The NSVZ relations for $\mathcal{N} = 1$ supersymmetric theories with multiple gauge couplings

D.S. Korneev,^a D.V. Plotnikov,^b K.V. Stepanyantz^b and N.A. Tereshina^b

^a*RWTH Aachen University, Faculty of Mathematics, Computer Science and Natural Science, Institute for Theoretical Particle Physics and Cosmology, 52062, Aachen, Germany*

^b*Moscow State University, Faculty of Physics, Department of Theoretical Physics, 119991, Moscow, Russia*

E-mail: dmitry.korneev@rwth-aachen.de, dim.plotnikov2013@yandex.ru, stepan@m9com.ru, tereshinanata2013@yandex.ru

ABSTRACT: We investigate the NSVZ relations for $\mathcal{N} = 1$ supersymmetric gauge theories with multiple gauge couplings. As examples, we consider MSSM and the flipped SU(5) model, for which they easily reproduce the results for the two-loop β -functions. For $\mathcal{N} = 1$ SQCD interacting with the Abelian gauge superfield we demonstrate that the NSVZ-like equation for the Adler D -function follows from the NSVZ relations. Also we derive all-loop equations describing how the NSVZ equations for theories with multiple gauge couplings change under finite renormalizations. They allow describing a continuous set of NSVZ schemes in which the exact NSVZ β -functions are valid for all gauge coupling constants. Very likely, this class includes the HD+MSL scheme, which is obtained if a theory is regularized by Higher covariant Derivatives and divergences are removed by Minimal Subtractions of Logarithms. That is why we also discuss how one can construct the higher derivative regularization for theories with multiple gauge couplings. Presumably, this regularization allows to derive the NSVZ equations for such theories in all loops. In this paper we make the first step of this derivation, namely, the NSVZ equations for theories with multiple gauge couplings are rewritten in a new form which relates the β -functions to the anomalous dimensions of the quantum gauge superfields, of the Faddeev-Popov ghosts, and of the matter superfields. The equivalence of this new form to the original NSVZ relations follows from the extension of the non-renormalization theorem for the triple gauge-ghost vertices, which is also derived in this paper.

KEYWORDS: Renormalization Group, Renormalization Regularization and Renormalons, Supersymmetric Gauge Theory

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1 Introduction

Although supersymmetry has not yet been discovered in direct experiments, there are some indications that physics beyond the Standard model is supersymmetric [1]. These indications are obtained from the analysis of various quantum corrections, which are compared with experimental data and theoretical considerations. Certainly, the most convincing

evidence in favor of supersymmetry is the unification of gauge coupling constants in supersymmetric extensions of the Standard model [2–4]. The evolution of couplings is encoded in the β -functions. To see that the gauge coupling unification becomes much better in the supersymmetric case, it is sufficient to consider the lowest (one-loop) contributions to the β -functions. However, the higher order corrections are also very important. For $\mathcal{N} = 1$ supersymmetric gauge theories there is an exact expression for the β -function, which relates it to the anomalous dimension of the matter superfields [5–8] and is usually called “the exact NSVZ β -function”. As a rule, this equation was analysed for theories with a simple gauge group. This implies that in this case there is the only gauge coupling constant α . If Yukawa couplings are denoted by λ , then the NSVZ equation for a theory with a single gauge coupling constant is written in the form

$$\beta(\alpha, \lambda) = -\frac{\alpha^2 \left(3C_2 - T(R) + C(R)_i^j \gamma_j^i(\alpha, \lambda)/r \right)}{2\pi(1 - C_2\alpha/2\pi)}, \quad (1.1)$$

where $r = \dim G$ is a dimension of the gauge group,

$$\text{tr}(T^A T^B) \equiv T(R)\delta^{AB}; \quad (T^A T^A)_i^j \equiv C(R)_i^j; \quad C_2 \equiv T(\text{Adj}). \quad (1.2)$$

Here T^A denotes the generators of the representation in which the chiral matter superfields lie. They should be distinguished from the generators of the fundamental representation denoted by t^A , which are assumed to be normalized by the condition

$$\text{tr}(t^A t^B) = \frac{1}{2}\delta^{AB}. \quad (1.3)$$

From the NSVZ equation we see that the all-loop β -function for the pure $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theory is given by the geometric series [6]. Moreover, it is possible to prove [9, 10] that the finiteness of $\mathcal{N} = 4$ SYM [11–15] and the finiteness of $\mathcal{N} = 2$ supersymmetric gauge theories beyond the one-loop approximation [14–16] also follow from eq. (1.1).¹ Note that due to the $\mathcal{N} = 2$ non-renormalization theorem it is possible to construct $\mathcal{N} = 2$ supersymmetric theories finite in all loops [21] choosing a gauge group and a representation for the matter superfields in such a way that the one-loop β -function vanishes.² The NSVZ relation can also be applied for investigating $\mathcal{N} = 1$ finite theories [25–29] (see [30] for a recent review). In particular, it allows to prove vanishing of the $(L+1)$ -loop β -function for theories finite in L -loops [31, 32] in a simple way [33]. The exact NSVZ β -function is also useful for studying fixed points in $\mathcal{N} = 1$ supersymmetric theories, see, e.g., [34, 35]. Moreover, in softly broken supersymmetric theories the renormalization of the gaugino masses is described by the NSVZ-like relations [36–38].

It is important that the NSVZ equation is valid only for certain (NSVZ) renormalization prescriptions, which constitute a continuous set [39–41]. In particular, it does not hold in the $\overline{\text{DR}}$ -scheme [42–46], when the theory is regularized by dimensional reduction [47] and

¹Note that in the latter case a manifestly $\mathcal{N} = 2$ quantization procedure is needed. It can be constructed with the help of the harmonic superspace [17–19] and an invariant regularization [20].

²Some finiteness preserving terms which break extended supersymmetry have been constructed in [22–24].

divergences are removed by modified minimal subtraction [48]. Also it is not valid in the MOM-scheme [49, 50]. Certainly, it is possible to construct an NSVZ scheme by making specially tuned finite renormalizations in each order of the perturbation theory [42–46], see also [51, 52]. However, there is a simple renormalization prescription giving some NSVZ schemes in all orders. To obtain it, one should regularize a theory by higher covariant derivatives [53, 54] in a supersymmetric way [55, 56]. It is important that this regularization also includes the insertion of the Pauli-Villars determinants for removing one-loop divergences which survive after adding a term with higher derivatives to the action [57]. In the supersymmetric case such a construction can be found in refs. [58, 59]. With the Higher covariant Derivative regularization an NSVZ scheme can be obtained if divergences are removed by Minimal Subtractions of Logarithms or, in other words, in the HD+MSL scheme. In this scheme only powers of $\ln \Lambda/\mu$ are included into renormalization constants, where Λ is the dimensionful cut-off parameter of the regularized theory and μ is the renormalization point [60, 61]. Equivalently, the HD+MSL scheme can be introduced by imposing certain boundary conditions on renormalization constants [62]. Note that minimal subtractions of logarithms can supplement various versions of the higher derivative regularization (which differ in the form of the higher derivative terms and in the Pauli-Villars masses), so that the HD+MSL prescription in general produces a certain set of NSVZ schemes.

The all-loop proof that the HD+MSL prescription gives some NSVZ schemes [63] is based on the all-loop derivation of the NSVZ equation in refs. [63–65], see also ref. [66].³ The NSVZ equation is obtained as follows:

1. According to [64] the triple gauge-ghost vertices (with two ghost external lines and one external line of the quantum gauge superfield) are finite in all orders. (This statement has been verified in the one- and two-loop approximations by explicit calculations made in [64, 72].)
2. Using the finiteness of the triple gauge-ghost vertices the NSVZ β -function can be rewritten in an equivalent form [64], which relates the β -function to the anomalous dimensions of the quantum gauge superfield, of the Faddeev-Popov ghosts, and of the matter superfields,

$$\frac{\beta(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi} \left(3C_2 - T(R) - 2C_2\gamma_c(\alpha, \lambda) - 2C_2\gamma_V(\alpha, \lambda) + \frac{1}{r}C(R)_i^j\gamma_j^i(\alpha, \lambda) \right). \quad (1.4)$$

3. In [65] it was proved that with the higher covariant derivative regularization the integrals giving the β -function are integrals of double total derivatives in the momentum space.⁴

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³Earlier a similar derivation has been made for $\mathcal{N} = 1$ SQED [67], see also [68]. It allowed proving that in this case NSVZ schemes are obtained under the HD+MSL [62] and on-shell [69] prescriptions. In the corresponding softly broken theory the NSVZ-like equation for the renormalization of the photino mass also holds in the HD+MSL scheme [70, 71].

⁴This fact was first noted in calculating the lowest quantum corrections in [73] (the factorization into total derivatives) and [74] (the factorization into double total derivatives). Subsequently, it has been confirmed by numerous multiloop calculations for various supersymmetric theories, see, e.g., [75–81]. It should also be noted that in the case of using dimensional reduction the structure of loop integrals is different [82].

4. The integrals of double total derivatives are nontrivial due to singularities of the integrands, which appear due to the identity

$$\frac{\partial^2}{\partial Q^\mu \partial Q_\mu} \left(\frac{1}{Q^2} \right) = -4\pi^2 \delta^4(Q). \quad (1.5)$$

The sum of these singularities has been calculated in [63]. (The sums of ghost and matter singularities have also been calculated by a different method in ref. [66].) As the result, the NSVZ equation (1.4) is obtained for the renormalization group functions (RGFs) defined in terms of the bare couplings for an *arbitrary* renormalization prescription supplementing the higher covariant derivative regularization.

5. For RGFs defined in terms of the renormalized couplings some NSVZ schemes are obtained using the fact that in the HD+MSL scheme RGFs defined in terms of the bare couplings coincide with the ones defined in terms of the renormalized couplings up to a formal change of the argument [62].

Although the NSVZ equation was usually studied for theories with a single gauge coupling constant, its analogs can be written for theories with multiple gauge couplings (in which gauge groups are given by direct products of simple and/or U(1) factors) [83]. For example, the all-loop equations describing the renormalization of the Minimal Supersymmetric Standard Model (MSSM) couplings written in [84] are actually the integrated form of the NSVZ relations in this case. It was verified [85] that they really agree with the explicit two-loop calculations. The NSVZ and NSVZ-like relations were applied in MSSM and its extensions for investigating various quantum corrections [86, 87], although for obtaining the higher-loop results in the $\overline{\text{DR}}$ -scheme a proper tuning of a subtraction scheme was certainly needed. Also the NSVZ equations were used for studying the reduction of couplings in theories with multiple gauge couplings, see [30, 88, 89] and references therein. In refs. [90, 91] the equation similar to the Abelian NSVZ relation [92, 93] was written for the Adler D -function [94] in $\mathcal{N} = 1$ supersymmetric quantum chromodynamics (SQCD) interacting with the Abelian gauge superfield. (The theory considered in [90, 91] has the gauge group $\text{SU}(N) \times \text{U}(1)$ and, therefore, two gauge coupling constants.) Thus, the NSVZ equations are important for phenomenologically interesting theories with multiple gauge couplings and deserve a more detailed consideration in this case. In particular, it is necessary to derive them using the methods of the perturbation theory and specify renormalization prescriptions under which they are valid. In this paper we start this investigation considering as examples MSSM (see, e.g., [1]) and the flipped SU(5) Grand Unification Theory (GUT) [95–98].

The paper is organized as follows. In section 2 we recall a form of the NSVZ relations for theories with multiple gauge couplings. In the next section 3 these equations are written for $\mathcal{N} = 1$ SQCD interacting with the $\mathcal{N} = 1$ supersymmetric Abelian gauge superfield, MSSM, and the flipped SU(5) model. The scheme dependence of the NSVZ relations for theories with multiple gauge couplings is investigated in section 4. We present equations describing how they change under finite renormalizations of couplings and matter superfields, and construct finite renormalizations which transfer an NSVZ scheme into

another NSVZ scheme. In particular, this implies that the NSVZ schemes constitute a continuous sets. In theories with a single gauge coupling some schemes of this class in all loops are obtained with the help of the HD+MSL prescription. That is why in section 5 we explain how one can regularize theories with multiple gauge couplings by higher covariant derivatives. The general construction is illustrated by the examples of MSSM and the flipped SU(5) model. This regularization can be used a starting point for the all-loop derivation of the NSVZ equations for the theories under consideration. In theories with a single gauge coupling such a derivation produces a new form of the NSVZ equation (1.4). Its generalization to the case of multiple gauge couplings is constructed in section 6 using the extension of the non-renormalization theorem for the triple gauge-ghost vertices, which is also derived in this section.

2 The NSVZ equations for theories with multiple gauge couplings

In this paper we will consider a general renormalizable $\mathcal{N} = 1$ supersymmetric theory with the gauge group

$$G = G_1 \times G_2 \times \dots \times G_n, \quad (2.1)$$

where for each $K = 1, \dots, n$ the subgroup G_K is either a simple compact group or U(1). The chiral matter superfields can be split into sets (numerated by the index a) in such a way that each of these sets transforms under certain irreducible representations R_{aK} of the simple subgroups G_K or has certain charges q_{aK} with respect to $G_K = \text{U}(1)$. This implies that for a fixed a the matter superfields are also numerated by indices corresponding to various subgroups G_K . It is convenient to introduce the index i numerating all matter superfields ϕ_i as the set

$$i = \{a; i_1, i_2, \dots, i_n\} \equiv \{a; i_a\}. \quad (2.2)$$

It is important that for different values of a the sets of $i_1, \dots, i_n$ values are in general different.

In this notation the generators of the gauge group G corresponding to the subgroup G_K can be written as

$$\left(T^{A_K}\right)_i^j = \delta_a^b \cdot \delta_{i_1}^{j_1} \dots \delta_{i_{K-1}}^{j_{K-1}} \left(T_a^{A_K}\right)_{i_K}^{j_K} \delta_{i_{K+1}}^{j_{K+1}} \dots \delta_{i_n}^{j_n}, \quad (2.3)$$

where $\left(T_a^{A_K}\right)_{i_K}^{j_K}$ are either the generators of G_K in the representation R_{aK} for simple subgroups G_K or the charges of the superfields $\phi_i \equiv \phi_{a; i_1 i_2 \dots i_n}$ with respect to $G_K = \text{U}(1)$. Note that we always assume that the generators are normalized by the conditions

$$\text{tr} \left(t^{A_K} t^{B_K} \right) = \frac{1}{2} \delta^{A_K B_K} \quad \text{for simple subgroups;} \quad (2.4)$$

$$\left(T^{A_K}\right)_{i_K}^{j_K} \rightarrow q_K \quad \text{for U(1) subgroups,} \quad (2.5)$$

where t^{A_K} are generators of the fundamental representation of a simple subgroup G_K .

$\mathcal{N} = 1$ supersymmetric theories with the gauge group (2.1) in the massless limit are described by the action

$$S = \sum_{K=1}^n \text{Re} \frac{1}{4} \int d^4x d^2\theta (W^a)^{A_K} (W_a)^{A_K} + \frac{1}{4} \int d^4x d^4\theta \phi^{*i} (e^{2V})_i^j \phi_j + \left(\frac{1}{6} \lambda_0^{ijk} \int d^4x d^2\theta \phi_i \phi_j \phi_k + \text{c.c.} \right), \quad (2.6)$$

which is written in terms of $\mathcal{N} = 1$ superfields. Such a formulation is especially convenient, because in this case $\mathcal{N} = 1$ supersymmetry is a manifest symmetry of the theory. Using a proper regularization and quantization procedure [99–101], $\mathcal{N} = 1$ supersymmetry can also be made manifest at all steps of calculating quantum corrections.

In eq. (2.6) V denotes the gauge superfield. In the matter part of the action it is given by the expression

$$V_i^j = \sum_{K=1}^n e_{0K} V^{A_K} (T^{A_K})_i^j, \quad (2.7)$$

where the bare gauge coupling constants are denoted by e_{0K} . Below we will also use the notation

$$V_K \equiv \begin{cases} e_{0K} V^{A_K} t^{A_K} & (\text{or } e_{0K} V^{A_K} T^{A_K}) & \text{if } G_K \text{ is non-Abelian;} \\ V^{A_K} & & \text{if } G_K = \text{U}(1). \end{cases} \quad (2.8)$$

The gauge superfield strengths corresponding to various subgroups G_K in the direct product (2.1) are defined as

$$e_{0K} (W_a)^{A_K} (t^{A_K})_{i_k}^{j_k} \equiv \frac{1}{8} \bar{D}^2 \left[\exp \left(-2e_{0K} V^{A_K} t^{A_K} \right) D_a \exp \left(2e_{0K} V^{A_K} t^{A_K} \right) \right]_{i_k}^{j_k} \quad \text{for simple subgroups;} \quad (2.9)$$

$$(W_a)^{A_K} \equiv \frac{1}{4} \bar{D}^2 D_a V^{A_K} \quad \text{for U}(1) \text{ subgroups.} \quad (2.10)$$

All expressions $(W^a)^{A_K} (W_a)^{A_K}$ are invariant under gauge transformations of the whole group G , so that there are n invariants of this structure. Therefore, the theory (2.6) has n gauge coupling constants $\alpha_K = e_K^2/4\pi$, each of them corresponding to a certain subgroup of the group G . In terms of the renormalized couplings the corresponding n β -functions are defined by the equations

$$\beta_K(\alpha, \lambda) \equiv \left. \frac{d\alpha_K}{d \ln \mu} \right|_{\alpha_0, \lambda_0 = \text{const}}, \quad (2.11)$$

where the subscript 0 marks the bare gauge and Yukawa couplings, and μ stands for the renormalization point.

Taking into account the chiral matter superfields ϕ_a belong to irreducible representations of all simple subgroups G_K , the renormalization constants for them have no indices,

$$(\phi_{a,R})_{i_1 i_2 \dots i_K} = (Z_a)^{1/2} (\phi_a)_{i_1 i_2 \dots i_K}, \quad (2.12)$$

where the subscript R denotes the renormalized superfields. In terms of the renormalized coupling constant the corresponding anomalous dimensions are defined by the equation

$$\gamma_a(\alpha, \lambda) \equiv \left. \frac{d \ln Z_a}{d \ln \mu} \right|_{\alpha_0, \lambda_0 = \text{const}}. \quad (2.13)$$

Earlier we denoted the generators of the representation R_{aK} by T_a^{AK} . This implies that

$$(T_a^{AK} T_a^{AK})_{i_K}{}^{j_K} = C(R_{aK}) \delta_{i_K}{}^{j_K}. \quad (2.14)$$

It is important that the right hand side of this equation is proportional to the δ -symbol because, by construction, the representation R_{aK} is irreducible. (If $G_K = \text{U}(1)$, then $T_a^{AK} \rightarrow q_{aK}$ are numbers.) Also it is useful to define the analog of $T(R)$ with the help of the equation

$$T_K(R_{aK}) \delta^{AKBK} \equiv (T_a^{AK} T_a^{BK})_{i_K}{}^{i_K}. \quad (2.15)$$

Then the matter contribution to the one-loop β -function corresponding to the coupling constant α_K will be proportional to

$$\mathbf{T}_K(R) = \sum_a \mathbf{T}_{aK}, \quad (2.16)$$

where we introduced the notation

$$\mathbf{T}_{aK} = \begin{cases} \delta_{i_1}{}^{i_1} \dots \delta_{i_{K-1}}{}^{i_{K-1}} T_K(R_{aK}) \delta_{i_{K+1}}{}^{i_{K+1}} \dots \delta_{i_n}{}^{i_n} & \text{if } G_K \text{ is simple;} \\ \delta_{i_1}{}^{i_1} \dots \delta_{i_{K-1}}{}^{i_{K-1}} q_{aK}^2 \delta_{i_{K+1}}{}^{i_{K+1}} \dots \delta_{i_n}{}^{i_n} & \text{if } G_K = \text{U}(1). \end{cases} \quad (2.17)$$

Really, calculating the function β_K in the one-loop approximation it is necessary to sum over all indices of chiral superfields corresponding to all subgroups except for G_K , while the trace of two generators of the subgroup G_K gives the factor (2.15).

It is reasonable to suggest (see, e.g., [83]) that the exact β -function for a coupling constant corresponding to a factor G_K in the gauge group G is given by the expression

$$\frac{\beta_K(\alpha, \lambda)}{\alpha_K^2} = -\frac{1}{2\pi(1 - C_2(G_K)\alpha_K/2\pi)} \left[3C_2(G_K) - \sum_a \mathbf{T}_{aK} (1 - \gamma_a(\alpha, \lambda)) \right]. \quad (2.18)$$

(Note that this equation is also valid even if the subgroup G_K coincides with $\text{U}(1)$. Certainly, in this case it is necessary to set $C_2(G_K) = 0$.) Really, if G is a simple group, then the representation to which the matter superfields belong can be presented as a direct sum of R_a , so that

$$\begin{aligned} R &= \sum_a R_a; & \mathbf{T}_{aK} &\rightarrow T(R_a) \equiv \mathbf{T}_a; & T(R) &= \sum_a T(R_a); \\ \frac{1}{r} C(R)_i{}^j \gamma_j{}^i(\alpha, \lambda) &= \frac{1}{r} \sum_a C(R_a) \gamma_a(\alpha, \lambda) \delta_i{}^i = \sum_a T(R_a) \gamma_a(\alpha, \lambda). \end{aligned} \quad (2.19)$$

Therefore, the NSVZ relation (1.1) in this case can be written in the form

$$\frac{\beta(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi(1 - C_2\alpha/2\pi)} \left[3C_2 - \sum_a \mathbf{T}_a (1 - \gamma_a(\alpha, \lambda)) \right], \quad (2.20)$$

which is similar to eq. (2.18). Thus, eq. (2.18) is a natural generalization of eq. (1.1) to the case of $\mathcal{N} = 1$ supersymmetric theories with multiple gauge couplings.

3 Examples

3.1 $\mathcal{N} = 1$ SQCD + SQED

As the simplest example we consider $\mathcal{N} = 1$ SQCD with N_f flavors interacting with the electromagnetic field in the supersymmetric way. This theory is based on the group $G \times U(1)$ and contains N_f quark flavors in a certain irreducible representation R of the group G . (For “true” SQCD $G = SU(3)$, $N_f = 6$, and R is the fundamental representation.) Each flavor is composed from two chiral matter superfields ϕ_a and $\tilde{\phi}_a$ in the representations R and $\bar{R}$, respectively, with the opposite $U(1)$ charges q_a and $-q_a$. At the classical level the action in the massless limit is given by the expression

$$S = \frac{1}{2g^2} \text{Re tr} \int d^4x d^2\theta W^a W_a + \frac{1}{4e^2} \text{Re} \int d^4x d^2\theta \mathbf{W}^a \mathbf{W}_a + \sum_{a=1}^{N_f} \frac{1}{4} \int d^4x d^4\theta \left(\phi_a^+ e^{2V+2q_a \mathbf{V}} \phi_a + \tilde{\phi}_a^+ e^{-2V-2q_a \mathbf{V}} \tilde{\phi}_a \right), \quad (3.1)$$

where V and $\mathbf{V}$ are the gauge superfields corresponding to the subgroups G and $U(1)$, respectively, and the supersymmetric gauge superfield strengths are defined as

$$W_a = \frac{1}{8} \bar{D}^2 \left(e^{-2V} D_a e^{2V} \right); \quad \mathbf{W}_a = \frac{1}{4} \bar{D}^2 D_a \mathbf{V}. \quad (3.2)$$

Evidently, there are two coupling constants in the theory. The analog of the strong coupling constant is $\alpha_s \equiv g^2/4\pi$, while the analog of the electromagnetic coupling constant is $\alpha \equiv e^2/4\pi$. We will denote the corresponding β -functions by $\beta_s(\alpha_s, \alpha)$ and $\beta(\alpha_s, \alpha)$, respectively.

To construct NSVZ expressions for both these β functions, we need the values $\mathbf{T}_{aK}$ defined by eq. (2.17). It is easy to see that

$$\mathbf{T}_{\phi_a, G} = \mathbf{T}_{\tilde{\phi}_a, G} = T(R); \quad \mathbf{T}_{\phi_a, U(1)} = \mathbf{T}_{\tilde{\phi}_a, U(1)} = q_a^2 \dim R. \quad (3.3)$$

Substituting these expressions into eq. (2.18) and using the notation $C_2 = C_2(G)$ we obtain the exact expressions for the β -functions corresponding to the strong and electromagnetic coupling constants,

$$\frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1 - C_2\alpha_s/2\pi)} \left[3C_2 - 2T(R) \sum_{a=1}^{N_f} (1 - \gamma_a(\alpha_s, \alpha)) \right]; \quad (3.4)$$

$$\frac{\beta(\alpha_s, \alpha)}{\alpha^2} = \frac{1}{\pi} \dim R \sum_{a=1}^{N_f} q_a^2 (1 - \gamma_a(\alpha_s, \alpha)). \quad (3.5)$$

Certainly, they are valid only under certain renormalization prescriptions, which will be discussed below and presumably include the HD+MSL scheme.

Earlier the theory (3.1) was considered in refs. [90, 91] (see also [102]), where the corrections to the electromagnetic coupling constant generated by loops of quarks, gluons, and their superpartners were analyzed. These corrections are encoded in the Adler

D -function [94], for which an all-order expression has been obtained. The D -function calculated in [90, 91] is very similar to the β -function for the electromagnetic coupling constant. Up to a normalization factor, the Adler function can be obtained from the function $\beta(\alpha_s, \alpha)/\alpha^2$ by taking the limit $\alpha \rightarrow 0$,

$$D(\alpha_s) = \frac{3\pi}{2} \lim_{\alpha \rightarrow 0} \frac{\beta(\alpha_s, \alpha)}{\alpha^2} = \frac{3}{2} \dim R \sum_{a=1}^{N_f} q_a^2 \left(1 - \lim_{\alpha \rightarrow 0} \gamma_a(\alpha_s, \alpha)\right). \quad (3.6)$$

Note that in the limit $\alpha \rightarrow 0$ all chiral matter superfields have the same anomalous dimension,

$$\lim_{\alpha \rightarrow 0} \gamma_a(\alpha_s, \alpha) = \gamma(\alpha_s). \quad (3.7)$$

Really, the anomalous dimensions of various flavors are different because the electromagnetic charges q_a are different for different flavors. In the limit $\alpha \rightarrow 0$ it is necessary to omit all superdiagrams containing propagators of the electromagnetic gauge superfield $\mathbf{V}$, so that the dependence on the electromagnetic charges disappears. Therefore, using eq. (3.7) the exact Adler function (3.6) can be rewritten in the form

$$D(\alpha_s) = \frac{3}{2} \dim R \sum_{a=1}^{N_f} q_a^2 \left(1 - \gamma(\alpha_s)\right). \quad (3.8)$$

This expression reproduces the result of refs. [90, 91] in the particular case when $G = \text{SU}(N)$ and R is the fundamental representation (so that $\dim R = N$).⁵ Also eq. (3.8) agrees with the result of ref. [102] if we take into account that for an irreducible representation $\gamma_i^j = \delta_i^j \cdot \gamma$. A detailed analysis under what renormalization prescriptions eq. (3.8) is valid has been done in ref. [52]. In particular, it holds in the HD+MSL scheme [102].

3.2 NSVZ relations for MSSM

As another illustration, we consider NSVZ equations for MSSM. This model is a softly broken $\mathcal{N} = 1$ supersymmetric theory with the gauge group

$$G = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y. \quad (3.9)$$

Certainly, this implies that there are three gauge coupling constants

$$\alpha_3 = \frac{e_3^2}{4\pi}; \quad \alpha_2 = \frac{e_2^2}{4\pi}; \quad \alpha_1 = \frac{5}{3} \cdot \frac{e_1^2}{4\pi}. \quad (3.10)$$

Due to the factor $5/3$ in the definition of α_1 the gauge coupling unification condition is written in the simple form $\alpha_1 = \alpha_2 = \alpha_3$. Actually, this factor encodes the value of the Weinberg angle at the scale of Grand Unification, $\sin^2 \theta_W = 3/8$.

The chiral matter superfields of MSSM include three generations of quarks and leptons and two Higgs doublets

$$H_u = \begin{pmatrix} H_{u1} \\ H_{u2} \end{pmatrix}; \quad H_d = \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix}. \quad (3.11)$$

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⁵In the two-loop approximation it gives the expression coinciding with the one obtained in ref. [103].

a	Q_1, Q_2, Q_3	U_1, U_2, U_3	D_1, D_2, D_3	L_1, L_2, L_3	E_1, E_2, E_3	H_u	H_d
SU(3)	$\bar{3}$	3	3	1	1	1	1
SU(2)	2	1	1	2	1	2	2
U(1) _Y	-1/6	2/3	-1/3	1/2	-1	-1/2	1/2

Table 1. Chiral matter superfields in MSSM and their quantum numbers with respect to $SU(3) \times SU(2) \times U(1)$. The subscripts 1, 2, and 3 numerate generations.

a	Q_1, Q_2, Q_3	U_1, U_2, U_3	D_1, D_2, D_3	L_1, L_2, L_3	E_1, E_2, E_3	H_u	H_d
SU(3)	1	1/2	1/2	0	0	0	0
SU(2)	3/2	0	0	1/2	0	1/2	1/2
U(1) _Y	1/6	4/3	1/3	1/2	1	1/2	1/2

Table 2. Values of T_{aK} for all MSSM superfields. The index a numerates superfields, and $K = SU(3), SU(2), U(1)$.

The model under consideration does not contain right neutrinos. This is not essential for us, because they acquire large masses of the GUT scale order and do not affect running of couplings at low energies. Therefore, each generation includes the chiral superfields U_a, D_a, E ,

$$Q^a = \begin{pmatrix} \tilde{U}^a \\ \tilde{D}^a \end{pmatrix}, \quad \text{and} \quad L = \begin{pmatrix} \tilde{N} \\ \tilde{E} \end{pmatrix}. \quad (3.12)$$

The right upper quarks, right down quarks, and right charged leptons are components of U_a, D_a , and E , respectively. The superfields U_a and D_a lie in the fundamental representation of $SU(3)$, while E -s are $SU(3)$ singlets. All superfields containing right quarks and leptons are $SU(2)$ singlets. The (charge conjugated) left quarks and leptons are components of Q^a and L , respectively. These superfields transform under the fundamental representation of $SU(2)$. With respect to $SU(3)$ the superfields Q^a and L belong to the antifundamental and trivial representations, respectively. All chiral matter superfields nontrivially transform under $U(1)$. The corresponding charges (i.e., hypercharges) are presented in table 1 together with the $SU(3)$ and $SU(2)$ representation. The subscripts 1, 2, and 3 in this table correspond to the generations. The index a numerates the chiral matter superfields in MSSM that transform under irreducible representations of $SU(3)$ and $SU(2)$.

Values of T_{aK} for all a and $K = SU(3), SU(2), U(1)$ calculated according to the definition (2.17) are presented in table 2. Note that for quarks and leptons they correspond to the superfields of a single generation. Using these values it is easy to calculate the (well-known) coefficients which are present in the one-loop MSSM β -functions,

$$3C_2(SU(3)) - T_{SU(3)}(R) = 9 - 3\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 3; \quad (3.13)$$

$$3C_2(SU(2)) - T_{SU(2)}(R) = 6 - \left[3\left(\frac{3}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2}\right] = -1; \quad (3.14)$$

$$-T_{U(1)}(R) = -\left[3\left(\frac{1}{6} + \frac{4}{3} + \frac{1}{3} + \frac{1}{2} + 1\right) + \frac{1}{2} + \frac{1}{2}\right] = -11. \quad (3.15)$$

However, we are interested in exact all-order expressions for the β -functions, which can be constructed with the help of eq. (2.18). Using the data of table 2 from this equation we conclude that they can be written as

$$\frac{\beta_3(\alpha, \lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1 - 3\alpha_3/2\pi)} \left[3 + \sum_{I=1}^3 \left(\gamma_{Q_I}(\alpha, \lambda) + \frac{1}{2}\gamma_{U_I}(\alpha, \lambda) + \frac{1}{2}\gamma_{D_I}(\alpha, \lambda) \right) \right]; \quad (3.16)$$

$$\begin{aligned} \frac{\beta_2(\alpha, \lambda)}{\alpha_2^2} = & -\frac{1}{2\pi(1 - \alpha_2/\pi)} \left[-1 + \sum_{I=1}^3 \left(\frac{3}{2}\gamma_{Q_I}(\alpha, \lambda) + \frac{1}{2}\gamma_{L_I}(\alpha, \lambda) \right) \right. \\ & \left. + \frac{1}{2}\gamma_{H_u}(\alpha, \lambda) + \frac{1}{2}\gamma_{H_d}(\alpha, \lambda) \right]; \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{\beta_1(\alpha, \lambda)}{\alpha_1^2} = & -\frac{3}{5} \cdot \frac{1}{2\pi} \left[-11 + \sum_{I=1}^3 \left(\frac{1}{6}\gamma_{Q_I}(\alpha, \lambda) + \frac{4}{3}\gamma_{U_I}(\alpha, \lambda) + \frac{1}{3}\gamma_{D_I}(\alpha, \lambda) \right) \right. \\ & \left. + \frac{1}{2}\gamma_{L_I}(\alpha, \lambda) + \gamma_{E_I}(\alpha, \lambda) \right] + \frac{1}{2}\gamma_{H_u}(\alpha, \lambda) + \frac{1}{2}\gamma_{H_d}(\alpha, \lambda), \end{aligned} \quad (3.18)$$

where the index I numerates generations. Note that the factor $3/5$ in the last equation appears due to the factor $5/3$ in the definition of α_1 , see eq. (3.10). Certainly, eqs. (3.16)–(3.18) are in agreement with the analogous equations written in [84] in a different form.

For completeness, in appendix A.1 we verify that these equations reproduce the correct two-loop expressions for the MSSM β -functions (which was first made in [85] starting from the result of ref. [84]). For this purpose in appendix A.1 we listed the one-loop contributions to the anomalous dimensions entering these equations and substitute them into the NSVZ relations. The result of this calculation exactly coincides with the known expressions for the two-loop β -functions, see, e.g., [104]. Certainly, the coincidence occurs due to the scheme independence of the two-loop contributions to the gauge β -functions, which will be discussed below in section 4.1.

3.3 NSVZ relations for the flipped SU(5) model

As another interesting example we consider the so-called flipped SU(5) model [95–98]. This model is based on the gauge group $SU(5) \times U(1)$ and, therefore, contains two coupling constants. We will denote them by α_5 and α_1 . Certainly, the normalization of α_1 should be fixed by a certain condition, which will be described below.

The matter content of the flipped SU(5) theory includes three generations of quarks and leptons which are placed into the representations

$$3 \times \left(\overline{10}(1) + 5(-3) + 1(5) \right). \quad (3.19)$$

In our notation the first symbol denotes an SU(5) representation, while a U(1) charge is written in brackets. Unlike the usual SU(5) model, in the flipped SU(5) model the charged leptons are singlets with respect to SU(5), the representation $\overline{10}$ contains D and N (instead

of U and E , respectively), and the representation 5 contains U (instead of D),

$$\overline{10}^{ij} \sim \begin{pmatrix} 0 & D_3 & -D_2 & \tilde{U}^1 & \tilde{D}^1 \\ -D_3 & 0 & D_1 & \tilde{U}^2 & \tilde{D}^2 \\ D_2 & -D_1 & 0 & \tilde{U}^3 & \tilde{D}^3 \\ -\tilde{U}^1 & -\tilde{U}^2 & -\tilde{U}^3 & 0 & N \\ -\tilde{D}^1 & -\tilde{D}^2 & -\tilde{D}^3 & -N & 0 \end{pmatrix}; \quad 5_i \sim \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \tilde{E} \\ -\tilde{N} \end{pmatrix}; \quad 1 \sim E. \quad (3.20)$$

The $SU(5) \times U(1)$ symmetry can be broken down to $SU(3) \times SU(2) \times U(1)_Y$ by the vacuum expectation values of two chiral Higgs superfields H and $\tilde{H}$ in the representations $10(-1)$ and $\overline{10}(1)$, respectively. The group $U(1)_Y$ is obtained as a superposition of the $SU(5)$ transformations with

$$\omega_5 = \exp \left\{ \frac{i\alpha_Y}{30} \begin{pmatrix} 2 \cdot 1_3 & 0 \\ 0 & -3 \cdot 1_2 \end{pmatrix} \right\} \quad (3.21)$$

(where 1_3 and 1_2 are 3×3 and 2×2 identity matrices, respectively) and the $U(1)$ transformations with $\omega_1 = \exp(-iq\alpha_Y/5)$, where q is a $U(1)$ charge of the corresponding superfield normalized as in eq. (3.19). Also it is necessary to include Higgs superfields responsible for breaking $SU(2) \times U(1)_Y$ down to $U(1)_{\text{em}}$. This is done by introducing two chiral superfields h and $\tilde{h}$ in the representations $5(2)$ and $\bar{5}(-2)$, respectively. Moreover, (for $N_G = 3$ generations) the model includes $N_G + 1 = 4$ chiral superfields ϕ_R which are singlets with respect to $SU(5)$ and have the $U(1)$ charges equal to 0. The superpotential of the resulting theory is presented in appendix A.2.

Note that the normalization of the $U(1)$ charge adopted in eq. (3.19) is not convenient for physical applications. To construct a more convenient prescription, we recall that the considered model can naturally be embedded into the $SO(10)$ theory, which has a single coupling constant. Therefore, a better requirement for fixing a normalization of α_1 is obtained by the requirement that the gauge couplings are unified as $\alpha_1 = \alpha_5$. To find the relation between these two different normalization conditions, we note that the generators of the $SU(5)$ subgroup embedded into $SO(10)$ and normalized by eq. (1.3) are

$$\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & t_r^A \\ -t_r^A & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} T(t_r^A); \quad \frac{1}{\sqrt{2}} \begin{pmatrix} t_i^A & 0 \\ 0 & t_i^A \end{pmatrix} = \frac{1}{\sqrt{2}} T(t_i^A), \quad (3.22)$$

In this equation t_r^A and t_i^A (with $A = 1, \dots, 12$) denote real and purely imaginary generators of the $SU(5)$ fundamental representation, respectively. This implies that $e_5 = e_{10}/\sqrt{2}$. The generator of the $U(1)$ subgroup embedded into $SO(10)$, again, normalized by eq. (1.3) is written as

$$-\frac{i}{\sqrt{20}} \begin{pmatrix} 0 & -1_5 \\ 1_5 & 0 \end{pmatrix}, \quad (3.23)$$

where 1_5 is the 5×5 identity matrix. The image of this generator in the spinor representation (see, e.g., [1]) multiplied by e_{10} acting on the $SU(5)$ representations $\overline{10}$, 5, and 1 gives

$$\frac{e_{10}}{\sqrt{80}} = \frac{e_5}{\sqrt{40}}, \quad -\frac{3e_{10}}{\sqrt{80}} = -\frac{3e_5}{\sqrt{40}}, \quad \text{and} \quad \frac{5e_{10}}{\sqrt{80}} = \frac{5e_5}{\sqrt{40}}, \quad (3.24)$$

a	$\overline{10}_1, \overline{10}_2, \overline{10}_3$	$5_1, 5_2, 5_3$	E_1, E_2, E_3	$H, \tilde{H}$	$h, \tilde{h}$	$\phi_1, \phi_2, \phi_3, \phi_4$
SU(5)	3/2	1/2	0	3/2	1/2	0
U(1)	10	45	25	10	20	0

Table 3. Values of $\mathbf{T}_{aK}$ (where $K = \text{SU}(5), \text{U}(1)$) for the chiral matter superfields of the flipped SU(5) model.

respectively. This implies that if normalize α_1 by the condition $\alpha_1 = \alpha_5$ for the SO(10) symmetric theory, then the U(1) coupling constant should be defined as

$$\alpha_1 \equiv 40 \cdot \frac{e_1^2}{4\pi} \quad (3.25)$$

provided the U(1) charges in units of e_1 are given by eq. (3.19).

The coefficients $\mathbf{T}_{aK}$ for the flipped SU(5) model are presented in table 3. Note that for the representation $\overline{10}$ the generators are written as

$$\left(T_{\overline{10}}^A\right)^{ij}_{kl} = -\frac{1}{2} \left[\delta_k^i (t^A)_l^j - \delta_k^j (t^A)_l^i - \delta_l^i (t^A)_k^j + \delta_l^j (t^A)_k^i \right], \quad (3.26)$$

where $(t^A)_i^j$ are the generators of the fundamental representation, so that

$$\text{tr} \left(T_{\overline{10}}^A T_{\overline{10}}^B \right) = \left(T_{\overline{10}}^A \right)^{ij}_{kl} \left(T_{\overline{10}}^B \right)^{kl}_{ij} = 3 \text{tr} (t^A t^B) = \frac{3}{2} \delta^{AB}. \quad (3.27)$$

Certainly, the number 3/2 here and similar numbers for other representations are well-known (see, e.g., [105]), but the explicit calculation is more visual.

Substituting the values $\mathbf{T}_{aK}$ presented in table 3 into eq. (2.18) and taking into account the factor 40 in eq. (3.25) we obtain the exact β -functions for the flipped SU(5) model in the form

$$\begin{aligned} \frac{\beta_5(\alpha, \lambda)}{\alpha_5^2} = & -\frac{1}{2\pi(1 - 5\alpha_5/2\pi)} \left[5 + \sum_{I=1}^3 \left(\frac{3}{2} \gamma_{\overline{10}_I}(\alpha, \lambda) + \frac{1}{2} \gamma_{5_I}(\alpha, \lambda) \right) + \frac{3}{2} \gamma_H(\alpha, \lambda) \right. \\ & \left. + \frac{3}{2} \gamma_{\tilde{H}}(\alpha, \lambda) + \frac{1}{2} \gamma_h(\alpha, \lambda) + \frac{1}{2} \gamma_{\tilde{h}}(\alpha, \lambda) \right], \end{aligned} \quad (3.28)$$

$$\begin{aligned} \frac{\beta_1(\alpha, \lambda)}{\alpha_1^2} = & \frac{1}{8} \cdot \frac{1}{2\pi} \left[60 - \sum_{I=1}^3 \left(2\gamma_{\overline{10}_I}(\alpha, \lambda) + 9\gamma_{5_I}(\alpha, \lambda) + 5\gamma_{E_I}(\alpha, \lambda) \right) - 2\gamma_H(\alpha, \lambda) \right. \\ & \left. - 2\gamma_{\tilde{H}}(\alpha, \lambda) - 4\gamma_h(\alpha, \lambda) - 4\gamma_{\tilde{h}}(\alpha, \lambda) \right]. \end{aligned} \quad (3.29)$$

(Certainly, these equations are written for the region with unbroken $\text{SU}(5) \times \text{U}(1)$ symmetry, i.e., above all thresholds.)

In the one-loop approximation eqs. (3.28) and (3.29) exactly agree with the results of ref. [98]. For calculating the two-loop contributions to the β -function we need to know the one-loop expressions for the anomalous dimensions present in these equations. They

are collected in appendix A.2. Substituting them into the exact NSVZ relations after some transformations we obtain the two-loop β -functions, which are also presented in appendix A.2. Their parts which do not contain Yukawa couplings agree with the corresponding expressions presented in [98]. (Unfortunately, we did not find the expressions for the parts containing the Yukawa couplings for this model.) Therefore, we obtain a nontrivial evidence that the NSVZ equations are also valid for theories with multiple gauge couplings.

4 NSVZ relations and the ambiguity of choosing a renormalization prescription

4.1 Scheme dependence of NSVZ relations for theories with multiple gauge couplings

It is well known [42–46] that the NSVZ relation does not hold for an arbitrary renormalization prescription, and even the most popular $\overline{\text{DR}}$ scheme is not NSVZ. The scheme dependence of the NSVZ equation was analysed in [106] for theories with a single (gauge) coupling constant. For theories with simple gauge groups containing Yukawa couplings a similar analysis has been done in [49]. In this paper we extend these results to the case of theories with multiple gauge couplings.

It is known [107] that various renormalization schemes are related by finite renormalizations, which can be written in the form

$$\alpha'_K = \alpha'_K(\alpha, \lambda); \quad \lambda' = \lambda'(\alpha, \lambda); \quad Z'_a(\alpha', \lambda', \ln \Lambda/\mu) = z_a(\alpha, \lambda) Z_a(\alpha, \lambda, \ln \Lambda/\mu), \quad (4.1)$$

where $\alpha'(\alpha, \lambda)$, $\lambda'(\alpha, \lambda)$, and $z_a(\alpha, \lambda)$ are finite functions. Note that here we will not assume that the finite renormalization of Yukawa couplings is related to the finite renormalization of matter superfields.

Under the finite renormalization (4.1) the β -functions change as [108, 109]

$$\beta'_K(\alpha', \lambda') = \frac{\partial \alpha'_K}{\partial \alpha_L} \beta_L(\alpha, \lambda) + \frac{\partial \alpha'_K}{\partial \lambda^{ijk}} (\beta_\lambda)^{ijk}(\alpha, \lambda) + \frac{\partial \alpha'_K}{\partial \lambda_{ijk}^*} (\beta_\lambda^*)_{ijk}(\alpha, \lambda), \quad (4.2)$$

where the summation over the index L is performed from 1 to n (see eq. (2.1)), and the Yukawa β -function is defined as

$$(\beta_\lambda)^{ijk}(\alpha, \lambda) \equiv \left. \frac{d\lambda^{ijk}}{d \ln \mu} \right|_{\alpha_0, \lambda_0 = \text{const}}. \quad (4.3)$$

For theories with a single gauge coupling the two-loop β -function is scheme independent [109]. For multicharge theories containing Yukawa couplings the Yukawa β -functions are scheme-dependent starting from the two-loop approximation. However, the gauge β -functions even for theories with multiple gauge couplings are scheme-independent in the two-loop approximation [104]. To prove this statement carefully, we should take into account that finite renormalizations should have the same structure as the corresponding quantum corrections [42, 50, 110]. In particular, this implies that the new gauge couplings α'_K are related to the original ones by the equations

$$\frac{1}{\alpha'_K} = \frac{1}{\alpha_K} + C_K + O(\alpha, \lambda^2), \quad (4.4)$$

where C_K are some constants. Consequently, the lowest term in the expression for α'_K contains only α_K^2 ,

$$\alpha'_K = \alpha_K - C_K \alpha_K^2 + O(\alpha^3, \alpha^2 \lambda^2). \quad (4.5)$$

Substituting this expression into eq. (4.2) and taking into account that in the one-loop approximation $\beta_K \sim \alpha_K^2$ and that the one-loop expression for $(\beta_\lambda)^{ijk}$ is given by a sum of terms proportional to $\alpha\lambda$ and λ^3 we obtain

$$\begin{aligned} \beta'_K \left(\alpha_K - C_K \alpha_K^2 + O(\alpha^3, \alpha^2 \lambda^2), \lambda + O(\alpha\lambda, \lambda^3) \right) \\ = (1 - 2C_K \alpha_K) \beta_K(\alpha, \lambda) + O(\alpha^4, \alpha^3 \lambda^2, \alpha^2 \lambda^4). \end{aligned} \quad (4.6)$$

In the lowest approximation each β -function can be presented in the form

$$\beta_K(\alpha, \lambda) = \frac{\alpha_K^2}{\pi} \beta_{0K} + \beta_{2\text{-loop}, K}(\alpha, \lambda) + O(\alpha^4, \alpha^3 \lambda^2, \alpha^2 \lambda^4), \quad (4.7)$$

where β_{0K} are numerical constants and $\beta_{2\text{-loop}, K}(\alpha, \lambda)$ contains terms proportional to α^3 and $\alpha^2 \lambda^2$. Substituting the expression (4.7) into eq. (4.6) we see that both sides of the resulting equation have the same dependence on the constants C_K ,

$$\begin{aligned} \frac{\alpha_K^2}{\pi} \beta_{0K} (1 - 2C_K \alpha_K) + \beta'_{2\text{-loop}, K}(\alpha, \lambda) + O(\alpha^4, \alpha^3 \lambda^2, \alpha^2 \lambda^4) \\ = \frac{\alpha_K^2}{\pi} \beta_{0K} (1 - 2C_K \alpha_K) + \beta_{2\text{-loop}, K}(\alpha, \lambda) + O(\alpha^4, \alpha^3 \lambda^2, \alpha^2 \lambda^4). \end{aligned} \quad (4.8)$$

From this equation we conclude that the two-loop gauge β -functions are scheme-independent even for theories with multiple gauge couplings,⁶

$$\beta'_{2\text{-loop}, K}(\alpha, \lambda) = \beta_{2\text{-loop}, K}(\alpha, \lambda). \quad (4.9)$$

Under the finite renormalization (4.1) the anomalous dimension of chiral matter superfields changes as

$$\gamma'_a(\alpha', \lambda') = \gamma_a(\alpha, \lambda) + \frac{\partial \ln z_a}{\partial \alpha_L} \beta_L(\alpha, \lambda) + \frac{\partial \ln z_a}{\partial \lambda^{ijk}} (\beta_\lambda)^{ijk}(\alpha, \lambda) + \frac{\partial \ln z_a}{\partial \lambda_{ijk}^*} (\beta_\lambda^*)_{ijk}(\alpha, \lambda). \quad (4.10)$$

Evidently, in the one-loop approximation the anomalous dimension is scheme-independent. This implies that the NSVZ equations (2.18) do not depend on a renormalization prescription up to the order $O(\alpha, \lambda^2)$ inclusive exactly as for theories with a single gauge coupling constant.

Next, let us construct an equation which describes how the NSVZ equations change under the finite renormalization (4.1). For this purpose we first write the transformations inverse to (4.2) and (4.10), which are obtained by making the replacement $\alpha'_K \leftrightarrow \alpha_K$, $\lambda' \leftrightarrow \lambda$, $z_a \rightarrow (z_a)^{-1}$. We will assume that original RGFs (without primes) satisfy the NSVZ

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⁶If finite renormalizations are not restricted by the perturbative structure of quantum corrections, then the two-loop β -functions will be scheme-dependent [111].

equation (2.18). In this equation we express them in terms of the new RGFs (denoted by primes). This gives the equation

$$\begin{aligned}
 & \frac{\partial \alpha_K}{\partial \alpha'_L} \beta'_L(\alpha', \lambda') + \frac{\partial \alpha_K}{\partial \lambda'^{ijk}} (\beta'_\lambda)^{ijk}(\alpha', \lambda') + \frac{\partial \alpha_K}{\partial \lambda'^{*}_{ijk}} (\beta'^{*}_\lambda)_{ijk}(\alpha', \lambda') \\
 &= \beta_K(\alpha, \lambda) \\
 &= -\frac{\alpha_K^2}{2\pi(1 - C_2(G_K)\alpha_K/2\pi)} \left[3C_2(G_K) - \mathbf{T}_K(R) + \sum_a \mathbf{T}_{aK} \gamma_a(\alpha, \lambda) \right] \\
 &= -\frac{\alpha_K^2}{2\pi(1 - C_2(G_K)\alpha_K/2\pi)} \left[3C_2(G_K) - \mathbf{T}_K(R) + \sum_a \mathbf{T}_{aK} \left(\gamma'_a(\alpha', \lambda') \right. \right. \\
 & \quad \left. \left. - \frac{\partial \ln z_a}{\partial \alpha'_L} \beta'_L(\alpha', \lambda') - \frac{\partial \ln z_a}{\partial \lambda'^{ijk}} (\beta'_\lambda)^{ijk}(\alpha', \lambda') - \frac{\partial \ln z_a}{\partial \lambda'^{*}_{ijk}} (\beta'^{*}_\lambda)_{ijk}(\alpha', \lambda') \right) \right]. \quad (4.11)
 \end{aligned}$$

Solving it for $\beta'_K(\alpha', \lambda')$ we obtain the equation relating new RGFs,

$$\beta'_K(\alpha', \lambda') = (\mathcal{M}^{-1})_K{}^L \mathcal{P}_L, \quad (4.12)$$

where the summation over L is performed from 1 to n , and we use the notations

$$\begin{aligned}
 \mathcal{M}_K{}^L &\equiv \left(1 - \frac{C_2(G_K)\alpha_K}{2\pi} \right) \frac{\partial \alpha_K}{\partial \alpha'_L} - \frac{\alpha_K^2}{2\pi} \sum_a \mathbf{T}_{aK} \frac{\partial \ln z_a}{\partial \alpha'_L}; \quad (4.13) \\
 \mathcal{P}_L &\equiv -\frac{\alpha_L^2}{2\pi} \left\{ 3C_2(G_L) - \mathbf{T}_L(R) + \sum_a \mathbf{T}_{aL} \left(\gamma'_a(\alpha', \lambda') - \frac{\partial \ln z_a}{\partial \lambda'^{ijk}} (\beta'_\lambda)^{ijk}(\alpha', \lambda') \right. \right. \\
 & \quad \left. \left. - \frac{\partial \ln z_a}{\partial \lambda'^{*}_{ijk}} (\beta'^{*}_\lambda)_{ijk}(\alpha', \lambda') \right) \right\} \\
 & \quad - \left(1 - \frac{C_2(G_L)\alpha_L}{2\pi} \right) \left\{ \frac{\partial \alpha_L}{\partial \lambda'^{ijk}} (\beta'_\lambda)^{ijk}(\alpha', \lambda') + \frac{\partial \alpha_L}{\partial \lambda'^{*}_{ijk}} (\beta'^{*}_\lambda)_{ijk}(\alpha', \lambda') \right\}. \quad (4.14)
 \end{aligned}$$

For theories with a single gauge coupling constant this equation agrees with the results of refs. [49, 106]. From eq. (4.12) we see that under arbitrary finite renormalizations the NSVZ equation does not preserve its form. However, as we will see in the next section, there is a certain class of finite renormalizations which do not break the form of the NSVZ relation.

4.2 A class of the NSVZ schemes

Even for $\mathcal{N} = 1$ SQED there is a continuous set of the NSVZ renormalization schemes [39]. For non-Abelian theories with a single coupling a similar class of the NSVZ schemes was obtained in [40]. In this section we construct a generalization of these results to the case of theories containing multiple gauge couplings and Yukawa couplings. For this purpose we rewrite eq. (2.18) in the equivalent form

$$\frac{\beta_K(\alpha, \lambda)}{\alpha_K^2} \left(1 - C_2(G_K) \frac{\alpha_K}{2\pi} \right) = -\frac{1}{2\pi} \left[3C_2(G_K) - \mathbf{T}_K(R) + \sum_a \mathbf{T}_{aK} \gamma_a(\alpha, \lambda) \right]. \quad (4.15)$$

Using the definitions of the β -functions and the anomalous dimensions of matter superfields (eqs. (2.11) and (2.13), respectively) this equation can be presented in the form [8]

$$\frac{d}{d \ln \mu} \left\{ \frac{1}{\alpha_K} + \frac{C_2(G_K)}{2\pi} \ln \alpha_K - \frac{1}{2\pi} \left(3C_2(G_K) - \mathbf{T}_K(R) \right) \ln \frac{\mu}{\Lambda} - \frac{1}{2\pi} \sum_a \mathbf{T}_{aK} \ln Z_a \right\} = 0. \quad (4.16)$$

Two different renormalization schemes can always be related by a finite renormalization [107], which for the considered theory can be written in the form (4.1). Let us assume that the NSVZ equations (2.18) are valid for both original and new couplings. This implies that eq. (4.16) also holds for the new couplings. Therefore, they satisfy the equation

$$\frac{d}{d \ln \mu} \left\{ \frac{1}{\alpha'_K} + \frac{C_2(G_K)}{2\pi} \ln \alpha'_K - \frac{1}{2\pi} \left(3C_2(G_K) - \mathbf{T}_K(R) \right) \ln \frac{\mu}{\Lambda} - \frac{1}{2\pi} \sum_a \mathbf{T}_{aK} \ln Z'_a \right\} = 0. \quad (4.17)$$

Subtracting eq. (4.16) from eq. (4.17) we obtain the equation which contains only functions which describe the considered finite renormalization,

$$\frac{d}{d \ln \mu} \left\{ \frac{1}{\alpha'_K} - \frac{1}{\alpha_K} + \frac{C_2(G_K)}{2\pi} \ln \frac{\alpha'_K}{\alpha_K} - \frac{1}{2\pi} \sum_a \mathbf{T}_{aK} \ln z_a \right\} = 0. \quad (4.18)$$

We will be interested only in trivial solutions of this equation, which are obtained if the expressions in the curly brackets (numerated by the index K) are constants and do not depend on couplings. Denoting them by B_K we derive the system of equations describing finite renormalizations which do not take out of the class of the NSVZ schemes for the considered theory,

$$\frac{1}{\alpha'_K} - \frac{1}{\alpha_K} + \frac{C_2(G_K)}{2\pi} \ln \frac{\alpha'_K}{\alpha_K} - \frac{1}{2\pi} \sum_a \mathbf{T}_{aK} \ln z_a = B_K. \quad (4.19)$$

Evidently, this class is parameterized by m arbitrary functions z_a and n arbitrary constants B_K , where m is the number of the chiral matter superfields ϕ_a , and n is the number of factors in the product (2.1).

Note that if B_K defined as the left hand side of eq. (4.19) are not constants, then the matrices $\mathcal{M}_K^L$ and $\mathcal{P}_L$ (given by eqs. (4.13) and (4.14), respectively) can be written as

$$\mathcal{M}_K^L = \left(\frac{\alpha_K}{\alpha'_K} \right)^2 \delta_{KL} \left(1 - \frac{C_2(G_K)}{2\pi} \alpha'_K \right) + \alpha_K^2 \frac{\partial B_K}{\partial \alpha'_L}; \quad (4.20)$$

$$\begin{aligned} \mathcal{P}_L = & -\frac{\alpha_L^2}{2\pi} \left\{ 3C_2(G_L) - \mathbf{T}_L(R) + \sum_a \mathbf{T}_{aL} \gamma'_a(\alpha', \lambda') \right\} \\ & -\alpha_L^2 \left\{ \frac{\partial B_L}{\partial \lambda'^{ijk}} (\beta'_\lambda)^{ijk}(\alpha', \lambda') + \frac{\partial B_L}{\partial \lambda'^{*}_{ijk}} (\beta'^{*}_\lambda)_{ijk}(\alpha', \lambda') \right\}. \end{aligned} \quad (4.21)$$

From these equations and eq. (4.12) it is evident that the NSVZ equations really remain invariant under such finite renormalizations for that B_K are independent of couplings.

For theories with a single gauge coupling constant a certain subset of NSVZ schemes is produced by the HD+MSL prescription, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms. Presumably, this is also so for theories with multiple gauge couplings. Note that the HD+MSL

prescription is not unique, because minimal subtractions of logarithms can supplement various versions of the higher covariant derivative regularization. The explicit dependence of the two-loop anomalous dimension and of the three-loop β -function on regularization parameters for a general $\mathcal{N} = 1$ supersymmetric gauge theory with a single gauge coupling in the HD+MSL scheme can be found in ref. [112]. This dependence is not trivial, so that the HD+MSL schemes really give a continuous set of NSVZ schemes, which, in general, constitute a certain subset of the whole NSVZ scheme class.

5 The higher covariant derivative regularization for theories with multiple gauge couplings

5.1 General features of the higher derivative regularization in supersymmetric theories with multiple gauge couplings

For theories with a single gauge coupling constant some NSVZ schemes are given by the HD+MSL prescription in all loops [63–65]. Possibly, this is also so for theories with multiple gauge couplings. That is why it is expedient to construct the higher covariant derivative regularization for such theories. Very likely, if it is supplemented by minimal subtractions of logarithms, then eq. (2.18) is valid in all orders of the perturbation theory.

The versions of the higher covariant derivative regularization which were used for supersymmetric theories in refs. [58, 59] can be introduced only for theories satisfying the anomaly cancellation condition $\text{tr}(T^A\{T^B, T^C\}) = 0$ [72]. For theories considered in section 3 this condition really holds, so that the higher covariant derivative regularization can be constructed. Although it is made in a rather standard way, some features specific for the models under consideration should be described in more detail.

To introduce the higher covariant derivative regularization, first, we need to add a higher derivative term to the classical action [53, 54]. Next, one should insert into the generating functional the Pauli-Villars determinants, which cancel one-loop divergences that survived after this [57]. For quantizing the theory we will use the background field method [113–115] in the supersymmetric formulation [14, 99]. Also it is necessary to take into account that the quantum gauge superfield is renormalized nonlinearly [116–118], and that the nonlinear renormalization is really needed for cancelling certain divergences [119, 120] and for the renormalization group equations to be satisfied [121]. In principle, it can be reduced to a linear renormalization of an infinite set of parameters (which are very similar to the parameter in the gauge fixing term) included into a certain function $\mathcal{F}(V)$. Then the background superfield method and the nonlinear renormalization are introduced by making the substitutions

$$\begin{aligned} e^{2V_K} &\rightarrow e^{2\mathcal{F}_K(V_K)} e^{2\mathbf{V}_K} && \text{if } G_K \text{ is non-Abelian;} \\ V_K &\rightarrow V_K + \mathbf{V}_K && \text{if } G_K = \text{U}(1), \end{aligned} \quad (5.1)$$

where $\mathcal{F}_K(V_K) = e_{0K} \mathcal{F}_K(V_K)^{A_K} T^{A_K}$ and T^{A_K} are the generators of a relevant representation. Note that for Abelian gauge superfields there is no need to introduce the non-linear renormalization, so that in this case $\mathcal{F}_K(V_K) = V_K$. However, if a subgroup G_K is non-Abelian, then the corresponding function $\mathcal{F}_K(V_K)$ is not linear. For theories with a single

gauge coupling constant its explicit form in the lowest nontrivial approximation can be found in [119, 120]. Taking into account that gauge superfields corresponding to different subgroups G_K of the gauge group (2.1) commute, the resulting replacement for the gauge superfield (2.7) can be written as

$$e^{2V} \rightarrow e^{2\mathcal{F}(V)} e^{2V}. \quad (5.2)$$

After making this replacement and adding terms containing higher derivatives the gauge part of the action corresponding to a subgroup G_K is modified as

$$\begin{aligned} & \frac{1}{4} \text{Re} \int d^4x d^2\theta (W^a)^{A_K} (W_a)^{A_K} \\ & \rightarrow \frac{1}{4} \text{Re} \int d^4x d^2\theta (W^a)^{A_K} \left[\left(e^{-2V} e^{-2\mathcal{F}(V)} R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} W_a \right]^{A_K}, \end{aligned} \quad (5.3)$$

where the gauge superfield strength in the right hand side is defined by the equations

$$\begin{aligned} e_{0K} (W_a)^{A_K} t^{A_K} &\equiv \frac{1}{8} \bar{D}^2 \left(e^{-2V_K} e^{-2\mathcal{F}_K(V_K)} D_a (e^{2\mathcal{F}_K(V_K)} e^{2V_K}) \right) & \text{if } G_K \text{ is non-Abelian;} \\ (W_a)^{A_K} &= \frac{1}{4} \bar{D}^2 D_a V^{A_K} + \frac{1}{4} \bar{D}^2 D_a V^{A_K} & \text{if } G_K = \text{U}(1). \end{aligned} \quad (5.4)$$

(Certainly, if $G_K = \text{U}(1)$, the index A_K takes a single value, $V^{A_K} \rightarrow V_K$.)

The expression (5.3) contains the regulator function $R(x)$ rapidly increasing at infinity and satisfying the condition $R(0) = 1$. For simplicity, we use the same regulator function for all subgroups of the gauge group, although this is of course not necessary. In our notation the covariant derivatives present inside this function are defined as

$$\nabla_a = D_a; \quad \bar{\nabla}_{\dot{a}} = e^{2\mathcal{F}(V)} e^{2V} \bar{D}_{\dot{a}} e^{-2V} e^{-2\mathcal{F}(V)}. \quad (5.5)$$

The parameter Λ in the argument of the function R has the dimension of mass and plays the role of an ultraviolet cutoff.

Similarly, for all matter superfields it is necessary to make the substitution

$$\frac{1}{4} \sum_a \int d^4x d^4\theta \phi_a^+ e^{2V} \phi_a \rightarrow \frac{1}{4} \sum_a \int d^4x d^4\theta \phi_a^+ F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \phi_a, \quad (5.6)$$

where $F(x)$ is a regulator function which should rapidly grow at infinity and satisfy the condition $F(0) = 1$. As in the gauge part of the action, for simplicity, we use the same regulator function for all chiral matter superfields.

In the superpotential we do not introduce any functions with higher derivatives. The regularized action obtained after the modifications described above will be denoted by S_{reg} .

The gauge fixing terms should be added for all subgroups G_K ,

$$S_{\text{gf}} = - \sum_K \frac{1}{32\xi_{0K}} \int d^4x d^4\theta \nabla^2 V^{A_K} \left[K \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right)_{\text{Adj}} \bar{\nabla}^2 V \right]^{A_K}, \quad (5.7)$$

where $K(x)$ is another regulator function, ξ_{0K} are bare gauge parameters, and the background covariant derivatives are written as

$$\nabla_a = D_a; \quad \bar{\nabla}_{\dot{a}} = e^{2V} \bar{D}_{\dot{a}} e^{-2V}. \quad (5.8)$$

For non-Abelian subgroups G_K it is also necessary to introduce the Faddeev-Popov and Nielsen-Kallosh ghosts. Their actions are given by expressions

$$S_{\text{FP}} = \sum_{K, G_K \neq U(1)} \frac{1}{2} \int d^4x d^4\theta \frac{\partial \mathcal{F}_K^{-1}(\tilde{V}_K)^{A_K}}{\partial \tilde{V}^{B_K}} \Big|_{\tilde{V}_K = \mathcal{F}_K(V_K)} \left((e^{2V_K})_{\text{Adj}} \bar{c}_K + \bar{c}_K^+ \right)^{A_K} \\ \times \left\{ \left(\frac{\mathcal{F}_K(V_K)}{1 - e^{2\mathcal{F}_K(V_K)}} \right)_{\text{Adj}} c_K^+ + \left(\frac{\mathcal{F}_K(V_K)}{1 - e^{-2\mathcal{F}_K(V_K)}} \right)_{\text{Adj}} (e^{2V_K})_{\text{Adj}} c_K \right\}^{B_K}; \quad (5.9)$$

$$S_{\text{NK}} = \sum_{K, G_K \neq U(1)} \frac{1}{2e_{0K}^2} \text{tr} \int d^4x d^4\theta b_K^+ \left[K \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) \right]_{\text{Adj}} b_K. \quad (5.10)$$

It is important that the replacement $S \rightarrow S_{\text{reg}}$ removes divergences only beyond the one-loop approximation. A characteristic feature of the higher derivative regularization [122] is the presence of residual one-loop divergences, which should be regularized by inserting the Pauli-Villars determinants into the generating functional [57].

Following [58, 59], to cancel the one-loop (sub)divergences originating from gauge and ghost loops, for each non-Abelian subgroup G_K of the gauge group (2.1) we introduce three (commuting) chiral superfields $\varphi_{1,K}$, $\varphi_{2,K}$, and $\varphi_{3,K}$ in the adjoint representation. Then the one-loop divergences will be regularized if we insert into the generating functional the factor

$$\prod_{K, G_K \neq U(1)} \text{Det}^{-1}(PV, M_{\varphi,K}) = \int \prod_{K, G_K \neq U(1)} D\varphi_{1,K} D\varphi_{2,K} D\varphi_{3,K} e^{iS_{\varphi}}, \quad (5.11)$$

where

$$S_{\varphi} = \frac{1}{2} \sum_{K, G_K \neq U(1)} \text{tr} \left\{ \int d^4x d^4\theta \left[\varphi_{1,K}^+ \left(R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{1,K} \right. \right. \\ \left. \left. + \varphi_{2,K}^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{2,K} + \varphi_{3,K}^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{3,K} \right] \right. \\ \left. + \left[\int d^4x d^2\theta M_{\varphi,K} \left(\varphi_{1,K}^2 + \varphi_{2,K}^2 + \varphi_{3,K}^2 \right) + \text{c.c.} \right] \right\}. \quad (5.12)$$

Note that in this case

$$\left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{1,K} = e^{2\mathcal{F}(V)} e^{2V} \varphi_{1,K} e^{-2V} e^{-2\mathcal{F}(V)} \\ = e^{2\mathcal{F}_K(V_K)} e^{2V_K} \varphi_{1,K} e^{-2V_K} e^{-2\mathcal{F}_K(V_K)}. \quad (5.13)$$

Similar equations are also valid for the other superfields φ .

The (sub)divergences generated by a matter loop are regularized by inserting into the generating functional

$$\prod_K \text{Det}^{c_K}(PV, M_K) \quad (5.14)$$

$$= \prod_{K, G_K \neq \text{U}(1)} \left(\int D\Phi_K e^{iS_{\Phi,K}} \right)^{-c_K} \cdot \prod_{K, G_K = \text{U}(1)} \left(\int D\Phi_K D\tilde{\Phi}_K e^{iS_{\Phi,K}} \right)^{-c_K},$$

where c_K are certain numbers which will be specified below (see eq. (5.17)). For the non-Abelian subgroups G_K it is possible to choose the commuting chiral Pauli-Villars superfields Φ_K in the adjoint representation of the subgroup G_K . With respect to the other subgroups these superfields are singlets. Then, for regularizing one-loop matter (sub)divergences the actions for them should be taken in the form

$$S_{\Phi,K} = \frac{1}{2} \text{tr} \left\{ \int d^4x d^4\theta \Phi_K^+ \left(F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \Phi_K + \left[\int d^4x d^2\theta M_K \Phi_K^2 + \text{c.c.} \right] \right\}. \quad (5.15)$$

If a subgroup G_K coincides with $\text{U}(1)$, then we introduce two commuting chiral Pauli-Villars superfields Φ_K and $\tilde{\Phi}_K$ with the charges ± 1 with respect to the considered $\text{U}(1)$ subgroup. Again, with respect to the other subgroups they are invariant. The action for these superfields is given by the expression

$$S_{\Phi,K} = \frac{1}{4} \int d^4x d^4\theta \left[\Phi_K^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2e_{0K}(V_K + V_K)} \Phi_K \right. \\ \left. + \tilde{\Phi}_K^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{-2e_{0K}(V_K + V_K)} \tilde{\Phi}_K \right] + \left[\frac{1}{2} \int d^4x d^2\theta M_K \tilde{\Phi}_K \Phi_K + \text{c.c.} \right]. \quad (5.16)$$

The cancellation of the one-loop divergences and subdivergences occurs if the constants c_K in eq. (5.14) have the values

$$c_K = \begin{cases} \frac{\mathbf{T}_K(R)}{C_2(G_K)} & \text{for non-Abelian subgroups } G_K; \\ \frac{1}{2} \mathbf{T}_K(R) & \text{if } G_K = \text{U}(1). \end{cases} \quad (5.17)$$

(Note that the factor $1/2$ appears because in the Abelian case we introduce two Pauli-Villars superfields.) Also it is important that the masses of the Pauli-Villars superfields should be proportional to the dimensionful parameter Λ present in the higher derivative terms,

$$M_{\varphi,K} = a_{\varphi,K} \Lambda; \quad M_K = a_K \Lambda, \quad (5.18)$$

the coefficients $a_{\varphi,K}$ and a_K being independent of couplings.

5.2 The higher covariant derivative regularization for MSSM

As an example, we construct a version of the higher covariant derivative regularization for MSSM. In this case the regularized action is constructed exactly according to the prescriptions (5.3) and (5.6) described in the previous section. That is why below we will

not discuss this construction in detail. As for the Pauli-Villars determinants, to cancel the one-loop divergences coming from gauge and ghost loops, we insert into the generating functional

$$\text{Det}(PV, M_\varphi)^{-1} \equiv \int \prod_{K=\text{SU}(3), \text{SU}(2)} D\varphi_{1,K} D\varphi_{2,K} D\varphi_{3,K} \exp(iS_\varphi). \quad (5.19)$$

Here we introduced three commuting Pauli-Villars superfields for both non-Abelian subgroups (SU(3) and SU(2)) of the gauge group $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$. Evidently, for the U(1) subgroup there is no need to introduce such superfields. According to eq. (5.12) the action for them is written as

$$\begin{aligned} S_\varphi = & \frac{1}{2} \sum_{K=\text{SU}(3), \text{SU}(2)} \text{tr} \left\{ \int d^4x d^4\theta \left[\varphi_{1,K}^+ \left(R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{1,K} \right. \right. \\ & + \varphi_{2,K}^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{2,K} + \varphi_{3,K}^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_{3,K} \Big] \\ & \left. + \left[\int d^4x d^2\theta M_{\varphi,K} \left(\varphi_{1,K}^2 + \varphi_{2,K}^2 + \varphi_{3,K}^2 \right) + \text{c.c.} \right] \right\}. \end{aligned} \quad (5.20)$$

The corresponding masses are proportional to the parameter Λ in the higher derivative terms,

$$M_{\varphi, \text{SU}(3)} \equiv M_{\varphi,3} = a_{\varphi,3} \Lambda; \quad M_{\varphi, \text{SU}(2)} \equiv M_{\varphi,2} = a_{\varphi,2} \Lambda, \quad (5.21)$$

where the coefficients $a_{\varphi,3}$ and $a_{\varphi,2}$ are constants independent of couplings.

To cancel the one-loop (sub)divergences which appear from a loop of chiral matter superfields, we introduce the chiral Pauli-Villars superfields Φ_3 , Φ_2 , Φ_1 and $\tilde{\Phi}_1$. The superfield Φ_3 lies in the adjoint representation of SU(3) and is invariant under SU(2) and U(1) transformations. Similarly, the superfield Φ_2 lies in the adjoint representation of SU(2) and remains invariant under SU(3) and U(1) transformations. The superfields Φ_1 and $\tilde{\Phi}_1$ are singlets with respect to SU(3) and SU(2), but have nontrivial (± 1 in units of e_{01}) opposite U(1) charges. The gauge invariant actions for these superfields have the form

$$\begin{aligned} S_{\Phi,3} = & \frac{1}{2} \text{tr} \left\{ \int d^4x d^4\theta \Phi_3^+ \left(F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}_3(V_3)} e^{2V_3} \right)_{\text{Adj}} \Phi_3 \right. \\ & \left. + \left[\int d^4x d^2\theta M_3 \Phi_3^2 + \text{c.c.} \right] \right\}; \\ S_{\Phi,2} = & \frac{1}{2} \text{tr} \left\{ \int d^4x d^4\theta \Phi_2^+ \left(F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}_2(V_2)} e^{2V_2} \right)_{\text{Adj}} \Phi_2 \right. \\ & \left. + \left[\int d^4x d^2\theta M_2 \Phi_2^2 + \text{c.c.} \right] \right\}; \\ S_{\Phi,1} = & \frac{1}{4} \int d^4x d^4\theta \left[\Phi_1^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2e_{01}(V_1+V_1)} \Phi_1 + \tilde{\Phi}_1^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{-2e_{01}(V_1+V_1)} \tilde{\Phi}_1 \right] \\ & + \left[\frac{1}{2} \int d^4x d^2\theta M_1 \tilde{\Phi}_1 \Phi_1 + \text{c.c.} \right], \end{aligned} \quad (5.22)$$

where

$$M_1 = a_1 \Lambda; \quad M_2 = a_2 \Lambda; \quad M_3 = a_3 \Lambda, \quad (5.23)$$

and the coefficients $a_{1,2,3}$ do not depend on couplings. The (inverse) Pauli-Villars determinants are given by the expressions

$$\begin{aligned}\text{Det}^{-1}(PV, M_3) &= \int D\Phi_3 \exp(iS_{\Phi,3}); \\ \text{Det}^{-1}(PV, M_2) &= \int D\Phi_2 \exp(iS_{\Phi,2}); \\ \text{Det}^{-1}(PV, M_1) &= \int D\Phi_1 D\tilde{\Phi}_1 \exp(iS_{\Phi,1}).\end{aligned}\quad (5.24)$$

According to eq. (5.17), to find the degrees of these determinants in the generating functional, we need the coefficients $T_K(R)$ defined by eq. (2.16) and the constants $C_2(G_K)$ for the non-Abelian subgroups G_K . In the case of MSSM they have the following values:

$$T_{\text{SU}(3)}(R) = 3\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 6; \quad C_2(\text{SU}(3)) = 3; \quad (5.25)$$

$$T_{\text{SU}(2)}(R) = 3\left(\frac{3}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} = 7; \quad C_2(\text{SU}(2)) = 2; \quad (5.26)$$

$$T_{\text{U}(1)}(R) = 3\left(\frac{1}{6} + \frac{4}{3} + \frac{1}{3} + \frac{1}{2} + 1\right) + \frac{1}{2} + \frac{1}{2} = 11. \quad (5.27)$$

Therefore, using eq. (5.17) we conclude that the generating functional for the regularized theory is given by the expression

$$\begin{aligned}Z_{\text{MSSM}} &= \int D\mu \text{Det}^2(PV, M_3) \text{Det}^{7/2}(PV, M_2) \text{Det}^{11/2}(PV, M_1) \\ &\quad \times \exp\left(iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{FP}} + iS_{\text{NK}} + iS_{\varphi} + iS_{\text{sources}}\right),\end{aligned}\quad (5.28)$$

where S_{reg} is the regularized action, which in particular includes the higher derivative terms, S_{gf} is the gauge fixing action, S_{FP} and S_{NK} are actions for the Faddeev-Popov and Nielsen-Kallosh ghosts, respectively, S_{φ} is the Pauli-Villars action (5.20), and S_{sources} contains all relevant sources.

5.3 The higher covariant derivative regularization for the flipped SU(5) GUT

As another example we consider the flipped SU(5) model (for which the gauge group is $\text{SU}(5) \times \text{U}(1)$). The regularized action is again constructed according to the prescriptions (5.3) and (5.6) described in section 5.1. The one-loop divergences which remain after this are regularized by inserting the Pauli-Villars determinants.

To cancel the one-loop divergences generated by the gauge and ghost superfields, we use three chiral superfields $\varphi_{1,2,3}$ in the adjoint representation of SU(5) neutral with respect to U(1) and insert into the generating functional the corresponding (inverse) determinant

$$\text{Det}^{-1}(PV, M_{\varphi}) \equiv \int D\varphi_1 D\varphi_2 D\varphi_3 \exp(iS_{\varphi}), \quad (5.29)$$

where

$$\begin{aligned}S_{\varphi} &= \frac{1}{2} \text{tr} \left\{ \int d^4x d^4\theta \left[\varphi_1^+ \left(R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_1 + \varphi_2^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_2 \right. \right. \\ &\quad \left. \left. + \varphi_3^+ \left(e^{2\mathcal{F}(V)} e^{2V} \right)_{\text{Adj}} \varphi_3 \right] + \left[\int d^4x d^2\theta M_{\varphi} \left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \right) + \text{c.c.} \right] \right\}.\end{aligned}\quad (5.30)$$

The divergences coming from a loop of chiral matter superfields are regularized with the help of the (commuting) Pauli-Villars superfields Φ_5 , Φ_1 , and $\tilde{\Phi}_1$. The chiral superfield Φ_5 lies in the adjoint representation of $SU(5)$ and is neutral with respect to the $U(1)$ subgroup. The chiral superfields Φ_1 and $\tilde{\Phi}_1$ are $SU(5)$ singlets and have the opposite $U(1)$ charges $\pm e_{01}$, where e_{01} is normalized by eq. (3.19). The actions for these superfields are given by the expressions

$$\begin{aligned}
 S_{\Phi,5} &= \frac{1}{2} \text{tr} \left\{ \int d^4x d^4\theta \Phi_5^\dagger \left(F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2\mathcal{F}_5(V_5)} e^{2V_5} \right)_{\text{Adj}} \Phi_5 \right. \\
 &\quad \left. + \left[\int d^4x d^2\theta M_5 \Phi_5^2 + \text{c.c.} \right] \right\}; \\
 S_{\Phi,1} &= \frac{1}{4} \int d^4x d^4\theta \left[\Phi_1^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{2e_{01}(V_1+V_1)} \Phi_1 + \tilde{\Phi}_1^* F \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) e^{-2e_{01}(V_1+V_1)} \tilde{\Phi}_1 \right] \\
 &\quad + \left[\frac{1}{2} \int d^4x d^2\theta M_1 \tilde{\Phi}_1 \Phi_1 + \text{c.c.} \right]
 \end{aligned} \tag{5.31}$$

which are used in constructing the corresponding Pauli-Villars determinants

$$\begin{aligned}
 \text{Det}^{-1}(PV, M_5) &= \int D\Phi_5 \exp(iS_{\Phi,5}); \\
 \text{Det}^{-1}(PV, M_1) &= \int D\Phi_1 D\tilde{\Phi}_1 \exp(iS_{\Phi,1}).
 \end{aligned} \tag{5.32}$$

Standardly, the Pauli-Villars masses are proportional to the parameter Λ ,

$$M_\varphi = a_\varphi \Lambda; \quad M_1 = a_1 \Lambda; \quad M_5 = a_5 \Lambda, \tag{5.33}$$

and the corresponding ratios are independent of couplings.

To find the degrees of the Pauli-Villars determinants, we need the values

$$\mathbf{T}_{SU(5)}(R) = 3\left(\frac{3}{2} + \frac{1}{2}\right) + 2\left(\frac{3}{2} + \frac{1}{2}\right) = 10; \quad C_2(SU(5)) = 5; \tag{5.34}$$

$$\mathbf{T}_{U(1)}(R) = 3(10 + 45 + 25) + 2(10 + 20) = 300. \tag{5.35}$$

Then, according to the prescription described in section 5.1, the generating functional for the regularized theory can be written as

$$\begin{aligned}
 Z_{\text{flipped } SU(5)} &= \int D\mu \text{Det}^2(PV, M_5) \text{Det}^{150}(PV, M_1) \\
 &\quad \times \exp \left(iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{FP}} + iS_{\text{NK}} + iS_\varphi + iS_{\text{sources}} \right).
 \end{aligned} \tag{5.36}$$

6 Non-renormalization of the triple gauge-ghost vertices and a new form of the NSVZ equation

A very important ingredient of the all-loop derivation of the NSVZ β -function for theories with a single gauge coupling is a new form of the NSVZ equation [64]. It relates the β -function in a certain loop to the anomalous dimensions of the quantum gauge superfield,

of the Faddeev-Popov ghosts, and of the matter superfields in the previous loop. It is this equation that appears after summing singular contributions [63] produced by integrals of double total derivatives which determine the β -function in the supersymmetric case [65]. In this section we construct such a form of the NSVZ β -function for theories with multiple gauge couplings. This is done using the non-renormalization theorem for the triple gauge-ghost vertices which is proved for these theories with the help of the Slavnov-Taylor identities [123, 124]. The proof is very similar to the one discussed in [64] for theories with a single gauge coupling, so that here we will describe it more briefly.⁷ Note that for theories with multiple gauge couplings we will consider only the triple gauge-ghost vertices in which external lines correspond to the same simple subgroup G_K . If the external lines correspond to different subgroups, then the divergent contributions are forbidden by the renormalizability.

6.1 Slavnov-Taylor identities for the triple gauge-ghost vertices in theories with multiple gauge couplings

In the case of using the background superfield method the original gauge invariance of a classical theory generates two types of transformations. The background gauge invariance remains a manifest symmetry of the effective action, while the quantum gauge invariance is broken down to the BRST transformations [128, 129] by the gauge fixing procedure. In the supersymmetric case the BRST transformations are written as [116, 130, 131]

$$\begin{aligned}
 \delta V^{A_K} &= -\varepsilon_K \frac{\partial \mathcal{F}_K^{-1}(\tilde{V}_K)^{A_K}}{\partial \tilde{V}^{B_K}} \Big|_{\tilde{V}_K = \mathcal{F}_K(V_K)} \\
 &\quad \times \left\{ \left(\frac{\mathcal{F}_K(V_K)}{1 - e^{2\mathcal{F}_K(V_K)}} \right)_{\text{Adj}} c_K^+ + \left(\frac{\mathcal{F}_K(V_K)}{1 - e^{-2\mathcal{F}_K(V_K)}} \right)_{\text{Adj}} (e^{2V_K})_{\text{Adj}} c_K \right\}^{B_K}; \\
 \delta \phi &= \sum_K \varepsilon_K c_K \phi \equiv \sum_K \delta_K \phi; & \delta V_K &= 0; & \delta b_K &= 0; \\
 \delta \bar{c}_K &= -\frac{1}{16\xi_{0K}} \varepsilon_K \bar{D}^2 \left[\left(e^{-2V_K} K \left(-\frac{\nabla^2 \bar{\nabla}^2}{16\Lambda^2} \right) \right)_{\text{Adj}} \nabla^2 V_K \right]; & \delta c_K &= \varepsilon_K c_K^2; \\
 \delta c_K^+ &= -\frac{1}{16\xi_{0K}} \varepsilon_K D^2 \left[K \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right)_{\text{Adj}} \bar{\nabla}^2 V_K \right]; & \delta c_K^+ &= \varepsilon_K (c_K^+)^2, \quad (6.1)
 \end{aligned}$$

where ε_K are anticommuting parameters which do not depend on the coordinates. Note that for the gauge and matter superfields the BRST transformations are reduced to the quantum gauge transformations for which the parameters corresponding to non-Abelian subgroups G_K are given by the chiral superfields

$$A_K = \varepsilon_K c_K; \quad A_K^+ = -\varepsilon_K c_K^+. \quad (6.2)$$

This implies that any gauge invariant expression is also BRST invariant.

⁷For some theories formulated in terms of usual fields a similar statement is valid in the Landau gauge [125, 126], see also [127] for the verification by an explicit four-loop calculation.

The BRST invariance leads to the Slavnov-Taylor identities [123, 124], which are a key ingredient for proving the finiteness of the triple gauge-ghost vertices. To obtain these identities, we make a change of variables coinciding with the transformations (6.1) corresponding to the subgroup G_K in the generating functional Z . The total action of the theory

$$S_{\text{total}} = S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} \quad (6.3)$$

is invariant under the transformations (6.1). By construction, the Pauli-Villars determinants are invariant under both background and quantum gauge transformations. As a consequence of the latter invariance, they are also BRST invariant. Therefore, only the source term

$$\begin{aligned} S_{\text{sources}} = \sum_K \bigg\{ & \int d^4x d^4\theta V^{AK} J^{AK} + \int d^4x d^2\theta \left(j_c^{AK} c^{AK} + \bar{j}_c^{AK} \bar{c}^{AK} \right) \\ & + \int d^4x d^2\bar{\theta} \left(c^{+AK} j_c^{+AK} + \bar{c}^{+AK} \bar{j}_c^{+AK} \right) \bigg\} \\ & + \sum_a \bigg\{ \int d^4x d^2\theta j^{ia} \phi_{ia} + \int d^4x d^2\bar{\theta} j_{ia}^* \phi^{*ia} \bigg\} \end{aligned} \quad (6.4)$$

in the generating functional is not BRST invariant. Using the standard technique (see, e.g., [122]) a set of the generating Slavnov-Taylor identities in the considered case can be written as

$$\begin{aligned} & \int d^4x d^4\theta_x \frac{\delta\Gamma}{\delta V_x^{AK}} \langle \delta V_x^{AK} \rangle \\ & + \int d^4x d^2\theta_x \left(\langle \delta \bar{c}_x^{AK} \rangle \frac{\delta\Gamma}{\delta \bar{c}_x^{AK}} + \langle \delta c_x^{AK} \rangle \frac{\delta\Gamma}{\delta c_x^{AK}} + \langle \delta_K \phi_{ia,x} \rangle \frac{\delta\Gamma}{\delta \phi_{ia,x}} \right) \\ & + \int d^4x d^2\bar{\theta}_x \left(\langle \delta \bar{c}_x^{+AK} \rangle \frac{\delta\Gamma}{\delta \bar{c}_x^{+AK}} + \langle \delta c_x^{+AK} \rangle \frac{\delta\Gamma}{\delta c_x^{+AK}} + \langle \delta_K \phi_x^{*ia} \rangle \frac{\delta\Gamma}{\delta \phi_x^{*ia}} \right) = 0. \end{aligned} \quad (6.5)$$

Each of these identities corresponds to a certain non-Abelian subgroup of the gauge group. The (super)fields in eq. (6.5) are not set to 0, so that it is possible to differentiate these equalities and construct an infinite set of identities relating various Green functions.

For deriving the non-renormalization theorem we need the identity which is obtained after differentiating eq. (6.5) with respect to $\bar{c}_y^{+BK}$, c_z^{CK} , and c_w^{DK} . After this we set all superfields to 0. In theories with multiple gauge couplings the ghost number conservations hold for each simple subgroup G_K of the gauge group G . Therefore, the numbers of ghosts and antighosts corresponding to any non-Abelian G_K should be equal in each nontrivial Green function. Taking this into account we obtain the Slavnov-Taylor identity

$$\begin{aligned} & \left\{ \int d^4x d^4\theta_x \left(\frac{\delta^3\Gamma}{\delta \bar{c}_y^{+BK} \delta V_x^{AK} \delta c_z^{CK}} \cdot \frac{\delta}{\delta c_w^{DK}} \langle \delta V_x^{AK} \rangle - \frac{\delta^3\Gamma}{\delta \bar{c}_y^{+BK} \delta V_x^{AK} \delta c_w^{DK}} \cdot \frac{\delta}{\delta c_z^{CK}} \langle \delta V_x^{AK} \rangle \right) \right. \\ & \left. - \int d^4x d^2\theta_x \frac{\delta^2\Gamma}{\delta \bar{c}_y^{+BK} \delta c_x^{AK}} \cdot \frac{\delta^2}{\delta c_z^{CK} \delta c_w^{DK}} \langle \delta c_x^{AK} \rangle \right\} \Big|_{\text{fields}=0} = 0. \end{aligned} \quad (6.6)$$

(Terms containing matter Green functions vanish due to the global Z_3 -symmetry $\phi_j \rightarrow e^{2\pi i k/3} \phi_j$ of the massless theory.)

The Green functions entering this equation can be expressed in terms of certain functions depending on external momenta. To construct the corresponding expressions, we need to involve chirality and symmetry considerations. For instance, taking into account that the ghost superfields are chiral or antichiral we can write the corresponding two-point Green functions in the form

$$\begin{aligned} \left. \frac{\delta^2 \Gamma}{\delta \bar{c}_y^{+B_K} \delta c_x^{A_K}} \right|_{\text{fields}=0} &= -\frac{D_y^2 \bar{D}_x^2}{16} G_{c_K} \delta_{xy}^8 \delta_{A_K B_K}; \\ \left. \frac{\delta^2 \Gamma}{\delta \bar{c}_y^{B_K} \delta c_x^{+A_K}} \right|_{\text{fields}=0} &= \frac{\bar{D}_y^2 D_x^2}{16} G_{c_K} \delta_{xy}^8 \delta_{A_K B_K}, \end{aligned} \quad (6.7)$$

where the functions G_{c_K} depend on couplings and ∂^2/Λ^2 . Evidently, all two-point functions in which ghosts correspond to different subgroups vanish due to the symmetries responsible for the ghost number conservations. Note that in theories with multiple gauge couplings the functions G_{c_K} are in general different for different K .

The derivative of $\langle \delta V_x^{A_K} \rangle$ with respect to the ghost superfield can also be expressed in terms of the function G_{c_K} . For this purpose we will use the identity obtained with the help of the substitution $\bar{c}^{+B_K} \rightarrow \bar{c}^{+B_K} + a^{+B_K}$ (where a^{+B_K} is an arbitrary antichiral superfield) in the generating functional Z . In terms of the effective action the result can be written as the identity

$$\varepsilon_K \frac{\delta \Gamma}{\delta \bar{c}_x^{+B_K}} = \frac{1}{4} D^2 \langle \delta V_x^{B_K} \rangle. \quad (6.8)$$

Again, in this equation (super)fields are not set to 0, so that it can be differentiated. This implies that eq. (6.8) generates an infinite set of identities relating various Green functions. In particular, if we differentiate eq. (6.8) with respect to the ghost superfield $c_y^{A_K}$ and take into account chirality consideration, then the result can be written in the form

$$\left. \frac{\delta}{\delta c_y^{A_K}} \langle \delta V_x^{B_K} \rangle \right|_{\text{fields}=0} = -\varepsilon_K \cdot \frac{1}{4} G_{c_K} \bar{D}^2 \delta_{xy}^8 \delta_{A_K B_K}. \quad (6.9)$$

Next, using chirality considerations we write down all possible structures that can appear in the part of the effective action corresponding to the triple $V\bar{c}^+c$ vertex,

$$\begin{aligned} \frac{i}{4} \sum_K e_{0K} f^{A_K B_K C_K} \int d^4 \theta \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \bar{c}^{+A_K}(p+q, \theta) \left[s_K(p, q) \partial^2 \Pi_{1/2} V^{B_K}(-p, \theta) \right. \\ \left. + \mathcal{S}_{\mu, K}(p, q) (\gamma^\mu)_{\dot{a}}^b D_b \bar{D}^{\dot{a}} V^{B_K}(-p, \theta) + \mathcal{S}_K(p, q) V^{B_K}(-p, \theta) \right] c^{C_K}(-q, \theta), \end{aligned} \quad (6.10)$$

where $f^{A_K B_K C_K}$ denotes structure constants of the subgroup G_K . The functions s_K , $\mathcal{S}_{\mu, K}$, and $\mathcal{S}_K$ have the dimensions m^{-2} , m^{-1} , and m^0 , respectively. Note that in [64] similar functions were denoted by f , F_μ , and F , but here (following ref. [72]) we use other letters in order to distinguish them from the higher derivative regulator function F , see eq. (5.6).

From eq. (6.10) we obtain the explicit expression for the three-point $V\bar{c}^+c$ vertex,

$$\begin{aligned} & \left. \frac{\delta^3 \Gamma}{\delta \bar{c}_x^{+A_K} \delta V_y^{B_K} \delta c_z^{C_K}} \right|_{\text{fields}=0} \\ &= -\frac{i e_{0K}}{16} f^{A_K B_K C_K} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left(s_K(p, q) \partial^2 \Pi_{1/2} \right. \\ & \quad \left. - \mathcal{S}_{\mu, K}(p, q) (\gamma^\mu)_{\dot{a}}{}^b \bar{D}^{\dot{a}} D_b + \mathcal{S}_K(p, q) \right)_y \left(D_x^2 \delta_{xy}^8(q+p) \bar{D}_z^2 \delta_{yz}^8(q) \right), \end{aligned} \quad (6.11)$$

where we introduced the notation

$$\delta_{xy}^8(q) \equiv \delta^4(\theta_x - \theta_y) e^{iq_\alpha(x^\alpha - y^\alpha)}. \quad (6.12)$$

Also eq. (6.6) contains the second derivative of the expression

$$\langle \delta c^{A_K} \rangle = \varepsilon_K \cdot \frac{i}{2} e_{0K} f^{A_K B_K C_K} \langle c^{B_K} c^{C_K} \rangle \quad (6.13)$$

with respect to the ghost superfields. It is convenient to introduce the auxiliary source term

$$-\frac{1}{2} \sum_K e_{0K} \int d^4 x d^2 \theta f^{A_K B_K C_K} \mathcal{J}^{A_K} c^{B_K} c^{C_K} + \text{c.c.}, \quad (6.14)$$

which is invariant under the BRST transformations due to their nilpotency. This term allows presenting the considered correlator as the third derivative of the effective action with respect to the source $\mathcal{J}$ and two ghost superfields. Involving chirality considerations the corresponding contribution to the effective action can be written in the form

$$-\frac{1}{2} \sum_K e_{0K} f^{A_K B_K C_K} \int d^2 \theta \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} c^{A_K}(q+p, \theta) c^{B_K}(-q, \theta) \mathcal{J}^{C_K} H_K(p, q), \quad (6.15)$$

where $H_K(p, q)$ are dimensionless functions. Due to the symmetry with respect to permutations of the ghost superfields they satisfy the equations

$$H_K(p, q) = H_K(p, -q - p). \quad (6.16)$$

Then the correlator entering eq. (6.6) can be expressed in terms of these functions,

$$\begin{aligned} & \left. \frac{\delta^2}{\delta c_z^{C_K} \delta c_w^{D_K}} \langle \delta c_y^{B_K} \rangle \right|_{\text{fields}=0} \\ &= -i \varepsilon_K \cdot \left. \frac{\delta^3 \Gamma}{\delta c_z^{C_K} \delta c_w^{D_K} \delta \mathcal{J}_y^{B_K}} \right|_{\text{fields}=0} \\ &= -\frac{i e_{0K} \varepsilon_K}{4} f^{B_K C_K D_K} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} H_K(p, q) \bar{D}_z^2 \delta_{zy}^8(q+p) \bar{D}_w^2 \delta_{yw}^8(q). \end{aligned} \quad (6.17)$$

Substituting the expressions (6.7), (6.9), (6.11), and (6.17) into the Slavnov-Taylor identity (6.6) after some transformations we rewrite this equation in the equivalent form

$$G_{c_K}(q) \mathcal{S}_K(q, p) + G_{c_K}(p) \mathcal{S}_K(p, q) = 2 G_{c_K}(q+p) H_K(-q-p, q), \quad (6.18)$$

where, for simplicity, the argument of the functions G_c is denoted by q instead of $-q^2/\Lambda^2$. We see that for theories with multiple gauge couplings the number of such equations is equal to the number of non-Abelian subgroups G_K in the direct product (2.1).

Eq. (6.18) is the most convenient form of the Slavnov-Taylor identity for the $V\bar{c}^+c$ vertex. In the next section it will be used for proving its finiteness in all orders of the perturbation theory.

6.2 All-loop finiteness of the triple gauge-ghost vertices

To prove the all-loop finiteness of the triple gauge-ghost vertices in which all external lines correspond to the same non-Abelian subgroup G_K , we first prove the finiteness of the functions H_K defined by eq. (6.15). Each of these functions is contributed by superdiagrams in which one external line corresponds to the source $\mathcal{J}$, and the other two correspond to the Faddeev-Popov ghost c_K . The vertex with the external source is given by the expression (6.14). If both ghost superfields in eq. (6.14) produce propagators, then the supergraph contributing to the function H_K contains

$$\int d^4y d^2\theta_y \mathcal{J}_y^{A_K} \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2} \delta_{y1}^8 \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2} \delta_{y2}^8, \quad (6.19)$$

where two ghost propagators connect the superspace point y to the superspace points 1 and 2. With the help of the identity

$$\int d^4x d^4\theta_x = -\frac{1}{2} \int d^4x d^2\theta_x \bar{D}_x^2 \quad (6.20)$$

the expression (6.19) can be presented in the form of the integral over the full superspace,

$$-2 \int d^4y d^4\theta_y \mathcal{J}_y^{A_K} \cdot \frac{D_y^2}{4\partial^2} \delta_{y1}^8 \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2} \delta_{y2}^8. \quad (6.21)$$

Evidently, if (only) one ghost superfield in the vertex (6.14) corresponds to an external line, then the vertex can also be presented as such an integral. (Two external ghosts in eq. (6.14) can appear only in the tree contribution to the function H_K .) This implies that all vertices in the considered supergraph can be written as integrals over the full superspace. Therefore, it is possible to apply standard rules for calculating supergraphs. It is well known (see, e.g., [100]) that the resulting expression is also given by an integral over the full superspace. However, all external lines (which correspond to $\mathcal{J}_K$ and two c_K) are chiral. Therefore, due to eq. (6.20) a nontrivial result can be obtained only if at least two right spinor covariant derivatives act on the external lines. If we take into account that the function H_K is dimensionless, this implies that the dimension of the remaining loop integral is m^{-2} . In other words, the degree of divergence is equal to (-2) , so that this integral is finite in the ultraviolet region. Due to the renormalizability of the considered theory possible subdivergences are removed after the renormalization in the previous orders. Thus, the functions H_K (for any K such that $G_K \neq \text{U}(1)$) appear to UV finite.

Next, we involve the Slavnov-Taylor identity (6.18) multiplied by the ghost renormalization constant Z_{c_K} . Using the finiteness of the renormalized Green function

$$(G_{c_K})_R(\alpha, \lambda, q^2/\mu^2) = \lim_{\Lambda \rightarrow \infty} Z_{c_K}(\alpha, \lambda, \Lambda/\mu) G_{c_K}(\alpha_0, \lambda_0, q^2/\Lambda^2) \quad (6.22)$$

we see that

$$\lim_{\Lambda \rightarrow \infty} \left. \frac{d(Z_{c_K} G_{c_K})}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}} = 0. \quad (6.23)$$

Therefore, differentiating eq. (6.18) multiplied by Z_{c_K} with respect to $\ln \Lambda$ and keeping in mind the finiteness of the function H_K we obtain

$$\begin{aligned} 0 &= \lim_{\Lambda \rightarrow \infty} \frac{d}{d \ln \Lambda} \left(Z_{c_K} G_{c_K}(q) \mathcal{S}_K(q, p) + Z_{c_K} G_{c_K}(p) \mathcal{S}_K(p, q) \right) \Big|_{\alpha, \lambda = \text{const}} \\ &= \lim_{\Lambda \rightarrow \infty} \left(Z_{c_K} G_{c_K}(q) \frac{d}{d \ln \Lambda} \mathcal{S}_K(q, p) + Z_{c_K} G_{c_K}(p) \frac{d}{d \ln \Lambda} \mathcal{S}_K(p, q) \right). \end{aligned} \quad (6.24)$$

In this equation we set $p = -q$ and take into account that $G_{c_K}(-q) = G_{c_K}(q)$ because this function depends only on q^2/Λ^2 . The result can be written in the form

$$\lim_{\Lambda \rightarrow \infty} \frac{d}{d \ln \Lambda} \left(\mathcal{S}_K(q, -q) + \mathcal{S}_K(-q, q) \right) = 0. \quad (6.25)$$

The function $\mathcal{S}_K(-q, q)$ also depends only on q^2/Λ^2 , so that from the above equation we conclude that it is finite in the ultraviolet region.

In our notation the renormalization constants for the gauge coupling constants, for the quantum gauge superfields, and for the Faddeev-Popov ghosts are defined by the equations

$$\frac{1}{\alpha_{0K}} = \frac{Z_{\alpha_K}}{\alpha_K}; \quad V_K = Z_{V_K} Z_{\alpha_K}^{-1/2} (V_K)_R; \quad \bar{c}_K c_K = Z_{c_K} Z_{\alpha_K}^{-1} (\bar{c}_K)_R (c_K)_R, \quad (6.26)$$

respectively. Taking into account that

$$V_K = e_{0K} V^{AK} T^{AK}; \quad \bar{c}_K = e_{0K} \bar{c}^{AK} T^{AK}; \quad c_K = e_{0K} c^{AK} T^{AK}; \quad (6.27)$$

$$(V_K)_R = e_K \left(V^{AK} \right)_R T^{AK}; \quad (\bar{c}_K)_R = e_K \left(\bar{c}^{AK} \right)_R T^{AK}; \quad (c_K)_R = e_K \left(c^{AK} \right)_R T^{AK} \quad (6.28)$$

we see that the components of the quantum gauge superfields and of the Faddeev-Popov ghosts are renormalized as

$$V^{AK} = Z_{V_K} \left(V^{AK} \right)_R; \quad \bar{c}^{AK} c^{BK} = Z_{c_K} \left(\bar{c}^{AK} \right)_R \left(c^{BK} \right)_R. \quad (6.29)$$

Using these equations one can obtain the renormalization constants for the triple gauge-ghost vertices from eq. (6.10). Certainly they are the same for all 4 types of the considered vertices and for all functions entering eq. (6.10). For the vertex corresponding to the subgroup G_K the renormalization constant is equal to $Z_{\alpha_K}^{-1/2} Z_{c_K} Z_{V_K}$, for example,

$$(\mathcal{S}_K)_R(p, q) = \lim_{\Lambda \rightarrow \infty} Z_{\alpha_K}^{-1/2} Z_{c_K} Z_{V_K} \mathcal{S}_K(p, q). \quad (6.30)$$

The renormalized function in the left hand side of this equation is UV finite by definition. Certainly, this is so for the particular case $p = -q$. From the other side, as we discussed above, the function $\mathcal{S}_K(-q, q)$ is not divergent in the ultraviolet region. Therefore, the corresponding renormalization constant should be finite,

$$\frac{d}{d \ln \Lambda} \left(Z_{\alpha_K}^{-1/2} Z_{c_K} Z_{V_K} \right) = 0. \quad (6.31)$$

This implies that it is possible (although not necessary) to choose a renormalization scheme in which the renormalization constants satisfy the relation

$$Z_{\alpha_K}^{-1/2} Z_{c_K} Z_{V_K} = 1. \quad (6.32)$$

Evidently, this equation holds in the HD+MSL scheme, because in this case various Z -s include only powers of $\ln \Lambda/\mu$, while all finite constants vanish.

6.3 A new form of the NSVZ β -function for theories with multiple gauge couplings

Let us derive a new form of the NSVZ equation for theories with multiple gauge couplings making the transformations analogous to the ones proposed in ref. [64]. However, here we will use RGFs defined in terms of the renormalized coupling assuming that eq. (6.32) is satisfied. Certainly, for RGFs defined in terms of the bare couplings a similar result is also valid. (In this case it is sufficient to use eq. (6.31), while eq. (6.32) is not needed.)

First, we equivalently rewrite the NSVZ equation (2.18) in the form

$$\frac{\beta_K(\alpha, \lambda)}{\alpha_K^2} = -\frac{1}{2\pi} \left[3C_2(G_K) - \sum_a \mathbf{T}_{aK} (1 - \gamma_a(\alpha, \lambda)) \right] + \frac{C_2(G_K)}{2\pi} \cdot \frac{\beta_K(\alpha, \lambda)}{\alpha_K}. \quad (6.33)$$

It is convenient to express the function $\beta_K(\alpha, \lambda)$ in the right hand of this equation in terms of the renormalization constant for the gauge coupling α_K ,

$$\beta_K(\alpha, \lambda) = \frac{d\alpha_K}{d\ln \mu} \Big|_{\alpha_0, \lambda_0 = \text{const}} = \alpha_K \frac{d \ln Z_{\alpha_K}}{d \ln \mu} \Big|_{\alpha_0, \lambda_0 = \text{const}}. \quad (6.34)$$

This renormalization constant can in turn be related to the renormalization constants of the quantum gauge superfield and of the Faddeev-Popov ghosts corresponding to the subgroup G_K with the help of equation (6.32). As a result, the β -function for a certain gauge coupling can be expressed in terms of the anomalous dimensions for the corresponding quantum gauge superfield and Faddeev-Popov ghosts,

$$\beta_K(\alpha, \lambda) = 2\alpha_K \frac{d}{d \ln \mu} \ln (Z_{V_K} Z_{c_K}) = 2\alpha_K (\gamma_{V_K}(\alpha, \lambda) + \gamma_{c_K}(\alpha, \lambda)), \quad (6.35)$$

where

$$\gamma_{V_K}(\alpha, \lambda) = \frac{d \ln Z_{V_K}}{d \ln \mu} \Big|_{\alpha_0, \lambda_0 = \text{const}}; \quad \gamma_{c_K}(\alpha, \lambda) = \frac{d \ln Z_{c_K}}{d \ln \mu} \Big|_{\alpha_0, \lambda_0 = \text{const}}. \quad (6.36)$$

Substituting the expression (6.35) for the function β_K into the right hand side of eq. (6.33) we obtain the equivalent form the NSVZ relation

$$\frac{\beta_K(\alpha, \lambda)}{\alpha_K^2} = -\frac{1}{2\pi} \left[C_2(G_K) (3 - 2\gamma_{V_K}(\alpha, \lambda) - 2\gamma_{c_K}(\alpha, \lambda)) - \sum_a \mathbf{T}_{aK} (1 - \gamma_a(\alpha, \lambda)) \right], \quad (6.37)$$

which generalizes a similar equation constructed in [64] for theories with a single gauge coupling constant.

Unlike the original NSVZ relation (2.18), this form of the NSVZ equation has a simple graphical interpretation. If we consider a vacuum supergraph (which does not contain external lines), then the corresponding contribution to the function β_K is produced by the sum of all superdiagrams which are obtained by attaching two external lines of the background gauge superfield V_K . From the other side, cutting internal lines we produce superdiagrams contributing to anomalous dimensions of various quantum superfields. Eq. (6.37) relates these contributions to the β -function.⁸ However, it is worth to note that for the quantum gauge superfields only cuts of the V_K -propagators contribute to the function β_K . Possibly, this can be explained with the help of the method for constructing integrals of double total derivatives proposed in [65]. We also hope that the generalization of the results obtained in [63] allows deriving eq. (6.37) in a way analogous to the case of theories with a single gauge coupling constant.

7 Conclusion

In this paper we investigated the NSVZ equations for $\mathcal{N} = 1$ supersymmetric models with multiple gauge couplings. In these theories the gauge group is a direct product of simple subgroups and $U(1)$ factors. Certainly, there are a lot of various phenomenologically interesting examples of such theories, for instance, MSSM or flipped $SU(5)$ GUT considered in this paper in detail. In theories with multiple gauge couplings a number of β -functions is equal to a number of (simple and $U(1)$) factors in the gauge group. We assume that in all loops each of these β -functions satisfies the NSVZ equations (2.18) and (6.37) in a certain subtraction scheme. A strong evidence in favor of this is that in MSSM and flipped $SU(5)$ GUT predictions of the NSVZ equations for the two-loop β -functions exactly coincide with the corresponding (well-known) results obtained by direct calculations made in $\overline{DR}$ -scheme. Certainly, this coincidence occurs because the two-loop gauge β -functions are scheme independent.⁹ From the NSVZ equations for theories with multiple gauge couplings it is also possible to derive the exact NSVZ-like equation for the Adler D -function in $\mathcal{N} = 1$ SQCD, which was first obtained in [90, 91].

It is highly probable that the NSVZ scheme for theories with multiple gauge couplings is given by the HD+MSL prescription as for theories with a simple gauge group [63]. HD means that a theory should be regularized by Higher covariant Derivatives. This regularization in particular includes insertion of the Pauli-Villars determinants for removing one-loop divergences. Note that the ratios of the Pauli-Villars masses to the dimensionful regularization parameter (Λ) should be independent of couplings. MSL indicates that divergences are removed with the help of Minimal Subtractions of Logarithms, when only powers of $\ln \Lambda/\mu$ are included into renormalization constants. Therefore, for investigating the NSVZ equations it is highly desirable to construct a version of the higher covariant derivative regularization applicable for theories multiple gauge couplings. This was also

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⁸For the quantum gauge superfield this is correct only if the corresponding contribution to the two-point Green function is transversal. Certainly, due to the Slavnov-Taylor identities this is always so for the sum of all supergraphs.

⁹Note that the Yukawa β -functions are scheme dependent starting from the two-loop approximation.

done in this paper for the general case and was illustrated by the examples of MSSM and flipped SU(5) GUT. Also the higher covariant derivative regularization is a very important ingredient needed for the all-loop perturbative derivation of the NSVZ β -function. According to [63–66], summing the perturbation series we do not obtain the original form of the NSVZ equation. Instead of it we obtain the relation between the L -loop β -function and the $(L-1)$ -loop anomalous dimensions of the quantum gauge superfield, of the Faddeev-Popov ghosts, and of the matter superfields. The original NSVZ equation can be derived from it with the help of the non-renormalization theorem for the triple gauge-ghost vertices. In this paper we generalized this theorem to the case of theories with multiple gauge couplings and obtained from it the new form of the NSVZ equations for theories with multiple gauge couplings, which is given by eq. (6.37). We hope to finalize the all-loop perturbative derivation of these equations in the forthcoming publications.

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A NSVZ equations in the order $O(\alpha, \lambda^2)$

In this appendix we present expressions for the (scheme-independent) one-loop anomalous dimensions for all chiral matter superfields in MSSM and flipped SU(5) GUT. We substitute them into the NSVZ equations and obtain two-loop expressions for the β -functions. As we discussed in section 4.1, they do not depend on a renormalization prescription, so that it is possible to compare them with the known results obtained in the $\overline{\text{DR}}$ -scheme. Below we demonstrate the agreement of the results derived by these two methods. (For MSSM this was first done in [85].) This agreement can be considered as a strong evidence that the NSVZ equations discussed in this paper are really valid for theories with multiple gauge couplings.

A.1 MSSM

Let us construct two-loop contributions to the β -functions of MSSM starting from the exact β -functions (3.16) – (3.18). For this purpose we need expressions for the one-loop anomalous dimensions of all chiral matter superfields. For a theory with a simple gauge group the one-loop anomalous dimension is given by the expression

$$\gamma_i^j(\alpha, \lambda) = -\frac{\alpha}{\pi} C(R)_i^j + \frac{1}{4\pi^2} \lambda_{imn}^* \lambda^{jmn} + O(\alpha^2, \alpha\lambda^2, \lambda^4). \quad (\text{A.1})$$

To generalize this result to the case of theories with multiple gauge couplings, we first discuss the structure of the matrix $\lambda_{imn}^* \lambda^{jmn}$. It is evidently Hermitian and invariant under the transformations of the gauge group. If matter superfields lie in an irreducible

representation of a simple gauge group, then this matrix is proportional to δ_i^j . However, in the general case it is not so. Nevertheless, making a proper unitary rotation in the generation space it is always possible to diagonalize the matrix $\lambda_{imn}^* \lambda^{jmn}$. We will always assume that this is done. When this matrix becomes diagonal, the superfields ϕ_a are its eigenvectors corresponding to the certain eigenvalues, which we will denote by $(\lambda^* \lambda)_a$. Then we see that for theories with multiple gauge couplings the one-loop expression for the anomalous dimension of the superfield ϕ_a generalizing eq. (A.1) can be written in the form

$$\gamma_a(\alpha, \lambda) = - \sum_K \frac{\alpha_K}{\pi} C(R_{aK}) + \frac{1}{4\pi^2} (\lambda^* \lambda)_a + O(\alpha^2, \alpha \lambda^2, \lambda^4), \quad (\text{A.2})$$

where the factor $C(R_{aK})$ is defined by eq. (2.14). These factors can be calculated as follows. If R is an irreducible representation of a simple group G , then

$$C(R) = \frac{1}{\dim R} \text{tr}(T^A T^A) = \frac{T(R) r}{\dim R}, \quad (\text{A.3})$$

where r is a dimension of the group G and $\dim R$ is a dimension of the representation R . In particular, for the (anti)fundamental representation of the group $\text{SU}(N)$

$$C(\text{fund. SU}(N)) = \frac{N^2 - 1}{2N}. \quad (\text{A.4})$$

For the $\text{U}(1)$ subgroups $C(R_{aK})$ should be replaced by q_a^2 , where q_a is a charge of the superfield ϕ_a with respect to the considered $\text{U}(1)$ subgroup.

Because eq. (A.2) contains the Yukawa couplings, for calculating one-loop anomalous dimensions we also need the explicit expression for a part of the MSSM action containing the superpotential. It is written in the form

$$\Delta S = \frac{1}{2} \int d^4x d^2\theta W(\phi_a) + \text{c.c.}, \quad (\text{A.5})$$

where the superpotential is given by the expression

$$\begin{aligned} W(\phi_a) = & (Y_U)_{IJ} (\tilde{U} \tilde{D})_I^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{u1} \\ H_{u2} \end{pmatrix} U_{aJ} + (Y_D)_{IJ} (\tilde{U} \tilde{D})_I^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} D_{aJ} \\ & + (Y_E)_{IJ} (\tilde{N} \tilde{E})_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} E_J + \boldsymbol{\mu} (H_{u1} \ H_{u2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix}. \end{aligned} \quad (\text{A.6})$$

In this expression the indices $I, J = 1, 2, 3$ numerate generations of elementary particles. The Yukawa couplings λ^{ijk} can be written in terms of the 3×3 Yukawa matrices $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$. The parameter $\boldsymbol{\mu}$ (which should be distinguished from the normalization point μ) has the dimension of mass and is not essential for calculating the considered renormalization group functions.

Now, it is possible to obtain all expressions for the anomalous dimensions of the chiral matter superfields starting from eq. (A.2). Certainly, constructing them we need to take into account the factor $5/3$ present in the definition of the constant α_1 , see eq. (3.10). The

result can be written in the form (see, e.g., [86])

$$\gamma_Q(\alpha, Y) = -\frac{\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{8\pi^2} (Y_U Y_U^+ + Y_D Y_D^+) + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.7})$$

$$\gamma_U(\alpha, Y) = -\frac{4\alpha_1}{15\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{4\pi^2} Y_U^+ Y_U + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.8})$$

$$\gamma_D(\alpha, Y) = -\frac{\alpha_1}{15\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{4\pi^2} Y_D^+ Y_D + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.9})$$

$$\gamma_L(\alpha, Y) = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{1}{8\pi^2} Y_E Y_E^+ + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.10})$$

$$\gamma_E(\alpha, Y) = -\frac{3\alpha_1}{5\pi} + \frac{1}{4\pi^2} Y_E^+ Y_E + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.11})$$

$$\gamma_{H_u}(\alpha, Y) = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{3}{8\pi^2} \text{tr}(Y_U^+ Y_U) + O(\alpha^2, \alpha Y^2, Y^4); \quad (\text{A.12})$$

$$\gamma_{H_d}(\alpha, Y) = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{1}{8\pi^2} \text{tr}(3 Y_D^+ Y_D + Y_E^+ Y_E) + O(\alpha^2, \alpha Y^2, Y^4). \quad (\text{A.13})$$

Note that in our conventions the matrices $Y_U^+ Y_U$, $Y_D^+ Y_D$, $Y_E^+ Y_E$ should be made diagonal by specially constructed unitary rotations of the chiral superfields U , D , and E which act on the indices numerating generations. Similarly, the matrices $Y_U Y_U^+ + Y_D Y_D^+$ and $Y_E Y_E^+$ should be made diagonal by proper unitary rotations of the superfields Q and L , respectively.¹⁰ In eqs. (A.7)–(A.13) we certainly assume that for calculating anomalous dimensions of the quark and lepton superfields it is necessary to take a diagonal element of a proper product of the Yukawa matrices which corresponds to the considered generation.

Substituting eqs. (A.7)–(A.13) into the NSVZ equations (3.16)–(3.18) after some simple algebra we obtain the two-loop MSSM β -functions,

$$\begin{aligned} \frac{\beta_3(\alpha, Y)}{\alpha_3^2} = & -\frac{1}{2\pi} \left[3 - \frac{11\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{7\alpha_3}{2\pi} \right. \\ & \left. + \frac{1}{8\pi^2} \text{tr}(2 Y_U^+ Y_U + 2 Y_D^+ Y_D) \right] + O(\alpha^2, \alpha Y^2, Y^4); \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \frac{\beta_2(\alpha, Y)}{\alpha_2^2} = & -\frac{1}{2\pi} \left[-1 - \frac{9\alpha_1}{20\pi} - \frac{25\alpha_2}{4\pi} - \frac{6\alpha_3}{\pi} \right. \\ & \left. + \frac{1}{8\pi^2} \text{tr}(3 Y_U^+ Y_U + 3 Y_D^+ Y_D + Y_E^+ Y_E) \right] + O(\alpha^2, \alpha Y^2, Y^4); \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \frac{\beta_1(\alpha, Y)}{\alpha_1^2} = & -\frac{1}{2\pi} \left[-\frac{33}{5} - \frac{199\alpha_1}{100\pi} - \frac{27\alpha_2}{20\pi} - \frac{22\alpha_3}{5\pi} \right. \\ & \left. + \frac{1}{8\pi^2} \cdot \frac{1}{5} \text{tr}(13 Y_U^+ Y_U + 7 Y_D^+ Y_D + 9 Y_E^+ Y_E) \right] + O(\alpha^2, \alpha Y^2, Y^4). \end{aligned} \quad (\text{A.16})$$

They exactly coincide with the known expressions (which can be found in, e.g., [104]).

A.2 The flipped SU(5) model

Now let us calculate two-loop β -functions for the flipped SU(5) model by substituting one-loop anomalous dimensions of various chiral superfields into the NSVZ equations. For

¹⁰Usually such rotations are chosen in such a way that certain Yukawa matrices will be diagonal, real, and positive. The convention adopted in this paper certainly differs from it.

completeness, here we consider the model containing N_G generations of the matter superfields in the representation $\overline{10}(1) + 5(-3) + 1(5)$, N_H Higgs superfields in the representation $10(-1) + \overline{10}(1)$, N_h Higgs superfields in the representation $5(2) + \overline{5}(-2)$, and $N_G + 1$ singlets ϕ . The theory considered in section 3.3 corresponds to

$$N_G = 3; \quad N_H = 1; \quad N_h = 1. \quad (\text{A.17})$$

Below indices of the chiral superfields $\overline{10}^{ij}$, 5_i , and E which numerate the generations will be denoted by the letters $I, J = 1, \dots, N_G$. Similarly, the Higgs superfields h_i and $\tilde{h}^i$ will be numerated by the letters $K, L = 1, \dots, N_h$. For the Higgs superfields H_{ij} and $\tilde{H}^{ij}$ we will use the letters $M, N = 1, \dots, N_H$. The singlets will be numerated by the letters $R, S, T = 1, \dots, N_G + 1$. The superpotential of the model has the form [96]

$$\begin{aligned} W(\phi_a) = & (\lambda_1)^{IJK} \varepsilon_{ijklm} (\overline{10}^{ij})_I (\overline{10}^{kl})_J (\tilde{h}^m)_K + (\lambda_2)^{IJK} (\overline{10}^{ij})_I (5_i)_J (h_j)_K \\ & + (\lambda_3)^{IJK} (5_i)_I E_J (\tilde{h}^i)_K + (\lambda_4)^{MNK} \varepsilon_{ijklm} (\tilde{H}^{ij})_M (\tilde{H}^{kl})_N (\tilde{h}^m)_K \\ & + (\lambda_5)^{MNK} \varepsilon^{ijklm} (H_{ij})_M (H_{kl})_N (h_m)_K + (\lambda_6)^{IMR} (\overline{10}^{ij})_I (H_{ij})_M \phi_R \\ & + (\lambda_7)^{KLR} (h_i)_K (\tilde{h}^i)_L \phi_R + (\lambda_8)^{RST} \phi_R \phi_S \phi_T. \end{aligned} \quad (\text{A.18})$$

Note that in our notation the superfields in the representations 10 and $\overline{10}$ are normalized in such a way that their kinetic terms contain the coefficient $1/4$. For instance, in the classical action the term quadratic in the superfield H_{ij} (without the gauge superfields) has the form

$$\frac{1}{4} \int d^4x d^4\theta (H^*)^{ij} H_{ij}. \quad (\text{A.19})$$

To avoid double summations, one can rescale all superfields in the representations 10 and $\overline{10}$ in $1/\sqrt{2}$ times, e.g., $H_{ij} \rightarrow H_{ij}/\sqrt{2}$, etc.

If N_G , N_H , and N_h are not fixed, then from eq. (2.18) and table 3 (and also using eq. (3.25)) we obtain the exact β -functions

$$\begin{aligned} \frac{\beta_5(\alpha, \lambda)}{\alpha_5^2} = & -\frac{1}{2\pi(1 - 5\alpha_5/2\pi)} \left[15 - 2N_G - 3N_H - N_h + \sum_{I=1}^{N_G} \left(\frac{3}{2} \gamma_{\overline{10}_I}(\alpha, \lambda) + \frac{1}{2} \gamma_{5_I}(\alpha, \lambda) \right) \right. \\ & \left. + \frac{3}{2} \sum_{M=1}^{N_H} \left(\gamma_{H_M}(\alpha, \lambda) + \gamma_{\tilde{H}_M}(\alpha, \lambda) \right) + \frac{1}{2} \sum_{K=1}^{N_h} \left(\gamma_{h_K}(\alpha, \lambda) + \gamma_{\tilde{h}_K}(\alpha, \lambda) \right) \right]; \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned} \frac{\beta_1(\alpha, \lambda)}{\alpha_1^2} = & \frac{1}{2\pi} \left[2N_G + \frac{1}{2} N_H + N_h - \frac{1}{8} \sum_{I=1}^{N_G} \left(2\gamma_{\overline{10}_I}(\alpha, \lambda) + 9\gamma_{5_I}(\alpha, \lambda) + 5\gamma_{E_I}(\alpha, \lambda) \right) \right. \\ & \left. - \frac{1}{4} \sum_{M=1}^{N_H} \left(\gamma_{H_M}(\alpha, \lambda) + \gamma_{\tilde{H}_M}(\alpha, \lambda) \right) - \frac{1}{2} \sum_{K=1}^{N_h} \left(\gamma_{h_K}(\alpha, \lambda) + \gamma_{\tilde{h}_K}(\alpha, \lambda) \right) \right]. \end{aligned} \quad (\text{A.21})$$

To construct the anomalous dimensions entering these equations, we use eq. (A.2). With the help of eq. (A.3) we see that the relevant coefficients $C(R_{aK})$ present in the part of this equation containing the gauge coupling constants have the form

$$C(5) = C(\overline{5}) = \frac{1}{2} \cdot \frac{24}{5} = \frac{12}{5}; \quad C(10) = C(\overline{10}) = \frac{3}{2} \cdot \frac{24}{10} = \frac{18}{5}. \quad (\text{A.22})$$

Using these values after some calculations we obtain the anomalous dimensions for all chiral superfields of the model. In the expressions for them listed below, the summation is not performed over the indices indicated in bold letters,

$$\begin{aligned}\gamma_{\overline{10}_I}(\alpha, \lambda) = & -\frac{\alpha_1}{40\pi} - \frac{18\alpha_5}{5\pi} + \frac{6}{\pi^2} (\lambda_1^*)_{IJK} (\lambda_1)^{IJK} + \frac{1}{8\pi^2} (\lambda_2^*)_{IJK} (\lambda_2)^{IJK} \\ & + \frac{1}{8\pi^2} (\lambda_6^*)_{IMR} (\lambda_6)^{IMR} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.23})$$

$$\begin{aligned}\gamma_{5_I}(\alpha, \lambda) = & -\frac{9\alpha_1}{40\pi} - \frac{12\alpha_5}{5\pi} + \frac{1}{4\pi^2} (\lambda_2^*)_{JIK} (\lambda_2)^{JIK} \\ & + \frac{1}{8\pi^2} (\lambda_3^*)_{IJK} (\lambda_3)^{IJK} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.24})$$

$$\gamma_{E_I}(\alpha, \lambda) = -\frac{5\alpha_1}{8\pi} + \frac{5}{8\pi^2} (\lambda_3^*)_{JIK} (\lambda_3)^{JIK} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \quad (\text{A.25})$$

$$\begin{aligned}\gamma_{H_M}(\alpha, \lambda) = & -\frac{\alpha_1}{40\pi} - \frac{18\alpha_5}{5\pi} + \frac{6}{\pi^2} (\lambda_5^*)_{MNK} (\lambda_5)^{MNK} \\ & + \frac{1}{8\pi^2} (\lambda_6^*)_{IMR} (\lambda_6)^{IMR} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.26})$$

$$\gamma_{\tilde{H}_M}(\alpha, \lambda) = -\frac{\alpha_1}{40\pi} - \frac{18\alpha_5}{5\pi} + \frac{6}{\pi^2} (\lambda_4^*)_{MNK} (\lambda_4)^{MNK} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \quad (\text{A.27})$$

$$\begin{aligned}\gamma_{h_K}(\alpha, \lambda) = & -\frac{\alpha_1}{10\pi} - \frac{12\alpha_5}{5\pi} + \frac{1}{4\pi^2} (\lambda_2^*)_{IJK} (\lambda_2)^{IJK} + \frac{6}{\pi^2} (\lambda_5^*)_{MNK} (\lambda_5)^{MNK} \\ & + \frac{1}{8\pi^2} (\lambda_7^*)_{KLR} (\lambda_7)^{KLR} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned}\gamma_{\tilde{h}_K}(\alpha, \lambda) = & -\frac{\alpha_1}{10\pi} - \frac{12\alpha_5}{5\pi} + \frac{6}{\pi^2} (\lambda_1^*)_{IJK} (\lambda_1)^{IJK} \\ & + \frac{1}{8\pi^2} (\lambda_3^*)_{IJK} (\lambda_3)^{IJK} + \frac{6}{\pi^2} (\lambda_4^*)_{MNK} (\lambda_4)^{MNK} \\ & + \frac{1}{8\pi^2} (\lambda_7^*)_{LKR} (\lambda_7)^{LKR} + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned}\gamma_{\phi_R}(\alpha, \lambda) = & \frac{5}{4\pi^2} (\lambda_6^*)_{IMR} (\lambda_6)^{IMR} + \frac{5}{8\pi^2} (\lambda_7^*)_{KLR} (\lambda_7)^{KLR} \\ & + \frac{9}{4\pi^2} (\lambda_8^*)_{RST} (\lambda_8)^{RST} + O(\alpha^2, \alpha\lambda^2, \lambda^4). \end{aligned} \quad (\text{A.30})$$

Substituting these anomalous dimensions into the NSVZ equations (A.20) and (A.21) we obtain the two-loop expressions for the β -functions of the model under consideration,

$$\begin{aligned}\frac{\beta_5(\alpha, \lambda)}{\alpha_5^2} = & -\frac{1}{2\pi} \left[15 - 2N_G - 3N_H - N_h + \frac{75\alpha_5}{2\pi} - N_G \left(\frac{3\alpha_1}{20\pi} + \frac{58\alpha_5}{5\pi} \right) \right. \\ & - N_H \left(\frac{3\alpha_1}{40\pi} + \frac{183\alpha_5}{10\pi} \right) - N_h \left(\frac{\alpha_1}{10\pi} + \frac{49\alpha_5}{10\pi} \right) + \frac{12}{\pi^2} (\lambda_1^*)_{IJK} (\lambda_1)^{IJK} \\ & + \frac{7}{16\pi^2} (\lambda_2^*)_{IJK} (\lambda_2)^{IJK} + \frac{1}{8\pi^2} (\lambda_3^*)_{IJK} (\lambda_3)^{IJK} + \frac{12}{\pi^2} (\lambda_4^*)_{MNK} (\lambda_4)^{MNK} \\ & + \frac{12}{\pi^2} (\lambda_5^*)_{MNK} (\lambda_5)^{MNK} + \frac{3}{8\pi^2} (\lambda_6^*)_{IMR} (\lambda_6)^{IMR} + \frac{1}{8\pi^2} (\lambda_7^*)_{KLR} (\lambda_7)^{KLR} \left. \right] \\ & + O(\alpha^2, \alpha\lambda^2, \lambda^4); \end{aligned} \quad (\text{A.31})$$

$$\begin{aligned}
\frac{\beta_1(\alpha, \lambda)}{\alpha_1^2} = & -\frac{1}{2\pi} \left[-2N_G - \frac{1}{2}N_H - N_h - N_G \left(\frac{13\alpha_1}{20\pi} + \frac{18\alpha_5}{5\pi} \right) - N_H \left(\frac{\alpha_1}{80\pi} + \frac{9\alpha_5}{5\pi} \right) \right. \\
& - N_h \left(\frac{\alpha_1}{10\pi} + \frac{12\alpha_5}{5\pi} \right) + \frac{9}{2\pi^2} (\lambda_1^*)_{IJK} (\lambda_1)^{IJK} + \frac{7}{16\pi^2} (\lambda_2^*)_{IJK} (\lambda_2)^{IJK} \\
& + \frac{19}{32\pi^2} (\lambda_3^*)_{IJK} (\lambda_3)^{IJK} + \frac{9}{2\pi^2} (\lambda_4^*)_{MNK} (\lambda_4)^{MNK} + \frac{9}{2\pi^2} (\lambda_5^*)_{MNK} (\lambda_5)^{MNK} \\
& \left. + \frac{1}{16\pi^2} (\lambda_6^*)_{IMR} (\lambda_6)^{IMR} + \frac{1}{8\pi^2} (\lambda_7^*)_{KLR} (\lambda_7)^{KLR} \right] + O(\alpha^2, \alpha\lambda^2, \lambda^4). \quad (\text{A.32})
\end{aligned}$$

The gauge parts of these expressions exactly coincide with the ones presented in [98]. (N_5 and N_{10} used in [98] are related to N_h and N_H used in this paper by the equations $N_5 = 2N_h$ and $N_{10} = 2N_H$.) As for the Yukawa parts, we did not manage to find them in earlier papers.

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