$$

and (ii) it is invariant under the gauge transformations (3.44)

$$\mathfrak{W}_{\alpha(n)}(\delta_\Lambda H) = 0. \quad (3.45b)$$

The superprojectors (3.30) can be used to recast the super-Cotton tensors (3.43) in the simple form

$$\mathfrak{W}_{\alpha(n)}(H) = \frac{1}{(2n+1)^n} \prod_{t=0}^{\lceil n/2 \rceil - 1} (\mathbb{F} + M_{(t,n)}^{(-)}) \prod_{t=1}^{\lfloor n/2 \rfloor} (\mathbb{F} - M_{(t,n)}^{(+)}) \mathbf{\Pi}_{[n]}^\perp H_{\alpha(n)}, \quad (3.46)$$

where $M_{(t,n)}^{(\pm)}$ denotes the partial pseudo-masses (3.17). In the flat superspace limit, $\mathcal{S} \rightarrow 0$, the super-Cotton tensor (3.46) reduces to those given in [39, 48]. Expressing $\mathfrak{W}_{\alpha(n)}(H)$ in the form (3.46) is beneficial for the following reasons: (i) transversality of $\mathfrak{W}_{\alpha(n)}(H)$ is manifest on account of property (3.27); (ii) gauge invariance is also manifest as a consequence of (3.41); and (iii) in the transverse gauge

$$H_{\alpha(n)} \equiv H_{\alpha(n)}^T, \quad \mathcal{D}^\beta H_{\beta\alpha(n-1)}^T = 0, \quad (3.47)$$

it follows from (3.22c) that $\mathfrak{W}_{\alpha(n)}(H)$ factorises as follows

$$\mathfrak{W}_{\alpha(n)}(H^T) = \frac{1}{(2n+1)^n} \prod_{t=0}^{\lceil n/2 \rceil - 1} (\mathbb{F} + M_{(t,n)}^{(-)}) \prod_{t=1}^{\lfloor n/2 \rfloor} (\mathbb{F} - M_{(t,n)}^{(+)}) H_{\alpha(n)}^T. \quad (3.48)$$

From the above observations, it follows that the action [43, 44] for the superconformal higher-spin prepotential $H_{\alpha(n)}$

$$\mathbb{S}_{\text{SCHS}}^{(n)}[H] = -\frac{i^n}{2^{\lfloor n/2 \rfloor + 1}} \int d^3x d^2\theta E H^{\alpha(n)} \mathfrak{W}_{\alpha(n)}(H), \quad E^{-1} = \text{Ber}(E_A^M), \quad (3.49)$$

is manifestly gauge-invariant. In the transverse gauge (3.47), the kinetic operator in (3.49) factorises into wave operators associated with partially massless superfields of all depths, in accordance with (3.48).

4 Conclusion

Given a maximally symmetric spacetime, the unitary irreducible representations of its isometry algebra may be realised on the space of tensor fields satisfying certain differential constraints. The purpose of a spin projection operator is to take an unconstrained field, which describes a multiplet of irreducible representations, and return the component corresponding to the irreducible representation with maximal spin.¹² In this paper we have derived the spin projection operators for fields of arbitrary rank on AdS_3 space and their extensions to $\mathcal{N} = 1$ AdS superspace. We leave generalisations of our results to the (p, q) AdS superspaces [46] with $\mathcal{N} = p + q > 1$ for future work.

Making use of the (super)spin projection operators, we obtained new representations for the linearised higher-spin (super)Cotton tensors and the corresponding (super)conformal actions in AdS_3 . The significance of these new realisations is that the following properties are each made manifest: (i) gauge invariance; (ii) transversality; and (iii) factorisation. We also show that the poles of the (super)projectors are intimately related to partially massless (super)fields. This property was first established in the case of AdS_4 (super)space in [19, 20], and appears to be a universal feature of the (super)projectors. It would be interesting to verify this in the case of AdS_d with $d > 4$.

As compared with previous approaches in AdS_4 (super)space [19, 20], a novel feature of the spin projectors derived here is that they are formulated entirely in terms of Casimir operators of the AdS_3 algebra.¹³ Studying their zero curvature limit has allowed us to obtain new realisations of the spin projection operators in $3d$ Minkowski space in terms of only the Pauli-Lubanski scalar and the momentum squared operator. This idea may be straightforwardly applied to the case of $4d$ Minkowski space to derive new realisations of the Behrends-Fronsdal projectors.

In particular, let us define the square of the Pauli-Lubanski vector,

$$\mathbb{W}^2 = \mathbb{W}^a \mathbb{W}_a, \quad \mathbb{W}_a := -\frac{1}{2} \varepsilon_{abcd} M^{bc} \partial^d. \quad (4.1)$$

On the field $\phi_{\alpha(m)\dot{\alpha}(n)}$ of Lorentz type $(\frac{m}{2}, \frac{n}{2})$, it may be shown that $\mathbb{W}^2$ assumes the form (see, e.g. [49])

$$\mathbb{W}^2 \phi_{\alpha(m)\dot{\alpha}(n)} = s(s+1) \square \phi_{\alpha(m)\dot{\alpha}(n)} + mn \partial_{\alpha\dot{\alpha}} \partial^{\beta\dot{\beta}} \phi_{\alpha(m-1)\dot{\beta}(n-1)\dot{\beta}}, \quad (4.2)$$

where we have defined $s := \frac{1}{2}(m+n)$. On any transverse field $\psi_{\alpha(m)\dot{\alpha}(n)}$ this reduces to $(\mathbb{W}^2 - s(s+1)\square)\psi_{\alpha(m)\dot{\alpha}(n)} = 0$. It is possible to express the Behrends-Fronsdal spin projection operators $\Pi_{(m,n)}^\perp$ solely in terms of the Casimir operators $\mathbb{W}^2$ and $\square$ of the $4d$

¹²In three dimensions, in order to single out an irreducible representation, one needs to bisect the spin projector into helicity projectors.

¹³We were not able to obtain expressions for the superspin projection operators in $\text{AdS}^{3|2}$ which involve only Casimir operators.

Poincaré algebra as follows¹⁴

$$\Pi_{(m,n)}^\perp \phi_{\alpha(m)\dot{\alpha}(n)} = \frac{m!}{(m+n)!n!} \frac{1}{\square^n} \prod_{j=0}^{n-1} \left(\mathbb{W}^2 - (s-j)(s-j-1)\square \right) \phi_{\alpha(m)\dot{\alpha}(n)} \quad (4.3a)$$

$$= \frac{n!}{(m+n)!m!} \frac{1}{\square^m} \prod_{j=0}^{m-1} \left(\mathbb{W}^2 - (s-j)(s-j-1)\square \right) \phi_{\alpha(m)\dot{\alpha}(n)}. \quad (4.3b)$$

The operators $\Pi_{(m,n)}^\perp$ satisfy the four dimensional analogues of the properties (1.2).

In a similar fashion, it should be possible to obtain new realisations for the AdS_4 spin projection operators of [20] in terms of the Casimir operators of the algebra $\mathfrak{so}(3,2)$. In this case, $\square$ should be replaced with the quadratic Casimir operator

$$\mathbb{Q} := \square_{\text{AdS}} - \mathcal{S}^2 (M^2 + \bar{M}^2), \quad M^2 := M^{\alpha\beta} M_{\alpha\beta}, \quad \bar{M}^2 := \bar{M}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}. \quad (4.4)$$

Finally, the role of $\mathbb{W}^2$ will be played by the quartic Casimir operator $\mathbb{W}_{\text{AdS}}^2$,¹⁵

$$\begin{aligned} \mathbb{W}_{\text{AdS}}^2 := & -\frac{1}{2} (\mathbb{Q} + 2\mathcal{S}^2) (M^2 + \bar{M}^2) + \mathcal{D}^{\alpha\dot{\alpha}} \mathcal{D}^{\beta\dot{\beta}} M_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}} \\ & - \frac{1}{4} \mathcal{S}^2 (M^2 M^2 + \bar{M}^2 \bar{M}^2 + 6M^2 \bar{M}^2). \end{aligned} \quad (4.5)$$

Both operators commute with the AdS_4 covariant derivative $[\mathbb{Q}, \mathcal{D}_{\alpha\dot{\alpha}}] = [\mathbb{W}_{\text{AdS}}^2, \mathcal{D}_{\alpha\dot{\alpha}}] = 0$.

Note added in proof. When $m = n = s$, the spin projection operator (4.3) takes the form

$$\Pi_{(s,s)}^\perp \equiv \Pi_{(s)}^\perp = \frac{1}{\square^s (2s)!} \prod_{j=0}^{s-1} \left(\mathbb{W}^2 - j(j+1)\square \right). \quad (4.6)$$

In this case, it may be shown that $\Pi_{(s)}^\perp$ annihilates any field $\phi_{\alpha(s')\dot{\alpha}(s')}$ of lower rank:

$$\Pi_{(s)}^\perp \phi_{\alpha(s')\dot{\alpha}(s')} = 0, \quad s' < s. \quad (4.7)$$

Let us comment on the implications of (4.7) on fields with vectorial indices. Consider a field $\mathbf{h}_{a_1 \dots a_s}$ which is totally symmetric in its vector indices and has a non-zero trace

$$\mathbf{h}_{a_1 \dots a_s} = \mathbf{h}_{(a_1 \dots a_s)} \equiv \mathbf{h}_{a(s)}, \quad \eta^{bc} \mathbf{h}_{bca(s-2)} \neq 0, \quad (4.8)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. Upon converting to $4d$ two component spinor notation, see e.g. [49] for the details, $\mathbf{h}_{a(s)}$ decomposes into irreducible $\text{SL}(2, \mathbb{C})$ fields as follows

$$\mathbf{h}_{\alpha_1 \dot{\alpha}_1, \dots, \alpha_s \dot{\alpha}_s} := (\sigma^{a_1})_{\alpha_1 \dot{\alpha}_1} \cdots (\sigma^{a_s})_{\alpha_s \dot{\alpha}_s} \mathbf{h}_{a_1 \dots a_s} = h_{\alpha(s)\dot{\alpha}(s)} + \cdots \quad (4.9)$$

¹⁴These expressions may be easily converted to vector or four component notation.

¹⁵Here we use the convention $[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = -2\mathcal{S}^2 (\varepsilon_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta})$, where $\mathcal{S}^2$ is related to the AdS_4 scalar curvature via $R = -12\mathcal{S}^2$.

Here $h_{\alpha(s)\dot{\alpha}(s)}$ is associated with the traceless part of $\mathbf{h}_{a(s)}$, whilst the $+\dots$ represent lower-rank fields $h_{\alpha(s')\dot{\alpha}(s')}$ associated with the trace of $\mathbf{h}_{a(s)}$. From (4.7) it follows that the operator $\Pi_{(s)}^\perp$ selects the transverse and traceless (TT) component of $\mathbf{h}_{a(s)}$,

$$\partial^b h_{ba(s-1)}^{\text{TT}} = 0, \quad \eta^{bc} h_{bca(s-2)}^{\text{TT}} = 0, \quad h_{a(s)}^{\text{TT}} := \Pi_{(s)}^\perp \mathbf{h}_{a(s)}. \quad (4.10)$$

Therefore, the spin- s projection operator (4.6) is a TT projector when acting on a rank- s field which is symmetric and traceful in its vectorial indices. Similar conclusions hold in the three dimensional case. This is because the spin projection operators (2.21) and (2.35) in AdS_3 (and hence also those in $\mathbb{M}^3$ given by eq. (2.77)) satisfy a property analogous to (4.7), as pointed out in eqs. (2.29) and (2.41).

Acknowledgments

The work of DH is supported by the Jean Rogerson Postgraduate Scholarship and an Australian Government Research Training Program Scholarship. The work of SMK is supported in part by the Australian Research Council, project No. DP200101944. The work of MP is supported by the Hackett Postgraduate Scholarship UWA, under the Australian Government Research Training Program.

A Notation and conventions

We follow the notation and conventions adopted in [50]. In particular, the Minkowski metric is $\eta_{ab} = \text{diag}(-1, 1, 1)$. The spinor indices are raised and lowered using the $\text{SL}(2, \mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\gamma} \varepsilon_{\gamma\beta} = \delta_\beta^\alpha \quad (A.1)$$

by the standard rule:

$$\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta. \quad (A.2)$$

We make use of real gamma-matrices, $\gamma_a := ((\gamma_a)_\alpha^\beta)$, which obey the algebra

$$\gamma_a \gamma_b = \eta_{ab} \mathbb{1} + \varepsilon_{abc} \gamma^c, \quad (A.3)$$

where the Levi-Civita tensor is normalised as $\varepsilon^{012} = -\varepsilon_{012} = 1$. Given a three-vector V_a , it can be equivalently described by a symmetric second-rank spinor $V_{\alpha\beta}$ defined as

$$V_{\alpha\beta} := (\gamma^a)_{\alpha\beta} V_a = V_{\beta\alpha}, \quad V_a = -\frac{1}{2} (\gamma_a)^{\alpha\beta} V_{\alpha\beta}. \quad (A.4)$$

Any antisymmetric tensor $F_{ab} = -F_{ba}$ is Hodge-dual to a three-vector F_a , specifically

$$F_a = \frac{1}{2} \varepsilon_{abc} F^{bc}, \quad F_{ab} = -\varepsilon_{abc} F^c. \quad (A.5)$$

Then, the symmetric spinor $F_{\alpha\beta} = F_{\beta\alpha}$, which is associated with F_a , can equivalently be defined in terms of F_{ab} :

$$F_{\alpha\beta} := (\gamma^a)_{\alpha\beta} F_a = \frac{1}{2} (\gamma^a)_{\alpha\beta} \varepsilon_{abc} F^{bc}. \quad (\text{A.6})$$

These three algebraic objects, F_a , F_{ab} and $F_{\alpha\beta}$, are in one-to-one correspondence to each other, $F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha\beta}$. The corresponding inner products are related to each other as follows:

$$-F^a G_a = \frac{1}{2} F^{ab} G_{ab} = \frac{1}{2} F^{\alpha\beta} G_{\alpha\beta}. \quad (\text{A.7})$$

The Lorentz generators with two vector indices ($M_{ab} = -M_{ba}$), one vector index (M_a) and two spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$) are related to each other by the rules: $M_a = \frac{1}{2} \varepsilon_{abc} M^{bc}$ and $M_{\alpha\beta} = (\gamma^a)_{\alpha\beta} M_a$. These generators act on a vector V_c and a spinor Ψ_γ as follows:

$$M_{ab} V_c = 2\eta_{c[a} V_{b]}, \quad M_{\alpha\beta} \Psi_\gamma = \varepsilon_{\gamma(\alpha} \Psi_{\beta)}. \quad (\text{A.8})$$

The following identities hold:

$$M_{\alpha_1}{}^\beta \Phi_{\beta\alpha_2\dots\alpha_n} = -\frac{1}{2}(n+2)\Phi_{\alpha(n)}, \quad (\text{A.9a})$$

$$M^{\beta\gamma} M_{\beta\gamma} \Phi_{\alpha(n)} = -\frac{1}{2}n(n+2)\Phi_{\alpha(n)}. \quad (\text{A.9b})$$

B Generating function formalism

We employ the generating function formalism which was developed in [22]. Within this framework, a one-to-one correspondence between a homogenous polynomial $\phi_{(n)}(\Upsilon)$ of degree n and a rank- n spinor field $\phi_{\alpha(n)}$ is established via the rule

$$\phi_{(n)}(\Upsilon) := \Upsilon^{\alpha_1} \dots \Upsilon^{\alpha_n} \phi_{\alpha(n)}. \quad (\text{B.1})$$

Here, we have introduced the commuting real auxiliary variables Υ^α , which are inert under the action of the Lorentz generators $M_{\alpha\beta}$.

Making use of the auxiliary fields Υ^α , and their corresponding partial derivatives, $\partial_\beta := \frac{\partial}{\partial \Upsilon^\beta}$, we can realise the AdS_3 derivatives as index-free operators on the space of homogenous polynomials of degree n . We introduce the differential operators which increase and decrease the degree of homogeneity by 2, 0 and -2 respectively:

$$\mathcal{D}_{(2)} := \Upsilon^\alpha \Upsilon^\beta \mathcal{D}_{\alpha\beta}, \quad \mathcal{D}_{(0)} := \Upsilon^\alpha \mathcal{D}_\alpha{}^\beta \partial_\beta, \quad \mathcal{D}_{(-2)} := \mathcal{D}^{\alpha\beta} \partial_\alpha \partial_\beta. \quad (\text{B.2})$$

Note that the action of $\mathcal{D}_{(0)}$ is equivalent to that of the Casimir operator $\mathcal{F}$.

Making use of the algebra (2.2), one can derive the important identities

$$[\mathcal{D}_{(2)}, \mathcal{D}_{(-2)}^t] \phi_{(n)} = 4t(n-t+2)(\mathcal{Q} - \tau_{(t,n+2)} \mathcal{S}^2) \mathcal{D}_{(-2)}^{t-1} \phi_{(n)}, \quad (\text{B.3a})$$

$$[\mathcal{D}_{(-2)}, \mathcal{D}_{(2)}^t] \phi_{(n)} = -4t(n+t)(\mathcal{Q} - \tau_{(t,n+2t)} \mathcal{S}^2) \mathcal{D}_{(2)}^{t-1} \phi_{(n)}, \quad (\text{B.3b})$$

$$\mathcal{D}_{(2)}^t \mathcal{D}_{(-2)}^t \phi_{(n)} = \prod_{j=0}^{t-1} \left(\mathcal{F}^2 - (n-2j)^2 (\mathcal{Q} - (n-2j-2)(n-2j+2) \mathcal{S}^2) \right) \phi_{(n)}, \quad (\text{B.3c})$$

via induction on t . Here $\mathcal{Q}$ and $\mathcal{F}$ are the quadratic Casimir operators (2.3) and $\tau_{(t,n)}$ are the partially massless values (2.14).

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