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Boundary states, overlaps, nesting and bootstrapping AdS/dCFT

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Abstract: Integrable boundary states can be built up from pair annihilation amplitudes called K-matrices. These amplitudes are related to mirror reflections and they both satisfy Yang Baxter equations, which can be twisted or untwisted. We relate these two notions to each other and show how they are fixed by the unbroken symmetries, which, together with the full symmetry, must form symmetric pairs. We show that the twisted nature of the K-matrix implies specific selection rules for the overlaps. If the Bethe roots of the same type are paired the overlap is called chiral, otherwise it is achiral and they correspond to untwisted and twisted K-matrices, respectively. We use these findings to develop a nesting procedure for K-matrices, which provides the factorizing overlaps for higher rank algebras automatically. We apply these methods for the calculation of the simplest asymptotic all-loop 1-point functions in AdS/dCFT. In doing so we classify the solutions of the YBE for the K-matrices with centrally extended $\mathfrak{su}(2|2)_c$ symmetry and calculate the generic overlaps in terms of Bethe roots and ratio of Gaudin determinants.

Keywords: Integrable Field Theories, AdS-CFT Correspondence, Boundary Quantum Field Theory, Bethe Ansatz

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1 Introduction

Recently there have been renewed interest and relevant progress in calculating overlaps between periodic multiparticle states and integrable boundary states. They appear in quite distinct parts of theoretical physics including statistical physics and the gauge/string duality.

In statistical physics there are significant activities in analyzing the behavior of both integrable and non-integrable systems after a quantum quench [1]. In a typical situation a parameter of the Hamiltonian is suddenly changed implying that the ground-state of the pre-quenched Hamiltonian is no longer an eigenstate of the post-quench Hamiltonian. The after quench time evolution can be fully described once the overlaps of this initial state with the eigenstates of the post-quench Hamiltonian are known. Integrable quenches are those when the initial state is an integrable boundary state [2–4].

In the AdS/CFT correspondence there are at least two places where overlaps appeared so far. Recently a new class of three-point functions were investigated in the AdS_5/CFT_4 correspondence involving a local gauge invariant single trace operator and two determinant operators dual to maximal giant gravitons [5, 6]. The authors showed that the three-point function can be calculated as an overlap between the finite volume multiparticle state and a finite volume integrable boundary state.

In the other, much more investigated, application a codimension one defect is introduced in the gauge theory, which breaks part of the gauge symmetry of the model. As a result some scalar fields develop vacuum expectation values and one-point functions of local gauge invariant operators can be non zero. Their space-time dependence is fixed by the unbroken conformal symmetry up to an operator dependent normalization constant.

As the operators are normalized by their two point functions far away from the defect this coefficient is a physical quantity. Since in the integrable description of the AdS/CFT correspondence local gauge-invariant operators are related to finite volume multiparticle states their one-point functions can be interpreted as finite volume overlaps with a boundary state created by the defect. Much progress has been achieved so far which included D3-D5 defects [7–9] and D3-D7 defects [10, 11]. Most of the analysis is restricted however for the leading order results in the coupling and only subsectors for the whole theory, although some partial one-loop results are also available in a diagonal subsector [12, 13]. In [14] the authors proposed an all loop formula for the one point functions in the $\mathfrak{su}(2)$ sector in the asymptotic domain, i.e. neglecting wrapping corrections. One of the aim of our paper is to go beyond these results and try to bootstrap the overlaps to be valid for any couplings and any sectors in the asymptotic domain.

Motivated by the above mentioned applications there were already significant activities and progress in calculating the matrix elements of integrable boundary states and eigenstates of the transfer matrix for integrable spin chains. The calculations of these on-shell overlaps have been already performed for various setups. In [15, 16] the authors investigated the Neél state in the XXZ spin chain and derived that the on-shell overlaps are non-vanishing only when the Bethe roots are paired and they can be written as the product of one particle overlap functions and a ratio of Gaudin-like determinants. This finding turned out to be true for several other models when the boundary states were integrable. The integrability condition in the XXZ spin chain was proposed in [2]. In [17] the author proposed a same type of formula for arbitrary states, built from solutions of the boundary Yang Baxter equation (two-site states), and proved its validity numerically. For matrix product states it was proposed that the on-shell overlap formulas cannot be written in the previous product form, rather a sum of those and this claim was verified numerically for the XXX spin chain [8]. The integrability condition can be generalized to nested systems and there are some known overlap formulas for integrable states in these cases. In [3, 4, 18] there are numerically verified formulas for two-site and matrix product states in the $\mathfrak{su}(3)$ spin chain. The authors found that the non-vanishing overlaps require pair structures for the two types of Bethe roots. In [9, 11] there are numerically verified overlap formulas for $\mathfrak{so}(6)$ spin chains and each type of Bethe roots have their pair structure for non-vanishing overlaps. In [5] the authors investigated a different boundary state of the $\mathfrak{so}(6)$ spin chain and found a novel pair structure which was different from what had been found so far. All of these results are in specific low rank models and are not exactly derived[1]. Another aim of our paper is to perform a systematic study of higher rank spin chains, derive the pair structure of their Bethe roots and develop a nesting procedure which could provide the factorizing overlaps. In doing so we found that the unbroken symmetries play a crucial role. This residual symmetries together with the original symmetry must form a symmetric pair and the nature of these maximal subalgebras are intimately related to the nature of the pair structures whose knowledge is essential in formulating the right nesting.

[1]In a recent paper [19] derived some overlap formulas for non-nested system but it is not obvious whether this can be generalized to the nested ones.

Keeping both the spin chain and AdS/dCFT applications in mind we try to be as general as required. In order to make the presentations lighter new notions and methods are demonstrated in various examples.

The rest of the paper is organized as follows: in the next section we introduce integrable boundary states and their relations to boundary reflections emphasizing that for non-crossing invariant theories they are not equivalent. We then turn to analyze overlaps between integrable boundary states and periodic states. Periodic states are the eigenstates of the transfer matrices, which can be constructed from scattering- or spin chain R-matrices. We first recall the eigenvalues and Bethe root structures for $\mathfrak{su}(N)$, $\mathfrak{so}(4)$, $\mathfrak{so}(6)$ and $\mathfrak{su}(2|2)_c$ chains, as they will be relevant in what follows. We then analyze the consequences of the integrability requirement for the root structure and conclude that for algebras with a nontrivial Dynkin diagram symmetry roots can be paired in a chiral and an achiral way. In the next section we investigate integrable two sites boundary states related to solutions of the KYBE, the YBE for boundary states. We observe that there are two types of solutions of the KYBE with quite distinct symmetries, which we relate to the chirality/achirality of the overlaps. Keeping also the AdS/dCFT applications in mind we derive the most general bosonic solution of the KYBE for the centrally extended $\mathfrak{su}(2|2)_c$ algebra, and identify its symmetries. In the next section we invent a version of the nesting, which enables us to calculate the overlaps in various spin chains including the $\mathfrak{su}(2|2)_c$ symmetric ones, relevant for AdS/dCFT. Symmetry argumentations combined with nesting and the selection rules for roots can be used to investigate the possible K-matrices for the AdS/dCFT correspondence. We close this section with an all-loop asymptotic proposal for the simplest D3-D5 1-point functions. Finally we conclude and provide a list of open problems. Technical details are relegated to appendices.

2 Boundary states, K- and boundary Yang-Baxter equations

There are two ways to place an integrable boundary in a two dimensional system: it can be placed either in space or in time. We formulate these two cases at such a level of generality which can cover both the AdS/CFT scattering matrix and all rational spin chains.

2.1 Boundary states and KYBE

If the boundary is placed in time it serves as an initial or finite state. The final state annihilates pairs of particles, while the initial state creates those. In a QFT an integrable boundary state is annihilated by the infinitely many parity odd charges of the theory [20]. This in particular implies that both the initial and the final boundary states can be described by the two particle K-matrix and one is related to the other by conjugation. We will focus on a final state, which can be depicted on the left of figure 1.

The K-matrix $K^{\alpha\beta}_{a\dot{b}}(p)$ is the amplitude of annihilating a pair of particles with labels $a, \dot{b}$ and momenta $p, -p$ and having boundary degrees of freedom α and β on the two ends. To keep the discussion on a general level we allowed that particles with momentum p and $-p$ transform w.r.t. different representations (of the same dimension) which we differentiated by a dot on the index.

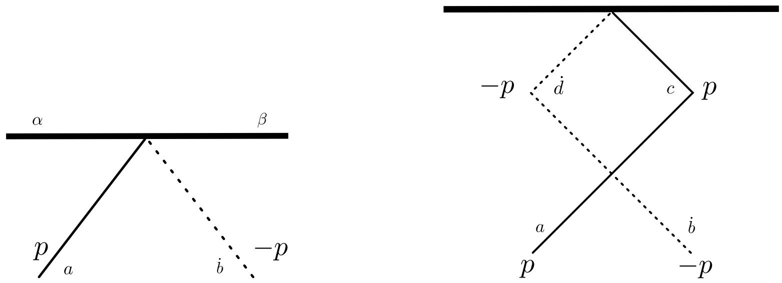

Figure 1. Graphical representation of the K-matrix and its crossing property. The boundary state might have an inner degree of freedom, which is labeled by Greek letters. It might also annihilate particles from different representations, which is indicated by straight and dashed lines and by a dot on the indices of the dashed particles. One typical example is when it annihilates a pair consisting of a particle and an anti-particle, which transform w.r.t. a representation and its contragradient representation, respectively. Such a case can appear for instance in $\mathfrak{su}(N)$ spin chains.

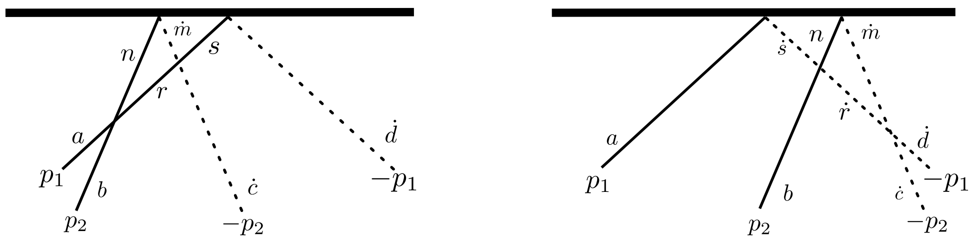

Figure 2. K-matrix YBE from shifting particle lines. Dashed lines and dotted indices might transform w.r.t. different representations. If they are different the KYBE is called twisted, otherwise it is called untwisted.

In integrable theories particle-trajectories can be shifted without altering the amplitudes, see figure 1. As a consequence, the K-matrix satisfies the crossing equation

$$K^{\alpha\beta}_{a\dot b}(p) = S^{c\dot d}_{a\dot b}(p,-p)K^{\alpha\beta}_{\dot d c}(-p)\,; \qquad K_{1\dot 2}(p) = S_{1\dot 2}(p,-p)K_{\dot 2 1}(-p) \tag{2.1}$$

where we also introduced a compact notation by indicating only in which place and representation the scattering S- and K-matrices act, i.e ($p \to 1, -p \to 2$) and suppressed to write out explicitly the boundary degrees of freedom.

Two pairs of particles can be annihilated in two different ways, see figure 2 leading to the Yang-Baxter equation for the K-matrix (KYBE):

$$S^{rn}_{ab}(p_1,p_2)K^{\alpha\beta}_{n\dot m}(p_2)S^{s\dot m}_{r\dot c}(p_1,-p_2)K^{\beta\gamma}_{s\dot d}(p_1) = K^{\alpha\beta}_{a\dot s}(p_1)S^{n\dot s}_{b\dot r}(p_2,-p_1)K^{\beta\gamma}_{n\dot m}(p_2)S^{\dot m\dot r}_{\dot c\dot d}(-p_2,-p_1) \tag{2.2}$$

or alternatively

$$K_{3\dot 4}(p_2)K_{1\dot 2}(p_1)S_{1\dot 4}(p_1,-p_2)S_{13}(p_1,p_2) = K_{1\dot 2}(p_1)K_{3\dot 4}(p_2)S_{3\dot 2}(p_2,-p_1)S_{\dot 4\dot 2}(-p_2,-p_1) \tag{2.3}$$

where again for each particle we associated a vector space as ($p_1 \to 1, -p_1 \to 2, p_2 \to 3, -p_2 \to 4$). If the dotted representation is really different from the undotted one the KYBE is called *twisted*. If the two representations are the same, the dots can be neglected and the KYBE is called *untwisted*.

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Thanks to integrability the general multiparticle annihilation process can be written in terms of the two particle annihilation amplitudes and the two particle scattering matrices. In particular, in terms of the ZF operators, the boundary state has an exponential form

$$\langle B| = \langle 0| \exp\left\{ \int_{-\infty}^{\infty} \frac{dp}{4\pi} K_{AB}(p) Z^A(-p) Z^B(p) \right\} \tag{2.4}$$

where $Z^A(p)$ is the operator, which annihilates a particle of type A and momentum p, and these operators form the ZF algebra

$$Z^A(p_1) Z^B(p_2) = S^{AB}_{CD}(p_1, p_2) Z^D(p_2) Z^C(p_1) \tag{2.5}$$

The boundary state contains the contributions of all possible particles, which we indicated by summing over indices of all types. Typically either $K^{\alpha\beta}_{ab} = 0 = K^{\alpha\beta}_{\dot{a}\dot{b}}$ or $K^{\alpha\beta}_{a\dot{b}} = 0 = K^{\alpha\beta}_{\dot{a}b}$, which is related to the fact that $S^{\dot{c}d}_{ab} = 0$. Consistency of the boundary state, i.e. invariance for $p \to -p$ and uniqueness of the exponentials, implies the crossing property and the KYBE.

2.2 Reflection matrices and BYBE

Alternatively, we can place the boundary in space and characterize it by specifying how particles scatter off it. Integrable boundaries have infinitely many parity even conserved charges. As a consequence, multiparticle reflections factorize into one-particle reflections and pairwise scatterings, thus it is enough to determine the one-particle reflections. This reflection amplitude is not independent from the K-matrix above. Indeed, one can perform a rotation in exchanging the role of space and time, which gives rise to the mirror theory. In non-relativistic theories the mirror dispersion relation $\tilde{E}(\tilde{p})$, obtained by analytical continuation, can be different from the original one $E(p)$. Here and from now on we indicate mirror quantities by tilde. Typically we parametrize the dispersion relation with a generalized rapidity parameter $p(z), E(z)$, and then the scattering matrix depends on these rapidities $S(z_1, z_2)$. For a properly chosen rapidity parameter the mirror rapidity, $\tilde{z}$ is simply the shifted version of the original one $z = \tilde{z} + \frac{\omega}{2}$. The notation indicates that $z \to z + \omega$ is the crossing transformation which maps $E \to -E$ and $p \to -p$ and replaces particles with antiparticles[2]. Thus going from the original theory to the mirror theory is half of a crossing transformation. The mirror dispersion relation is $\tilde{E}(\tilde{z}) = -ip(\tilde{z} + \frac{\omega}{2})$, $\tilde{p}(\tilde{z}) = -iE(\tilde{z} + \frac{\omega}{2})$, while the mirror scattering matrix is $\tilde{S}(\tilde{z}_1, \tilde{z}_2) = S(\tilde{z}_1 + \frac{\omega}{2}, \tilde{z}_2 + \frac{\omega}{2})$. In the following we show that the KYBE is equivalent to the boundary YBE (BYBE) for the reflection matrix in this mirror theory. In doing so we suppress to write out the spectator boundary degrees of freedom. The index form of the KYBE, (2.2), can be rewritten by introducing the charge conjugation matrix C and its inverse $C^{n\bar{m}} C_{\bar{m}k} = \delta^n_k$ which intertwine between the particle and antiparticle representations, as

$$\begin{aligned} S^{rn}_{ab}(z_1, z_2) C^{\bar{m}\dot{m}} K_{n\dot{m}}(z_2) S^{s\dot{m}}_{r\dot{c}}(z_1, -z_2) C_{\dot{m}\bar{m}} K_{s\dot{d}}(z_1) C^{\bar{c}\dot{c}} C^{\bar{d}\dot{d}} = \\ C^{\bar{c}\dot{c}} C^{\bar{d}\dot{d}} K_{a\dot{s}}(z_1) C^{\dot{s}\bar{s}} C_{\bar{s}\dot{s}} S^{n\dot{s}}_{b\dot{r}}(z_2, -z_1) K_{n\dot{m}}(z_2) C^{\dot{m}\bar{m}} C_{\bar{m}\dot{m}} S^{\dot{m}\dot{r}}_{\dot{c}\dot{d}}(-z_2, -z_1) \end{aligned} \tag{2.6}$$

[2]In relativistic theories z is the rapidity and $\omega = i\pi$, while for the AdS_5/CFT_4 integrable model z is a torus variable and $\omega = \omega_2$ [21]. For physical processes both in the theory and its mirror version the rapidity variables are real. Since ω is imaginary the $z \to \tilde{z} + \frac{\omega}{2}$ transformation involves an analytical continuation.

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Let us introduce the matrix $K(z)$ with indices: $C^{\bar{d}\dot{d}}K_{s\dot{d}}(z) = K_s^{\bar{d}}(z)$. It connects particles to twisted antiparticles labeled by bar, i.e. to the contragradient representation of the dotted representation. In terms of these quantities the KYBE takes a form

$$S_{12}(z_1,z_2)K_2(z_2)C_2S^{t_2}_{1\dot{2}}(z_1,-z_2)C_2^{-1}K_1(z_1) = K_1(z_1)C_1S^{t_1}_{2\dot{1}}(z_2,-z_1)C_1K_2(z_2)S_{\bar{2}\bar{1}}(-z_2,-z_1) \tag{2.7}$$

Using the crossing symmetry of the S-matrix $C_1^{-1}S^{t_1}_{1\dot{2}}(z_1,z_2)C_1 = S_{\bar{2}1}(z_2,z_1+\omega)$ and parity invariance $S_{21}(-z_2,-z_1) = S_{12}(z_1,z_2)$ the equation takes the form:

$$S_{12}(z_1,z_2)K_2(z_2)S_{\bar{2}1}(-z_2+\omega,z_1)K_1(z_1) = K_1(z_1)S_{\bar{1}2}(-z_1+\omega,z_2)K_2(z_2)S_{\bar{1}\bar{2}}(z_1,z_2) \tag{2.8}$$

We now use analytical continuation to go to the mirror theory, together with relabeling and a parity transformation $z_1 \to \frac{\omega}{2} - \tilde{z}_2$ and $z_2 \to \frac{\omega}{2} - \tilde{z}_1$, to reach

$$\tilde{S}_{12}(\tilde{z}_1,\tilde{z}_2)\tilde{R}_1(\tilde{z}_1)\tilde{S}_{2\bar{1}}(\tilde{z}_2,-\tilde{z}_1)\tilde{R}_2(\tilde{z}_2) = \tilde{R}_2(\tilde{z}_2)\tilde{S}_{1\bar{2}}(\tilde{z}_1,-\tilde{z}_2)\tilde{R}_1(\tilde{z}_1)\tilde{S}_{\bar{2}\bar{1}}(-\tilde{z}_2,-\tilde{z}_1) \tag{2.9}$$

where the mirror S-matrix was used (which is also parity invariant) and we introduced a mirror reflection matrix $\tilde{R}(\tilde{z}) = K(\frac{\omega}{2} - \tilde{z})$, see also figure 3:

$$\tilde{R}_a^{\bar{b}}(\tilde{z}) = C^{\bar{b}\dot{b}}K_{a\dot{b}}\left(\frac{\omega}{2} - \tilde{z}\right) \tag{2.10}$$

Thus we can conclude that if the K-matrix satisfies the KYBE, (2.3), then the mirror reflection factor defined by (2.10) satisfies the BYBE. If the representation with the bar is different from the one without the bar, the BYBE is called *twisted*, otherwise it is called *untwisted*. Let us point out that the mirror BYBE is not always equivalent to the BYBE in the original theory. It can differ in two ways. For AdS/CFT, particles and antiparticles are in the same representation but the dispersion relation and the scattering matrix is not relativistic invariant, thus $\tilde{S}$ and S are different. For rational spin chains and corresponding quantum field theories particles and antiparticles can transform w.r.t. different representations, thus an untwisted KYBE can results in twisted BYBE and vica versa. We elaborate on the possible cases in section 5.

We obtained the mirror BYBE from the KYBE by a mirror and a parity transformation: $z \to \frac{\omega}{2} - \tilde{z}$. Graphically the resulting equation can be depicted as on figure 4.

The mirror reflection matrix satisfies the unitarity relation

$$\tilde{R}_i^j(\tilde{z})\tilde{R}_j^k(-\tilde{z}) = \delta_i^k\,; \qquad \tilde{R}_1(\tilde{z})\tilde{R}_1(-\tilde{z}) = \mathbb{I}. \tag{2.11}$$

In summarizing, in quantum field theories the boundary state in the physical theory can be represented by the K-matrix, which satisfies the crossing equation and the KYBE. It is related to a reflection matrix of the mirror theory as (2.10), which satisfies unitarity.

Let us finally note that we have exactly the same equations for integrable spin chains, where the R-matrix plays the role of the scattering matrix and the K-matrix, solution of the boundary YBE is the reflection amplitude.

So far our considerations were in the infinite volume setting. In practical applications, however the boundary states are in finite volume and we are interested in the overlap of the finite volume boundary state and the finite volume multiparticle states. In the following we recall the finite volume periodic spectrum in various models we need.

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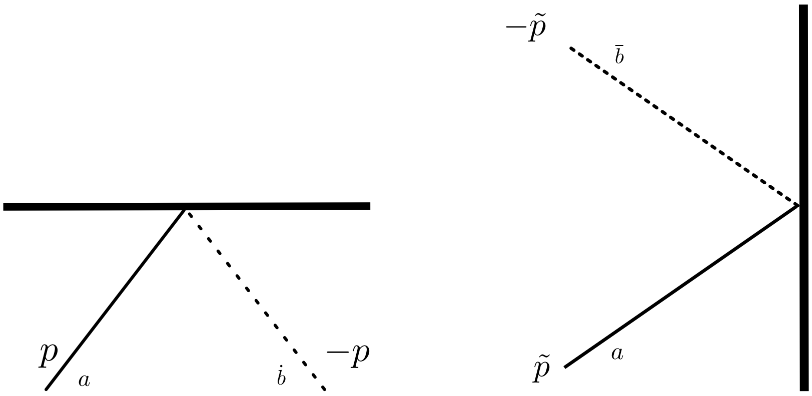

Figure 3. Reflection matrix in the mirror theory as obtained by a mirror (rotation by $\pi/2$) and a parity transformation: $z \to \frac{\omega}{2} - \tilde{z}$. The K-matrix connects the a representation and a dotted one, while the mirror reflection connects the same representation and the contragradient of the dotted one.

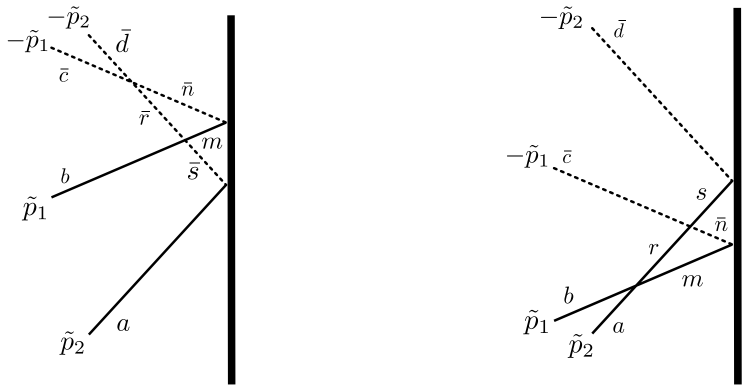

Figure 4. Boundary Yang Baxter equation in the mirror model. Dashed line indicates that the representation of the reflected particle can be different from the original one. If it is different, the BYBE is called twisted, otherwise it is called untwisted.

3 Spin chains and asymptotic spectrum

Integrable spin chains are interesting in their own rights, but they also have a direct connection to integrable QFTs. For any integrable QFT with inner degrees of freedom the scattering matrix has a scalar factor S_0 and a matrix part, R:

$$S(z_1, z_2) = S_0(z_1, z_2) R(z_1, z_2) \tag{3.1}$$

The matrix part satisfies the YBE and can be considered as an R-matrix of an integrable spin chain. The large volume (asymptotic) spectrum of the QFT, neglecting exponentially small volume corrections, is simply the infinite volume spectrum $E_n(L) = \sum_i E(z_i)$ only the momenta are quantized. This momentum quantization can be determined from the eigenvalues of the transfer matrix of the spin chain $t(z, \{z_i\})$, which is the trace of the monodromy matrix, built from the R-matrices as

$$t(z, \{z_i\}) = \mathrm{Tr}_0(T_0(z))\,; \qquad T_0(z) = R_{0L}(z, z_L) \dots R_{01}(z, z_1) \tag{3.2}$$

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Here 0 labels an auxiliary particle with rapidity z, whose representation space is traced over. This space can carry the same representation as the physical particle or some different representations. Transfer matrices for different representations and spectral parameters commute with each other and can be diagonalized in a spectral parameter independent basis.

In the following we recall the results of the nested Bethe ansatz for the rational $\mathfrak{su}(N)$, $\mathfrak{so}(4)$ and $\mathfrak{so}(6)$ spin chains, together with the centrally extended $\mathfrak{su}(2|2)_c$, which will be relevant for the later investigations. In spin chains the spectral parameter is traditionally denoted by u, which is not necessarily the rapidity, it might be some non-trivial function of it $u(z)$.

In the AdS_5/CFT_4 correspondence spin chains appear also on the Yang-Mills side. Indeed, the one-loop dilatation operator of the scaling dimensions of local operators can be related to an $\mathfrak{su}(2,2|4)$ nearest neighbors spin chain [22]. At higher loops the interaction range in the spin chain gets extended and spoils the $\mathfrak{su}(2,2|4)$ structure. What carries over is the nested Bethe ansatz obtained by choosing a pseudo vacuum. This pseudo vacuum breaks the $\mathfrak{su}(2,2|4)$ symmetry to $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$, which gets central extended at higher loop orders.

3.1 Spectrum of the $\mathfrak{su}(N)$ spin chain

In the $\mathfrak{su}(N)$ symmetric spin chains the R-matrix is a function of the differences of the spectral parameters and has the form

$$R(u_1, u_2) \equiv R(u_1 - u_2) = \mathbf{1} + \frac{1}{u_1 - u_2}\mathbf{P} \tag{3.3}$$

where $\mathbf{1}$ is the identity and $\mathbf{P}$ is the permutation operator acting on N-dimensional spaces, carrying the fundamental representations. The eigenvectors of the transfer matrix can be parametrized by Bethe roots $|u_1^{(a)}, \dots u_{n_a}^{(a)}\rangle \equiv |\mathbf{u}^{(a)}\rangle$, with $a = 1, \dots, N-1$ and the eigenvalues can be built up from the elementary building blocks z_k:

$$t(u)\big|\mathbf{u}^{(a)}\big\rangle = \Lambda(u)\big|\mathbf{u}^{(a)}\big\rangle\,; \qquad \Lambda(u) = \sum_{k=1}^{N} \frac{Q_{k-1}^{[k+1]}(u)}{Q_{k-1}^{[k-1]}(u)} \frac{Q_k^{[k-2]}(u)}{Q_k^{[k]}(u)} \tag{3.4}$$

where $f^{[k]}(u) = f(u + \frac{k}{2})$ [23]. The blocks, z_k, contribute also to the eigenvalues of the transfer matrices, where the auxiliary representation corresponds to a rectangular Young diagram and can be written in terms of the Q-functions which encode the Bethe roots

$$Q_0(u) = u^L\,, \qquad Q_k(u) = \prod_{i=1}^{n_k}\left(u - u_i^{(k)}\right), \qquad k = 1, \dots, N-1,\,; \qquad Q_N(u) = 1. \tag{3.5}$$

Bethe roots can be obtained from the Bethe Ansatz equations, which arise by demanding the regularity of the transfer matrix at $u = u_i^{(k)}$.

3.2 Spectrum of the $\mathfrak{so}(4)$ spin chain

The R-matrix in the $\mathfrak{so}(4)$ spin chain can be written as

$$R(u) = \mathbf{1} + \frac{1}{u}\mathbf{P} - \frac{1}{1+u}\mathbf{K}. \tag{3.6}$$

where $\mathbf{K}$ is the trace operator, $K_{ij}^{kl} = \delta_{ij}\delta^{kl}$. As the $\mathfrak{so}(4)$ Lie algebra can be written as $\mathfrak{so}(4) \equiv \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ the R-matrix has a factorized form

$$R(u) \cong \frac{u}{u+1}\left(\mathbf{1} + \frac{1}{u}\mathbf{P}\right) \otimes \left(\mathbf{1} + \frac{1}{u}\mathbf{P}\right). \tag{3.7}$$

which carries over to the transfer matrix and the Bethe roots

$$t(u) = \left(\frac{u}{u+1}\right)^L t_+(u) \otimes t_-(u), \qquad\qquad \big|\mathbf{u}_+, \mathbf{u}_-\big\rangle = \big|\mathbf{u}_+\big\rangle \otimes \big|\mathbf{u}_-\big\rangle, \tag{3.8}$$

such that the eigenvalues can be written in terms of the $\mathfrak{su}(2)$ Q-functions as

$$\Lambda(u) = \left(\frac{u}{u+1}\right)^L \Lambda_+(u)\Lambda_-(u)\,; \qquad \Lambda_\pm(u) = \left(\frac{u+1}{u}\right)^L \frac{Q_\pm^{[-1]}(u)}{Q_\pm^{[1]}(u)} + \frac{Q_\pm^{[3]}(u)}{Q_\pm^{[1]}(u)} \tag{3.9}$$

3.3 Spectrum of the $\mathfrak{so}(6)$ spin chain

The $\mathfrak{so}(6)$ symmetric R-matrix can be written as

$$R(u) = \mathbf{1} + \frac{1}{u}\mathbf{P} - \frac{1}{2+u}\mathbf{K}. \tag{3.10}$$

The eigenvalue of the transfer matrix with the fundamental representations of $\mathfrak{so}(6)$ can be written in terms of three types of Bethe roots with $Q_1, Q_\pm$ as follows [24]:

$$\Lambda(u) = \frac{Q_0^{[2]}}{Q_0}\frac{Q_1^{[-1]}}{Q_1^{[1]}} + \frac{Q_1^{[3]}}{Q_1^{[1]}}\frac{Q_+}{Q_+^{[2]}}\frac{Q_-}{Q_-^{[2]}} + \frac{Q_+}{Q_+^{[2]}}\frac{Q_-^{[4]}}{Q_-^{[2]}} + \frac{Q_0^{[2]}}{Q_0^{[4]}}\frac{Q_1^{[5]}}{Q_1^{[3]}} + \frac{Q_1^{[1]}}{Q_1^{[3]}}\frac{Q_+^{[4]}}{Q_+^{[2]}}\frac{Q_-^{[4]}}{Q_-^{[2]}} + \frac{Q_+^{[4]}}{Q_+^{[2]}}\frac{Q_-}{Q_-^{[2]}} \tag{3.11}$$

where $Q_0 = u^L$ and we did not write out explicitly their argument, which is u.

3.4 Spectrum of the $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$ spin chain

This is the spin chain which appears in the asymptotic limit of the spectral problem in the AdS_5/CFT_4 correspondence. The R-matrix is invariant under the centrally extended $\mathfrak{su}(2|2)_c$ algebra. Details about notations and conventions can be found in appendix A. As this is a superalgebra the transfer matrix is the supertrace of the product of graded R-matrices. The full R-matrix is the tensor product of two copies of the $\mathfrak{su}(2|2)_c R$-matrices, thus the transfer matrix has a factorized form

$$t(u) = t_+(u)t_-(u) \tag{3.12}$$

Each transfer matrix is related to an $\mathfrak{su}(2|2)_c$ symmetry and their eigenvectors and eigenvalues can be written in terms of y-roots $\{y_k\}_{k=1\ldots N}$ and w-roots $\{w\}_{l=1\ldots M}$

$$t_\pm(u)|\mathbf{y}_\pm, \mathbf{w}_\pm\rangle = \Lambda_\pm(u)|\mathbf{y}_\pm, \mathbf{w}_\pm\rangle \tag{3.13}$$

These are usually referred to as the left and the right wings. Focusing only on one of them the eigenvalue on a $2L$ long chain with inhomogeneities $\{p_i\}_{i=1\ldots 2L}$ takes the form [25]

$$\Lambda(u) = e^{-i\frac{p(u)}{2}(N-2L)}\frac{\mathcal{R}^{(+)[1]}}{\mathcal{R}^{(+)[-1]}}\left\{\frac{\mathcal{R}^{(-)[1]}\mathcal{R}_y^{[-1]}}{\mathcal{R}^{(+)[1]}\mathcal{R}_y^{[1]}} - \frac{\mathcal{R}_y^{[-1]}Q_w^{[2]}}{\mathcal{R}_y^{[1]}Q_w} - \frac{\mathcal{B}_y^{[1]}Q_w^{[-2]}}{\mathcal{B}_y^{[-1]}Q_w} + \frac{\mathcal{B}^{(+)[-1]}\mathcal{B}_y^{[1]}}{\mathcal{B}^{(-)[-1]}\mathcal{B}_y^{[-1]}}\right\} \tag{3.14}$$

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where

$$\mathcal{R}_y(u) = \prod_{j=1}^{N}(x(u) - y_j)\,; \qquad Q_w(u) = \prod_{l=1}^{M}(u - w_l)\,; \qquad \mathcal{R}^{(\pm)}(u) = \prod_{i=1}^{2L}(x(u) - x^{\pm}(p_i)) \tag{3.15}$$

and the $\mathcal{B}$ quantities can be obtained from the $\mathcal{R}$-s by replacing $x(u)$ with $1/x(u)$:

$$\mathcal{B}_y(u) = \prod_{j=1}^{N}(1/x(u) - y_j)\,; \qquad \mathcal{B}^{(\pm)}(u) = \prod_{i=1}^{2L}(1/x(u) - x^{\pm}(p_i)) \tag{3.16}$$

Shifts are understood as $x^{[\pm 1]}(u) \equiv x^{\pm}(u) = x(u \pm \frac{i}{2g})$ and we assumed that the total momentum vanishes: $\sum_i p_i = 0$. Bethe ansatz equations for the roots can be obtained from the regularity of the transfer matrix at $x^+(u) = y_j$ and $u = w_l$, see (6.38).

4 Selection rules for integrable overlaps

Boundary states can be analyzed by computing the overlaps with bulk states. Nonzero overlaps require a pair structure and in the following we elaborate on the possible structures. In particular, we analyze the $\mathfrak{su}(N)$, $\mathfrak{so}(4)$, $\mathfrak{so}(6)$ and $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$ spin chains. Similarly to the large volume QFT spectrum the large volume overlaps are basically the overlaps in the corresponding spin chains, i.e. the matrix element of a boundary state $\langle \Psi |$ with the eigenstates of the transfer matrix. By generalizing the notion of integrable boundaries from QFT to spin chains the authors of [2] came up with the definition of an integrable boundary state. This is a state which is annihilated by the odd conserved charges of the theory. Since the transfer matrix $t(u)$ generates the conserved charges the integrability requirement translates into

$$\langle \Psi | \, t(u) = \langle \Psi | \, \Pi t(u) \Pi \tag{4.1}$$

where Π is the space reflection operator [26].

In the following we refine this definition and analyze its consequences for the allowed pair structures appearing in the spin chains introduced above.

4.1 Pair structure in the $\mathfrak{su}(N)$ spin chain

The parity transformation $\Pi t(u) \Pi$ reverses the order in the product of R-matrices in the definition of the transfer matrix:

$$\Pi t(u) \Pi = \mathrm{Tr}_0 R_{01}(u) \dots R_{0L}(u).$$

Since the R-matrix with fundamental and with anti-fundamental representations are related by crossing symmetry

$$\bar{R}_{01}(u) = R_{01}^{t_0}(-u - N/2) = R_{01}^{t_1}(-u - N/2) \tag{4.2}$$

where t_0 and t_1 denote transposition in spaces 0 and 1, respectively, the parity transformed transfer matrix can be related to the transfer matrix where the auxiliary space is the anti-fundamental representation

$$\Pi t(u) \Pi = \mathrm{Tr}_0 \bar{R}_{0L}(-u - N/2) \dots \bar{R}_{01}(-u - N/2) = \bar{t}(-u - N/2) \tag{4.3}$$

The eigenvalues of the anti-fundamental transfer matrix has the same structure as the fundamental one and can be written in terms of the same Q-functions:

$$\bar{t}(u)\big|\mathbf{u}^{(a)}\big\rangle = \bar{\Lambda}(u)\big|\mathbf{u}^{(a)}\big\rangle; \qquad \bar{\Lambda}(u) = \sum_{k=1}^{N} \frac{Q_{k-1}^{[N-k-1]}(u)}{Q_{k-1}^{[N-k+1]}(u)} \frac{Q_{k}^{[N-k+2]}(u)}{Q_{k}^{[N-k]}(u)}, \tag{4.4}$$

We now investigate the overlap of an integrable boundary state and the Bethe state. In doing so we insert the transfer matrices into the overlap

$$\Lambda(u)\big\langle\Psi\big|\mathbf{u}^{(a)}\big\rangle = \big\langle\Psi\big|t(u)\big|\mathbf{u}^{(a)}\big\rangle = \big\langle\Psi\big|\bar{t}(-u-N/2)\big|\mathbf{u}^{(a)}\big\rangle = \bar{\Lambda}(-u-N/2)\big\langle\Psi\big|\mathbf{u}^{(a)}\big\rangle, \tag{4.5}$$

Thus the non-vanishing overlap requires $\Lambda(u) = \bar{\Lambda}(-u-N/2)$. This actually implies the same relation for the fused transfer matrices and leads to similar relations to each building block, z_k. This is equivalent to $Q_k(u) = Q_k(-u)$, which implies that each type of root must have the following pair structure:

$$\mathbf{u}^{(a)} = \left\{u_1^{(a)}, -u_1^{(a)}, \dots, u_{n_a/2}^{(a)}, -u_{n_a/2}^{(a)}\right\}, \qquad \text{for all } a = 1,\dots,N-1. \tag{4.6}$$

where here and from now on it is understood that for odd n_a we have a zero rapidity. In the following we show that in the other models the pair structure can be even richer.

4.2 Pair structure in the $\mathfrak{so}(4)$ spin chain

The integrability condition involves the space reflected transfer matrix, which can be related by the crossing symmetry to the original transfer matrix as

$$\Pi t(u)\Pi = t(-u-1). \tag{4.7}$$

Inserting the transfer matrix into the matrix element $\langle\Psi\big|\mathbf{u}_+,\mathbf{u}_-\rangle$ we can easily conclude that the non-vanishing overlap requires that

$$\Lambda(u) \equiv \left(\frac{u}{u+1}\right)^L \Lambda_+(u)\Lambda_-(u) = \Lambda(-u-1). \tag{4.8}$$

Now this integrability requirement (4.7) can be satisfied in *two different* ways:

$$Q_\pm(u) = Q_\pm(-u) \qquad \text{or} \qquad Q_+(u) = Q_-(-u) \tag{4.9}$$

Accordingly, we can have two different pair structures, what we call *chiral* and *achiral*:

1. Chiral pair structure, where

$$\mathbf{u}^{(\pm)} = \left\{u_1^{(\pm)}, -u_1^{(\pm)}, \dots, u_{n_\pm/2}^{(\pm)}, -u_{n_\pm/2}^{(\pm)}\right\}. \tag{4.10}$$

2. Achiral pair structure, where $(n_+ = n_- = n)$

$$\mathbf{u}^{(+)} = \{+u_1, +u_2, \dots, +u_{n-1}, +u_n\} = -\mathbf{u}^{(-)} = -\{-u_1, -u_2, \dots, -u_{n-1}, -u_n\}. \tag{4.11}$$

Thus the naive generalization of the integrability condition is too "weak" since it does not fix the pair structure uniquely. The reason is that we did not use the "elementary" transfer matrix, rather the product of two elementary ones. Based on the "elementary" transfer matrices $t_\pm(u)$, we can define two types of integrable states:

$$\text{chiral integrable state:}\ \ \langle\Psi|\, t_\pm(u) = \langle\Psi|\, \Pi t_\pm(u)\Pi. \tag{4.12}$$

$$\text{achiral integrable state:}\ \ \langle\Psi|\, t_+(u) = \langle\Psi|\, \Pi t_-(u)\Pi. \tag{4.13}$$

After a simple calculation one can check that the non-vanishing overlaps of chiral and achiral integrable states require chiral and achiral pair structures, respectively.

4.3 Pair structure in the $\mathfrak{so}(6)$ spin chain

From the naive integrability condition (4.1), one can derive the following requirement for the eigenvalues of the Bethe states with non-vanishing overlaps

$$\Lambda(u) = \Lambda(-u-2). \tag{4.14}$$

Similarly to the $\mathfrak{so}(4)$ model the integrability condition (4.14) can be satisfied in two alternative ways:

1. Chiral pair structure, where

$$\mathbf{u}^{(1)} = \left\{u_1^{(1)}, -u_1^{(1)}, \dots, u_{n_1/2}^{(1)}, -u_{n_1/2}^{(1)}\right\}; \quad \mathbf{u}^{(\pm)} = \left\{u_1^{(\pm)}, -u_1^{(\pm)}, \dots, u_{n_\pm/2}^{(\pm)}, -u_{n_\pm/2}^{(\pm)}\right\}, \tag{4.15}$$

2. Achiral pair structure, where $(n_+ = n_- = n)$

$$\mathbf{u}^{(1)} = \left\{u_1^{(1)}, -u_1^{(1)}, \dots, u_{n_1/2}^{(1)}, -u_{n_1/2}^{(1)}\right\}; \quad \mathbf{u}^{(+)} = \{+u_1, +u_2, \dots, +u_{n-1}, +u_n\} = -\mathbf{u}^{(-)}, \tag{4.16}$$

The reason why the integrability condition did not fix completely the pair structure is similar to the $\mathfrak{so}(4)$ case. Namely, we did not use the elementary transfer matrices to relate the transfer matrix and the parity transformed one. The "elementary" transfer matrices $t^{(\pm)}(u)$ corresponds to auxiliary spaces carrying the spinor representations of the SO(6) group. Let us define these transfer matrices as

$$t^{(\pm)}(u) = \mathrm{Tr}_0 R_{0L}^{(\pm)}(u)\dots R_{01}^{(\pm)}(u); \qquad R^{(+)}(u) = \mathbf{1} + \frac{1}{u+1-\frac{1}{2}} e_{ij}\otimes E_{ji} \tag{4.17}$$

$$R^{(-)}(u) = \mathbf{1} - \frac{1}{u+1+\frac{1}{2}} e_{ij}\otimes E_{ij} \tag{4.18}$$

where e_{ij} and E_{ij} are the defining and the six dimensional representations of $\mathfrak{gl}(4)$. The two representations, and such a way the two R-matrices, are connected by crossing symmetry

$$R^{(+)t}(-u-2) = R^{(-)}(u) \tag{4.19}$$

which implies the connection between the space reflected transfer matrices

$$\Pi t^{(\pm)}(u)\Pi = \mathrm{Tr}_0 R_{01}^{(\pm)t}(u)\dots R_{0L}^{(\pm)t}(u) = t^{(\mp)}(-u-2) \tag{4.20}$$

The eigenvalues of $t^{(\pm)}(u)$ can be written in terms of the Q-functions as

$$\begin{aligned}\Lambda^{(+)} &= \frac{Q_0^{[3]}}{Q_0^{[1]}}\frac{Q_+^{[-1]}}{Q_+^{[1]}} + \frac{Q_0^{[3]}}{Q_0^{[1]}}\frac{Q_1}{Q_1^{[2]}}\frac{Q_+^{[3]}}{Q_+^{[1]}} + \frac{Q_-^{[5]}}{Q_-^{[3]}} + \frac{Q_1^{[4]}}{Q_1^{[2]}}\frac{Q_-^{[1]}}{Q_-^{[3]}}\,;\\ \Lambda^{(-)} &= \frac{Q_0^{[1]}}{Q_0^{[3]}}\frac{Q_+^{[5]}}{Q_+^{[3]}} + \frac{Q_0^{[1]}}{Q_0^{[3]}}\frac{Q_1^{[4]}}{Q_1^{[2]}}\frac{Q_+^{[1]}}{Q_+^{[3]}} + \frac{Q_-^{[-1]}}{Q_-^{[1]}} + \frac{Q_1}{Q_1^{[2]}}\frac{Q_-^{[3]}}{Q_-^{[1]}}\end{aligned} \tag{4.21}$$

Using these formulas, together with the chiral/achiral pair structures, we can define two types of integrable initial states[3]:

$$\text{chiral integrable state:}\;\; \langle\Psi|\, t^{(\pm)}(u) = \langle\Psi|\,\Pi t^{(\pm)}(u)\Pi. \tag{4.22}$$

$$\text{achiral integrable state:}\;\; \langle\Psi|\, t^{(+)}(u) = \left(\frac{u+\frac{3}{2}}{u+\frac{1}{2}}\right)^L \langle\Psi|\,\Pi t^{(-)}(u)\Pi. \tag{4.23}$$

Let us finally note that the $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ spin chains are equivalent. The only difference between the model of this paragraph and the above investigated $\mathfrak{su}(4)$ model is that the quantum spaces are in different representations.

4.4 Pair structure in the $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$ spin chain

Similarly to the previous spin chains we can define two types of integrable states based on how we relate the transfer matrices of the two wings $t_\pm(u)$ to each other:

$$\text{chiral integrable state:}\;\; \langle\Psi|\, t_\pm(u) = \langle\Psi|\,\Pi t_\pm(u)\Pi. \tag{4.24}$$

$$\text{achiral integrable state:}\;\; \langle\Psi|\, t_+(u) = \langle\Psi|\,\Pi t_-(u)\Pi. \tag{4.25}$$

For the achiral integrable state roots of the left wing are opposite to the roots of the right wing:

$$\mathbf{y}_+ = -\mathbf{y}_-\,; \qquad \mathbf{w}_+ = -\mathbf{w}_- \tag{4.26}$$

while for chiral integrable states we have within each wing the following root structure:

$$\mathbf{y} = \{y_1, -y_1, \dots, y_{N/2}, -y_{N/2}\}\,; \qquad \mathbf{w} = \{w_1, -w_1, \dots, w_{M/2}, -w_{M/2}\} \tag{4.27}$$

4.5 General structures

Let us summarize our observations in the previous cases. We have seen that there exist two types of pair structures, chiral and achiral and for both integrability conditions can be defined. Based on these examples it is a natural assumption that the achiral structure is related to an outer automorphism of the symmetry algebra, i.e. to a symmetry of the Dynkin diagram. Non-trivial Dynkin diagram symmetries exist only for the $\mathfrak{sl}(N)$ and $\mathfrak{so}(2N)$ algebras, and for algebras being the direct sum of two identical copies. Therefore we expect that achiral pair structures can exist only for these cases. In the following we show how the chirality of the overlap can be read off from the symmetries of the K-matrices. A more detailed explanation in the case of rational spin chains can be found in appendix B.

[3] The particular prefactor is related to our normalization of the $R^{(\pm)}$-matrices. Clearly by renormalizing them the prefactor can be eliminated.

5 Integrable states from K-matrices and their symmetries

In the previous section we defined chiral and achiral integrable states, but it is not clear if they are realized at all. In [27] it was shown that a large class of integrable boundary states can be obtained from the solution of the KYBE as two-site and matrix product states

$$\langle\Psi| = \langle\psi_1|\otimes\langle\psi_2|\otimes\cdots\otimes\langle\psi_{L/2}|, \quad \langle\mathrm{MPS}| = \mathrm{Tr}\left[\omega_{i_1}\omega_{i_2}\dots\omega_{i_L}\right]\sum_{i_1,\dots,i_L}\langle i_1|\otimes\langle i_2|\otimes\cdots\otimes\langle i_L|, \tag{5.1}$$

where $\langle\psi_k|\in\mathcal{H}\otimes\mathcal{H}$ and $\langle i|\in\mathcal{H}$, $\omega_i\in\mathrm{End}(V)$, $\mathcal{H}$ being a one site Hilbert space and V is the boundary vector space. For two site states we can take

$$\langle\psi_i| = \langle a|\otimes\langle b|\,K_{ab}(u_i) \tag{5.2}$$

where $K(u)$ is the solution of the KYBE. Indeed, the integrability condition is satisfied since from the KYBE equation $\langle\Psi|\,K_0(u)T_0^{t_0}(u) = \langle\Psi|\,\Pi T_0(u)\Pi K_0(u)$ follows, which after inverting K_0 and tracing over the auxiliary space provides the required equation (4.1). Matrix product states can be obtained from specific solutions of the KYBE with inner degrees of freedom. Indeed if $K_{ab}^{\alpha\beta}$ factorizes as $K_{ab}^{\alpha\beta} = \omega_a^{\alpha\gamma}\omega_b^{\gamma\beta}$ then the MPS is integrable due to a similar argument.

In the following we classify the solutions of the KYBE or BYBE and reveal how the unbroken symmetry can be related to the chirality of the overlap. In rational spin chains when the boundary breaks the original symmetry algebra $\mathfrak{g}$ to $\mathfrak{h}$ then integrability requires that $(\mathfrak{g},\mathfrak{h})$ has to be a symmetric pair [28]. Thus the residual symmetry algebra has to form an invariant sub-algebra for a Lie algebra involution α i.e. $\mathfrak{h} := \{X\in\mathfrak{g}|\alpha(X) = X\}$. The residual symmetry $\mathfrak{h}$ is called *regular* if α is an inner involution, while *special* if it is an outer one. They are also related to an isometry of the Dynkin diagram. For inner involutions the symmetry is the identity, while for outer it acts non-trivially.

In the analysis of rational spin chains it was found that for regular residual symmetries the BYBEs are always untwisted. For special residual symmetries one can always find particles which reflect back into a different representation, such that their BYBE is twisted [28]. In this sense boundaries with regular residual symmetries can be called untwisted, while those with special symmetries as twisted. This so far concerned the BYBE. The chirality of the overlap, however is determined by the nature of the K-matrix and the twistedness of the KYBE. In order to understand this we analyze some examples.

5.1 K-matrices in the $\mathfrak{so}(4)$ spin chain

For the $\mathfrak{so}(4)\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ spin chain, there are two types of solutions of the KYBE, which by their nature can be called factorizing and non-factorizing. For the factorizing solutions, the K-matrix is factorized in the spinor basis as

$$K_{ab,a'b'}(u) = \sigma^i_{aa'}\sigma^j_{bb'}K_{ij}(u) = K^{(+)}_{ab}(u)K^{(-)}_{a'b'}(u) \tag{5.3}$$

where $K^{(\pm)}(u)$ are solutions of the $\mathfrak{su}(2)$ KYBE see in (6.7) and σ^i-s are the Pauli matrices for $i = 1,2,3$ and $\sigma^4 = iI$. The two-site state built from this K-matrix satisfy the chiral

$\mathfrak{so}(5)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(2) \oplus \mathfrak{so}(4)$	$\mathfrak{u}(3)$
special	special	regular	regular
$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} b & c & 0 & 0 & 0 & 0 \\ -c & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} d & u & 0 & 0 & 0 & 0 \\ -u & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & u & 0 & 0 \\ 0 & 0 & -u & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & u \\ 0 & 0 & 0 & 0 & -u & d \end{pmatrix}$
$\mathfrak{sp}(4)$	$\mathfrak{so}(4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$
chiral	chiral	achiral	achiral

Table 1. Nontrivial solutions of the KYBE for the $\mathfrak{so}(6)$ spin chain. The first line is the unbroken part of $\mathfrak{so}(6)$, together with the nature of their remaining symmetries. The parameters in the solutions are $a = \frac{u-1}{u+1}$, $b = \frac{\frac{1}{2}+d^2-u^2}{d^2+\left(\frac{1}{2}+u\right)^2}$, $c = \frac{du}{d^2+\left(\frac{1}{2}+u\right)^2}$ and d is a constant. In the third line the symmetries in the $\mathfrak{su}(4)$ language is shown, while the forth line reflects the type of the overlap.

integrability condition (4.12) thus factorizing K-matrices correspond to chiral overlaps. The residual symmetries of these K-matrices are factorized $H_+ \times H_-$ where $H_\pm$ can be independently either $\mathfrak{u}(1)$ or $\mathfrak{su}(2)$. The corresponding residual symmetry algebras are all regular.

The non-factorizing K-matrix reads as

$$K_{ab,a'b'}(u) = K^0_{ad'} K^0_{bc'} R^{c'd'}_{a'b'}(2u) \tag{5.4}$$

where $K^0 \in \mathrm{End}(\mathbb{C}^2)$. The two-site state built from this K-matrix satisfies the achiral integrability condition and the residual symmetry of K is the diagonal $\mathfrak{su}(2)_D \cong \mathfrak{so}(3)$, which is a special sub-algebra.

For the $\mathfrak{so}(4)$ algebra representations and conjugate representations are equivalent, thus the twisted nature of the BYBE and KYBE equations are equivalent, too. Here we found that regular residual symmetry algebras correspond to chiral, while special ones to achiral overlaps.

5.2 K-matrices in the $\mathfrak{so}(6)$ spin chain

For the $\mathfrak{so}(6)$ spin chain, there are five types of solutions of the KYBE [29]. They can be classified according to their symmetries. Since the unbroken symmetry together with $\mathfrak{so}(6)$ has to form a symmetric pair we have the following possibilities for the residual symmetry: $\mathfrak{so}(6)$, $\mathfrak{so}(5)$, $\mathfrak{so}(2) \oplus \mathfrak{so}(4)$, $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ or $\mathfrak{u}(3)$. The regular subalgebras are $\mathfrak{so}(6), \mathfrak{so}(2) \oplus \mathfrak{so}(4)$, $\mathfrak{u}(3)$ while the special ones are $\mathfrak{so}(5)$ and $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. The full symmetry is preserved by the identity, while the rest of the explicit K/R-matrices can be found in table (1).

In order to decide whether these K-matrices define chiral or achiral integrable states, we have to switch to the $\mathfrak{su}(4)$ description as we know that chiral boundary states connects the $\mathfrak{su}(4)$ fundamental representations to themselves, but achiral boundary states involves a conjugation. Since for $\mathfrak{su}(4)$ models particles and anti-particles transform in different representations there are two types of BYBE or KYBE, depending on how particles reflect

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Symmetry algebra	Type of sub-algebra	Explicit subalgebras	conjugation	Pair structure
$\mathfrak{gl}(N\|M)$	untwisted	$\mathfrak{gl}(n\|m)\oplus\mathfrak{gl}(N-n\|M-m)$	non-trivial	achiral
	twisted	$\mathfrak{osp}(N\|M)$	non-trivial	chiral
$\mathfrak{so}(4k)$	untwisted	$\mathfrak{so}(2l)\oplus\mathfrak{so}(4k-2l)$, $\mathfrak{u}(2k)$	trivial	chiral
	twisted	$\mathfrak{so}(2l+1)\oplus\mathfrak{so}(4k-2l-1)$	trivial	achiral
$\mathfrak{so}(4k+2)$	untwisted	$\mathfrak{so}(2l)\oplus\mathfrak{so}(4k+2-2l)$, $\mathfrak{u}(2k+1)$	non-trivial	achiral
	twisted	$\mathfrak{so}(2l+1)\oplus\mathfrak{so}(4k+1-2l)$	non-trivial	chiral

Table 2. Regular/special and chiral/achiral classification together with the conjugation properties.

back off the boundary or annihilated by the boundary state. If the particle reflects back as an antiparticle the boundary BYBE is a twisted one, with a special residual symmetry, but the KYBE is an untwisted one with a chiral overlap. If however the particle reflects back as a particle then the KYBE is a twisted one, it involves a conjugation and leads to achiral overlaps.

5.3 General Lie algebras

We have seen that the twisted nature of the residual symmetry, which is the same for the reflection matrix and the K-matrix is purely determined by the type of the BYBE, while the chiral nature of the overlap by the type of the KYBE. For algebras such as $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(2n)$ the Dynkin diagram has no nontrivial symmetry, all residual symmetries are regular. Both the BYBEs and the KYBEs are untwisted and the overlaps are chiral. For algebras $\mathfrak{su}(n)$ and $\mathfrak{so}(2n)$ the BYBE has the same nature as the symmetry. When switching to the KYBE we have to introduce a charge conjugation. This charge conjugation correspond also to a symmetry of the Dynkin diagram. For $\mathfrak{su}(n)$ and $\mathfrak{so}(4n+2)$ this is the same symmetry which defined the twisted BYBE. As a consequence the nature of the KYBE is just the opposite to that of the BYBE. This happens when the dotted representation is the contragradient in the boundary state. In case of $\mathfrak{so}(4n)$, however the charge conjugation is trivial and the nature of the BYBE and KYBE are the same. This is summarized in the table 2. We elaborate further on this in appendix B.

5.4 K-matrices in the $\mathfrak{su}(2|2)_c\oplus\mathfrak{su}(2|2)_c$ spin chain

Similarly to the $\mathfrak{so}(4)$ case the symmetry algebra is the direct sum of two identical algebras. As a consequence there are two types of solutions of the KYBE.

The general non-factorizing K-matrix was found in [5, 6] and involves the scattering matrix:

$$K_{ab,\dot{a}\dot{b}}(u)=K^0_{a\dot{d}}K^0_{b\dot{c}}R^{\dot{c}\dot{d}}_{\dot{a}\dot{b}}(u,-u) \tag{5.5}$$

where $K^0\in\mathrm{End}(\mathbb{C}^4)$ is related to the choice of the basis and R is the $\mathfrak{su}(2|2)_c$ invariant R-matrix. The two-site state built from this K-matrix satisfies the achiral integrability condition and the residual symmetry is the diagonal $\mathfrak{su}(2|2)_c$.

The other type of solutions are the factorizing ones

$$K_{ab,\dot{a}\dot{b}}(z)=K_0(z)K_{ab}(z)K_{\dot{a}\dot{b}}(z) \tag{5.6}$$

where $K_{ab}(z)$ solves the $\mathfrak{su}(2|2)_c$ KYBE

$$K_{34}(z_2)K_{12}(z_1)S_{14}(z_1,-z_2)S_{13}(z_1,z_2) = K_{12}(z_1)K_{34}(z_2)S_{32}(z_2,-z_1)S_{42}(-z_2,-z_1) \quad (5.7)$$

here we used the rapidity variable instead of the spectral parameter, but they can be used interchangeably. In the following we solve this equation and classify all solutions. Motivated by the results for $\mathfrak{su}(2|2)$ without central extension we search the solutions in the bosonic form

$$K(z) = K_0(z)\begin{pmatrix} k_1(z) & k_2(z) & 0 & 0 \\ k_3(z) & k_4(z) & 0 & 0 \\ 0 & 0 & h_1(z) & h_2(z) \\ 0 & 0 & h_3(z) & h_4(z) \end{pmatrix} \quad (5.8)$$

5.4.1 Leading order solution

Plugging back the ansatz into the KYBE and making a small g expansion one finds two classes of solutions, differing in which 2 by 2 block is nontrivial:

$$K(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & e^{i\frac{p(z)}{2}}h_1 & e^{i\frac{p(z)}{2}}h_2 \\ 0 & 0 & e^{i\frac{p(z)}{2}}h_3 & e^{i\frac{p(z)}{2}}h_4 \end{pmatrix}; \qquad K(z) = \begin{pmatrix} k_1 & k_2 & 0 & 0 \\ k_3 & k_4 & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{p(z)}{2}} \\ 0 & 0 & -e^{i\frac{p(z)}{2}} & 0 \end{pmatrix} \quad (5.9)$$

where $h_1, h_2 = h_3, h_4$ or $k_1, k_2 = k_3, k_4$ are arbitrary constants. Let us analyze the symmetries of these solutions.

5.4.2 Leading order symmetry

In the weak coupling, $g \to 0$, limit the AdS/CFT S-matrix is gauge-equivalent to the rational $\mathfrak{su}(2|2)$ S-matrix:

$$S(u) = uI_g + P \quad (5.10)$$

where I_g and P are the graded unity and permutation operators and the rapidity is $u = x^+ + \frac{1}{x^+} - \frac{i}{2g}$. The KYBE is equivalent to the twisted BYBE (or representation changing reflection equation). Our solutions are consistent with the solutions classified in [30], which consist of constant matrices of the form

$$K = \begin{pmatrix} V_a & 0 \\ 0 & V_s \end{pmatrix} \qquad \text{or} \qquad K = \begin{pmatrix} V_s & 0 \\ 0 & V_a \end{pmatrix} \quad (5.11)$$

where V_s and V_a are arbitrary symmetric and anti-symmetric 2 by 2 matrices. In order to determine the symmetry of the K-matrix we assume that V_s and V_a are invertible and focus on the first case. The symmetry transformations should commute with the scattering matrix and annihilate the K-matrix

$$[\Delta(M), S(u)] = 0\,; \quad \Delta(M)K(u) = 0\,; \qquad \Delta(M) = (M \otimes I + I \otimes M) \quad (5.12)$$

where $\otimes$ denotes the graded tensor product for which $(a \otimes b)\,(c \otimes d) = (-1)^{[b][c]}\, ac \otimes bd$. Transformations commuting with the S-matrix (5.10) form the $\mathfrak{gl}(2|2)$ algebra, while those which leave the K-matrix invariant form a super Lie sub-algebra $\mathfrak{h}$. In order to identify its defining relations we elaborate (5.12)

$$
\begin{aligned}
\left[(M\otimes I+I\otimes M)\,K\right]_{ij} &= \sum_k M_{ik}K_{kj}+\sum_l(-1)^{[i][j]+[i][l]}M_{jl}K_{il}= \\
&= \sum_k M_{ik}K_{kj}+\sum_l(-1)^{[l][j]+[l]}M_{jl}K_{il}=\left[MK+KM^{st}\right]_{ij}
\end{aligned}
\tag{5.13}
$$

where we used that K is bosonic as a matrix and the super transpose reads as $\left[M^{st}\right]_{ij}=(-1)^{[i][j]+[i]}M_{ji}$. Therefore the symmetry algebra is the $\mathfrak{osp}(2|2)$ algebra, which is usually defined by the relation

$$
M+KM^{st}K^{-1}=0\,;\qquad M=\begin{pmatrix} R & S\\ Q & L\end{pmatrix}
\tag{5.14}
$$

and, in the particular parametrization, looks like

$$
R+V_sR^tV_s^{-1}=0\,;\qquad L+V_aL^tV_a^{-1}=0\,;\qquad Q-V_aS^tV_s^{-1}=0
\tag{5.15}
$$

From these relations it is obvious that the two bosonic sub-algebras of $\mathfrak{osp}(2|2)$ are $\mathfrak{so}(2)$ and $\mathfrak{sp}(2)\cong\mathfrak{su}(2)$.

5.4.3 All loop solutions

In order to find the all loop solutions one starts with the one-loop form and promotes the constants to non-trivial functions at higher orders in g. We no longer demand that $k_2(z)=k_3(z)$ at higher orders. Equations having particles only with bosonic labels fixes the ratio of $k_1(z)$ and $k_4(z)$ to be a constant. One then selects simple looking equations including $k_1(z_1),k_1(z_2)$ and $k_3(z_1),k_3(z_2)$. These equations can be evaluated at any z_2 leading to equations for $k_1(z_1)$ and $k_3(z_1)$. Particularly simple choice is $z_2=0$, although one should be careful as both $x^\pm(z)$ go to 0 for $z\to 0$, with the ratio being 1. Similar equation can be derived for $k_2(z_1)$ including $k_3(z_1)$ and $k_1(z_1)$, which finally can be solved leading to the most general 3 parameter family of solutions

$$
\begin{aligned}
k_{1,4}(z) &= k_{1,4}\frac{x^+(z)(1+x^-(z)x^+(z))}{x^-(z)(A+x^+(z)^2)}\,;\qquad A=k_1k_4-k_2^2 \\
k_2(z) &= \frac{x^+(z)(k_2-x^+(z)+x^-(z)(A+k_2x^+(z))}{x^-(z)(A+x^+(z)^2)}\,; \\
k_3(z) &= \frac{x^+(z)(k_2+x^+(z)+x^-(z)(-A+k_2x^+(z))}{x^-(z)(A+x^+(z)^2)}
\end{aligned}
\tag{5.16}
$$

We can use the bosonic $\mathfrak{sl}(2)$ symmetries of the S-matrix to bring the solution into some canonical form. The three parameter family of such symmetries transform the solutions as $k_1\to e^ak_1;k_4\to e^{-a}k_4$, or as $k_1\to a(2k_2+ak_4);k_2\to k_2+ak_4$ or finally as $k_2\to k_2+ak_1;k_4\to k_4+a(2k_2+ak_1)$ and leaves A invariant. These transformations can be used to arrange $k_1=k_4=0$ or $k_2=0;k_1=k_4$. We can thus observe that basically we have only one free parameter in the solution.

In order to completely fix this solution one has to fix the overall scalar factor, which, from the unitarity and crossing unitarity equations, satisfies the equations

$$K_0\left(z+\frac{\omega_2}{2}\right)K_0\left(-z+\frac{\omega_2}{2}\right)=e^{ip(z+\frac{\omega_2}{2})}\,;\quad K_0(z)=S_0(z,-z)K_0(-z)e^{-2ip(z)}\left(\frac{A+x^+(z)^2}{A+x^-(z)^2}\right)^2 \tag{5.17}$$

Shifting variables in the first equation we might use $K_0(z+\omega_2)K_0(-z)=1$ instead. For the moment we cannot see how these equations could be easily solved for generic A.

The other type of solution takes the form

$$\begin{aligned} h_{1,4}(z) &= h_{1,4}\frac{x^-(z)(1+x^-(z)x^+(z))}{x^+(z)(A+x^+(z)^2)}\,; \qquad A=h_1h_4-h_2^2 \\ h_2(z) &= \frac{x^-(z)(h_2+x^-(z)+x^+(z)(-A+h_2x^-(z))}{x^+(z)(A+x^+(z)^2)}\,; \\ h_3(z) &= \frac{x^-(z)(h_2-x^-(z)+x^+(z)(A+h_2x^-(z))}{x^+(z)(A+x^+(z)^2)} \end{aligned} \tag{5.18}$$

Let us now identify the symmetries of these solutions.

5.4.4 All loop symmetry

In the following we show that the solutions above can be obtained from a centrally extended $\mathfrak{osp}(2|2)_c$ symmetry. In doing so we generalize the $\mathfrak{osp}(2|2)$ embedding to the centrally extended version of $\mathfrak{su}(2|2)$. See the appendix for the details of the defining relation of the centrally extended $\mathfrak{su}(2|2)_c$ algebra. Following (5.15) we define the fermionic generators of $\mathfrak{osp}(2|2)_c$ as

$$\tilde{\mathbb{Q}}_\alpha{}^a=\mathbb{Q}_\alpha{}^a+\epsilon_{\alpha\beta}s^{ab}\mathbb{Q}^{\dagger\,\beta}_b \tag{5.19}$$

where s^{ab} is any symmetric invertible matrix. These generators have the following anti-commutation relations

$$\begin{aligned} \left\{\tilde{\mathbb{Q}}^a_\alpha,\tilde{\mathbb{Q}}^b_\beta\right\} &= \left\{\mathbb{Q}^a_\alpha+\epsilon_{\alpha\gamma}s^{ac}\mathbb{Q}^{\dagger\gamma}_c,\mathbb{Q}^b_\beta+\epsilon_{\beta\delta}s^{bd}\mathbb{Q}^{\dagger\delta}_d\right\}= \\ &=\epsilon_{\alpha\beta}\left(s^{ac}\mathbb{R}^b_c-s^{bd}\mathbb{R}^a_d\right)+s^{ab}\left(\epsilon_{\beta\delta}\mathbb{L}^\delta_\alpha+\epsilon_{\alpha\gamma}\mathbb{L}^\gamma_\beta\right)+\epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}+\epsilon_{\alpha\beta}s^{ac}s^{bd}\epsilon_{cd}\mathbb{C}^\dagger= \\ &=\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{\mathbb{R}}+s^{ab}\left(\epsilon_{\beta\delta}\mathbb{L}^\delta_\alpha+\epsilon_{\alpha\gamma}\mathbb{L}^\gamma_\beta\right)+\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{\mathbb{C}}. \end{aligned} \tag{5.20}$$

where

$$\tilde{\mathbb{R}}=s^{ab}\mathbb{R}_b{}^c\epsilon_{ac}\,;\qquad \tilde{\mathbb{C}}=\mathbb{C}+\det(s^{ab})\mathbb{C}^\dagger \tag{5.21}$$

Therefore the generators $\tilde{\mathbb{R}},\mathbb{L}_\alpha{}^\beta,\tilde{\mathbb{Q}}_\alpha{}^a,\tilde{\mathbb{C}}$ form a centrally extended $\mathfrak{osp}(2|2)_c$ algebra:

$$\left[\tilde{\mathbb{R}},\tilde{\mathbb{Q}}^a_\alpha\right]=-s^{ab}\epsilon_{bc}\tilde{\mathbb{Q}}^c_\alpha\,;\quad \left[\mathbb{L}^\beta_\alpha,\mathbb{J}_\gamma\right]=\delta^\beta_\gamma\mathbb{J}_\alpha-\frac{1}{2}\delta^\beta_\alpha\mathbb{J}_\gamma\,;\quad \left[\mathbb{L}^\beta_\alpha,\mathbb{J}^\gamma\right]=-\delta^\gamma_\alpha\mathbb{J}^\beta+\frac{1}{2}\delta^\beta_\alpha\mathbb{J}^\gamma, \tag{5.22}$$

$$\left\{\tilde{\mathbb{Q}}^a_\alpha,\tilde{\mathbb{Q}}^b_\beta\right\}=\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{\mathbb{R}}+s^{ab}\left(\epsilon_{\beta\delta}\mathbb{L}^\delta_\alpha+\epsilon_{\alpha\gamma}\mathbb{L}^\gamma_\beta\right)+\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{\mathbb{C}}. \tag{5.23}$$

Notice that $\mathcal{V}(p)\otimes\mathcal{V}(-p)$ is a representation of the non centrally extended $\mathfrak{osp}(2|2)$ since $\tilde{\mathbb{C}}\cdot\mathcal{V}(p)\otimes\mathcal{V}(-p)=0$.

The fundamental S-matrix $S(p_1,p_2):\mathcal{V}(p_1)\otimes\mathcal{V}(p_2)\to\mathcal{V}(p_2)\otimes\mathcal{V}(p_1)$ commutes with the conserved charges

$$\Delta(\mathbb{J})S(p_1,p_2)=S(p_1,p_2)\Delta^{op}(\mathbb{J}) \tag{5.24}$$

for all $\mathbb{J}\in\mathfrak{su}(2|2)_c$. Let us assume that the K-matrix $K(p)\in\mathcal{V}(p)\otimes\mathcal{V}(-p)$ has $\mathfrak{osp}(2|2)$ symmetry i.e.

$$K(p)\Delta^{op}(\mathbb{J})=0. \tag{5.25}$$

For simplicity, we fix the embedding as

$$s^{ab}=\begin{pmatrix}0 & s\\ s & 0\end{pmatrix}. \tag{5.26}$$

By using bosonic generators the equation (5.25) fixes the tensor structure of $K(p)$ as

$$K(p)=e(p)\langle e_1|\otimes\langle e_2|+f(p)\langle e_2|\otimes\langle e_1|+\langle e_3|\otimes\langle e_4|-\langle e_4|\otimes\langle e_3|. \tag{5.27}$$

The fermionic generators completely fix $e(p)$ and $f(p)$ as follows: by applying $\tilde{\mathbb{Q}}_3{}^1=\mathbb{Q}_3{}^1+s\mathbb{Q}_2^{\dagger\,4}$ we obtain

$$K(p)\Delta^{op}\left(\mathbb{Q}_3{}^1\right)=e^{-ip/4}\left[\left(e(p)b(-p)-a(p)\right)\langle e_1|\otimes\langle e_4|+\left(f(p)b(p)-a(-p)\right)\langle e_4|\otimes\langle e_1|\right] \tag{5.28}$$

$$K(p)\Delta^{op}\left(\mathbb{Q}_2^{\dagger\,4}\right)=e^{ip/4}\left[\left(e(p)d(-p)-c(p)\right)\langle e_1|\otimes\langle e_4|+\left(f(p)d(p)-c(-p)\right)\langle e_4|\otimes\langle e_1|\right] \tag{5.29}$$

where we used that it is a graded tensor product in moving the fermionic generators through $|e_3\rangle$ and $|e_4\rangle$. Therefore

$$e(p)=\frac{s^{-1}a(p)e^{-ip/4}+c(p)e^{ip/4}}{-s^{-1}b(p)e^{-ip/4}+d(p)e^{ip/4}},\qquad f(p)=\frac{s^{-1}a(p)e^{-ip/4}-c(p)e^{ip/4}}{s^{-1}b(p)e^{-ip/4}+d(p)e^{ip/4}} \tag{5.30}$$

where we further used that $a(-p)=a(p)=d(p), b(-p)=-b(p)=-c(p)$. We obtain the same relations for the other fermionic generators. Using the explicit forms of $a(p)$ and $b(p)$ we obtain

$$e(p)=e^{ip/2}\frac{s^{-1}x^- -1}{x^+ + s^{-1}},\qquad f(p)=e^{ip/2}\frac{1+s^{-1}x^-}{x^+ - s^{-1}}. \tag{5.31}$$

These agree with the solution of the KYBE, once $k_1=k_4=0$ and $k_2=s^{-1}$ is chosen.

In summarizing, we found that the factorizing bosonic solutions of the KYBE must have $\mathfrak{osp}(2|2)_c$ symmetry. The 3 parameters in the solutions are related how the boundary $\mathfrak{osp}(2|2)_c$ symmetry is embedded into the centrally extended $\mathfrak{su}(2|2)_c$ bulk symmetry.

6 Asymptotic overlaps and nesting for *K*-matrices

In this section we demonstrate how K-matrices can be defined for various levels of the nesting and how this ideas can be used to calculate factorizing overlaps.

In calculating the spectrum of a spin chain with a higher rank symmetry we typically use the nesting method. This means that we start with an R-matrix with symmetry $\mathfrak{g}$ and some representation where one site states can be labeled as $i = 1, \dots, N$. (For simplicity we can assume an $\mathfrak{su}(N)$ spin chain). In diagonalizing the transfer matrix we first choose a pseudo vacuum, typically picking one of the indices say 1, assuming that the state $|0\rangle = |1\rangle^{\otimes L}$ is an eigenstate of the transfer matrix. R_{11}^{11} describes the diagonal scattering of these excitations. Then we introduce L_1 excitations with labels $j = 2, \dots, N$ over this pseudovacuum with rapidity $u^{(1)}$

$$\big|u_1^{(1)}, \dots, u_{L_1}^{(1)}\big\rangle_{j_1 \dots j_N}; \qquad j_i = 2, \dots, N \tag{6.1}$$

These excitations propagate over the pseudo vacuum through a diagonal scattering, R_{1j}^{1j}, but scatter on themselves non-trivially with a reduced $R^{(1)}$-matrix, which we calculate from the exchange relation

$$\big|u_1^{(1)}, u_2^{(1)}\big\rangle_{ab} = R^{(1)dc}_{\ \ ab}(u_1^{(1)} - u_2^{(1)})\big|u_2^{(1)}, u_1^{(1)}\big\rangle_{cd} \tag{6.2}$$

In the next step we repeat the procedure for this $R^{(1)}$-matrix having symmetry $\mathfrak{g}^{(1)} \subset \mathfrak{g}$, by choosing a second level pseudo vacuum, say $|2\rangle^{\otimes L_1}$ and diagonal scattering $R_{22}^{(1)22}$ among themselves and second level excitations $k = 3, \dots, N$ with diagonal propagation $R_{2j}^{(2)2j}$ and non-diagonal scatterings $R^{(3)}$. We then carry on this procedure until it terminates. As a result the spectrum of the transfer matrix is described in terms of particles with rapidities $u_j^{(a)}$ of various nesting levels $a = 1, \dots, N-1$, which scatter on each other diagonally as obtained at each step of the nesting.

The aim of this section is to develop a similar procedure for K-matrices and overlaps. Our procedure is a recursive one which can determine not only the nested K-matrices, but also the generic overlaps if they are factorizing. In doing so we assume that the square of the overlap[4] has the following "factorizing" form

$$\frac{|\langle\Psi|\mathbf{u}^{(a)}\rangle|^2}{\langle\mathbf{u}^{(a)}|\mathbf{u}^{(a)}\rangle} = \prod_{a,i} h^{(a)}(u_i^{(a)})\frac{G^+}{G^-} \tag{6.3}$$

where the norm of the state is given by the Gaudin determinant, $\langle\mathbf{u}^{(a)}|\mathbf{u}^{(a)}\rangle = G$ which, for states with a pair structure, can be written into a factorized form $G = G^+G^-$ and the overlap function of the nested excitations at level a are denoted by $h^{(a)}$. Let us note that in the thermodynamic limit (number of sites goes to infinity) the ratio of determinant cancels and we can calculate systematically the overlaps in this limit. As a first step we normalize the K-matrix appearing in the boundary state $\langle\Psi|$ for the pseudo vacuum by dividing with K_{11} in order to ensure a normalized boundary state $\langle\Psi^{(1)}|0^{(1)}\rangle = 1$, i.e. a normalized overlap with the first level pseudo vacuum $|0\rangle \equiv |0^{(1)}\rangle$. Clearly we have to choose the pseudo vacuum, such that it has a nonzero overlap with the boundary state. We will comment on the importance of the choice of the pseudo vacuum later. Then the

[4]By the abuse of terminology sometimes we call the squares as overlaps.

idea is to extract the nested level K-matrix in the limit as

$$K^{(1)}_{ab}(u^{(1)}) = \lim_{L\to\infty} \frac{\langle\Psi^{(1)}\big|\big|u^{(1)}, -u^{(1)}\big\rangle_{ab}}{\sqrt{{}_{ab}\big\langle u^{(1)}, -u^{(1)}\big|u^{(1)}, -u^{(1)}\big\rangle_{ab}}} \tag{6.4}$$

This K-matrix, by construction, satisfies the

$$K^{(1)}_{ab}(u^{(1)}) = R^{(1)cd}_{ab} K^{(1)}_{dc}(-u^{(1)}) \tag{6.5}$$

crossing equation. The one-particle overlap of the first level magnon is simply

$$h^{(1)}(u^{(1)}) = k^{(1)}(u^{(1)})k^{(1)}(u^{(1)})^* ; \qquad k^{(1)}(u^{(1)}) = K^{(1)}_{22}(u^{(1)}) \tag{6.6}$$

By dividing $K^{(1)}(u^{(1)})$ with $k^{(1)}(u^{(1)})$ we can build up $\langle\Psi^{(2)}|$ which has a normalized overlap with the second level pseudo vacuum $\langle\Psi^{(2)}|0^{(2)}\rangle = 1$. We then can proceed with the nesting at the second level. This procedure ends up with the nested K-matrices and overlaps $h^{(a)}$. We demonstrate this method in the following on various models. We use coordinate space Bethe vectors to calculate the overlap, while explicit formulas and technical details are relegated to appendix C.

In the following we verify this method by reconstructing previously known results for XXX, $\mathfrak{su}(3)$ and $\mathfrak{so}(6)$ spin chains [9, 17, 18]. After the verification, we propose a new overlap formula for the $\mathfrak{su}(2|2)_c$ spin chain using our nesting method.

6.1 Overlaps in the XXX spin chain

Let us start with the general integrable two-site state of the XXX spin chain of size L. The elements of the normalized boundary state, which solves the $\mathfrak{su}(2)$ KYBE, are:

$$\begin{aligned} &K_{11} = 1 ; \quad K_{12} = -ie^{-\theta}(\cosh\beta + 2\alpha\sinh\beta) ; \quad K_{21} = -ie^{-\theta}(\cosh\beta - 2\alpha\sinh\beta) ; \\ &K_{22} = -e^{-2\theta}. \end{aligned} \tag{6.7}$$

where α, β and θ are arbitrary parameters. We are interested in the overlap of the integrable state $\langle\Psi^{(1)}|$ with a two magnon state, built over the pseudo-vacuum $|1\rangle^{\otimes L}$ in the large L limit. As we have only one nesting level we denote $u^{(1)}$ by u and the scalar $R^{(1)}$ by S. In coordinate space Bethe ansatz the two magnon state is a plane wave of the form

$$|u, -u\rangle = \sum_{n_1=1}^{L} \sum_{n_2=n_1+1}^{L} \left(e^{ip(n_1-n_2)} + e^{-ip(n_1-n_2)}S(2u)\right)|n_1 n_2\rangle \tag{6.8}$$

where

$$p = -i\log\frac{u - i/2}{u + i/2}, \qquad S(u) = \frac{u + i}{u - i}. \tag{6.9}$$

and $|n_1 n_2\rangle$ represents a state, in which sites n_1 and n_2 are in state 2. This state is not symmetric in the rapidities, it satisfies $|u, -u\rangle = S(2u)| - u, u\rangle$. It is also not normalized, the norm is proportional to L in the large L limit. In calculating the overlap we have to analyze carefully the parity of n_1 and n_2 and their relations. The result from appendix C is

$$\begin{aligned}\langle\Psi^{(1)}|u,-u\rangle =& (\Sigma(p)+\Sigma(-p)S(2u))\left(K_{12}^2+K_{21}^2+\left(e^{ip}+e^{-ip}\right)K_{12}K_{21}\right)\\ &+\frac{L}{2}\left(e^{-ip}+e^{+ip}S(2u)\right)K_{22}.\end{aligned} \tag{6.10}$$

where in the asymptotic limit ($L\to\infty$), after proper regularization, $\Sigma(p)$ can be written as

$$\Sigma(p)=\frac{L}{2}\frac{1}{e^{2ip}-1}. \tag{6.11}$$

By substituting into (6.10) and dividing by the norm of the state in the $L\to\infty$ limit we obtain that

$$K^{(1)}(u)=k(u)=\frac{1}{L}\langle\Psi|u,-u\rangle=e^{-2\theta}\sinh(\beta)^2\frac{u^2+\alpha^2}{u(u-i/2)}. \tag{6.12}$$

We can calculate the normalized overlap square leading to

$$h(u)=k(u)k^*(u)=e^{-4\theta}\sinh(\beta)^4\frac{(u^2+\alpha^2)^2}{u^2(u^2+\frac{1}{4})} \tag{6.13}$$

which is the known one particle overlap function of the XXX spin chain [17]. Thus we provided an alternative calculation of those results. In the following we check how nesting works. For this we first analyze an $\mathfrak{su}(3)$ spin chain with $\mathfrak{so}(3)$ symmetry.

6.2 Overlaps in $\mathfrak{su}(3)$ spin chains with $\mathfrak{so}(3)$ symmetry

We analyze a two site state and a matrix product state for this model.

6.2.1 Two-site state

We take the integrable two-site state to be

$$\langle\Psi|=(\langle 1|\otimes\langle 1|+\langle 2|\otimes\langle 2|+\langle 3|\otimes\langle 3|)^{\otimes L/2} \tag{6.14}$$

We choose the pseudo-vacuum as $|1\rangle^{\otimes L}$ and introduce excitations with labels 2 and 3. We would like to calculate the two-site K-matrix, $K^{(1)}(u^{(1)})$ of these $\mathfrak{su}(2)$ excitations. We read off the $K^{(1)}$-matrix from an overlap with a two magnon state in appendix C

$$K_{ab}^{(1)}(u^{(1)}):=\frac{1}{L}\left\langle\Psi\middle|u^{(1)},-u^{(1)}\right\rangle=\frac{1}{2}\left(e^{-ip}\delta_{ab}+e^{ip}R_{ab}^{(1)cc}(2u^{(1)})\right)=\frac{u^{(1)}}{u^{(1)}-i/2}\delta_{ab} \tag{6.15}$$

By defining

$$k^{(1)}(u):=K_{11}^{(1)}(u)=\frac{u}{u-i/2} \tag{6.16}$$

we can see that it is related as $h^{(1)}(u)=k^{(1)}(u)k^{(1)}(u^*)^*=\frac{u^2}{u^2+1/4}$ to the one particle overlap function of the $u^{(1)}$ magnons which agrees with [3, 4]. By normalizing with this factor the second level state

$$\psi_{ab}^{(2)}:=\frac{K_{ab}^{(1)}(u)}{k^{(1)}(u)}=\delta_{ab} \tag{6.17}$$

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is an SU(2) integrable initial state with $\alpha = 0$ and $\beta = \theta = i\pi/2$ from which the k-function turns out to be $k^{(2)}(u^{(2)}) = \frac{u^{(2)}}{(u^{(2)}-i/2)}$ (see (6.13)). Notice that $h^{(2)}(u) = k^{(2)}(u)k^{(2)}(u^*)^* = \frac{u^2}{u^2+1/4}$ is the one particle overlap function of the $u^{(2)}$ magnons, see [3, 4].

6.2.2 Matrix product state with Pauli matrices

Here we show that similar ideas can be used also for boundaries with inner degrees of freedom. Let the MPS be

$$ {}^{\alpha,\beta}\langle \mathrm{MPS}| = \Big[(\langle 1|\,\sigma_1 + \langle 2|\,\sigma_2 + \langle 3|\,\sigma_3)^{\otimes L} \Big]^{\alpha,\beta} \tag{6.18} $$

where $\alpha, \beta = 1, 2$ are the "inner" indexes of the Pauli matrices. The pseudo vacuum is $|1\rangle^{\otimes L}$ and we calculate the overlap ${}^{\alpha,\beta}\langle \mathrm{MPS}|\, u, -u\rangle_{ab}$. From the results of the appendix C we can see that the overlap is diagonal in α and β:

$$ {}^{1,1}\langle \mathrm{MPS}\left| u^{(1)}, -u^{(1)}\right\rangle_{a,b} = K^{(1)+}_{ab}(u^{(1)})\,; \qquad {}^{2,2}\langle \mathrm{MPS}\left| u^{(1)}, -u^{(1)}\right\rangle_{a,b} = K^{(1)-}_{ab}(u^{(1)}) \tag{6.19} $$

with

$$ K^{(1)\pm}(u) = \frac{1}{u}\begin{pmatrix} u+\frac{i}{2} & \pm\frac{1}{2} \\ \mp\frac{1}{2} & u+\frac{i}{2} \end{pmatrix}\,; \qquad k^{(1)}(u) = K^{(1)\pm}_{11}(u). \tag{6.20} $$

Notice again that $h^{(1)}(u) = \frac{u^2+1/4}{u^2}$ and $\psi^{(2)\pm}_{ab}$s are integrable SU(2) states for inhomogeneous spin chains. Taking the homogeneous limit, we obtain an integrable two site state with the parameters

$$ \theta^{\pm} = i\frac{\pi}{2}, \qquad \beta^{\pm} = i\frac{\pi}{2}, \qquad \alpha^{\pm} = \pm\frac{1}{2} \tag{6.21} $$

therefore $h^{(2)}(u) = \frac{u^2+1/4}{u^2}$. We can see that $h^{(1)}(u)$ and $h^{(2)}(u)$ agree with [18].

6.3 Overlaps in $\mathfrak{so}(6)$ spin chains with $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ symmetry

This model is relevant for the weak coupling limit of AdS/dCFT. The six dimensional one site Hilbert space is parametrized by ϕ_i, $i = 1, \ldots, 6$ and we introduce the notation

$$ Z = \frac{1}{\sqrt{2}}\left(\phi_5 + i\phi_6\right)\,, \qquad \bar{Z} = \frac{1}{\sqrt{2}}\left(\phi_5 - i\phi_6\right) \tag{6.22} $$

We are going to analyze a two site state and a matrix product state and point out the importance of the right choice for the pseudovacuum.

6.3.1 Two-site state

Let the two-site state be

$$ \langle \Psi| = \left(Z\otimes Z + \bar{Z}\otimes\bar{Z} + \phi_1\otimes\phi_1 - \phi_2\otimes\phi_2 + \phi_3\otimes\phi_3 - \phi_4\otimes\phi_4\right)^{\otimes L/2} \tag{6.23} $$

We choose the pseudo vacuum as $Z^{\otimes L}$ and excitations are labeled with $a, b = 1, 2, 3, 4$. The excitations have an $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry. The overlap with a two magnon state $\left|u^{(1)}, -u^{(1)}\right\rangle_{ab}$ is calculated in the appendix C to be

$$K^{(1)}_{ab}(u^{(1)}) = \frac{1}{L}\left\langle\Psi\,\middle|u^{(1)}, -u^{(1)}\right\rangle_{ab} = \frac{u^{(1)}}{u^{(1)} - i/2} F_{ab}\,; \qquad F = \mathrm{diag}(1, -1, 1, -1) \tag{6.24}$$

This K-matrix satisfies the KYBE and has a factorized form in the spinor basis

$$K^{(1)}(u) \cong \frac{u}{u - i/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = k^{(1)}(u)\psi^{(L)} \otimes \psi^{(R)}. \tag{6.25}$$

Notice that $h^{(1)}(u) = \frac{u^2}{u^2+1/4}$ is the one particle overlap function of the $u^{(1)}$ magnons and $\psi^{(L/R)}$ are $\mathfrak{su}(2)$ integrable initial states for which the k-functions are $k^{(L/R)}(u) = \frac{u}{(u-i/2)}$ (see (6.13)). Clearly $h^{(L/R)}(u) = k^{(L/R)}(u)k^{(L/R)}(u^*)^* = \frac{u^2}{u^2+1/4}$ is the one particle overlap function of the $u^{(L/R)}$ magnons. We can see that $h^{(1)}(u)$ and $h^{(2)}(u)$ agree with [11].

Let us emphasize that the choice of the direction of the pseudo vacuum is crucial in obtaining and integrable K-matrix in the nesting. Indeed, the boundary state has the symmetry $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, which is broken to $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ by choosing the pseudo vacuum as $Z^{\otimes L}$ and $\mathfrak{so}(4)$ excitations over it. Since $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ is an integrable residual symmetry of the $\mathfrak{so}(4)$ model the excitation K-matrix satisfies the KYBE. Rotating the pseudovacuum or equivalently the two site state as

$$\phi_3 \to \cos(\alpha)\phi_3 + \sin(\alpha)\phi_5\,; \qquad \phi_5 \to -\sin(\alpha)\phi_3 + \cos(\alpha)\phi_5$$

the full symmetry of the boundary state $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is unchanged, however the excitation symmetry becomes only $\mathfrak{so}(2)$ which is no longer an integrable residual symmetry of the $\mathfrak{so}(4)$ model. The excitation K-matrix also has an $\mathfrak{so}(2)$ symmetry only, therefore it cannot be a solution of the KYBE, which can be checked by an explicit calculation.

Thus the excitation K-matrix is meaningful only with the proper pseudo vacuum. This is not at all surprising from the boundary nested BA point of view, since the correct choice of pseudo vacuum was important when the boundary was in space. For reflections in space, the excitation reflection matrices can be defined only when the labels of the pseudo vacuum reflects to themselves on the boundary. In [31] it was shown that the nesting of the $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ symmetric reflection matrix was related to following symmetry breaking

$$(\mathfrak{so}(6), \mathfrak{so}(3) \oplus \mathfrak{so}(3)) \longrightarrow (\mathfrak{so}(4), \mathfrak{so}(3))\,.$$

These two integrable symmetries are related to special residual symmetries. In contrast, we have just seen above that when the boundary is in time then the nesting of the residual symmetries is

$$(\mathfrak{so}(6), \mathfrak{so}(3) \oplus \mathfrak{so}(3)) \longrightarrow (\mathfrak{so}(4), \mathfrak{so}(2) \oplus \mathfrak{so}(2))$$

which are both related to chiral overlaps. In this example, for space boundaries and reflections nesting preserves the regular/special residual symmetries, while for time boundaries and overlaps the chiral/achiral nature of the pairings. This indicates that the nesting of the K-matrix can be different when the boundaries are in time or in space.

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6.3.2 Matrix product state with Pauli matrices

Let the MPS be

$$
\begin{aligned}
{}^{\alpha,\beta}\langle \mathrm{MPS}| &= \left[\left(\sqrt{2}\phi_1\sigma_1+\sqrt{2}\phi_3\sigma_2+\sqrt{2}\phi_5\sigma_3\right)^{\otimes L}\right]^{\alpha,\beta} \\
&= \left[\left(\sqrt{2}\phi_1\sigma_1+\sqrt{2}\phi_3\sigma_2+(Z+\bar{Z})\sigma_3\right)^{\otimes L}\right]^{\alpha,\beta}.
\end{aligned}
\tag{6.26}
$$

We take the pseudo vacuum and the excitations as before. The overlaps, calculated in the appendix C, turns out to be diagonal in α and β with non-vanishing components

$$
{}^{1,1}\langle \mathrm{MPS}\,|n_1 n_2\rangle_{a,b} = K^{(1)+}_{ab}(u)\,; \qquad {}^{2,2}\langle \mathrm{MPS}\,|n_1 n_2\rangle_{a,b} = K^{(1)-}_{ab}(u) \tag{6.27}
$$

where the K-matrices can be written as

$$
K^{(1)\pm}(u) = \begin{pmatrix}
\frac{u^2+iu-1/2}{u(u+i/2)} & 0 & \mp\frac{1}{u} & 0 \\
0 & -\frac{u+i}{u+i/2} & 0 & 0 \\
\pm\frac{1}{u} & 0 & \frac{u^2+iu-1/2}{u(u+i/2)} & 0 \\
0 & 0 & 0 & -\frac{u+i}{u+i/2}
\end{pmatrix} \tag{6.28}
$$

It is factorized in the spinor basis:

$$
K^{(1)\pm}(u) = \frac{u+i/2}{u}\begin{pmatrix} 1 & \pm\frac{1}{2u+i} \\ \mp\frac{1}{2u+i} & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \pm\frac{1}{2u+i} \\ \mp\frac{1}{2u+i} & 1 \end{pmatrix}. \tag{6.29}
$$

showing that it is an integrable K-matrix. We can define the k-function and the $\mathfrak{su}(2)$ two-site states as

$$
k^{(1)}(u) = \frac{u+i/2}{u}\,; \quad \psi^{(L/R)\pm}_{ab} = \left.\begin{pmatrix} 1 & \pm\frac{1}{2u+i} \\ \mp\frac{1}{2u+i} & 1 \end{pmatrix}\right|_{u=0} = \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} \tag{6.30}
$$

Notice that $h^{(1)}(u) = \frac{u^2+1/4}{u^2}$ is the one particle overlap function for the $u^{(1)}$ magnons and $\psi^{(2)\pm}_{ab}$s are integrable initial states for which the k-function is (see (6.13)) $k^{(2)}(u) = k^{(2)\pm}(u) = \frac{(u+i/2)}{u}$. Clearly $h^{(2)}(u) = \frac{u^2+1/4}{u^2}$ is the one particle overlap function for the $u^{(2)}$ magnons. We can see that $h^{(1)}(u)$ and $h^{(2)}(u)$ agree with [9].

6.4 Overlaps in $\mathfrak{su}(2|2)_c$ spin chains

We would like to analyze an all loop two-site integrable state which in the weakly coupled limit reproduces the result in the previous section (6.30). This can be obtained from

$$
K(p) = \begin{pmatrix}
k_1(p) & k_2(p) & 0 & 0 \\
k_3(p) & k_4(p) & 0 & 0 \\
0 & 0 & 0 & e^{i\frac{p}{2}} \\
0 & 0 & -e^{i\frac{p}{2}} & 0
\end{pmatrix} \tag{6.31}
$$

by choosing $k_1 = k_4 = s(g) = g^{-1} + \dots$ and $k_2 = 0$. One can indeed check that the weak coupling expansion of $K(p)$ reproduces $\psi^{(L/R)}_{ab}(u)$ in the upper 2 by 2 block and gives zero elsewhere. The all coupling integrable boundary state for $2L$ sites then takes the form

$$\langle\Psi| = \langle K(p_1)| \otimes \dots \otimes \langle K(p_L)| \; ; \qquad \langle K(p)| = K_{ij}(p) \langle i| \otimes \langle j| \mathbb{I}_g. \tag{6.32}$$

The pseudo-vacuum is going to be $|1\rangle^{\otimes 2L}$ and excitations are labeled by 3, 4 and 2. States of the nested Bethe ansatz are labeled by $|\mathbf{y}, \mathbf{w}\rangle$ or $|\mathbf{p}, \mathbf{y}, \mathbf{w}\rangle$ if we want to emphasize the dependence on the inhomogeneities $\mathbf{p}$. The normalized overlap square is assumed to factorize as

$$\frac{|\langle\Psi|\,\mathbf{p},\mathbf{y},\mathbf{w}\rangle|^2}{\langle\mathbf{p},\mathbf{y},\mathbf{w}\,|\mathbf{p},\mathbf{y},\mathbf{w}\rangle} = \prod_{i=1}^{L} h^p(p_i) \prod_{i=1}^{N/2} h^y(v_i) \prod_{i=1}^{M/2} h^w(w_i) \frac{G^+}{G^-}. \tag{6.33}$$

As a first step we renormalize the boundary state by pulling out the overlap of the pseudovacuum:

$$\frac{K(p)}{K_{11}(p)}\,; \qquad h^p(p) = |K_{1,1}(p)|^2 \tag{6.34}$$

The corresponding state is denoted by $\langle\Psi^{(1)}|$. In the following let us use special inhomogeneities $p_{2k-1} = p$ and $p_{2k} = p$ for $k = 1, \dots L$ as the boundary overlaps does not depend on the inhomogeneities. These inhomogeneities merely influence how the second level excitations are propagating and do not effect the overlaps. Using these special momenta we can calculate the two particle overlap in the asymptotic limit. Using the explicit form of the two-particle coordinate space Bethe vectors [32] we obtain in appendix C that

$$K^{(1)}_{\alpha\beta}(y) = \frac{\left|\langle\Psi^{(1)}\,|y,-y\rangle_{\alpha,\beta}\right|^2}{_{\alpha\beta}\langle y,-y\,|y,-y\rangle_{\alpha,\beta}} = \frac{4g^2}{s^2}\frac{(y^2+s^2)^2}{y^2+4g^2(y^2+1)^2}\epsilon_{\alpha\beta} \tag{6.35}$$

Observe that we obtained a Dimer state, $\theta = 0, \beta = \frac{i\pi}{2}, \alpha \to \infty$, for the inhomogeneous $\mathfrak{su}(2)$ model at the second level of the nesting. For this state the overlap can be written as (the number of magnons has to equal to the half length of the spin chain i.e. $N = 2M$) [33]

$$\frac{|\langle Dimer\,|\mathbf{y},\mathbf{w}\rangle|^2}{\langle\mathbf{y},\mathbf{w}\,|\mathbf{y},\mathbf{w}\rangle} = (-1)^{N/2}\prod_{i=1}^{N/2}(v_i^2+\frac{1}{4g^2})\prod_{i=1}^{\lfloor N/4\rfloor}\frac{1}{w_i^2\left(w_i^2+\frac{1}{4g^2}\right)}\frac{\hat{G}^+}{\hat{G}^-}, \quad v_i = y_i + y_i^{-1} \tag{6.36}$$

with the determinant corresponding to the subchains. Using the results above, the proposed overlap formula is

$$\frac{|\langle\Psi|\,\mathbf{p},\mathbf{y},\mathbf{w}\rangle|^2}{\langle\mathbf{p},\mathbf{y},\mathbf{w}\,|\mathbf{p},\mathbf{y},\mathbf{w}\rangle} = \prod_{i=1}^{L}|K_{1,1}(p_i)|^2\prod_{i=1}^{N/2}\frac{-(y_i^2+s^2)^2}{s^2y_i^2}\prod_{i=1}^{\lfloor N/4\rfloor}\frac{1}{w_i^2\left(w_i^2+\frac{1}{4g^2}\right)}\frac{G^+}{G^-}. \tag{6.37}$$

where the determinants $G^{\pm}$ can be written in terms of the roots as follows. Let us parametrize the Bethe Ansatz equations as

$$e^{i\phi_{v_j}} = 1 = \prod_{i=1}^{2L} e^{-ip_i/2} \frac{y_j - x_i^+}{y_j - x_i^-} \prod_{k=1}^{M} \frac{v_j - w_k + \frac{i}{2g}}{v_j - w_k - \frac{i}{2g}}, \quad \text{for all} \quad j = 1, \ldots, N, \tag{6.38}$$

$$e^{i\phi_{w_j}} = 1 = \prod_{i=1}^{N} \frac{w_j - v_i + \frac{i}{2g}}{w_j - v_i - \frac{i}{2g}} \prod_{\substack{l=1 \\ l \neq j}}^{M} \frac{w_j - w_l - \frac{i}{g}}{w_j - w_l + \frac{i}{g}}, \quad \text{for all} \quad j = 1, \ldots, M, \tag{6.39}$$

where $v_i = y_i + y_i^{-1}$. Using the rescaled Bethe roots $\hat{v}_i = g v_i$ and $\hat{w}_i = g w_i$, one can define the Gaudin determinant as

$$G = \begin{vmatrix} \left(\partial_{\hat{v}_i} \phi_{v_j}\right)_{N\times N} & \left(\partial_{\hat{v}_i} \phi_{w_j}\right)_{N\times M} \\ \left(\partial_{\hat{w}_i} \phi_{v_j}\right)_{M\times N} & \left(\partial_{\hat{w}_i} \phi_{w_j}\right)_{M\times M} \end{vmatrix} = G_+ G_-. \tag{6.40}$$

For the definition of G_+ and G_-, we have to separate two cases.

M is even. For even M, the pair structure is

$$\mathbf{p} = \left\{\mathbf{p}^+, \mathbf{p}^-\right\}, \qquad \mathbf{v} = \left\{\mathbf{v}^+, \mathbf{v}^-\right\}, \qquad \mathbf{w} = \left\{\mathbf{w}^+, \mathbf{w}^-\right\}, \tag{6.41}$$

where $p_i^+ = -p_i^-$, $v_i^+ = -v_i^-$ and $w_i^+ = -w_i^-$. The Gaudin-like determinants can be written as

$$G_{\pm} = \begin{vmatrix} \left(\partial_{\hat{v}_i^+} \phi_{v_j^+} \pm \partial_{\hat{v}_i^+} \phi_{v_j^-}\right)_{N/2\times N/2} & \left(\partial_{\hat{v}_i^+} \phi_{w_j^+} \pm \partial_{\hat{v}_i^+} \phi_{w_j^-}\right)_{N/2\times M/2} \\ \left(\partial_{\hat{w}_i^+} \phi_{v_j^+} \pm \partial_{\hat{w}_i^+} \phi_{v_j^-}\right)_{M/2\times N/2} & \left(\partial_{\hat{w}_i^+} \phi_{w_j^+} \pm \partial_{\hat{w}_i^+} \phi_{w_j^-}\right)_{M/2\times M/2} \end{vmatrix}. \tag{6.42}$$

where $\partial_v = \frac{y^2}{y^2-1}\partial_y$. The Gaudin matrix is factorized as $G = G_+ G_-$.

M is odd. For odd M, the pair structure is

$$\mathbf{p} = \left\{\mathbf{p}^+, \mathbf{p}^-\right\}, \qquad \mathbf{v} = \left\{\mathbf{v}^+, \mathbf{v}^-\right\}, \qquad \mathbf{w} = \left\{\mathbf{w}^+, \mathbf{w}^-, w^0\right\}, \tag{6.43}$$

where $p_i^+ = -p_i^-$, $v_i^+ = -v_i^-$, $w_i^+ = -w_i^-$ and $w^0 = 0$. The Gaudin-like determinants can be written in this case as

$$G_+ = \begin{vmatrix} \left(\partial_{\hat{v}_i^+} \phi_{v_j^+} + \partial_{\hat{v}_i^+} \phi_{v_j^-}\right)_{N/2\times N/2} & \left(\partial_{\hat{v}_i^+} \phi_{w_j^+} + \partial_{\hat{v}_i^+} \phi_{w_j^-}\right)_{N/2\times \lfloor M/2\rfloor} & 2\left(\partial_{\hat{v}_i^+} \phi_{w^0}\right)_{N/2\times 1} \\ \left(\partial_{\hat{w}_i^+} \phi_{v_j^+} + \partial_{\hat{w}_i^+} \phi_{v_j^-}\right)_{\lfloor M/2\rfloor\times N/2} & \left(\partial_{\hat{w}_i^+} \phi_{w_j^+} + \partial_{\hat{w}_i^+} \phi_{w_j^-}\right)_{\lfloor M/2\rfloor\times \lfloor M/2\rfloor} & 2\left(\partial_{\hat{w}_i^+} \phi_{w^0}\right)_{\lfloor M/2\rfloor\times 1} \\ \left(\partial_{\hat{v}_i^+} \phi_{w^0}\right)_{1\times N/2} & \left(\partial_{\hat{w}_i^+} \phi_{w^0}\right)_{1\times \lfloor M/2\rfloor} & \partial_{\hat{w}^0} \phi_{w^0} \end{vmatrix} \tag{6.44}$$

and G_- is the same as above. The Gaudin determinant is again factorized as $G = G_+ G_-$.

We extensively tested these formulas for various sizes $2L = 4, 6, 8$ and $M = 2N = 2, 4$, numerically, by specifying $\mathbf{p}$ and keeping s generic. In all the cases we found perfect agreement.

6.5 Summary

So far we have conjectured K-matrices for nested excitation from two particle states. The question is how we can generalize the ideas for more excitations and whether the obtained K-matrices are integrable. We elaborate on this in the following.

Let us assume we start the nesting with the top level excited Bethe states and denote them as

$$\left|u_1,\dots,u_N\right\rangle_{a_1\dots a_N}. \tag{6.45}$$

This state satisfies the following exchange relation

$$\left|\dots,u_i,u_{i+1},\dots\right\rangle_{\dots a_i a_{i+1}\dots} = S^{bc}_{a_i a_{i+1}}(u_1-u_2)\left|\dots,u_{i+1},u_i,\dots\right\rangle_{\dots cb\dots} \tag{6.46}$$

where $S(u)$ is the scattering matrix of the excitations which satisfy the YBE and the unitarity relation and derives from the R-matrix of the spin chain. We can define the N-particle K-matrix as the matrix element

$$K_{a_1b_1\dots a_{N/2}b_{N/2}}(u_1,\dots,u_{N/2}) = \left\langle\Psi\middle|u_1,-u_1,\dots,u_{N/2},-u_{N/2}\right\rangle_{a_1b_1\dots a_{N/2}b_{N/2}}. \tag{6.47}$$

From the exchange relation (6.46) we can derive that the four-particle K-matrix automatically satisfy the following equation

$$S_{13}(u_1-u_2)S_{14}(u_1+u_2)K_{3412}(u_2,u_1) = S_{42}(u_1-u_2)S_{32}(u_1+u_2)K_{1234}(u_1,u_2). \tag{6.48}$$

In order to connect to the previous investigations we take the $L\to\infty$ limit. From equation (6.48), we can see that if the N-particle K-matrix factorizes into the product of two-particle K-matrices in the $L\to\infty$ limit as

$$K_{a_1b_1\dots a_{N/2}b_{N/2}}(u_1,\dots,u_{N/2}) = K_{a_1b_1}(u_1)\dots K_{a_{N/2}b_{N/2}}(u_{N/2}) \tag{6.49}$$

then the two-particle K-matrix automatically satisfies the KYBE. From the integrability point of view, naively one could think that the question is whether the four-particle K-matrices satisfy the KYBE or not. However, we can see that this comes from the construction of the Bethe states and the real question is whether the N-particle K-matrix factorizes into two-particle K-matrices in the $L\to\infty$ limit or not. This question is not easy to decide and it is not obvious at all how it is connected to the integrability of the state $\left\langle\Psi\right|$.

We have already seen that the proper direction of the pseudo vacuum in the nesting is relevant, which can be supported by symmetry arguments. Thus we can formulate necessary requirements for the existence of integrable nested K-matrices.

Recall that an integrable state can be labeled by a symmetric pair $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{g}$ is the symmetry algebra of the spin chain and $\mathfrak{h}$ is the residual symmetry which annihilates the state. For the top-level excitations, these symmetries are reduced to $(\mathfrak{g}^{(1)},\mathfrak{h}^{(1)})$. If the excitation K-matrix satisfies the KYBE (i.e. it is factorized, then $(\mathfrak{g}^{(1)},\mathfrak{h}^{(1)})$ has to be a symmetric pair, too. The consistency of the pair structures requires that both symmetries $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}^{(1)},\mathfrak{h}^{(1)})$ have to belong to the same chiral or achiral pair structures. Since the pair $(\mathfrak{g}^{(1)},\mathfrak{h}^{(1)})$ depends on the choice of the pseudo-vacuum the factorizability depends on the choice of the pseudo vacuum, too. We can repeat the analysis for each next nesting level.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	symmetries
D3	●	●	●	●							
D5	●	●	●		●	●	●				$\mathfrak{osp}(4\|4)$
D7	●	●	●		●	●	●	●	●		$\mathfrak{so}(2,3) \times \mathfrak{so}(5)$

Table 3. Various brane structures on $AdS_5 \times S^5$ and their symmetries.

7 Application to one-point functions in AdS/CFT

In culminating all the investigations in the previous sections we investigate the defect $\mathcal{N} = 4$ SYM with the aim of providing all loop asymptotic one-point functions for local gauge invariant operators. There are two types of defects which are integrable for the scalar sector at tree level. Their gravity duals belong the following D-brane configurations:

1. D5-brane which wraps around AdS_4 and S_2 (see table 3).

2. D7-brane which wraps around AdS_4 and S_4 (see table 3).

The symmetry of the $D5$ brane is $\mathfrak{osp}(4|4)$ while that of the $D7$ brane is $\mathfrak{so}(2,3) \times \mathfrak{so}(5)$. Since in the integrability argumentations the Lorentzian signature is irrelevant we do not write it out explicitly, i.e. we write $\mathfrak{su}(2,2) \equiv \mathfrak{su}(4)$ and $\mathfrak{so}(2,3) \equiv \mathfrak{so}(5)$. At week coupling the spectrum of single trace operators can be described by an $\mathfrak{gl}(4|4)$ spin chain. The tree level one-point functions can be obtained from overlaps between one-loop Bethe states and boundary states. Since integrable states belong to symmetric pairs we can easily decide which configurations can be integrable. The pair $(\mathfrak{gl}(4|4), \mathfrak{osp}(4|4))$ is a symmetric pair but $(\mathfrak{gl}(4|4), \mathfrak{so}(5) \oplus \mathfrak{so}(5))$ is not, therefore this argument suggests that the D7 configuration may be not integrable for the full spectrum. In the following we investigate only the D5 defect configuration.

Since the D5 defect corresponds to the symmetric pair $(\mathfrak{gl}(4|4), \mathfrak{osp}(4|4))$ it must have chiral pair structures. As a consequence the excitation K-matrix must have the same chiral pair structure, too. For the AdS_5/CFT_4 S-matrix, there are factorisable and non-factorisable K-matrices which have chiral and achiral pair structures, respectively. Thus we have to use the factorisable K-matrix which have residual symmetry $\mathfrak{osp}(2|2) \oplus \mathfrak{osp}(2|2)$. This is exactly what we determined in section 5. This argument is used the integrability requirement only. Let us continue now with the explicit symmetries of the top level excitations.

The bosonic symmetries of the D5 defect are $\mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where $\mathfrak{so}(5)$ comes form the conformal while $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ from the R-symmetry. Let us continue with the symmetries of the excitations over the pseudo-vacuum $\mathrm{Tr} Z^L$. This pseudo-vacuum breaks the conformal symmetry to Lorentz symmetry $\mathfrak{so}(1,3) \equiv \mathfrak{so}(4)$ and the R-symmetry to $\mathfrak{so}(4)$. The D5 defect breaks the Lorentz symmetry to $\mathfrak{so}(1,2) \equiv \mathfrak{su}(2)$ which is the diagonal algebra of the Lorentz algebra $\mathfrak{so}(1,3) \equiv \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The residual R-symmetry depends on the orientation D-brane. It can be $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$, $\mathfrak{so}(3)$ or $\mathfrak{so}(2)$. Clearly only the residual R-symmetry $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ is consistent with the symmetry algebra $\mathfrak{osp}(2|2) \oplus \mathfrak{osp}(2|2)$ which

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is obtained from the integrability argument. We can see however that the residual Lorentz symmetry, the diagonal $\mathfrak{su}(2)$, is not factorized as it is required. This problem is similar to choosing a pseudo vacuum with a non-proper direction (see subsection 6.3.1). The problem can be cured by global conformal transformations. From symmetry argumentations we saw that if and integrable K-matrix exists then the defect must respect the full Lorentz symmetry, therefore we have to use a symmetry transformation which transforms the plane defect to a spherical one. This can be done by using special conformal transformations.

Now let us focus on the tree level overlaps. In the SYM theory with defects some of the scalar fields require nonzero vacuum expectation values. In the simplest case they are

$$\phi_i(x) = \frac{\sigma_i}{x_4}\,; \quad i = 1, 2, 3 \qquad\qquad \phi_i(x) = 0\,; \quad i = 4, 5, 6. \tag{7.1}$$

At tree level the excitation K-matrix in the $\mathfrak{so}(2,4)$ sector is zero and the K-matrix of the $\mathfrak{so}(6)$ sector is given in (6.29). In the inner boundary space in which the Pauli matrices act it is diagonal, thus a direct sum of two scalar K-matrices. In order to get these scalar K-matrices we have to put together the $\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$ subsectors. This means we have to find rational scalar $\mathfrak{su}(2|2)$ K-matrices with bosonic symmetry $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. This symmetry constrains the matrix structure as

$$\begin{pmatrix} 1 & B & 0 & 0 \\ -B & 1 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & -A & 0 \end{pmatrix}; \qquad \begin{array}{l} A = 0 \\ B = \frac{c}{i-2u} \end{array}. \tag{7.2}$$

Since the tree level K-matrix is zero at the $\mathfrak{so}(2,4)$ sector, we have to choose $A = 0$. Substituting to the KYBE, we obtain that $B(u) = \frac{c}{i+2u}$ is the most general solution. From comparing to (6.29) we have to choose $c = \pm 1$, where the signs are related to the inner degrees of freedom. In extending this tree level K-matrix for AdS/dCFT at all loops we have to choose a solution of the KYBE equation of the form:

$$K(p) = \begin{pmatrix} k_1(p) & k_2(p,s) & 0 & 0 \\ k_2(p) & k_4(p,s) & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{p}{2}} \\ 0 & 0 & -e^{i\frac{p}{2}} & 0 \end{pmatrix}; \qquad \begin{array}{l} k_1 = k_4 = \frac{s x^+(1+x^+x^-)}{x^-(s^2+(x^+)^2)} \\ k_2 = \frac{x^+(s^2x^- - x^+)}{x^-(s^2+(x^+)^2)} \\ k_3 = \frac{x^+(x^+ - s^2x^-)}{x^-(s^2+(x^+)^2)} \end{array} \tag{7.3}$$

Indeed, choosing $s = \pm\frac{1}{2}g^{-1} + O(1)$ and expanding at weak coupling we reproduce the tree level result. Unfortunately we cannot see at the moment how the function $s(g)$ could be fixed from some symmetry argumentations. One possible way is to compare the all loop asymptotic overlaps with explicit string theory or higher loop YM calculations.

The full $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$ scalar K-matrix can be written as a tensor product of two identical copies of (7.3) as written in (5.6). The full transfer matrix also has a factorized form (3.12) and each factor can be diagonalized independently. The eigenvalues can be written in terms of Bethe roots (3.13). We denote the Bethe roots of the left wing by $\mathbf{y}^{(1)}, \mathbf{w}^{(1)}$, while that of the right wing by $\mathbf{y}^{(2)}, \mathbf{w}^{(2)}$. Within each wing they all satisfy their own Bethe ansatz equations (6.38). The full transfer matrix eigenvalue depends additionally on the physical momenta which connect the Bethe roots on the two sides by the middle node BA equation

$$e^{i\phi_{p_j}} = 1 = e^{ip_j(L-N+\frac{M_1+M_2}{2}+1)} \prod_{i=1;i\neq j}^{N} e^{ip_i} \frac{x_j^+ - x_i^-}{x_j^- - x_i^+} \frac{1-\frac{1}{x_j^+ x_i^-}}{1-\frac{1}{x_j^- x_i^+}} \sigma(p_j,p_i)^2 \prod_{\nu=1}^{2} \prod_{k=1}^{M_1} \frac{x_j^- - y_k^{(\nu)}}{x_j^+ - y_k^{(\nu)}}, \quad \text{for all} \quad j=1,\dots,N. \tag{7.4}$$

where the number of p, $y^{(\alpha)}, w^{(\alpha)}$ variables are N, M_α, K_α, respectively [25]. Let us denote the eigenvector of the full transfer as $|\mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\rangle$. Based on the overlap formula we obtained in the previous section (6.37) we conjecture the overlap for the full spectrum to take the form

$$\begin{aligned} &\frac{\left|\langle\Psi|\ \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\rangle\right|^2}{\langle\mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\ |\mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\rangle} \qquad (7.5)\\ &= \prod_{i=1}^{N/2} |K_0(p_1)|^2 |k_1(p_i)|^4 \prod_{\nu=1}^{2} \prod_{i=1}^{M_\nu/2} \frac{-((y_i^{(\nu)})^2+s^2)^2}{s^2(y_i^{(\nu)})^2} \prod_{i=1}^{\lfloor M_\nu/4\rfloor} \frac{1}{(w_i^{(\nu)})^2\left((w_i^{(\nu)})^2+\frac{1}{4g^2}\right)} \frac{G_+}{G_-} \end{aligned}$$

where the new determinants involve differentiation w.r.t. the momenta, too:

$$G_\pm = \left(\partial_{\hat{U}_i^+}\phi_{U_j^+} \pm \partial_{\hat{U}_i^+}\phi_{U_j^-}\right) \tag{7.6}$$

Here $\mathbf{U}^+ = \left\{\mathbf{p}^+, \mathbf{v}^{(\alpha)+}, \mathbf{w}^{(\alpha)+}\right\}$ collects all the variables and $\hat{\mathbf{U}} = \left\{\mathbf{u}^+, \hat{\mathbf{w}}^{(\alpha)+}, \hat{\mathbf{w}}^{(\alpha)+}\right\}$ is the collection of the properly normalized rapidities. In particular u_i is the rapidity parameter belonging to p_i. For simplicity we assumed that K_1, K_2 are even.

This is the contribution of the scalar K-matrices, out of which we have two in the simplest case. If their s-parameter is the opposite of each other their contributions just double.

8 Conclusions

In this paper we analyzed integrable boundary states, overlaps, nesting and their applications in bootstrapping the simplest asymptotic all loop 1-point functions in AdS/dCFT. We started by formulating the YBE for the boundary state (KYBE) which can annihilate pairs of particles corresponding to different representations. We called this boundary state twisted in order to distinguish from the one which annihilates particles in the same representation, which is called untwisted. This twisting is related to a symmetry of the Dynkin diagram of the full symmetry, which can be charge conjugation (exchanging representation with contragradient representation) or some other involutions of the algebra. We then showed that for each solution of the KYBE one can associate a reflection matrix in the mirror theory, which solves the boundary YBE. Since the crossing involves a charge conjugation the BYBE is untwisted only if the conjugated twisted transformation is a trivial one, otherwise it is twisted.

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We then turned to the investigation of the overlap of the finite size boundary state with a periodic multiparticle state, which is an eigenstate of the transfer matrix. Eigenvalues and eigenvectors can be formulated in terms of Bethe roots, which satisfy the Bethe equations, following from the regularity of the transfer matrix. The usual definition of the integrable finite size boundary states demands that the difference of the transfer matrix and the parity transformed transfer matrix annihilates the boundary state. We used this definition to derive a pair structure between Bethe roots. In doing so we observed that the definition does not fix uniquely the type of the pair structure. For theories with extra symmetries, corresponding to symmetries of the Dynkin diagram, two different pair structures are allowed. If the Bethe roots are paired within each type we called the overlap chiral. If however, different Bethe roots, related by the Dynkin symmetry were paired, we called the overlap achiral.

We then focused on boundary states which are built up from K-matrices, solutions of the KYBE. We could relate the chirality of the overlap to the twisted nature of the KYBE, i.e. twisted KYBE leads to achiral overlaps, while untwisted equations to chiral ones. The chirality property of the overlap can be related to the unbroken symmetries. These symmetries are the same for K-matrices and the corresponding mirror reflections, and has been classified for spin chains. It was found that the residual symmetry of the BYBE together with the symmetry of the bulk theory must form a symmetric pair. These symmetric pairs are classified and they all are related to involutions of the algebra. It was found that for inner involutions the BYBE equations are not twisted, while for outer involutions they are twisted and the twist is related to a nontrivial symmetry of the Dynkin diagram. We then analyzed in detail how the twisted nature of the BYBE, can be related to the twisted nature of the KYBE. As a result we could relate the chirality of the overlap to the type of the unbroken symmetry. This was all crucial for formulating the nesting program for overlaps. This analysis was for spin chains, but we wanted also to see how it extends to AdS/CFT. For this reason we determined the most general solution of the KYBE for the centrally extended $\mathfrak{su}(2|2)_c$ scattering matrix. We found that it has three parameters, out of which two can be transformed away. The corresponding solution had an $\mathfrak{os}(2|2)_c$ symmetry, which together with $\mathfrak{su}(2|2)_c$ formed a symmetric pair, the only one of this kind.

The next step was the calculation of the overlap formulas. In doing so we suggested a completely new and original way how nesting could be used for overlaps and K-matrices. The framework was the nested Bethe ansatz, in which at each step a pseudo vacuum is chosen and excitation with smaller symmetry and reduced scattering matrix is identified. We followed the same idea and defined the nested K-matrices by the overlap of the infinite volume two particle state with the boundary state. This two particle state was a coordinate space BA eigenstate of the nested excitations. In choosing the right pseudo vacuum and excitations the residual symmetries played a crucial role. Indeed, the boundary state determines the symmetries of the problem, which fixes the chirality of the overlaps. At each step we have to choose such a pseudo vacuum whose symmetry is in the same chirality class. We tested these ideas for various spin chains relevant also for 1-point functions for $D5$ branes in AdS/dCFT. We then carried out this program for the newly calculated $\mathfrak{su}(2|2)_c$ K-matrix.

As an application of our results we investigated the symmetries of various D-branes in the AdS/dCFT setting. We found that the D5 brane have the chance to be integrable at leading order in the coupling for the whole theory, not only for a subsector. We identified the leading order K-matrix, in case of the simplest defect with matrix product states of Pauli matrices. The symmetry investigations suggested that in order the whole theory be integrable at finite couplings the symmetry related to the Lorentz transformations has to be enhanced. We speculated about a mechanism, how this can happen. If it happens our proposal describes the asymptotic 1-point functions for any couplings and any sector of the theory. We thus spelled out our conjectured overlaps corresponding to this K-matrix in terms of the Bethe roots of the $\mathfrak{su}(2|2)_c$ algebra together with the ratio of Gaudin determinants. We tested this formula for various sizes and Bethe roots and found convincing evidence of its correctness.

Our solution for the K-matrix and overlaps is just the first step in solving completely the 1-point functions in AdS/dCFT. First of all we found a free parameter in the solution which should be related to some physical data. Also we did not calculate the prefactor of the solution, which should be fixed from unitarity of the mirror reflection factor and crossing symmetry of the K-matrix. In order to perform these tasks many new data are needed for AdS/dCFT both from the weak and strong coupling type. They should extend the available overlaps both in the coupling and also for larger sectors of the theory.

The situation we analyzed was the simplest possible boundary state as far as the boundary degrees of freedom was concerned. It would be interesting to understand how it could be extended for higher dimensional representations and boundary spaces. Particularly interesting problem is the calculation of the overlaps from the CFT side. Especially the D7 brane, which seems to be integrable at 1 loop for the subsector considered so far, but does not seem to be integrable from our point of view.

Our analysis provides an asymptotic overlap, which does not include finite size wrapping effects. These finite size effects are due to virtual particles and can be dealt with using the thermodynamic Bethe ansatz [5, 6]. Recent works [34–37] on excited state g-functions can help to include such corrections also in our case.

Our developments are relevant also for spin chains. The new nesting procedure we initiated was tested only in a few examples. It would be very nice to extend systematically the calculations for any algebras and for all boundary conditions.

Note added. While our draft was finalized the paper [38] appeared on the arxiv, with partially overlapping results. The paper [38] has two parts: the first calculates the 1-point functions of BPS operators for the D5 system using supersymmetric localization. The second deals with non-BPS operators in the bootstrap setting. By identifying the unbroken symmetries the authors calculate the solution of the KYBE, which is relevant for the defect problem and use crossing symmetry and unitarity to fix the scalar factor. In order to apply for AdS/dCFT and to fix the CDD ambiguity they analyze the asymptotic overlaps in the $\mathfrak{su}(2)$ sector. By comparing with 1-loop calculations they fix the scalar factor and use excited boundary states to describe $\mathfrak{su}(2)$ overlaps for a large class of defects.

Our paper analyzed AdS/dCFT and spin chains in the same time, and formulated a nesting program how the generic overlaps can be calculated. We found two types of

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solutions of the KYBE, and calculated the generic asymptotic overlaps for all sectors, which is relevant for the simplest AdS/dCFT setting.

We are overlapping with [38] with the calculation of the K-matrix from symmetries. Indeed our K-matrix (5.31) is equivalent to their (4.30), once we change the sign of the momentum originating from a different definition of the S-matrix and take into account that their S-matrix uses a different phase factor for bosons.

All other results we obtained are independent. Actually their results nicely extends ours by determining the scalar factor and by fixing our free parameter in the solution. By combining the two results the asymptotic all loop 1-point functions for all sectors are available now.

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A Notations and conventions for $\mathfrak{su}(2|2)_c$

In this appendix we summarize in a selfcontained way the notations and convention we used for the AdS_5/CFT_4 integrable model thorough the paper. This model has an $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c$ symmetry, where the centrally extended $\mathfrak{su}(2|2)_c$ algebra is defined by the relations

$$\begin{aligned}
&\left[\mathbb{R}_a{}^b, \mathbb{J}_c\right] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c, && \left[\mathbb{L}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma,\\
&\left[\mathbb{R}_a{}^b, \mathbb{J}^c\right] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c, && \left[\mathbb{L}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma,\\
&\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}, && \left\{\mathbb{Q}_a^{\dagger\,\alpha}, \mathbb{Q}_b^{\dagger\,\beta}\right\} = \epsilon_{ab}\epsilon^{\alpha\beta}\mathbb{C}^\dagger,\\
&\left\{\mathbb{Q}_\alpha{}^b, \mathbb{Q}_a^{\dagger\,\beta}\right\} = \delta_\alpha^\beta \mathbb{R}_a{}^b + \delta_a^b \mathbb{L}_\alpha{}^\beta + \frac{1}{2}\delta_\alpha^\beta\delta_a^b \mathbb{H}
\end{aligned} \tag{A.1}$$

and the central charges are related to the worldsheet momenta as $\mathbb{C}=\mathbb{C}^\dagger=ig\left(e^{-i\mathbb{P}/2}-e^{i\mathbb{P}/2}\right)$. The algebra form a bialgebra with the following co-product:

$$\Delta(\mathbb{J}) = \mathbb{J}\otimes\mathbb{U}^{-[\mathbb{J}]} + \mathbb{U}^{[\mathbb{J}]}\otimes\mathbb{J} \tag{A.2}$$

where $\mathbb{U} = e^{i\mathbb{P}/4}$ and $\left[\mathbb{R}_a{}^b\right] = \left[\mathbb{L}_\alpha{}^\beta\right] = [\mathbb{H}] = 0$, $[\mathbb{Q}_\alpha{}^a] = 1$, $\left[\mathbb{Q}_a^{\dagger\,\beta}\right] = -1$, $[\mathbb{C}] = 2$, $\left[\mathbb{C}^\dagger\right] = -2$. Let us choose the parameterization of the defining representation $\mathcal{V}(p)$ as

$$\begin{aligned}
&\mathbb{Q}_\alpha{}^a\left|e_b\right\rangle = a(p)\delta_b^a\left|e_\alpha\right\rangle, && \mathbb{Q}_a^{\dagger\,\alpha}\left|e_b\right\rangle = c(p)\epsilon_{ab}\epsilon^{\alpha\beta}\left|e_\beta\right\rangle,\\
&\mathbb{Q}_\alpha{}^a\left|e_\beta\right\rangle = b(p)\epsilon_{\alpha\beta}\epsilon^{ab}\left|e_b\right\rangle, && \mathbb{Q}_a^{\dagger\,\alpha}\left|e_\beta\right\rangle = d(p)\delta_\beta^\alpha\left|e_a\right\rangle
\end{aligned} \tag{A.3}$$

where bosonic labels are $a, b = 1, 2$, fermionic ones are $\alpha, \beta = 3, 4$ and we take the symmetric choice

$$\begin{aligned}
&a(p) = d(p) = \eta(p) = \sqrt{ig(x^-(p) - x^+(p))},\\
&b(p) = -\frac{1}{x^-(p)}e^{-ip/2}\eta(p) = c(p) = -\frac{1}{x^+(p)}e^{ip/2}\eta(p)
\end{aligned} \tag{A.4}$$

with

$$x^{\pm}(p) = e^{\pm i\frac{p}{2}} \frac{1+\sqrt{1+16g^2\sin^2\frac{p}{2}}}{4g\sin\frac{p}{2}} \tag{A.5}$$

and g is related to the t' Hooft coupling of the $\mathcal{N}=4$ SYM theory as $g=\frac{\sqrt{\lambda}}{4\pi}$. The dispersion relation follows from the algebra

$$e^{ip} = \frac{x^+(p)}{x^-(p)}\,; \quad E(p) = ig\left(x^-(p) - \frac{1}{x^-(p)} - x^+(p) + \frac{1}{x^+(p)}\right) = \sqrt{1+16g^2\sin^2\frac{p}{2}} \tag{A.6}$$

By demanding that the fundamental S-matrix $S(p_1,p_2): \mathcal{V}(p_1)\otimes\mathcal{V}(p_2) \to \mathcal{V}(p_2)\otimes\mathcal{V}(p_1)$ commutes with the conserved charges

$$\Delta(\mathbb{J})S(p_1,p_2) = S(p_1,p_2)\Delta^{op}(\mathbb{J}) \tag{A.7}$$

for all $\mathbb{J}\in\mathfrak{su}(2|2)_c$ we can obtain [21]:

$$\begin{aligned}
&S^{aa}_{aa} = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,; \qquad S^{ba}_{ab} = -\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,; \qquad S^{ab}_{ab} = S^{aa}_{aa} - S^{ba}_{ab} \\
&S^{\alpha\alpha}_{\alpha\alpha} = -1\,; \qquad S^{\beta\alpha}_{\alpha\beta} = -\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\,; \qquad S^{\alpha\beta}_{\alpha\beta} = S^{\alpha\alpha}_{\alpha\alpha} - S^{\beta\alpha}_{\alpha\beta} \\
&S^{a\alpha}_{a\alpha} = \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde\eta_1}\,; \quad S^{\alpha a}_{a\alpha} = \frac{x_2^- - x_2^+}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde\eta_2}\,; \quad S^{a\alpha}_{\alpha a} = \frac{x_1^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_2}{\tilde\eta_1}\,; \quad S^{\alpha a}_{\alpha a} = \frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\frac{\eta_2}{\tilde\eta_2} \\
&S^{\alpha\beta}_{ab} = -\epsilon_{ab}\epsilon^{\alpha\beta}\frac{ix_1^- x_2^-(x_2^+ - x_1^+)\eta_1\eta_2}{x_1^+ x_2^+(x_2^+ - x_1^-)(1 - x_1^- x_2^-)}\,; \qquad S^{ab}_{\alpha\beta} = -\epsilon_{\alpha\beta}\epsilon^{ab}\frac{i(x_1^+ - x_1^-)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_2^+ - x_1^-)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}
\end{aligned} \tag{A.8}$$

where $a,b=1,2$; $a\neq b$, while $\alpha,\beta=3,4$; $\alpha\neq\beta$ and we have streamlined the notations as $S^{aa}_{aa}\equiv S^{aa}_{aa}(p_1,p_2)$, $x_i^{\pm}=x^{\pm}(p_i)$ and $\eta_1 = e^{ip_1/4}e^{ip_2/2}\eta(p_1)$, $\eta_2 = e^{ip_2/4}\eta(p_2)$, $\tilde\eta_1 = e^{ip_1/4}\eta(p_1)$ and $\tilde\eta_2 = e^{ip_1/2}e^{ip_2/4}\eta(p_2)$.

The energy and momentum can be parametrized in terms of the torus variable z using Jacobi elliptic functions:

$$p(z) = 2\mathrm{am}(z,k)\,; \qquad E(z) = \mathrm{dn}(z,k)\,; \qquad k=-16g^2 \tag{A.9}$$

where the rapidity torus has two periods $2\omega_1 = 4K(k)$ and $2\omega_2 = 4iK(1-k)-4K(k)$, with $K(k)$ being the elliptic K function. Crossing transformation and reflections are easy to implement on the torus:

$$x^{\pm}(z+\omega_2) = \frac{1}{x^{\pm}(z)}\,; \qquad x^{\pm}(-z) = -x^{\mp}(z) \tag{A.10}$$

where explicitly

$$x^{\pm}(z) = \frac{1}{2g}\left(\frac{\mathrm{cn}z}{\mathrm{sn}z} \pm i\right)(1+\mathrm{dn}z) \tag{A.11}$$

In the weak coupling limit the spectral parameter is also useful. It is defined as

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g} \tag{A.12}$$

$\mathfrak{su}(N)$			$\mathfrak{so}(2N)$		
residual symmetry	type	reflection	residual symmetry	type	reflection
$\mathfrak{su}(M)\oplus\mathfrak{su}(N-M)\oplus\mathfrak{u}(1)$	regular	untwisted	$\mathfrak{so}(2m)\oplus\mathfrak{so}(2N-2m)$	regular	untwisted
$\mathfrak{so}(N)$	special	twisted	$\mathfrak{u}(N)$	regular	untwisted
$\mathfrak{sp}(N)$	special	twisted	$\mathfrak{so}(2m+1)\oplus\mathfrak{so}(2N-2m-1)$	special	twisted

Table 4. Classification of the solution of the BYBEs together with their symmetries.

Thus we can define $x(u)$ such that

$$x^{\pm}(u)=x\left(u\pm\frac{i}{2g}\right) \tag{A.13}$$

All of the three: the momentum, p, the spectral parameter u or the torus rapidity z can be used to label the particle's representation.

B KYBE, symmetries and selection rules for spin chains

In this appendix we focus on rational spin chains of type $\mathfrak{su}(N)$ and $\mathfrak{so}(2N)$ and investigate the relations between the type of the residual symmetries, YBE and chirality of the overlaps. The R-matrix of the $\mathfrak{su}(N)$ model involves the identity and the permutation (3.3), while that of the $\mathfrak{so}(2N)$ contains additionally the trace operator (3.10). There are two types of BYBEs: the untwisted one

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v)=K_2(v)R_{12}(u+v)K_1(u)R_{12}(u-v) \tag{B.1}$$

which exist both for $\mathfrak{su}(N)$ and $\mathfrak{so}(2N)$ and the twisted one

$$R_{12}(u-v)K_1(u)\bar{R}_{12}(u+v)K_2(v)=K_2(v)\bar{R}_{12}(u+v)K_1(u)R_{12}(u-v) \tag{B.2}$$

which exist only for the $\mathfrak{su}(N)$ R-matrix. Here $\bar{R}_{12}(u)=R_{\bar{1}2}(u)=R_{1\bar{2}}(u)$ describes the scattering of the fundamental representation and the antifundamental and $R_{12}(u)=R_{\bar{1}\bar{2}}(u)$. The classification of the solutions of these equations together with the type of the residual symmetry is shown in table 4.

In the following we investigate the boundary states belonging to these BYBEs.

B.1 Connection between the reflection and KYBE for rational spin chains

The K-matrices can be used to define integrable two-site states:

$$\langle\Psi|=\langle\psi|^{\otimes L/2}\,;\qquad \langle\psi|=\langle a|\otimes\langle b|\,K^{ab}(0). \tag{B.3}$$

If the K- matrix is not twisted the KYBE ensures that the boundary state satisfies the following requirement

$$\langle\Psi|\,K_0(u)T_0(u)=\langle\Psi|\,\Pi T_0(u)\Pi K_0(u)\,;\qquad T_0(u)=R_{0L}(u)R_{0,L-1}(u)\dots R_{02}(u)R_{01}(u) \tag{B.4}$$

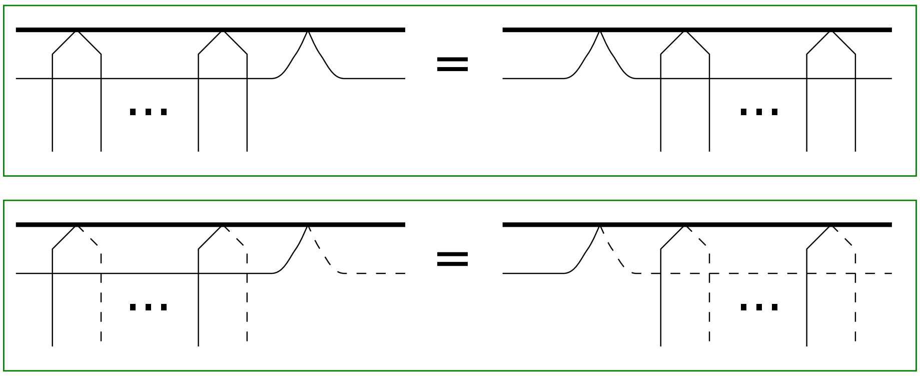

Figure 5. The proof of (B.4).

If however, the K-matrix is a twisted one we have to introduce a spin chain with alternating inhomogeneities. In this spin chain we can introduce two monodromy matrices

$$T_0(u) = \bar{R}_{0L}(u)R_{0,L-1}(u)\dots\bar{R}_{02}(u)R_{01}(u), \tag{B.5}$$

$$\bar{T}_0(u) = R_{0L}(u)\bar{R}_{0,L-1}(u)\dots R_{02}(u)\bar{R}_{01}(u), \tag{B.6}$$

such that the boundary state satisfies the relation

$$\langle\Psi|\, K_0(u)T_0(u) = \langle\Psi|\, \Pi\bar{T}_0(u)\Pi K_0(u) \tag{B.7}$$

The proof of (B.4) and (B.7) is shown in figure 5. From (B.4) and (B.7), the following integrability conditions follow

$$\text{untwisted: } \langle\Psi|\, t(u) = \langle\Psi|\, \Pi t(u)\Pi\,; \qquad \text{twisted: } \langle\Psi|\, t(u) = \langle\Psi|\, \Pi\bar{t}(u)\Pi$$

The untwisted case is simpler. It is a natural assumption that the space reflection exchanges the sign of the Bethe roots $\Pi\left|\mathbf{u}^{(a)}\right\rangle = \left|-\mathbf{u}^{(a)}\right\rangle$. This assumption was shown in (4.1) from the explicit form of the transfer matrices' eigenvalues. The non-vanishing overlap implies that the eigenvalues satisfy

$$\Lambda\left(u|\mathbf{u}^{(a)}\right) = \Lambda\left(u|-\mathbf{u}^{(a)}\right)\;; \qquad t(u)\left|\mathbf{u}^{(a)}\right\rangle = \Lambda\left(u|\mathbf{u}^{(a)}\right)\left|\mathbf{u}^{(a)}\right\rangle. \tag{B.8}$$

The eigenvalue is symmetric in each type of Bethe roots therefore this equation implies

$$\mathbf{u}^{(a)} = \left\{u_1^{(a)}, -u_1^{(a)}, \dots, u_{N_a/2}^{(a)}, -u_{N_a/2}^{(a)}\right\} \tag{B.9}$$

the chirality of the overlaps.

There can be other symmetries of the transfer matrix eigenvalue beyond the permutation of the Bethe roots with the same type. These are the symmetries of the Dynkin diagram which leave invariant the representations of the quantum and auxiliary spaces. It can be illustrated with the Dynkin diagram which is decorated by the used representations. We use only the fundamental representations which is identified to one of the Dynkin node. The red note indicates the auxiliary representation. The additional black nodes are connected to the Dynkin nodes of the quantum spaces. The first box in figure 6 shows the

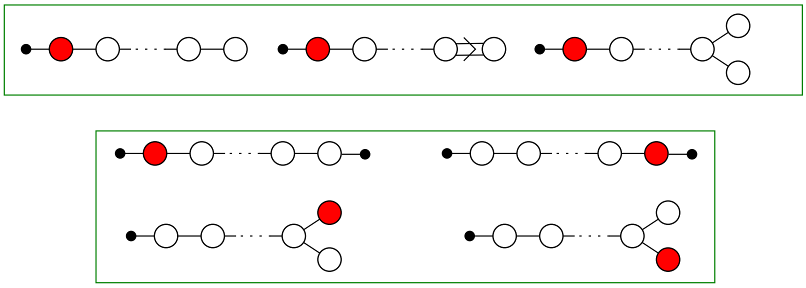

Figure 6. Decorated Dynkin diagrams.

decorated Dynkin diagrams of the untwisted cases. We can see that the $\mathfrak{su}(N)$ and the $\mathfrak{so}(2n+1)$ cases has no additional symmetry therefore the symmetric pairs

$$(\mathfrak{su}(N),\mathfrak{so}(N))\,;\qquad(\mathfrak{su}(2n),\mathfrak{sp}(2n))\,;\qquad(\mathfrak{so}(2n+1),\mathfrak{so}(M)\oplus\mathfrak{so}(2n+1-M))\tag{B.10}$$

has pair structure (B.9) i.e. these are chiral symmetric pairs.

In the case of the decorated $\mathfrak{so}(2n)$ diagram there is a symmetry which interchanges the roots $\mathbf{u}^{+}$ and $\mathbf{u}^{-}$, therefore for the $\mathfrak{so}(2n)$ spin chain there can be two types of pair structures

- chiral pair structure for which

$$\begin{aligned}\mathbf{u}^{(a)}&=\left\{u_1^{(a)},-u_1^{(a)},\dots,u_{m_a/2}^{(a)},-u_{m_a/2}^{(a)}\right\},\quad\text{for } a=1,\dots n-2\,;\\ \mathbf{u}^{(\pm)}&=\left\{u_1^{(\pm)},-u_1^{(\pm)},\dots,u_{m_\pm/2}^{(\pm)},-u_{m_\pm/2}^{(\pm)}\right\}.\end{aligned}\tag{B.11}$$

- achiral pair structure for which

$$\mathbf{u}^{(a)}=\left\{u_1^{(a)},-u_1^{(a)},\dots,u_{m_a/2}^{(a)},-u_{m_a/2}^{(a)}\right\},\qquad\text{for } a=1,\dots n-2\,;\qquad\mathbf{u}^{(+)}=-\mathbf{u}^{(-)}\tag{B.12}$$

Before we decide which pair structures belong to the concrete examples, let us continue with the twisted case for the $\mathfrak{su}(N)$ model, which belongs to the symmetric pair $(\mathfrak{su}(N),\mathfrak{su}(M)\oplus\mathfrak{su}(N-M)\oplus\mathfrak{u}(1))$. In this case the quantum space is an alternate tensor product of the particles and antiparticles and there are two transfer matrices $t(u)$ and $\bar{t}(u)$ where the auxiliary spaces are particle and antiparticle, respectively. They are diagonalizable simultaneously and let $\left|\mathbf{u}^{(a)}\right\rangle$ be a Bethe state for which

$$t(u)\left|\mathbf{u}^{(a)}\right\rangle=\Lambda\left(u|\mathbf{u}^{(a)}\right)\left|\mathbf{u}^{(a)}\right\rangle\,;\qquad\bar{t}(u)\left|\mathbf{u}^{(a)}\right\rangle=\bar{\Lambda}\left(u|\mathbf{u}^{(a)}\right)\left|\mathbf{u}^{(a)}\right\rangle.\tag{B.13}$$

The eigenvalue of the transfer matrices can be written as [23]

$$\Lambda(u)=\sum_{k=1}^{N}\frac{Q_{k-1}^{[k+1]}(u)}{Q_{k-1}^{[k-1]}(u)}\frac{Q_{k}^{[k-2]}(u)}{Q_{k}^{[k]}(u)}\,;\qquad\bar{\Lambda}(u)=\sum_{k=1}^{N}\frac{Q_{k-1}^{[N-k-1]}(u)}{Q_{k-1}^{[N-k+1]}(u)}\frac{Q_{k}^{[N-k+2]}(u)}{Q_{k}^{[N-k]}(u)},\tag{B.14}$$

where

$$Q_0(u) = Q_N(u) = u^{L/2}. \tag{B.15}$$

Using this explicit form we can show that

$$t(u)\Pi\left|\mathbf{u}^{(a)}\right\rangle = \Pi\bar{t}(-u-N/2)\left|\mathbf{u}^{(a)}\right\rangle = \bar{\Lambda}\left(-u-N/2|\mathbf{u}^{(a)}\right)\Pi\left|\mathbf{u}^{(a)}\right\rangle = \Lambda\left(u|-\mathbf{u}^{(a)}\right)\Pi\left|\mathbf{u}^{(a)}\right\rangle \tag{B.16}$$

therefore the parity transformation acts on the Bethe roots as $\Pi\left|\mathbf{u}^{(a)}\right\rangle = \left|-\mathbf{u}^{(a)}\right\rangle$. The non-vanishing overlap implies that

$$\Lambda\left(u|\mathbf{u}^{(a)}\right) = \bar{\Lambda}\left(u|-\mathbf{u}^{(a)}\right). \tag{B.17}$$

The extended Dynkin diagrams of these eigenvalues is in the first row of the second box of figure 6. Now there are two additional black nodes since there are particles and antiparticles in the quantum space. We can see that these Diagram is connected by the original Dynkin diagram isomorphism which transforms the Bethe roots as $\tilde{\mathbf{u}}^{(a)} = \mathbf{u}^{(N-a)}$ therefore it implies that

$$\bar{\Lambda}\left(u|\mathbf{u}^{(a)}\right) = \Lambda\left(u|\tilde{\mathbf{u}}^{(a)}\right) \quad\longrightarrow\quad \Lambda\left(u|\mathbf{u}^{(a)}\right) = \Lambda\left(u|-\tilde{\mathbf{u}}^{(a)}\right). \tag{B.18}$$

Since there is no symmetry of the extended diagram of Λ the sets $\mathbf{u}^{(a)}$, $-\tilde{\mathbf{u}}^{(a)}$ have to be the same which means that the symmetric pair $(\mathfrak{su}(N), \mathfrak{su}(M)\oplus\mathfrak{su}(N-M)\oplus\mathfrak{u}(1))$ belongs to an achiral pair structure i.e.

$$\mathbf{u}^{(a)} = \left\{+u_1^{(a)}, +u_2^{(a)}, \ldots, +u_{m_a-1}^{(a)}, +u_{m_a}^{(a)}\right\} = -\mathbf{u}^{(N-a)}, \qquad \text{for } a = 1, \ldots, \left\lfloor\frac{N-1}{2}\right\rfloor \tag{B.19}$$

and

$$\mathbf{u}^{(N/2)} = \left\{u_1^{(N/2)}, -u_1^{(N/2)}, \ldots, u_{m_{N/2}/2}^{(N/2)}, -u_{m_{N/2}/2}^{(N/2)}\right\}. \tag{B.20}$$

We can see that in the $\mathfrak{su}(N)$ case we could decide which symmetric pair is chiral or achiral because the auxiliary space representation breaks the symmetry of the original Dynkin diagram. We can do the same $\mathfrak{so}(2n)$ by choosing another auxiliary space, such as the spinor representations, as they are not invariant under the Dynkin diagram symmetry. The corresponding transfer matrices are denoted by $t^{(+)}(u)$ and $t^{(-)}(u)$. Using these matrices, we can define two integrability conditions

$$\langle\Psi|\, t^{(+)}(u) = \langle\Psi|\, \Pi t^{(+)}(u)\Pi, \tag{B.21}$$

$$\langle\Psi|\, t^{(+)}(u) = \langle\Psi|\, \Pi t^{(-)}(u)\Pi. \tag{B.22}$$

From the previous argument, we can see that the conditions (B.21) and (B.22) lead to chiral and achiral pair structures, respectively.

In figure 7 we can see the KYBEs for the K-matrices which solves the integrable conditions (B.21) and (B.22). The red and blue lines belongs two the spinor representations. The first line and second lines belong to (B.21) and (B.22), respectively. We recall that switching from reflection to K-matrices we have to use a conjugation, which changes the representations to their contragradients. We know that

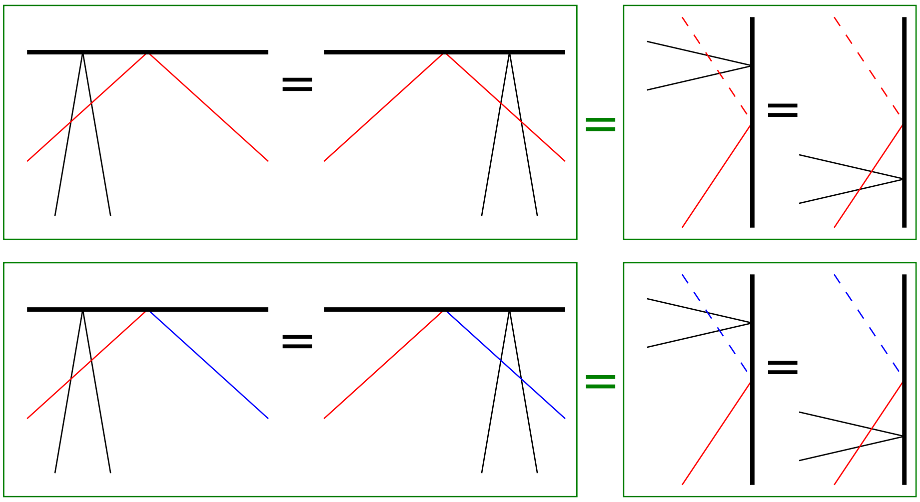

Figure 7. Reflection equations for $\mathfrak{so}(2n)$ model. Black line: vector rep. Red and blue line: spinor reps. Dashed lines: contragradient reps.

- For $\mathfrak{so}(4n)$, the spinor representations are pseudo-reals therefore the dashed red line is equivalent to the simple red line. Which means that the first and the second lines describe non representation changing and representation changing reflections, i.e. untwisted and twisted reflections, respectively.
- For $\mathfrak{so}(4n+2)$, the spinor representations are contragradients of each other therefore the dashed blue line is equivalent to the simple red line. Which means that the first and the second lines describe representation changing and non representation changing reflections, i.e. twisted and untwisted reflections, respectively.

The table 2 summarize the results of this section and naturally extends them for $\mathfrak{gl}(N|M)$.

C Calculation of two particle overlaps

In this appendix we elaborate on the calculation of the overlaps in the $L \to \infty$ limit using two particle coordinate Bethe ansatz state of section 6. Bethe ansatz states represent a plane wave containing two particles with momentum p and $-p$. We are not going to impose periodicity for the wave functions, thus our states are of-shell Bethe states. Actually in the $L \to \infty$ limit off-shell and on-shell states become equivalent.

C.1 XXX spin chain

Let us start with the general integrable two-site state of the XXX spin chain of size L. We take the boundary state with $\langle\Psi| = \langle\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}|$ and calculate the overlap of the integrable state $\langle\Psi|$ with a two magnon state, built over the pseudo-vacuum $|1\rangle^{\otimes L}$ in the large L limit. In coordinate space Bethe ansatz the two magnon state is a plane wave of the form

$$|u, -u\rangle = \sum_{n_1=1}^{L} \sum_{n_2=n_1+1}^{L} \left(e^{ip(n_1-n_2)} + e^{-ip(n_1-n_2)} S(2u) \right) |n_1 n_2\rangle \tag{C.1}$$

where$p=-i\log\frac{u-i/2}{u+i/2}$ and $S(u)=\frac{u+i}{u-i}$. The state $|n_1n_2\rangle$ has excitation 2 at sites n_1 and n_2. To obtain the overlap $\langle\Psi\,|u,-u\rangle$ we have to use the following elementary overlaps

$$\langle\Psi\,|n_1n_2\rangle=\begin{cases}\psi_{12}^2 & \text{if } n_1,n_2 \text{ are even}\\ \psi_{21}^2 & \text{if } n_1,n_2 \text{ are odd}\\ \psi_{12}\psi_{21} & \text{if } n_1 \text{ is even and } n_2 \text{ is odd}\\ \psi_{21}\psi_{12} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even and } n_2-n_1>1\\ \psi_{22} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even and } n_2-n_1=1\end{cases}\tag{C.2}$$

By plugging back these into the overlap $\langle\Psi\,|u,-u\rangle$ we obtain

$$\begin{aligned}\langle\Psi\,|u,-u\rangle=&\sum_{m_1=1}^{L/2}\sum_{m_2=m_1+1}^{L/2}\left(e^{ip(2m_1-2m_2)}+e^{-ip(2m_1-2m_2)}S(2u)\right)\psi_{12}^2+\\&+\sum_{m_1=1}^{L/2}\sum_{m_2=m_1+1}^{L/2}\left(e^{ip(2m_1-2m_2)}+e^{-ip(2m_1-2m_2)}S(2u)\right)\psi_{21}^2+\\&+\sum_{m_1=1}^{L/2}\sum_{m_2=m_1+1}^{L/2}\left(e^{ip(2m_1-2m_2+1)}+e^{-ip(2m_1-2m_2+1)}S(2u)\right)\psi_{12}\psi_{21}+\\&+\sum_{m_1=1}^{L/2}\sum_{m_2=m_1+1}^{L/2}\left(e^{ip(2m_1-2m_2-1)}+e^{-ip(2m_1-2m_2-1)}S(2u)\right)\psi_{21}\psi_{12}+\\&+\sum_{m=1}^{L/2}\left(e^{-ip}+e^{+ip}S(2u)\right)\psi_{22}.\end{aligned}\tag{C.3}$$

It is convenient to introduce the following quantity

$$\Sigma(p)=\sum_{m_1=1}^{L/2}\sum_{m_2=m_1+1}^{L/2}e^{ip(2m_1-2m_2)}.\tag{C.4}$$

with which the overlap (C.3) can be written as

$$\begin{aligned}\langle\Psi\,|u,-u\rangle=&\left(\Sigma(p)+\Sigma(-p)S(2u)\right)\left(\psi_{12}^2+\psi_{21}^2+\left(e^{ip}+e^{-ip}\right)\psi_{12}\psi_{21}\right)\\&+\frac{L}{2}\left(e^{-ip}+e^{+ip}S(2u)\right)\psi_{22}.\end{aligned}\tag{C.5}$$

C.2 SU(3) spin chain with SO(3) symmetry

We analyze a two site state and a matrix product state for these models.

C.2.1 Two-site state

Let the two-site state be

$$\langle\Psi|=\left(\langle1|\otimes\langle1|+\langle2|\otimes\langle2|+\langle3|\otimes\langle3|\right)^{\otimes L/2}\tag{C.6}$$

We choose the pseudo-vacuum as $|3\rangle^{\otimes L}$ and introduce excitations with labels 1 and 2. The two magnon state can be written as

$$|u,-u\rangle_{ab} = \sum_{n_1=1}^{L}\sum_{n_2=n_1+1}^{L}\left(e^{ip(n_1-n_2)}\,|n_1 n_2\rangle_{ab} + e^{-ip(n_1-n_2)}S_{ab}^{cd}(2u)\,|n_1 n_2\rangle_{cd}\right) \tag{C.7}$$

where $|n_1 n_2\rangle_{ab}$ represent state a at site n_1 and b at site n_2, while $a,b,c,d = 1,2$. The scattering matrix of the top level Bethe ansatz excitations is

$$S(u) = \frac{i}{u-i}\mathbf{1} + \frac{u}{u-i}\mathbf{P} \tag{C.8}$$

with normalization $S_{11}^{11} = 1$. We have to calculate the following scalar product

$$\langle\Psi\,|n_1 n_2\rangle_{ab} = \begin{cases}\delta_{ab} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even and } n_2 - n_1 = 1\\ 0 & \text{otherwise}\end{cases} \tag{C.9}$$

By dividing with the asymptotic norm of the states we obtain the K-matrix of the top level excitations:

$$K_{ab}^{(1)}(u) := \frac{1}{L}\langle\Psi\,|u,-u\rangle_{ab} = \frac{1}{2}\left(e^{-ip}\delta_{ab} + e^{ip}S_{ab}^{cc}(2u)\right) = \frac{u}{u-i/2}\delta_{ab} \tag{C.10}$$

C.2.2 Matrix product state with Pauli matrices

Let the MPS be

$${}^{\alpha,\beta}\langle\text{MPS}| = \left[(\langle 1|\,\sigma_1 + \langle 2|\,\sigma_2 + \langle 3|\,\sigma_3)^{\otimes L}\right]^{\alpha,\beta} \tag{C.11}$$

where $\alpha,\beta = 1,2$ are the "inner" indexes of the Pauli matrices. The pseudo vacuum is $|3\rangle^{\otimes L}$ and we calculate the overlap ${}^{\alpha,\beta}\langle\text{MPS}|\,u,-u\rangle_{ab}$. The elementary overlaps with the states $|n_1 n_2\rangle_{ab}$ can be written as

$${}^{\alpha,\beta}\langle\text{MPS}\,|n_1 n_2\rangle_{a,b} = \begin{cases}(\sigma_3\sigma_a\sigma_3\sigma_b)^{\alpha\beta} = -\,(\sigma_a\sigma_b)^{\alpha\beta} & \text{if } n_1, n_2 \text{ are even}\\ (\sigma_a\sigma_3\sigma_b\sigma_3)^{\alpha\beta} = -\,(\sigma_a\sigma_b)^{\alpha\beta} & \text{if } n_1, n_2 \text{ are odd}\\ (\sigma_3\sigma_a\sigma_b\sigma_3)^{\alpha\beta} = (\sigma_a\sigma_b)^{\alpha\beta} & \text{if } n_1 \text{ is even and } n_2 \text{ is odd}\\ (\sigma_a\sigma_b\sigma_3\sigma_3)^{\alpha\beta} = (\sigma_a\sigma_b)^{\alpha\beta} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even}\end{cases} \tag{C.12}$$

Let us introduce the following notation $F_{ab}^{\alpha\beta} = (\sigma_a\sigma_b)^{\alpha\beta}$. The full overlap thus can be written as

$$\begin{aligned}{}^{\alpha,\beta}\langle\text{MPS}\,|u,-u\rangle_{a,b} = &\left(\Sigma(p)F_{ab}^{\alpha\beta} + \Sigma(-p)S_{ab}^{cd}(2u)F_{cd}^{\alpha\beta}\right)\left(e^{ip} + e^{-ip} - 2\right) +\\ &+\frac{L}{2}\left(e^{-ip}F_{ab}^{\alpha\beta} + e^{+ip}S_{ab}^{cd}(2u)F_{cd}^{\alpha\beta}\right)\end{aligned} \tag{C.13}$$

This overlap is diagonal in α and β. The remaining components can be written after normalizing with L^{-1} as

$${}^{1,1}\langle\text{MPS}\,|u,-u\rangle_{a,b} = K_{ab}^{(1)+}(u)\,; \qquad {}^{2,2}\langle\text{MPS}\,|u,-u\rangle_{a,b} = K_{ab}^{(1)-}(u) \tag{C.14}$$

with

$$K^{(1)\pm}(u) = \frac{1}{u}\begin{pmatrix} u+\frac{i}{2} & \pm\frac{1}{2}\\ \mp\frac{1}{2} & u+\frac{i}{2}\end{pmatrix} = k^{(1)}(u)\psi^{(2)\pm}\,; \qquad k^{(1)}(u) = K_{11}^{(1)\pm}(u). \tag{C.15}$$

C.3 $\mathfrak{so}(6)$ spin chains with $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetry symmetry

The six dimensional one site Hilbert space is parametrized by ϕ_i, $i=1,\dots,6$ and we introduce the notation

$$Z=\frac{1}{\sqrt{2}}\left(\phi_5+i\phi_6\right)\,,\qquad \bar{Z}=\frac{1}{\sqrt{2}}\left(\phi_5-i\phi_6\right) \tag{C.16}$$

We are going to analyze a two site state and a matrix product state.

C.3.1 Two-site state

Let the two-site state be

$$\langle\Psi|=\left(Z\otimes Z+\bar{Z}\otimes\bar{Z}+\phi_1\otimes\phi_1-\phi_2\otimes\phi_2+\phi_3\otimes\phi_3-\phi_4\otimes\phi_4\right)^{\otimes L/2} \tag{C.17}$$

We choose the pseudo vacuum as $Z^{\otimes L}$ and then the two magnon state can be written as

$$|u,-u\rangle_{ab}=\sum_{n_1=1}^{L}\sum_{n_2=n_1+1}^{L}\left(e^{ip(n_1-n_2)}\left|n_1n_2\right\rangle_{ab}+e^{-ip(n_1-n_2)}S_{ab}^{cd}(2u)\left|n_1n_2\right\rangle_{cd}\right)+\delta_{ab}e(2u)\sum_{n=1}^{L}|n\rangle \tag{C.18}$$

where the excitations are labeled with $a,b,c,d=1,2,3,4$ and

$$S(u)=\frac{i}{u-i}\mathbf{1}+\frac{u}{u-i}\mathbf{P}-i\frac{u}{(u-i)(u+i)}\mathbf{K},\quad e(u)=-\frac{u}{u+i}, \tag{C.19}$$

while $|n\rangle=\left|Z\dots Z\bar{Z}Z\dots Z\right\rangle$. We have to calculate the following elementary scalar products

$$\langle\Psi\left|n_1n_2\right\rangle_{ab}=\begin{cases}F_{ab} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even and } n_2-n_1=1\\ 0 & \text{otherwise}\end{cases} \tag{C.20}$$

where

$$F=\operatorname{diag}(1,-1,1,-1) \tag{C.21}$$

Since $\langle\Psi\left|n\right\rangle=0$ the K-matrix of the top level $\mathfrak{so}(4)$ excitations is

$$K_{ab}^{(1)}(u)=\frac{1}{L}\left\langle\Psi\left|u,-u\right\rangle_{ab}=\frac{1}{2}\left(e^{-ip}F_{ab}+e^{ip}S_{ab}^{cd}(2u)F_{cd}\right)=\frac{u}{u-i/2}F_{ab} \tag{C.22}$$

C.3.2 Matrix product state with Pauli matrices

Let the MPS be

$$\begin{aligned}{}^{\alpha,\beta}\langle\mathrm{MPS}|&=\left[\left(\sqrt{2}\phi_1\sigma_1+\sqrt{2}\phi_3\sigma_2+\sqrt{2}\phi_5\sigma_3\right)^{\otimes L}\right]^{\alpha,\beta}\\&=\left[\left(\sqrt{2}\phi_1\sigma_1+\sqrt{2}\phi_3\sigma_2+(Z+\bar{Z})\sigma_3\right)^{\otimes L}\right]^{\alpha,\beta}.\end{aligned} \tag{C.23}$$

We take the pseudo vacuum and the excitations as before. The overlaps with the states $\left|n_1n_2\right\rangle_{ab}$ can be written similarly as (C.12) and for $|n\rangle$ we obtain

$${}^{\alpha,\beta}\langle\mathrm{MPS}\left|n\right\rangle=\delta^{\alpha\beta} \tag{C.24}$$

The full overlap can be written as

$$
\begin{aligned}
{}^{\alpha,\beta}\langle \mathrm{MPS}\,|u,-u\rangle_{a,b} = &\left(\Sigma(p)F^{\alpha\beta}_{ab} + \Sigma(-p)S^{cd}_{ab}(2u)F^{\alpha\beta}_{cd}\right)\left(e^{ip}+e^{-ip}-2\right)+ \\
&+\frac{L}{2}\left(e^{-ip}F^{\alpha\beta}_{ab}+e^{+ip}S^{cd}_{ab}(2u)F^{\alpha\beta}_{cd}\right)+L\delta_{ab}\delta^{\alpha\beta}e(2u).
\end{aligned}
\tag{C.25}
$$

This overlap is diagonal in α and β. The remaining components can be written again as

$$
{}^{1,1}\langle \mathrm{MPS}\,|n_1 n_2\rangle_{a,b} = K^{(1)+}_{ab}(u)\,; \qquad {}^{2,2}\langle \mathrm{MPS}\,|n_1 n_2\rangle_{a,b} = K^{(1)-}_{ab}(u) \tag{C.26}
$$

where the K-matrices can be written as

$$
K^{(1)\pm}(u) = \begin{pmatrix} \frac{u^2+iu-1/2}{u(u+i/2)} & 0 & \mp\frac{1}{u} & 0 \\ 0 & -\frac{u+i}{u+i/2} & 0 & 0 \\ \pm\frac{1}{u} & 0 & \frac{u^2+iu-1/2}{u(u+i/2)} & 0 \\ 0 & 0 & 0 & -\frac{u+i}{u+i/2} \end{pmatrix} \tag{C.27}
$$

C.4 Two-site state in $\mathfrak{su}(2|2)_c$ with $\mathfrak{osp}(2|2)_c$ symmetry

Let the parameters of the K-matrix be $k_1 = k_4 = s$, $k_2 = 0$ and the boundary state take the form

$$
\langle B| = \langle K(p_1)| \otimes \cdots \otimes \langle K(p_L)|\,; \qquad \langle K(p)| = K_{i,j}(p)\,\langle i| \otimes \langle j|\,\mathbb{I}_g. \tag{C.28}
$$

The pseudovacuum is $|1\rangle^{\otimes 2L}$ and excitations are labeled with $3, 4$ and 2. The two-particle Bethe state in coordinate space can be written as [32]

$$
\begin{aligned}
|y_1,y_2\rangle_{\alpha,\beta} = &\sum_{1\le n_1<n_2\le 2L} \Psi_{n_1}(y_1)\Psi_{n_2}(y_2)\,|n_1,n_2\rangle_{\alpha,\beta} - \Psi_{n_1}(y_2)\Psi_{n_2}(y_1)R^{\gamma\delta}_{\alpha\beta}(y_1,y_2)\,|n_1,n_2\rangle_{\gamma,\delta} \\
&+\epsilon_{\alpha\beta}\sum_{1\le n\le 2L}\Psi_n(y_1)\Psi_n(y_2)g_n(y_1,y_2)\,|n\rangle
\end{aligned}
\tag{C.29}
$$

where α, β=3,4, and R is the R-matrix of the XXX model

$$
R(y_1,y_2) = \frac{-i/g}{v_1-v_2-i/g}\mathbf{1} + \frac{v_1-v_2}{v_1-v_2-i/g}\mathbf{P}\,; \qquad v_i = y_i + 1/y_i \tag{C.30}
$$

The basis vectors are

$$
|n_1,n_2\rangle_{\alpha,\beta} = |1,\ldots,1,\alpha,1,\ldots 1,\beta,1\ldots,1\rangle\,,\,; \qquad |n\rangle = |1,\ldots,1,2,1,\ldots 1\rangle \tag{C.31}
$$

while the wave functions and S-matrices read as

$$
\Psi_n(y) = \psi_n(y)\prod_{k=1}^{n-1} S(y,x_k)\,; \qquad S(y,x_k) = \frac{y-x_k^+}{y-x_k^-}e^{-ip_k/2}, \tag{C.32}
$$

$$
g_n(y_1,y_2) = e^{-ip_n/2}\frac{y_1y_2 - x_n^+x_n^-}{y_1y_2x_n^-}\frac{y_1-y_2}{v_1-v_2-i/g}\,; \qquad \psi_n(y) = e^{-ip_n/4}\frac{y\sqrt{i(x_n^- - x_n^+)}}{y-x_n^-}. \tag{C.33}
$$

where $\mathbb{I}_g$ is the graded identity: the product of the permutation and the graded permutation. Graded permutation picks up a minus sign, whenever two fermions are interchanged. We normalize the K function as

$$\langle K(p)|\, K(p)\rangle = 1. \tag{C.34}$$

Let us assume that the overlap with a general Bethe state looks in the $L \to \infty$ limit looks like as[5]

$$\frac{|\langle B|\, \mathbf{p}, \mathbf{y}, \mathbf{w}\rangle|^2}{\langle \mathbf{p}, \mathbf{y}, \mathbf{w}\, |\mathbf{p}, \mathbf{y}, \mathbf{w}\rangle} = \prod_{i=1}^{L} h^p(p_i) \prod_{i=1}^{N/2} h^y(v_i) \prod_{i=1}^{M/2} h^w(w_i) \tag{C.35}$$

First we define the renormalized boundary state $\langle \bar{B}|$ for which the K-matrix is

$$\bar{K}(p) = \frac{K(p)}{K_{1,1}(p)}. \tag{C.36}$$

For this boundary state the overlap with pseudo-vacuum is 1 therefore

$$h^p(p) = |K_{1,1}(p)|^2 \tag{C.37}$$

and the remaining overlap is

$$\frac{|\langle \bar{K}|\, \mathbf{p}, \mathbf{y}, \mathbf{w}\rangle|^2}{\langle \mathbf{p}, \mathbf{y}, \mathbf{w}\, |\mathbf{p}, \mathbf{y}, \mathbf{w}\rangle} = \prod_{i=1}^{N/2} h^y(v_i) \prod_{i=1}^{M/2} h^w(w_i). \tag{C.38}$$

This is basically the nesting for the boundary states and overlaps. In the following let us use special inhomogeneities

$$p_{2k-1} = p, \qquad p_{2k} = -p, \qquad \text{for } k = 1, \dots, L. \tag{C.39}$$

as the boundary overlaps does not depend on the inhomogeneities. Due to the special form of the boundary state nonzero contributions comes only from states $n_1 = 2m - 1$ and $n_2 = 2m$ and n arbitrarily:

$$\begin{aligned} \langle \bar{B}\,|y, -y\rangle_{\alpha,\beta} = -\epsilon_{\alpha\beta} L K_{34}(p) \Bigg[& \psi_{2m-1}(y)\psi_{2m}(-y) S(-y, x_{2m-1}) \\ & + \frac{2v + \frac{i}{g}}{2v - \frac{i}{g}} \psi_{2m-1}(-y)\psi_{2m}(y) S(y, x_{2m-1}) \Bigg] \\ + \epsilon_{\alpha\beta} L K_{12}(p) \big[& \psi_{2m-1}(y)\psi_{2m-1}(-y) g_{2m-1}(y, -y) \\ & - \psi_{2m}(-y)\psi_{2m}(y) S(-y, x_{2m-1}) S(y, x_{2m-1}) g_{2m}(y, -y) \big] \end{aligned} \tag{C.40}$$

here we used that

$$S(y, x_{2k-1}) S(y, x_{2k}) S(-y, x_{2k-1}) S(-y, x_{2k}) = 1 \tag{C.41}$$

[5] One should actually put the ratio of Gaudin type determinant at each step of the nesting. In this appendix we do not write them out as they are irrelevant for nested K-matrices.

Dividing by the leading order norm of the state leads to

$$\frac{\left|\langle \bar{B}\,|y,-y\rangle\right|^2}{\langle y_1,y_2\,|y,-y\rangle_{\alpha,\beta}} = \frac{4g^2}{s^2}\frac{(y^2+s^2)^2}{y^2+4g^2(y^2+1)^2}\epsilon_{\alpha\beta} \tag{C.42}$$

and the boundary state at the nested level is the SU(2) dimer state, what we already know.

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