Symmetry enhancements in 7d heterotic strings

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ABSTRACT: We use a moduli space exploration algorithm to produce a complete list of maximally enhanced gauge groups that are realized in the heterotic string in 7d, encompassing the usual Narain component, and five other components with rank reduction realized via nontrivial holonomy triples. Using lattice embedding techniques we find an explicit match with the mechanism of singularity freezing in M-theory on K3. The complete global data for each gauge group is explicitly given.

KEYWORDS: Gauge Symmetry, String Duality, Superstring Vacua, Superstrings and Heterotic Strings

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1 Introduction

The classification of gauge symmetries that can arise in the string landscape is an important problem that is comparatively easy to address for vacua with high supersymmetry. A particularly interesting set of vacua of this kind is provided by heterotic strings compactified on T^d , realizing models with 16 supercharges and gauge groups of rank $r = 16 + d$. Although the basic mechanism governing the allowed gauge groups was determined a long time ago by Narain [1], the full classification of all possibilities, together with their respective moduli, had not been made except for the case $d = 1$ [2] in the subsequent years. For $d = 2$ the list of possible gauge groups was implicitly known due to the duality between the heterotic string on T^2 and F -theory on elliptically fibered K3 surfaces, whose gauge groups were classified in [3, 4].

Carrying out an exhaustive classification of allowed gauge groups in $d = 2$ was the main goal of [5], where the results were found to match exactly with those of [4] but also the moduli for a representative model with each gauge group were given. It turned out that the most effective strategy for doing the classification consists on moving from singular points of maximal enhancement in moduli space to others via manipulations of the root lattices in a controlled way, in a process which is best described as an exploration algorithm.

This algorithm is at heart a tool for finding embeddings of lattices into other lattices, which is precisely the context in which the mechanism of gauge symmetry enhancement is best formulated for the heterotic string on T^d . However it turns out that other components in the moduli space of theories with 16 supercharges can be treated in this way. By means of the asymmetric orbifold construction of [6], theories with 16 supercharges but with gauge symmetry with reduced rank can be obtained. In particular, one finds the so called CHL string component [7, 8] in 9d, for which the momentum lattice was constructed by Mikhailov in [9]. In 8d the story is the same, but in 7d one finds four extra components (six including the Narain and CHL component). These were constructed together with their momentum lattices in [10]. Classifying the possible gauge groups in the CHL string by means of the exploration algorithm was the main goal of [11], where the respective topologies of the groups were obtained using results of [12].

In this paper we extend this work to the six aforementioned components in 7d. To this end it is necessary to state precisely how the enhanced symmetry groups can be obtained from the momentum lattices, which we do by a natural generalization of the case for the CHL string [9, 11, 12]. We see that the lattice alone is not sufficient to determine the allowed gauge groups, but rather one must impose a constraint in the embeddings characterized by an integer, which comes from the string theory but is ad hoc from the point of view of the lattice (see Proposition 3.3). Implementing this constraint in our algorithm we obtain a list of maximally enhanced gauge algebras for each component. We find respectively 1035, 407, 50, 9, 3, and 3 such algebras for the $\mathbb{Z}_m$ -triples, respectively with $m = 1, 2, 3, 4, 5, 6$. If we distinguish the enhancements by the global data of the gauge group, the corresponding numbers are 1232, 429, 52, 18, 3 and 3.

On the other hand it is well known that the heterotic string on T^3 is dual to M-theory on K3. Gauge groups with reduced rank are realized in the later when there are so-called

partially frozen singularities on the K3 [10, 13, 14]. It is then natural to ask how this mechanism of partial freezing appears in the heterotic string. We study this problem by exploiting relations between the reduced rank momentum lattices and the Narain lattice and find a match with the known results in the M-theory side. General freezing rules involving the topology of the gauge groups are obtained, generalizing the results of [12] for the 8d CHL string.

This paper is organized as follows. In section 2 we review the construction of rank reduced heterotic theories in nine to seven dimensions, emphasizing the role of outer automorphisms of the gauge lattice in the framework of asymmetric orbifolds. Then in section 3 we state the criteria for gauge groups being realized in the relevant theories in terms of lattice embeddings, and briefly review how the exploration algorithm works. The problem of singularity freezing is studied in section 4. Finally, the main results obtained with the exploration algorithm are presented and discussed in section 5. In appendix A we leave some comments regarding the role of some technicalities of lattice embeddings in the heterotic string, which may help the reader who is not used to thinking in these terms.

2 Basic constructions with rank reduction

In this section we review how rank reduced theories with 16 supercharges are constructed from the heterotic string in nine to seven dimensions. The idea is to get an intuitive understanding of these constructions through the manipulation of Dynkin Diagrams, illustrating the asymmetric orbifold construction with an outer automorphism. This complements the more general (and abstract) treatment in [10]. We go through the CHL string, the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic theory compactification without vector structure and the $\mathbb{Z}_m$ -triples.

2.1 CHL string

The CHL string in 9d can be realized as the $E_8 \times E_8$ heterotic string compactified on an orbifold of a circle involving the outer automorphism θ which exchanges both E_8 's and a half-period shift a along the circle [8]. The resulting target space has an holonomy θ along the compact direction which breaks the gauge group $E_8 \times E_8$ to its diagonal E_8 . The shift a obstructs the recovery of the broken E_8 in the twisted sector and so it ensures that the rank of the total gauge group is reduced.

Since θ is an outer automorphism of a gauge group, its implementation as an orbifold symmetry naturally leads to a picture of Dynkin Diagram folding. In the case of the CHL string, one “folds one E_8 into the other”, and finds that the gauge group of the resulting theory is E_8 (with an extra $U(1)$ for arbitrary radius). Turning on a Wilson line does not change this picture since it must break both E_8 's in the same way, and one then just folds one of the broken groups into the other.

Even though the length of a root is not by itself a meaningful concept, it is helpful to think that the nodes that get superposed in folding a diagram correspond to shortened roots. The reason is that this maps naturally to an increase in the level of the associated gauge algebra by a factor equal to the order of the automorphism θ . In this case, the $E_8 \times E_8$ at level 1 becomes an E_8 at level 2. On the other hand, connected diagrams

containing invariant nodes correspond to algebras at level 1. In the 9d CHL string there are no states in the gauge sector invariant under the orbifold symmetry, and so there are no gauge groups at level 1. Compactifying on a circle to 8d, one gets an extra $SU(2)$ at the self-dual radius which is unaffected by the folding and finds that indeed there are level 1 gauge symmetries (namely symplectic algebras of rank ≤ 10).

The main idea here is that using the symmetry (a, θ) one constructs a vacuum of the heterotic string with an holonomy that in particular projects out Cartan generator states. Such an holonomy can not be implemented in the theory by merely turning on Wilson lines, as outer automorphisms are not connected to the identity element in the gauge group. However, the set of holonomies that can be obtained by orbifolding the target manifold is larger and includes those of this type. Together with the diagram folding picture, this story generalizes to the other constructions reviewed below.

2.2 Compactification without vector structure

There is a theory dual to the 8d CHL string which is obtained from the $Spin(32)/\mathbb{Z}_2$ heterotic string by compactifying it on a T^2 without vector structure [15]. The basic idea is that the spectrum of the 10d theory does not contain vector representations of $Spin(32)$, and so one should consider topologies of the gauge bundle which do not admit such representations. An obstruction of this type is measured by a mod two cohomology class $\tilde{w}_2$, analogous to the second Stieffel-Whitney class w_2 which obstructs spin structure.

This compactification is characterized by the fact that the two holonomies g_1, g_2 on the torus commute as elements of $Spin(32)/\mathbb{Z}_2$, but do *not* commute when lifted to elements of the double cover $Spin(32)$. In other words, the commutator of these holonomies is lifted to a nontrivial element in $Spin(32)$ which is identified with the identity upon quotienting by one of the spinor classes in its center. The lifting $Spin(32)/\mathbb{Z}_2 \rightarrow Spin(32)$ is therefore obstructed and no vector representations are allowed.

Two such holonomies can not be put simultaneously on a maximal torus of the gauge group. Similarly to the CHL string, one of them has to be realized by orbifolding the theory. The difference in this case is that the 10d gauge group $Spin(32)/\mathbb{Z}_2$ does not have any outer automorphism. One can however turn on a Wilson line along one of the compact directions such that from the point of view of the remaining dimensions the gauge group is actually broken to one which does in fact have an outer automorphism. Concretely, we turn on a Wilson line $A = (\frac{1}{2}^8, 0^8)$ which breaks $Spin(32)/\mathbb{Z}_2 \rightarrow Spin(16)^2/\mathbb{Z}_2$. This can be represented diagrammatically as

$$\begin{array}{ccc}
 \text{Linear chain of 16 white nodes with a black node at the end} & \xrightarrow{A} & \text{Two chains of 8 white nodes each, with a black node at the top connecting them} \\
 & \xrightarrow{\theta} & \text{Single chain of 8 white nodes at the bottom}
 \end{array}
 \tag{2.1}$$

where the white nodes are simple roots and the black nodes represent the fundamental weight which generates the $\mathbb{Z}_2$ in each case. We see that the r.h.s. corresponds to a group

with outer automorphism θ . Orbifolding the theory by this symmetry and a half period shift along the second compact direction we obtain a theory with gauge group $\text{Spin}(16) \times \text{U}(1)^2$ (for arbitrary values of the torus metric and B-field). We note that the fundamental weight gets projected out by the orbifold symmetry, but the gauge group is $\text{Spin}(16)/\mathbb{Z}_2$ [11, 12].

The commutator of the holonomies chosen is the exponential of

$$A - \theta(A) = \left(\frac{1}{2}^8, 0^8\right) - \left(0^8, \frac{1}{2}^8\right) = \left(\frac{1}{2}^8, -\frac{1}{2}^8\right), \quad (2.2)$$

which does not yield the identity in $\text{Spin}(32)$ but rather the element which gets identified with it in $\text{Spin}(32)/\mathbb{Z}_2$. This corresponds to the discussion above. More generally one can deform this Wilson line by adding vectors symmetric in the first and last eight components, i.e. those of the form (δ, δ) , as to respect condition (2.2). One can also turn on another Wilson line A' in the second compact direction such that $\theta(A') = A'$, since the product of two holonomies on the same direction should commute. Together with deformations of the metric and the B-field we reach other points in moduli space exhibiting different gauge symmetries (classified in [11]). This moduli space is equivalent to that of the 8d CHL string, where the equivalence is given by T-duality [10].

2.3 Holonomy triples in 7d

The basic idea behind the construction just described can be applied to the heterotic string on a circle and further compactifying two dimensions on a torus. This comes from the fact that there are various 9d gauge groups analogous to the 10d $\text{Spin}(32)/\mathbb{Z}_2$. It is enough to consider the following five:

$$\frac{(\text{E}_7 \times \text{SU}(2))^2}{\mathbb{Z}_2}, \quad \frac{(\text{E}_6 \times \text{SU}(3))^2}{\mathbb{Z}_3}, \quad \frac{(\text{Spin}(10) \times \text{SU}(4))^2}{\mathbb{Z}_4}, \quad \frac{\text{SU}(5)^4}{\mathbb{Z}_5}, \quad \frac{(\text{SU}(2) \times \text{SU}(3) \times \text{SU}(6))^2}{\mathbb{Z}_6}. \quad (2.3)$$

These correspond to breakings of $\text{E}_8 \times \text{E}_8$ by a Wilson line A , so that it is most natural to work in the framework of the $\text{E}_8 \times \text{E}_8$ string. Natural choices for these Wilson lines are, respectively,

$$A = \begin{cases} (0^6, -\frac{1}{2}, \frac{1}{2}) \times (\frac{1}{2}, -\frac{1}{2}, 0^6) & (\mathbb{Z}_2) \\ (0^5, -\frac{1}{3}, \frac{2}{3}) \times (\frac{2}{3}, -\frac{2}{3}, 0^5) & (\mathbb{Z}_3) \\ (0^4, -\frac{1}{4}, \frac{3}{4}) \times (-\frac{3}{4}, \frac{1}{4}, 0^4) & (\mathbb{Z}_4) \\ (0^3, -\frac{1}{5}, \frac{4}{5}) \times (-\frac{4}{5}, \frac{1}{5}, 0^3) & (\mathbb{Z}_5) \\ (0^2, -\frac{1}{6}, \frac{5}{6}) \times (-\frac{5}{6}, \frac{1}{6}, 0^2) & (\mathbb{Z}_6) \end{cases}. \quad (2.4)$$

The $\mathbb{Z}_m$'s correspond not only to the fundamental group of each broken gauge group but also to the cyclic group generated by the outer automorphism θ to be implemented. The name ‘ $\mathbb{Z}_m$ -triple’ refers to this group together with the three holonomies consisting of (2.4) and the pair analogous to the one discussed in the previous section, which we now discuss.

2.3.1 $\mathbb{Z}_2$ -triple

Consider first the $\mathbb{Z}_2$ -triple. From the point of view of the T^2 on which the 9d theory is compactified, the gauge group is $(\text{E}_7 \times \text{SU}(2))^2/\mathbb{Z}_2$, which indeed has an order two outer automorphism, exchanging the $\text{E}_7 \times \text{SU}(2)$ factors. However, using this symmetry

to orbifold the theory just gives us the CHL string, as discussed in section 2.1. Consider instead turning on a Wilson line A' on one of the T^2 directions (x^1), of the form

$$A' = (0^5, -\frac{1}{2}, \frac{1}{2}, 0) \times (0, -\frac{1}{2}, \frac{1}{2}, 0^5). \quad (2.5)$$

It has the effect of further breaking the gauge group to $(E_6 \times U(1)^2)^2$. From the point of view of the other T^2 direction (x^2), the gauge group has then an order 2 outer automorphism corresponding to the symmetry of each E_6 diagram. To get a consistent theory (meaning that the partition function is modular invariant), however, we have to take into account how the orbifold symmetry acts on the 16 internal directions and not only the 12 corresponding to the E_6 's. Fortunately, it is not hard to find such a consistent automorphism. One just has to take the one corresponding to the symmetry of the affine diagram of the original gauge algebra $2E_7 + 2A_1$:

(2.6)

It can then be shown that, together with an order 2 shift in x^2 , one obtains a consistent theory with an holonomy that breaks 8 Cartan generators, and the gauge group is $F_4 \times F_4$ at level 1 times $U(1)^3$, for arbitrary metric and B-field. The former is due to the automorphism having an associated projector $P_\theta = 1 + \theta$ of rank 8. The later comes from the fact that each E_6 folds into an F_4 , where two nodes are left invariant (cf. discussion in section 2.1). As in the previous construction, we can represent this breaking diagrammatically:

(2.7)

Let us now consider the commutator of the holonomies along the T^2 . We find that

$$\theta(A') - A' = (0^5, 1, -1, 0) \times (0, 1, -1, 0^5), \quad (2.8)$$

which is just the fundamental weight represented as a black node in the above diagram. Its exponential is a nontrivial element of $(E_7 \times SU(2)^2)$ which gets identified with the identity in the quotient $(E_7 \times SU(2)^2)/\mathbb{Z}_2$, mirroring the situation in the compactification without vector structure as expected. One may also deform the Wilson lines along all directions by adding vectors invariant under θ . This restriction reduces the degrees of freedom of the theory with respect to the Narain moduli space in the appropriate way.

Finally we note that here we have obtained a particular gauge group, $F_4 \times F_4 \times U(1)^3$, out of the many possibilities that exist in the moduli space of the theory. The general construction carried out in [10] leads to a momentum lattice analogous to the Narain lattice, with which we may systematically explore this moduli space (as we discuss in next

section). In this case, the momentum lattice is just the Mikhailov lattice in 7d and the theory is equivalent to the 7d CHL string. We emphasize that the $\mathbb{Z}_2$ -triple does not involve the exchange of the E_8 's (or subgroups thereof), and so strictly speaking it does not correspond to the CHL string. Indeed, one can construct the CHL string but not the $\mathbb{Z}_2$ -triple in 9d. When they exist, they are equivalent by T-duality.

2.3.2 $\mathbb{Z}_3$ -triple

Starting in the $\mathbb{Z}_3$ -triple we find genuinely new rank-reduced moduli space components with respect to the 8d case. Here the gauge group from the point of view of the T^2 is $(E_6 \times SU(3))^2/\mathbb{Z}_3$. We turn on a Wilson line along x^1 of the form

$$A' = (0^4, -\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, 0) \times (0, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, 0^4). \quad (2.9)$$

This breaks the gauge group to $(SO(8) \times U(1)^4)^2$. To get the rank 3 automorphism we again consider the symmetry of the affine diagram of the original group:

This descends to the triality of each $SO(8)$ and folds them into $G_2 \times G_2$ at level 1. The projector $P_\theta = 1 + \theta + \theta^2$ is of rank 4, eliminating 12 Cartan generators, and so the resulting gauge group is $G_2 \times G_2 \times U(1)^3$ for arbitrary metric and B-field. Again, the orbifold includes an order 3 shift in x^2 . The corresponding breaking diagram is

The commutator of A' and θ is given by

$$\theta(A') - A' = (0^4, 1, -1, 0^2) \times (0^2, 1, -1, 0^4), \quad (2.12)$$

corresponding to the weight represented by a the black node in the diagram above, and the story is the same as before for the $\mathbb{Z}_2$ -triple. In this case one can deform the three Wilson lines with four degrees of freedom each, which is the rank of the projector P_θ . Together with the nine degrees of freedom coming from the metric and B-field, the dimension of the moduli space is 21, and its local geometry is given by the coset

$$SO(7, 3, \mathbb{R})/(SO(7, \mathbb{R}) \times SO(3, \mathbb{R})). \quad (2.13)$$

In [10] it was proposed that the global structure is given by the automorphism group of the momentum lattice of the theory, which was determined to be

$$\Lambda_3 = \Pi_{3,3} \oplus A_2 \oplus A_2, \quad (2.14)$$

extending the results for the first two components of the moduli space where the Narain and the Mikhailov lattice respectively play this role.

2.3.3 $\mathbb{Z}_4$ -triple

For the $\mathbb{Z}_4$ -triple we start with the 9d gauge group $(\text{Spin}(10) \times \text{SU}(4))^2/\mathbb{Z}_4$ and turn on the Wilson line

$$A' = \frac{1}{8}(-1, -1, -3, -3, 5, 3, 1, -1) \times (1, -1, -3, -5, 3, 3, 1, 1), \quad (2.15)$$

which breaks it to $(\text{SU}(2)^8/\mathbb{Z}_2) \times \text{U}(1)^8$. The affine diagram of the original group has an order 4 symmetry:

The surviving $\text{SU}(2)$'s under the action of A' correspond to the outermost nodes of the affine $\text{Spin}(10)$'s, and they get identified under θ into $\text{SU}(2) \times \text{SU}(2)$ at level 4. The rank of the projector $P_\theta = 1 + \theta + \theta^2 + \theta^3$ is 2, and so 14 Cartan generators are eliminated. There is again an order 4 shift in x^2 in the orbifold symmetry, and we get the gauge group $(\text{SU}(2) \times \text{SU}(2)/\mathbb{Z}_2 \times \mathbb{Z}_2) \times \text{U}(1)^3$ for generic metric and B-field. Here the $\text{SU}(2)^2$ factor is quotiented by its center, a fact which is derived at the end of section 4.3. The breaking diagram is

The nodes in yellow are just the lowest roots of each $\text{Spin}(10)$. Note that the black node in the r.h.s. represents a different weight than the one in the l.h.s. Indeed, it corresponds to an order two element in the center of $\text{SU}(2)^8$, while the former corresponds to an order four element in the l.h.s. group.

We find that

$$\theta(A') - A' = (0^3, 1, -1, 0^3) \times (0^3, 1, -1, 0^3), \quad (2.18)$$

which is the weight in the l.h.s. of the diagram above. The moduli space is of dimension 15, locally of the form

$$\text{SO}(5, 3, \mathbb{R})/(\text{SO}(5, \mathbb{R}) \times \text{SO}(3, \mathbb{R})), \quad (2.19)$$

and the momentum lattice is

$$\Lambda_4 = \Pi_{3,3} \oplus A_1 \oplus A_1. \quad (2.20)$$

2.3.4 $\mathbb{Z}_5$ and $\mathbb{Z}_6$ -triples

For the $\mathbb{Z}_5$ -triple we use Wilson line

$$A' = \frac{1}{10}(-5, 1, -3, 7, 5, 3, 1, -1) \times (1, -1, -3, -5, -7, 3, -1, 5), \quad (2.21)$$

which breaks $SU(5)^4/\mathbb{Z}_5$ to $U(1)^{16}$. The automorphism θ corresponds to the symmetry

and has projector $P_\theta = 0$. The rank of the gauge group is reduced by a factor of 16 and only the Cartans coming from the T^3 compactification are present. We have that

$$\theta(A') - A' = (0^2, 1, -1, 0^4) \times (0^4, 1, -1, 0^2), \quad (2.23)$$

which is the weight associated to the $\mathbb{Z}_5$ quotient. The moduli space has dimension 9 and is locally of the form

$$SO(3, 3, \mathbb{R}) / (SO(3, \mathbb{R}) \times SO(3, \mathbb{R})), \quad (2.24)$$

and the momentum lattice is just $\Pi_{3,3}$.

The story for the $\mathbb{Z}_6$ -triple is basically the same, the only differences being that the Wilson line used is

$$A' = \frac{1}{12}(1, -5, 7, 5, 3, 1, -1, -3) \times (3, 1, -1, -3, -5, 7, 5, -1), \quad (2.25)$$

the automorphism θ corresponds to the symmetry of the affine $(SU(2) \times SU(3) \times SU(6))^2$ diagram,

and

$$\theta(A') - A' = (0, 1, -1, 0^5) \times (0^5, 1, -1, 0^1). \quad (2.27)$$

As in the previous case there are no Wilson line degrees of freedom, and the local and global data for the moduli space are the same. One should note however that the groups which are realized at level 5 in the $\mathbb{Z}_5$ -triple are realized in this case at level 6. Indeed, this information is not contained implicitly in the momentum lattice.

3 7d heterotic string and momentum lattices

Here we explain the basic machinery of how gauge symmetry groups can be obtained from the momentum lattices corresponding to certain 7d heterotic string compactifications with 16 supercharges. These include the Narain lattice for T^3 compactifications, the Mikhailov lattice for the 7d CHL string, and the four extra momentum lattices for components with further rank reduction obtained in [10].

3.1 The Narain construction

It was shown in [1] that the perturbative spectrum of the heterotic string on T^d can be put in correspondence with an even self-dual Lorentzian lattice $\text{II}_{16+d,d}$ of signature $(+^{16+d}, -^d)$. This lattice is spanned by vectors $(P, p_L; p_R)$, where P is the left gauge lattice momentum and $p_{L,R}$ are the right and left internal space momenta.

The only massless states in the spectrum have $p_R = 0$, and those which realize the adjoint representation of the gauge algebra $\mathfrak{g}$ also have $P^2 + p_L^2 = 2$. They correspond therefore to a set of length $\sqrt{2}$ vectors in $\text{II}_{16+d,d}$ spanning a positive definite sublattice L , which is just the root lattice of $\mathfrak{g}$. The question of what gauge algebras can be realized in the theory is then equivalent to the question of what root lattices L can be embedded in the Narain lattice. Note that this embedding has to be such that the intersection of the real span of L with $\text{II}_{16+d,d}$ does not contain a larger root lattice L' , since this would leave out extra states that do form part of the massless spectrum. We can be more precise about the relation between gauge symmetries and lattice embeddings and in the way gain more information. As discussed in [11], relaxing the condition $P^2 + p_L^2 = 2$ while keeping $p_R = 0$ defines an overlattice $M \supseteq L$ corresponding to the weight lattice of the global gauge group G . In this case, M is such that the intersection of its real span with $\text{II}_{16+d,d}$ is M itself, i.e. it is *primitively* embedded in $\text{II}_{16+d,d}$ (see appendix A.1 for an extended discussion regarding these embeddings). The full statement regarding the possibility of some gauge group G being realized in the heterotic string on T^d is as follows:

Proposition 1. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq 16 + d$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group. $G \times \text{U}(1)^{16+d-r}$ is realized in the heterotic string on T^d as a gauge symmetry group if and only if its weight lattice M admits a primitive embedding in the Narain lattice $\text{II}_{16+d,d}$ such that the vectors in M of length $\sqrt{2}$ are roots.*

At the end of the day, the classification of the possible gauge groups that can be obtained in the heterotic string on T^d turns out to be a (conceptually) simple problem of lattice embeddings.

3.1.1 Exploration algorithm

There exist many useful theorems and techniques, mainly due to Nikulin [16, 17], for determining if some lattice M admits a primitive embedding in another lattice Λ . However, even if they can yield insight into the structure of the theory, they do not by themselves give an efficient method for obtaining a thorough classification of the allowed gauge groups. We are therefore led to develop more constructive methods which can be easily turned into computer algorithms.

Such an algorithm was presented in [5], and it works roughly as follows (see [11] for a detailed explanation):

1. Take a point in moduli space of T^d compactifications with maximally enhanced gauge group, say $\text{Spin}(32 + 2d)$, such that the embedding of the root lattice $L \hookrightarrow \text{II}_{16+d,d}$ is explicitly known.

2. Break the gauge group by removing a simple root. This relaxes a constraint on the moduli such that the semisimple rank reduced part of the gauge group generically corresponds to a d -dimensional subvariety V of the moduli space.
3. Enhance the group with a different simple root than the one previously removed. This generically selects a different point of maximal enhancement contained in V .

This procedure is repeated for different choices of breakings-enhancements for a given starting point, and then repeated again starting from the newly found gauge groups, up until it does not yield new results. It is natural to assume that all points of maximal enhancement in moduli space can be reached in this way, and it is in fact true for the cases $d = 1, 2$ [5].

Non-maximal enhancements can be obtained from the maximal ones by simply removing an arbitrary number of roots. Remarkably, for $d = 1, 2$ there are respectively only two gauge groups which can not be obtained in this way, namely $\text{Spin}(16)^2/\mathbb{Z}_2$ for $d = 1$ and $\text{Spin}(8)^4/\mathbb{Z}_2^2$ for $d = 2$. This pattern suggest that for $d = 3$ this will be the case for $\text{Spin}(4)^8/\mathbb{Z}_2^4 \simeq \text{SU}(2)^{16}/\mathbb{Z}_2^4$, but our results indicate that this group is not actually realized in the theory, and so the likelihood for an analog of the two aforementioned groups in $d = 3$ is very low.

In this paper we are interested in the possible gauge groups that can be realized in the heterotic string on T^3 . Using the exploration algorithm just described, we have collected a set of points of maximal enhancement characterized by their root lattices L , i.e. their gauge algebras $\mathfrak{g}$. For each point we compute the weight lattice M and from it the generators of the fundamental group H , using the methods described in [11]. The results are presented in section 5.

3.2 The CHL string and Mikhailov lattice

Now we wish to extend the discussion of the previous subsection to the CHL string on T^d , which can be realized as an asymmetric orbifold of the heterotic string on T^d . The analog of the Narain lattice for this theory was constructed by Mikhailov in [9] and can be written as

$$\Pi_{(d)} = \Pi_{d-1,d-1}(2) \oplus \Pi_{1,1} \oplus \text{E}_8, \quad (3.1)$$

where the (2) indicates that $\Pi_{d-1,d-1}$ is scaled by a factor of $\sqrt{2}$. Depending on the dimension d , this lattice may be rewritten in different ways using lattice isomorphisms. For $d = 3$, we have

$$\Pi_{2,2}(2) \oplus \Pi_{1,1} \oplus \text{E}_8 \simeq \Pi_{3,3} \oplus \text{D}_4 \oplus \text{D}_4 \simeq \Pi_{3,3} \oplus \text{F}_4 \oplus \text{F}_4. \quad (3.2)$$

Here we have used the root lattice isomorphism $\text{D}_4 \simeq \text{F}_4$ (the corresponding root *systems* are of course not isomorphic, see appendix A.2) to reflect the fact that the ‘canonical’ point in the theory has gauge algebra 2F_4 and not 2D_4 , as shown in section 2.3.1.

The relation between lattice embeddings and realizability of gauge groups in the CHL string is more complicated than for the usual heterotic string on tori. In the latter, the roots of the gauge algebra correspond to the length $\sqrt{2}$ vectors in some positive definite

lattice Λ primitively embedded into $\Pi_{16+d,d}$. In the CHL string the mass formulas are such that it is also possible for some but not all vectors of length 2 to give roots. In order for such a vector v to correspond to a root, it must satisfy the condition that its inner product with all other vectors in the whole Mikhailov lattice is even [9]. In this case we say that v is a level 2 vector (not to be confused with the level of the Kac-Moody algebra for the gauge group). More generally, a vector v in a lattice Λ is said to be at embedding level ℓ if the product of v with every vector in Λ is divisible by ℓ .

On the other hand, the statement that the global structure of the gauge group is given by the primitively weight overlattice M does not generalize to the case where the momentum lattice is not self-dual and the gauge algebras are not of ADE type. The problem of obtaining this global data was studied in detail in [12]. It was shown in particular that the fundamental group $\pi_1(G)$ of the gauge group G is given by the quotient of the cocharacter lattice $M^\vee$ and the coroot lattice $L^\vee$ where the latter is embedded in the dual momentum lattice $\Pi_{(d)}^*$ and the former is the corresponding overlattice which is primitively embedded in $\Pi_{(d)}^*$.

One strategy to obtain all the possible gauge groups in the theory is to apply the exploration algorithm described above to the dual lattice $\Pi_{(d)}^*$ (which usually has to be rescaled to be made even) and compute the lattices L and M the same way as for the Narain lattice, but dualizing the algebra $\mathfrak{g} \rightarrow \mathfrak{g}^\vee$ at the end. It can be shown that the embedding level condition for vectors to be roots are the same as for the original lattice $\Pi_{(d)}$. This corresponds to the method employed in [11] to obtain the list of gauge groups for the CHL string in 8d.

Having dealt with this subtlety, a statement generalizing proposition 3.1 for the usual heterotic string to the CHL string on T^d can be made as follows:

Proposition 2. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq d + 8$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group. $G \times U(1)^{d+8-r}$ is realized in the CHL string on T^d as a gauge symmetry group if and only if the weight lattice $M^\vee$ of the dual group $G^\vee$ admits a primitive embedding in the dual Mikhailov lattice $\Pi_{(d)}^*(2)$ such that the vectors in $M^\vee$ of length $\sqrt{2}\ell$ at embedding level $\ell = 1, 2$ in $\Pi_{(d)}^*(2)$ belong to $L^\vee$.*

We see that the embedding level ℓ plays an important role in the theory, allowing to treat the problem of finding the possible gauge groups without reference to the string theory itself, as in the case of the original heterotic string.

Finally let us recall that the simple factors in G have associated Kac-Moody algebras at level $\mathfrak{m} = 1, 2$ where $2/\mathfrak{m}$ is the squared length of the corresponding longest root. For $d = 2$ there are only ADE groups at level 2 and symplectic groups at level 1 (including $\text{Sp}(1) = \text{SU}(2)$). This moduli space was exhaustively explored in [11] using an extension of the algorithm discussed in section 3.1.1. For $d = 3$ there are more interesting possibilities including B_3 and F_4 at level 1.

3.3 Momentum lattices from triples

Let us now turn to the $\mathbb{Z}_m$ -triples reviewed in section 2.3. The respective momentum lattices are given in table 1, where we also show the rank reduction of the respective

m	Λ_m	Frozen Singularity	r_-
1	$\text{II}_{3,3} \oplus \text{E}_8 \oplus \text{E}_8$	$\emptyset$	0
2	$\text{II}_{3,3} \oplus \text{F}_4 \oplus \text{F}_4$	$\text{D}_4 \oplus \text{D}_4$	8
3	$\text{II}_{3,3} \oplus \text{G}_2 \oplus \text{G}_2$	$\text{E}_6 \oplus \text{E}_6$	12
4	$\text{II}_{3,3} \oplus \text{A}_1 \oplus \text{A}_1$	$\text{E}_7 \oplus \text{E}_7$	14
5	$\text{II}_{3,3}$	$\text{E}_8 \oplus \text{E}_8$	16
6	$\text{II}_{3,3}$	$\text{E}_8 \oplus \text{E}_8$	16

Table 1. Momentum lattices Λ_m for the moduli spaces of heterotic $\mathbb{Z}_m$ -triples. The gauge group rank for $m = 1$ is 19, which is just the Narain component. The case $m = 2$ is dual to but not the same as the CHL component [10]. The frozen singularities correspond to the orthogonal complements of $\Lambda_m \hookrightarrow \text{II}_{19,3}$.

gauge groups. Here again we have chosen to write the lattices in terms of the canonical point groups using the lattice isomorphisms $\text{D}_4 \simeq \text{F}_4$ and $\text{A}_2 \simeq \text{G}_2$. We also record the frozen singularity for each lattice Λ_m , which in this context corresponds to the orthogonal complement of the embedding $\Lambda_m \hookrightarrow \text{II}_{19,3}$. This point is discussed in more detail in the next section.

It is natural to ask whether we can extend propositions 3.1 and 3.2 to these lattices. An obvious ansatz is the following:

Proposition 3. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq r_m$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group, and $r_m = 19, 11, 7, 5, 3, 3$ respectively for $m = 1, \dots, 6$. $G \times \text{U}(1)^{19-r_m}$ is realized in the $\mathbb{Z}_m$ -triple as a gauge symmetry group if and only if the weight lattice $M^\vee$ of the dual group $G^\vee$ admits a primitive embedding in the dual momentum lattice $\Lambda_m^*(m)$ such that the vectors in $M^\vee$ of length $\sqrt{2\ell}$ at embedding level $\ell = 1, m$ in $\Lambda_m^*(m)$ belong to $L^\vee$. Simple factors are realized at level $\mathfrak{m} = 2m/\alpha_{\text{long}}^2$, where α_{long} is a long root in $L \hookrightarrow \Lambda_m$.*

The key ingredient is that the vectors of length $\sqrt{2m}$ at embedding level m correspond to massless states and give e.g. long roots for non-ADE gauge groups. This can in fact be explicitly proved in the particular construction used in [10] to obtain the momentum lattices. This roughly corresponds to the fact that in this construction there is a rescaling by a factor of $\sqrt{m}$ involved, such that the product of long roots, coming from invariant states in the parent theory of the orbifold, with all other vectors is scaled by a factor of m . We will however confirm this for the general case by showing in section 4 that assuming this ansatz one can reproduce the mechanism of singularity freezing in the dual M-theory on K3 from the heterotic side.

We note however that there is a subtlety in the case $m = 4$ that makes some gauge groups not conform to Proposition 3.3. The lattice $\Lambda_4 = \text{II}_{3,3} \oplus \text{A}_1 \oplus \text{A}_1$ has dual lattice $\Lambda_4^*(4) = \text{II}_{3,3}(4) \oplus \text{A}_1 \oplus \text{A}_1$. The “canonical” maximal enhancement in the theory has root lattice $L = 5\text{A}_1$, so that the coroot lattice $L^\vee$ found in $\Lambda_4^*(4)$ should be $L^\vee = 5\text{A}_1(4)$. The canonical point in this lattice, however, has lattice $L^\vee = 3\text{A}_1(4) \oplus 2\text{A}_1$, which does not match with the correct coroot lattice. Indeed, the fundamentally correct approach is not to

naively explore the dual lattices in the same way as the original ones. One should instead explicitly look for the coroot lattices of those root lattices obtained from the original lattice and take every other vector in its overlattice as part of the cocharacter lattice. We note however that both procedures give the exact same results except in the $m = 4$ case where these two A_1 's are involved.

An extension of the exploration algorithm used for the CHL string to these lattices is straightforward and produces the results presented in section 5.2. In section 4.3 we will see that these can be reproduced by applying an appropriate projection map to the Narain component.

4 Frozen singularities from the heterotic side

It was already noted by Mikhailov in [9] that the momentum lattice for the CHL string is primitively embedded in the Narain lattice such that its orthogonal complement corresponds to the frozen singularity on the dual F/M-theories on K3 (for $d = 2, 3$, respectively). This observation was extended in [10] to the $\mathbb{Z}_m$ -triples in 7d. Here we make use of it together with Proposition 3.3 to determine precisely how the ADE singularities are partially frozen (usually to give non-ADE algebras) and recover the known “freezing rules” on the K3 side.

4.1 Freezing rules in 8d

Let us first demonstrate the general method of obtaining the freezing rules in the $d = 2$ case, which map gauge groups in the Narain component to the CHL component of the moduli space.

We start by considering an embedding of the Mikhailov lattice $\text{II}_{2,2} \oplus D_8 \simeq \text{II}_{2,2} \oplus C_8$ into the Narain lattice $\text{II}_{18,2}$. This is done in practice by taking the orthogonal complement of any primitively embedded D_8 lattice in $\text{II}_{18,2}$, which is unique modulo automorphisms of the later. We then consider in turn an embedding of a C_n root lattice in the Mikhailov sublattice (cf. Proposition 3.2), which will therefore be also embedded in the Narain lattice,

$$C_n \hookrightarrow \text{II}_{2,2} \oplus C_8 \hookrightarrow \text{II}_{18,2}, \quad n \leq 10. \quad (4.1)$$

This will correspond to an embedding $C_n \oplus D_8 \hookrightarrow \text{II}_{18,2}$ which will however neither be primitive, nor conform to the rules of Proposition 3.1 due to the long roots. It does however define an $(n + 8)$ -plane in the ambient space of $\text{II}_{18,2}$ which in turn defines some primitively embedded weight lattice M . One may chose to focus only on the root sublattice $L \subseteq W$, which is enough to make the comparison with singularity freezing in F-theory. In this case we find that the naive embedding of $C_n \oplus D_8$ inherited from the Mikhailov lattice and the frozen singularity there corresponds an actual embedding

$$D_{n+8} \hookrightarrow \text{II}_{18,2}, \quad (4.2)$$

which may require extra weights (but not roots) to be made primitive. With this we recover the freezing rule for F-theory on K3 in the reverse. Indeed, applying these rules to all the possible gauge algebras in the Narain component gives those in the CHL component [11, 18].

4.2 Freezing rules in 7d

In 7d there are more possibilities for freezing singularities, each one defining a different momentum lattice as shown in table 1. The process outlined above can be repeated in this case and we obtain the following patterns

$$\begin{aligned}
 m = 2 : \quad & C_p + C_q \rightarrow D_{p+4} + D_{q+4}, \quad p, q \geq 0, \\
 & C_p + F_q \rightarrow D_{p+4} + E_{q+4}, \quad p \geq 0, \quad q = 2, 3, 4, \\
 & F_p + F_q \rightarrow E_{p+4} + E_{q+4}, \quad p, q = 2, 3, 4, \\
 m = 3 : \quad & G_p + G_q \rightarrow E_{6+p} + E_{6+q}, \quad p, q = 0, 1, 2, \\
 m = 4 : \quad & A_p + A_q \rightarrow E_{7+p} + E_{7+q}, \quad p, q = 0, 1, \\
 m = 5, 6 : \quad & \emptyset \rightarrow E_8 + E_8,
 \end{aligned} \tag{4.3}$$

where we have defined

$$C_1 \equiv A_1, \quad F_2 \equiv A_2, \quad F_3 \equiv B_3, \quad G_1 \equiv A_1, \tag{4.4}$$

with the r.h.s. algebras always at level 1. We note that for $m = 4$, the A_1 's are not uniquely embedded in the momentum lattice $\Pi_{3,3} \oplus A_1 \oplus A_1$. The ones appearing in the above formulas correspond to those explicitly shown in this lattice, while those embedded in $\Pi_{3,3}$ remain unaffected upon unfreezing. However, all of them are realized at level 4.

The converse rules agree perfectly with the freezing mechanism in M-theory on K3 [10, 14]. When applied to the enhancements found in the Narain moduli space one reproduces the results, at the level of the algebras, obtained with the exploration algorithm applied to the remaining momentum lattices, as expected.

4.3 Full projection map

It was shown in [12] that the list of gauge groups found in the heterotic string on T^2 , together with their fundamental groups, can be projected to that of the 8d CHL string, generalizing the freezing rules for the algebras discussed above. Namely, consider a gauge group, obtained from the Narain lattice, of the form

$$G = \tilde{G}/H = G_1 \times \cdots \times G_s \times \text{Spin}(2n+16)/H, \tag{4.5}$$

where H is generated by an element $k = (k_1, \dots, k_s, \hat{k})$ of the center $Z(\tilde{G})$. The corresponding group in the CHL string will be of the form

$$G' = G_1 \times \cdots \times G_s \times \text{Sp}(n)/H', \tag{4.6}$$

with H' generated by the element $k' = (k_1, \dots, k_s, \hat{k}')$ of the center $Z(\tilde{G}')$. As can be expected, only the contribution of the partially frozen factor will change. Indeed the center of $\text{Spin}(2n+16)$ and that of $\text{Sp}(n)$ are different. For n odd, we have $\hat{k} \in \mathbb{Z}_4$ and $\hat{k}' \in \mathbb{Z}_2$, and the projection reads

$$\hat{k} \rightarrow \hat{k}' = \hat{k} \pmod{2} \quad (\{0, 1, 2, 3\} \rightarrow \{0, 1, 0, 1\}), \quad n = \text{odd}. \tag{4.7}$$

For n even, we have $\hat{k} \equiv (\hat{k}^{(1)}, \hat{k}^{(2)}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and again $\hat{k}' \in \mathbb{Z}_2$, and the projection reads

$$\hat{k} \rightarrow \hat{k}' = \hat{k}^{(1)} + \hat{k}^{(2)} \pmod{2} \quad (\{0, s, c, v\} \rightarrow \{0, 1, 1, 0\}) , \quad n = \text{even} , \quad (4.8)$$

where $\{0, s, c, v\} \equiv \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. As a simple example, the gauge group $\text{Spin}(32)/\mathbb{Z}_2$ is mapped to $\text{Sp}(8)/\mathbb{Z}_2$ [11, 12], since the quotient of the former corresponds to a spinor class in the center. In the case that $n = 0$, we lose a simple factor and $(k_1, \dots, k_s, \hat{k})$ goes to $(k_1, \dots, k_s)$.

This map can be directly generalized to all the different components in the moduli space of 7d theories treated here. Similarly, only the contributions to the fundamental group coming from the partially frozen factors change. In the 7d CHL string the rules for going from D_{n+4} to C_n are equivalent to those for going from D_{n+8} to C_n described above. For example, we find that $(\text{Spin}(24)/\mathbb{Z}_2) \times \text{Spin}(14)$ maps to $(\text{Sp}(8)/\mathbb{Z}_2) \times \text{Sp}(2)$. For the freezing $E_{4+n} \rightarrow F_n$ (cf. (4.3)), the center of the gauge group is unaltered and so is the corresponding contribution to the fundamental group, i.e. $\hat{k} \rightarrow \hat{k}' = \hat{k}$. This is also true for the freezing $E_{6+n} \rightarrow G_n$ in the $m = 3$ case.

For $m = 4$, however, the only partial freezing $E_8 \rightarrow A_1$ does present a change in the center of the gauge group. Our results indicate that the contribution to the fundamental group changes from $\hat{k} = 0$ to $\hat{k}' = 1$. In other words, $E_8 \rightarrow \text{SU}(2)/\mathbb{Z}_2 \simeq \text{SO}(3)$. This can be seen from the fact that the coroot lattice of the rightmost A_1 's in $\Lambda_4 = \text{II}_{3,3} \oplus A_1 \oplus A_1$ embeds into $\Lambda_4^* = \text{II}_{3,3}^* \oplus A_1^* \oplus A_1^*$ such that its real span contains the fundamental weights of each A_1 .

For $m = 5, 6$, the rule $E_8 \rightarrow \emptyset$ has no effect on the fundamental group other than shortening $(k_1, \dots, k_s, \hat{k})$ to $(k_1, \dots, k_s)$. With these generalized freezing rules, one can project the enhancements in the Narain component of the moduli space to the other five components treated in this paper to reproduce the results found with our exploration algorithm.

5 Classification of gauge groups

Now we present the main results of this work and expand in the methods used to obtain them. The full tables with maximal enhancements and their global data are given in appendix B. Here we give tables with the counting of the different gauge symmetries which are realized in each component.

5.1 Narain component

Obtaining the gauge groups for the Narain component is done with a straightforward extension of the original exploration algorithm developed in [5]. Here we have however also computed the complete global data for each group, giving the explicit generators for the fundamental groups using the methods of [11] based on [12]. We have for example the gauge group (# 421 of table 4)

$$\frac{\text{SU}(8) \times \text{SU}(8) \times \text{Spin}(10)}{\mathbb{Z}_8}, \quad (5.1)$$

Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Z}_3^2$	$\mathbb{Z}_2\mathbb{Z}_4$	$\mathbb{Z}_7$	$\mathbb{Z}_2\mathbb{Z}_6$	$\mathbb{Z}_4^2$	$\mathbb{Z}_8$	Algebras	Groups
19	652	381	68	51	37	5	1	6	16	3	2	1	2	1	2	1035	1232
18	852	492	89	52	35	9	1	4	10	3	6	1	1	1	1	1180	1557
17	827	442	73	39	23	8	1	2	4	2	3					1024	1424
16	694	334	47	25	12	4	1	1	1	1	1					793	1121
15	528	217	24	12	4	2	1									567	788
14	389	128	11	6	1	1										403	536
13	272	66	3	2												276	343
12	192	33	1	1												193	227
11	128	14														128	142
10	88	6														88	94
9	57	2														57	59
8	39	1														39	40
7	24															24	24
6	16															16	16
5	9															9	9
4	6															6	6
3	3															3	3
2	2															2	2
1	1															1	1
All	4779	2116	316	188	112	29	5	13	31	9	12	2	3	2	3	5844	7624

Table 2. Number of algebras and groups of each rank with a certain fundamental group for the heterotic string on T^3 .

where the fundamental group $\mathbb{Z}_8$ is generated by the element $(1, 1, 3)$ of the center $\mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4$ of the universal cover $SU(8) \times SU(8) \times Spin(10)$.

All the maximally enhanced groups in this component are listed in table 4 in appendix B.1. The data includes the ADE type of the gauge group and the corresponding fundamental group. The generators of the fundamental group are listed in table 9 in appendix B.2. For each generator we give a sequence of numbers representing the contribution from the center of each simple factor. In the example just given, the generator is 113. Note that the ordering of the sequence corresponds to the ordering of the listed ADE type. To properly read the sequence one must write expressions of the form $A_3^2 D_4^3$ as $(A_3, A_3, D_4, D_4, D_4)$, e.g, assigning each number in the sequence to each subsequent ADE factor. For D_{2n} factors there are four order two elements in the center denoted v , c , s and 1 , corresponding to the vector class, spinor classes and the identity, respectively. Note that in some cases the fundamental group has more than one generator.

The total number of distinct gauge algebras and distinct gauge groups for different ranks of the semisimple part are listed in table 2. These have been obtained by deleting nodes in the Dynkin Diagrams of the maximally enhanced groups, and we assume that this gives all the possibilities, as discussed in section 3.1.1.

We note that there are many cases in which two gauge groups have isomorphic fundamental groups with inequivalent inclusions in the center of the universal covering (meaning that they are not related by outer automorphisms of the group, as is the case e.g. for

$\mathbb{Z}_2$ triple										$\mathbb{Z}_4$ triple					
Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	Algebras	Groups	Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	Algebras	Groups
11	224	143	44	7	3	7	1	407	429	5	5	10	3	9	18
10	307	192	51	5	3	8	1	473	567	4	5	7	2	6	14
9	284	161	37	2	2	4	1	372	491	3	3	4	1	3	8
8	214	101	18	1	1	2		244	337	2	2	2	1	2	5
7	137	45	5					143	187	1	1	1		1	2
6	84	17	1					85	102	All	16	24	7	21	47
5	46	4						46	50	$\mathbb{Z}_3$ triple					
4	26	1						26	27	Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_3$	Algebras	Groups
3	12							12	12	7	41	6	5	50	52
2	6							6	6	6	37	5	4	41	46
1	2							2	2	5	24	2	2	24	28
All	1342	664	156	15	9	21	3	1816	2210	4	15	1	1	15	17
										3	8			8	8
										2	5			5	5
										1	2			2	2
										All	132	14	12	145	158
										$\mathbb{Z}_5$ and $\mathbb{Z}_6$ triples					
										Rank	1		Algebras		Groups
										3	3		3		3
										2	2		2		2
										1	1		1		1
										All	6		6		6

Table 3. Number of algebras and groups of each rank with a certain fundamental group for the heterotic $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$ triples.

$\text{SO}(2n)$ versus $\text{Spin}(2n)/\mathbb{Z}_2$ for $n \neq 4$). These are not distinguished in table 4, so that the numbering goes only up to 1163. The inequivalence is taken into account in table 9 by putting primes on the corresponding numbering.

5.2 Triples

The results for the components of the moduli space with rank reduction are obtained by an extension of the exploration algorithm taking into account Proposition 3.3. The gauge groups are recorded in tables 5 to 8 in appendix B.1, while the generators for the fundamental groups are recorded in tables 10 and 11 in appendix B.2. In the case of the $\mathbb{Z}_5$ and $\mathbb{Z}_6$ -triples all of the gauge groups are simply connected and so no global data is required to specify them. The data is presented with the same conventions as for the Narain component, together with the notation defined in eq. (4.4). As explained in section 4.3, all the gauge groups for the non-trivial $\mathbb{Z}_m$ triples can be obtained from those of the Narain component using a projection map generalizing the one obtained in [12] for the 8d CHL string. The total number of distinct gauge algebras and distinct gauge groups are listed in table 3.

6 Conclusions

In this paper we have advanced the classification of possible gauge groups realized in the string landscape to three-dimensional internal manifolds when the number of supercharges is 16. This was done by finding embeddings of weight lattices into the momentum lattices constructed in [10], taking into account an extra constraint on the role of the lattice vectors as stated in Proposition 3.3. We have however ignored other components with further rank reduction (see e.g. [19–21]), which to our knowledge have so far no heterotic description.

We have also studied the mechanism of partial singularity freezing found in M-theory on K3 from the heterotic side, using lattice embedding techniques. We have also constructed the more general freezing rules which indicate how the topology of the gauge group changes,

generalizing the results of [12] for the 8d CHL string. It is clear that this mechanism can be generalized in the heterotic string to compactifications to lower dimensions, e.g. on T^4 . We plan on investigating this point further in a future work.

Finally, we note that our results may serve to test Swampland conjectures [22], which are also easier to study in high dimensional theories with high supersymmetry (see e.g. [18, 21, 23]). More generally it would be interesting to determine with more confidence if there are or not more components in the moduli space of heterotic strings with 16 supercharges using the techniques of asymmetric orbifolds. Recent results [18, 21] indicate that this is not the case in 9d and 8d, but the 7d case remains in question.

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A Comments on lattices technicalities

In this appendix we discuss some technicalities regarding lattice embeddings in the context of the heterotic string and its compactifications considered in the paper, namely the concept of a primitive embedding and of lattice isomorphisms.

A.1 Primitive embeddings

The Narain moduli space is usually understood as the space of embeddings of the even self-dual Lorentzian lattice $\text{II}_{16+d,d}$ into an ambient space $\mathbb{R}^{16+d,d}$. The roots of the gauge group are associated to lattice vectors $(P, p_L; p_R)$ subject to the constraints $P^2 + p_L^2 = 2$ and $p_R = 0$, hence they are precisely the vectors of squared norm 2 which lie in the positive definite subspace $\mathbb{R}^{16+d}$. When one rotates and/or boosts the lattice by an $\text{SO}(d+16, d)$ transformation, one moves through the moduli space and the set of vectors intersecting $\mathbb{R}^{16+d}$ changes. This corresponds to changes in the spectrum, particularly in the gauge symmetry group.

Conversely we may think of this problem as fixing the Narain lattice itself and transforming instead the ambient space. Intuitively we can think of rotating the positive-definite $(16+d)$ -plane in relation to the lattice, whose origin it intersects. This intuition can be formalized by noting that the moduli space is indeed (locally) a Grassmann space:

$$\mathcal{M} = \text{SO}(16+d, d)/\text{SO}(16+d) \times \text{SO}(d) = \text{Gr}(16+d, d). \quad (\text{A.1})$$

Each model in moduli space corresponds therefore to a choice of a $(16+d)$ -plane whose intersection with the Narain lattice yields the data for the gauge group. Indeed, this

intersection contains vectors which are not necessarily constrained to have squared norm 2, which in the case of not being sums of roots will correspond to weights. Because of this we refer to the intersecting lattice as M .

The weight lattice M is an example of a primitive embedding. Technically, a primitive embedding of a lattice L into another lattice M is such that the intersection of the real span of L with M is L itself. A non-primitive embedding is provided by any proper sublattice M' of M of the same rank, since its real span will contain vectors of M not in M' . An example of this is the embedding of the lattice D_8 into E_8 , as the real span of D_8 contains the spinor weight which extends it to E_8 .

From this discussion we see that the natural way of thinking about sublattices of the Narain lattice (and other momentum lattices) is by considering primitive embeddings, which are in general not root lattices, but *weight* lattices. There are of course other possibilities such as lattices with no roots and hence no weights, but we do not consider these cases as they play no role in the classification of the gauge algebra performed here.

A.2 Lattice isomorphisms

In this paper we have chosen to describe the momentum lattices in ways that naturally reflect the gauge groups that they yield in the respective string theory. The simplest example is that of the Mikhailov lattice in 8d, which was originally written as

$$\Pi_{1,1}(2) \oplus \Pi_{1,1} \oplus E_8 \simeq \Pi_{2,2} \oplus D_8. \quad (\text{A.2})$$

We write it instead as

$$\Pi_{2,2} \oplus C_8, \quad (\text{A.3})$$

where the lattice isomorphism $D_n \simeq C_n$ has been employed. Indeed, both lattices are exactly the same, but the *algebras* are not. For instance, one finds the lattice C_8 in the Narain moduli space, but its long root does not give a massless state and so the gauge algebra is D_8 . In the CHL string however one can obtain such a massless state and indeed realize the gauge algebra C_8 .

For the CHL string in 7d we similarly use the lattice isomorphism $D_4 \simeq F_4$, and the story is similar to the 8d case just discussed. This extends to the $\mathbb{Z}_3$ -triple where one finds the gauge algebra $2G_2$ at the canonical point in moduli space and not $2A_2$.

B Maximal enhancements for 7d heterotic string

In this appendix we record the maximally enhanced gauge groups realized in the 7d $\mathbb{Z}_m$ -triples constructed from the heterotic string. The algebras and the fundamental groups are presented in appendix B.1, while the generators of the fundamental group are presented in appendix B.2. The way in which the data is encoded is explained in section 5.

B.1 Maximally enhanced algebras and fundamental groups

1 $A_1 A_3^6$	$\mathbb{Z}_4^2$	48 $A_2^4 A_5 A_6$	$\mathbb{Z}_3$	95 $A_1^2 A_5^2 A_7$	$\mathbb{Z}_2$	142 $A_1^2 A_4 A_5 A_8$	189 $A_1^4 A_6 A_9$	$\mathbb{Z}_2$
2 $A_3^5 A_4$	$\mathbb{Z}_4$	49 $A_1 A_2^3 A_3 A_5 A_6$		96 $A_1^2 A_5^2 A_7$	$\mathbb{Z}_2^2$	143 $A_2 A_4 A_5 A_8$	190 $A_1^2 A_2 A_6 A_9$	
3 $A_1 A_2^3 A_3^2 A_4^2$		50 $A_1^2 A_2^3 A_5 A_6$	$\mathbb{Z}_2$	97 $A_2 A_5^2 A_7$		144 $A_2 A_4 A_5 A_8$	191 $A_2^2 A_6 A_9$	
4 $A_1^2 A_3^3 A_4^2$	$\mathbb{Z}_2$	51 $A_2 A_2^3 A_5 A_6$		98 $A_1^2 A_2^2 A_6 A_7$		145 $A_1 A_5^2 A_8$	192 $A_1 A_3 A_6 A_9$	
5 $A_2 A_3^3 A_4^2$		52 $A_1^2 A_2 A_4 A_5 A_6$		99 $A_3^3 A_6 A_7$		146 $A_1^3 A_2 A_6 A_8$	193 $A_1 A_3 A_6 A_9$	$\mathbb{Z}_2$
6 $A_2^2 A_3 A_4^3$		53 $A_2^2 A_4 A_5 A_6$		100 $A_1^3 A_3 A_6 A_7$	$\mathbb{Z}_2$	147 $A_1 A_2^2 A_6 A_8$	194 $A_4 A_6 A_9$	
7 $A_1^3 A_4^4$	$\mathbb{Z}_5$	54 $A_1 A_3 A_4 A_5 A_6$		101 $A_1 A_2 A_3 A_6 A_7$		148 $A_1^3 A_6 A_8$	195 $A_1^3 A_7 A_9$	$\mathbb{Z}_2$
8 $A_1 A_2 A_4^4$	$\mathbb{Z}_5$	55 $A_4^2 A_5 A_6$		102 $A_2^3 A_6 A_7$		149 $A_2 A_3 A_6 A_8$	196 $A_1 A_2 A_7 A_9$	
9 $A_3 A_4^4$	$\mathbb{Z}_5$	56 $A_1^3 A_5^2 A_6$	$\mathbb{Z}_2$	103 $A_2^3 A_6 A_7$	$\mathbb{Z}_2$	150 $A_1 A_4 A_6 A_8$	197 $A_3 A_7 A_9$	
10 $A_2^4 A_3^2 A_5$	$\mathbb{Z}_3$	57 $A_1 A_2 A_5^2 A_6$		104 $A_1^2 A_4 A_6 A_7$		151 $A_5 A_6 A_8$	198 $A_1^2 A_8 A_9$	
11 $A_1^2 A_2 A_3^2 A_4 A_5$	$\mathbb{Z}_2$	58 $A_3 A_5^2 A_6$		105 $A_2 A_4 A_6 A_7$		152 $A_1^4 A_7 A_8$	199 $A_2 A_8 A_9$	$\mathbb{Z}_2$
12 $A_2^2 A_3^2 A_4 A_5$		59 $A_3 A_5^2 A_6$	$\mathbb{Z}_2$	106 $A_1 A_5 A_6 A_7$		153 $A_1^2 A_2 A_7 A_8$	200 $A_1 A_9^2$	
13 $A_1 A_3^3 A_4 A_5$	$\mathbb{Z}_2$	60 $A_1^3 A_5^2 A_6^2$		107 $A_1 A_5 A_6 A_7$	$\mathbb{Z}_2$	154 $A_2^2 A_7 A_8$	201 $A_1 A_9^2$	$\mathbb{Z}_5$
14 $A_1^3 A_3 A_4^2 A_5$	$\mathbb{Z}_2$	61 $A_1 A_3^2 A_6^2$		108 $A_2^2 A_7$		155 $A_1 A_3 A_7 A_8$	202 $A_1^2 A_2 A_3 A_{10}$	
15 $A_1 A_2 A_3 A_4^2 A_5$		62 $A_1^2 A_2 A_3 A_6^2$		109 $A_1^2 A_7^2$	$\mathbb{Z}_2 \mathbb{Z}_4$	156 $A_4 A_7 A_8$	203 $A_3^3 A_{10}$	
16 $A_2^2 A_3^2 A_5$		63 $A_2^2 A_3 A_6^2$		110 $A_1^3 A_2 A_7^2$	$\mathbb{Z}_4$	157 $A_3^3 A_8^2$	204 $A_1 A_2 A_3^3 A_{10}$	
17 $A_1^3 A_4^3 A_5$		64 $A_1 A_3^2 A_6^2$		111 $A_1 A_2^2 A_7^2$		158 $A_1^3 A_8^2$	205 $A_3^3 A_{10}$	$\mathbb{Z}_3$
18 $A_1 A_2^4 A_5$	$\mathbb{Z}_3^2$	65 $A_1^3 A_4 A_6^2$		112 $A_1 A_2^2 A_7^2$	$\mathbb{Z}_2$	159 $A_1 A_2 A_8^2$	206 $A_1^3 A_2 A_4 A_{10}$	
19 $A_1^2 A_2^2 A_3 A_5^2$	$\mathbb{Z}_6$	66 $A_1 A_2 A_4 A_6^2$		113 $A_1^2 A_3 A_7^2$	$\mathbb{Z}_2$	160 $A_1 A_2 A_8^2$	207 $A_1 A_2^2 A_4 A_{10}$	$\mathbb{Z}_3$
20 $A_2^3 A_3 A_5^2$	$\mathbb{Z}_3$	67 $A_3 A_4 A_6^2$		114 $A_2^2 A_3 A_7^2$	$\mathbb{Z}_2^2$	161 $A_3 A_8^2$	208 $A_1^2 A_3 A_4 A_{10}$	
21 $A_2^3 A_3 A_5^2$	$\mathbb{Z}_6$	68 $A_1^2 A_5 A_6^2$		115 $A_1^2 A_3 A_7^2$	$\mathbb{Z}_2 \mathbb{Z}_4$	162 $A_3 A_8^2$	209 $A_2 A_3 A_4 A_{10}$	$\mathbb{Z}_3$
22 $A_1^3 A_3^2 A_5^2$	$\mathbb{Z}_2^2$	69 $A_2 A_5 A_6^2$		116 $A_1^2 A_3 A_7^2$	$\mathbb{Z}_8$	163 $A_1^3 A_2^3 A_3 A_9$	210 $A_1 A_4^2 A_{10}$	$\mathbb{Z}_2$
23 $A_1 A_2 A_3^2 A_5^2$	$\mathbb{Z}_2$	70 $A_1 A_6^3$	$\mathbb{Z}_7$	117 $A_2 A_3 A_7^2$		164 $A_1 A_2^3 A_3 A_9$	211 $A_1^2 A_2 A_5 A_{10}$	
24 $A_3^3 A_5^2$	$\mathbb{Z}_2$	71 $A_1^2 A_2^2 A_3 A_7$		118 $A_2 A_3 A_7^2$	$\mathbb{Z}_2$	165 $A_1 A_2^3 A_3 A_9$	212 $A_2^2 A_5 A_{10}$	$\mathbb{Z}_2$
25 $A_1 A_2^3 A_4 A_5^2$	$\mathbb{Z}_3$	72 $A_1^2 A_3^3 A_7$	$\mathbb{Z}_2 \mathbb{Z}_4$	119 $A_2 A_3 A_7^2$	$\mathbb{Z}_4$	166 $A_1^2 A_2^3 A_3 A_9$	213 $A_1 A_3 A_5 A_{10}$	$\mathbb{Z}_2$
26 $A_1^2 A_3 A_4 A_5^2$	$\mathbb{Z}_2$	73 $A_1 A_2 A_3^3 A_7$	$\mathbb{Z}_4$	120 $A_1 A_4 A_7^2$		167 $A_2^2 A_3 A_9$	214 $A_4 A_5 A_{10}$	
27 $A_2 A_3 A_4 A_5^2$		74 $A_4^4 A_7$	$\mathbb{Z}_4$	121 $A_1 A_4 A_7^2$	$\mathbb{Z}_2$	168 $A_1 A_3^3 A_9$	215 $A_1^3 A_6 A_{10}$	$\mathbb{Z}_2$
28 $A_2 A_3 A_4 A_5^2$	$\mathbb{Z}_2$	75 $A_4^4 A_7$	$\mathbb{Z}_2 \mathbb{Z}_4$	122 $A_5 A_7^2$		169 $A_1^4 A_2 A_4 A_9$	216 $A_1 A_2 A_6 A_{10}$	$\mathbb{Z}_2$
29 $A_1 A_2^2 A_5^2$		76 $A_1^3 A_2 A_3 A_4 A_7$	$\mathbb{Z}_2$	123 $A_5 A_7^2$	$\mathbb{Z}_2$	170 $A_1^2 A_2^2 A_4 A_9$	217 $A_3 A_6 A_{10}$	
30 $A_1^4 A_5^3$	$\mathbb{Z}_2 \mathbb{Z}_6$	77 $A_1 A_2^2 A_3 A_4 A_7$		124 $A_1^2 A_2^3 A_3 A_8$	$\mathbb{Z}_3$	171 $A_1^3 A_4 A_9$	218 $A_1^2 A_7 A_{10}$	
31 $A_1^2 A_2 A_5^3$	$\mathbb{Z}_6$	78 $A_1^2 A_2^3 A_4 A_7$	$\mathbb{Z}_2$	125 $A_1^2 A_3 A_8$	$\mathbb{Z}_3$	172 $A_1^3 A_3 A_4 A_9$	219 $A_2 A_7 A_{10}$	$\mathbb{Z}_2$
32 $A_2^2 A_5^3$	$\mathbb{Z}_3$	79 $A_1^2 A_3^2 A_4 A_7$	$\mathbb{Z}_4$	126 $A_1 A_2^2 A_3 A_8$		173 $A_1 A_2 A_3 A_4 A_9$	220 $A_1 A_8 A_{10}$	
33 $A_2^2 A_5^3$	$\mathbb{Z}_3^2$	80 $A_2 A_3^2 A_4 A_7$		127 $A_1^2 A_3^3 A_8$	$\mathbb{Z}_2$	174 $A_1 A_2 A_3 A_4 A_9$	221 $A_9 A_{10}$	$\mathbb{Z}_2$
34 $A_1 A_3 A_5^3$	$\mathbb{Z}_2$	81 $A_2 A_3^2 A_4 A_7$	$\mathbb{Z}_2$	128 $A_2 A_3^3 A_8$		175 $A_3^4 A_9$	222 $A_1^4 A_2^2 A_{11}$	$\mathbb{Z}_6$
35 $A_1 A_3 A_5^3$	$\mathbb{Z}_6$	82 $A_1^4 A_4 A_7$	$\mathbb{Z}_2$	129 $A_1 A_2^3 A_4 A_8$	$\mathbb{Z}_3$	176 $A_1^2 A_4 A_9$	223 $A_1^2 A_2^3 A_{11}$	$\mathbb{Z}_3$
36 $A_4 A_5^3$		83 $A_1^2 A_2^2 A_4 A_7$		130 $A_1^2 A_2 A_3 A_4 A_8$		177 $A_1^2 A_4 A_9$	224 $A_1^2 A_3^2 A_{11}$	$\mathbb{Z}_6$
37 $A_4 A_5^3$	$\mathbb{Z}_3$	84 $A_2^2 A_4 A_7$		131 $A_2^2 A_3 A_4 A_8$		178 $A_2 A_4^2 A_9$	225 $A_4^2 A_{11}$	$\mathbb{Z}_3$
38 $A_1 A_2^3 A_3^2 A_6$		85 $A_1 A_3 A_4^2 A_7$		132 $A_1 A_2^3 A_4 A_8$		179 $A_1^3 A_2 A_5 A_9$	226 $A_1^3 A_2 A_3 A_{11}$	$\mathbb{Z}_2$
39 $A_1^2 A_2^3 A_3 A_4 A_6$		86 $A_1^4 A_3 A_5 A_7$	$\mathbb{Z}_2^2$	133 $A_1^3 A_4^2 A_8$		180 $A_1 A_2^2 A_5 A_9$	227 $A_1^3 A_2 A_3 A_{11}$	$\mathbb{Z}_4$
40 $A_2^3 A_3 A_4 A_6$		87 $A_1^2 A_2 A_3 A_5 A_7$	$\mathbb{Z}_2$	134 $A_1 A_2 A_4^2 A_8$		181 $A_1 A_2^2 A_5 A_9$	228 $A_1 A_2^2 A_3 A_{11}$	
41 $A_1 A_2 A_2^3 A_4 A_6$		88 $A_2^2 A_3 A_5 A_7$		135 $A_3 A_4^2 A_8$		182 $A_1^2 A_3 A_5 A_9$	229 $A_1 A_2^2 A_3 A_{11}$	$\mathbb{Z}_2$
42 $A_3^3 A_4 A_6$		89 $A_1 A_2^3 A_5 A_7$	$\mathbb{Z}_2$	136 $A_1^2 A_2^2 A_5 A_8$	$\mathbb{Z}_3$	183 $A_2 A_3 A_5 A_9$	230 $A_1 A_2^2 A_3 A_{11}$	$\mathbb{Z}_3$
43 $A_1^3 A_2 A_4^2 A_6$		90 $A_1 A_2^3 A_5 A_7$	$\mathbb{Z}_2^2$	137 $A_3^3 A_5 A_8$	$\mathbb{Z}_3$	184 $A_2 A_3 A_5 A_9$	231 $A_1 A_2^2 A_3 A_{11}$	$\mathbb{Z}_6$
44 $A_1 A_2^2 A_4^2 A_6$		91 $A_1^3 A_4 A_5 A_7$	$\mathbb{Z}_2$	138 $A_1^3 A_3 A_5 A_8$	$\mathbb{Z}_2$	185 $A_1 A_4 A_5 A_9$	232 $A_1^2 A_3^2 A_{11}$	$\mathbb{Z}_2$
45 $A_2^2 A_3 A_4^2 A_6$		92 $A_1 A_2 A_4 A_5 A_7$		139 $A_1 A_2 A_3 A_5 A_8$		186 $A_1 A_4 A_5 A_9$	233 $A_1^2 A_3^3 A_{11}$	$\mathbb{Z}_4$
46 $A_2 A_3 A_4^2 A_6$		93 $A_1 A_2 A_4 A_5 A_7$	$\mathbb{Z}_2$	140 $A_1 A_2 A_3 A_5 A_8$	$\mathbb{Z}_3$	187 $A_5^2 A_9$	234 $A_2 A_3^2 A_{11}$	
47 $A_1 A_4^3 A_6$		94 $A_3 A_4 A_5 A_7$		141 $A_3^2 A_5 A_8$		188 $A_5^2 A_9$	235 $A_2 A_3^2 A_{11}$	$\mathbb{Z}_2$

236 $A_2 A_3^2 A_{11}$	Z_4	283 $A_6 A_{13}$		330 $A_2 A_3 A_4 A_6 D_4$		377 $A_2 A_3^3 D_4^2$	Z_2^2	424 $A_1^2 A_4 A_8 D_5$	
237 $A_1^4 A_4 A_{11}$	Z_2	284 $A_1^3 A_2 A_{14}$		331 $A_1 A_4^2 A_6 D_4$		378 $A_1 A_5^2 D_4^2$	Z_2^2	425 $A_2 A_4 A_8 D_5$	
238 $A_1^2 A_2 A_4 A_{11}$		285 $A_1^3 A_2 A_{14}$	Z_3	332 $A_1 A_3 A_5 A_6 D_4$	Z_2	379 $A_5^2 A_7 D_4^2$	Z_2	426 $A_1 A_5 A_8 D_5$	
239 $A_1^2 A_2 A_4 A_{11}$	Z_2	286 $A_1 A_2^2 A_{14}$		333 $A_4 A_5 A_6 D_4$		380 $A_1^3 D_4^4$	Z_2^4	427 $A_6 A_8 D_5$	
240 $A_2^2 A_4 A_{11}$		287 $A_1 A_2^2 A_{14}$	Z_3	334 $A_1 A_2 A_6^2 D_4$		381 $A_1 A_2^2 A_3^3 D_5$	Z_4	428 $A_1^3 A_2 A_9 D_5$	Z_2
241 $A_2^2 A_4 A_{11}$	Z_3	288 $A_1^2 A_3 A_{14}$		335 $A_3 A_6^2 D_4$		382 $A_2 A_3^2 D_5$	Z_4	429 $A_1 A_2^2 A_9 D_5$	
242 $A_1 A_3 A_4 A_{11}$		289 $A_2 A_3 A_{14}$		336 $A_1^2 A_3 A_7 D_4$	Z_2	383 $A_2^2 A_3^2 A_4 D_5$		430 $A_1 A_2^2 A_9 D_5$	Z_2
243 $A_1 A_3 A_4 A_{11}$	Z_2	290 $A_2 A_3 A_{14}$	Z_3	337 $A_1 A_2^2 A_3 A_7 D_4$	Z_2	384 $A_1 A_3^2 A_4 D_5$	Z_4	431 $A_1^2 A_3 A_9 D_5$	Z_2
244 $A_4^2 A_{11}$		291 $A_1 A_4 A_{14}$		338 $A_1^2 A_3^2 A_7 D_4$	Z_2^2	385 $A_1^2 A_2^2 A_4^2 D_5$		432 $A_2 A_3 A_9 D_5$	
245 $A_1^3 A_5 A_{11}$	Z_2	292 $A_5 A_{14}$		339 $A_1^2 A_3^2 A_7 D_4$	$Z_2 Z_4$	386 $A_1 A_2 A_3 A_4^2 D_5$		433 $A_1 A_4 A_9 D_5$	
246 $A_1^3 A_5 A_{11}$	Z_6	293 $A_1^4 A_{15}$	Z_4	340 $A_2 A_3^2 A_7 D_4$	Z_2	387 $A_1^2 A_3^2 D_5$		434 $A_1 A_4 A_9 D_5$	Z_2
247 $A_1 A_2 A_5 A_{11}$		294 $A_1^2 A_2 A_{15}$		341 $A_2 A_3^2 A_7 D_4$	Z_4	388 $A_1^3 A_3^2 A_5 D_5$	Z_2^2	435 $A_5 A_9 D_5$	
248 $A_1 A_2 A_5 A_{11}$	Z_3	295 $A_1^2 A_2 A_{15}$	Z_2	342 $A_1^2 A_2 A_4 A_7 D_4$	Z_2	389 $A_1 A_2 A_3^2 A_5 D_5$	Z_2	436 $A_5 A_9 D_5$	Z_2
249 $A_3 A_5 A_{11}$		296 $A_1^2 A_2 A_{15}$	Z_4	343 $A_2^2 A_4 A_7 D_4$		390 $A_1^2 A_3 A_4 A_5 D_5$	Z_2	437 $A_1^2 A_2 A_{10} D_5$	
250 $A_3 A_5 A_{11}$	Z_2	297 $A_2^2 A_{15}$		344 $A_1 A_3 A_4 A_7 D_4$	Z_2	391 $A_2 A_3 A_4 A_5 D_5$		438 $A_2^2 A_{10} D_5$	
251 $A_3 A_5 A_{11}$		298 $A_2^2 A_{15}$	Z_2	345 $A_1^3 A_5 A_7 D_4$	Z_2^2	392 $A_1 A_2^2 A_5 D_5$		439 $A_1 A_3 A_{10} D_5$	
252 $A_3 A_5 A_{11}$	Z_6	299 $A_1 A_3 A_{15}$		346 $A_1 A_2 A_5 A_7 D_4$	Z_2	393 $A_1^4 A_5^2 D_5$	Z_2^2	440 $A_4 A_{10} D_5$	
253 $A_1^2 A_6 A_{11}$		300 $A_1 A_3 A_{15}$	Z_2	347 $A_3 A_5 A_7 D_4$	Z_2	394 $A_2^2 A_5^2 D_5$	Z_3	441 $A_1^3 A_{11} D_5$	Z_2
254 $A_1^2 A_6 A_{11}$	Z_2	301 $A_1 A_3 A_{15}$	Z_4	348 $A_1^2 A_6 A_7 D_4$	Z_2	395 $A_2^2 A_5^2 D_5$	Z_6	442 $A_1^3 A_{11} D_5$	Z_4
255 $A_2 A_6 A_{11}$		302 $A_4 A_{15}$		349 $A_2 A_6 A_7 D_4$		396 $A_1 A_3 A_5^2 D_5$	Z_2	443 $A_1 A_2 A_{11} D_5$	
256 $A_1 A_7 A_{11}$		303 $A_4 A_{15}$	Z_2	350 $A_1 A_7^2 D_4$	Z_4	397 $A_4 A_5^2 D_5$		444 $A_1 A_2 A_{11} D_5$	Z_2
257 $A_8 A_{11}$		304 $A_1^3 A_{16}$		351 $A_1 A_1^3 A_8 D_4$	Z_3	398 $A_4 A_5^2 D_5$	Z_2	445 $A_1 A_2 A_{11} D_5$	Z_4
258 $A_1^3 A_2^2 A_{12}$		305 $A_1 A_2 A_{16}$		352 $A_1 A_2 A_4 A_8 D_4$		399 $A_1 A_2^2 A_3 A_6 D_5$		446 $A_3 A_{11} D_5$	Z_2
259 $A_1 A_1^3 A_{12}$		306 $A_3 A_{16}$		353 $A_2 A_5 A_8 D_4$	Z_3	400 $A_2 A_3^2 A_6 D_5$		447 $A_3 A_{11} D_5$	Z_4
260 $A_1^2 A_2 A_3 A_{12}$		307 $A_2^2 A_{17}$		354 $A_1 A_6 A_8 D_4$		401 $A_1^2 A_2 A_4 A_6 D_5$		448 $A_1^2 A_{12} D_5$	
261 $A_2^2 A_3 A_{12}$		308 $A_1^2 A_{17}$	Z_3	355 $A_1^2 A_2 A_9 D_4$	Z_2	402 $A_2^2 A_4 A_6 D_5$		449 $A_2 A_{12} D_5$	
262 $A_1 A_2^2 A_{12}$		309 $A_2 A_{17}$		356 $A_1 A_2 A_3 A_9 D_4$	Z_2	403 $A_1 A_3 A_4 A_6 D_5$		450 $A_1 A_{13} D_5$	
263 $A_1^3 A_4 A_{12}$		310 $A_2 A_{17}$	Z_3	357 $A_1^2 A_4 A_9 D_4$	Z_2	404 $A_4 A_6 D_5$		451 $A_1 A_{13} D_5$	Z_2
264 $A_1 A_2 A_4 A_{12}$		311 $A_1 A_{18}$		358 $A_2 A_4 A_9 D_4$		405 $A_1 A_2 A_5 A_6 D_5$		452 $A_{14} D_5$	
265 $A_3 A_4 A_{12}$		312 A_{19}		359 $A_1 A_5 A_9 D_4$	Z_2	406 $A_3 A_5 A_6 D_5$		453 $A_2^2 A_3^2 D_4 D_5$	Z_2
266 $A_1^2 A_5 A_{12}$		313 $A_3^2 D_4$	$Z_2 Z_4$	360 $A_6 A_9 D_4$		407 $A_1^2 A_6^2 D_5$		454 $A_2^2 A_4 D_4 D_5$	
267 $A_2 A_5 A_{12}$		314 $A_2 A_3^3 A_4 D_4$	Z_2	361 $A_1 A_2^2 A_{10} D_4$		408 $A_2 A_6^2 D_5$		455 $A_1 A_2^2 A_5 D_4 D_5$	Z_2
268 $A_1 A_6 A_{12}$		315 $A_1 A_2^3 A_4^2 D_4$		362 $A_2 A_3 A_{10} D_4$		409 $A_1^3 A_2^2 A_7 D_5$	Z_2	456 $A_1^2 A_3 A_5 D_4 D_5$	Z_2^2
269 $A_7 A_{12}$		316 $A_2^2 A_3 A_4^2 D_4$		363 $A_1 A_4 A_{10} D_4$		410 $A_1^2 A_2 A_3 A_7 D_5$	Z_2	457 $A_1 A_4 A_5 D_4 D_5$	Z_2
270 $A_1^4 A_2 A_{13}$	Z_2	317 $A_1 A_2 A_4^3 D_4$		364 $A_5 A_{10} D_4$		411 $A_1^2 A_2 A_3 A_7 D_5$	Z_4	458 $A_5^2 D_4 D_5$	Z_2
271 $A_1^2 A_2^2 A_{13}$		318 $A_1^2 A_2 A_3^2 A_5 D_4$	Z_2^2	365 $A_2^2 A_2 A_{11} D_4$	Z_2	412 $A_2^2 A_3 A_7 D_5$		459 $A_2^2 A_6 D_4 D_5$	
272 $A_1^2 A_2^2 A_{13}$	Z_2	319 $A_1 A_3^3 A_5 D_4$	Z_2^2	366 $A_2^2 A_{11} D_4$	Z_3	413 $A_2^2 A_3 A_7 D_5$	Z_2	460 $A_1 A_2 A_7 D_4 D_5$	Z_2
273 $A_2^3 A_{13}$		320 $A_1 A_2 A_3 A_4 A_5 D_4$	Z_2	367 $A_2^2 A_{11} D_4$	Z_6	414 $A_1 A_3^2 A_7 D_5$	Z_4	461 $A_1 A_9 D_4 D_5$	Z_2
274 $A_1^3 A_3 A_{13}$	Z_2	321 $A_2 A_2^2 A_5 D_4$		368 $A_1 A_3 A_{11} D_4$	Z_2	415 $A_1^3 A_4 A_7 D_5$	Z_2	462 $A_1^2 A_3^2 A_3 D_5^2$	Z_2
275 $A_1 A_2 A_3 A_{13}$		322 $A_1^3 A_2 A_5^2 D_4$	Z_2^2	369 $A_1 A_3 A_{11} D_4$	Z_4	416 $A_1 A_2 A_4 A_7 D_5$		463 $A_1 A_2 A_3^2 D_5^2$	Z_4
276 $A_1 A_2 A_3 A_{13}$	Z_2	323 $A_1 A_2^2 A_5^2 D_4$	Z_6	370 $A_4 A_{11} D_4$	Z_2	417 $A_3 A_4 A_7 D_5$	Z_2	464 $A_1 A_2^2 A_4 D_5^2$	
277 $A_2^2 A_{13}$		324 $A_1^2 A_3 A_5^2 D_4$	Z_2^2	371 $A_1 A_2 A_{12} D_4$		418 $A_1^2 A_5 A_7 D_5$	Z_2	465 $A_1 A_4^2 D_5^2$	
278 $A_1^2 A_4 A_{13}$		325 $A_1 A_4 A_5^2 D_4$	Z_2	372 $A_1^2 A_{13} D_4$	Z_2	419 $A_2 A_5 A_7 D_5$		466 $A_1^2 A_2 A_5 D_5^2$	Z_2
279 $A_1^2 A_4 A_{13}$	Z_2	326 $A_5^3 D_4$	Z_6	373 $A_2 A_{13} D_4$		420 $A_1 A_6 A_7 D_5$		467 $A_1 A_3 A_5 D_5^2$	Z_2
280 $A_2 A_4 A_{13}$		327 $A_2^3 D_4$	$Z_2 Z_6$	374 $A_1 A_{14} D_4$		421 $A_2^2 D_5$	Z_8	468 $A_4 A_5 D_5^2$	
281 $A_1 A_5 A_{13}$		328 $A_3^3 A_6 D_4$	Z_2	375 $A_{15} D_4$	Z_4	422 $A_2^3 A_8 D_5$	Z_3	469 $A_1 A_2 A_6 D_5^2$	
282 $A_1 A_5 A_{13}$	Z_2	329 $A_1 A_2^2 A_4 A_6 D_4$		376 $A_1^2 A_3^2 D_4$	Z_2^3	423 $A_1 A_2 A_3 A_8 D_5$		470 $A_1^2 A_7 D_5^2$	Z_2

471 $A_1^2 A_7 D_5^2$	Z_4	518 $A_5 A_8 D_6$		565 $A_1 D_6^3$	Z_2^2	612 $A_2^2 A_4 D_4 D_7$		659 $A_2^2 A_7 D_8$	Z_2
472 $A_2 A_7 D_5^2$	Z_4	519 $A_1^2 A_2 A_9 D_6$	Z_2	566 $A_1 A_2 A_3^3 D_7$	Z_4	613 $A_1 A_2 A_5 D_4 D_7$	Z_2	660 $A_1 A_3 A_7 D_8$	Z_2
473 $A_1 A_8 D_5^2$		520 $A_2^2 A_9 D_6$		567 $A_3^4 D_7$	Z_4	614 $A_2 A_6 D_4 D_7$		661 $A_4 A_7 D_8$	Z_2
474 $A_9 D_5^2$		521 $A_2^2 A_9 D_6$	Z_2	568 $A_1^2 A_3^3 A_4 D_7$		615 $A_1 A_7 D_4 D_7$	Z_2	662 $A_1 A_2 A_8 D_8$	
475 $A_1^2 A_2 D_5^3$	Z_2	522 $A_1 A_3 A_9 D_6$	Z_2	569 $A_1 A_2^2 A_3 A_4 D_7$		616 $A_1 A_3^2 D_5 D_7$		663 $A_1^2 A_9 D_8$	Z_2
476 $A_4 D_5^3$		523 $A_4 A_9 D_6$		570 $A_1^2 A_3^2 A_4 D_7$	Z_2	617 $A_2^2 A_3 D_5 D_7$		664 $A_2 A_9 D_8$	
477 $D_4 D_5^3$	Z_2	524 $A_4 A_9 D_6$	Z_2	571 $A_2 A_3^2 A_4 D_7$		618 $A_1 A_2 A_4 D_5 D_7$		665 $A_1 A_{10} D_8$	
478 $A_1 A_3^2 A_2^2 D_6$	Z_2	525 $A_1 A_2 A_{10} D_6$		572 $A_1^2 A_2 A_4^2 D_7$		619 $A_1^2 A_5 D_5 D_7$	Z_2	666 $A_{11} D_8$	Z_2
479 $A_3^2 A_3 A_4 D_6$		526 $A_3 A_{10} D_6$		573 $A_2^2 A_4^2 D_7$		620 $A_2 A_5 D_5 D_7$		667 $A_1^2 A_2 A_3 D_4 D_8$	Z_2^2
480 $A_1 A_2 A_3^2 A_4 D_6$	Z_2	527 $A_1^2 A_{11} D_6$	Z_2	574 $A_1 A_3 A_4^2 D_7$		621 $A_1 A_6 D_5 D_7$		668 $A_2^2 A_3 D_4 D_8$	Z_2
481 $A_3^2 A_4 D_6$	Z_2	528 $A_2 A_{11} D_6$		575 $A_4^3 D_7$		622 $A_7 D_5 D_7$	Z_2	669 $A_1 A_3^2 D_4 D_8$	Z_2^2
482 $A_1 A_2^2 A_2 D_6$		529 $A_2 A_{11} D_6$	Z_2	576 $A_1^2 A_2 A_3 A_5 D_7$	Z_2	623 $A_1 A_2 A_3 D_6 D_7$	Z_2	670 $A_1^2 A_5 D_4 D_8$	Z_2^2
483 $A_2 A_3 A_4^2 D_6$		530 $A_1 A_{12} D_6$		577 $A_1 A_3^2 A_5 D_7$	Z_2	624 $A_2^2 D_6 D_7$	Z_2	671 $A_1^2 A_2^2 D_5 D_8$	Z_2
484 $A_2^2 A_5 D_6$	Z_3	531 $A_{13} D_6$		578 $A_3^3 A_4 A_5 D_7$	Z_3	625 $A_2 A_4 D_6 D_7$		672 $A_3^3 A_3 D_5 D_8$	Z_2^2
485 $A_1^3 A_2 A_3 A_5 D_6$	Z_2^2	532 $A_3^3 A_3^2 D_4 D_6$	Z_2^3	579 $A_1 A_2 A_4 A_5 D_7$		626 $A_1 A_5 D_6 D_7$	Z_2	673 $A_1 A_2 A_3 D_5 D_8$	Z_2
486 $A_1 A_2^2 A_3 A_5 D_6$	Z_2	533 $A_1 A_2 A_3^2 D_4 D_6$	Z_2^2	580 $A_3 A_4 A_5 D_7$		627 $A_6 D_6 D_7$		674 $A_1^2 A_4 D_5 D_8$	Z_2
487 $A_2^2 A_3^2 A_5 D_6$	Z_2^2	534 $A_1 A_3 A_5 D_4 D_6$	Z_2^2	581 $A_1^2 A_3^2 D_7$	Z_2	628 $A_1 D_5 D_6 D_7$	Z_2	675 $A_1^2 A_5 D_5 D_8$	Z_2
488 $A_2 A_3^2 A_5 D_6$	Z_2	535 $A_4 A_5 D_4 D_6$	Z_2	582 $A_1^2 A_3^2 A_6 D_7$		629 $D_6^2 D_7$	Z_2	676 $A_1^2 D_4 D_5 D_8$	Z_2^2
489 $A_1^2 A_2 A_4 A_5 D_6$	Z_2	536 $A_2 A_7 D_4 D_6$	Z_2	583 $A_1 A_2 A_3 A_6 D_7$		630 $A_1 A_2^2 D_7^2$		677 $A_1 D_5^2 D_8$	Z_2
490 $A_2^2 A_4 A_5 D_6$		537 $A_9 D_4 D_6$	Z_2	584 $A_3^2 A_6 D_7$		631 $A_1^2 A_3 D_7^2$	Z_2	678 $A_3^3 A_2 D_6 D_8$	Z_2^2
491 $A_1 A_3 A_4 A_5 D_6$	Z_2	538 $A_1^2 A_3 D_4^2 D_6$	Z_2^3	585 $A_1^2 A_4 A_6 D_7$		632 $A_2 A_3 D_7^2$		679 $A_1 A_2^2 D_6 D_8$	Z_2
492 $A_1^2 A_5 D_6$		539 $A_1 A_2^2 A_3 D_5 D_6$	Z_2	586 $A_2 A_4 A_6 D_7$		633 $A_1 A_4 D_7^2$		680 $A_1^2 A_3 D_6 D_8$	Z_2^2
493 $A_3^2 A_5 D_6$	Z_2^2	540 $A_2^2 A_3^2 D_5 D_6$	Z_2^2	587 $A_1 A_5 A_6 D_7$		634 $A_5 D_7^2$		681 $A_2 A_3 D_6 D_8$	Z_2
494 $A_1 A_2 A_5^2 D_6$	Z_2	541 $A_2^2 A_4 D_5 D_6$		588 $A_6^2 D_7$		635 $A_1^3 A_2 A_3^2 D_8$	Z_2^2	682 $A_1 A_4 D_6 D_8$	Z_2
495 $A_3 A_5^2 D_6$	Z_2	542 $A_1 A_3 A_4 D_5 D_6$	Z_2	589 $A_3^3 A_2 A_7 D_7$	Z_2	636 $A_1 A_2^2 A_3^2 D_8$	Z_2	683 $A_5 D_6 D_8$	Z_2
496 $A_3 A_5^2 D_6$	Z_2^2	543 $A_4^2 D_5 D_6$		590 $A_1 A_2^2 A_7 D_7$		637 $A_2^2 A_3^2 D_8$	Z_2^2	684 $A_1 D_4 D_6 D_8$	Z_2^2
497 $A_1 A_3^2 A_6 D_6$		544 $A_3^3 A_5 D_5 D_6$	Z_2^2	591 $A_1^2 A_3 A_7 D_7$	Z_2	638 $A_2 A_3^2 D_8$	Z_2	685 $D_5 D_6 D_8$	Z_2
498 $A_2^2 A_3 A_6 D_6$		545 $A_1 A_2 A_5 D_5 D_6$	Z_2	592 $A_2^2 A_3 A_7 D_7$	Z_4	639 $A_1^2 A_2 A_3 A_4 D_8$	Z_2	686 $A_1^2 A_2 D_7 D_8$	Z_2
499 $A_1 A_3^2 A_6 D_6$	Z_2	546 $A_3 A_5 D_5 D_6$	Z_2	593 $A_2 A_3 A_7 D_7$		640 $A_2^2 A_3 A_4 D_8$		687 $A_2^2 D_7 D_8$	
500 $A_1 A_2 A_4 A_6 D_6$		547 $A_2 A_6 D_5 D_6$		594 $A_2 A_3 A_7 D_7$	Z_2	641 $A_1 A_2^2 A_4 D_8$	Z_2	688 $A_1 A_3 D_7 D_8$	Z_2
501 $A_3 A_4 A_6 D_6$		548 $A_1 A_7 D_5 D_6$	Z_2	595 $A_2 A_3 A_7 D_7$	Z_4	642 $A_1 A_2 A_4^2 D_8$		689 $A_3^3 D_8^2$	Z_2^2
502 $A_1^2 A_5 A_6 D_6$	Z_2	549 $A_8 D_5 D_6$		596 $A_1 A_4 A_7 D_7$		643 $A_1^4 A_2 A_5 D_8$	Z_2^2	690 $A_1 A_2 D_8^2$	Z_2
503 $A_2 A_5 A_6 D_6$		550 $A_1 A_3 D_4 D_5 D_6$	Z_2^2	597 $A_5 A_7 D_7$		644 $A_1^3 A_3 A_5 D_8$	Z_2^2	691 $A_3 D_8^2$	Z_2^2
504 $A_1 A_6^2 D_6$		551 $A_1 A_2 D_5^2 D_6$	Z_2	598 $A_1^2 A_2 A_8 D_7$		645 $A_1 A_2 A_3 A_5 D_8$	Z_2	692 $A_1 A_3^2 A_3 D_9$	
505 $A_1^2 A_2^2 A_7 D_6$	Z_2	552 $A_1^4 A_3 D_6^2$	Z_2^3	599 $A_1 A_3 A_8 D_7$		646 $A_2^2 A_5 D_8$	Z_2	693 $A_2^2 A_3^2 D_9$	
506 $A_3^2 A_7 D_6$		553 $A_2^2 A_2 A_3 D_6^2$	Z_2^2	600 $A_4 A_8 D_7$		647 $A_1^2 A_4 A_5 D_8$	Z_2	694 $A_2^2 A_2^2 A_4 D_9$	
507 $A_3^2 A_3 A_7 D_6$	Z_2^2	554 $A_2^2 A_3 D_6^2$	Z_2	601 $A_1^3 A_9 D_7$	Z_2	648 $A_2 A_4 A_5 D_8$		695 $A_3^2 A_4 D_9$	
508 $A_1 A_2 A_3 A_7 D_6$	Z_2	555 $A_1 A_3^2 D_6^2$	Z_2^2	602 $A_1 A_2 A_9 D_7$		649 $A_1 A_5^2 D_8$	Z_2	696 $A_1 A_2 A_3 A_4 D_9$	
509 $A_3^2 A_7 D_6$	Z_2	556 $A_3 A_4 D_6^2$	Z_2	603 $A_1 A_2 A_9 D_7$	Z_2	650 $A_1 A_3^2 D_8$	Z_2^2	697 $A_2^2 A_2^2 D_9$	
510 $A_3^2 A_7 D_6$	Z_4	557 $A_1^2 A_5 D_6^2$	Z_2^2	604 $A_3 A_9 D_7$		651 $A_1 A_2^2 A_6 D_8$		698 $A_2 A_4^2 D_9$	
511 $A_1^2 A_4 A_7 D_6$	Z_2	558 $A_2 A_5 D_6^2$	Z_2	605 $A_1^2 A_{10} D_7$		652 $A_1^2 A_3 A_6 D_8$	Z_2	699 $A_1 A_2^2 A_5 D_9$	
512 $A_2 A_4 A_7 D_6$		559 $A_7 D_6^2$	Z_2	606 $A_2 A_{10} D_7$		653 $A_2 A_3 A_6 D_8$		700 $A_1^2 A_3 A_5 D_9$	Z_2
513 $A_1 A_5 A_7 D_6$	Z_2	560 $A_1^3 D_4 D_6^2$	Z_2^3	607 $A_1 A_{11} D_7$		654 $A_1 A_4 A_6 D_8$		701 $A_2 A_3 A_5 D_9$	
514 $A_6 A_7 D_6$		561 $A_1 A_2 D_4 D_6^2$	Z_2^2	608 $A_1 A_{11} D_7$	Z_2	655 $A_5 A_6 D_8$		702 $A_1 A_4 A_5 D_9$	
515 $A_1 A_2^2 A_8 D_6$		562 $A_3 D_4 D_6^2$	Z_2^2	609 $A_1 A_{11} D_7$	Z_4	656 $A_1^4 A_7 D_8$	Z_2^2	703 $A_2^2 D_9$	
516 $A_2 A_3 A_8 D_6$		563 $A_2^2 D_5 D_6^2$	Z_2^2	610 $A_{12} D_7$		657 $A_1^2 A_2 A_7 D_8$	Z_2	704 $A_5^2 D_9$	Z_2
517 $A_1 A_4 A_8 D_6$		564 $A_2 D_5 D_6^2$	Z_2	611 $A_2 A_3^2 D_4 D_7$	Z_2	658 $A_2^2 A_7 D_8$		705 $A_1^2 A_2 A_6 D_9$	

706 $A_2^2 A_6 D_9$	753 $A_1 A_4 D_4 D_{10}$ $\mathbb{Z}_2$	800 $D_7 D_{12}$ $\mathbb{Z}_2$	847 $A_3 A_5^2 E_6$	894 $A_1 A_2^2 D_4 E_6$
707 $A_1 A_3 A_6 D_9$	754 $A_5 D_4 D_{10}$ $\mathbb{Z}_2$	801 $A_1^2 A_2^2 D_{13}$	848 $A_3 A_5^2 E_6$ $\mathbb{Z}_2$	895 $A_2^2 A_5 D_4 E_6$ $\mathbb{Z}_3$
708 $A_4 A_6 D_9$	755 $A_1^2 A_2 D_5 D_{10}$ $\mathbb{Z}_2$	802 $A_1 A_2 A_3 D_{13}$	849 $A_3 A_5^2 E_6$ $\mathbb{Z}_3$	896 $A_1 A_3 A_5 D_4 E_6$ $\mathbb{Z}_2$
709 $A_1^3 A_7 D_9$ $\mathbb{Z}_2$	756 $A_2^2 D_5 D_{10}$	803 $A_1^2 A_4 D_{13}$	850 $A_3 A_5^2 E_6$ $\mathbb{Z}_6$	897 $A_4 A_5 D_4 E_6$
710 $A_1 A_2 A_7 D_9$	757 $A_1 A_3 D_5 D_{10}$ $\mathbb{Z}_2$	804 $A_2 A_4 D_{13}$	851 $A_1^2 A_2 A_3 A_6 E_6$	898 $A_1 A_2 A_6 D_4 E_6$
711 $A_3 A_7 D_9$ $\mathbb{Z}_2$	758 $A_4 D_5 D_{10}$	805 $A_1 A_5 D_{13}$	852 $A_2^2 A_3 A_6 E_6$	899 $A_1^2 A_7 D_4 E_6$ $\mathbb{Z}_2$
712 $A_1^2 A_8 D_9$	759 $A_1^3 D_6 D_{10}$ $\mathbb{Z}_2^2$	806 $A_6 D_{13}$	853 $A_1 A_3^2 A_6 E_6$	900 $A_1 A_8 D_4 E_6$
713 $A_2 A_8 D_9$	760 $A_1 A_2 D_6 D_{10}$ $\mathbb{Z}_2$	807 $A_1 D_5 D_{13}$	854 $A_1^3 A_4 A_6 E_6$	901 $A_9 D_4 E_6$
714 $A_1 A_9 D_9$	761 $A_3 D_6 D_{10}$ $\mathbb{Z}_2$	808 $D_6 D_{13}$	855 $A_1 A_2 A_4 A_6 E_6$	902 $A_1^2 A_3^2 D_5 E_6$ $\mathbb{Z}_2$
715 $A_1 A_9 D_9$ $\mathbb{Z}_2$	762 $A_1^2 D_7 D_{10}$ $\mathbb{Z}_2$	809 $A_1^3 A_2 D_{14}$ $\mathbb{Z}_2$	856 $A_3 A_4 A_6 E_6$	903 $A_1^2 A_2 A_4 D_5 E_6$
716 $A_{10} D_9$	763 $A_2 D_7 D_{10}$	810 $A_1 A_2^2 D_{14}$	857 $A_1^2 A_5 A_6 E_6$	904 $A_1 A_3 A_4 D_5 E_6$
717 $A_1 A_5 D_4 D_9$ $\mathbb{Z}_2$	764 $A_1 D_8 D_{10}$ $\mathbb{Z}_2$	811 $A_1 A_2^2 D_{14}$ $\mathbb{Z}_2$	858 $A_2 A_5 A_6 E_6$	905 $A_2^2 D_5 E_6$
718 $A_1 A_2^2 D_5 D_9$	765 $D_9 D_{10}$	812 $A_1^2 A_3 D_{14}$ $\mathbb{Z}_2$	859 $A_1 A_6^2 E_6$	906 $A_1^3 A_5 D_5 E_6$ $\mathbb{Z}_2$
719 $A_1 A_4 D_5 D_9$	766 $A_1 A_2^2 A_3 D_{11}$	813 $A_2 A_3 D_{14}$	860 $A_1^4 A_2 A_7 E_6$ $\mathbb{Z}_2$	907 $A_3 A_5 D_5 E_6$
720 $A_5 D_5 D_9$	767 $A_1^2 A_3^2 D_{11}$ $\mathbb{Z}_2$	814 $A_1 A_4 D_{14}$	861 $A_1^3 A_3 A_7 E_6$ $\mathbb{Z}_2$	908 $A_1^2 A_6 D_5 E_6$
721 $D_5^2 D_9$	768 $A_2 A_3^2 D_{11}$	815 $A_1 A_4 D_{14}$ $\mathbb{Z}_2$	862 $A_1 A_2 A_3 A_7 E_6$	909 $A_2 A_6 D_5 E_6$
722 $A_2^2 D_6 D_9$	769 $A_1^2 A_2 A_4 D_{11}$	816 $A_5 D_{14}$	863 $A_1^2 A_4 A_7 E_6$	910 $A_1 A_7 D_5 E_6$
723 $A_1 A_3 D_6 D_9$ $\mathbb{Z}_2$	770 $A_2^2 A_4 D_{11}$	817 $A_1 D_4 D_{14}$ $\mathbb{Z}_2$	864 $A_2 A_4 A_7 E_6$	911 $A_8 D_5 E_6$
724 $A_4 D_6 D_9$	771 $A_1 A_3 A_4 D_{11}$	818 $D_5 D_{14}$	865 $A_1 A_5 A_7 E_6$	912 $A_4 D_4 D_5 E_6$
725 $A_1 A_2 D_7 D_9$	772 $A_4^2 D_{11}$	819 $A_1^2 A_2 D_{15}$	866 $A_1 A_5 A_7 E_6$ $\mathbb{Z}_2$	913 $A_1 A_2 D_5^2 E_6$
726 $A_1^2 D_8 D_9$ $\mathbb{Z}_2$	773 $A_1^3 A_5 D_{11}$ $\mathbb{Z}_2$	820 $A_2^2 D_{15}$	867 $A_6 A_7 E_6$	914 $A_3 D_5^2 E_6$
727 $A_1 D_9^2$	774 $A_1 A_2 A_5 D_{11}$	821 $A_1 A_3 D_{15}$	868 $A_1^3 A_2 A_8 E_6$ $\mathbb{Z}_3$	915 $A_2^2 A_3 D_6 E_6$
728 $A_1^3 A_3^2 D_{10}$ $\mathbb{Z}_2$	775 $A_3 A_5 D_{11}$	822 $A_4 D_{15}$	869 $A_1 A_2^2 A_8 E_6$ $\mathbb{Z}_3$	916 $A_1 A_3^2 D_6 E_6$ $\mathbb{Z}_2$
729 $A_1^2 A_2^2 A_3 D_{10}$ $\mathbb{Z}_2$	776 $A_1^2 A_6 D_{11}$	823 $A_3^2 D_{16}$ $\mathbb{Z}_2$	870 $A_1^2 A_3 A_8 E_6$	917 $A_1 A_2 A_4 D_6 E_6$
730 $A_1^3 A_3^2 D_{10}$ $\mathbb{Z}_2^2$	777 $A_2 A_6 D_{11}$	824 $A_1 A_2 D_{16}$	871 $A_2 A_3 A_8 E_6$	918 $A_3 A_4 D_6 E_6$
731 $A_1 A_2 A_3^2 D_{10}$ $\mathbb{Z}_2$	778 $A_1 A_7 D_{11}$	825 $A_1 A_2 D_{16}$ $\mathbb{Z}_2$	872 $A_2 A_3 A_8 E_6$ $\mathbb{Z}_3$	919 $A_1^2 A_5 D_6 E_6$ $\mathbb{Z}_2$
732 $A_3^3 D_{10}$ $\mathbb{Z}_2$	779 $A_8 D_{11}$	826 $A_3 D_{16}$ $\mathbb{Z}_2$	873 $A_1 A_4 A_8 E_6$	920 $A_2 A_5 D_6 E_6$
733 $A_1^2 A_2 A_4 D_{10}$ $\mathbb{Z}_2$	780 $A_2^2 D_4 D_{11}$	827 $A_1^2 D_{17}$	874 $A_5 A_8 E_6$	921 $A_1 A_6 D_6 E_6$
734 $A_1 A_2^2 A_4 D_{10}$	781 $A_1 A_2 D_5 D_{11}$	828 $A_2 D_{17}$	875 $A_1^4 A_9 E_6$ $\mathbb{Z}_2$	922 $A_7 D_6 E_6$
735 $A_1^2 A_3 A_4 D_{10}$ $\mathbb{Z}_2$	782 $A_2 D_6 D_{11}$	829 $A_1 D_{18}$	876 $A_1^2 A_2 A_9 E_6$	923 $A_2 D_5 D_6 E_6$
736 $A_2 A_3 A_4 D_{10}$	783 $A_1 D_7 D_{11}$	830 D_{19}	877 $A_2^2 A_9 E_6$	924 $A_1^2 A_3^2 D_7 E_6$
737 $A_1 A_2^2 D_{10}$	784 $A_1^3 A_2^2 D_{12}$ $\mathbb{Z}_2$	831 $A_1^2 A_2 A_3^3 E_6$ $\mathbb{Z}_2$	878 $A_1 A_3 A_9 E_6$	925 $A_1 A_2 A_3 D_7 E_6$
738 $A_1^4 A_5 D_{10}$ $\mathbb{Z}_2^2$	785 $A_1 A_2^3 D_{12}$	832 $A_2^2 A_3^3 E_6$	879 $A_1 A_3 A_9 E_6$ $\mathbb{Z}_2$	926 $A_1^2 A_4 D_7 E_6$
739 $A_1^2 A_2 A_5 D_{10}$ $\mathbb{Z}_2$	786 $A_1^2 A_2 A_3 D_{12}$ $\mathbb{Z}_2$	833 $A_1 A_2 A_3^2 A_4 E_6$	880 $A_4 A_9 E_6$	927 $A_2 A_4 D_7 E_6$
740 $A_1 A_3 A_5 D_{10}$ $\mathbb{Z}_2$	787 $A_2^2 A_3 D_{12}$	834 $A_1^3 A_2 A_4^2 E_6$	881 $A_1^3 A_{10} E_6$	928 $A_1 A_5 D_7 E_6$
741 $A_1 A_3 A_5 D_{10}$ $\mathbb{Z}_2^2$	788 $A_2^2 A_3 D_{12}$ $\mathbb{Z}_2$	835 $A_1^2 A_3 A_4^2 E_6$	882 $A_1 A_2 A_{10} E_6$	929 $A_6 D_7 E_6$
742 $A_4 A_5 D_{10}$	789 $A_1 A_3^2 D_{12}$ $\mathbb{Z}_2$	836 $A_2 A_3 A_4^2 E_6$	883 $A_3 A_{10} E_6$	930 $A_1 D_5 D_7 E_6$
743 $A_4 A_5 D_{10}$ $\mathbb{Z}_2$	790 $A_1^3 A_4 D_{12}$ $\mathbb{Z}_2$	837 $A_1 A_4^3 E_6$	884 $A_1^2 A_{11} E_6$	931 $D_6 D_7 E_6$
744 $A_1^3 A_6 D_{10}$ $\mathbb{Z}_2$	791 $A_1 A_2 A_4 D_{12}$	838 $A_2^4 A_5 E_6$ $\mathbb{Z}_3^2$	885 $A_1^2 A_{11} E_6$ $\mathbb{Z}_2$	932 $A_1^2 A_3 D_8 E_6$ $\mathbb{Z}_2$
745 $A_1 A_2 A_6 D_{10}$	792 $A_3 A_4 D_{12}$ $\mathbb{Z}_2$	839 $A_1 A_2^2 A_3 A_5 E_6$ $\mathbb{Z}_3$	886 $A_1^2 A_{11} E_6$ $\mathbb{Z}_3$	933 $A_1 A_4 D_8 E_6$
746 $A_3 A_6 D_{10}$	793 $A_1^2 A_5 D_{12}$ $\mathbb{Z}_2$	840 $A_1^2 A_3^2 A_5 E_6$ $\mathbb{Z}_2$	887 $A_1^2 A_{11} E_6$ $\mathbb{Z}_6$	934 $A_5 D_8 E_6$
747 $A_1^2 A_7 D_{10}$ $\mathbb{Z}_2$	794 $A_2 A_5 D_{12}$	841 $A_2 A_3^2 A_5 E_6$	888 $A_2 A_{11} E_6$	935 $D_5 D_8 E_6$
748 $A_2 A_7 D_{10}$	795 $A_1 A_6 D_{12}$	842 $A_2^2 A_4 A_5 E_6$ $\mathbb{Z}_3$	889 $A_2 A_{11} E_6$ $\mathbb{Z}_3$	936 $A_1^2 A_2 D_9 E_6$
749 $A_1 A_8 D_{10}$	796 $A_1 A_2 D_4 D_{12}$ $\mathbb{Z}_2$	843 $A_1 A_3 A_4 A_5 E_6$	890 $A_1 A_{12} E_6$	937 $A_2^2 D_9 E_6$
750 $A_9 D_{10}$	797 $A_1^2 D_5 D_{12}$ $\mathbb{Z}_2$	844 $A_2^2 A_5 E_6$	891 $A_{13} E_6$	938 $A_1 A_3 D_9 E_6$
751 $A_1 A_2^2 D_4 D_{10}$ $\mathbb{Z}_2$	798 $A_2 D_5 D_{12}$ $\mathbb{Z}_2$	845 $A_1^3 A_5^2 E_6$ $\mathbb{Z}_6$	892 $A_1 A_4^2 D_4 E_6$ $\mathbb{Z}_3$	939 $A_4 D_9 E_6$
752 $A_1^2 A_3 D_4 D_{10}$ $\mathbb{Z}_2^2$	799 $A_1 D_6 D_{12}$ $\mathbb{Z}_2$	846 $A_1 A_2 A_5^2 E_6$ $\mathbb{Z}_3$	893 $A_1 A_2^2 A_4 D_4 E_6$	940 $D_4 D_9 E_6$

941	$A_3^3 D_{10} E_6$	Z_2	979	$A_3 A_4 A_5 E_7$	Z_2	1017	$A_1 A_2 A_5 D_4 E_7$	Z_2	1055	$D_5 D_7 E_7$		1093	$A_1^2 A_2 A_3 A_4 E_8$		1131	$A_2 A_4 D_5 E_8$
942	$A_1 A_2 D_{10} E_6$		980	$A_1^2 A_5^2 E_7$	Z_2	1018	$A_3 A_5 D_4 E_7$	Z_2	1056	$A_1^4 D_8 E_7$	Z_2^2	1094	$A_2^2 A_3 A_4 E_8$		1132	$A_1 A_5 D_5 E_8$
943	$A_3 D_{10} E_6$		981	$A_2 A_5^2 E_7$		1019	$A_2 A_6 D_4 E_7$		1057	$A_1^2 A_2 D_8 E_7$	Z_2	1095	$A_1 A_3^2 A_4 E_8$		1133	$A_6 D_5 E_8$
944	$A_1^2 D_{11} E_6$		982	$A_1^2 A_4^2 A_6 E_7$		1020	$A_1 A_7 D_4 E_7$	Z_2	1058	$A_2^2 D_8 E_7$		1096	$A_1^3 A_4^2 E_8$		1134	$A_1 D_5^2 E_8$
945	$A_2 D_{11} E_6$		983	$A_3^2 A_6 E_7$		1021	$A_8 D_4 E_7$		1059	$A_1 A_3 D_8 E_7$	Z_2	1097	$A_1 A_2 A_4^2 E_8$		1135	$A_1 A_2^2 D_6 E_8$
946	$A_1 D_{12} E_6$		984	$A_1 A_2 A_3 A_6 E_7$		1022	$A_1^3 A_2^2 D_5 E_7$	Z_2	1060	$A_4 D_8 E_7$		1098	$A_3 A_4^2 E_8$		1136	$A_2 A_3 D_6 E_8$
947	$D_{13} E_6$		985	$A_3^2 A_6 E_7$		1023	$A_1^2 A_2 A_3 D_5 E_7$	Z_2	1061	$A_1 A_2 D_9 E_7$		1099	$A_1^3 A_3 A_5 E_8$	Z_2	1137	$A_1 A_4 D_6 E_8$
948	$A_1^3 A_5^2 E_6^2$	Z_3	986	$A_1^2 A_4 A_6 E_7$		1024	$A_2^2 A_3 D_5 E_7$		1062	$A_3 D_9 E_7$		1100	$A_1 A_2 A_3 A_5 E_8$		1138	$A_5 D_6 E_8$
949	$A_2^2 A_3 E_6^2$	Z_3	987	$A_2 A_4 A_6 E_7$		1025	$A_1 A_3^2 D_5 E_7$	Z_2	1063	$A_1^2 D_{10} E_7$	Z_2	1101	$A_3^2 A_5 E_8$		1139	$D_5 D_6 E_8$
950	$A_1 A_3^2 E_6^2$		988	$A_1 A_5 A_6 E_7$		1026	$A_1 A_2 A_4 D_5 E_7$		1064	$A_2 D_{10} E_7$		1102	$A_1^2 A_4 A_5 E_8$		1140	$A_1^2 A_2 D_7 E_8$
951	$A_1^2 A_4 E_6^2$		989	$A_5^2 E_7$		1027	$A_3 A_4 D_5 E_7$		1065	$A_2 D_{10} E_7$	Z_2	1103	$A_2 A_4 A_5 E_8$		1141	$A_2^2 D_7 E_8$
952	$A_3 A_4 E_6^2$		990	$A_1^3 A_2 A_7 E_7$	Z_2	1028	$A_1^2 A_5 D_5 E_7$	Z_2	1066	$A_1 D_{11} E_7$		1104	$A_1 A_5^2 E_8$		1142	$A_1 A_3 D_7 E_8$
953	$A_1^2 A_5 E_6^2$	Z_3	991	$A_1 A_5^2 A_7 E_7$		1029	$A_2 A_5 D_5 E_7$		1067	$D_{12} E_7$		1105	$A_1^3 A_2 A_6 E_8$		1143	$A_4 D_7 E_8$
954	$A_2 A_5 E_6^2$	Z_3	992	$A_1 A_5^2 A_7 E_7$	Z_2	1030	$A_2 A_5 D_5 E_7$	Z_2	1068	$A_1^3 A_3 E_6 E_7$	Z_2	1106	$A_1^2 A_3 A_6 E_8$		1144	$A_1 A_2 D_8 E_8$
955	$A_1 A_6 E_6^2$		993	$A_2^2 A_3 A_7 E_7$	Z_2	1031	$A_1 A_6 D_5 E_7$		1069	$A_1 A_2 A_3 E_6 E_7$		1107	$A_1^2 A_3 A_6 E_8$		1145	$A_1^2 D_9 E_8$
956	$A_7 E_6^2$		994	$A_2 A_3 A_7 E_7$		1032	$A_7 D_5 E_7$		1070	$A_3^2 E_6 E_7$		1108	$A_2 A_3 A_6 E_8$		1146	$A_2 D_9 E_8$
957	$A_1 A_2 D_4 E_6^2$		995	$A_1 A_4 A_7 E_7$		1033	$A_1 A_2 D_4 D_5 E_7$	Z_2	1071	$A_1^2 A_4 E_6 E_7$		1109	$A_1 A_4 A_6 E_8$		1147	$A_1 D_{10} E_8$
958	$A_3 D_4 E_6^2$		996	$A_1 A_4 A_7 E_7$	Z_2	1034	$A_1^2 D_5^2 E_7$	Z_2	1072	$A_2 A_4 E_6 E_7$		1110	$A_5 A_6 E_8$		1148	$D_{11} E_8$
959	$A_1^2 D_5 E_6^2$		997	$A_5 A_7 E_7$		1035	$A_2 D_5^2 E_7$		1073	$A_1 A_5 E_6 E_7$		1111	$A_1^4 A_7 E_8$	Z_2	1149	$A_1^3 A_2 E_6 E_8$
960	$A_1 D_6 E_6^2$		998	$A_1^2 A_2 A_8 E_7$		1036	$A_2^2 A_2^2 D_6 E_7$	Z_2	1074	$A_6 E_6 E_7$		1112	$A_1^2 A_2 A_7 E_8$		1150	$A_2 A_3 E_6 E_8$
961	$D_7 E_6^2$		999	$A_2^2 A_8 E_7$		1037	$A_2^2 D_6 E_7$		1075	$A_2 D_4 E_6 E_7$		1113	$A_2^2 A_7 E_8$		1151	$A_2 A_3 E_6 E_8$
962	$A_1 E_6^3$	Z_3	1000	$A_1 A_3 A_8 E_7$		1038	$A_1^3 A_3 D_6 E_7$	Z_2^2	1076	$A_1 D_5 E_6 E_7$		1114	$A_1 A_3 A_7 E_8$		1152	$A_1 A_4 E_6 E_8$
963	$A_1^2 A_2^2 A_3 E_7$	Z_2	1001	$A_4 A_8 E_7$		1039	$A_1 A_2 A_3 D_6 E_7$	Z_2	1077	$D_6 E_6 E_7$		1115	$A_4 A_7 E_8$		1153	$A_5 E_6 E_8$
964	$A_1^3 A_5 E_7$		1002	$A_1^3 A_9 E_7$	Z_2	1040	$A_3^2 D_6 E_7$	Z_2	1078	$E_6^2 E_7$		1116	$A_1^4 A_8 E_8$		1154	$A_1 D_4 E_6 E_8$
965	$A_1 A_2^2 A_3 A_4 E_7$		1003	$A_1 A_2 A_9 E_7$		1041	$A_1^2 A_4 D_6 E_7$	Z_2	1079	$A_1^3 A_2 E_7^2$	Z_2	1117	$A_1 A_2 A_8 E_8$		1155	$D_5 E_6 E_8$
966	$A_1^2 A_3 A_4 E_7$	Z_2	1004	$A_1 A_2 A_9 E_7$	Z_2	1042	$A_2 A_4 D_6 E_7$		1080	$A_1 A_2^2 E_7^2$		1118	$A_3 A_8 E_8$		1156	$A_1^2 A_2 E_7 E_8$
967	$A_2 A_3^2 A_4 E_7$		1005	$A_3 A_9 E_7$		1043	$A_1 A_5 D_6 E_7$	Z_2	1081	$A_1^2 A_3 E_7^2$	Z_2	1119	$A_1^2 A_9 E_8$		1157	$A_2^2 E_7 E_8$
968	$A_1^2 A_2 A_4 E_7$		1006	$A_3 A_9 E_7$	Z_2	1044	$A_6 D_6 E_7$		1082	$A_2 A_3 E_7^2$		1120	$A_2 A_9 E_8$		1158	$A_1 A_3 E_7 E_8$
969	$A_2^2 A_4 E_7$		1007	$A_1^2 A_{10} E_7$		1045	$A_2 D_4 D_6 E_7$	Z_2	1083	$A_2 A_3 E_7^2$	Z_2	1121	$A_1 A_{10} E_8$		1159	$A_4 E_7 E_8$
970	$A_1 A_3 A_4^2 E_7$		1008	$A_2 A_{10} E_7$		1046	$A_1 D_5 D_6 E_7$	Z_2	1084	$A_1 A_4 E_7^2$		1122	$A_{11} E_8$		1160	$D_4 E_7 E_8$
971	$A_1^4 A_5 A_7 E_7$	Z_2^2	1009	$A_1 A_{11} E_7$		1047	$D_6^2 E_7$	Z_2	1085	$A_5 E_7^2$		1123	$A_1 A_3^2 D_4 E_8$		1161	$A_1^3 E_8^2$
972	$A_1^2 A_2 A_3 A_5 E_7$	Z_2	1010	$A_{12} E_7$		1048	$A_1 A_5^2 D_7 E_7$		1086	$A_1 D_4 E_7^2$	Z_2	1124	$A_2^2 A_3 D_4 E_8$		1162	$A_1 A_2 E_8^2$
973	$A_2^2 A_3 A_5 E_7$		1011	$A_1 A_2^2 A_3 D_4 E_7$	Z_2	1049	$A_1^2 A_3 D_7 E_7$	Z_2	1087	$D_5 E_7^2$		1125	$A_1 A_2 A_4 D_4 E_8$		1163	$A_3 E_8^2$
974	$A_2^2 A_3 A_5 E_7$	Z_2	1012	$A_1^2 A_3^2 D_4 E_7$	Z_2^2	1050	$A_2 A_3 D_7 E_7$		1088	$D_5 E_7^2$	Z_2	1126	$A_2 A_5 D_4 E_8$			
975	$A_1 A_3^2 A_5 E_7$	Z_2	1013	$A_2^2 A_4 D_4 E_7$		1051	$A_1 A_4 D_7 E_7$		1089	$A_1 A_2^2 A_3^2 E_8$		1127	$A_1 A_6 D_4 E_8$			
976	$A_1^3 A_4 A_5 E_7$	Z_2	1014	$A_1 A_3 A_4 D_4 E_7$	Z_2	1052	$A_5 D_7 E_7$		1090	$A_1^2 A_3^2 E_8$	Z_2	1128	$A_1^2 A_2^2 D_5 E_8$			
977	$A_1 A_2 A_4 A_5 E_7$		1015	$A_4^2 D_4 E_7$		1053	$A_5 D_7 E_7$	Z_2	1091	$A_2 A_3^2 E_8$		1129	$A_1 A_2 A_3 D_5 E_8$			
978	$A_3 A_4 A_5 E_7$		1016	$A_1^3 A_5 D_4 E_7$	Z_2^2	1054	$A_1 D_4 D_7 E_7$	Z_2	1092	$A_1^3 A_2^2 A_4 E_8$		1130	$A_1^2 A_4 D_5 E_8$			

 Table 4. Algebras of maximal rank for the heterotic string on T^3 .

1	$A_1^2 A_3^3$	Z_3^3	48	$A_3 A_4 C_2^2$	Z_2	95	$A_1 A_5 C_5$	Z_2	142	$A_4 C_1^2 D_5$		189	$C_1 C_3 E_7$		236	$A_1^2 C_7 F_2$	
2	$A_2 A_3^3$	Z_2^2	49	$A_1^2 A_5 C_2^2$	Z_2^2	96	$A_1 A_2^2 C_1 C_5$		143	$A_1 A_3 C_2 D_5$	Z_2^2	190	$A_1 C_1^2 E_8$		237	$A_2 C_7 F_2$	
3	$A_1 A_5^2$	Z_2^2	50	$A_2 A_5 C_2^2$	Z_2	97	$A_1 A_4 C_1 C_5$		144	$A_1 A_2 C_1 C_2 D_5$	Z_2	191	$C_1 C_2 E_8$		238	$A_1 C_8 F_2$	
4	$A_2^2 A_7$	Z_2	51	$A_7 C_2^2$	Z_2	98	$A_5 C_1 C_5$		145	$A_1^2 C_2^2 D_5$	Z_2^2	192	$A_1 A_4^2 F_2$	Z_3	239	$C_9 F_2$	
5	$A_2^2 A_3^2 C_1$	Z_2	52	$A_2 A_3^2 C_3$	Z_2	99	$A_2^2 C_2 C_5$		146	$A_2 C_2^2 D_5$	Z_2	193	$A_1 A_2^2 A_4 F_2$		240	$A_4 C_1 D_4 F_2$	
6	$A_3^2 A_4 C_1$		53	$A_2^2 A_4 C_3$		100	$A_1 A_3 C_2 C_5$	Z_2	147	$A_1 C_2 C_3 D_5$	Z_2	194	$A_1 A_4^2 F_2$		241	$C_5 D_4 F_2$	
7	$A_1 A_2^2 A_5 C_1$	Z_2	54	$A_1 A_2 A_5 C_3$	Z_2	101	$A_4 C_2 C_5$		148	$A_1^2 C_4 D_5$	Z_2^2	195	$A_2^2 A_5 F_2$	Z_3	242	$A_4 D_5 F_2$	
8	$A_1^2 A_3 A_5 C_1$	Z_2^2	55	$A_2 A_6 C_3$		102	$A_1 A_2 C_3 C_5$		149	$A_1 C_1 C_4 D_5$	Z_2	196	$A_1 A_3 A_5 F_2$	Z_2	243	$A_1 A_2 C_1 D_5 F_2$	
9	$A_1 A_4 A_5 C_1$	Z_2	56	$A_1 A_7 C_3$	Z_2	103	$A_1^2 C_4 C_5$	Z_2	150	$C_2 C_4 D_5$	Z_2	197	$A_4 A_5 F_2$		244	$A_3 C_1 D_5 F_2$	
10	$A_2^2 C_1$	Z_2	57	$A_1 A_3^2 C_1 C_3$		104	$A_1 C_5^2$		151	$C_1 C_5 D_5$		198	$A_1 A_2 A_6 F_2$		245	$A_2 C_2 D_5 F_2$	
11	$A_2^2 A_6 C_1$		58	$A_2^2 A_3 C_1 C_3$		105	$A_1 A_2^2 C_6$	Z_2	152	$C_1^2 D_4 D_5$	Z_2	199	$A_1^2 A_7 F_2$	Z_2	246	$A_1 C_3 D_5 F_2$	
12	$A_1 A_2 A_7 C_1$	Z_2	59	$A_1 A_2 A_4 C_1 C_3$		106	$A_1^2 A_3 C_6$	Z_2^2	153	$C_1 D_5^2$	Z_2	200	$A_1 A_8 F_2$		247	$C_4 D_5 F_2$	
13	$A_1 A_9 C_1$	Z_2	60	$A_1^2 A_5 C_1 C_3$	Z_2	107	$A_1 A_4 C_6$	Z_2	154	$A_1^2 A_3 D_6$	Z_2^3	201	$A_9 F_2$		248	$A_2 C_1 D_6 F_2$	
14	$A_1^2 A_2^2 A_3 C_1^2$	Z_2	61	$A_2 A_5 C_1 C_3$		108	$A_5 C_6$	Z_2	155	$A_1 A_3 C_1 D_6$	Z_2^2	202	$A_1^2 A_3^2 C_1 F_2$	Z_2	249	$C_3 D_6 F_2$	
15	$A_1 A_2 A_3^2 C_1^2$	Z_4	62	$A_1 A_6 C_1 C_3$		109	$A_1^2 A_2 C_1 C_6$	Z_2	156	$A_1 A_2 C_1^2 D_6$	Z_2	203	$A_1^2 A_2 A_4 C_1 F_2$		250	$A_1 C_1 D_7 F_2$	
16	$A_1 A_2^2 A_4 C_1^2$		63	$A_7 C_1 C_3$	Z_2	110	$A_2^2 C_1 C_6$		157	$A_3^2 C_2 D_6$	Z_2^3	204	$A_1 A_3 A_4 C_1 F_2$		251	$C_2 D_7 F_2$	
17	$A_1 A_4 C_1^2$		64	$A_1 A_2 A_3 C_2 C_3$	Z_2	111	$A_1 A_3 C_1 C_6$	Z_2	158	$A_1 A_2 C_2 D_6$	Z_2^2	205	$A_4^2 C_1 F_2$		252	$C_1 D_8 F_2$	
18	$A_2^2 A_2 A_5 C_1^2$	Z_2	65	$A_3^2 C_2 C_3$	Z_2	112	$A_4 C_1 C_6$		159	$A_3 C_2 D_6$	Z_2^2	206	$A_1^3 A_5 C_1 F_2$	Z_2	253	$D_9 F_2$	
19	$A_1 A_3 A_5 C_1^2$	Z_2	66	$A_2 A_4 C_2 C_3$		113	$A_1^2 C_2 C_6$	Z_2^2	160	$A_2^2 C_1 C_2 D_6$	Z_2^2	207	$A_3 A_5 C_1 F_2$		254	$A_1 A_2 E_6 F_2$	
20	$A_4 A_5 C_1^2$		67	$A_1 A_5 C_2 C_3$	Z_2	114	$A_1 A_2 C_2 C_6$	Z_2	161	$A_2 C_1 C_2 D_6$	Z_2	208	$A_1^2 A_6 C_1 F_2$		255	$A_3 E_6 F_2$	
21	$A_1 A_2 A_6 C_1^2$		68	$A_6 C_2 C_3$		115	$A_3 C_2 C_6$	Z_2	162	$A_1 C_2 D_6$	Z_2^2	209	$A_2 A_6 C_1 F_2$		256	$A_1^2 C_1 E_6 F_2$	
22	$A_1^2 A_7 C_1^2$	Z_2	69	$A_1 A_2^2 C_3^2$		116	$A_1^2 C_3 C_6$	Z_2	163	$A_1 C_1 C_3 D_6$	Z_2	210	$A_1 A_7 C_1 F_2$		257	$A_1 C_2 E_6 F_2$	
23	$A_1^2 A_7 C_1^2$	Z_4	70	$A_1^2 A_3 C_3^2$	Z_2	117	$A_2 C_3 C_6$		164	$C_2 C_3 D_6$	Z_2	211	$A_8 C_1 F_2$		258	$C_3 E_6 F_2$	
24	$A_2 A_7 C_1^2$	Z_4	71	$A_2 A_3 C_3^2$		118	$A_1 C_4 C_6$	Z_2	165	$A_1 C_4 D_6$	Z_2^2	212	$A_2^2 A_3 C_2 F_2$		259	$A_2 E_7 F_2$	
25	$A_1 A_8 C_1^2$		72	$A_1 A_4 C_3^2$		119	$C_5 C_6$		166	$C_1 C_4 D_6$	Z_2	213	$A_1 A_2^2 C_2 F_2$	Z_2	260	$A_1 C_1 E_7 F_2$	
26	$A_9 C_1^2$		73	$A_5 C_3^2$		120	$A_2^2 C_7$		167	$A_1 C_1 C_2 D_7$	Z_2	214	$A_1 A_2 A_4 C_2 F_2$		261	$C_2 E_7 F_2$	
27	$A_1^2 A_3^2 C_2$	Z_3^3	74	$A_1^2 A_2 A_3 C_4$	Z_2^2	121	$A_1 A_2 C_1 C_7$		168	$C_3^2 D_7$	Z_2	215	$A_3 A_4 C_2 F_2$		262	$A_1 E_8 F_2$	
28	$A_1 A_2 A_3^2 C_2$	Z_2^2	75	$A_2^2 A_3 C_4$	Z_2	122	$A_2 C_2 C_7$		169	$A_2^2 C_1 D_8$	Z_2^2	216	$A_1^2 A_5 C_2 F_2$	Z_2	263	$C_1 E_8 F_2$	
29	$A_1 A_3 A_5 C_2$	Z_2^2	76	$A_1 A_3^2 C_4$	Z_2^2	123	$A_1 C_3 C_7$		170	$A_1 C_1^2 D_8$	Z_2	217	$A_2 A_5 C_2 F_2$		264	$A_1^3 A_2^2 F_2^2$	Z_3
30	$A_4 A_5 C_2$	Z_2	77	$A_1^2 A_5 C_4$	Z_2^2	124	$A_1 A_2 C_8$	Z_2	171	$A_1 C_2 D_8$	Z_2^2	218	$A_1 A_6 C_2 F_2$	Z_2^2	265	$A_2^2 A_3 F_2^2$	Z_3
31	$A_2 A_7 C_2$	Z_2	78	$A_1^2 A_2^2 C_1 C_4$	Z_2	125	$A_1^2 C_1 C_8$	Z_2	172	$C_1 C_2 D_8$	Z_2	219	$A_7 C_2 F_2$		266	$A_1 A_3^2 F_2^2$	
32	$A_9 C_2$	Z_2	79	$A_1^3 A_3 C_1 C_4$	Z_2^2	126	$A_2 C_1 C_8$	Z_2	173	$C_1^2 D_9$		220	$A_1^2 A_2^2 C_3 F_2$		267	$A_1^3 A_4 F_2^2$	
33	$A_1 A_2^2 A_3 C_1 C_2$	Z_2	80	$A_1 A_2 A_3 C_1 C_4$	Z_2	127	$A_1 C_2 C_8$	Z_2	174	$A_4 C_1 E_6$		221	$A_1 A_2 A_3 C_3 F_2$		268	$A_3 A_4 F_2^2$	
34	$A_1^2 A_3^2 C_1 C_2$	Z_2^2	81	$A_1^2 A_4 C_1 C_4$	Z_2	128	$C_3 C_8$	Z_2	175	$A_1 A_2 C_1^2 E_6$		222	$A_1^2 A_4 C_3 F_2$		269	$A_1^2 A_5 F_2^2$	Z_3
35	$A_2^2 A_4 C_1 C_2$		82	$A_1 A_5 C_1 C_4$	Z_2	129	$A_1 C_1 C_9$		176	$A_3 C_1^2 E_6$		223	$A_2 A_4 C_3 F_2$		270	$A_2 A_5 F_2^2$	Z_3
36	$A_1 A_3 A_4 C_1 C_2$	Z_2	83	$A_1^3 A_2 C_2 C_4$	Z_2^2	130	$C_2 C_9$		177	$A_2 C_1 C_2 E_6$		224	$A_1 A_5 C_3 F_2$		271	$A_1 A_6 F_2^2$	
37	$A_1^2 C_1 C_2$		84	$A_1 A_2^2 C_2 C_4$	Z_2	131	$A_1 C_{10}$	Z_2	178	$A_1 C_1 C_3 E_6$		225	$A_6 C_3 F_2$		272	$A_7 F_2^2$	
38	$A_1^2 A_5 C_1 C_2$	Z_2^2	85	$A_1^2 A_3 C_2 C_4$	Z_2^2	132	$C_1 C_{10}$		179	$C_2 C_3 E_6$		226	$A_2^2 A_3 C_4 F_2$	Z_2	273	$A_1 A_2 D_4 F_2^2$	
39	$A_1 A_2 A_5 C_1 C_2$	Z_2	86	$A_2 A_3 C_2 C_4$	Z_2	133	$A_1^2 A_3 C_2 D_4$	Z_2^3	180	$C_1 C_4 E_6$		227	$A_1 A_4 C_4 F_2$		274	$A_3 D_4 F_2^2$	
40	$A_3 A_5 C_1 C_2$	Z_2	87	$A_1 A_4 C_2 C_4$	Z_2	134	$A_1 A_3 C_1 C_2 D_4$	Z_2^2	181	$C_5 E_6$		228	$A_5 C_4 F_2$		275	$A_1^2 D_5 F_2^2$	
41	$A_2 A_6 C_1 C_2$		88	$A_5 C_2 C_4$	Z_2	135	$A_1^3 C_2 D_4$	Z_2^3	182	$A_1 A_2 C_1 E_7$	Z_2	229	$A_1^2 A_2 C_5 F_2$		276	$A_1 D_6 F_2^2$	
42	$A_1 A_7 C_1 C_2$	Z_2	89	$A_1^2 A_2 C_3 C_4$	Z_2	136	$A_1 A_2 C_2^2 D_4$	Z_2^2	183	$A_1^2 C_1^2 E_7$	Z_2^2	230	$A_2^2 C_5 F_2$		277	$D_7 F_2^2$	
43	$A_8 C_1 C_2$		90	$A_2^2 C_3 C_4$		137	$A_3 C_2^2 D_4$	Z_2^2	184	$A_2 C_1^2 E_7$	Z_2^2	231	$A_1 A_3 C_5 F_2$		278	$A_1 E_6 F_2^2$	Z_3
44	$A_1^4 A_3 C_2^2$	Z_3^2	91	$A_1 A_3 C_3 C_4$	Z_2	138	$A_1^2 C_1 C_4 D_4$	Z_2^2	185	$A_2 C_2 E_7$	Z_2	232	$A_4 C_5 F_2$		279	$E_7 F_2^2$	
45	$A_1^2 A_2 A_3 C_2^2$	Z_2^2	92	$A_1^3 C_4^2$	Z_2^2	139	$A_1 C_2 C_4 D_4$	Z_2^2	186	$A_1 C_1 C_2 E_7$	Z_2	233	$A_1^3 C_6 F_2$	Z_2	280	$A_1 A_2^2 A_3 F_3$	Z_2
46	$A_2^2 A_3 C_2^2$	Z_2	93	$A_1 A_2 C_4^2$	Z_2	140	$A_1^3 D_4^2$	Z_2^4	187	$C_2^2 E_7$	Z_2	234	$A_1 A_2 C_6 F_2$		281	$A_1^2 A_3^2 F_3$	Z_2^2
47	$A_1 A_3^2 C_2^2$	Z_2^2	94	$A_3 C_4^2$	Z_2^2	141	$A_1^2 A_2 C_1^2 D_5$	Z_2	188	$A_1 C_3 E_7$	Z_2	235	$A_3 C_6 F_2$		282	$A_2^2 A_4 F_3$	

283	$A_1 A_3 A_4 F_3$	Z_2	307	$A_1^2 A_4 C_2 F_3$	Z_2	331	$A_1 C_3 D_4 F_3$	Z_2	355	$A_6 F_2 F_3$	379	$A_1 A_5 C_1 F_4$	403	$A_1 A_4 F_2 F_4$
284	$A_2^2 F_3$		308	$A_2 A_4 C_2 F_3$		332	$A_1 A_2 D_5 F_3$	Z_2	356	$A_2 D_4 F_2 F_3$	380	$A_6 C_1 F_4$	404	$A_5 F_2 F_4$
285	$A_1^3 A_5 F_3$	Z_2^2	309	$A_1 A_5 C_2 F_3$	Z_2	333	$A_2^2 C_1 D_5 F_3$	Z_2	357	$A_1 D_5 F_2 F_3$	381	$A_1 A_2^2 C_2 F_4$	405	$A_1 D_4 F_2 F_4$
286	$A_1 A_2 A_5 F_3$	Z_2	310	$A_6 C_2 F_3$		334	$A_2 C_1 D_5 F_3$		358	$D_6 F_2 F_3$	382	$A_2 A_3 C_2 F_4$	406	$D_5 F_2 F_4$
287	$A_3 A_5 F_3$	Z_2	311	$A_1 A_2^2 C_3 F_3$		335	$A_1 C_2 D_5 F_3$	Z_2	359	$E_6 F_2 F_3$	383	$A_1 A_4 C_2 F_4$	407	$A_1^2 A_2 F_3 F_4$
288	$A_2 A_6 F_3$		312	$A_7^2 A_3 C_3 F_3$	Z_2	336	$C_3 D_5 F_3$		360	$A_1^2 A_2 F_3^2$	Z_2	$A_5 C_2 F_4$	408	$A_2^2 F_3 F_4$
289	$A_1 A_7 F_3$	Z_2	313	$A_2 A_3 C_3 F_3$		337	$A_2 D_6 F_3$	Z_2	361	$A_1 A_2^2 F_3^2$		$A_1^2 A_2 C_3 F_4$	409	$A_1 A_3 F_3 F_4$
290	$A_8 F_3$		314	$A_1 A_4 C_3 F_3$		338	$A_1 C_1 D_6 F_3$	Z_2	362	$A_7^2 A_3 F_3^2$	Z_2	$A_2^2 C_3 F_4$	410	$A_4 F_3 F_4$
291	$A_1^3 A_2^2 C_1 F_3$	Z_2	315	$A_5 C_3 F_3$		339	$C_2 D_6 F_3$	Z_2	363	$A_2 A_3 F_3^2$		$A_1 A_3 C_3 F_4$	411	$D_4 F_3 F_4$
292	$A_7^2 A_2 A_3 C_1 F_3$	Z_2	316	$A_5 C_3 F_3$	Z_2	340	$A_1 D_7 F_3$	Z_2	364	$A_2 A_3 F_3^2$	Z_2	$A_4 C_3 F_4$	412	$A_3^2 F_4$
293	$A_2^2 A_3 C_1 F_3$		317	$A_1^4 C_4 F_3$	Z_2^2	341	$C_1 D_7 F_3$		365	$A_1 A_4 F_3^2$		$A_1 A_2 C_4 F_4$	413	$A_1 A_2 F_4^2$
294	$A_1 A_2^2 C_1 F_3$	Z_2	318	$A_7^2 A_2 C_4 F_3$	Z_2	342	$A_2 E_6 F_3$		366	$A_5 F_3^2$		$A_7^2 C_5 F_4$	414	$A_3 F_4^2$
295	$A_1 A_2 A_4 C_1 F_3$		319	$A_2^2 C_4 F_3$		343	$A_1 C_1 E_6 F_3$		367	$A_1 D_4 F_3^2$	Z_2	$A_2 C_5 F_4$		
296	$A_3 A_4 C_1 F_3$		320	$A_1 A_3 C_4 F_3$	Z_2	344	$C_2 E_6 F_3$		368	$D_5 F_3^2$		$A_1 C_6 F_4$	392	
297	$A_1^2 A_5 C_1 F_3$	Z_2	321	$A_4 C_4 F_3$		345	$A_1 E_7 F_3$	Z_2	369	$D_5 F_3^2$	Z_2	$C_7 F_4$	393	
298	$A_2 A_5 C_1 F_3$		322	$A_1 A_2 C_5 F_3$		346	$C_1 E_7 F_3$		370	$A_1 A_2^2 F_4$		$A_1 C_1 D_5 F_4$	394	
299	$A_2 A_5 C_1 F_3$	Z_2	323	$A_3 C_5 F_3$		347	$C_1 E_7 F_3$	Z_2	371	$A_2^2 A_3 F_4$		$C_2 D_5 F_4$	395	
300	$A_1 A_6 C_1 F_3$		324	$A_1^2 C_6 F_3$	Z_2	348	$E_8 F_3$		372	$A_1 A_2 A_4 F_4$		$C_1 D_6 F_4$	396	
301	$A_7 C_1 F_3$		325	$A_2 C_6 F_3$		349	$A_1^2 A_3 F_2 F_3$	Z_2	373	$A_2 A_5 F_4$		$A_1 E_6 F_4$	397	
302	$A_1^2 A_2^2 C_2 F_3$	Z_2	326	$A_2 C_6 F_3$	Z_2	350	$A_1 A_2 A_3 F_2 F_3$		374	$A_1 A_6 F_4$		$C_1 E_6 F_4$	398	
303	$A_2^3 C_2 F_3$		327	$A_1 C_7 F_3$		351	$A_2^2 F_2 F_3$		375	$A_7^2 A_2^2 C_1 F_4$		$E_7 F_4$	399	
304	$A_1^3 A_3 C_2 F_3$	Z_2^2	328	$C_8 F_3$		352	$A_2^2 A_4 F_2 F_3$		376	$A_1 A_2 A_3 C_1 F_4$		$A_1^3 A_2 F_2 F_4$	400	
305	$A_1 A_2 A_3 C_2 F_3$	Z_2	329	$A_1 A_2 C_1 D_4 F_3$	Z_2	353	$A_2 A_4 F_2 F_3$		377	$A_7^2 A_4 C_1 F_4$		$A_7^2 A_3 F_2 F_4$	401	
306	$A_3^2 C_2 F_3$	Z_2	330	$A_2 C_2 D_4 F_3$	Z_2	354	$A_1 A_5 F_2 F_3$		378	$A_2 A_4 C_1 F_4$		$A_2 A_3 F_2 F_4$	402	

 Table 5. Algebras of maximal rank for the heterotic Z_2 triple.

1 $A_1^3 A_2^2 Z_3$	7 $A_2 A_5 Z_3$	13 $A_1 D_6$	19 $A_2^2 G_1$	25 $A_1 D_5 G_1$	31 $A_2 A_3 G_1^2$	37 $D_5 G_1^2 Z_2$	43 $A_1 D_4 G_2$	49 $D_4 G_1 G_2$
2 $A_2^2 A_3 Z_3$	8 $A_1 A_6$	14 D_7	20 $A_1^2 A_4 G_1$	26 $D_6 G_1$	32 $A_2 A_3 G_1^2 Z_2$	38 $A_1^3 A_2 G_2$	44 $D_5 G_2$	50 $A_1^2 G_2^2$
3 $A_1 A_2^3$	9 A_7	15 $A_1 E_6$	21 $A_2 A_4 G_1$	27 $E_6 G_1$	33 $A_1 A_4 G_1^2$	39 $A_7^2 A_3 G_2$	45 $A_1^2 A_2 G_1 G_2$	51 $A_1 A_2 G_2^2$
4 $A_1^3 A_4$	10 $A_1 A_2 D_4$	16 E_7	22 $A_1 A_5 G_1$	28 $A_1^3 A_2 G_1^2 Z_2$	34 $A_5 G_1^2$	40 $A_2 A_3 G_2$	46 $A_2^2 G_1 G_2$	52 $A_3 G_2^2$
5 $A_3 A_4$	11 $A_3 D_4$	17 $A_1^3 A_3 G_1$	23 $A_6 G_1$	29 $A_1 A_2^2 G_1^2$	35 $A_1 D_4 G_1^2 Z_2$	41 $A_1 A_4 G_2$	47 $A_1 A_3 G_1 G_2$	
6 $A_1^2 A_5 Z_3$	12 $A_1^2 D_5$	18 $A_1 A_2 A_3 G_1$	24 $A_2 D_4 G_1$	30 $A_1^2 A_3 G_1^2 Z_2$	36 $D_5 G_1^2$	42 $A_5 G_2$	48 $A_4 G_1 G_2$	

 Table 6. Algebras of maximal rank for the heterotic Z_3 triple.

1 $A_1^5 Z_2^2$	3 $A_1^3 A_2 Z_2^2$	5 $A_1 A_2^2 Z_2$	7 $A_1^2 A_3 Z_2^2$	9 $A_2 A_3 Z_2$	11 $A_1 A_4 Z_2$	13 $A_1 D_4 Z_2$	15 $D_5 Z_2$
2 $A_1^3 A_2 Z_2$	4 $A_1 A_2^2$	6 $A_1^2 A_3 Z_2$	8 $A_2 A_3$	10 $A_1 A_4$	12 A_5	14 D_5	

 Table 7. Algebras of maximal rank for the heterotic Z_4 triple.

1 A_1^3
2 $A_1 A_2$
3 A_3

 Table 8. Algebras of maximal rank for the heterotic Z_5 and Z_6 triples.

B.2 Fundamental group generators

4	$A_1^2 A_3^3 A_4^2$	Z_2	1122200	172	$A_1^3 A_3 A_4 A_9$	Z_2	001205	342	$A_1^2 A_2 A_4 A_7 D_4$	Z_2	11004s	455	$A_1 A_2^2 A_5 D_4 D_5$	Z_2	1003s2
11	$A_1^2 A_2 A_3^2 A_4 A_5$	Z_2	0102203	172'	$A_1^3 A_3 A_4 A_9$	Z_2	111005	344	$A_1 A_3 A_4 A_7 D_4$	Z_2	0204s	457	$A_1 A_4 A_5 D_4 D_5$	Z_2	103s2
13	$A_1 A_3^3 A_4 A_5$	Z_2	102203	174	$A_1 A_2 A_3 A_4 A_9$	Z_2	10205	346	$A_1 A_2 A_5 A_7 D_4$	Z_2	10340	458	$A_5^2 D_4 D_5$	Z_2	3302
14	$A_1^3 A_3 A_4^2 A_5$	Z_2	1112003	179	$A_1^3 A_2 A_5 A_9$	Z_2	000035	347	$A_3 A_5 A_7 D_4$	Z_2	204s	458'	$A_5^2 D_4 D_5$	Z_2	33s0
23	$A_1 A_2 A_3^2 A_5^2$	Z_2	000233	179'	$A_1^3 A_2 A_5 A_9$	Z_2	111005	348	$A_1^2 A_6 A_7 D_4$	Z_2	1104s	460	$A_1 A_2 A_7 D_4 D_5$	Z_2	004s2
23'	$A_1 A_2 A_3^2 A_5^2$	Z_2	102203	181	$A_1 A_2^2 A_5 A_9$	Z_2	00035	355	$A_1^2 A_2 A_9 D_4$	Z_2	01005s	461	$A_1 A_9 D_4 D_5$	Z_2	1502
24	$A_3^3 A_5^2$	Z_2	00233	182	$A_1^2 A_3 A_5 A_9$	Z_2	00035	356	$A_1 A_2 A_3 A_9 D_4$	Z_2	1005s	461'	$A_1 A_9 D_4 D_5$	Z_2	15s0
26	$A_1^2 A_3 A_4 A_5^2$	Z_2	002033	182'	$A_1^2 A_3 A_5 A_9$	Z_2	01205	356'	$A_1 A_2 A_3 A_9 D_4$	Z_2	10250	462	$A_1^2 A_2^3 A_3 D_5^2$	Z_2	1100222
26'	$A_1^2 A_3 A_4 A_5^2$	Z_2	110033	184	$A_2 A_3 A_5 A_9$	Z_2	0035	357	$A_1^2 A_4 A_9 D_4$	Z_2	0105s	466	$A_1^2 A_2 A_5 D_5^2$	Z_2	010322
28	$A_2 A_3 A_4 A_5^2$	Z_2	02033	186	$A_1 A_4 A_5 A_9$	Z_2	0035	359	$A_1 A_5 A_9 D_4$	Z_2	0350	467	$A_1 A_3 A_5 D_5^2$	Z_2	12302
34	$A_1 A_3 A_5^3$	Z_2	02033	188	$A_5^2 A_9$	Z_2	035	359'	$A_1 A_5 A_9 D_4$	Z_2	105s	470	$A_1^2 A_7 D_5^2$	Z_2	11402
50	$A_1^2 A_3^2 A_5 A_6$	Z_2	012230	189	$A_1^4 A_6 A_9$	Z_2	011105	365	$A_1^2 A_2 A_{11} D_4$	Z_2	0006s	475	$A_1^2 A_2 D_5^3$	Z_2	110222
56	$A_1^3 A_5^2 A_6$	Z_2	011330	193	$A_1 A_3 A_6 A_9$	Z_2	1205	365'	$A_1^2 A_2 A_{11} D_4$	Z_2	11060	477	$D_4 D_5^3$	Z_2	s222
59	$A_3 A_5^2 A_6$	Z_2	2330	195	$A_1^3 A_7 A_9$	Z_2	11105	368	$A_1 A_3 A_{11} D_4$	Z_2	006s	478	$A_1 A_3^2 A_3^2 D_6$	Z_2	100022s
76	$A_1^3 A_2 A_3 A_4 A_7$	Z_2	0110204	226	$A_1^3 A_2 A_3 A_{11}$	Z_2	011006	370	$A_4 A_{11} D_4$	Z_2	06s	480	$A_1 A_2 A_3^2 A_4 D_6$	Z_2	10220s
78	$A_1^2 A_3^2 A_4 A_7$	Z_2	110204	229	$A_1 A_3^2 A_3 A_{11}$	Z_2	00026	372	$A_1^2 A_{13} D_4$	Z_2	0170	481	$A_3^3 A_4 D_6$	Z_2	2220v
81	$A_2 A_3^2 A_4 A_7$	Z_2	02204	232	$A_2^2 A_3^2 A_{11}$	Z_2	11006	379	$A_2^2 A_7 D_4^2$	Z_2	004ss	486	$A_1 A_2^2 A_3 A_5 D_6$	Z_2	00023s
82	$A_1^4 A_4 A_7$	Z_2	1111004	235	$A_2 A_3^2 A_{11}$	Z_2	0026	389	$A_1 A_2 A_3^2 A_5 D_5$	Z_2	102230	488	$A_2 A_3^2 A_5 D_6$	Z_2	0023s
87	$A_1^2 A_2 A_3 A_5 A_7$	Z_2	010034	237	$A_1^4 A_4 A_{11}$	Z_2	001106	390	$A_1^2 A_3 A_4 A_5 D_5$	Z_2	012032	489	$A_1^2 A_2 A_4 A_5 D_6$	Z_2	11003s
87'	$A_1^2 A_2 A_3 A_5 A_7$	Z_2	110204	239	$A_1^2 A_2 A_4 A_{11}$	Z_2	11006	396	$A_1 A_3 A_5^2 D_5$	Z_2	00332	491	$A_1 A_3 A_4 A_5 D_6$	Z_2	0203s
89	$A_1 A_3^2 A_5 A_7$	Z_2	10034	243	$A_1 A_3 A_4 A_{11}$	Z_2	0206	396'	$A_1 A_3 A_5^2 D_5$	Z_2	02330	491'	$A_1 A_3 A_4 A_5 D_6$	Z_2	1203v
89	$A_1 A_3^2 A_5 A_7$	Z_2	12230	245	$A_3^3 A_5 A_{11}$	Z_2	01106	396''	$A_1 A_3 A_5^2 D_5$	Z_2	12032	494	$A_1 A_2 A_5^2 D_6$	Z_2	0033v
91	$A_3^3 A_4 A_5 A_7$	Z_2	001034	250	$A_3 A_5 A_{11}$	Z_2	206	398	$A_4 A_5^2 D_5$	Z_2	0332	495	$A_3 A_5^2 D_6$	Z_2	033v
93	$A_1 A_2 A_4 A_5 A_7$	Z_2	10034	254	$A_1^2 A_6 A_{11}$	Z_2	1106	409	$A_1^3 A_2^2 A_7 D_5$	Z_2	0110042	495'	$A_3 A_5^2 D_6$	Z_2	203s
95	$A_1^2 A_2^2 A_7$	Z_2	01034	270	$A_1^4 A_2 A_{13}$	Z_2	000107	410	$A_1^2 A_2 A_3 A_7 D_5$	Z_2	110240	495''	$A_3 A_5^2 D_6$	Z_2	2330
95'	$A_1^2 A_2^2 A_7$	Z_2	11330	272	$A_1^2 A_2^2 A_{13}$	Z_2	01007	413	$A_2^2 A_3 A_7 D_5$	Z_2	00242	499	$A_1 A_3^2 A_6 D_6$	Z_2	1220s
100	$A_1^3 A_3 A_6 A_7$	Z_2	011204	274	$A_1^3 A_3 A_{13}$	Z_2	00107	415	$A_1^3 A_4 A_7 D_5$	Z_2	011042	502	$A_1^2 A_5 A_6 D_6$	Z_2	1130s
103	$A_3^2 A_6 A_7$	Z_2	2204	276	$A_1 A_2 A_3 A_{13}$	Z_2	1007	417	$A_3 A_4 A_7 D_5$	Z_2	2042	505	$A_1^2 A_3^2 A_7 D_6$	Z_2	01004s
107	$A_1 A_5 A_6 A_7$	Z_2	1304	279	$A_1^2 A_4 A_{13}$	Z_2	0107	418	$A_1^2 A_5 A_7 D_5$	Z_2	01340	508	$A_1 A_2 A_3 A_7 D_6$	Z_2	0024v
112	$A_1 A_2^2 A_7^2$	Z_2	00044	282	$A_1 A_5 A_{13}$	Z_2	107	418'	$A_1^2 A_5 A_7 D_5$	Z_2	11042	508'	$A_1 A_2 A_3 A_7 D_6$	Z_2	1004s
113	$A_1^2 A_3 A_7^2$	Z_2	11204	295	$A_2^2 A_2 A_{15}$	Z_2	0008	428	$A_1^3 A_2 A_9 D_5$	Z_2	001052	509	$A_3^2 A_7 D_6$	Z_2	024v
118	$A_2 A_3 A_7^2$	Z_2	0044	298	$A_2^2 A_{15}$	Z_2	008	428'	$A_1^3 A_2 A_9 D_5$	Z_2	111050	511	$A_1^2 A_4 A_7 D_6$	Z_2	0104s
121	$A_1 A_4 A_7^2$	Z_2	0044	300	$A_1 A_3 A_{15}$	Z_2	008	430	$A_1 A_2^2 A_9 D_5$	Z_2	10052	511'	$A_1^2 A_4 A_7 D_6$	Z_2	1104v
123	$A_5 A_7^2$	Z_2	044	303	$A_4 A_{15}$	Z_2	08	431	$A_1^3 A_3 A_9 D_5$	Z_2	01052	513	$A_1 A_5 A_7 D_6$	Z_2	104s
127	$A_1^2 A_3^3 A_8$	Z_2	112220	314	$A_2 A_3^3 A_4 D_4$	Z_2	02220s	431'	$A_1^2 A_3 A_9 D_5$	Z_2	01250	513'	$A_1 A_5 A_7 D_6$	Z_2	1340
138	$A_1^3 A_3 A_5 A_8$	Z_2	111230	320	$A_1 A_2 A_3 A_4 A_5 D_4$	Z_2	10203s	434	$A_1 A_4 A_9 D_5$	Z_2	1052	519	$A_1^2 A_2 A_9 D_6$	Z_2	0005s
152	$A_1^4 A_7 A_8$	Z_2	111140	325	$A_1 A_4 A_5^2 D_4$	Z_2	0033s	436	$A_5 A_9 D_5$	Z_2	350	519'	$A_1^2 A_2 A_9 D_6$	Z_2	0105v
163	$A_1^3 A_2^2 A_3 A_9$	Z_2	0010025	328	$A_3^3 A_6 D_4$	Z_2	2220s	441	$A_3^3 A_{11} D_5$	Z_2	01160	521	$A_2^2 A_9 D_6$	Z_2	005s
165	$A_1 A_3^2 A_3 A_9$	Z_2	100025	332	$A_1 A_3 A_5 A_6 D_4$	Z_2	1230s	444	$A_1 A_2 A_{11} D_5$	Z_2	0062	522	$A_1 A_3 A_9 D_6$	Z_2	005s
166	$A_1^2 A_2 A_3^2 A_9$	Z_2	010025	336	$A_1^2 A_3^2 A_7 D_4$	Z_2	110004s	446	$A_3 A_{11} D_5$	Z_2	062	522'	$A_1 A_3 A_9 D_6$	Z_2	105v
168	$A_1 A_3^3 A_9$	Z_2	10025	337	$A_1 A_2^2 A_3 A_7 D_4$	Z_2	00024s	451	$A_1 A_{13} D_5$	Z_2	170	522''	$A_1 A_3 A_9 D_6$	Z_2	1250
169	$A_1^4 A_2 A_4 A_9$	Z_2	0111005	340	$A_2 A_3^2 A_7 D_4$	Z_2	0024s	453	$A_2^2 A_3^2 D_4 D_5$	Z_2	0022s2	524	$A_4 A_9 D_6$	Z_2	05s

527	$A_1^2 A_{11} D_6$	Z_2	006v	626	$A_1 A_5 D_6 D_7$	Z_2	13v2	711	$A_3 A_7 D_9$	Z_2	242	798	$A_2 D_5 D_{12}$	Z_2	02s
527'	$A_1^2 A_{11} D_6$	Z_2	1160	628	$A_1 D_5 D_6 D_7$	Z_2	12s2	715	$A_1 A_9 D_9$	Z_2	152	799	$A_1 D_6 D_{12}$	Z_2	0vs
529	$A_2 A_{11} D_6$	Z_2	06v	629	$D_6^2 D_7$	Z_2	ss2	717	$A_1 A_5 D_4 D_9$	Z_2	13s2	800	$D_7 D_{12}$	Z_2	2s
535	$A_4 A_5 D_4 D_6$	Z_2	03ss	631	$A_1^2 A_3 D_7^2$	Z_2	11222	723	$A_1 A_3 D_6 D_9$	Z_2	12s2	809	$A_1^3 A_2 D_{14}$	Z_2	0010s
536	$A_2 A_7 D_4 D_6$	Z_2	04sv	636	$A_1 A_2^2 A_3^2 D_8$	Z_2	00022s	726	$A_1^2 D_8 D_9$	Z_2	11s2	811	$A_1 A_2^2 D_{14}$	Z_2	100s
537	$A_9 D_4 D_6$	Z_2	50s	638	$A_2 A_3^3 D_8$	Z_2	0022s	728	$A_1^3 A_2^3 D_{10}$	Z_2	111000s	812	$A_1^2 A_3 D_{14}$	Z_2	010s
539	$A_1 A_2^2 A_3 D_5 D_6$	Z_2	10022s	638'	$A_2 A_3^3 D_8$	Z_2	0222v	729	$A_1^2 A_2^2 A_3 D_{10}$	Z_2	01002s	815	$A_1 A_4 D_{14}$	Z_2	10s
542	$A_1 A_3 A_4 D_5 D_6$	Z_2	1202s	639	$A_1^2 A_2 A_3 A_4 D_8$	Z_2	11020s	731	$A_1 A_2 A_3^2 D_{10}$	Z_2	1002s	817	$A_1 D_4 D_{14}$	Z_2	10s
545	$A_1 A_2 A_5 D_5 D_6$	Z_2	0032s	641	$A_1 A_3^2 A_4 D_8$	Z_2	0220s	732	$A_3^3 D_{10}$	Z_2	222v	823	$A_1^3 D_{16}$	Z_2	000s
546	$A_3 A_5 D_5 D_6$	Z_2	032s	645	$A_1 A_2 A_3 A_5 D_8$	Z_2	1003s	733	$A_1^3 A_2 A_4 D_{10}$	Z_2	11100s	825	$A_1 A_2 D_{16}$	Z_2	00s
546'	$A_3 A_5 D_5 D_6$	Z_2	230s	645'	$A_1 A_2 A_3 A_5 D_8$	Z_2	1023v	735	$A_1^2 A_3 A_4 D_{10}$	Z_2	0120s	826	$A_3 D_{16}$	Z_2	0s
548	$A_1 A_7 D_5 D_6$	Z_2	042v	646	$A_3^2 A_5 D_8$	Z_2	220s	739	$A_1^2 A_2 A_5 D_{10}$	Z_2	0003s	831	$A_1^2 A_2 A_3^3 E_6$	Z_2	1102220
548'	$A_1 A_7 D_5 D_6$	Z_2	140s	647	$A_2^2 A_4 A_5 D_8$	Z_2	0103s	740	$A_1 A_3 A_5 D_{10}$	Z_2	003s	840	$A_1^2 A_3^2 A_5 E_6$	Z_2	012230
551	$A_1 A_2 D_6^2 D_6$	Z_2	1022s	649	$A_1 A_2^2 D_8$	Z_2	033v	740'	$A_1 A_3 A_5 D_{10}$	Z_2	120s	848	$A_3 A_5^2 E_6$	Z_2	2330
554	$A_2^2 A_3 D_6^2$	Z_2	002ss	649'	$A_1 A_5^2 D_8$	Z_2	103s	740''	$A_1 A_3 A_5 D_{10}$	Z_2	123v	860	$A_1^4 A_2 A_7 E_6$	Z_2	1111040
556	$A_3 A_4 D_6^2$	Z_2	20ss	652	$A_2^2 A_3 A_6 D_8$	Z_2	1120s	743	$A_4 A_5 D_{10}$	Z_2	03s	861	$A_1^3 A_3 A_7 E_6$	Z_2	011240
558	$A_2 A_5 D_6^2$	Z_2	03sv	657	$A_2^2 A_2 A_7 D_8$	Z_2	0004s	744	$A_3^3 A_6 D_{10}$	Z_2	1110s	866	$A_1 A_5 A_7 E_6$	Z_2	1340
559	$A_7 D_6^2$	Z_2	4vv	657'	$A_1^2 A_2 A_7 D_8$	Z_2	1104v	747	$A_1^2 A_7 D_{10}$	Z_2	114v	875	$A_1^4 A_9 E_6$	Z_2	011150
564	$A_2 D_6^2 D_6^2$	Z_2	02ss	659	$A_2^2 A_7 D_8$	Z_2	004s	751	$A_1 A_2^2 D_4 D_{10}$	Z_2	100ss	879	$A_1 A_3 A_9 E_6$	Z_2	1250
570	$A_1^2 A_3^2 A_4 D_7$	Z_2	112202	660	$A_1 A_3 A_7 D_8$	Z_2	004s	753	$A_1 A_4 D_4 D_{10}$	Z_2	10ss	885	$A_1^2 A_{11} E_6$	Z_2	1160
576	$A_1^2 A_2 A_3 A_5 D_7$	Z_2	010232	660'	$A_1 A_3 A_7 D_8$	Z_2	024v	754	$A_5 D_4 D_{10}$	Z_2	30s	896	$A_1 A_3 A_5 D_4 E_6$	Z_2	123s0
577	$A_1 A_2^2 A_5 D_7$	Z_2	10232	661	$A_4 A_7 D_8$	Z_2	04s	755	$A_1^2 A_2 D_5 D_{10}$	Z_2	0102s	899	$A_2^2 A_7 D_4 E_6$	Z_2	114s0
577'	$A_1 A_2^2 A_5 D_7$	Z_2	12230	663	$A_2^2 A_9 D_8$	Z_2	015v	757	$A_1 A_3 D_5 D_{10}$	Z_2	102s	902	$A_1^2 A_3^2 D_5 E_6$	Z_2	112220
578	$A_1^3 A_4 A_5 D_7$	Z_2	111032	666	$A_{11} D_8$	Z_2	6v	757'	$A_1 A_3 D_5 D_{10}$	Z_2	120s	906	$A_1^3 A_5 D_5 E_6$	Z_2	111320
581	$A_1^2 A_2^2 D_7$	Z_2	00332	668	$A_2^2 A_3 D_4 D_8$	Z_2	002ss	760	$A_1 A_2 D_6 D_{10}$	Z_2	00ss	916	$A_1 A_3^2 D_6 E_6$	Z_2	122s0
581'	$A_1^2 A_2^2 D_7$	Z_2	11330	671	$A_1^2 A_2^2 D_5 D_8$	Z_2	11002s	760'	$A_1 A_2 D_6 D_{10}$	Z_2	10vs	919	$A_1^2 A_5 D_6 E_6$	Z_2	113s0
589	$A_1^3 A_2 A_7 D_7$	Z_2	011042	673	$A_1 A_2 A_3 D_5 D_8$	Z_2	0022s	761	$A_3 D_6 D_{10}$	Z_2	0ss	932	$A_1^2 A_3 D_8 E_6$	Z_2	112s0
591	$A_1^2 A_3 A_7 D_7$	Z_2	11042	674	$A_1^2 A_4 D_5 D_8$	Z_2	1102s	762	$A_1^2 D_7 D_{10}$	Z_2	012s	941	$A_1^3 D_{10} E_6$	Z_2	111s0
591'	$A_1^2 A_3 A_7 D_7$	Z_2	11240	675	$A_1 A_5 D_5 D_8$	Z_2	130s	764	$A_1 D_8 D_{10}$	Z_2	1vs	963	$A_1^2 A_2^2 A_3^2 E_7$	Z_2	0100221
594	$A_2 A_3 A_7 D_7$	Z_2	0242	675'	$A_1 A_5 D_5 D_8$	Z_2	132v	767	$A_1^2 A_3^2 D_{11}$	Z_2	11222	966	$A_1^2 A_3^2 A_4 E_7$	Z_2	012201
601	$A_1^3 A_9 D_7$	Z_2	00152	677	$A_1 D_5^2 D_8$	Z_2	022s	773	$A_1^3 A_5 D_{11}$	Z_2	11132	972	$A_1^2 A_2 A_3 A_5 E_7$	Z_2	000231
601'	$A_1^3 A_9 D_7$	Z_2	11150	679	$A_1 A_2^2 D_6 D_8$	Z_2	100ss	784	$A_1^3 A_2^2 D_{12}$	Z_2	01100s	974	$A_2^2 A_3 A_5 E_7$	Z_2	00231
603	$A_1 A_2 A_9 D_7$	Z_2	1052	681	$A_2 A_3 D_6 D_8$	Z_2	02vs	786	$A_1^2 A_2 A_3 D_{12}$	Z_2	0002s	975	$A_1 A_3^2 A_5 E_7$	Z_2	00231
608	$A_1 A_{11} D_7$	Z_2	062	682	$A_1 A_4 D_6 D_8$	Z_2	10ss	786'	$A_1^2 A_2 A_3 D_{12}$	Z_2	1100s	975'	$A_1 A_3^2 A_5 E_7$	Z_2	12201
611	$A_2 A_3^2 D_4 D_7$	Z_2	022s2	683	$A_5 D_6 D_8$	Z_2	3sv	788	$A_2^2 A_3 D_{12}$	Z_2	002s	975''	$A_1 A_3^2 A_5 E_7$	Z_2	12230
613	$A_1 A_2 A_5 D_4 D_7$	Z_2	103s2	685	$D_5 D_6 D_8$	Z_2	2vs	789	$A_1 A_3^2 D_{12}$	Z_2	002s	976	$A_1^3 A_4 A_5 E_7$	Z_2	011031
615	$A_1 A_7 D_4 D_7$	Z_2	04s2	686	$A_1^2 A_2 D_7 D_8$	Z_2	1102s	790	$A_1^3 A_4 D_{12}$	Z_2	0110s	979	$A_3 A_4 A_5 E_7$	Z_2	2031
619	$A_1^2 A_5 D_5 D_7$	Z_2	01322	688	$A_1 A_3 D_7 D_8$	Z_2	022s	792	$A_3 A_4 D_{12}$	Z_2	20s	980	$A_1^2 A_5^2 E_7$	Z_2	11031
622	$A_7 D_5 D_7$	Z_2	422	690	$A_1 A_2 D_8^2$	Z_2	00ss	793	$A_2^2 A_5 D_{12}$	Z_2	110s	980'	$A_1^2 A_5^2 E_7$	Z_2	11330
623	$A_1 A_2 A_3 D_6 D_7$	Z_2	102s2	700	$A_1^2 A_3 A_5 D_9$	Z_2	01232	796	$A_1 A_2 D_4 D_{12}$	Z_2	00ss	990	$A_1^3 A_2 A_7 E_7$	Z_2	001041
624	$A_3^2 D_6 D_7$	Z_2	22v2	704	$A_5^2 D_9$	Z_2	332	797	$A_2^2 D_5 D_{12}$	Z_2	002s	992	$A_1 A_2^2 A_7 E_7$	Z_2	10041
626'	$A_1 A_5 D_6 D_7$	Z_2	03s2	709	$A_1^3 A_7 D_9$	Z_2	01142	797'	$A_1^2 D_5 D_{12}$	Z_2	110s	993	$A_1^2 A_3 A_7 E_7$	Z_2	01041

993'	$A_1^2 A_3 A_7 E_7$	Z_2	11240	1086	$A_1 D_4 E_7^2$	Z_2	0s11	839	$A_1 A_2^2 A_3 A_5 E_6$	Z_3	011021	442	$A_1^3 A_{11} D_5$	Z_4	00131
996	$A_1 A_4 A_7 E_7$	Z_2	1041	1088	$D_5 E_7^2$	Z_2	211	842	$A_2^2 A_4 A_5 E_6$	Z_3	11021	445	$A_1 A_2 A_{11} D_5$	Z_4	1031
1002	$A_1^3 A_9 E_7$	Z_2	00051	1090	$A_1^2 A_3^3 E_8$	Z_2	112220	846	$A_1 A_2 A_5^2 E_6$	Z_3	00221	447	$A_3 A_{11} D_5$	Z_4	132
1002'	$A_1^3 A_9 E_7$	Z_2	11150	1099	$A_1^3 A_3 A_5 E_8$	Z_2	111230	849	$A_3 A_5^2 E_6$	Z_3	0221	463	$A_1 A_2 A_3^2 D_5^2$	Z_4	001111
1004	$A_1 A_2 A_9 E_7$	Z_2	0051	1111	$A_1^4 A_7 E_8$	Z_2	111140	868	$A_1^3 A_2 A_8 E_6$	Z_3	000131	471	$A_1^2 A_7 D_5^2$	Z_4	00211
1006	$A_3 A_9 E_7$	Z_2	051	10	$A_4^4 A_3^2 A_5$	Z_3	1111002	869	$A_1 A_2^2 A_8 E_6$	Z_3	00131	472	$A_2 A_7 D_5^2$	Z_4	0211
1011	$A_1 A_2^2 A_3 D_4 E_7$	Z_2	1002s1	20	$A_2^3 A_3 A_5^2$	Z_3	011022	872	$A_2 A_3 A_8 E_6$	Z_3	1031	510	$A_3^2 A_7 D_6$	Z_4	112v
1014	$A_1 A_3 A_4 D_4 E_7$	Z_2	120s1	25	$A_1 A_2^2 A_4 A_5^2$	Z_3	011022	886	$A_1^2 A_{11} E_6$	Z_3	0041	566	$A_1 A_2 A_3^3 D_7$	Z_4	001111
1017	$A_1 A_2 A_5 D_4 E_7$	Z_2	003s1	32	$A_2^2 A_5^3$	Z_3	00222	889	$A_2 A_{11} E_6$	Z_3	041	567	$A_3^4 D_7$	Z_4	01111
1018	$A_3 A_5 D_4 E_7$	Z_2	03s1	32'	$A_2^2 A_5^3$	Z_3	11022	892	$A_1 A_2^4 D_4 E_6$	Z_3	0111101	567'	$A_3^4 D_7$	Z_4	11112
1018'	$A_3 A_5 D_4 E_7$	Z_2	2301	37	$A_4 A_5^3$	Z_3	0222	895	$A_2^3 A_5 D_4 E_6$	Z_3	11201	592	$A_1^2 A_3 A_7 D_7$	Z_4	00121
1020	$A_1 A_7 D_4 E_7$	Z_2	1401	48	$A_5^4 A_5 A_6$	Z_3	111120	948	$A_1^3 A_2^2 E_6^2$	Z_3	0001111	595	$A_2 A_3 A_7 D_7$	Z_4	0121
1022	$A_1^3 A_2^2 D_5 E_7$	Z_2	1110021	124	$A_1^2 A_3^2 A_3 A_8$	Z_3	0011103	949	$A_2^3 A_3 E_6^2$	Z_3	11011	609	$A_1 A_{11} D_7$	Z_4	031
1023	$A_1^2 A_2 A_3 D_5 E_7$	Z_2	010221	125	$A_2^4 A_3 A_8$	Z_3	011103	953	$A_1^2 A_5 E_6^2$	Z_3	00211	7	$A_1^3 A_4^4$	Z_5	0001122
1025	$A_1 A_3^2 D_5 E_7$	Z_2	12201	129	$A_1 A_3^2 A_4 A_8$	Z_3	011103	954	$A_2 A_5 E_6^2$	Z_3	0211	8	$A_1 A_2 A_4^4$	Z_5	001122
1028	$A_1^2 A_5 D_5 E_7$	Z_2	00321	136	$A_2^2 A_2^2 A_5 A_8$	Z_3	000123	962	$A_1 E_6^3$	Z_3	0111	9	$A_3 A_4^4$	Z_5	01122
1028'	$A_1^2 A_5 D_5 E_7$	Z_2	11301	137	$A_2^3 A_5 A_8$	Z_3	00123	2	$A_3^5 A_4$	Z_4	211110	177	$A_1^2 A_2^4 A_9$	Z_5	00114
1030	$A_2 A_5 D_5 E_7$	Z_2	0321	137'	$A_2^3 A_5 A_8$	Z_3	11103	71	$A_1^2 A_2^2 A_3^2 A_7$	Z_4	1100112	178	$A_2 A_4^4 A_9$	Z_5	0114
1033	$A_1 A_2 D_4 D_5 E_7$	Z_2	10s21	140	$A_1 A_2 A_3 A_5 A_8$	Z_3	01023	73	$A_1 A_2 A_3^3 A_7$	Z_4	002112	201	$A_1 A_2^2$	Z_5	024
1034	$A_1^2 D_5^2 E_7$	Z_2	01221	144	$A_2 A_4 A_5 A_8$	Z_3	1023	74	$A_3^4 A_7$	Z_4	02112	19	$A_1^2 A_2^2 A_3 A_5^2$	Z_6	0011255
1036	$A_1^2 A_2^2 D_6 E_7$	Z_2	1100s1	158	$A_1^3 A_8^2$	Z_3	00033	79	$A_2^2 A_3^2 A_4 A_7$	Z_4	111102	21	$A_2^3 A_3 A_5^2$	Z_6	011255
1039	$A_1 A_2 A_3 D_6 E_7$	Z_2	002s1	160	$A_1 A_2 A_8^2$	Z_3	0033	110	$A_1^3 A_2 A_7^2$	Z_4	011022	31	$A_1^2 A_2 A_5^3$	Z_6	110255
1040	$A_3^2 D_6 E_7$	Z_2	02s1	162	$A_3 A_8^2$	Z_3	033	119	$A_2 A_3 A_7^2$	Z_4	0222	35	$A_1 A_3 A_5^3$	Z_6	02255
1041	$A_1^2 A_4 D_6 E_7$	Z_2	110s1	223	$A_1^2 A_2^3 A_{11}$	Z_3	000114	227	$A_1^3 A_2 A_3 A_{11}$	Z_4	011013	222	$A_1^4 A_2^2 A_{11}$	Z_6	00111110
1043	$A_1 A_5 D_6 E_7$	Z_2	03v1	225	$A_2^4 A_{11}$	Z_3	00114	233	$A_1^2 A_3^2 A_{11}$	Z_4	00213	224	$A_1^2 A_3^2 A_{11}$	Z_6	1101110
1045	$A_2 D_4 D_6 E_7$	Z_2	0ss1	230	$A_1 A_2^2 A_3 A_{11}$	Z_3	01104	233'	$A_1^2 A_3^2 A_{11}$	Z_4	11013	231	$A_1 A_2^2 A_3 A_{11}$	Z_6	011210
1046	$A_1 D_5 D_6 E_7$	Z_2	02s1	241	$A_2^2 A_4 A_{11}$	Z_3	1104	236	$A_2 A_3^2 A_{11}$	Z_4	0213	246	$A_1^3 A_5 A_{11}$	Z_6	011210
1047	$D_6^2 E_7$	Z_2	sv1	248	$A_1 A_2 A_5 A_{11}$	Z_3	0024	293	$A_4^4 A_{15}$	Z_4	00114	252	$A_3 A_5 A_{11}$	Z_6	2210
1049	$A_1^2 A_3 D_7 E_7$	Z_2	01221	251	$A_3 A_5 A_{11}$	Z_3	024	296	$A_1^2 A_2 A_{15}$	Z_4	1104	323	$A_1 A_2^2 A_5^2 D_4$	Z_6	01111s
1053	$A_5 D_7 E_7$	Z_2	321	285	$A_1^3 A_2 A_{14}$	Z_3	00015	301	$A_1 A_3 A_{15}$	Z_4	024	326	$A_5^3 D_4$	Z_6	211s
1054	$A_1 D_4 D_7 E_7$	Z_2	1s21	287	$A_1 A_2^2 A_{14}$	Z_3	0015	341	$A_2 A_3^2 A_7 D_4$	Z_4	0112s	367	$A_2^2 A_{11} D_4$	Z_6	112s
1057	$A_1^2 A_2 D_8 E_7$	Z_2	010s1	290	$A_2 A_3 A_{14}$	Z_3	105	350	$A_1 A_7^2 D_4$	Z_4	022s	395	$A_2^2 A_5^2 D_5$	Z_6	11112
1059	$A_1 A_3 D_8 E_7$	Z_2	10s1	308	$A_1^2 A_{17}$	Z_3	006	369	$A_1 A_3 A_{11} D_4$	Z_4	013s	845	$A_1^3 A_5^2 E_6$	Z_6	011112
1059'	$A_1 A_3 D_8 E_7$	Z_2	12v1	310	$A_2 A_{17}$	Z_3	06	375	$A_{15} D_4$	Z_4	4s	850	$A_3 A_5^2 E_6$	Z_6	2112
1063	$A_2^2 D_{10} E_7$	Z_2	00s1	351	$A_1 A_2^3 A_8 D_4$	Z_3	011130	381	$A_1 A_2^2 A_3^3 D_5$	Z_4	1001111	887	$A_1^2 A_{11} E_6$	Z_6	1122
1065	$A_2 D_{10} E_7$	Z_2	0s1	353	$A_2 A_5 A_8 D_4$	Z_3	1230	382	$A_2 A_4^4 D_5$	Z_4	011112	70	$A_1 A_6^3$	Z_7	0123
1068	$A_1^3 A_3 E_6 E_7$	Z_2	111201	366	$A_2^2 A_{11} D_4$	Z_3	1140	384	$A_1 A_3^3 A_4 D_5$	Z_4	111101	116	$A_1^2 A_3 A_7^2$	Z_8	01115
1079	$A_1^3 A_2 E_7^2$	Z_2	011011	394	$A_2^2 A_5^2 D_5$	Z_3	11220	411	$A_1^2 A_2 A_3 A_7 D_5$	Z_4	010121	421	$A_7^2 D_5$	Z_8	131
1081	$A_1^2 A_3 E_7^2$	Z_2	00211	422	$A_2^2 A_8 D_5$	Z_3	11130	414	$A_1 A_5^2 A_7 D_5$	Z_4	01122				
1083	$A_2 A_3 E_7^2$	Z_2	0211	484	$A_2^4 A_5 D_6$	Z_3	111120	414'	$A_1 A_3^2 A_7 D_5$	Z_4	10121				

22	$A_1^3 A_3^2 A_5^2$	Z_2 0012203 Z_2 0102230	493	$A_1^3 A_5^2 D_6$	Z_2 00033v Z_2 01103s	635	$A_1^3 A_2 A_3^2 D_8$	Z_2 011002s Z_2 101020c	741	$A_1 A_3 A_5 D_{10}$	Z_2 003s Z_2 120c
86	$A_1^4 A_3 A_5 A_7$	Z_2 0001034 Z_2 0110204	493'	$A_1^3 A_5^2 D_6$	Z_2 01103s Z_2 10130s	637	$A_1^2 A_3^3 D_8$	Z_2 00022s Z_2 11200s	752	$A_1^2 A_3 D_4 D_{10}$	Z_2 010ss Z_2 1020c
90	$A_1 A_3^2 A_5 A_7$	Z_2 02204 Z_2 10034	496	$A_3 A_5^2 D_6$	Z_2 033v Z_2 203s	643	$A_1^4 A_2 A_5 D_8$	Z_2 000103s Z_2 111003v	759	$A_1^3 D_6 D_{10}$	Z_2 000ss Z_2 1110c
96	$A_1^2 A_5^2 A_7$	Z_2 01034 Z_2 10304	507	$A_1^3 A_3 A_7 D_6$	Z_2 00104s Z_2 110240	644	$A_1^3 A_3 A_5 D_8$	Z_2 00103s Z_2 01023v	971	$A_1^4 A_3 A_5 E_7$	Z_2 0011031 Z_2 1101201
114	$A_1^2 A_3 A_7^2$	Z_2 00044 Z_2 11204	533	$A_1 A_2 A_3^2 D_4 D_6$	Z_2 0022sv Z_2 1002cs	644'	$A_1^3 A_3 A_5 D_8$	Z_2 00103s Z_2 11020s	1012	$A_1^2 A_3^2 D_4 E_7$	Z_2 0102s1 Z_2 1020c1
318	$A_1^2 A_2 A_3^2 A_5 D_4$	Z_2 010023s Z_2 100203c	534	$A_1 A_3 A_5 D_4 D_6$	Z_2 003ss Z_2 120cs	650	$A_1 A_5^2 D_8$	Z_2 033v Z_2 103s	1016	$A_1^3 A_5 D_4 E_7$	Z_2 0003s1 Z_2 1110c1
319	$A_1 A_3^3 A_5 D_4$	Z_2 02220s Z_2 10023s	534'	$A_1 A_3 A_5 D_4 D_6$	Z_2 003ss Z_2 120sc	656	$A_1^4 A_7 D_8$	Z_2 00004s Z_2 11110s	1038	$A_1^3 A_3 D_6 E_7$	Z_2 0012v1 Z_2 1100s1
322	$A_1^3 A_2 A_5^2 D_4$	Z_2 000033s Z_2 111003c	534''	$A_1 A_3 A_5 D_4 D_6$	Z_2 0230s Z_2 103sv	667	$A_1^2 A_2 A_3 D_4 D_8$	Z_2 0002ss Z_2 1100cc	1056	$A_1^4 D_8 E_7$	Z_2 0001s1 Z_2 1110v1
324	$A_1^2 A_3 A_5^2 D_4$	Z_2 00033s Z_2 01203c	540	$A_1^2 A_3^2 D_5 D_6$	Z_2 01022s Z_2 10202c	669	$A_1 A_3^2 D_4 D_8$	Z_2 002ss Z_2 020cc	313	$A_3^5 D_4$	Z_2 20022s Z_4 01111s
324'	$A_1^3 A_3 A_5^2 D_4$	Z_2 01203s Z_2 10230s	544	$A_1^3 A_5 D_5 D_6$	Z_2 00032s Z_2 11102c	670	$A_1^2 A_5 D_4 D_8$	Z_2 0130s Z_2 103sv	339	$A_1^2 A_3^2 A_7 D_4$	Z_2 11004s Z_4 111160
338	$A_1^2 A_3^2 A_7 D_4$	Z_2 00024s Z_2 112040	544'	$A_1^3 A_5 D_5 D_6$	Z_2 00132v Z_2 11030s	672	$A_1^3 A_3 D_5 D_8$	Z_2 01102s Z_2 10120c	30	$A_1^4 A_5^3$	Z_2 0011033 Z_6 0101114
345	$A_1^3 A_5 A_7 D_4$	Z_2 001340 Z_2 11004s	550	$A_1 A_3 D_4 D_5 D_6$	Z_2 02s2v Z_2 10c2s	676	$A_1^2 D_4 D_5 D_8$	Z_2 00s2s Z_2 11c0c	327	$A_3^3 D_4$	Z_2 033s Z_6 112c
377	$A_2 A_3^3 D_4^2$	Z_2 0022ss Z_2 0202cc	553	$A_1^2 A_2 A_3 D_6^2$	Z_2 0102sv Z_2 1002vs	678	$A_1^3 A_2 D_6 D_8$	Z_2 0010ss Z_2 1100vc	18	$A_1 A_2^4 A_5^2$	Z_3 0001122 Z_3 0110024
378	$A_1 A_5^2 D_4^2$	Z_2 0330s Z_2 103sc	555	$A_1 A_3^2 D_6^2$	Z_2 002ss Z_2 020cc	680	$A_1^2 A_3 D_6 D_8$	Z_2 002vs Z_2 010sc	33	$A_2^2 A_5^3$	Z_3 00222 Z_3 11024
388	$A_1^3 A_3^2 A_5 D_5$	Z_2 0010232 Z_2 1102202	555'	$A_1 A_3^2 D_6^2$	Z_2 002ss Z_2 120sv	680'	$A_1^2 A_3 D_6 D_8$	Z_2 010ss Z_2 102sv	838	$A_2^4 A_5 E_6$	Z_3 001121 Z_3 110022
393	$A_1^4 A_5^2 D_5$	Z_2 0011330 Z_2 1101032	557	$A_1^2 A_5 D_6^2$	Z_2 003sv Z_2 110ss	684	$A_1 D_4 D_6 D_8$	Z_2 0svs Z_2 10sc	72	$A_1^3 A_3^3 A_7$	Z_4 0002112 Z_2 0112004
456	$A_1^2 A_3 A_5 D_4 D_5$	Z_2 0103s2 Z_2 1023c0	561	$A_1 A_2 D_4 D_6^2$	Z_2 00ss Z_2 10ccv	689	$A_1^3 D_8^2$	Z_2 000ss Z_2 011cv	75	$A_3^4 A_7$	Z_4 02112 Z_2 22004
485	$A_1^3 A_2 A_3 A_5 D_6$	Z_2 001023v Z_2 110003s	562	$A_3 D_4 D_6^2$	Z_2 0sss Z_2 20cc	691	$A_3 D_8^2$	Z_2 0ss Z_2 2cv	109	$A_1^5 A_7^2$	Z_4 0001122 Z_2 0111104
487	$A_1^2 A_3^2 A_5 D_6$	Z_2 00023s Z_2 01203v	563	$A_1^3 D_5 D_6^2$	Z_2 012sv Z_2 102vs	730	$A_1^3 A_3^2 D_{10}$	Z_2 00102s Z_2 01020c	115	$A_1^2 A_3 A_7^2$	Z_4 00222 Z_2 11204
487'	$A_1^2 A_3^2 A_5 D_6$	Z_2 01220s Z_2 102230	565	$A_1 D_6^3$	Z_2 0ssv Z_2 0cvs	738	$A_1^4 A_5 D_{10}$	Z_2 00003s Z_2 01110c	1	$A_1 A_3^6$	Z_4 0011112 Z_4 0101231

376	$A_1^2 A_3^3 D_4^2$	Z_2 00022ss Z_2 00202cc Z_2 11002vv	538	$A_1^2 A_3 D_4^2 D_6$	Z_2 002ssv Z_2 010scs Z_2 100csc	560	$A_1^3 D_4 D_6^2$	Z_2 000sss Z_2 001ccv Z_2 1100cc
532	$A_1^3 A_3^2 D_4 D_6$	Z_2 00102ss Z_2 01020cs Z_2 100220c	552	$A_1^4 A_3 D_6^2$	Z_2 00002ss Z_2 00110cc Z_2 11010sv	380	$A_1^3 D_4^4$	Z_2 000ssss Z_2 000cccc Z_2 0110scv Z_2 1010cvs

Table 9. Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic string on T^3 .

4	$A_2^2 A_7$	Z_2	004	82	$A_1 A_5 C_1 C_4$	Z_2	1300	161	$A_2 C_1 C_2 D_6$	Z_2	001s	312	$A_1^2 A_3 C_3 F_3$	Z_2	01201
5	$A_2^2 A_3^2 C_1$	Z_2	00220	82'	$A_1 A_5 C_1 C_4$	Z_2	1301	163	$A_1 C_1 C_3 D_6$	Z_2	100s	316	$A_5 C_3 F_3$	Z_2	301
7	$A_1 A_2^2 A_5 C_1$	Z_2	10030	84	$A_1 A_2^2 C_2 C_4$	Z_2	10011	164	$C_2 C_3 D_6$	Z_2	10s	318	$A_1^2 A_2 C_4 F_3$	Z_2	01011
9	$A_1 A_4 A_5 C_1$	Z_2	1030	86	$A_2 A_3 C_2 C_4$	Z_2	0201	166	$C_1 C_4 D_6$	Z_2	01v	320	$A_1 A_3 C_4 F_3$	Z_2	1011
10	$A_5^2 C_1$	Z_2	330	87	$A_1 A_4 C_2 C_4$	Z_2	1011	167	$A_1 C_1 C_2 D_7$	Z_2	1012	320'	$A_1 A_3 C_4 F_3$	Z_2	1201
12	$A_1 A_2 A_7 C_1$	Z_2	0040	88	$A_5 C_2 C_4$	Z_2	310	168	$C_2^2 D_7$	Z_2	112	324	$A_1^2 C_6 F_3$	Z_2	0011
13	$A_1 A_9 C_1$	Z_2	150	89	$A_1^2 A_2 C_3 C_4$	Z_2	11001	170	$A_1 C_1^2 D_8$	Z_2	000s	326	$A_2 C_6 F_3$	Z_2	011
14	$A_1^2 A_2^2 A_3 C_1^2$	Z_2	1100200	91	$A_1 A_3 C_3 C_4$	Z_2	0201	172	$C_1 C_2 D_8$	Z_2	00s	329	$A_1 A_2 C_1 D_4 F_3$	Z_2	100s1
18	$A_1^2 A_2 A_5 C_1^2$	Z_2	010300	93	$A_1 A_2 C_4^2$	Z_2	0011	182	$A_1 A_2 C_1 E_7$	Z_2	1001	330	$A_2 C_2 D_4 F_3$	Z_2	01s1
19	$A_1 A_3 A_5 C_1^2$	Z_2	12300	95	$A_1 A_5 C_5$	Z_2	130	183	$A_1^2 C_1^2 E_7$	Z_2	01001	331	$A_1 C_3 D_4 F_3$	Z_2	10s1
22	$A_1^2 A_7 C_1^2$	Z_2	11400	100	$A_1 A_3 C_2 C_5$	Z_2	1210	185	$A_2 C_2 E_7$	Z_2	011	332	$A_1 A_2 D_5 F_3$	Z_2	1021
30	$A_4 A_5 C_2$	Z_2	031	103	$A_1^2 C_4 C_5$	Z_2	1110	186	$A_1 C_1 C_2 E_7$	Z_2	0011	333	$A_1^2 C_1 D_5 F_3$	Z_2	01021
31	$A_2 A_7 C_2$	Z_2	040	105	$A_1 A_2^2 C_6$	Z_2	1001	187	$C_2^2 E_7$	Z_2	011	335	$A_1 C_2 D_5 F_3$	Z_2	0121
32	$A_9 C_2$	Z_2	51	107	$A_1 A_4 C_6$	Z_2	101	188	$A_1 C_3 E_7$	Z_2	101	337	$A_2 D_6 F_3$	Z_2	0s1
33	$A_1 A_2^2 A_3 C_1 C_2$	Z_2	100201	108	$A_5 C_6$	Z_2	31	196	$A_1 A_3 A_5 F_2$	Z_2	1230	338	$A_1 C_1 D_6 F_3$	Z_2	00s1
36	$A_1 A_3 A_4 C_1 C_2$	Z_2	12001	109	$A_1^2 A_2 C_1 C_6$	Z_2	01001	199	$A_1^2 A_7 F_2$	Z_2	1140	339	$C_2 D_6 F_3$	Z_2	0s1
39	$A_1 A_2 A_5 C_1 C_2$	Z_2	00301	111	$A_1 A_3 C_1 C_6$	Z_2	1001	202	$A_1^2 A_3^2 C_1 F_2$	Z_2	112200	339'	$C_2 D_6 F_3$	Z_2	1v1
40	$A_3 A_5 C_1 C_2$	Z_2	0301	111'	$A_1 A_3 C_1 C_6$	Z_2	1201	206	$A_1^3 A_5 C_1 F_2$	Z_2	111300	340	$A_1 D_7 F_3$	Z_2	121
40'	$A_3 A_5 C_1 C_2$	Z_2	2301	114	$A_1 A_2 C_2 C_6$	Z_2	0011	213	$A_1 A_3^2 C_2 F_2$	Z_2	12210	345	$A_1 E_7 F_3$	Z_2	011
42	$A_1 A_7 C_1 C_2$	Z_2	0400	114'	$A_1 A_2 C_2 C_6$	Z_2	1001	216	$A_1^2 A_5 C_2 F_2$	Z_2	11310	347	$C_1 E_7 F_3$	Z_2	011
42'	$A_1 A_7 C_1 C_2$	Z_2	1401	115	$A_3 C_2 C_6$	Z_2	011	226	$A_1^2 A_3 C_4 F_2$	Z_2	11210	349	$A_1^3 A_3 F_2 F_3$	Z_2	111201
46	$A_2^2 A_3 C_2^2$	Z_2	00211	116	$A_1^2 C_3 C_6$	Z_2	0101	233	$A_1^3 C_6 F_2$	Z_2	11110	360	$A_1^3 A_2 F_3^2$	Z_2	011011
48	$A_3 A_4 C_2^2$	Z_2	2011	118	$A_1 C_4 C_6$	Z_2	101	280	$A_1 A_2^2 A_3 F_3$	Z_2	10021	362	$A_1^2 A_3 F_3^2$	Z_2	00211
50	$A_2 A_5 C_2^2$	Z_2	0301	124	$A_1 A_2 C_8$	Z_2	001	283	$A_1 A_3 A_4 F_3$	Z_2	1201	364	$A_2 A_3 F_3^2$	Z_2	0211
51	$A_7 C_2^2$	Z_2	400	125	$A_1^2 C_1 C_8$	Z_2	0001	286	$A_1 A_2 A_5 F_3$	Z_2	0031	367	$A_1 D_4 F_3^2$	Z_2	0s11
52	$A_2 A_3^2 C_3$	Z_2	0220	125'	$A_1^2 C_1 C_8$	Z_2	1101	287	$A_3 A_5 F_3$	Z_2	031	369	$D_5 F_3^2$	Z_2	211
54	$A_1 A_2 A_5 C_3$	Z_2	1030	126	$A_2 C_1 C_8$	Z_2	001	287'	$A_3 A_5 F_3$	Z_2	231	192	$A_1 A_2^4 F_2$	Z_3	011111
56	$A_1 A_7 C_3$	Z_2	040	127	$A_1 C_2 C_8$	Z_2	001	289	$A_1 A_7 F_3$	Z_2	141	195	$A_2^2 A_5 F_2$	Z_3	1121
60	$A_1^2 A_5 C_1 C_3$	Z_2	01300	128	$C_3 C_8$	Z_2	01	291	$A_1^3 A_2^2 C_1 F_3$	Z_2	1110001	264	$A_1^3 A_2^2 F_2^2$	Z_3	0001111
63	$A_7 C_1 C_3$	Z_2	400	131	$A_1 C_{10}$	Z_2	11	292	$A_1^2 A_2 A_3 C_1 F_3$	Z_2	010201	265	$A_2^2 A_3 F_2^2$	Z_3	11011
64	$A_1 A_2 A_3 C_2 C_3$	Z_2	10210	141	$A_1^2 A_2 C_1^2 D_5$	Z_2	110002	294	$A_1 A_3^2 C_1 F_3$	Z_2	12201	269	$A_1^2 A_5 F_2^2$	Z_3	00211
65	$A_3^2 C_2 C_3$	Z_2	2200	144	$A_1 A_2 C_1 C_2 D_5$	Z_2	10012	297	$A_1^2 A_5 C_1 F_3$	Z_2	00301	270	$A_2 A_5 F_2^2$	Z_3	0211
67	$A_1 A_5 C_2 C_3$	Z_2	0310	146	$A_2 C_2^2 D_5$	Z_2	0112	297'	$A_1^2 A_5 C_1 F_3$	Z_2	11301	278	$A_1 E_6 F_2^2$	Z_3	0111
67'	$A_1 A_5 C_2 C_3$	Z_2	1300	147	$A_1 C_2 C_3 D_5$	Z_2	1102	299	$A_2 A_5 C_1 F_3$	Z_2	0301	15	$A_1 A_2 A_3^2 C_1^2$	Z_4	001111
70	$A_1^2 A_3 C_3^2$	Z_2	11200	149	$A_1 C_1 C_4 D_5$	Z_2	0012	302	$A_1^2 A_2^2 C_2 F_3$	Z_2	110011	23	$A_1^2 A_7 C_1^2$	Z_4	00211
75	$A_2^2 A_3 C_4$	Z_2	0021	150	$C_2 C_4 D_5$	Z_2	012	305	$A_1 A_2 A_3 C_2 F_3$	Z_2	00211	24	$A_2 A_7 C_1^2$	Z_4	0211
78	$A_1^2 A_2^2 C_1 C_4$	Z_2	110001	152	$C_1^2 D_4 D_5$	Z_2	00s2	306	$A_3^2 C_2 F_3$	Z_2	0211				
80	$A_1 A_2 A_3 C_1 C_4$	Z_2	00201	153	$C_1 D_5^2$	Z_2	022	307	$A_1^2 A_4 C_2 F_3$	Z_2	11011				
81	$A_1^2 A_4 C_1 C_4$	Z_2	11001	156	$A_1 A_2 C_1^2 D_6$	Z_2	1000s	309	$A_1 A_5 C_2 F_3$	Z_2	0301				

2	$A_2 A_3^3$	$\mathbb{Z}_2$ 0022 $\mathbb{Z}_2$ 0202	47'	$A_1 A_3^2 C_2^2$	$\mathbb{Z}_2$ 00211 $\mathbb{Z}_2$ 12001	106	$A_1^2 A_3 C_6$	$\mathbb{Z}_2$ 0101 $\mathbb{Z}_2$ 1021	158	$A_1 A_2 C_2 D_6$	$\mathbb{Z}_2$ 001s $\mathbb{Z}_2$ 100c
3	$A_1 A_5^2$	$\mathbb{Z}_2$ 033 $\mathbb{Z}_2$ 103	49	$A_1^2 A_5 C_2^2$	$\mathbb{Z}_2$ 00301 $\mathbb{Z}_2$ 11011	113	$A_1^3 C_2 C_6$	$\mathbb{Z}_2$ 00011 $\mathbb{Z}_2$ 11110	159	$A_3 C_2 D_6$	$\mathbb{Z}_2$ 01s $\mathbb{Z}_2$ 20v
8	$A_1^2 A_3 A_5 C_1$	$\mathbb{Z}_2$ 01030 $\mathbb{Z}_2$ 10230	74	$A_1^2 A_2 A_3 C_4$	$\mathbb{Z}_2$ 00021 $\mathbb{Z}_2$ 11001	134	$A_1 A_3 C_1 C_2 D_4$	$\mathbb{Z}_2$ 0200s $\mathbb{Z}_2$ 1001c	160	$A_1^2 C_1 C_2 D_6$	$\mathbb{Z}_2$ 0100s $\mathbb{Z}_2$ 1001v
28	$A_1 A_2 A_3^2 C_2$	$\mathbb{Z}_2$ 00220 $\mathbb{Z}_2$ 10021	76	$A_1 A_3^2 C_4$	$\mathbb{Z}_2$ 0021 $\mathbb{Z}_2$ 0201	136	$A_1 A_2 C_2^2 D_4$	$\mathbb{Z}_2$ 0011s $\mathbb{Z}_2$ 1001c	162	$A_1 C_2^2 D_6$	$\mathbb{Z}_2$ 001s $\mathbb{Z}_2$ 010c
29	$A_1 A_3 A_5 C_2$	$\mathbb{Z}_2$ 0031 $\mathbb{Z}_2$ 1201	77	$A_1^2 A_5 C_4$	$\mathbb{Z}_2$ 0130 $\mathbb{Z}_2$ 1031	137	$A_3 C_2^2 D_4$	$\mathbb{Z}_2$ 011s $\mathbb{Z}_2$ 200s	165	$A_1 C_4 D_6$	$\mathbb{Z}_2$ 01v $\mathbb{Z}_2$ 10s
29'	$A_1 A_3 A_5 C_2$	$\mathbb{Z}_2$ 0231 $\mathbb{Z}_2$ 1030	79	$A_1^3 A_3 C_1 C_4$	$\mathbb{Z}_2$ 011001 $\mathbb{Z}_2$ 101200	138	$A_1^2 C_1 C_4 D_4$	$\mathbb{Z}_2$ 0001s $\mathbb{Z}_2$ 1100c	169	$A_1^2 C_1 D_8$	$\mathbb{Z}_2$ 000s $\mathbb{Z}_2$ 110c
34	$A_1^2 A_3^2 C_1 C_2$	$\mathbb{Z}_2$ 010201 $\mathbb{Z}_2$ 102001	83	$A_1^3 A_2 C_2 C_4$	$\mathbb{Z}_2$ 001011 $\mathbb{Z}_2$ 110001	139	$A_1 C_2 C_4 D_4$	$\mathbb{Z}_2$ 001s $\mathbb{Z}_2$ 110s	171	$A_1 C_2 D_8$	$\mathbb{Z}_2$ 00s $\mathbb{Z}_2$ 11c
38	$A_1^3 A_5 C_1 C_2$	$\mathbb{Z}_2$ 000301 $\mathbb{Z}_2$ 111001	85	$A_1^2 A_3 C_2 C_4$	$\mathbb{Z}_2$ 00201 $\mathbb{Z}_2$ 01011	143	$A_1 A_3 C_2 D_5$	$\mathbb{Z}_2$ 0202 $\mathbb{Z}_2$ 1012	281	$A_1^2 A_3^2 F_3$	$\mathbb{Z}_2$ 01021 $\mathbb{Z}_2$ 10201
38'	$A_1^3 A_5 C_1 C_2$	$\mathbb{Z}_2$ 001300 $\mathbb{Z}_2$ 110301	85'	$A_1^2 A_3 C_2 C_4$	$\mathbb{Z}_2$ 01011 $\mathbb{Z}_2$ 10210	145	$A_1^2 C_2^2 D_5$	$\mathbb{Z}_2$ 01012 $\mathbb{Z}_2$ 10102	285	$A_1^3 A_5 F_3$	$\mathbb{Z}_2$ 00031 $\mathbb{Z}_2$ 11101
45	$A_1^2 A_2 A_3 C_2^2$	$\mathbb{Z}_2$ 010201 $\mathbb{Z}_2$ 100210	92	$A_1^3 C_4^2$	$\mathbb{Z}_2$ 00011 $\mathbb{Z}_2$ 01101	148	$A_1^2 C_4 D_5$	$\mathbb{Z}_2$ 0012 $\mathbb{Z}_2$ 1102	304	$A_1^3 A_3 C_2 F_3$	$\mathbb{Z}_2$ 001201 $\mathbb{Z}_2$ 110011
47	$A_1 A_3^2 C_2^2$	$\mathbb{Z}_2$ 00211 $\mathbb{Z}_2$ 02011	94	$A_3 C_4^2$	$\mathbb{Z}_2$ 011 $\mathbb{Z}_2$ 201	155	$A_1 A_3 C_1 D_6$	$\mathbb{Z}_2$ 020v $\mathbb{Z}_2$ 100s	317	$A_1^4 C_4 F_3$	$\mathbb{Z}_2$ 000111 $\mathbb{Z}_2$ 111001

1	$A_1^2 A_3^3$	$\mathbb{Z}_2$ 00022 $\mathbb{Z}_2$ 00202 $\mathbb{Z}_2$ 11002	44	$A_1^4 A_3 C_2^2$	$\mathbb{Z}_2$ 0000211 $\mathbb{Z}_2$ 0011011 $\mathbb{Z}_2$ 1101001	135	$A_1^3 C_2^2 D_4$	$\mathbb{Z}_2$ 00011s $\mathbb{Z}_2$ 00101c $\mathbb{Z}_2$ 11000s	157	$A_1^3 C_2 D_6$	$\mathbb{Z}_2$ 0001s $\mathbb{Z}_2$ 0010c $\mathbb{Z}_2$ 1100v
27	$A_1^3 A_3^2 C_2$	$\mathbb{Z}_2$ 001021 $\mathbb{Z}_2$ 010201 $\mathbb{Z}_2$ 100221	133	$A_1^2 A_3 C_2 D_4$	$\mathbb{Z}_2$ 0020s $\mathbb{Z}_2$ 0101s $\mathbb{Z}_2$ 1001c	154	$A_1^2 A_3 D_6$	$\mathbb{Z}_2$ 002v $\mathbb{Z}_2$ 010s $\mathbb{Z}_2$ 100c	140	$A_1^3 D_4^2$	$\mathbb{Z}_2$ 000ss $\mathbb{Z}_2$ 000cc $\mathbb{Z}_2$ 0110s $\mathbb{Z}_2$ 1010c

Table 10. Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_2$ triple.

17	$A_1^3 A_3 G_1$	$\mathbb{Z}_2$ 11121	32	$A_2 A_3 G_1^2$	$\mathbb{Z}_2$ 0211	1	$A_1^3 A_2^2 \mathbb{Z}_3$	00011	7	$A_2 A_5 \mathbb{Z}_3$	02
28	$A_1^3 A_2 G_1^2$	$\mathbb{Z}_2$ 011011	35	$A_1 D_4 G_1^2$	$\mathbb{Z}_2$ 0s11	2	$A_2^2 A_3 \mathbb{Z}_3$	110	15	$A_1 E_6 \mathbb{Z}_3$	01
30	$A_1^2 A_3 G_1^2$	$\mathbb{Z}_2$ 00211	37	$D_5 G_1^2$	$\mathbb{Z}_2$ 211	6	$A_1^2 A_5 \mathbb{Z}_3$	002			

Table 11. Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_3$ triple.

2	$A_1^3 A_2 \mathbb{Z}_2$	0110	5	$A_1 A_2^2 \mathbb{Z}_2$	100	6'	$A_1^2 A_3 \mathbb{Z}_2$	010	11	$A_1 A_4 \mathbb{Z}_2$	10	13'	$A_1 D_4 \mathbb{Z}_2$	10	1	A_1^5	$\mathbb{Z}_2$ 00001 $\mathbb{Z}_2$ 00010	7	$A_1^2 A_3$	$\mathbb{Z}_2$ 010 $\mathbb{Z}_2$ 100
2'	$A_1^3 A_2 \mathbb{Z}_2$	0010	6	$A_1^2 A_3 \mathbb{Z}_2$	002	9	$A_2 A_3 \mathbb{Z}_2$	02	13	$A_1 D_4 \mathbb{Z}_2$	0s	15	D_5	$\mathbb{Z}_2$ 2	3	$A_1^3 A_2$	$\mathbb{Z}_2$ 0010 $\mathbb{Z}_2$ 0100			

Table 12. Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_4$ triple.

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