Bootstrapping octagons in reduced kinematics from A_2 cluster algebras

Song He,^{a,b,c,d,e} Zhenjie Li,^{a,e} Yichao Tang^{a,e,f} and Qinglin Yang^{a,e}

^aCAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

^bPeng Huanwu Center for Fundamental Theory, Hefei, Anhui 230026, China

^cSchool of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China

^dICTP-AP International Centre for Theoretical Physics Asia-Pacific, Beijing/Hangzhou, China

^eSchool of Physical Sciences, University of Chinese Academy of Sciences, No. 19A Yuquan Road, Beijing 100049, China

^fSchool of Physics, Peking University, Beijing 100871, China

E-mail: songhe@itp.ac.cn, lizhenjie@itp.ac.cn, tangyichao@itp.ac.cn, yangqinglin@itp.ac.cn

ABSTRACT: Multi-loop scattering amplitudes/null polygonal Wilson loops in $\mathcal{N} = 4$ super-Yang-Mills are known to simplify significantly in reduced kinematics, where external legs/edges lie in an $1 + 1$ dimensional subspace of Minkowski spacetime (or boundary of the AdS_3 subspace). Since the edges of a $2n$ -gon with even and odd labels go along two different null directions, the kinematics is reduced to two copies of $G(2, n)/T \sim A_{n-3}$. In the simplest octagon case, we conjecture that all loop amplitudes and Feynman integrals are given in terms of two overlapping A_2 functions (a special case of two-dimensional harmonic polylogarithms): in addition to the letters $v, 1+v, w, 1+w$ of $A_1 \times A_1$, there are two letters $v-w, 1-vw$ mixing the two sectors but they never appear together in the same term; these are the reduced version of four-mass-box algebraic letters. Evidence supporting our conjecture includes all known octagon amplitudes as well as new computations of multi-loop integrals in reduced kinematics. By leveraging this alphabet and conditions on first and last entries, we initiate a bootstrap program in reduced kinematics: within the remarkably simple space of overlapping A_2 functions, we easily obtain octagon amplitudes up to two-loop NMHV and three-loop MHV. We also briefly comment on the generalization to $2n$ -gons in terms of A_2 functions and beyond.

KEYWORDS: Scattering Amplitudes, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

ARXIV EPRINT: [2106.03709](https://arxiv.org/abs/2106.03709)

Contents

1	Introduction and review	1
1.1	Review of $R^{1,1}$ kinematics	3
2	Octagons: two overlapping A_2's and cluster bootstrap	6
2.1	Octagon integrals from two overlapping A_2 's	8
2.2	Cluster bootstrap for octagons	12
2.2.1	Construction of the cluster function space	12
2.2.2	$k + \ell = 2$	14
2.2.3	$k + \ell = 3$	15
3	Towards $2n$-gons: A_2 functions and beyond	17
4	Conclusion and discussions	19

1 Introduction and review

Recent years have witnessed enormous progress in unravelling rich mathematical structures of scattering amplitudes in Quantum Field Theory (QFT). A remarkable example is $\mathcal{N} = 4$ super-Yang-Mills (SYM), especially the theory in the planar limit: the all-loop integrand has been determined purely with on-shell data [1] and reformulated geometrically in terms of the positive Grassmannian [2] and the amplituhedron [3]; the (post-integration) amplitudes have been determined to very high loop orders for six and seven points (see [4–15] and the review [16]). The starting point of hexagon and heptagon bootstrap is the observation [17] that the *symbol alphabet* [18, 19] of six- and seven-point amplitudes are dictated by cluster algebras (see [20, 21]) naturally associated with $G(4, 6)/T$ and $G(4, 7)/T$, i.e., A_3 and E_6 , respectively [22]. Starting at eight points, $G(4, n)/T$ cluster algebras are infinite and the (finite) symbol alphabets involve algebraic letters that go beyond the usual cluster coordinates. Recently, the two-loop NMHV amplitudes¹ have been computed for $n = 8$ [24] and higher [25] using the method of $\bar{Q}$ equations [26], and the alphabet has been explained using tropical positive Grassmannians [27–29] (see also [30]) as well as Yangian invariants and plabic graphs [31–33]. There has also been new progress on the cluster algebra structures for individual Feynman integrals in $\mathcal{N} = 4$ SYM [34, 35] and in a broader context [36].

Despite the hidden simplicity and rich structures of $\mathcal{N} = 4$ SYM, it quickly becomes too difficult to compute or understand scattering amplitudes and Feynman integrals in an analytic form, as the number of legs and loops increase. For example, beyond seven points, the functions for two-loop NMHV amplitudes and double-pentagon integrals, whose

¹As this paper is being prepared, the 3-loop MHV octagon amplitude has also been computed using the $\bar{Q}$ equation in [23].

symbols involve algebraic letters [25, 37], have not been determined so far (let alone those at higher loops). Moreover, the origin of these algebraic letters, the finer structures of the symbol such as cluster adjacency [11, 38], and the patterns of algebraic words [24, 25] have not been understood either. Given these limitations, it has become increasingly difficult to understand the cluster algebras and algebraic letters underlying these multi-leg amplitudes and integrals, or to compute/bootstrap them using such information.

This is one of the motivations for considering restricted kinematical configurations where scattering amplitudes and Feynman integrals simplify. In this paper, we consider the so-called $R^{1,1}$ kinematics, where external momenta or Wilson-loop edges lie in an $1+1$ dimensional subspace of the Minkowski spacetime. It is well known that MHV amplitudes/bosonic Wilson loops simplify a lot in $R^{1,1}$ kinematics. In fact, $R^{1,1}$ Wilson loops were first computed at strong coupling via AdS/CFT correspondence [39, 40] since the minimal surfaces simplify greatly in the AdS_3 subspace; later they were computed at weak coupling as well [41, 42]. In [43], this was extended to non-MHV amplitudes (supersymmetrically) reduced to $R^{1,1}$ kinematics: by combining the all-loop $\bar{Q}$ equations [26] and a collinear-soft uplifting formalism [44], compact analytic formulas with $k + \ell = 3$ (i.e., one-loop N^2 MHV, two-loop NMHV and three-loop MHV) were obtained for the first time for octagons and essentially extends to all multiplicities! We believe that these results have only revealed a tiny portion of the richness of scattering amplitudes in $R^{1,1}$ kinematics: not only have we obtained very compact formulas with huge simplifications at higher loops and deeper into the non-MHV sectors, but we also see non-trivial structures. Therefore, although simpler, $R^{1,1}$ kinematics still exhibits considerable complexity, making it an ideal laboratory for studying $\mathcal{N} = 4$ SYM.

We remark that the non-trivial structures in $R^{1,1}$ amplitudes observed in [43] nicely reflect the algebraic letters and cluster algebra structures of their 4d counterparts. Unlike the $k + \ell = 2$ case, general $R^{1,1}$ amplitudes are “non-factorizable” in the sense that there are terms containing conformal cross-ratios from both even and odd sectors.² These non-trivial “mixing” factors first appear as poles in $R^{1,1}$ leading singularities/Yangian invariants of one-loop N^2 MHV octagon, and they emerge as “mixing letters” at higher loops in general. We will see that, while rational letters of $G_+(4, 2n)/T$ reduce to letters of two copies of $G_+(2, n)/T \sim A_{n-3}$ (for even and odd sectors), algebraic letters reduce to the mixing letters in $R^{1,1}$ kinematics. For example, all rational letters of the octagon reduce to $A_1^2 := A_1 \times A_1$ letters $v, 1+v$ ($w, 1+w$) with cross-ratios in the odd (even) sector, and algebraic singularities reduce to $v - w$ and $1 - vw$. The $R^{1,1}$ octagon with these 6 letters is given in terms of the well-known two-dimensional harmonic polylogarithms (2dHPL) [45], or more precisely, two A_2 functions with an overlapping A_1^2 part, as we will explain shortly (see [46] for some earlier results).

Besides amplitudes, we will see even more significant simplifications of certain multi-loop, dual conformally invariant (DCI) Feynman integrals (originally proposed in [35, 47]) in $R^{1,1}$ kinematics, which also support our conjectures about the underlying cluster al-

²For $R^{1,1}$ kinematics, a null polygon must have an even number of edges, which we denote as $2n$, and take a zigzag shape where edges with even and odd labels go along the two light-like directions, as we further explain below.

gebra structure. It is well known that box-ladder integrals can be naturally embedded in $R^{1,1}$ kinematics where they are given by A_1^2 functions. Penta-box ladder integrals, which are A_3 functions in 4d, become in $R^{1,1}$ kinematics the simplest A_1^2 functions, $\text{Li}_\ell(1+v)\text{Li}_\ell(1+w)$, at ℓ loops [35]. We will illustrate our point with more non-trivial examples of eight-point double-pentagon and ladder integrals (similar to those considered in [48], but with more general numerators) involving square roots which are generally difficult to evaluate in 4d; in $R^{1,1}$ kinematics the $d\log$ recursion and differential equations they satisfy become much simpler, and it turns out they are all given in terms of A_2 functions at most! Moreover, just like amplitudes at $k + \ell = 3$ [43], we will show that mixing letters can only appear on the third entry for these integrals, indicating a similar pattern for the algebraic words in 4d.

In addition to this simple alphabet, we propose physical conditions on the first two entries and $\bar{Q}$ -implied conditions on the last two (one) entries for MHV (NMHV) amplitudes, which exclude mixing letters for these entries. Based on these considerations, we will initiate a $R^{1,1}$ octagon bootstrap program. We will first show how to construct these overlapping A_2 functions at any given weight (whose dimension admits a beautiful recursion relation), and find that only a small number of new functions are needed at low weight. We then locate amplitudes with $k + \ell = 2, 3$, which have been obtained using $\bar{Q}$ equations in [43], in the cluster function space. We find that $k + \ell = 2$ amplitudes can be fixed by dihedral symmetry and collinear limits, while for $k + \ell = 3$, we need additionally the soft-collinear operator-product-expansion (OPE) for Wilson loops [49–52] up to sub-leading orders.

We also briefly comment on the higher-point amplitudes in $R^{1,1}$ kinematics, which can be uplifted from lower-point building blocks due to strong constraints under collinear-soft limits [43, 44]. In a precise sense, it has been shown in [43] for $k + \ell \leq 3$ that the “most complicated part” of higher-point amplitudes can be uplifted from octagons up to a “remainder” that does not contain mixing letters. This implies that for higher multiplicities with $k + \ell \leq 3$, all we need in $R^{1,1}$ kinematics are simply A_2 functions (one for each square root of four-mass box, with one mixing letter), and A_{n-3}^2 for the remainder which is free of mixing letters. This is consistent with what we have observed in 4d: algebraic words organize themselves according to square roots of four-mass boxes, with a “remainder” which contains only rational letters. However, for $k + \ell > 3$ more complicated mixing letters, which correspond to more complicated algebraic letters in 4d, may appear.

1.1 Review of $R^{1,1}$ kinematics

Let us briefly review some basic facts about $R^{1,1}$ kinematics [43]. Since the edges of the zigzag-shaped $2n$ -gon go along two null directions $x^\pm$ (see figure 1 for the octagon), it is convenient to parametrize the momentum twistors [53] as

$$Z_{2i-1} = (\lambda_{2i-1}^1, 0, \lambda_{2i-1}^2, 0), \quad Z_{2i} = (0, \tilde{\lambda}_{2i}^1, 0, \tilde{\lambda}_{2i}^2), \quad (1.1)$$

which amounts to the reduction $G_+(4, 2n) \rightarrow G_+^{\text{even}}(2, n) \times G_+^{\text{odd}}(2, n)$. Meanwhile, the conformal group $\text{SL}(4)$ reduces to $\text{SL}(2) \times \text{SL}(2)$. The kinematics is encoded in even and odd $\text{SL}(2)$ -invariants: $\langle i j \rangle$ (resp. $[i j]$) for odd (resp. even) i, j , which are in fact

Figure 1. Octagon Wilson loops=8-point amplitudes in $R^{1,1}$ kinematics.

one-dimensional distances along the null direction (in figure 1, odd/even direction is vertical/horizontal). Any $SL(4)$ -invariant expression in four dimensions reduces, and the only four-brackets that remain non-vanishing are those that involve two odd and two even labels, e.g. for i, k odd and j, l even we have $\langle ijkl \rangle = \langle ik \rangle [jl]$.

The most general cross-ratios in $R^{1,1}$ kinematics are those for the odd and even $A_{n-3} \sim G_+(2, n)/T$ cluster algebras: we define $u_{i,j,k,l}^{2d} := \frac{[ij][kl]}{[ik][jl]}$ in the even sector and $[] \rightarrow \langle \rangle$ in the odd sector. Any four-dimensional cross-ratio factorizes into a product of even and odd cross-ratios: for i, j, k, l all of the same parity, the 4d cross-ratio reduces as (our convention is $x_a \sim (a - 1a)$)

$$u_{i,j,k,l}^{4d} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \rightarrow u_{i-1,j-1,k-1,l-1}^{2d} u_{i,j,k,l}^{2d}. \quad (1.2)$$

Amplitudes in $R^{1,1}$ kinematics are trivial for $2n < 8$: for $2n = 6$ (hexagon), the all-loop BDS-normalized amplitude $R_{6,0} = R_{6,1}/R_{6,1}^{\text{tree}} = 1$. The first non-trivial case is $2n = 8$ (octagon), which corresponds to $A_1^{\text{even}} \times A_1^{\text{odd}}$: using the notation of [43], we define *positive* variables for the odd and even sectors as:

$$v = \frac{u_{1,3,5,7}^{2d}}{1 - u_{1,3,5,7}^{2d}} = \frac{\langle 13 \rangle \langle 57 \rangle}{\langle 17 \rangle \langle 35 \rangle} \in [0, \infty), \quad w = \frac{u_{2,4,6,8}^{2d}}{1 - u_{2,4,6,8}^{2d}} = \frac{[24][68]}{[28][46]} \in [0, \infty), \quad (1.3)$$

and the letters of these two A_1 's are simply $v, 1+v$ and $w, 1+w$. For the $2n$ -gon, we have $n(n-3)/2$ multiplicatively independent letters for A_{n-3}^{even} , and same for A_{n-3}^{odd} .³ When reduced to $R^{1,1}$ kinematics, we expect that any cluster variable of $G_+(4, 2n)/T$ simply becomes a monomial of these $n(n-3)$ letters. This is the key reason for the simplification in $R^{1,1}$ kinematics. Starting at $2n = 8$ (and $k + \ell > 2$), algebraic letters appear in 4d involving square roots of Gram determinants. These, as we will see, reduce to new letters that mix the odd and even sectors.

³They can be chosen as the *dihedral coordinates* [54] of $\mathcal{M}_{0,n}$, which satisfy the so-called *u*-equations [55, 56], and they form a multiplicative basis for all possible cross-ratios in that sector. For $2n = 8$, the two *u* variables (which add up to 1) in the odd sector are, $v/(1+v)$, $1/(1+v)$, and the two in the even sector are $w/(1+w)$, $1/(1+w)$ (all between 0 and 1).

Next, we quickly review the tree amplitudes and leading singularities (or Yangian invariants) in $R^{1,1}$ kinematics obtained by supersymmetric reduction from 4d. We reduce the fermionic part of the super-twistors, χ_i 's, in the same way as the bosonic part in (1.1). Thus, the (dual) superconformal group reduces to $SL(2|2) \times SL(2|2)$. The fermionic parts in any Yangian invariant must factorize into two sectors, so the basic $SL(2|2)$ -invariants are (for i, j, k odd)

$$(ijk) := \frac{\delta^{0||2}(\langle i j \rangle \chi_k + \langle j k \rangle \chi_i + \langle k i \rangle \chi_j)}{\langle i j \rangle \langle j k \rangle \langle k i \rangle}, \quad (1.4)$$

and similarly, $[ijk]$ for even labels with $\langle \rangle \rightarrow []$. Apart from the MHV case (which is by definition $R_{n,0} = 1$), the simplest tree amplitude is the NMHV hexagon $R_{6,1}^{\text{tree}} = -(135)[246]$, and in [43] all tree amplitudes have been obtained as degree- $(k+k)$ polynomials of basic R-invariants with unit coefficients, using a remarkably simple BCFW recursion directly in $R^{1,1}$ kinematics.

Note that in $R^{1,1}$ kinematics the highest k possible is $k = n - 2$, in which case the tree amplitude takes the simplest form: it is given by the product of $n-2$ even R-invariants and $n-2$ odd ones, which has a beautiful geometric interpretation. Replacing χ_a by dZ_a [57, 58] in the basic R-invariants (ijk) , it becomes the canonical form $(ijk) \rightarrow d \log \frac{\langle ij \rangle}{\langle ik \rangle} d \log \frac{\langle jk \rangle}{\langle ik \rangle}$ of a triangle with vertices Z_i, Z_j, Z_k . For R-invariants of higher k , we have the k -th power of the 2-form of an n -gon, which is a well-known result of the $m = 2$ tree amplituhedron [3]. For example, for $2n = 8, k = 2$ we have

$$R_{8,2}^{\text{tree}} = (135)(357)[246][468] = \left(\frac{(135) + (357)}{2} \right)^2 \left(\frac{[246] + [468]}{2} \right)^2 =: (1357)[2468]. \quad (1.5)$$

Leading singularities, on the other hand, usually do not have smooth 2d limits (although any complete amplitude/WL must have), and it is still an important open question how to compute them directly in $R^{1,1}$ kinematics.⁴ Here, we content ourselves by simply summarizing the results for octagons. For NMHV, we have 3 independent R-invariants in each sector: (1) $:= (135)$ and its 3 cyclic rotations in the odd sector satisfy $(1) - (3) + (5) - (7) = 0$, and similarly for [2] $:= [246]$ etc. in the even sector with $[2] - [4] + [6] - [8] = 0$. We can rewrite the $3 \times 3 = 8 + 1$ independent Yangian invariants using the length-8 cyclic orbit of (1)[2] together with $R_{8,1}^{\text{tree}} = (3)[6] - (3)[4] - (1)[8] - (5)[6]$. Accompanying them are two independent functions of v, w , so that to all loops we have

$$R_{8,1} = (([2](3) + [6](7))f_{8,1}^1(v, w) + 3 \text{ cyclic}) + R_{8,1}^{\text{tree}} f_{8,1}^2(v, w). \quad (1.6)$$

For $N^2\text{MHV}$ octagons, since all Yangian invariants are proportional to $R_{8,2}^{\text{tree}}$ above, they differ by possible prefactors (functions of v, w) which *mix* the odd and even sectors. A general argument based on positroid cells of $G_+(4, 8)$ concludes that there are only 4 pre-factors:

$$\frac{v}{v-w}, \quad \frac{vw}{1-vw}, \quad \frac{w}{v-w}, \quad \frac{1}{1-vw}, \quad (1.7)$$

⁴One can define the $R^{1,1}$ amplituhedron by reducing external data to $R^{1,1}$ kinematics, but individual Yangian invariants generally do not have smooth limits. Many $4k$ -dim positroid cells, which give leading singularities/Yangian invariants in 4d, do not have well-defined 2d limits.

which (up to a sign) are related by cyclic rotations. The non-trivial linear combinations $\frac{v+w}{v-w}$ and $\frac{1+vw}{1-vw}$ give exactly the coefficients of the two four-mass-boxes in $R^{1,1}$. To all loops, no other leading singularities can appear for N^2 MHV octagon. We write

$$R_{8,2} = R_{8,2}^{\text{tree}} \left(\frac{v}{v-w} F_{8,2}(v, w) + \text{cyclic} \right). \quad (1.8)$$

2 Octagons: two overlapping A_2 's and cluster bootstrap

In this section we consider the main object of interest, the octagon ($2n = 8$): we provide strong evidence for its remarkably simple alphabet, which can be used to efficiently bootstrap and constrain higher-loop amplitudes and Feynman integrals. It has already been observed in [43] that only 6 letters, $v, 1+v, w, 1+w, v-w, 1-vw$ appear for $k+\ell = 3$; the first four are from A_1^2 , and the last two mix the even and odd sectors. It is easy to see that these mixing letters are $R^{1,1}$ avatars of algebraic singularities in 4d containing Gram-determinant square roots for four-mass boxes. For octagons, the two mixing letters correspond respectively to the two four-mass configurations (x_1, x_3, x_5, x_7) and (x_2, x_4, x_6, x_8) (these vertices are labelled by blue and red x 's respectively in figure 1). More precisely, with $U := u_{a,b,c,d}^{4d}$ and $V := u_{b,c,d,a}^{4d}$ for (x_a, x_b, x_c, x_d) , in $R^{1,1}$ kinematics the square root $\Delta_{a,b,c,d} := \sqrt{(1+U-V)^2 - 4UV}$ reduces to⁵

$$\Delta_{2,4,6,8} = \frac{|v-w|}{(1+v)(1+w)}, \quad \Delta_{1,3,5,7} = \frac{|1-vw|}{(1+v)(1+w)}. \quad (2.1)$$

Equation (1.8) is a preview of these mixing letters as non-trivial poles for (the leading singularities of) one-loop N^2 MHV; they really start to appear in the third symbol-entry of two-loop NMHV and three-loop MHV.

Note that the six letters form a C_2 cluster algebra: with $z_1 = v$, $z_2 = -v/w$, the alphabet becomes $\{z_1, 1+z_1, z_2, 1+z_2, z_1-z_2, z_1^2+z_2\}$, which is in accordance with the notation of [36]. Thus as we mentioned, all octagon functions belong to the class of 2dHPL. However, octagon functions are very special 2dHPL since the two mixing letters never appear in the same term, which follows from the fact that in 4d the two square roots cannot appear together. For this reason, we will refer to the alphabet as two overlapping A_2 's (rather than a C_2): one with $v, 1+v, w, 1+w, v-w$ and the other with $1-vw$ instead. The second A_2 (associated with $\Delta_{1,3,5,7}$) is related to the first one (associated with $\Delta_{2,4,6,8}$) by a cyclic rotation $i \rightarrow i-1$, or equivalently $v \rightarrow 1/w$ and $w \rightarrow v$, so that the letters become $1/w, 1+1/w, v, 1+v, 1/w-v = (1-vw)/w$.

Now we state our main **conjecture** of the section:

To all loops, the octagon $R_{8,k}$ with $k = 0, 1, 2$ can be written as linear combinations of two A_2 cluster functions, sharing the letters $v, w, 1+v, 1+w$ but one has $v-w$ in addition and the other has $1-vw$, and the coefficients are given by $R^{1,1}$ leading singularities.

⁵We label any four-mass box by the 4 off-shell dual points, x_a, x_b, x_c, x_d , which differ from the notation in [59] and [43]: e.g. for the octagon, the 2 four-mass boxes are swapped.

As a refinement, we further conjecture that *the first two entries and last two entries for MHV (and last entry for NMHV) are free of mixing letters, i.e. they are from A_1^2* . The first-two-entry conditions follow from physical discontinuity (for the first entry) and Steinmann relations [9], so that only four-mass boxes (containing Li_2) and degenerations (only logarithmic functions) are allowed in the first two entries. Using $\bar{Q}$ equations in $R^{1,1}$ kinematics [43], the last two entries for MHV are determined by leading singularities and last entries of NMHV decagon, which are in turn determined by leading singularities of $N^2\text{MHV}$ amplitudes. These derivations are independent of loop orders, and we find the last two entries for MHV and the last one entry for NMHV are all free of mixing letters.

Before providing explicit results of octagon amplitudes/integrals which support our conjecture, let us give some general arguments and explanations. First, the A_1^2 part represents the $R^{1,1}$ reduction of all possible cluster variables of $G_+(4,8)/T$. We have explicitly evaluated the 440 letters provided in [28], and find that in $R^{1,1}$ kinematics they all become monomials of $v, w, 1+v, 1+w$.

Furthermore, for either four-mass box, (x_2, x_4, x_6, x_8) (x_1, x_3, x_5, x_7) of the octagon, *any* algebraic letter can be written as $\mathbf{x} - \alpha_{\pm}$ with the two roots satisfying $\alpha_+ \alpha_- = U$, $(1 - \alpha_+)(1 - \alpha_-) = V$, where $\mathbf{x}$ is a DCI, rational function of Plücker coordinates. Our conjecture states that this combination must reduce to either $v - w$ or $1 - vw$ (up to prefactors that are monomials of letters in A_1^2). There are two cases: for the four-mass boxes appeared in one-loop $N^2\text{MHV}$, e.g. $F(2, 4, 6, 8)$, we have “trivial” algebraic letters with $\mathbf{x} = 0$ or 1 , which do not lead to any mixing (still in A_1^2); the symbol $\mathcal{S}[F(2, 4, 6, 8)]$ reduces to [43]:

$$\frac{1}{2} \left(U \otimes \frac{1 - \alpha_-}{1 - \alpha_+} + V \otimes \frac{\alpha_+}{\alpha_-} \right) = \frac{1}{2} \left(vw \otimes \frac{1 + v}{1 + w} - (1 + v)(1 + w) \otimes \frac{v}{w} \right). \quad (2.2)$$

For non-trivial algebraic letters, it is remarkable that in $R^{1,1}$ kinematics the rational function $\mathbf{x}$ always degenerates to α_+ or α_- , thus the only non-trivial combination is $\alpha_+ - \alpha_- = \Delta$, exactly producing the two mixing letters in (2.1). We have explicitly checked that this is true for the $9 + 9$ independent algebraic letters, first computed for the two-loop NMHV octagon [24].

Let us quickly review octagons with $k + \ell \leq 3$ computed using $\bar{Q}$ equations in [43]. For $k + \ell = 2$, i.e. one-loop NMHV and two-loop MHV, only $A_1^2 = \{v, 1+v, w, 1+w\}$ appear (and they are logarithmic functions of weight 2 and 4 respectively), which is consistent with the fact that they have no algebraic singularities. For $k + \ell = 3$, the first appearance of algebraic singularities is in the pre-factor of one-loop $N^2\text{MHV}$: in front of $F(2, 4, 6, 8)$ we have a pole at $\Delta_{2,4,6,8} \propto v - w = 0$, and for $F(1, 3, 5, 7)$ a pole at $\Delta_{1,3,5,7} \propto 1 - vw = 0$; this is consistent with the fact that as $v - w$ (or $1 - vw$) flips sign, so does $F(2, 4, 6, 8)$ (or $F(1, 3, 5, 7)$). Remarkably, these prefactors become mixing letters for higher-loop amplitudes. For both two-loop NMHV and three-loop MHV, the complete results are recorded in [43]: for the former, it is worth noticing that while $f_{8,1}^2$ is an A_1^2 function, $f_{8,1}^1$ is given by the sum of two A_2 functions, and the same is true for $R_{8,0}^{3\text{-loop}}$.

Furthermore, it has been noted in appendix D of [43] that mixing letters only appear on the third entry. For $f_{8,1}^2$ this is implied by our refined conjecture that first two entries and last entry are free of them, and for $R_{8,0}^{3\text{-loop}}$ the situation is even better since they are also absent in the fourth entry. In both cases, the first two entries in front of $v - w$ is exactly the symbol of $F(2, 4, 6, 8)$ in (2.2) (similarly for $1 - vw$ we find that of $F(1, 3, 5, 7)$); these can be seen by taking $(2, 1, 1)$ - and $(2, 1, 3)$ -coproducts respectively. The former is fully consistent with our results in 4d [24] since non-trivial algebraic letters only appear on the third entry (accompanied by four-mass boxes in the first two entries). The latter provides a very important prediction for $R_{8,0}^{3\text{-loop}}$ in 4d, namely non-trivial algebraic letters can only appear on the third entry as well!

In general, we expect that by using $\bar{Q}$ equations, one can predict such mixing letters for MHV and NMHV amplitudes from NMHV and N^2 MHV amplitudes at one-lower loop order. The derivation from two to three loops in [43] could be formulated more generally for higher loops if we assume the alphabet of NMHV decagon (as we will explain in the next section). This argument should provide a proof that only $v - w$ and $1 - vw$ can appear for higher loop octagons. For now, we content ourselves with explicit results for amplitudes (up to $k + \ell = 3$) and a series of two-loop and all-loop Feynman integrals.

2.1 Octagon integrals from two overlapping A_2 's

Although amplitudes/WL up to $k + \ell = 3$ provides strong evidence for our conjecture, since currently we have no analytic access to higher loops, it is desirable to see if we have more examples supporting our conjecture. This is why we turn to studying Feynman integrals, including all finite double-pentagon integrals for two-loop MHV and NMHV amplitudes, as well as certain ladder integrals to all loops. Their evaluations strongly support our conjecture: all of them are given in terms of two overlapping A_2 functions or even simpler functions e.g. one A_2 function, A_1^2 function or even just a single A_1 function. Moreover, they are constrained by such cluster algebra structures so strongly that we find large “degeneracy” in functions they evaluate to.

One loop. Let us begin by considering the one-loop case. It turns out we have exactly two kinds of weight-2 functions (linear combinations of two overlapping A_2 functions, having only physical discontinuities and satisfying Steinmann relations), all of them A_1^2 functions: two four-mass boxes and 10 logarithmic functions. We record the four-mass box $F(2, 4, 6, 8)$ (the other four-mass box $F(1, 3, 5, 7)$ is its cyclic image) whose symbol is given in (2.2):

$$F(2, 4, 6, 8) = L(w) - L(v) + \frac{1}{2} \log(v) \log(1+w) - \frac{1}{2} \log(w) \log(1+v), \quad \text{for } v > w, \quad (2.3)$$

with $v \leftrightarrow w$ if $v < w$, where $L(x) := \text{Li}_2(-x) + \frac{1}{2} \log(x) \log(1+x) + \frac{\pi^2}{12}$ is the Rogers dilogarithm.

Figure 2. Double pentagon integrals.

Let us take a look at some examples. Consider the two natural chiral pentagons needed for MHV and NMHV amplitudes with different numerators [59], which are clearly different from four-mass boxes in 4d,

$$\mathbf{N}_1 = \langle \bar{\ell} i \cap \bar{j} \rangle, \quad \mathbf{N}_2 = \frac{1}{2} \langle \bar{\ell} i \cap (jj+1((ikk+1) \cap X)) - (kk+1((ijj+1) \cap X)) \rangle.$$

For octagon $R^{1,1}$ kinematics, we can choose $i = 1$, $j = 4$, and $X = (67)$ in the first case, and $i = 1$, $j = 2$ and $X = (45)$ in the second case. Remarkably, they both reduce to the above two kinds of functions: the first pentagon integral evaluates to $I_{\text{pen.}} = \log(1 + v) \log(1 + w)$, while the second one becomes $F(1, 3, 5, 7)$. The fact that different integrals in 4d reduces to the same overlapping A_2 functions in $R^{1,1}$ illustrates how constraining such cluster algebra structures are. It is also interesting that although in 4d both type of pentagons contribute to one-loop NMHV ratio functions, the final result in $R^{1,1}$ contains logarithmic functions only (four-mass boxes, or Li_2 functions, cancel out and only start to contribute for $k + \ell \geq 2$ amplitudes).

Two loops. Now we move to two loops. Naively there are more than one hundred different weight-4 functions (see below), but we restrict ourselves to double pentagon integrals needed for MHV and NMHV amplitudes, where we have certain numerators for both loops (see [59, 60] for details), as shown in figure 2.

We use $\mathbf{N}_r$ for $r = 1, 2$ to denote the two types of numerators for these pentagons. For double pentagons needed for MHV amplitudes, $I_{\text{dp}}^1(i, j, k, l)$ with $\mathbf{N}_1 = \langle \bar{\ell} i \cap \bar{j} \rangle$ and $\mathbf{N}'_1 = \langle \ell' \bar{k} \cap \bar{\ell} \rangle$, there are five inequivalent topologies which are finite for $2n = 8$: $(i, j, k, l) = (1, 3, 4, 7), (1, 3, 4, 8), (1, 3, 5, 7), (1, 3, 6, 8)$ and $(1, 4, 5, 8)$. It is clear that $(1, 3, 4, 7)$ and $(1, 3, 5, 7)$ vanish in $R^{1,1}$ (even without putting the overall numerator $\langle 1347 \rangle = \langle 1357 \rangle = 0$), and the rest have been evaluated in [37]. We find that $(1, 3, 4, 8)$ is given by a simple A_1 function, while $(1, 3, 6, 8)$ and $(1, 4, 5, 8)$ are both A_2 functions. The reason why $I_{\text{dp}}^1(1, 3, 4, 8)$

is only A_1 is that, since it does not depend on the dual point x_6 , only three momentum twistors with even labels can appear in the final answer, which are insufficient to form a cross ratio, so that $I_{\text{dp}}^1(1, 3, 4, 8)$ only depends on v . For the two A_2 functions, explicit computations show that their $(2, 1, 1)$ -coproducts are:

$$\Delta_{2,1,1}(I_{\text{dp}}^1(1, 4, 5, 8)) = F(2, 4, 6, 8) \otimes (v-w) \otimes \frac{v}{w} + \text{non-mixing part} \quad (2.4)$$

$$\Delta_{2,1,1}(I_{\text{dp}}^1(1, 3, 6, 8)) = F(2, 4, 6, 8) \otimes (v-w) \otimes \frac{1+v}{1+w} + \text{non-mixing part} \quad (2.5)$$

which support our conjecture since the two integrals both have four off-shell dual points $\{x_2, x_4, x_6, x_8\}$. After a cyclic rotation, $w \rightarrow \frac{1}{v}$ and $v \rightarrow w$, $\{x_2, x_4, x_6, x_8\}$ turn into $\{x_1, x_3, x_5, x_7\}$ and the mixing letter $v-w$ becomes $1-vw$ as we expected. Furthermore, after two cyclic rotations, $v \rightarrow \frac{1}{v}$ and $w \rightarrow \frac{1}{w}$, the mixing part of $\Delta_{2,1,1}(I_{\text{dp}}^1(1, 4, 5, 8))$ comes back to itself, while the mixing part of $\Delta_{2,1,1}(I_{\text{dp}}^1(1, 3, 6, 8))$ becomes that of $\Delta_{2,1,1}(I_{\text{dp}}^1(1, 3, 6, 8)) + \Delta_{2,1,1}(I_{\text{dp}}^1(1, 4, 5, 8))$.

For another type of double pentagons $I_{\text{dp}}^2(i, j, k, l, m)$ with numerators $\mathbf{N}_1 = \langle \bar{\ell} i \cap \bar{j} \rangle$ and $\mathbf{N}_2 = \frac{1}{2} \langle \ell \bar{l} \cap (mm+1((lkk+1) \cap (ij))) - (kk+1((lmm+1) \cap (ij))) \rangle$ (figure 2(b)), we have 22 different integrals, 13 of which are non-trivial in $R^{1,1}$ kinematics. Note that as $\mathbf{N}_2$ greatly simplifies in $R^{1,1}$ kinematics, some of these integrals evaluate to the same results as the previous integrals. For instance, $I_{\text{dp}}^2(1, 3, 3, 6, 8)$ is equal to a penta-box integral. One nontrivial integral is figure 2(c). Contrary to $I_{\text{dp}}^1(1, 3, 5, 7)$, now the numerator $\mathbf{N}_2 \propto \langle \ell' 57 \rangle$ in $R^{1,1}$, which gives a non-vanishing result. This integral depends on all 8 dual points $x_1 \sim x_8$, indicating that its symbol has both mixing letters. Explicit computation shows

$$\begin{aligned} \Delta_{2,1,1}(I_{\text{dp}}^2(1, 3, 4, 6, 7)) \\ = F(2, 4, 6, 8) \otimes (v-w) \otimes \frac{1+v}{1+w} + \left(v \rightarrow \frac{1}{w}, w \rightarrow v \right) + \text{non-mixing part}, \end{aligned} \quad (2.6)$$

where $(v \rightarrow \frac{1}{w}, w \rightarrow v)$ means a cyclical rotation in the reverse direction. Indeed, the integral is a cluster function with two overlapping A_2 's, as we would expect for the most general octagon in $R^{1,1}$.

All-loop ladders. Finally we move to some all-loop integrals, and the simplest case with algebraic singularity is the double-penta-ladder which we denote as $I^{(L)}(1, 4, 5, 8)$; for $L = 2$ it reduces to the double-pentagon integral $I_{\text{dp}}^1(1, 4, 5, 8)$ but one can naturally extend it to all loops by inserting a box ladder in between. The integral is expected to contain only one square root $\Delta_{2,4,6,8}$, and in 4d the calculation based on $d \log$ recursion is quite involved due to the need of rationalization (no result beyond $L = 2$ has been obtained so far). We show that in $R^{1,1}$, the $d \log$ recursion becomes trivial to solve, and the integral evaluates to a single A_2 function with the mixing letter $v - w$ only appearing on the third entry.

By reducing the result of [47] to $R^{1,1}$ or directly write down $d \log$ forms, we see that the $d \log$ recursion takes the remarkably simple form:

$$I^{(L+1)}(v, w) = \int_1^\infty d \log t \int_1^\infty d \log s I^{(L)}(vt, ws), \quad (2.7)$$

where the source of the recursion is the one-loop hexagon:

$$I^{(1)} = \frac{v}{v-w} J(v, w) + \frac{w}{w-v} J(w, v), \quad (2.8)$$

with $J(v, w) := -2\text{Li}_2\left(\frac{1}{1+v}\right) + \log\left(\frac{v}{1+v}\right) \log(w(1+v))$. We see that, similarly to the $k + \ell = 3$ amplitude, beautifully the mixing letter $v - w$ first appears as a prefactor of the one-loop result, which manifests the symmetry under $v \leftrightarrow w$. Conversely, if we rewrite the recursion as

$$I^{(L+1)}(v, w) = \int_v^\infty d\log t \int_w^\infty d\log s I^{(L)}(t, s),$$

and it is easy to get a beautiful second-order differential equation

$$I^{(L)} = v\partial_v w\partial_w I^{(L+1)}. \quad (2.9)$$

The $d\log$ recursion, or differential equation, allows us to directly compute $I^{(L)}(1, 4, 5, 8)$ to all loops (including odd-weight objects in between). From weight 2 to 3, one can absorb the prefactor to get a more non-trivial $d\log$ form (analog of the rationalization we did in 4d), and starting from there we always have pure functions, which trivializes any further rationalization encountered for higher loops in 4d.

This computation reveals a remarkable phenomenon for the algebraic letters generalizing what we have seen at two loops: the mixing letter $v - w$ appears *only* on the third entry, with the first two accompanying it given by the symbol of four-mass box, and subsequent ones only containing v/w (recall that the alphabet of $I^{(L)}$ is $\{v, 1+v, w, 1+w, v-w\}$). Let us first show that in the last two entries, only v and w are allowed. We note that

$$v\partial_v I^{(L+1)} = - \int_w^\infty d\log s I^{(L)}(v, s) \quad \text{and} \quad w\partial_w I^{(L+1)} = - \int_v^\infty d\log t I^{(L)}(t, w)$$

are pure functions for $L \geq 2$. Now suppose the symbol of $I^{(L)}$ ($L \geq 2$) takes the form

$$\mathcal{S}(I^{(L)}) = S_w^{(L)} \otimes w + S_v^{(L)} \otimes v + S_{1+w}^{(L)} \otimes (1+w) + S_{1+v}^{(L)} \otimes (1+v) + S_{v-w}^{(L)} \otimes (v-w),$$

and we obtain

$$w\partial_w I^{(L)} = S_w^{(L)} + \frac{w}{1+w} S_{1+w}^{(L)} - \frac{w}{v-w} S_{v-w}^{(L)}.$$

Since $w\partial_w I^{(L)}$ is pure, $S_{1+w}^{(L)}$ and $S_{v-w}^{(L)}$ must vanish; symmetrically, $S_{1+v}^{(L)}$ vanishes because $v\partial_v I^{(L)}$ is pure. Therefore, we have proven that the last entries of $I^{(L)}$ for $L \geq 2$ can only be v or w . In fact, one can further prove that for $L > 2$, only v and w occur at the last two entries.

For the mixing part with $v - w$ on the third entry, by the symmetry between v and w and $\mathcal{S}(F(v, w)) = -\mathcal{S}(F(w, v))$ for four-mass box $F(2, 4, 6, 8) = F(v, w)$, the last entry for $L = 2$ can only be v/w . We will not give the detail of the proof here, but after some algebra one can show that by further integrations from $L = 2$, the mixing part of $\Delta_{2,1,2L-3}(I^{(L)})$ for $L \geq 2$ takes the remarkably simple form:

$$F(v, w) \otimes (v-w) \otimes \frac{1}{(2L-3)!} \log^{2L-3} \left(\frac{v}{w} \right). \quad (2.10)$$

2.2 Cluster bootstrap for octagons

Based on the alphabet, we are ready to consider the cluster bootstrap for octagons. We emphasize that these octagon amplitudes in $R^{1,1}$ kinematics with $k + \ell \leq 3$ have been obtained in [43] so our results here are not new. However, our main point is to illustrate how simple the relevant octagon space has become and, even in the absence of more OPE data, one can locate these amplitudes in the space with relative ease. We will construct the necessary space of cluster functions while ignoring all multi-zeta-values so that we are effectively working at the symbol level. Any beyond-symbol-ambiguities can be easily fixed in the end.

2.2.1 Construction of the cluster function space

The construction of integrable symbols can be done recursively: at weight k , we consider all integrable symbols of weight $(k-1)$ tensored with elements in the alphabet, and impose integrability conditions on the final two entries. The ansatz reads

$$\sum_{i,j} c_{ij} S_i^{(k-1)} \otimes l_j,$$

where $l_{j=1,\dots,n_1}$ denote the letters (n_1 is the number of letters, or dimension of the cluster algebra), and $S_{i=1,\dots,N_{k-1}}^{(k-1)}$ denote the independent integrable symbols of weight $(k-1)$. The integrability condition for the last two entries reads

$$\sum_{i,j,m} c_{ij} S_{i;m}^{(k-2)} d \log l_m \wedge d \log l_j = 0, \quad \text{where } S_i^{(k-1)} = \sum_m S_{i;m}^{(k-2)} \otimes l_m, \quad (2.11)$$

where $S_{i;m}^{(k-2)}$ are linear combinations of weight- $(k-2)$ integrable symbols.

Before we describe the A_1^2 and A_2 cluster function spaces of interest, we point out a remarkable fact concerning the number N_k of independent weight- k integrable symbols corresponding to a given alphabet. In general, it is an open question how to compute N_k without explicitly solving the above ansatz at each weight. However, for cluster algebra A_r with no constraints on the alphabet $\{z_i, 1+z_i, z_j-z_i | 1 \leq i < j \leq r\}$, there exists a nice recursion relation:

$$\forall k \geq 1: \quad N_k = n_1 N_{k-1} - n_2 N_{k-2} + \dots + (-)^{r-1} n_r N_{k-r}. \quad (2.12)$$

Here, n_m is the number of independent $d \log m$ -forms, $N_0 = 1$, and we have defined $N_{k < 0} := 0$ in order to apply it to cases with $k < r$. To see this, define the space $W^{(k,m)} = F^{(k-m)} \otimes_{\mathbb{Q}} \Omega_{\log}^{(m)}$ of $d \log m$ -forms with coefficients being equivalence classes of weight- $(k-m)$ pure functions under the symbol map. A theorem (see [54, 61]) states that for such A_r alphabets, the following sequence is exact:

$$0 \rightarrow W^{(k,0)} \xrightarrow{d} \dots \xrightarrow{d} W^{(k,r)} \xrightarrow{d} \underbrace{W^{(k,r+1)}}_{=0} \xrightarrow{d} \dots \xrightarrow{d} \underbrace{W^{(k,k)}}_{=0} \rightarrow 0. \quad (2.13)$$

The recursion relation follows by noticing $N_k = \dim F^{(k)}$ and $n_m = \dim \Omega_{\log}^{(m)}$.

In fact, the recursion relation also holds for “nice” degenerations of A_r alphabets. By this, we mean that the imposed constraints commute with the anti-derivative operator. For instance, A_1^2 is simply A_2 with the letter $v-w$ forbidden, but the anti-derivative of every object in $W^{(k,m)}$ satisfying this constraint clearly satisfies the same constraint, so that exactness is not spoiled, as long as we only count the $d \log$ forms without $d \log(v-w)$ in computing n_m .

The above counting works for type- A cases (and degenerations) where no further constraints have been imposed. When we do impose constraints on possible first entries, things become more subtle due to their possible effect on n_m . Fortunately for us, we are concerned with algebras of rank $n = 2$ and need only first-entries conditions with $k' = 2$, so that starting at $k - r \geq k'$ or $k \geq 4$, the coefficients n_m are not affected by such changes several “entries” ago.⁶ Checking $k = 3$ explicitly, we conclude that our constraints only change N_1 and N_2 , but do not affect the recursion for $k \geq 3$:

- For A_1^2 , the recursion reads $N_k = 4N_{k-1} - 4N_{k-2}$ with $k \geq 3$;
- For A_2 , the recursion reads $N_k = 5N_{k-1} - 6N_{k-2}$ with $k \geq 3$.

For A_1^2 and A_2 spaces where only $v, 1+v, w, 1+w$ are allowed in the first two entries, we have $N_1 = 4$ and $N_2 = 12$. Thus we find that $N_k(A_1^2) = (k+1)2^k$, and $N_k(A_2) = 4 \times 3^{k-1}$. The space from which we bootstrap the amplitudes is spanned by all integrable symbols from two overlapping such A_2 ’s, thus the number at weight k is given by

$$2N_k(A_2) - N_k(A_1^2) = 8 \times 3^{k-1} - (k+1) \times 2^k. \quad (2.14)$$

Indeed we have explicitly checked that there are 4, 12, 40, 136, 456, 1496, 4808, 15192 such integrable symbols at weight $k = 1, 2, \dots, 8$. Given the symbols, it is straightforward to integrate them to obtain actual 2dHPL functions.

In order to present these functions at each weight in a concise way, we note that it suffices to record only the new functions at each weight, i.e. those that are not products of lower-weight functions. In practice this can be done by projecting out all functions whose symbol contain shuffle products of lower-weight ones, and we list all new functions up to weight $k = 6$ in a supplementary notebook file attached to this paper.

Let us start with weight 2, where as mentioned, only 2 new functions (in addition to the 10 log log functions) are needed, which can be chosen to be $\text{Li}_2(-v)$ and $\text{Li}_2(-w)$. At weight 3, we have 12 new functions in total: 4 of which do not contain mixing letters, i.e. they are A_1^2 functions, and they can be chosen as $\text{Li}_3(-v)$, $\text{Li}_3(1+v)$, $\text{Li}_3(-w)$, $\text{Li}_3(1+w)$; 4 of them contain letter $v - w$, which are

$$\begin{aligned} &G_{1,0,0}\left(\frac{v}{w}\right), G_{w,0,-1}(v) + \log\left(1 - \frac{v}{w}\right) G_{-1,0}(w) - \log(w) G_{w,-1}(v), \\ &G_{w,-1,0}(v) - \log(w+1) G_{w,0}(v) + \log\left(1 - \frac{v}{w}\right) G_{0,-1}(w), \\ &G_{w,-1,-1}(v) + \frac{1}{2} \log(w+1)^2 \log\left(1 - \frac{v}{w}\right) - \log(w+1) G_{w,-1}(v) \end{aligned} \quad (2.15)$$

⁶Note that such counting only applies to this specific alphabet of type A with linear letters, thus for the familiar A_3 alphabet of six-point amplitudes with physical first-entry conditions, where the letters are not linear, our result cannot be applied directly.

and the remaining four containing $1-vw$ can be obtained by cyclic rotation $v \rightarrow 1/w, w \rightarrow v$ (in total we have 12 new functions). We have included terms that are given by products of lower-weight functions to ensure that only non-mixing letters appear in the first two entries.

Now we move to weight 4, 5, 6 etc., and again we can write down new A_1^2 functions and genuine A_2 functions with mixing letters separately. For example at weight 4, there are 6 A_1^2 functions that are new, which can be chosen as $\text{Li}_4(-v), \text{Li}_4(1+v), \text{Li}_4(1+1/v)$ and $(v \leftrightarrow w)$; there are 12 new functions which contain the letter $v-w$, and another 12 obtained by cyclic rotation (containing the letter $1-vw$), thus in total we have $6 + 12 + 12 = 30$ new functions. Similarly at weight 5, we have 12 new A_1^2 functions, 36 new functions with $v-w$ and 36 with $1-vw$, thus $12 + 36 + 36 = 84$ new functions in total; at weight 6 we find $18 + 98 + 98 = 214$ new functions. To recover the entire function space at each weight, one includes products of lower-weight new functions, e.g. at weight 3 in addition to the 12 new functions, we have $4(4+1)(4+2)/3! = 20 \log^3$ functions and $4 \times 2 = 8 \log \text{Li}_2$ functions, thus recovering the 40-dimensional space for $k=3$ as mentioned above. Note that one can also trivially include possible constants such as multi-zeta-values and obtain the full space covering beyond-symbol ambiguities.

We have not attempted to apply cluster adjacency or extended-Steinmann relations: since the two mixing letters never appear in the same term (let alone next to each other), such constraints can only be studied for the two A_2 's separately, but no such constraints has become relevant for the octagon functions we studied, at least not in terms of v, w variables.

With the function space at hand, we are able to bootstrap octagon amplitudes. We will first determine the symbol of amplitudes and fix beyond-symbol-ambiguities in the end. We arrange our results according to the level $k+\ell$, as $\bar{Q}$ equations relate ℓ -loop $N^k\text{MHV}$ amplitudes on the same level. For the $k+\ell=2$ case which turns out to contain only logarithmic terms, symmetry and collinear conditions suffice to determine the answer. For $k+\ell=3$, more conditions are needed, such as those imposed by soft-collinear OPE [50]. We summarize our results as follows.

2.2.2 $k+\ell=2$

At level $k+\ell=2$, the dimensions of integrable symbol spaces where no mixing letters appear in the first two entries are listed in the first line of table 1. As more constraints are imposed, the dimensions gradually reduce to 1.

1-loop NMHV. From (1.6), the symbols to be determined are $\mathcal{S}(f_{8,1}^1(v, w))$ and $\mathcal{S}(f_{8,1}^2(v, w))$, which satisfy the following symmetry and collinear conditions:

$$\begin{aligned} f_{8,1}^1(v, w) &= f_{8,1}^1(w, v), & f_{8,1}^2(v, w) &= f_{8,1}^2(w, v) = f_{8,1}^2(1/v, w), \\ \lim_{v \rightarrow 0} f_{8,1}^1(v, w) &= 0, & \lim_{v \rightarrow \infty} (f_{8,1}^2(v, w) - f_{8,1}^1(v, w) - f_{8,1}^1(v, 1/w)) &= 1. \end{aligned} \quad (2.16)$$

These suffice to fix the 1-loop NMHV amplitude.

2-loop MHV. The symbol to be determined is $\mathcal{S}(R_{8,0}^{(2)}(v, w))$, which satisfies the following symmetry and collinear conditions:

$$R_{8,0}^{(2)}(v, w) = R_{8,0}^{(2)}(w, 1/v) = R_{8,0}^{(2)}(1/v, w), \quad \lim_{v \rightarrow 0} R_{8,0}^{(2)}(v, w) = 1. \quad (2.17)$$

	1-loop NMHV	2-loop MHV
Last entries	24	136
Symmetry	10	15
Collinear	1	1

Table 1. Dimensions of integrable symbol spaces at level $k + \ell = 2$.

	1-loop N ² MHV	2-loop NMHV	3-loop MHV
Last entries	11	272	1496
Collinear	3	21	40
Four-mass	1		
OPE leading	1	7	21
OPE sub-leading		1	1

Table 2. Dimensions of integrable symbol spaces at level $k + \ell = 3$.

Greater efficiency is achieved by restricting the space of integrable symbols using $\bar{Q}$ equations, according to the refined version of our conjecture. In particular, for the 2-loop MHV amplitude, mixing letters are not allowed in the 3rd and 4th entries. These conditions are enough to fix the 2-loop MHV amplitude.

2.2.3 $k + \ell = 3$

At level $k + \ell = 3$, the dimensions of integrable symbol spaces satisfying various constraints are listed in table 2.

1-loop N²MHV. From (1.8), the symbol to be determined is $\mathcal{S}(F_{8,2}(v, w))$, which by itself does not satisfy any particular constraints. However, from our analysis above we see that all prefactors are of the form $\frac{v}{v-w}$. Thus, we may construct from $\mathcal{S}(F_{8,2}(v, w))$ an ansatz⁷ $\mathcal{S}(R_{8,2}^{(1)}(v, w))$:

$$\mathcal{S}(R_{8,2}^{(1)}(v, w)) = R_{8,2}^{\text{tree}}(v, w) \left(\frac{v}{v-w} \mathcal{S}(F_{8,2}(v, w)) + \text{cyclic} \right). \quad (2.18)$$

Now, we can impose the symmetry and collinear conditions:

$$R_{8,2}^{(1)}(v, w) = R_{8,2}^{(1)}(w, 1/v) = R_{8,2}^{(1)}(1/v, w), \quad \lim_{v \rightarrow 0} R_{8,2}^{(1)}(v, w) = 0. \quad (2.19)$$

To proceed, we impose the constraint that the soft-collinear OPE of our ansatz should match known results. Recall that the soft-collinear OPE is essentially a Taylor expansion

⁷As mentioned above, there are 12 integrable symbols at weight-2, so the ansatz for $\mathcal{S}(F_{8,2}(v, w))$ should have 12 free parameters to begin with. However, after constructing $\mathcal{S}(R_{8,2}^{(1)}(v, w))$, one of these immediately drops out by symmetry, hence the 11 in the table.

of the analytic part after factoring out the non-analytic part:

$$\begin{aligned}\mathcal{F}(v, w) &= \overbrace{F_m(v, w) \log^m v}^{\text{leading order of OPE}} + F_{m-1}(v, w) \log^{m-1} v + \cdots + F_0(v, w), \quad \text{where } F_i \text{ is analytic,} \\ &= \left(\sum_{n=0}^{\infty} f_{mn}(w) v^n \right) \log^m v + \cdots + \left(\sum_{n=0}^{\infty} f_{0n}(w) v^n \right).\end{aligned}$$

Notice that the prefactors are part of the functions $F_i(v, w)$, which are still analytic at $v = 0$.

At this stage, we should construct the full function space including constants and compute collinear-soft OPE expansion from there. However, we find it more convenient to compute discontinuities at the symbol level, and fix constants later. Therefore, we work “modulo beyond-symbol-ambiguities” — start by computing the symbol of $\mathcal{S}(F_i)$ for $i = 0, 1, \dots, m$; then, uplift symbols to functions F_i and expand to $f_{ij}(w)$ to compare with known results.

What are the “known results”? For the leading order of OPE in the MHV case up to 3 loops, the result has been computed from first principles [50]. For higher k , a first-principle calculation is not present, but we take the equivalent step by comparing with known results of the amplitude. For instance, $\mathcal{S}(R_{8,2}^{(1)}(v, w))$ should have the following leading ($m = 1$) OPE behavior [43]:

$$\mathcal{S}(f_{10}(w)) = 0, \quad (2.20)$$

$$\mathcal{S}(f_{1n}(w)) = \frac{(-1)^n}{n} + (-1)^n \sum_{i=1}^{n-1} \frac{(-w)^i + (-w)^{-i}}{n-i} + \frac{1}{w^n} [1+w] + w^n \left[\frac{1+w}{w} \right], \quad (2.21)$$

where $[\dots]$ denotes the tensor produced by the symbol map, so that $[1+w]$ in $\mathcal{S}(f_{1n}(w))$ simply means $\log(1+w)$ in $f_{1n}(w)$. This enables us to fix $\mathcal{S}(R_{8,2}^{(1)}(v, w))$.

As a side note, we can impose a different constraint for the last step. From the box expansion of 1-loop amplitudes, it is clear that the residue at the pole $(1-vw)$ must be proportional to the corresponding four-mass box:

$$\lim_{v \rightarrow 1/w} (1-vw) R_{8,2}^{(1)}(v, w) \propto F(1, 3, 5, 7). \quad (2.22)$$

This also enables us to fix $\mathcal{S}(R_{8,2}^{(1)}(v, w))$.

2-loop NMHV. The symbols to be determined are again $\mathcal{S}(f_{8,1}^1)$ and $\mathcal{S}(f_{8,1}^2)$, which satisfy the symmetry and collinear conditions as in the 1-loop case. According to our conjecture, $\bar{Q}$ equations exclude mixing letters from the last entry. Using these conditions alone, the space of integrable symbols is reduced to 21 dimensions.

The leading ($m = 1$) OPE behavior should match the known results [43]:

$$\mathcal{S}(f_{10}(w)) = 0, \quad (2.23)$$

$$\mathcal{S}(f_{1n}(w)) = \frac{2(-1)^n}{n} [(1+w) \otimes (1+w)] - \frac{(-1)^n}{n^2} [1+w] + \frac{w^n}{n^2} \left[\frac{w}{1+w} \right] - \frac{(-1)^n}{n^2} \sum_{i=1}^{n-1} \frac{(-w)^i}{n-i}. \quad (2.24)$$

This further reduces the space to 7 dimensions.

Next, it is natural to use the sub-leading order of OPE, which has not been computed from first principles. Luckily for us, the 2-loop NMHV amplitude itself is known exactly, ready for direct comparison. This way, we are able to test the constraining power of the sub-leading order of OPE without computing it from first principles. The result is satisfying: the 2-loop NMHV amplitude is completely determined by these conditions.

3-loop MHV. The symbol to be determined is again $\mathcal{S}(R_{8,0}^{(2)}(v,w))$, which satisfies the symmetry and collinear conditions as in the 2-loop case. According to our conjecture, mixing letters are not allowed on the last two symbol entries. These constraints reduce the dimension of the space of integrable symbols to 40.

In exactly the same way as the 2-loop NMHV case, imposing leading order OPE constraints reduces the dimension to 21, and imposing sub-leading order OPE constraints finishes the job!

3 Towards $2n$ -gons: A_2 functions and beyond

Having discussed octagon amplitudes and integrals, it is natural to wonder what we can say about amplitudes of higher points, for $2n = 10, 12, \dots$. As mentioned earlier, a remarkable property of $R^{1,1}$ kinematics is that we can apply the *collinear-soft uplift* to obtain higher-point amplitudes from lower objects, which are to be viewed as a function of off-shell points rather than polygons [43, 44]. The uplifting begins with a function S_8 which depends on four (off-shell) points, but may involve higher partial amplitudes like S_{10}, S_{12} etc., depending on the loop order:

$$R_n = \sum_{a \triangleleft b \triangleleft c \triangleleft d} (-)^{a+b+c+d} S_8(x_a, x_b, x_c, x_d) + \text{contributions from } S_{10}, S_{12} \text{ etc.} \quad (3.1)$$

In the first term we sum over all four-mass boxes with labels $1 \leq a, b, c, d \leq n$, where $a \triangleleft b$ means that the indices should be separated by at least 2, $a \leq b - 2$ (it is understood that d and a must be separated also, e.g. when $a = 1$ we must have $d \leq n - 1$). For example, for $2n = 8$ we have two such S_8 contributions

$$R_8 = S_8(x_2, x_4, x_6, x_8) + S_8(x_1, x_3, x_5, x_7), \quad (3.2)$$

and for $2n = 10, 12, \dots$, we have 25, 105, $\dots$ S_8 contributions, one for each four-mass box. Remarkably for $k + \ell = 2$, the S_8 contributions are all one can have, i.e. $S_{10} = S_{12} = \dots = 0$; for $k + \ell = 3$, we find non-vanishing remainders up to S_{12} , but very nicely they consist of simple logarithmic and dilogarithmic functions without any mixing letters, for two-loop NMHV and three-loop MHV, respectively [43]. Thus we see that, for $k + \ell \leq 3$, the amplitude is given by a sum of S_8 functions labelled by four-mass boxes, plus simpler remainders without any mixing letters!

This result for $k + \ell \leq 3$ immediately implies that up to three-loop MHV and two-loop NMHV, $2n$ -gon in $R^{1,1}$ kinematics consists of a collection of A_2 functions, one for each four-mass box (or S_8), and a A_{n-3}^2 function for the remainders which contains no mixing letters. Note that for $2n$ points, there are exactly $N_{4m}(n) \equiv n(n-3)(2n-5)(2n-7)/6$ four-mass

boxes (or S_8 's), e.g. $N_{4m} = 2, 25, 105, \dots$ for $n = 4, 5, 6, \dots$. For example, for $2n = 10$, there are 25 A_2 functions and a $A_2 \times A_2$ function (note different meanings of these A_2 's), thus 35 letters in total; for $2n = 12$, we have 123 letters, or 105 A_2 functions and a A_3^2 function. Purely for the alphabet, the $N_{4m}(n)$ A_2 's already include every letter in the A_{n-3}^2 part, but we emphasize that for $2n \geq 10$, in addition to the A_2 functions (one for each S_8), we still need an A_{n-3}^2 function for the remainder which cannot be split into various A_2 .

In general it would be more difficult to bootstrap for $2n \geq 10$ with this alphabet since the space of integrable symbol grows faster. Let us briefly comment on one loop (weight 2) case. Note that only the A_{n-3}^2 part can appear for one loop, and it is straightforward to enumerate weight-2 integrable symbols (or functions). There are two possibilities: Li_2 functions and log log ones; the former are single-variable functions which can be listed in even and odd sectors separately, and the latter are given by the symmetric part $\mathcal{S}[\log a \log b] = a \otimes b + b \otimes a$ with $a, b \in A_{n-3}^2$. Note that the point-count of the cluster configuration space of A_{n-3} on $\mathbb{F}_p$ is given by a polynomial $(p-2)(p-3) \cdots (p-n+2)$ for a generic prime number p , and we can infer the number of independent $d \log k$ -forms from the coefficient of p^{n-3-k} (up to a sign); in particular, the number of 1 forms is $n_1 := n(n-3)/2$ and the number of 2 forms is $n_2 = (3n^2 - n + 2)(n-3)(n-4)/24$.

It is clear that we have $n_1(2n_1 + 1)$ symmetric symbols (or log log functions), and the enumeration of Li_2 functions in each sector is as follows: from the n_1 letters of A_{n-3} , we can construct $\binom{n_1}{2}$ antisymmetric weight-2 symbols; on the other hand, integrability exactly imposes n_2 constraints; thus there are $\binom{n_1}{2} - n_2 = \binom{n-1}{3}$ antisymmetric integrable weight-2 symbols, or Li_2 functions.⁸ For example, for $2n = 8$ we have 1 + 1 Li_2 functions and 2×5 log log's, for $2n = 10$, we have 4 + 4 Li_2 's and 5×11 log log's, and for $2n = 12$, we have 10 + 10 Li_2 's and 9×19 log log's.

Besides amplitudes with $k + \ell \leq 3$, we find that simple Feynman integrals such as double pentagons with $2n \geq 10$ legs have similar alphabets in $R^{1,1}$ kinematics. We can see this already for integrals without four-mass square roots. The simplest $2n = 10$ point cases are the following two double-pentagon integrals (figure 3). Following the configuration of external legs, we see the left one depends on three cross ratios $\left\{ v = \frac{\langle 57 \rangle \langle 19 \rangle}{\langle 59 \rangle \langle 17 \rangle}, w_1 = \frac{[26][810]}{[68][210]}, w_2 = \frac{[410][68]}{[810][46]} \right\}$, while the right one $\left\{ v = \frac{\langle 17 \rangle \langle 35 \rangle}{\langle 13 \rangle \langle 57 \rangle}, w_1 = \frac{[26][410]}{[24][610]}, w_2 = \frac{[28][410]}{[24][810]} \right\}$. Explicit computation shows that the alphabet of the integrals are both $\{v, 1+v, w_1, w_2, 1+w_1, 1+w_2, 1-w_1w_2\}$. Hence both integrals are $A_2 \otimes A_1$ functions. For integrals with square roots, we find that in addition to such A_{n-3}^2 part, the part with algebraic letters again reduces to A_2 symbols with mixing letters, one for each square root. For example, for the most general double pentagon with $2n = 12$ points, its algebraic words become the sum of 16 A_2 symbols, of the form $\mathcal{S}(F(v, w)) \otimes (v - w) \otimes \frac{v}{w}$, where v, w are the $R^{1,1}$ cross-ratios of the corresponding four-mass box configuration.

What do we expect for higher k and/or higher loops? It is well known that even at two loops, amplitudes in 4d with $k \geq 3$ involve functions that go beyond multiple polylogarithms (MPL) [62, 63], but we do not know if this is the case in $R^{1,1}$ kinematics

⁸It is interesting that this is also the number of independent $R^{1,1}$ four-mass box functions in each sector, as one can shown by repeatedly using Abel identities for Rogers dilogarithms.

Figure 3. Two examples of 10pt integrals.

(we have not excluded the possibility that $R^{1,1}$ amplitudes still evaluate to MPL for $k \geq 3$). For $k \leq 2$, it is plausible that amplitudes evaluate to MPLs only; in these cases if one can prove that in 4d, all algebraic letters for amplitudes with $k \leq 2$ are still associated with square roots of four-mass boxes, then the above conjecture holds as well. Starting $k = 3$, it is likely that more complicated algebraic singularities in 4d lead to new letters in $R^{1,1}$ kinematics. In other words, for $k \geq 3$ it is possible that remainder functions S_{10}, S_{12} etc. contain new mixing letters associated with more legs (thus depending on more than just a pair of v and w), which represent more general algebraic letters in 4d. It would be extremely interesting to find amplitudes with such mixing letters, and study what kind of cluster functions, beyond A_2 and A_{n-3}^2 , are needed for these new beasts.

4 Conclusion and discussions

In this paper, we have studied scattering amplitudes and Feynman integrals in $\mathcal{N} = 4$ SYM in $R^{1,1}$ kinematics by exploring and leveraging their remarkably simple cluster-algebra structures. For the octagon, we have used the alphabet of two-overlapping A_2 's and conditions on first and last entries to bootstrap amplitudes up to $k + \ell = 3$. We have also studied various octagon integrals, which are all A_2 functions with interesting patterns of the symbol regarding such mixing letters; these allow us to predict patterns for algebraic letters in 4d. We have also discussed generalization to $2n$ -gon using A_2 functions and cluster functions involving more legs.

There are several natural directions for future investigations. Among others, it would be highly desirable to determine what functions appear for $R^{1,1}$ amplitudes with $2n \geq 10$ and $k \geq 3$. Some components of 4d super-amplitudes/WL, which are given by elliptic integrals and beyond e.g. “train tracks” in [64], may vanish when supersymmetrically reduced to $R^{1,1}$. For example, the famous “elliptic” component of two-loop N^3 MHV amplitude [62] vanishes since there are not enough η 's for it in supersymmetric $R^{1,1}$. It would be interesting to see if similar components which are given by functions beyond MPL survive the supersymmetric $R^{1,1}$ reduction. Relatedly, it is natural to expect that algebraic singularities beyond four-mass-box square roots still reduce to rational, mixing letters in $R^{1,1}$ kinematics for $2n \geq 10$, and it would be interesting to study what type of cluster algebras they belong to.

We have only scratched the surface of the analytic formulas for amplitudes and integrals in $R^{1,1}$ kinematics. Although in principle they can be obtained by reduction from 4d, the limit is very subtle and it is preferable to study them directly in 2d. This is already the case for leading singularities: the corresponding positroids in 4d usually do not have a well-defined 2d limit, which makes the reduction quite subtle even at one loop! We would like to develop a systematic way for classifying all Yangian invariants in $R^{1,1}$ kinematics: they must be given by products of even and odd $m = 2$ R-invariants, times possible prefactors that mix them. Based on this, one can study amplitudes/WL for $k + \ell \geq 3$ and Feynman integrals at higher loops, where $R^{1,1}$ kinematics serves as a perfect laboratory. We can hope to even address questions such as resummation of all-loop integrals/amplitudes, and make connection with strong coupling results [39, 40] as discussed in [43].

For bootstrapping higher-loop octagons, we remark that OPE has not been properly understood so far in $R^{1,1}$ kinematics (again the reduction to 2d is very subtle in the expansion), and it would be highly desirable to study this more systematically. Among other goals, combined with constraints from Yangian equations etc., it is possible to push the frontier of octagon bootstrap further. A very concrete target is the bootstrap of two-loop N^2 MHV octagon, where $\bar{Q}$ equations alone do not provide any additional constraints, but it is plausible that the first few orders of OPE data is sufficient to determine the amplitude. We can then go to higher points using collinear-soft uplift, and again by $\bar{Q}$ that would allow us to reach the land of $k + \ell = 4$ amplitudes, including three-loop NMHV and four-loop MHV cases.

Last but not least, there is the interesting question of constructing the space of integrable symbols and determining the dimension at each weight, for a given alphabet and possible constraints. We have only studied the simplest cases with *linear* alphabets of type A , where if we do not impose any other constraints, N_k can be determined purely from n_m (the number of independent $d \log m$ -forms) for $m \leq k$. It is already interesting to see that the recursion still works if we have imposed certain constraints for first k' entries. We suspect that exactly the same argument can be applied to any alphabet with linear letters (corresponding to hyperplane arrangements), and similar recursion relations should hold for those cases. We leave the question of extending these results to other alphabets (e.g., other finite-type cluster algebras) to future investigations.

Acknowledgments

It is a pleasure to thank Nima Arkani-Hamed, Livia Ferro, Johannes Henn, Yu-tin Huang, Chia-Kai Kuo and Chi Zhang for encouraging discussions, correspondence and collaborations on related projects. This research is supported in part by National Natural Science Foundation of China under Grant No. 11935013, 11947301, 12047502, 12047503.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar $N = 4$ SYM*, *JHEP* **01** (2011) 041 [[arXiv:1008.2958](#)] [[INSPIRE](#)].
- [2] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes*, Cambridge University Press (2016) [[DOI](#)] [[arXiv:1212.5605](#)] [[INSPIRE](#)].
- [3] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030 [[arXiv:1312.2007](#)] [[INSPIRE](#)].
- [4] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, *JHEP* **11** (2011) 023 [[arXiv:1108.4461](#)] [[INSPIRE](#)].
- [5] L.J. Dixon, J.M. Drummond, C. Duhr, M. von Hippel and J. Pennington, *Bootstrapping six-gluon scattering in planar $N = 4$ super-Yang-Mills theory*, *PoS LL2014* (2014) 077 [[arXiv:1407.4724](#)] [[INSPIRE](#)].
- [6] L.J. Dixon and M. von Hippel, *Bootstrapping an NMHV amplitude through three loops*, *JHEP* **10** (2014) 065 [[arXiv:1408.1505](#)] [[INSPIRE](#)].
- [7] J.M. Drummond, G. Papathanasiou and M. Spradlin, *A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon*, *JHEP* **03** (2015) 072 [[arXiv:1412.3763](#)] [[INSPIRE](#)].
- [8] L.J. Dixon, M. von Hippel and A.J. McLeod, *The four-loop six-gluon NMHV ratio function*, *JHEP* **01** (2016) 053 [[arXiv:1509.08127](#)] [[INSPIRE](#)].
- [9] S. Caron-Huot, L.J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude Using Steinmann Relations*, *Phys. Rev. Lett.* **117** (2016) 241601 [[arXiv:1609.00669](#)] [[INSPIRE](#)].
- [10] L.J. Dixon, J. Drummond, T. Harrington, A.J. McLeod, G. Papathanasiou and M. Spradlin, *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137 [[arXiv:1612.08976](#)] [[INSPIRE](#)].
- [11] J. Drummond, J. Foster, O. Gürdoğan and G. Papathanasiou, *Cluster adjacency and the four-loop NMHV heptagon*, *JHEP* **03** (2019) 087 [[arXiv:1812.04640](#)] [[INSPIRE](#)].
- [12] S. Caron-Huot, L.J. Dixon, F. Dulat, M. von Hippel, A.J. McLeod and G. Papathanasiou, *Six-Gluon amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory at six and seven loops*, *JHEP* **08** (2019) 016 [[arXiv:1903.10890](#)] [[INSPIRE](#)].
- [13] S. Caron-Huot, L.J. Dixon, F. Dulat, M. Von Hippel, A.J. McLeod and G. Papathanasiou, *The Cosmic Galois Group and Extended Steinmann Relations for Planar $\mathcal{N} = 4$ SYM Amplitudes*, *JHEP* **09** (2019) 061 [[arXiv:1906.07116](#)] [[INSPIRE](#)].
- [14] L.J. Dixon and Y.-T. Liu, *Lifting Heptagon Symbols to Functions*, *JHEP* **10** (2020) 031 [[arXiv:2007.12966](#)] [[INSPIRE](#)].
- [15] D. Chicherin, J. Henn and V. Mitev, *Bootstrapping pentagon functions*, *JHEP* **05** (2018) 164 [[arXiv:1712.09610](#)] [[INSPIRE](#)].

- [16] S. Caron-Huot et al., *The Steinmann Cluster Bootstrap for $N = 4$ Super Yang-Mills Amplitudes*, *PoS CORFU2019* (2020) 003 [[arXiv:2005.06735](#)] [[INSPIRE](#)].
- [17] J. Golden, A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Motivic Amplitudes and Cluster Coordinates*, *JHEP* **01** (2014) 091 [[arXiv:1305.1617](#)] [[INSPIRE](#)].
- [18] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [[arXiv:1006.5703](#)] [[INSPIRE](#)].
- [19] C. Duhr, H. Gangl and J.R. Rhodes, *From polygons and symbols to polylogarithmic functions*, *JHEP* **10** (2012) 075 [[arXiv:1110.0458](#)] [[INSPIRE](#)].
- [20] S. Fomin and A. Zelevinsky, *Cluster algebras I: foundations*, *J. Am. Math. Soc.* **15** (2002) 497.
- [21] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, *Invent. Math.* **154** (2003) 63.
- [22] D. Speyer and L. Williams, *The tropical totally positive grassmannian*, *J. Algeb. Comb.* **22** (2005) 189.
- [23] Z. Li and C. Zhang, *The Three-loop MHV Octagon from $\bar{Q}$ equations*, [arXiv:2110.00350](#) [[INSPIRE](#)].
- [24] S. He, Z. Li and C. Zhang, *Two-loop octagons, algebraic letters and $\bar{Q}$ equations*, *Phys. Rev. D* **101** (2020) 061701 [[arXiv:1911.01290](#)] [[INSPIRE](#)].
- [25] S. He, Z. Li and C. Zhang, *The symbol and alphabet of two-loop NMHV amplitudes from $\bar{Q}$ equations*, *JHEP* **03** (2021) 278 [[arXiv:2009.11471](#)] [[INSPIRE](#)].
- [26] S. Caron-Huot and S. He, *Jumpstarting the All-Loop S-matrix of Planar $N = 4$ Super Yang-Mills*, *JHEP* **07** (2012) 174 [[arXiv:1112.1060](#)] [[INSPIRE](#)].
- [27] J. Drummond, J. Foster, O. Gürdogan and C. Kalousios, *Algebraic singularities of scattering amplitudes from tropical geometry*, *JHEP* **04** (2021) 002 [[arXiv:1912.08217](#)] [[INSPIRE](#)].
- [28] N. Henke and G. Papathanasiou, *How tropical are seven- and eight-particle amplitudes?*, *JHEP* **08** (2020) 005 [[arXiv:1912.08254](#)] [[INSPIRE](#)].
- [29] N. Arkani-Hamed, T. Lam and M. Spradlin, *Non-perturbative geometries for planar $\mathcal{N} = 4$ SYM amplitudes*, *JHEP* **03** (2021) 065 [[arXiv:1912.08222](#)] [[INSPIRE](#)].
- [30] A. Herderschee, *Algebraic branch points at all loop orders from positive kinematics and wall crossing*, *JHEP* **07** (2021) 049 [[arXiv:2102.03611](#)] [[INSPIRE](#)].
- [31] J. Mago, A. Schreiber, M. Spradlin and A. Volovich, *Symbol alphabets from plabic graphs*, *JHEP* **10** (2020) 128 [[arXiv:2007.00646](#)] [[INSPIRE](#)].
- [32] S. He and Z. Li, *A note on letters of Yangian invariants*, *JHEP* **02** (2021) 155 [[arXiv:2007.01574](#)] [[INSPIRE](#)].
- [33] J. Mago, A. Schreiber, M. Spradlin, A.Y. Srikant and A. Volovich, *Symbol alphabets from plabic graphs II: rational letters*, *JHEP* **04** (2021) 056 [[arXiv:2012.15812](#)] [[INSPIRE](#)].
- [34] S. Caron-Huot, L.J. Dixon, M. von Hippel, A.J. McLeod and G. Papathanasiou, *The Double Pentagonal Integral to All Orders*, *JHEP* **07** (2018) 170 [[arXiv:1806.01361](#)] [[INSPIRE](#)].
- [35] S. He, Z. Li and Q. Yang, *Notes on cluster algebras and some all-loop Feynman integrals*, *JHEP* **06** (2021) 119 [[arXiv:2103.02796](#)] [[INSPIRE](#)].

- [36] D. Chicherin, J.M. Henn and G. Papathanasiou, *Cluster algebras for Feynman integrals*, *Phys. Rev. Lett.* **126** (2021) 091603 [[arXiv:2012.12285](#)] [[INSPIRE](#)].
- [37] S. He, Z. Li, Q. Yang and C. Zhang, *Feynman Integrals and Scattering Amplitudes from Wilson Loops*, *Phys. Rev. Lett.* **126** (2021) 231601 [[arXiv:2012.15042](#)] [[INSPIRE](#)].
- [38] J. Drummond, J. Foster and O. Gürdoğan, *Cluster Adjacency Properties of Scattering Amplitudes in $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **120** (2018) 161601 [[arXiv:1710.10953](#)] [[INSPIRE](#)].
- [39] L.F. Alday and J. Maldacena, *Minimal surfaces in AdS and the eight-gluon scattering amplitude at strong coupling*, [arXiv:0903.4707](#) [[INSPIRE](#)].
- [40] L.F. Alday and J. Maldacena, *Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space*, *JHEP* **11** (2009) 082 [[arXiv:0904.0663](#)] [[INSPIRE](#)].
- [41] V. Del Duca, C. Duhr and V.A. Smirnov, *A Two-Loop Octagon Wilson Loop in $N = 4$ SYM*, *JHEP* **09** (2010) 015 [[arXiv:1006.4127](#)] [[INSPIRE](#)].
- [42] P. Heslop and V.V. Khoze, *Analytic Results for MHV Wilson Loops*, *JHEP* **11** (2010) 035 [[arXiv:1007.1805](#)] [[INSPIRE](#)].
- [43] S. Caron-Huot and S. He, *Three-loop octagons and n -gons in maximally supersymmetric Yang-Mills theory*, *JHEP* **08** (2013) 101 [[arXiv:1305.2781](#)] [[INSPIRE](#)].
- [44] T. Goddard, P. Heslop and V.V. Khoze, *Uplifting Amplitudes in Special Kinematics*, *JHEP* **10** (2012) 041 [[arXiv:1205.3448](#)] [[INSPIRE](#)].
- [45] T. Gehrmann and E. Remiddi, *Numerical evaluation of two-dimensional harmonic polylogarithms*, *Comput. Phys. Commun.* **144** (2002) 200 [[hep-ph/0111255](#)] [[INSPIRE](#)].
- [46] M.A.C. Torres, *Cluster algebras in scattering amplitudes with special 2D kinematics*, *Eur. Phys. J. C* **74** (2014) 2757 [[arXiv:1310.6906](#)] [[INSPIRE](#)].
- [47] S. He, Z. Li, Y. Tang and Q. Yang, *The Wilson-loop $d \log$ representation for Feynman integrals*, *JHEP* **05** (2021) 052 [[arXiv:2012.13094](#)] [[INSPIRE](#)].
- [48] L. Ferro, *Differential equations for multi-loop integrals and two-dimensional kinematics*, *JHEP* **04** (2013) 160 [[arXiv:1204.1031](#)] [[INSPIRE](#)].
- [49] L.F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *An Operator Product Expansion for Polygonal null Wilson Loops*, *JHEP* **04** (2011) 088 [[arXiv:1006.2788](#)] [[INSPIRE](#)].
- [50] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Bootstrapping Null Polygon Wilson Loops*, *JHEP* **03** (2011) 092 [[arXiv:1010.5009](#)] [[INSPIRE](#)].
- [51] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, *JHEP* **12** (2011) 011 [[arXiv:1102.0062](#)] [[INSPIRE](#)].
- [52] B. Basso, A. Sever and P. Vieira, *Spacetime and Flux Tube S-Matrices at Finite Coupling for $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **111** (2013) 091602 [[arXiv:1303.1396](#)] [[INSPIRE](#)].
- [53] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [[arXiv:0905.1473](#)] [[INSPIRE](#)].
- [54] F.C.S. Brown, *Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$* , *Annales Sci. Ecole Norm. Sup.* **42** (2009) 371 [[math/0606419](#)] [[INSPIRE](#)].

- [55] N. Arkani-Hamed, Y. Bai, S. He and G. Yan, *Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet*, *JHEP* **05** (2018) 096 [[arXiv:1711.09102](#)] [[INSPIRE](#)].
- [56] N. Arkani-Hamed, S. He, T. Lam and H. Thomas, *Binary Geometries, Generalized Particles and Strings, and Cluster Algebras*, [arXiv:1912.11764](#) [[INSPIRE](#)].
- [57] N. Arkani-Hamed, H. Thomas and J. Trnka, *Unwinding the Amplituhedron in Binary*, *JHEP* **01** (2018) 016 [[arXiv:1704.05069](#)] [[INSPIRE](#)].
- [58] S. He and C. Zhang, *Notes on Scattering Amplitudes as Differential Forms*, *JHEP* **10** (2018) 054 [[arXiv:1807.11051](#)] [[INSPIRE](#)].
- [59] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo and J. Trnka, *Local Integrals for Planar Scattering Amplitudes*, *JHEP* **06** (2012) 125 [[arXiv:1012.6032](#)] [[INSPIRE](#)].
- [60] J.L. Bourjaily, A.J. McLeod, M. von Hippel and M. Wilhelm, *Rationalizing Loop Integration*, *JHEP* **08** (2018) 184 [[arXiv:1805.10281](#)] [[INSPIRE](#)].
- [61] F. Brown, *Iterated integrals in quantum field theory*, in *6th Summer School on Geometric and Topological Methods for Quantum Field Theory*, pp. 188–240 (2013) [[DOI](#)] [[INSPIRE](#)].
- [62] S. Caron-Huot and K.J. Larsen, *Uniqueness of two-loop master contours*, *JHEP* **10** (2012) 026 [[arXiv:1205.0801](#)] [[INSPIRE](#)].
- [63] J.L. Bourjaily, A.J. McLeod, M. Spradlin, M. von Hippel and M. Wilhelm, *Elliptic Double-Box Integrals: Massless Scattering Amplitudes beyond Polylogarithms*, *Phys. Rev. Lett.* **120** (2018) 121603 [[arXiv:1712.02785](#)] [[INSPIRE](#)].
- [64] J.L. Bourjaily, Y.-H. He, A.J. McLeod, M. Von Hippel and M. Wilhelm, *Traintracks through Calabi-Yau Manifolds: Scattering Amplitudes beyond Elliptic Polylogarithms*, *Phys. Rev. Lett.* **121** (2018) 071603 [[arXiv:1805.09326](#)] [[INSPIRE](#)].