

Symbol alphabets from plabic graphs

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ABSTRACT: Symbol alphabets of n -particle amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory are known to contain certain cluster variables of $\text{Gr}(4, n)$ as well as certain algebraic functions of cluster variables. In this paper we suggest an algorithm for computing these symbol alphabets from plabic graphs by solving matrix equations of the form $C \cdot Z = 0$ to associate functions on $\text{Gr}(m, n)$ to parameterizations of certain cells of $\text{Gr}(k, n)$ indexed by plabic graphs. For $m = 4$ and $n = 8$ we show that this association precisely reproduces the 18 algebraic symbol letters of the two-loop NMHV eight-particle amplitude from four plabic graphs.

KEYWORDS: Scattering Amplitudes, Supersymmetric Gauge Theory, Differential and Algebraic Geometry

ARXIV EPRINT: [2007.00646](https://arxiv.org/abs/2007.00646)

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1 Introduction

A central problem in studying the scattering amplitudes of planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory is to understand their analytic structure. Certain amplitudes are known or expected to be expressible in terms of generalized polylogarithm functions. The branch points of any such amplitude are encoded in its *symbol alphabet*—a finite collection of multiplicatively independent functions on kinematic space called *symbol letters* [1]. In [2] it was observed that for $n = 6, 7$, the symbol alphabet of all (then-known) n -particle amplitudes is the set of cluster variables [3, 4] of the $\text{Gr}(4, n)$ Grassmannian cluster algebra [5]. The hypothesis that this remains true to arbitrary loop order provides the bedrock underlying a bootstrap program that has enabled the computation of these amplitudes to impressively high loop order and remains supported by all available evidence (see [6] for a recent review).

For $n > 7$ the $\text{Gr}(4, n)$ cluster algebra has infinitely many cluster variables [4, 5]. While it has long been known that the symbol alphabets of some $n > 7$ amplitudes (such as the two-loop MHV amplitudes [7]) are given by finite subsets of cluster variables, there was no candidate guess for a “theory” to explain why amplitudes would select the subsets that they do. At the same time, it was expected [8, 9] that the symbol alphabets of even MHV amplitudes for $n > 7$ would generically require letters that are not cluster variables — specifically, that are algebraic functions of the Plücker coordinates on $\text{Gr}(4, n)$, of the type that appear in the one-loop four-mass box function [10, 11] (see appendix B). (Throughout this paper we use the adjective “algebraic” to specifically denote something that is algebraic but not rational.)

As often the case for amplitudes, guesses and expectations are valuable but explicit computations are king. Recently the two-loop eight-particle NMHV amplitude in SYM theory was computed [12], and it was found to have a 198-letter symbol alphabet that can be taken to consist of 180 cluster variables on $\text{Gr}(4, 8)$ and an additional 18 algebraic letters that involve square roots of four-mass box type. (Evidence for the former was presented in [9] based on an analysis of the Landau equations; the latter are consistent with the Landau analysis but less constrained by it.) The result of [12] provided the first concrete new data on symbol alphabets in SYM theory in over eight years. We will refer to this as “the eight-particle alphabet” in this paper since (turning again to hopeful speculation) it may turn out to be the complete symbol alphabet for all eight-particle amplitudes in SYM theory at all loop order.

A few recent papers have sought to explain or postdict the eight-particle symbol alphabet and to clarify its connection to the $\text{Gr}(4, 8)$ cluster algebra. In [13] polytopal realizations of certain compactifications of (the positive part of) the configuration space $\text{Conf}_8(\mathbb{P}^3)$ of eight particles in SYM theory were constructed. These naturally select certain finite subsets of cluster variables, including those in the eight-particle alphabet, and the square roots of four-mass box type make a natural appearance as well. At the same time, an equivalent but dual description, involving certain fans associated to the tropical totally positive Grassmannian [14], appeared simultaneously in [15, 16]. Moreover [15] proposed a construction that precisely computes the 18 algebraic letters of the eight-particle symbol alphabet by (roughly speaking) analyzing how the simplest candidate fan is embedded within the (infinite) $\text{Gr}(4, 8)$ cluster fan.

In this paper we show that the algebraic letters of the eight-particle symbol alphabet are precisely reproduced by an alternate construction that only requires solving a set of simple polynomial equations associated to certain plabic (planar bi-colored) graphs. Plabic graphs [17] are known to encode important properties of SYM amplitudes [18], in particular the singularities of integrands, and our work raises the possibility that symbol alphabets of SYM theory could also be encoded in certain plabic graphs. In section 2 we introduce our construction with a simple example and then complete the analysis for all graphs relevant to $\text{Gr}(4, 6)$ in section 3. In section 4 we consider an example where the construction yields non-cluster variables of $\text{Gr}(3, 6)$ and in section 5 we apply it to graphs that precisely reproduce the algebraic functions on $\text{Gr}(4, 8)$ that appear in the symbol of [12].

2 A motivational example

Motivated by [19], in this paper we consider solutions to sets of equations of the form

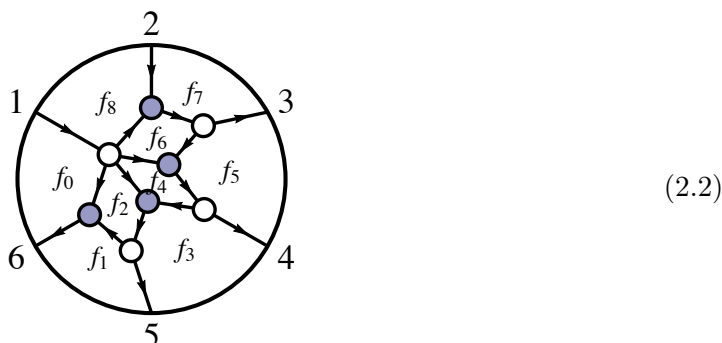
$$C \cdot Z = 0 \tag{2.1}$$

which are familiar from the study of several closely connected or essentially equivalent amplitude-related objects (leading singularities, Yangian invariants, on-shell forms; see for example [18, 20–23]).

For the application to SYM theory that will be the focus of this paper, Z is the $n \times 4$ matrix of momentum twistors describing the kinematics of an n -particle scattering event, but it is often instructive to allow Z to be $n \times m$ for general m .

The $k \times n$ matrix $C(f_0, \dots, f_d)$ in (2.1) parameterizes a d -dimensional cell of the totally non-negative Grassmannian $\text{Gr}(k, n)_{\geq 0}$. Specifically, we always take it to be the boundary measurement of a (reduced, perfectly oriented) plabic graph expressed in terms of the face weights f_α of the graph (see [17, 18]). One could equally well use edge weights, but using face weights allows us to further restrict our attention to bipartite graphs and to eliminate some redundancy; the only residual redundancy of face weights is that they satisfy $\prod_\alpha f_\alpha = 1$ for each graph.

For an illustrative example, consider



which affords us the opportunity to review the construction of the associated C -matrix from [17]. The graph is *perfectly oriented* because each black (white) vertex has all incident arrows but one pointing in (out). The graph has two sources $\{1, 2\}$ and four sinks $\{3, 4, 5, 6\}$ and we begin by forming a $2 \times (2 + 4)$ matrix with the 2×2 identity matrix occupying the source columns:

$$C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} & c_{15} & c_{16} \\ 0 & 1 & c_{23} & c_{24} & c_{25} & c_{26} \end{pmatrix}. \tag{2.3}$$

The remaining entries are given by

$$c_{ij} = (-1)^s \sum_{p:i \rightarrow j} \prod_{\alpha \in \widehat{p}} f_\alpha \tag{2.4}$$

where s is the number of sources strictly between i and j , the sum runs over all allowed paths p from i to j (allowed paths must traverse each edge only in the direction of its arrow) and the product runs over all faces α to the right of p , denoted by \widehat{p} . In this manner we find

$$\begin{aligned} c_{13} &= -f_0 f_1 f_2 f_3 f_4 f_5 f_6, & c_{23} &= f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8, \\ c_{14} &= -f_0 f_1 f_2 f_3 f_4 (1 + f_6), & c_{24} &= f_0 f_1 f_2 f_3 f_4 f_6 f_8, \\ c_{15} &= -f_0 f_1 f_2 (1 + f_4 + f_4 f_6), & c_{25} &= f_0 f_1 f_2 f_4 f_6 f_8, \\ c_{16} &= -f_0 (1 + f_2 + f_2 f_4 + f_2 f_4 f_6), & c_{26} &= f_0 f_2 f_4 f_6 f_8. \end{aligned} \tag{2.5}$$

Then for $m = 4$, (2.1) is a system of $2 \times 4 = 8$ equations for the eight independent face weights, which has the solution

$$\begin{aligned}
 f_0 &= -\frac{\langle 1234 \rangle}{\langle 2346 \rangle}, & f_1 &= -\frac{\langle 2346 \rangle}{\langle 2345 \rangle}, & f_2 &= \frac{\langle 2345 \rangle \langle 1236 \rangle}{\langle 1234 \rangle \langle 2356 \rangle}, \\
 f_3 &= -\frac{\langle 2356 \rangle}{\langle 2346 \rangle}, & f_4 &= \frac{\langle 2346 \rangle \langle 1256 \rangle}{\langle 2456 \rangle \langle 1236 \rangle}, & f_5 &= -\frac{\langle 2456 \rangle}{\langle 2356 \rangle}, \\
 f_6 &= \frac{\langle 2356 \rangle \langle 1456 \rangle}{\langle 3456 \rangle \langle 1256 \rangle}, & f_7 &= -\frac{\langle 3456 \rangle}{\langle 2456 \rangle}, & f_8 &= -\frac{\langle 2456 \rangle}{\langle 1456 \rangle},
 \end{aligned}
 \tag{2.6}$$

where $\langle ijkl \rangle = \det(Z_i Z_j Z_k Z_l)$ are Plücker coordinates on $\text{Gr}(4, 6)$.

We pause here to point out two features evident from (2.6). First, we see that on the solution of (2.1) each face weight evaluates (up to sign) to a product of powers of $\text{Gr}(4, 6)$ cluster variables, i.e. to a symbol letter of six-particle amplitudes in SYM theory [1]. Moreover, the cluster variables that appear ($\langle 2346 \rangle$, $\langle 2356 \rangle$, $\langle 2456 \rangle$, and the six frozen variables) constitute a single cluster of the $\text{Gr}(4, 6)$ algebra.

The fact that cluster variables of $\text{Gr}(m, n)$ seem to arise, at least in this example, raises the possibility that the symbol alphabets of amplitudes in SYM theory might be given more generally by the face weights of certain plabic graphs evaluated on solutions of $C \cdot Z = 0$. A necessary condition for this to have a chance of working is that the number of independent face weights should equal the number of equations (both eight in the above example); otherwise the equations would have no solutions or continuous families of solutions. For this reason we focus exclusively on graphs for which (2.1) admits isolated solutions for the face weights as functions of generic $n \times m$ Z -matrices; in particular this requires that $d = km$. In such cases the number of isolated solutions to (2.1) is called the *intersection number* of the graph.

The possible connection between plabic graphs and symbol alphabets is especially tantalizing because it manifestly has a chance to account for both issues raised in the introduction: (1) while the number of cluster variables of $\text{Gr}(4, n)$ is infinite for $n > 7$, the number of (reduced) plabic graphs is certainly finite for any fixed n , and (2) graphs with intersection number greater than 1 naturally provide candidate algebraic symbol letters. Our showcase example of (2) is presented in section 5.

3 Six-particle cluster variables

The problem formulated in the previous section can be considered for any k, m and n . In this section we thoroughly investigate the first case of direct relevance to the amplitudes of SYM theory: $m = 4$ and $n = 6$. Although this case is special for several reasons, it allows us to illustrate some concepts and terminology that will be used in later sections.

Modulo dihedral transformations on the six external points, there are a total of four different types of plabic graph to consider. We begin with the three graphs shown in figure 1 (a)–(c), which have $k = 2$. These all correspond to the top cell of $\text{Gr}(2, 6)_{\geq 0}$ and are related to each other by square moves. Specifically, performing a square move on f_2 of graph (a) yields graph (b), while performing a square move on f_4 of graph (a) yields graph

(c). This contrasts with more general cases, for example those considered in the next two sections, where we are in general interested in lower-dimensional cells.

The solution for the face weights of graph (a) (the same as (2.2)) were already given in (2.6), and those of graphs (b) and (c) are derived in (A.2) and (A.4) of appendix A. The latter two can alternatively be derived from the former via the square move rule (see [17, 18]). In particular, for graph (b) we have

$$\begin{aligned} f_0^{(b)} &= f_0^{(a)}(1 + f_2^{(a)}), & f_3^{(b)} &= f_3^{(a)}(1 + f_2^{(a)}), \\ f_1^{(b)} &= \frac{f_1^{(a)}}{1 + 1/f_2^{(a)}}, & f_2^{(b)} &= \frac{1}{f_2^{(a)}}, & f_4^{(b)} &= \frac{f_4^{(a)}}{1 + 1/f_2^{(a)}}, \end{aligned} \tag{3.1}$$

with f_5, f_6, f_7 and f_8 unchanged, while for graph (c) we have

$$\begin{aligned} f_2^{(c)} &= f_2^{(a)}(1 + f_4^{(a)}), & f_5^{(c)} &= f_5^{(a)}(1 + f_4^{(a)}), \\ f_3^{(c)} &= \frac{f_3^{(a)}}{1 + 1/f_4^{(a)}}, & f_4^{(c)} &= \frac{1}{f_4^{(a)}}, & f_6^{(c)} &= \frac{f_6^{(a)}}{1 + 1/f_4^{(a)}}, \end{aligned} \tag{3.2}$$

with f_0, f_1, f_7 and f_8 unchanged.

To every plabic graph one can naturally associate a quiver with nodes labeled by Plücker coordinates of $\text{Gr}(k, n)$. In figure 1 (d)–(f) we display these quivers for the graphs under consideration, following the source-labeling convention of [24, 25] (see also [26]). Because in this case each graph corresponds to the top cell of $\text{Gr}(2, 6)_{\geq 0}$, each labeled quiver is a seed of the $\text{Gr}(2, 6)$ cluster algebra. More generally we will have graphs corresponding to lower-dimensional cells, whose labeled quivers are seeds of subalgebras of $\text{Gr}(k, n)$.

Henceforth we refer to a labeled quiver associated to a plabic graph in this manner as an *input cluster*, taking the point of view that solving the equations $C \cdot Z = 0$ associates a collection of functions on $\text{Gr}(m, n)$ to every such input. As we now explain, for these three graphs there is also a natural way to graphically organize the structure of each of (2.6), (A.2) and (A.4) in terms of an *output cluster*, although in the next section we will see that this does not work for more general graphs.

First of all we note from (A.2) and (A.4) that, like what happened for graph (a) considered in the previous section, each face weight evaluates (up to sign) to a product of powers of $\text{Gr}(4, 6)$ cluster variables. Second, again we see that for each graph the collection of variables that appear precisely constitutes a single cluster of $\text{Gr}(4, 6)$: suppressing in each case the six frozen variables, we find $\langle 2346 \rangle, \langle 2356 \rangle$ and $\langle 2456 \rangle$ for graph (a); $\langle 1235 \rangle, \langle 2356 \rangle$ and $\langle 2456 \rangle$ for graph (b); and $\langle 1456 \rangle, \langle 2346 \rangle, \langle 2456 \rangle$ for graph (c). Finally, in each case there is a unique way to label the nodes of the quiver not with cluster variables of the “input” cluster algebra $\text{Gr}(2, 6)$, as we have done in figure 1 (d)–(f), but with cluster variables of the “output” cluster algebra $\text{Gr}(4, 6)$. We show these output clusters in figure 1 (g)–(i), using the convention that the face weight (also known as the cluster \mathcal{X} -variable) attached to node i is $\prod_j a_j^{b_{ji}}$ where b_{ji} is the (signed) number of arrows from j to i .

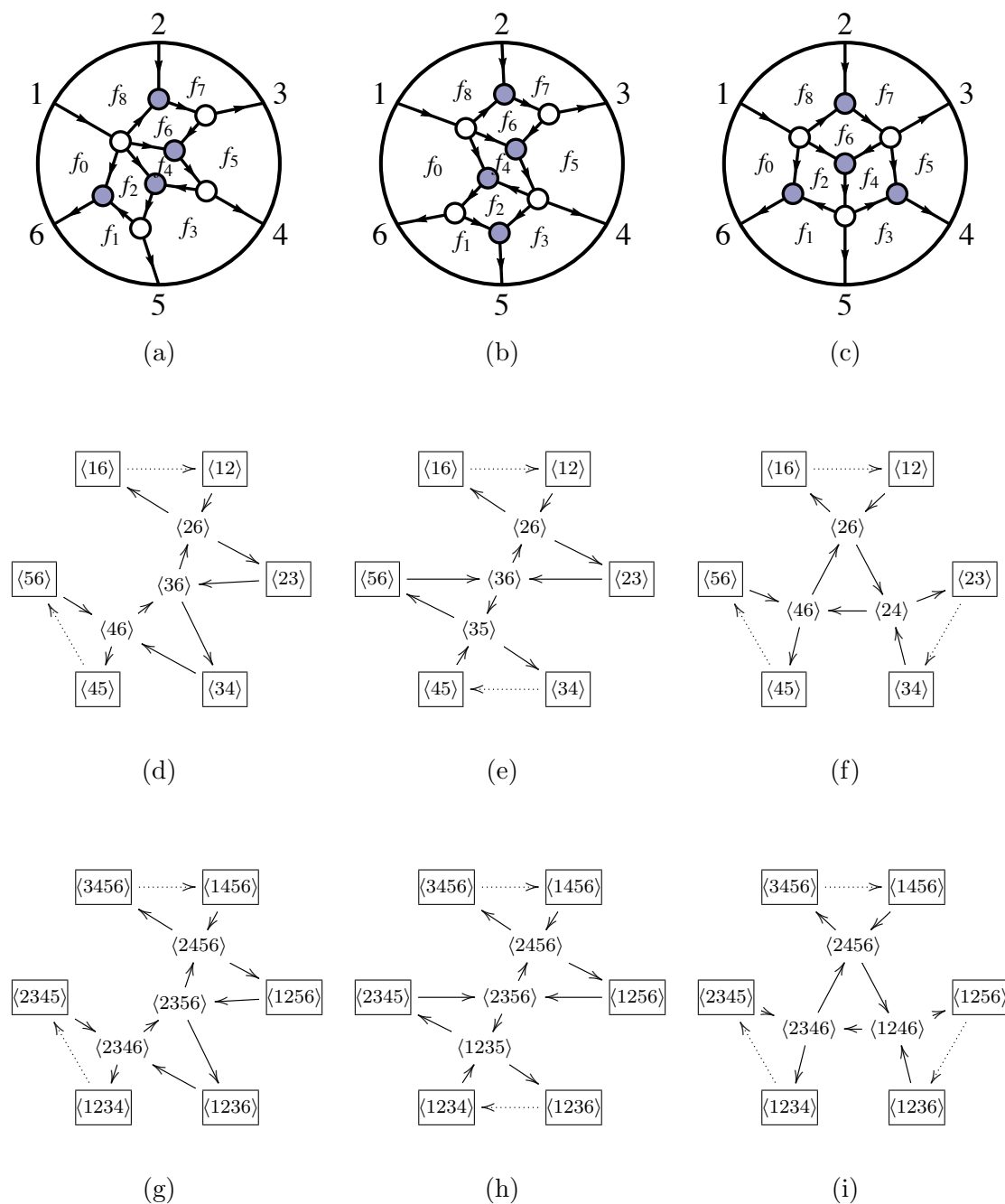
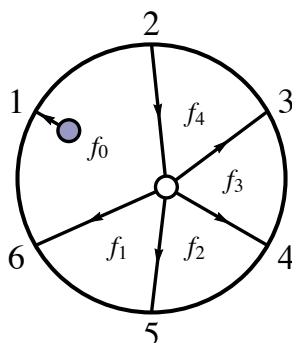


Figure 1. The three types of (reduced, perfectly orientable, bipartite) plabic graphs corresponding to km -dimensional cells of $\text{Gr}(k, n)_{\geq 0}$ for $k = 2$, $m = 4$ and $n = 6$ are shown in (a)–(c). The associated input and output clusters (see text) are shown in (d)–(f) and (g)–(i) respectively. Lines connecting two frozen nodes are usually omitted, but we include in (g)–(i) the dotted lines (having “black on the right” in the dual plabic graph) that encode (2.6), (A.2) and (A.4) (up to signs).

For the sake of completeness, we note that there is also (modulo \mathbb{Z}_6 cyclic transformations) a single relevant graph with $k = 1$:



for which the boundary measurement is

$$C = \begin{pmatrix} 0 & 1 & f_0 f_1 f_2 f_3 & f_0 f_1 f_2 & f_0 f_1 & f_0 \end{pmatrix} \tag{3.3}$$

and the solution to $C \cdot Z = 0$ is given by

$$f_0 = \frac{\langle 2345 \rangle}{\langle 3456 \rangle}, \quad f_1 = -\frac{\langle 2346 \rangle}{\langle 2345 \rangle}, \quad f_2 = -\frac{\langle 2356 \rangle}{\langle 2346 \rangle}, \quad f_3 = -\frac{\langle 2456 \rangle}{\langle 2356 \rangle}, \quad f_4 = -\frac{\langle 3456 \rangle}{\langle 2456 \rangle}. \tag{3.4}$$

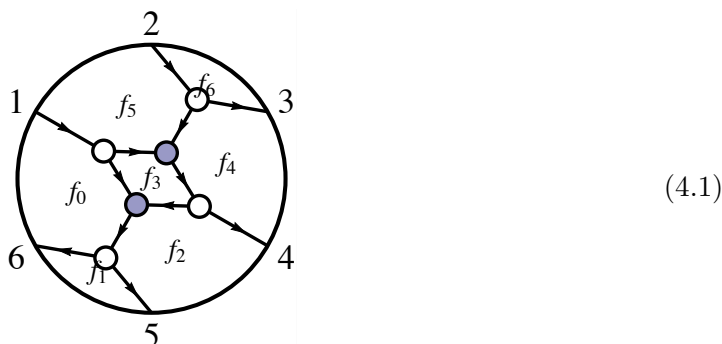
Again the face weights evaluate (up to signs) to simple ratios of $\text{Gr}(4, 6)$ cluster variables, though in this case both the input and output quivers are trivial. This graph is an example of the general feature that one can always uplift an n -point plabic graph relevant to our analysis to any value of $n' > n$ by inserting any number of black lollipops. (Graphs with white lollipops never admit solutions to $C \cdot Z = 0$ for generic Z .) In the language of symbology, this is in accord with the expectation that the symbol alphabet of an n' -particle amplitude always contains the $\mathbb{Z}_{n'}$ cyclic closure of the symbol alphabet of the corresponding n -particle amplitude.

In this section we have seen that solving $C \cdot Z = 0$ induces a map from clusters of $\text{Gr}(2, 6)$ (or subalgebras thereof) to clusters of $\text{Gr}(4, 6)$ (or subalgebras thereof). Of course, these two algebras are in any case naturally isomorphic. Although we leave a more detailed exposition for future work, we have also checked for $m = 2$ and $n \leq 10$ that every appropriate plabic graph of $\text{Gr}(k, n)$ maps to a cluster of $\text{Gr}(2, n)$ (or a subalgebra thereof), and moreover that this map is onto (every cluster of $\text{Gr}(2, n)$ is obtainable from some plabic graph of $\text{Gr}(k, n)$). However for $m > 2$ the situation is more complicated, as we see in the next section.

4 Towards non-cluster variables

Here we discuss some features of graphs for which the solution to $C \cdot Z = 0$ involves quantities that are not cluster variables of $\text{Gr}(m, n)$. A simple example for $k = 2, m = 3$,

$n = 6$ is the graph



whose boundary measurement has the form (2.3) with

$$\begin{aligned} c_{13} &= -0, & c_{15} &= -f_0 f_1 (1 + f_3), & c_{23} &= f_0 f_1 f_2 f_3 f_4 f_5, & c_{25} &= f_0 f_1 f_3 f_5, \\ c_{14} &= -f_0 f_1 f_2 f_3, & c_{16} &= -f_0 (1 + f_3), & c_{24} &= f_0 f_1 f_2 f_3 f_5, & c_{26} &= f_0 f_3 f_5. \end{aligned} \tag{4.2}$$

The solution to $C \cdot Z = 0$ is given by

$$\begin{aligned} f_0 &= \frac{\langle 123 \rangle \langle 145 \rangle}{\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle}, & f_1 &= -\frac{\langle 146 \rangle}{\langle 145 \rangle}, \\ f_2 &= \frac{\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle}{\langle 234 \rangle \langle 146 \rangle}, & f_3 &= -\frac{\langle 234 \rangle \langle 156 \rangle}{\langle 123 \rangle \langle 456 \rangle}, \\ f_4 &= -\frac{\langle 124 \rangle \langle 456 \rangle}{\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle}, & f_5 &= \frac{\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle}{\langle 134 \rangle \langle 156 \rangle}, \\ f_6 &= -\frac{\langle 134 \rangle}{\langle 124 \rangle}, \end{aligned} \tag{4.3}$$

which involves four mutable cluster variables $\langle 124 \rangle$, $\langle 134 \rangle$, $\langle 145 \rangle$, and $\langle 146 \rangle$ which comprise a cluster of the $\text{Gr}(3, 6)$ algebra, together with the quantity

$$\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle = \langle 123 \rangle \langle 456 \rangle - \langle 234 \rangle \langle 156 \rangle, \tag{4.4}$$

which is not a cluster variable of $\text{Gr}(3, 6)$.

We can gain some insight into the origin of (4.4) by considering what happens under a square move on f_3 . This transforms the face weights to

$$\begin{aligned} f_0 &= \frac{\langle 145 \rangle}{\langle 456 \rangle}, & f_1 &= -\frac{\langle 146 \rangle}{\langle 145 \rangle}, & f_2 &= -\frac{\langle 156 \rangle}{\langle 146 \rangle}, & f_3 &= -\frac{\langle 123 \rangle \langle 456 \rangle}{\langle 234 \rangle \langle 156 \rangle}, \\ f_4 &= -\frac{\langle 124 \rangle}{\langle 123 \rangle}, & f_5 &= -\frac{\langle 234 \rangle}{\langle 134 \rangle}, & f_6 &= -\frac{\langle 134 \rangle}{\langle 124 \rangle}, \end{aligned} \tag{4.5}$$

leaving four mutable cluster variables $\langle 124 \rangle$, $\langle 134 \rangle$, $\langle 145 \rangle$, and $\langle 146 \rangle$ which comprise a cluster of the $\text{Gr}(3, 6)$ algebra. However, it is not possible to associate a labeled “output” quiver to (4.5) in the usual way, as we did for the examples in the previous section.

To turn this story around: had we started not with (4.1) but with its square-moved partner, we would have encountered (4.5) and then noted that performing a square move back to (4.1) would necessarily introduce the multiplicative factor

$$1 + f_3 = -\frac{\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle}{\langle 234 \rangle \langle 156 \rangle} \tag{4.6}$$

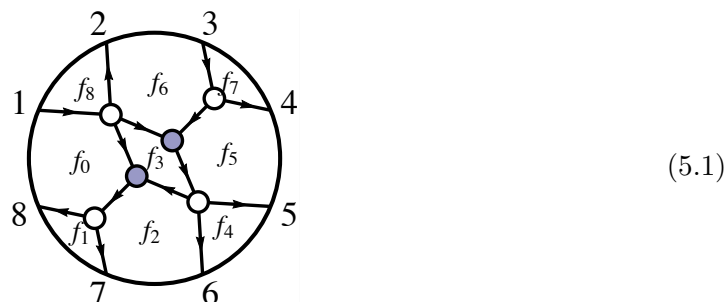
into four of the face weights.

The example considered in this section provides an opportunity to comment on the connection of our work to the study of cluster adjacency for Yangian invariants. In [27, 28] it was noted in several examples, and conjectured to be true in general, that the set of factors appearing in the denominator of any Yangian invariant with intersection number 1 are cluster variables of $\text{Gr}(4, n)$ that appear together in a cluster. This was proven to be true for all Yangian invariants in the $m = 2$ toy model of SYM theory in [29] and for all $m = 4$ N^2 MHV Yangian invariants in [30]. We recall from [18, 31] that the Yangian invariant associated to a plabic graph (or, to use essentially equivalent language, the form associated to an on-shell diagram) is given by $d \log f_1 \wedge \cdots \wedge d \log f_d$. One of our motivations for studying the conjecture that the face weights associated to any plabic graph always evaluate on the solution of $C \cdot Z = 0$ to products of powers of cluster variables was that it would immediately imply cluster adjacency for Yangian invariants. Although the graph (4.1) violates this stronger conjecture, it does not violate cluster adjacency because on-shell forms are invariant under square moves [18]. Therefore, even though $\langle 1 \times 4, 2 \times 3, 5 \times 6 \rangle$ appears in individual face weights of (4.3), it must drop out of the associated on-shell form because it is absent from (4.5).

5 Algebraic eight-particle symbol letters

One reason it is obvious that the solutions of $C \cdot Z = 0$ cannot always be written in terms of cluster variables of $\text{Gr}(m, n)$ is that for graphs with intersection number greater than 1 the solutions will necessarily involve algebraic functions of Plücker coordinates, whereas cluster variables are always rational.

The simplest example of this phenomenon occurs for $k = 2$, $m = 4$ and $n = 8$, for which there are four relevant plabic graphs in two cyclic classes. Let us start with



which has boundary measurement

$$C = \begin{pmatrix} 1 & c_{12} & 0 & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} \\ 0 & c_{32} & 1 & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} \end{pmatrix} \tag{5.2}$$

with

$$c_{12} = f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_7, \quad c_{14} = -0, \quad c_{15} = -f_0 f_1 f_2 f_3 f_4, \tag{5.3}$$

$$c_{16} = -f_0 f_1 f_2 f_3, \quad c_{17} = -f_0 f_1 (1 + f_3), \quad c_{18} = -f_0 (1 + f_3), \tag{5.4}$$

$$c_{32} = 0, \quad c_{34} = f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8, \quad c_{35} = f_0 f_1 f_2 f_3 f_4 f_6 f_8, \tag{5.5}$$

$$c_{36} = f_0 f_1 f_2 f_3 f_6 f_8, \quad c_{37} = f_0 f_1 f_3 f_6 f_8, \quad c_{38} = f_0 f_3 f_6 f_8. \tag{5.6}$$

The solution to $C \cdot Z = 0$ for generic $Z \in \text{Gr}(4, 8)$ can be written as

$$\begin{aligned}
 f_0 &= \sqrt{\frac{\langle 7(12)(34)(56) \rangle \langle 1234 \rangle}{a_5 \langle 2(34)(56)(78) \rangle \langle 3478 \rangle}}, & f_5 &= \sqrt{\frac{a_1 a_6 a_9 \langle 3(12)(56)(78) \rangle \langle 5678 \rangle}{a_4 a_7 \langle 6(12)(34)(78) \rangle \langle 3478 \rangle}}, \\
 f_1 &= -\sqrt{\frac{a_7 \langle 8(12)(34)(56) \rangle}{\langle 7(12)(34)(56) \rangle}}, & f_6 &= -\sqrt{\frac{a_3 \langle 1(34)(56)(78) \rangle \langle 3478 \rangle}{a_2 \langle 4(12)(56)(78) \rangle \langle 1278 \rangle}}, \\
 f_2 &= -\sqrt{\frac{a_4 \langle 5(12)(34)(78) \rangle \langle 3478 \rangle}{a_8 \langle 8(12)(34)(56) \rangle \langle 3456 \rangle}}, & f_7 &= -\sqrt{\frac{a_2 \langle 4(12)(56)(78) \rangle}{a_1 \langle 3(12)(56)(78) \rangle}}, \\
 f_3 &= \sqrt{\frac{a_8 \langle 1278 \rangle \langle 3456 \rangle}{a_9 \langle 1234 \rangle \langle 5678 \rangle}}, & f_8 &= -\sqrt{\frac{a_5 \langle 2(34)(56)(78) \rangle}{a_3 \langle 1(34)(56)(78) \rangle}}, \\
 f_4 &= -\sqrt{\frac{\langle 6(12)(34)(78) \rangle}{a_6 \langle 5(12)(34)(78) \rangle}},
 \end{aligned} \tag{5.7}$$

where

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle \tag{5.8}$$

and the nine a_i provide a (multiplicative) basis for the algebraic letters of the eight-particle symbol alphabet that contain the four-mass box square root $\sqrt{\Delta_{1357}}$, as reviewed in appendix B.

The nine face weights shown in (5.7) satisfy $\prod f_\alpha = 1$ so only eight are multiplicatively independent. It is easy to check that they remain multiplicatively independent if one sets all of the Plücker coordinates and brackets of the form (5.8) to one. This means that the projection of this eight-dimensional space onto the nine-dimensional space spanned by the nine algebraic letters is eight dimensional (and not smaller). We could try building an eight-particle alphabet by taking any subset of eight of the face weights as basis elements (i.e., letters), but we would always be one letter short of spanning the full algebraic space.

Fortunately there is a second plabic graph relevant to $\sqrt{\Delta_{1357}}$: the one obtained by performing a square move on f_3 of (5.1). As is by now familiar, performing the square move introduces one new multiplicative factor into the face weights:

$$1 + f_3 = \sqrt{\frac{\langle 1256 \rangle \langle 3478 \rangle}{a_9 \langle 1234 \rangle \langle 5678 \rangle}}, \tag{5.9}$$

which precisely supplies the ninth, missing letter! To summarize: the union of the nine face weights associated to the graph (5.1), and the nine associated to its square-move partner, multiplicatively span the nine-dimensional space of $\sqrt{\Delta_{1357}}$ -containing symbol letters in the eight-particle alphabet of [12].

The same story applies to the graphs obtained by cycling the external indices on (5.1) by one — their face weights provide all nine algebraic letters involving $\sqrt{\Delta_{2468}}$.

Of course it would be very interesting to thoroughly study the numerous plabic graphs relevant to $m = 4$, $n = 8$ that have intersection number 1. In particular it would be interesting to see if they encode all 180 of the rational (i.e., $\text{Gr}(4, 8)$ cluster variable)

symbol letters of [12], and whether they generate additional cluster variables such as those obtained from the constructions of [13, 15, 16].

Before concluding this section let us comment briefly on “ k ” since one may be confused why the plabic graph (5.1), which has $k = 2$ and is therefore associated to an N^2 MHV leading singularity, could be relevant for symbol alphabets of NMHV amplitudes. The symbol letters of an N^k MHV amplitude reveal all of its singularities, including multiple discontinuities that can be accessed only after a suitable analytic continuation. Physically these are computed by cuts involving lower-loop amplitudes that can have $k' > k$. Indeed the expectation that symbol letters of lower-loop higher- k amplitudes influence those of higher-loop lower- k amplitudes is manifest in the \overline{Q} -bar equation technology [7, 32, 33] underlying the computation of [12]. Moreover there is indirect evidence [34] that the symbol alphabet of the L -loop n -particle N^k MHV amplitude in SYM theory is independent of both k and L (beyond certain accidental shortenings that may occur for small k or L). This suggests that for the purpose of applying our construction to “the n -particle symbol alphabet” one should consider all relevant n -point plabic graphs regardless of k .

6 Discussion

The problem of “explaining” the symbol alphabets of n -particle amplitudes in SYM theory apparently requires, for $n > 7$, a mechanism for identifying finite sets of functions on $\text{Gr}(4, n)$ that include some subset of the cluster variables of the associated cluster algebra, together with certain non-cluster variables that are algebraic functions of the Plücker coordinates.

In this paper we have initiated the study of one candidate mechanism that manifestly satisfies both criteria and may be of independent mathematical interest. Specifically, to every (reduced, perfectly oriented) plabic graph of $\text{Gr}(k, n)_{\geq 0}$ that parameterizes a cell of dimension mk , one can naturally associate a collection of mk functions of Plücker coordinates on $\text{Gr}(m, n)$.

We have seen that for some graphs the output of this procedure is naturally associated to a seed of the $\text{Gr}(m, n)$ cluster algebra; for some graphs the output is a cluster’s worth of cluster variables that do not correspond to a seed but rather behave “badly” under mutations (this means they transform into things which are not cluster variables under square moves on the input plabic graph); and finally for some graphs the output involves non-cluster variables including, when the intersection number is greater than 1, algebraic functions.

We leave a more thorough investigation of this problem for future work. The “smoking gun” that this procedure may be relevant to symbol alphabets in SYM theory is provided by the example discussed in section 5, which successfully postdicts precisely the 18 multiplicatively independent algebraic letters that were recently found to appear in the two-loop eight-particle NMHV amplitude [12]. Our construction provides an alternative to the similar postdiction made recently in [15].

It is interesting to note that since for $m = 4$, $n = 8$ there are no other relevant plabic graphs having intersection number > 1 , beyond those already considered section 5, our

construction has no room for any additional algebraic letters for eight-particle amplitudes. Therefore, if it is true that the face weights of plabic graphs, evaluated on the locus $C \cdot Z = 0$, provide symbol alphabets for general amplitudes, then it necessarily follows that no eight-particle amplitude, at any loop order, can have any algebraic symbol letters beyond the 18 discovered in [12].

At first glance this rigidity seems to stand in contrast to the constructions of [13, 15, 16] which each involve some amount of choice — having to do with how coarse or fine one chooses one’s tropical fan, or equivalently how many factors to include in the Minkowski sum when building the dual polytope. But in fact our construction has a choice with a similar smell. When we say that we start with the C -matrix associated to a plabic graph, that automatically restricts us to very special clusters of $\text{Gr}(k, n)$ — those that contain only Plücker coordinates. Clusters containing more complicated, non-Plücker cluster variables are not associated to plabic graphs. One certainly could contemplate solving the $C \cdot Z = 0$ equations for C given by a “non-plabic” cluster parameterization of some cell of $\text{Gr}(k, n)_{\geq 0}$ and it would be interesting to map out the landscape of possibilities.

It has been a long-standing problem to understand the precise connection between the $\text{Gr}(k, n)$ cluster structure exhibited [18] at the level of integrands in SYM theory and the $\text{Gr}(4, n)$ cluster structure exhibited [2] by integrated amplitudes. It was pointed out in [19] that the $C \cdot Z = 0$ equations provide a concrete link between the two, and our results shed some initial light on this intriguing but still very mysterious problem. Since plabic graphs categorize the singularities of integrands of SYM theory, whereas symbol letters characterize the singularities of integrated amplitudes, perhaps one can think of the “input” and “output” clusters described in section 3 as “integrand” and “integrated” clusters. (See the last paragraph of section 5 for some comments on why k “disappears” upon integration.) Although we have seen that the latter are not in general clusters at all, the example of section 5 suggests that they may be even better: exactly what is needed for the symbol alphabets of SYM theory.

Note added. The preprint [35] appeared on arXiv shortly after and has significant overlap with the result presented in this note.

Acknowledgments

We are grateful to N. Arkani-Hamed for numerous encouraging discussions and to J. Bourjaily for helpful correspondence. This work was supported in part by the US Department of Energy under contract DE-SC0010010 Task A and by Simons Investigator Award #376208 (AV). Plabic graphs were drawn with the help of [36].

A Some six-particle details

Here we assemble some details of the calculation for graphs (b) and (c) of figure 1. The boundary measurement for graph (b) has the form (2.3) with

$$\begin{aligned}
 c_{13} &= -f_0 f_1 f_2 f_3 f_4 f_5 f_6, & c_{23} &= f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8, \\
 c_{14} &= -f_0 f_1 f_2 f_3 f_4 (1 + f_6), & c_{24} &= f_0 f_1 f_2 f_3 f_4 f_6 f_8, \\
 c_{15} &= -f_0 f_1 (1 + f_4 + f_2 f_4 + f_4 f_6 + f_2 f_4 f_6), & c_{25} &= f_0 f_1 f_4 f_6 f_8 (1 + f_2), \\
 c_{16} &= -f_0 (1 + f_4 + f_4 f_6), & c_{26} &= f_0 f_4 f_6 f_8,
 \end{aligned}
 \tag{A.1}$$

and the solution to $C \cdot Z = 0$ is given by

$$\begin{aligned}
 f_0^{(b)} &= -\frac{\langle 1235 \rangle}{\langle 2356 \rangle}, & f_1^{(b)} &= -\frac{\langle 1236 \rangle}{\langle 1235 \rangle}, & f_2^{(b)} &= \frac{\langle 1234 \rangle \langle 2356 \rangle}{\langle 2345 \rangle \langle 1236 \rangle}, \\
 f_3^{(b)} &= -\frac{\langle 1235 \rangle}{\langle 1234 \rangle}, & f_4^{(b)} &= \frac{\langle 2345 \rangle \langle 1256 \rangle}{\langle 1235 \rangle \langle 2456 \rangle}, & f_5^{(b)} &= -\frac{\langle 2456 \rangle}{\langle 2356 \rangle}, \\
 f_6^{(b)} &= \frac{\langle 2356 \rangle \langle 1456 \rangle}{\langle 3456 \rangle \langle 1256 \rangle}, & f_7^{(b)} &= -\frac{\langle 3456 \rangle}{\langle 2456 \rangle}, & f_8^{(b)} &= -\frac{\langle 2456 \rangle}{\langle 1456 \rangle}.
 \end{aligned}
 \tag{A.2}$$

The boundary measurement for graph (c) has

$$\begin{aligned}
 c_{13} &= -f_0 f_1 f_2 f_3 f_4 f_5 f_6, & c_{23} &= f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8, \\
 c_{14} &= -f_0 f_1 f_2 f_3 (1 + f_6 + f_4 f_6), & c_{24} &= f_0 f_1 f_2 f_3 f_6 f_8 (1 + f_4), \\
 c_{15} &= -f_0 f_1 f_2 (1 + f_6), & c_{25} &= f_0 f_1 f_2 f_6 f_8, \\
 c_{16} &= -f_0 (1 + f_2 + f_2 f_6), & c_{26} &= f_0 f_2 f_6 f_8,
 \end{aligned}
 \tag{A.3}$$

and the solution to $C \cdot Z = 0$ is

$$\begin{aligned}
 f_0^{(c)} &= -\frac{\langle 1234 \rangle}{\langle 2346 \rangle}, & f_1^{(c)} &= -\frac{\langle 2346 \rangle}{\langle 2345 \rangle}, & f_2^{(c)} &= \frac{\langle 2345 \rangle \langle 1246 \rangle}{\langle 1234 \rangle \langle 2456 \rangle}, \\
 f_3^{(c)} &= -\frac{\langle 1256 \rangle}{\langle 1246 \rangle}, & f_4^{(c)} &= \frac{\langle 2456 \rangle \langle 1236 \rangle}{\langle 2346 \rangle \langle 1256 \rangle}, & f_5^{(c)} &= -\frac{\langle 1246 \rangle}{\langle 1236 \rangle}, \\
 f_6^{(c)} &= \frac{\langle 1456 \rangle \langle 2346 \rangle}{\langle 3456 \rangle \langle 1246 \rangle}, & f_7^{(c)} &= -\frac{\langle 3456 \rangle}{\langle 2456 \rangle}, & f_8^{(c)} &= -\frac{\langle 2456 \rangle}{\langle 1456 \rangle}.
 \end{aligned}
 \tag{A.4}$$

B Notation for algebraic eight-particle symbol letters

Here we review some details from [12] to set the notation used in section 5. There are two basic square roots of four-mass box type that appear in symbol letters of eight-particle amplitudes. These are $\sqrt{\Delta_{1357}}$ and $\sqrt{\Delta_{2468}}$ with

$$\Delta_{1357} = (\langle 1256 \rangle \langle 3478 \rangle - \langle 1278 \rangle \langle 3456 \rangle - \langle 1234 \rangle \langle 5678 \rangle)^2 - 4 \langle 1234 \rangle \langle 3456 \rangle \langle 5678 \rangle \langle 1278 \rangle
 \tag{B.1}$$

and Δ_{2468} given by cycling every index by 1 (mod 8).

The eight-particle symbol alphabet can be written in terms of 180 $\text{Gr}(4, 8)$ cluster variables, plus 9 letters that are rational functions of Plücker coordinates and $\sqrt{\Delta_{1357}}$ and another 9 that are rational functions of Plücker coordinates and $\sqrt{\Delta_{2468}}$. We focus on the first 9 as the latter is a cyclic copy of the same story.

There are many different ways to write a basis for the eight-particle symbol alphabet as the various letters one can form satisfy numerous multiplicative identities among each other. For the sake of definiteness we use the basis provided in the ancillary Mathematica file attached to [12]. The choice of basis made there starts by defining

$$\begin{aligned} z &= \frac{1}{2} \left(1 + u - v + \sqrt{(1 - u - v)^2 - 4uv} \right), \\ \bar{z} &= \frac{1}{2} \left(1 + u - v - \sqrt{(1 - u - v)^2 - 4uv} \right) \end{aligned} \tag{B.2}$$

in terms of the familiar eight-particle cross ratios

$$u = \frac{\langle 1278 \rangle \langle 3456 \rangle}{\langle 1256 \rangle \langle 3478 \rangle}, \quad v = \frac{\langle 1234 \rangle \langle 5678 \rangle}{\langle 1256 \rangle \langle 3478 \rangle}. \tag{B.3}$$

Note that the square root appearing in (B.2) is

$$\sqrt{(1 - u - v)^2 - 4uv} = \frac{\sqrt{\Delta_{1357}}}{\langle 1256 \rangle \langle 3478 \rangle}. \tag{B.4}$$

Then a basis for the algebraic letters of the symbol alphabet is given by

$$\begin{aligned} a_1 &= \frac{x_a - z}{x_a - \bar{z}} \Big|_{i \rightarrow i+6}, & a_2 &= \frac{x_b - z}{x_b - \bar{z}} \Big|_{i \rightarrow i+6}, & a_3 &= -\frac{x_c - z}{x_c - \bar{z}} \Big|_{i \rightarrow i+6}, \\ a_4 &= -\frac{x_d - z}{x_d - \bar{z}} \Big|_{i \rightarrow i+4}, & a_5 &= -\frac{x_d - z}{x_d - \bar{z}} \Big|_{i \rightarrow i+6}, & a_6 &= \frac{x_e - z}{x_e - \bar{z}} \Big|_{i \rightarrow i+4}, \\ a_7 &= \frac{x_e - z}{x_e - \bar{z}} \Big|_{i \rightarrow i+6}, & a_8 &= \frac{z}{\bar{z}}, & a_9 &= \frac{1 - z}{1 - \bar{z}}, \end{aligned} \tag{B.5}$$

where the x 's are defined in (13) of [12]. While the overall sign of a symbol letter is irrelevant, we have taken the liberty of putting a minus sign in front of a_3 , a_4 and a_5 to ensure that each of the nine a_i , indeed each individual factor appearing in (5.7), is positive-valued for $Z \in \text{Gr}(4, 8)_{>0}$.

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