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Ward identity for loop level soft photon theorem for massless QED coupled to gravity

Sayali Atul Bhatkar

Indian Institute of Science Education and Research,
Homi Bhabha Rd, Pashan, Pune 411 008, India

E-mail: sayali014@gmail.com

Abstract: Motivated by Campiglia and Laddha [1], we show that the Sahoo-Sen soft photon theorem [2] for loop amplitudes is equivalent to an asymptotic conservation law. This asymptotic charge is directly related to the dressing of fields due to long range forces exclusively present in four spacetime dimensions. In presence of gravity, the new feature is that photons also acquire a dressing due to long range gravitational force and this dressing contributes to the asymptotic charge.

Keywords: Gauge Symmetry, Global Symmetries, Scattering Amplitudes

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1 Introduction and result

Asymptotic symmetries strongly constrain low energy physics of gauge theories [3–8]. Leading soft theorems are manifestations of asymptotic symmetries. Soft theorems are statements about universal properties of amplitudes in the limit when energy of some of the interacting massless particles is taken to be small [9–12].[1] The equivalence between the two was first demonstrated in the seminal paper [5]. Similar analysis was carried out for QED in [6–8]; it was shown that leading soft photon theorem is equivalent to the Ward identity of the so called large gauge transformations. Large gauge transformations constitute an infinite dimensional subgroup of U(1) gauge transformations.

Analogous investigations have been carried out to understand the possible symmetry origins of tree level subleading soft theorem. Ward identity corresponding to Low's subleading photon theorem has been studied in [14–16]. The symmetry underlying this Ward identity or its relation to U(1) gauge group is not clear. In [17], the authors proved an infinite hierarchy of asymptotic conservation laws for classical electromagnetism and showed

[1] Interested readers can look up the references of [2] and [13] for recent literature on Soft theorems.

that quantum version of the first of these laws is equivalent to Low's subleading soft photon theorem. The authors also provide evidence that suggests that this entire hierarchy is equivalent to the infinite hierarchy of tree level soft theorems proved in [18, 19]. Thus, tree level subleading soft theorems in QED can be related to asymptotic conservation laws though the question of existence of a well defined underlying symmetry still persists.

In this paper we are interested in studying the equivalence between soft theorems and asymptotic conservation laws in presence of loop corrections. The leading soft theorem is true to all loop orders and hence is an exact quantum statement. The Ward identity corresponding to large gauge transformations is also exact. Beyond the leading order, soft theorems receive non-trivial loop corrections in four spacetime dimensions as shown in [20–22]. A part of these loop corrections are divergent. In [23], the authors showed that these divergent terms can be absorbed by renormalising tree level Ward identity. In the seminal paper [2], the authors extended the regulating technique introduced in [24] and used it to show that loop effects lead to a new logarithmic soft theorem in four spacetime dimensions. Thus, the subleading soft theorem for loop amplitudes is very different from the tree level subleading soft theorem. This soft theorem is 1-loop exact.

A natural question arises at this point: is this soft theorem related to a new asymptotic symmetry? The first step in this direction was taken in [1]. The authors have provided evidence to show that the Sahoo-Sen soft theorem for massive scalar QED has an underlying conservation law. This is quite a remarkable result given the fact that the loop level soft factor has a very complicated structure [2]. The authors also established a correspondence between the loop level soft factor and the Fadeev-Kulish dressing of massive particles [26]. It must be however noted that the nature or existence of a well defined symmetry associated to this conservation law is not clear at this point.

In this paper, our aim is to show that the Sahoo-Sen soft photon theorem is equivalent to the asymptotic conservation law given in (2.20) for massless scalar QED in presence of dynamical gravity. Let us first quote the Sahoo-Sen soft photon theorem in presence of gravitational couplings and massless complex scalars [2]:

$$\mathcal{M}_{n+1}(p_i,k) = \frac{S_0}{\omega}\mathcal{M}_n(p_i) + S_{\log}\log\omega\mathcal{M}_n(p_i) + \dots\,,$$

here, $S_0 = \sum_i e_i \frac{\epsilon.p_i}{p_i.q}$ is the leading soft factor and

$$\begin{aligned} S_{\log} = {} & \frac{ig}{4\pi}\sum_{\substack{i,j;i\neq j\\ \eta_i\eta_j=1}} e_i\frac{\epsilon_\mu q_\rho}{p_i.q}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) - \frac{ig}{4\pi}\sum_i e_i\frac{\epsilon.p_i}{p_i.q}\sum_{j,\eta_j=1} q.p_j \\ & - \frac{1}{4\pi^2}\sum_{i,j;i\neq j} e_i\frac{\epsilon_\mu q_\rho}{p_i.q}\left[\frac{e_ie_j}{p_i.p_j}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) + g(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu)\log[p_i.p_j]\right]^{2} \\ & + \frac{g}{4\pi^2}\sum_i e_i\frac{\epsilon.p_i}{p_i.q}\sum_j q.p_j\log p_j.q. \end{aligned} \tag{1.1}$$

In above expression, ϵ is the polarisation vector for the soft photon and $k = \omega q$ is the

[2] Similar to the first electromagnetic term in this line, there could be a potential gravitational term:

$$-\frac{1}{4\pi^2}\sum_{i,j;i\neq j} g e_i\frac{\epsilon_\mu q_\rho}{p_i.q}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu)$$

that vanishes because of momentum conservation.

soft momentum. The indices i, j take values from 1 to n, where n is the number of hard particles. e_i, p_i denote the electric charge and momentum of i^{th} hard particle respectively. The momenta and charges are defined including η factors such that $\eta_i = 1(-1)$ for outgoing (incoming) particles. In above expression, we have introduced g to keep track of gravitational terms. We later set $g = 8\pi G$=1. An important point to note is that the presence of gravitational coupling modifies the soft photon theorem significantly.

We show that above soft theorem is equivalent to the asymptotic conservation law given in (5.2):

$$Q^{+}_{\text{1-loop}}[V^A_+]|_{\mathcal{I}^+_-} = Q^{-}_{\text{1-loop}}[V^A_-]|_{\mathcal{I}^-_+}. \tag{1.2}$$

$Q_{\text{1-loop}}$ gets contribution from dressing of free fields due to long range forces.

- The leading order dressing of massless scalar field is given by (3.3):

  $$\phi(x) = -\frac{i e^{ieA^1_r(\hat{x})\log r}}{8\pi^2 r}\int d\omega \left[b(\omega,\hat{x})e^{-i\omega u}e^{i\omega \log r \frac{h^1_{rr}(\hat{x})}{2}} - d^\dagger(\omega,\hat{x})e^{i\omega u}e^{-i\omega\log r\frac{h^1_{rr}(\hat{x})}{2}}\right].$$

  A^1_r defined in (2.8) is the electromagnetic dressing and h^1_{rr} defined in (2.9) is the gravitational dressing. The 1-loop charge receives contribution from both electromagnetic and gravitational dressing of massless scalar field. This contribution (given in (7.2)) can be schematically written as $Q_{\text{1-loop}} \sim \hat{S}_1[h^1_{rr} + A^1_r]$, where $\hat{S}_1$ closely resembles the tree level subleading soft operator.

- Photons also acquire gravitational dressing (4.10):

  $$A_\sigma(x) = -\frac{i}{8\pi^2 r}\int d\omega \left[a_\sigma(\omega,\hat{x})e^{-i\omega u}e^{i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}} - a^\dagger_\sigma(\omega,\hat{x})e^{i\omega u}e^{-i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}}\right].$$

  The leading order gravitational dressing factor i.e. the $\log r$ term is similar for both photon field and massless scalar field. The photon field acquires additional $\log\omega$ dressing and this additional dressing term contributes to the charge. This contribution (given in (7.2)) can be schematically written as $Q_{\text{1-loop}} \sim S_0 h^1_{rr}$, where S_0 is the leading soft factor.

- The two terms in the first line of (1.1) constitute the classical soft factor. The low energy expansion of classical radiative field is controlled by the classical soft factors [2, 29]. The first term in the first line is related to the '$\hat{S}_1 \overset{\text{class}}{h^1_{rr}}$'[3] term. This part of the charge is directly related to the asymptotic acceleration of massless scalar particles under the gravitational force. The second term in the first line is related to the '$S_0 \overset{\text{class}}{h^1_{rr}}$' term. This part of the charge corresponds to the late time acceleration of the soft photon under gravitational force. The last two lines are absent in soft classical radiation and represent purely quantum effects.

[3] $h^1_{rr} = \overset{\text{class}}{h^1_{rr}} + \overset{\text{quan}}{h^1_{rr}}$. Similarly $A^1_r = \overset{\text{class}}{A^1_r} + \overset{\text{quan}}{A^1_r}$.

- Let us switch off gravity for a moment and consider a purely electromagnetic setup. In [1], the authors discuss massive scalar particles in this setup. An interesting observation is that the classical part of the soft factor which is non zero for the massive case is absent for the massless case. This result comes out naturally from the charge perspective also. The expected classical contribution is '$\hat{S}_1 \overset{\text{class}}{A^1_r}$'. This classical mode i.e. $\overset{\text{class}}{A^1_r}$ given in (6.1) is trivial and there is no classical contribution to charge (in absense of gravity).

- It was noted in [2] that if we assume the momenta of the hard particles is $\mathcal{O}(\hbar^0)$, neither the classical nor the quantum soft factor has any power of $\hbar$. Thus, an intriguing aspect of the 'quantum' terms is that these terms are independent of $\hbar$. These terms do not trivially vanish in classical limit ($\hbar \to 0$). In [1], the authors pointed out that there is a discontinuity in the quantized photon field in the limit $\omega \to 0$ and derived the 'quantum' part of $\log \omega$ coefficient from this discontinuity. Classical solutions are continous in $\omega \to 0$. In our case, discontinuities of the quantum photon and graviton fields contribute to $\overset{\text{quan}}{A^1_r}$ and $\overset{\text{quan}}{h^1_{rr}}$ respectively. This discontinuity is absent for massless scalar field. The last two lines of (1.1) are obtained from $\hat{S}_1[\overset{\text{quan}}{h^1_{rr}} + \overset{\text{quan}}{A^1_r}]$ and $S_0 \overset{\text{quan}}{h^1_{rr}}$.

- For the massless case, the quantum contributions to the charge have divergent pieces arising from collinear configurations. Gravitational dressings of both massless scalar and photon fields have divergent pieces that cancel out in the total expression of the charge. The divergent part of electromagnetic dressing does not contribute to the charge. Thus, the charge is rendered finite.

2 Preliminaries

We consider a theory with a massless scalar ϕ minimally coupled to U(1) gauge field A_μ and gravitational field $g_{\mu\nu}$. So, our system is described by the action:

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + g^{\mu\nu}\left(D_\mu\phi\right)^*\left(D_\nu\phi\right) + \frac{1}{2}R\right], \tag{2.1}$$

where $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$ and $8\pi G = 1$.

We are interested in the asymptotic dynamics of above system. Massless particles end up at future null infinity ($r \to \infty$ with $t - r$ finite) which is represented as $\mathcal{I}^+$. To describe late time dynamics of massless fields, we need to use retarded co-ordinate system. The flat metric takes following form in this co-ordinate system ($u = t - r$):

$$ds^2 = -du^2 - 2dudr + r^2 2\gamma_{z\bar{z}}dzd\bar{z}; \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}.$$

We use $\hat{x}$ or $(z, \bar{z})$ interchangeably to describe points on S^2. An useful parametrisation of a 4 dimensional spacetime point is given by (Greek indices will be used to denote 4d

Cartesian components):

$$x^\mu = rq^\mu + ut^\mu, \quad q^\mu = (1, \hat{x}), \quad t^\mu = (1, \vec{0}). \tag{2.2}$$

Here, q^μ is a null vector that can be parameterised in terms of $(z, \bar{z})$ as

$$q = \frac{1}{1+z\bar{z}}\{1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\}.$$

Dynamics of scalar is given by

$$g^{\mu\nu} D_\mu D_\nu \phi(x) = 0. \tag{2.3}$$

Solution to this equation can be expanded around future null infinity. Using stationary phase approximation, we can obtain the leading order coefficient in asymptotic expansion for massless scalars. It is given by [6]

$$\phi(u, r, \hat{x}) = \frac{1}{r}\phi^1(u, \hat{x}) + \dots, \tag{2.4}$$

where

$$\phi^1(u, \hat{x}) = \frac{-i}{8\pi^2}\int d\omega [b(\omega, \hat{x}) e^{-i\omega u} - d^\dagger(\omega, \hat{x}) e^{i\omega u}]. \tag{2.5}$$

Next we turn to the gauge field. Choosing the covariant Lorenz gauge $\nabla_\mu A^\mu = 0$, Maxwell's equations reduce to

$$\Box A_\mu = -j_\mu + j_\mu^{\text{grav}}, \quad j_\mu = ie\left(\phi D_\mu \phi^* - \phi^* D_\mu \phi\right). \tag{2.6}$$

j_σ^{grav} represents the gravitational corrections and will be analysed in section 4. It should be noted that A_μ is used to denote the full solution including the homogenous part and inhomogenous terms coming from the U(1) current as well as gravitational coupling.

Using the fall offs given in (2.4) for massless scalars, we get following asymptotic behaviour for the current components:

$$j_u = \frac{j_u^2(u, \hat{x})}{r^2} + \dots, \quad j_A = \frac{j_A^2(u, \hat{x})}{r^2} + \dots, \quad j_r = \frac{j_r^4(u, \hat{x})}{r^4} + \dots, \quad (A = z, \bar{z}). \tag{2.7}$$

Above and henceforth, we denote the vector components on S^2 by capital latin alphabets. The asymptotic expansion of gauge field components that is consistent with above sources is:

$$\begin{aligned} A_r &= \frac{A_r^1(\hat{x})}{r} + A_r^{\log}(u, \hat{x})\frac{\log r}{r^2} + \dots, \qquad A_u = A_u^{\log}(u, \hat{x})\frac{\log r}{r} + \frac{A_u^1(u, \hat{x})}{r} + \dots, \\ A_A &= A_A^0(u, \hat{x}) + A_A^{\log}(u, \hat{x})\frac{\log r}{r} + \dots. \end{aligned} \tag{2.8}$$

Next let us consider the asymptotic behaviour of the gravitational field. We will work in the perturbative linear gravity regime where gravitational dynamics is confined to perturbations around flat space time: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the de Donder gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$,

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where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\sigma}{}_{\sigma}$. The metric field satisfies $\Box\bar{h}_{\mu\nu} = -2T_{\mu\nu}$. The metric field admits following expansion:[4]

$$\begin{aligned}
h_{rr} &= \frac{h^1_{rr}(\hat{x})}{r} + h^{\log}_{rr}(u,\hat{x})\frac{\log r}{r^2} + \dots, & h_{ur} &= \frac{h^1_{ur}(u,\hat{x})}{r} + h^{\log}_{ur}(u,\hat{x})\frac{\log r}{r^2} + \dots, \\
h_{uu} &= h^{\log}_{uu}(u,\hat{x})\frac{\log r}{r} + \frac{h^1_{uu}(u,\hat{x})}{r} + \dots, & h_{rA} &= h^0_{rA}(\hat{x}) + h^{\log}_{rA}(u,\hat{x})\frac{\log r}{r^2} + \dots, \\
h_{uA} &= h^0_{uA}(u,\hat{x}) + h^{\log}_{uA}(u,\hat{x})\frac{\log r}{r} + \dots, & h_{AB} &= r h^{-1}_{AB}(u,\hat{x}) + \log r\, h^{\log}_{AB}(u,\hat{x}) + \dots.
\end{aligned} \tag{2.9}$$

There exists similar asymptotic expansion of fields at past null infinity ($r \to \infty$ with $t + r$ finite) represented by $\mathcal{I}^-$.

2.1 Asymptotic conservation laws

Classical equations of motion can be used to derive conservation laws of the form:

$$Q^+[\lambda^+] \mid_{\mathcal{I}^+_-} = Q^-[\lambda^-] \mid_{\mathcal{I}^-_+}. \tag{2.10}$$

Here, $\mathcal{I}^+_-$ is the $u \to -\infty$ sphere of $\mathcal{I}^+$ and $\mathcal{I}^-_+$ is the $v \to \infty$ sphere of $\mathcal{I}^-$. The quantum version of above statements can be related to soft theorems. Here, λ^+ refers to an arbitrary parameter defined at $\mathcal{I}^+_-$. The parameter at $\mathcal{I}^-_+$ is related to it via antipodal map $\lambda^+(\hat{x}) = \lambda^-(-\hat{x})$. Thus, we have a conservation law for every possible choice of λ. In this section, we will review the leading conservation law for QED. (2.8) leads to following fall offs for the field strength:

$$\begin{aligned}
F_{ru} &= \frac{F^2_{ru}(u,\hat{x})}{r^2} + \dots, & F_{uA} &= F^0_{uA}(u,\hat{x}) + \dots, \\
F_{AB} &= F^0_{AB}(u,\hat{x}) + \dots, & F_{rA}{}^{5} &= \frac{F^2_{rA}(u,\hat{x})}{r^2} + \dots.
\end{aligned} \tag{2.11}$$

The Maxwell's equations given by $\nabla^\nu F_{\sigma\nu} = j_\sigma$ imply following equations for the coefficients:

$$\begin{aligned}
\partial_u F^2_{ru} + \partial_u D^B A^0_B &= j^2_u, \\
\partial_u F^2_{rA} - \frac{1}{2}\partial_A F^2_{ru} + \frac{1}{2}D^B F^0_{AB} &= \frac{1}{2}j^2_A.
\end{aligned} \tag{2.12}$$

Let us use above equations to the study the u-behaviour of the field strength components F^2_{ru} and F^2_{rA}. Around $|u| \to \infty$, the currents die stronger than any power law of u, so we can ignore the currents and the large u behaviour of F^2_{ru} gets fixed by A^0_A. Tree level soft theorems[6] dictate following behaviour for radiative data:

$$A^0_A|_{\mathcal{I}^+_-} = A^{0,0}_A(\hat{x})u^0 + \dots. \tag{2.13}$$

[4] Some of the coefficients are independent of u, this follows from the de Donder gauge condition itself.

[5] F_{rA} actually starts at $F_{rA} = F^{\log}_{rA}(u,\hat{x})\frac{\log r}{r^2} + \dots$ due to presence of massless fields. Maxwell's equations imply $\partial_u F^{\log}_{rA} = 0$, so we set this mode to 0.

[6] At tree level, the soft expansion is given by: $\tilde{A}^0_A \sim \sum_{m=-1}^{\infty} S_m \omega^m$.

'...' denote terms that fall off faster than any power law in u. Hence, around $u \to -\infty$, the field strength components admit following behaviour:

$$\begin{aligned} F^2_{ru}|_{\mathcal{I}^+_-} &= u^0 F^{2,0}_{ru}(\hat{x}) + \dots\,, \\ F^2_{rA}|_{\mathcal{I}^+_-} &= u F^{2,-1}_{rA}(\hat{x}) + u^0 F^{2,0}_{rA}(\hat{x}) + \dots\,. \end{aligned} \tag{2.14}$$

In [13], for specific physical processes, the author showed that following relation holds between asymptotic values of the fields:

$$F^{2,0}_{ru}(\hat{x})|_{\mathcal{I}^+_-} = F^{2,0}_{rv}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{2.15}$$

Above statement can be rewritten as conservation law for charges parameterized by a scalar function λ:

$$Q^+_{\text{lead}}[\lambda^+]|_{\mathcal{I}^+_-} = Q^-_{\text{lead}}[\lambda^-]|_{\mathcal{I}^-_+}. \tag{2.16}$$

$Q^+_{\text{lead}}[\lambda^+] = \int d^2z \lambda^+(\hat{x}) F^{2,0}_{ru}(\hat{x})$. Q^-_{lead} is defined analogously. And $\lambda^+(\hat{x}) = \lambda^-(-\hat{x})$. Above conservation law was proved for generic processes in [27]. The Ward identity for above charges is equivalent to the leading soft photon theorem [6, 13].

2.2 Outline of the paper

To study the $\log\omega$ soft theorem we need to incorporate the effect of long range forces on asymptotic dynamics. In absense of long range forces, asymptotic fields satisfy free equations of motion. Including the correction to the asymptotic dynamics due to long range interactions leads to dressing of the free fields.

- In sections 3 and 4, we discuss dressing of massless scalar field and photon field respectively. We show that as a result of these dressings, (2.13) is corrected to

$$A^0_A|_{\mathcal{I}^+_-} = A^{0,0}_A(\hat{x})u^0 + A^{0,1}_A(\hat{x})\frac{1}{u} + \dots\,. \tag{2.17}$$

  This shows that including the effect of long range forces changes the soft expansion of the gauge field non-trivially. The exact contribution from scalar dressing to $1/u$ term is given in (3.7). Similarly, the contribution from gravitational dressing of photon to $1/u$ term is given in (4.1).

Using (2.12), it can be shown that the $1/u$ leads to a $\log u$ term in (2.14). The exact relation is given in (5.9). Thus, we get:

$$F^2_{rA}|_{u\to-\infty} = uF^{2,-1}_{rA}(\hat{x}) + \log(-u)F^{2,\log}_{rA}(\hat{x}) + \dots\,. \tag{2.18}$$

We show in (A.14) of appendix A that expansion around the past null infinity is modified to:

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\left[v^0 F^{\log,0}_{rA}(\hat{x}) + \dots\right] + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{2.19}$$

Here, '...' denote terms that fall off faster than power law in v. This $\log r$ mode was missed in [1]. Hence, the conservation law proposed by [1] is not entirely correct.[7] Let us propose following conservation law for the logarithmic modes:[8]

$$F_{rA}^{2,\log}(\hat{x})|_{\mathcal{I}_-^+} = F_{rA}^{\log,0}(-\hat{x})|_{\mathcal{I}_+^-}. \tag{2.20}$$

We have checked (2.20) explicitly for classical processes with no incoming radiation.

- In section 5, we start with above conservation law and identify the soft and hard modes of the charge. We refer to it as 1-loop charge, since it is expected to be related to the $\log \omega$ soft theorem which is 1-loop exact. The expression of soft charge is given in (5.7). This operator isolates soft $\log \omega$ mode of the photon field. The expression of hard charge is given in (5.12) and (5.14). As discussed in section 1, the hard charge is given in terms of h^1_{rr} and A^1_r.
- We find the expression of h^1_{rr} and A^1_r modes in section 6. Each of these modes has a classical and a quantum part: $h^1_{rr} = \overset{\text{class}}{h^1_{rr}} + \overset{\text{quan}}{h^1_{rr}}$ and $A^1_r = \overset{\text{class}}{A^1_r} + \overset{\text{quan}}{A^1_r}$. In section 6.1, the classical modes are obtained by evolving the sources with retarded propagator. $\overset{\text{class}}{A^1_r}$ given in (6.1) turns out to be trivial.

  The quantum modes are slightly subtle. Discontinuity in $\omega \to 0$ leads to a $\log u$ term in C_{AB} as seen in (6.11) and (6.12). These modes contribute to $\overset{\text{quan}}{h^1_{rr}}$ via (6.13). $\overset{\text{quan}}{A^1_r}$ is obtained similarly.
- In section 7, we finally write down the Ward identity for the 1-loop charge and show its equivalence to the Sen-Sahoo soft theorem.

3 Dressing of massless scalar field

In this section we will study the dressing of free scalar fields under the effect of long range forces and find the resultant correction to the asymptotic field. We will show that long range forces produce a $1/u$ term in (2.13).

For massive fields the effect of long range forces can be obtained perturbatively by studying asymptotic potential order by order around $t \to \infty$. This leads to the well known Faddeev-Kulish dressing of massive scalars [26]. For massless scalars, the asymptotic states live at null infinity. So, we will study the corrections to the free equation of motion at null infinity. Massless scalars satisfy following equation:

$$g^{\mu\nu} D_\mu D_\nu \phi(x) = 0. \tag{3.1}$$

Let us expand above equation around future null infinity. Using the fall offs given in (2.8) and (2.9), we find that the leading order equation is (at $\mathcal{O}(\frac{1}{r^2})$):

$$-2\partial_u \partial_r \phi - \frac{2}{r}\partial_u \phi = \frac{h^1_{rr}(\hat{x})}{r}\partial_u^2 \phi - 2ie\frac{A^1_r(\hat{x})}{r}\partial_u \phi. \tag{3.2}$$

[7]We thank Arnab Priya Saha and Biswajit Sahoo for discussions about this point.
[8]We thank the authors of [1] for suggesting this new conservation law.

Thus, the leading order effect of long range forces on the massless field is given by h^1_{rr} and A^1_r. The solution of above equation is given by:

$$\phi(x) = -\frac{ie^{ieA^1_r(\hat{x})\log r}}{8\pi^2 r}\int d\omega \left[b(\omega,\hat{x})e^{-i\omega u}e^{i\omega \log\frac{r}{r_0}\frac{h^1_{rr}(\hat{x})}{2}} - d^\dagger(\omega,\hat{x})e^{i\omega u}e^{-i\omega\log\frac{r}{r_0}\frac{h^1_{rr}(\hat{x})}{2}}\right], \tag{3.3}$$

where, b and $d^\dagger$ are the free data for massless scalar. r_0 depends on bulk interactions, hence $r_0 \ll r$. For our analysis we set $r_0 = 1$. On quantisation, b can be interpreted as the annihilation operator for free particles while d would become the annihilation operator for free antiparticles (see (2.5)). From (3.3), we see that the leading order effect of long range forces is to associate a cloud of photons and gravitons to a free massless scalar particle. These dressing factors (h^1_{rr} and A^1_r) are analogous to the Fadeev-Kulish dressing of a free massive scalar particle. Next we find the correction to the U(1) current. Dressing of scalar field leads to a new logarithmic fall off in the current (2.7):

$$j_A = j^{\log}_A\frac{\log r}{r^2} + \frac{j^2_A}{r^2} + \dots, \tag{3.4}$$

where

$$j^{\log}_A = -\frac{1}{2}\partial_A h^1_{rr} j^2_u + 2e^2\partial_A A^1_r|\phi^1|^2. \tag{3.5}$$

Let us find the corrections to the gauge field due to these new logarithmic fall offs in the current. In Lorenz gauge, we have $\Box A_\mu = -j_\mu$. (This equation admits corrections due to gravity; gravitational corrections will be analysed in section 4). Using the retarded propagator, the solution to the gauge field is given by:

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x'\delta((x-x')^2)\Theta(t-t')j_\sigma(x'). \tag{3.6}$$

We will substitute the new logarithmic modes of the current in above expression and find the resultant contribution to the field. The details of the calculation have been relegated to appendix A.

We show that the log modes give rise to a $1/u$ term in A^0_A such that the coefficient is given by (A.11):

$$A^{0,1}_{\bar{z}}(\hat{x})|_{\text{scal}} = \frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty}du'\int_{S^2}d^2z'\frac{\epsilon^\mu_- q^\sigma}{q.q'}q'_{[\mu}D'^A q'_{\sigma]}j^{\log}_A. \tag{3.7}$$

We have added a subscript 'scal' to highlight that this contribution arises from scalar field dressing. In above expression we have used the following basis for polarisation vectors [6]:

$$\epsilon^\mu_- = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\bar{z}}[(1+z\bar{z})q^\mu], \quad \epsilon^\mu_+ = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}[(1+z\bar{z})q^\mu]. \tag{3.8}$$

The expression for A_z can be obtained from the expression for $A_{\bar{z}}$ by replacing ϵ_- by ϵ_+. The $1/u$-term has been discussed in the context of scattering of point particles in [30].

Let us recall (2.13). The u^0 term in this equation which is related to the leading soft theorem is unchanged by long range forces. We see that including the effect of long range forces introduces a new $1/u$-term given in (3.7) that is absent in (2.13). The $1/u$ term is $\mathcal{O}(e^3)$ (or $\mathcal{O}(eG)$ for gravitational correction). Thus the soft expansion of the gauge field changes non-trivially as we go to higher order in couplings.

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4 Dressing of gauge field

In this section, we study the effect of long range gravitational force on gauge fields. Before delving into the calculation, we state our result. The leading order correction to (2.13) as a result of coupling of photon with gravity is:

$$A_{\bar{z}}^{0,1}(\hat{x})|_{\text{grav dress}} = -\frac{1}{8\pi}\frac{\sqrt{2}}{1+z\bar{z}}h_{rr}^{1}(\hat{x})\lim_{\omega\to 0}\omega\left[a_{-}(\omega,\hat{x}) - a_{-}^{\dagger}(-\omega,\hat{x})\right]. \tag{4.1}$$

For $A_{z}^{0,1}$, we have to replace negative helicity operators with positive helicity operators.

Let us derive above expression. First we start with the homogenous equation $\Box A_{\mu}^{\text{hom}} = 0$. Asymptotically such a solution exhibits following form [6]:

$$A_{\sigma}^{\text{hom}}(u,r,\hat{x}) = -\frac{i}{8\pi^2 r}\int d\omega\left[a_{\sigma}(\omega,\hat{x})e^{-i\omega u} - a_{\sigma}^{\dagger}(\omega,\hat{x})e^{i\omega u}\right], \tag{4.2}$$

where $a_{\sigma} = \sum_{r=+,-}\epsilon_{\sigma}^{r}a_{r}$. Let us turn on the sources. Choosing the generalised Lorenz gauge $\nabla_{\mu}A^{\mu} = 0$, Maxwell's equations reduce to:

$$\nabla^2 A_{\mu} = -j_{\mu} + R_{\mu}{}^{\nu}A_{\nu}. \tag{4.3}$$

$R_{\mu\nu}$ is the Ricci tensor. Above equation can be written as: (Ignoring the U(1) current here.)

$$\Box A_{\sigma} = j_{\sigma}^{\text{grav}}, \tag{4.4}$$

where we have defined:

$$j_{\sigma}^{\text{grav}} = h^{\mu\nu}\partial_{\mu}\partial_{\nu}A_{\sigma} + \eta^{\mu\nu}\Gamma_{\mu\nu}^{\rho}\partial_{\rho}A_{\sigma} + 2\eta^{\mu\nu}\Gamma_{\mu\sigma}^{\rho}\partial_{\nu}A_{\rho} + \eta^{\mu\nu}A_{\lambda}\partial_{\mu}\Gamma_{\nu\sigma}^{\lambda} + [\partial_{\mu}\Gamma_{\nu\sigma}^{\mu} - \partial_{\nu}\Gamma_{\mu\sigma}^{\mu}]A^{\nu} + \mathcal{O}(G^2).$$

Since we are working perturbatively, j_{σ}^{grav} can be evaluated on the zeroth order solution. Using (2.9) and (4.2), we see that the source has following behaviour around future null infinity:

$$j_{\sigma}^{\text{grav}}(x) = \frac{1}{r^2}h_{rr}^{1}\partial_{u}^{2}A_{\sigma}^{1} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{4.5}$$

The $\mathcal{O}(\frac{1}{r^3})$ terms in $j_{\sigma}^{\text{grav}}(x')$ produce subleading corrections, hence are not relevant for our analysis. Analogous to the massless scalar equation (3.2), we get:

$$-2\partial_{u}\partial_{r}A_{\sigma} - \frac{2}{r}\partial_{u}A_{\sigma} = \frac{h_{rr}^{1}(\hat{x})}{r}\partial_{u}^{2}A_{\sigma}. \tag{4.6}$$

The solution to above equation is given by:

$$A_{\sigma}(u,\hat{x}) = -\frac{i}{8\pi^2}\int d\omega\left[a_{\sigma}(\omega,\hat{x})e^{-i\omega u}e^{i\omega\log r\frac{h_{rr}^{1}(\hat{x})}{2}} - a_{\sigma}^{\dagger}(\omega,\hat{x})e^{i\omega u}e^{-i\omega\log r\frac{h_{rr}^{1}(\hat{x})}{2}}\right]. \tag{4.7}$$

Thus, the $\log r$ dressing of photons is exactly similar to the $\log r$ dressing of massless scalars. This dressing does not contribute to the loop level charge. The contribution to the loop level charge comes from $1/u$ term. So, we need to check if the source (4.5) induces a $1/ru$ in A_μ. Using the Green's function for D'Alembertian operator:

$$A_\sigma^{\rm grav}(x) = -\frac{1}{2\pi}\int d^4x'\delta_+((x-x')^2)h^1_{rr}(z')\partial^2_{u'}A^1_\sigma(u',z').$$

We have used a superscript 'grav' to highlight the fact that this mode arises due to gravitational coupling. Taking the limit $r\to\infty$ with $u<r$:

$$\begin{aligned}A_\sigma^{\rm grav}(u,r,\hat{x}) &= -\frac{1}{4\pi r}\int du'dr'd^2z'\delta_+(u'+r'-u-\vec{x}'.\hat{x})h^1_{rr}(z')\partial^2_{u'}A^1_\sigma(u',z')+\mathcal{O}\left(\frac{1}{r^2}\right),\\ &= -\frac{1}{4\pi r}\partial_u\left[\int_{-\infty}^{\infty}du'\int_0^\infty dr'\int_{S^2}d^2z'\delta(u'+r'-u-\vec{x}'.\hat{x})h^1_{rr}(z')\partial_{u'}A^1_\sigma(u',z')\right].\end{aligned}$$

$\partial_{u'}A^0_\sigma$ vanishes for $|u'|>u_0$ as to the zeroth order the particles are free for $|u'|>u_0$, where, u_0 is some time scale that is set by short range interactions. We can use rotational symmetry to align $\hat{x}$ along z'-axis:

$$A_\sigma^{\rm grav}(u,r,\hat{x}) = -\frac{1}{2r}\partial_u\left[\int_{-u_0}^{u_0}du'\int_0^\infty dr'\int_{-1}^1 d\cos\theta'\frac{1}{r'}\delta\left(\cos\theta'-1+\frac{u-u'}{r'}\right)h^1_{rr}(\theta')\partial_{u'}A^1_\sigma(u',\theta')\right].$$

We will use the delta function to do the θ' integral. $\cos\theta'\epsilon[-1,1]$ leads to a bound on other integration variables. There are two allowed ranges: $u>u'$, $2r'>u-u'$; $u'>u$, $2r'<-(u'-u)$. The second range is inadmissible as r' needs to be positive. Also, r'-integral needs to be regulated with some IR cutoff.

$$A_\sigma^{\rm grav}(u,r,\hat{x}) = -\frac{1}{2r}\partial_u\left[\int_{-u_0}^{u_0}du'\int_{\frac{u-u'}{2}}^{R}\frac{dr'}{r'}h^1_{rr}(\theta')\partial_{u'}A^1_\sigma(u',\theta')|_{\cos\theta'=1-\frac{u-u'}{r'}}\right].$$

Taylor expanding the integrand around $\cos\theta'=1$, we get the leading order contribution in $u\to\infty$ limit to be:

$$A_\sigma^{\rm grav}(u,r,\hat{x}) = \frac{1}{2r}\left[\int_{-u_0}^{u_0}du'\frac{1}{u-u'}h^1_{rr}(\theta')\partial_{u'}A^1_\sigma(u',\theta')|_{\cos\theta'=1}\right]. \tag{4.8}$$

Above expression can be readily related to insertion of leading soft mode:

$$\begin{aligned}A_\sigma^{\rm grav}(u,r,\hat{x})_{u\to\infty} &= \frac{1}{2r}\frac{1}{u}h^1_{rr}(\hat{x})\int_{-u_0}^{u_0}du'\partial_{u'}A^1_\sigma(u',\hat{x}),\\ &= -\frac{1}{8\pi r}\frac{1}{u}h^1_{rr}(\hat{x})\lim_{\omega\to0}\omega\left[a_\sigma(\omega,\hat{x})-a^\dagger_\sigma(-\omega,\hat{x})\right].\end{aligned} \tag{4.9}$$

This term is due to acceleration of outgoing photons under gravitational field h^1_{rr}. We can arrive at (4.1) by co-ordinate transformation.

We can combine above expression with (4.7), to write:

$$A_\sigma(u,r,\hat{x}) = -\frac{i}{8\pi^2 r}\int d\omega\left[a_\sigma(\omega,\hat{x})e^{-i\omega u}e^{i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}} - a^\dagger_\sigma(\omega,\hat{x})e^{i\omega u}e^{-i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}}\right]. \tag{4.10}$$

Thus, we have obtained the dressing of photon due to the presence of long range gravitaional force. Similar to massless scalars, the dressing depends only on h^1_{rr}.

5 The asymptotic charge

In this section we will obtain the explicit expression for the 1-loop asymptotic charge. We start with the conservation equation (2.20):

$$F_{rA}^{2,\log}(\hat{x}) = F_{rA}^{\log,0}(-\hat{x}). \tag{5.1}$$

We recall that the l.h.s. is the coefficient of the $\frac{\log u}{r^2}$-mode present at the future. Similarly the r.h.s. is the coeffficient of the $\frac{\log r}{r^2}$ mode living at the past. We multiply above equation with an arbitrary parameter V^A and integrate over the sphere. We get

$$\int_{\mathcal{I}_-^+} d^2z V^A(\hat{x}) F_{rA}^{2,\log}(\hat{x}) = \int_{\mathcal{I}_+^-} d^2z V^A(-\hat{x}) F_{rA}^{\log,0}(-\hat{x}). \tag{5.2}$$

The charge at the future is defined by $Q^+_{\text{1-loop}}[V_+^A] = -\int d^2z V_+^A F_{rA}^{\log,0}|_{\mathcal{I}_-^+}$. The past charge is defined similarly. Our claim is that this conservation law reproduces the outgoing soft photon theorem given in (1.1). In the most general scenario there exists a $\log v$ mode at past. The $\log v$ mode corresponds to incoming soft photon and we have set these modes to zero. Similar conservation law that relates $\log v$ mode at $\mathcal{I}_+^-$ to $\log r$ mode at $\mathcal{I}_-^+$ reproduces the incoming soft theorem.

Let us study the future charge:

$$\begin{aligned} Q_+^{\text{1-loop}}[V] &= -\int d^2z V^A F_{rA}^{2,\log}|_{\mathcal{I}_-^+}, \\ &= u^2\partial_u^2 \int d^2z V^A F_{rA}^2|_{u\to-\infty}. \end{aligned}$$

The u-operator isolates the coefficient of the $\log u$ term of F_{rA}^2. We can rewrite the future charge as an integral over entire future null infinity minus the term at $\mathcal{I}_+^+$.

$$\begin{aligned} Q_+^{\text{1-loop}}[V] &= -\int_{-\infty}^{\infty} du' \int d^2z V^A \partial_u[u^2\partial_u^2 F_{rA}^2] - \int d^2z V^A F_{rA}^{2,\log}|_{\mathcal{I}_+^+}, \\ &:= Q_+^{\text{soft}}[V] + Q_+^{\text{hard}}[V]. \end{aligned} \tag{5.3}$$

This defines the soft and hard parts of asymptotic charge. We can simplify the soft charge expression further. Using Maxwell's equation (C.3) for $\partial_u F_{rA}^2$, we get:

$$Q_+^{\text{soft}} = -\frac{1}{2}\int_{-\infty}^{\infty} du' \int d^2z' V^A \partial_u\big[u^2\partial_u[\partial_A F_{ru}^2 - D^B F_{AB}^0 + j_A^2]\big] + \dots . \tag{5.4}$$

In above expression '...' arise due to the gravity corrections to Maxwell's equations. We have studied these terms explicitly in appendix C and we show that these corrections vanish. j_A^2 does not have a $1/u$-term, so j_A^2 also drops out of above expression and we get:

$$\begin{aligned} Q_+^{\text{soft}} &= \int_{-\infty}^{\infty} du' \int d^2z' \Big[V^z(\hat{x}')\partial_u[u^2\partial_u D_z D^{\bar{z}} A_{\bar{z}}^0(u,\hat{x}')] + z' \leftrightarrow \bar{z}'\Big], \\ &= \int_{-\infty}^{\infty} du' \int d^2z' \Big[D_z'^2 V^z \gamma^{z\bar{z}}\partial_u[u^2\partial_u A_{\bar{z}}^0(u,\hat{x}')] + z' \leftrightarrow \bar{z}'\Big]. \end{aligned} \tag{5.5}$$

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The last line was derived using integration by parts. Next it is instructive to go to the frequency space:

$$Q_+^{\text{soft}} = \int d^2z' \left[D_z'^2 V^z \gamma^{z\bar{z}} \lim_{\omega\to 0} \omega \partial_\omega^2 \omega \tilde{A}_{\bar{z}}^0(\omega, \hat{x}') + z' \leftrightarrow \bar{z}' \right].$$

The gauge field can be expressed in terms of Fock operators as:

$$\tilde{A}_{\bar{z}}^0(\omega, \hat{x}) = -i\sqrt{2}\frac{a_-(\omega, \hat{x})}{4\pi(1+z\bar{z})} \ldots \omega > 0, \tilde{A}_{\bar{z}}^0(\omega, \hat{x}) = i\sqrt{2}\frac{a_+^\dagger(-\omega, \hat{x})}{4\pi(1+z\bar{z})} \ldots \omega < 0. \tag{5.6}$$

Since, we have only outgoing radiation, the relevant part of the expression is:

$$Q_+^{\text{soft}} = -\frac{i}{4\pi}\int d^2z' \left[D_z'^2 V^z \sqrt{\gamma'^{z\bar{z}}} \lim_{\omega\to 0} \omega \partial_\omega^2 \omega a_-(\omega, \hat{x}') + z' \leftrightarrow \bar{z}' \right]. \tag{5.7}$$

Thus, this operator is related to zero energy photon modes. The ω-derivatives in particular isolate the coefficient of soft $\log\omega$ mode.

Next let us turn to the expression of future hard charge:

$$Q_+^{\text{hard}} = -\int d^2z' V^A F_{rA}^{2,\log}(\hat{x}').$$

Using (2.12), we have

$$\partial_u^2 F_{rA}^2 + \frac{1}{2}\partial_u \partial_A D^B A_B^0 + \frac{1}{2}\partial_u D^B F_{AB}^0 = \frac{1}{2}\partial_u j_A^2. \tag{5.8}$$

From above equation we get the precise relations:

$$F_{rz}^{2,\log} = -\gamma^{z\bar{z}} D_z^2 A_{\bar{z}}^{0,1} \text{ and } F_{r\bar{z}}^{2,\log} = -\gamma^{z\bar{z}} D_{\bar{z}}^2 A_z^{0,1}. \tag{5.9}$$

We recall that $A_A^{0,1}(\hat{x})$ denotes following mode in the gauge field: $A_A(x) \sim A_A^{0,0}(\hat{x}) + A_A^{0,1}(\hat{x})\frac{1}{u} + \ldots$. Using (5.9) in the expression for the hard charge, it can be written as

$$Q_+^{\text{hard}} = \int d^2z' V^z \gamma^{z\bar{z}} D_z^2 A_{\bar{z}}^{0,1}(\hat{x}') + \int d^2z' V^{\bar{z}} \gamma^{z\bar{z}} D_{\bar{z}}^2 A_z^{0,1}(\hat{x}') \tag{5.10}$$

To avoid unnecessary cluttering of equations we will work with $V^{\bar{z}} = 0$. Then we can integrate by parts to get following equation:

$$Q_+^{\text{hard}} = \int d^2z' D_z'^2 V^z \gamma^{z\bar{z}} A_{\bar{z}}^{0,1}(\hat{x}'). \tag{5.11}$$

We recall using (3.7), (3.5) and (4.1):

$$\begin{aligned} A_{\bar{z}}^{0,1}(\hat{x}) = \frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi} \int_{-\infty}^{\infty} du' \int d^2z' \frac{q^\mu \epsilon_-^\sigma}{q.q'} q'_{[\sigma} \partial'_{q^\mu]} \left[-\frac{1}{2} h_{rr}^1(\hat{x}') j_u^2(\hat{x}') + 2e^2 A_r^1(\hat{x}') |\phi^1(\hat{x}')|^2 \right] \\ - \frac{\sqrt{\gamma_{z\bar{z}}}}{8\pi} h_{rr}^1(\hat{x}) \lim_{\omega\to 0} \omega a_-(\omega, \hat{x}). \end{aligned} \tag{5.12}$$

Equations (5.11) and (5.12) provide us the expression of the future hard charge.

Let us turn to the expression of the past charge. We have:

$$Q_-^{\text{1-loop}}[V] = -\int d^2z V^A F_{rA}^{\log,0}|_{\mathcal{I}_+^-}.$$

We know from (A.14) that $F_{rA}^{\log,0}$ depends only on particle currents i.e. it has no contribution from radiation. Thus, at past the charge is entirely made of hard modes.

$$Q_-^{\text{1-loop}}[V] = -\int d^2z V^A F_{rA}^{\log,0}|_{\mathcal{I}_+^-} := Q_-^{\text{hard}}[V].$$

Thus as we had mentioned earlier we see that the conservation law that we have started with in (5.2), reproduces outgoing soft theorem. An analogous conservation law that relates $\log v$ mode at $\mathcal{I}_+^-$ to $\log r$ (a purely hard mode) at $\mathcal{I}_-^+$ will reproduce the incoming soft theorem.

Using (A.14), the charge at past can be recast as:

$$Q_-^{\text{hard}} = -\int d^2z' D_z'^2 V^z \gamma^{z\bar{z}} B^{\log}(\hat{x}'), \tag{5.13}$$

where,

$$B^{\log}(\hat{x}) = \frac{1}{4\pi} \frac{\sqrt{2}}{1+z\bar{z}} \int_{-\infty}^{\infty} dv' \int_{S^2} d^2z' \frac{q^\sigma \epsilon_-^\mu}{q.q'} q'_{[\mu}\partial'_{q^\sigma]} \left[-\frac{1}{2} h_{rr}^1(-\hat{x}') j_u^2(-\hat{x}') + 2e^2 A_r^1(-\hat{x}') |\phi^1(-\hat{x}')^2 \right]. \tag{5.14}$$

6 Expressions for h_{rr}^1 and A_r^1

In the preceding section, we studied the expression of the 1-loop asymptotic charge. The hard charges depend on h_{rr}^1 and A_r^1 via (5.12) and (5.14). In this section we will write down the expressions for h_{rr}^1 and A_r^1.

6.1 Classical part

We know that the solution for gauge field in Lorenz gauge is given by:

$$A_\mu(x^\mu)|_{\text{class}} = \frac{1}{2\pi} \int d^4x' \delta\big((x-x')^2\big) \Theta(t-t') j_\mu(x'),$$

where we have used the retarded propagator. The leading order term at large r is given by:

$$A_\mu(u,r,\hat{x})|_{\text{class}} = -\frac{1}{4\pi r} \int_{-\infty}^{\infty} du' \int d^2z' \frac{j_\mu^2(\hat{x}',u')}{q.q'}.$$

Above expression is consistent with the fall offs mentioned in (2.8). In particular we have:

$$\overset{\text{class}}{A_r^1}(\hat{x}) = \frac{1}{4\pi r} \int_{-\infty}^{\infty} du' \int d^2z' j_u^2(\hat{x}',u'). \tag{6.1}$$

This part of $A_r^1(x)$ is just a constant (i.e. independent of $u, \hat{x}$) hence does not contribute to the hard charge. Thus, the classical electromagnetic dressing is trivial which is consistent with the absence of classical $\log\omega$ term in soft electromagnetic radiation (in absence of gravitational coupling) [2].

Similarly, in De-Donder gauge, the metric perturbations satisfy $\Box\bar{h}_{\mu\nu} = -2T_{\mu\nu}$ with the solution given by:

$$\bar{h}_{\mu\nu}(x^\mu)|_{\text{class}} = \frac{1}{\pi}\int d^4x'\delta\big((x-x')^2\big)\Theta(t-t')T_{\mu\nu}(x').$$

The leading order solution around future null infinity is given by:

$$\bar{h}_{\mu\nu}(u,r,\hat{x})|_{\text{class}} = -\frac{1}{2\pi r}\int_{-\infty}^{\infty}du'\int d^2z'\frac{T^2_{\mu\nu}(\hat{x}',u')}{q.q'}.$$

Above expression is consistent with the fall offs mentioned in (2.9). A key point about the perturbations is that $\partial_u\bar{h}^1_{\mu\nu} = 0$. This kills off a lot of terms that would have otherwise been present in (3.2). Finally we have:

$$\overset{\text{class}}{h^1_{rr}}(\hat{x}) = -\frac{1}{2\pi r}\int_{-\infty}^{\infty}du'\int d^2z' q.q' T^2_{uu}(\hat{x}',u'). \tag{6.2}$$

6.2 Quantum part

Next we want to check if there is a part of h^1_{rr}, A^1_r that has not been captured by the retarded propagator. Let us work in $r, u\to\infty$ limit ($u<r$) where the sources have died down and we can use the homogenous solution. We will use Herdegen-like representation of $h_{\mu\nu}$. Herdegen representation [28] for photon is a way to write a generic homogenous solution for gauge field in Lorenz gauge in terms of free data A^0_A. Similarly, here we write a generic homogenous solution for metric field in De Donder gauge in terms of free data C_{AB}. ($C_{AB} = \lim_{r\to\infty}\frac{h_{AB}(x)}{r}$). (See appendix D for details):

$$\overset{\text{hom}}{h_{\mu\nu}}(x) = -\frac{1}{(4\pi)}\int d^2z'(1+z'\bar{z}')^2\Big[\epsilon^-_\mu\epsilon^-_\nu\dot{C}_{zz}(u=-x\cdot q',\hat{q}') + \epsilon^+_\mu\epsilon^+_\nu\dot{C}_{\bar{z}\bar{z}}(u=-x\cdot q',\hat{q}')\Big], \tag{6.3}$$

q'^μ is defined according to (2.2). From above expression it can be seen that $C_{zz}\sim\overset{\log}{C_{zz}}\log u$ gives rise to a $\frac{1}{r}$ term in $h_{\mu\nu}$. Let us find the h^1_{rr} term by co-ordinate transformation ($x^\mu = rq^\mu + \mathcal{O}(r^0)$). We will denote it with a 'quan' overtext. We have:

$$\overset{\text{quan}}{h^1_{rr}}(x) = \frac{1}{4\pi}\int d^2z'(1+z'\bar{z}')^2\frac{1}{q'.q}\left[\epsilon^-.q\epsilon^-.q\overset{\log}{C_{zz}}(\hat{x}') + \epsilon^+.q\epsilon^+.q\overset{\log}{C_{\bar{z}\bar{z}}}(\hat{x}')\right]. \tag{6.4}$$

Thus, we see that $\log u$ mode in C_{AB} contributes to h^1_{rr}. We will eventually see that the leading soft theorem itself implies existence of a $\log u$ mode.

The behaviour of the free data around $u\to\pm\infty$ dictated by tree level soft theorems is:

$$C_{AB} = D^{\pm}_{AB}(\hat{x})u^0 + \dots, \quad u\to\pm\infty. \tag{6.5}$$

Here, '...' denote any fall offs faster than power law fall off in u. Power law fall offs appear in above equation when we consider the effect of long range forces but these do not affect our analysis. Just by Fourier transform one can quickly check that $D^{\pm}_{AB}$ is related to the leading soft factor or see [29] for details. We will show that there is an additional term in (6.5):

$$C_{AB} = \overset{\log}{C}_{AB} \log|u| + D^{\pm}_{AB}(\hat{x}) u^0 + \dots , \quad u \to \pm\infty, \tag{6.6}$$

where, $\overset{\log}{C}_{AB}$ vanishes classically. An important point to note is that $\overset{\log}{C}_{AB}$ is not arbitrary but is fixed in terms of the leading soft factor i.e. $D^{\pm}_{AB}$, hence the free data for classical system is sufficient to describe the quantized system and we are not introducing new free data in the quantum system.

In [1], authors discussed the discontinuity in $\omega\tilde{A}_A$ as $\omega \to 0$ that is non trivial at quantum mechanical level. This discontinuity leads to a $\log|u|$ term in A_A.[9] We will discuss the gravitational analogue of this purely quantum $\log|u|$ mode. For scalars, $\omega\tilde{\phi}$ as $\omega \to 0$ is trivial. Hence there is no $\log|u|$ term for scalars.

Let us consider $C^{+}_{zz}(u,\hat{x})$ that has only positive frequencies. We know that around $\omega \sim 0$, the behaviour of the radiative data is given by $\tilde{C}^{+}_{zz} = \frac{1}{\omega}\tilde{C}^{+0}_{zz} + \dots$. This low energy behaviour dictates the large-u behaviour. Hence:

$$\begin{aligned} C^{+}_{zz}(u,\hat{x}) &= \frac{1}{2\pi}\int_0^{\infty} d\omega \left[\frac{1}{\omega}\tilde{C}^{+0}_{zz}(\hat{x}) + \dots\right] e^{-i\omega u}, \\ &= \frac{1}{2\pi}\log(u^{-1})\tilde{C}^{+0}_{zz}(\hat{x}) + \dots . \end{aligned} \tag{6.7}$$

Simlarly for negative frequencies, we have:

$$C^{-}_{zz}(u,\hat{x}) = -\frac{1}{2\pi}\log(u^{-1})\tilde{C}^{-0}_{zz}(\hat{x}) + \dots . \tag{6.8}$$

Collecting the positve and negative frequency terms we get:

$$\begin{aligned} C_{zz}(u,\hat{x}) &= -\frac{1}{2\pi}\left[\tilde{C}^{+0}_{zz}(\hat{x}) - \tilde{C}^{-0}_{zz}(\hat{x})\right]\log|u| + \dots , \\ &= -\frac{1}{2\pi}\lim_{\omega\to 0^{+}}\left[\omega\tilde{C}^{+}_{zz}(\omega,\hat{x}) + \omega\tilde{C}^{-}_{zz}(-\omega,\hat{x})\right]\log|u| + \dots . \end{aligned} \tag{6.9}$$

To find classical radiation, we use retarded propagators. For such solutions, $\omega\tilde{C}_{zz}$ is continuous at $\omega = 0$ [29] and the coefficient of $\log|u|$ term vanishes. It is important to note that the coefficient does not carry any factor of $\hbar$. Hence, the $\log|u|$ coefficient does not go to 0 just by taking $\hbar \to 0$. The coefficient vanishes when we demand retarded boundary conditions. We will see that it is non-trivial quantum mechanically. This is because of the fact that when we quantise the C_{AB} field, the positive frequencies involve annihilation operator while negative frequencies involve creation operator:

$$\tilde{C}_{zz}(\omega,\hat{x}) = \frac{-ic_{+}(\omega,\hat{x})}{2\pi(1+z\bar{z})^2}\dots \quad \omega > 0, \qquad \tilde{C}_{zz}(\omega,\hat{x}) = \frac{ic^{\dagger}_{-}(-\omega,\hat{x})}{2\pi(1+z\bar{z})^2}\dots \quad \omega < 0. \tag{6.10}$$

Thus, we get:

$$C_{zz}(u,\hat{x}) = \frac{i}{4\pi^2}\frac{1}{(1+z\bar{z})^2}\lim_{\omega\to 0}\omega\left[c_{+}(\omega,\hat{x}) + c^{\dagger}_{-}(-\omega,\hat{x})\right]\log|u| + \dots . \tag{6.11}$$

[9] It is interesting to note that this $\log u$ mode has appeared in equation (A.2) of [31].

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Similarly, for $C_{\bar{z}\bar{z}}$ we have,

$$\overset{\log}{C}_{\bar{z}\bar{z}}(\hat{x}) = \frac{i}{4\pi^2}\frac{1}{(1+z\bar{z})^2}\lim_{\omega\to 0}\omega\left[c_-(\omega,\hat{x}) + c_+^\dagger(-\omega,\hat{x})\right]. \tag{6.12}$$

We will see that above operators have non-trivial action when inserted in the expression for charge. Substituting for $\overset{\log}{C}_{AB}$ in the expression of $\overset{\text{quan}}{h^1_{rr}}$ given in (6.4):

$$\overset{\text{quan}}{h^1_{rr}}(x) = \frac{1}{4\pi}\int d^2z'(1+z'\bar{z}')^2\frac{1}{q'.q}\left[\epsilon^-.q\epsilon^-.q\overset{\log}{C}_{zz}(\hat{x}') + \epsilon^+.q\epsilon^+.q\overset{\log}{C}_{\bar{z}\bar{z}}(\hat{x}')\right]. \tag{6.13}$$

Next we need to do the sphere integral. We have relegated this calculation to appendix B and we will quote the results here. The finite part of the integral is:

$$< \text{out}|\overset{\text{quan}}{h^1_{rr}}(x)S|\text{in}> = -\frac{i}{2\pi^2}\sum_j (q.p_j)\log(q.p_j). \tag{6.14}$$

Next we repeat the calculation for gauge field and we get:

$$< \text{out}|\overset{\text{quan}}{A^1_r}(\hat{x})S|\text{in}> = -\frac{i}{4\pi^2}\sum_j e_j\log(q.p_j). \tag{6.15}$$

Finally, we have the complete expressions for h^1_{rr} and A^1_r which will be needed to evaluate the action of hard charge in the next section.

7 The Ward identity

The Ward identity for S matrix for the 1-loop asymptotic charge can be written down as:

$$\begin{aligned}&\left[Q^{\text{1-loop}}, S\right] = 0,\\ \Rightarrow &\left(Q^{\text{soft}}_+ S - SQ^{\text{soft}}_-\right) = -\left(Q^{\text{hard}}_+ S - SQ^{\text{hard}}_-\right).\end{aligned}$$

Using (5.11) and (5.13), we get

$$\left(Q^{\text{soft}}_+ S - SQ^{\text{soft}}_-\right) = -\int d^2z' D'^2_z V^z \gamma^{z\bar{z}}\left(A^{0,1}_{\bar{z}}(\hat{x}')S - SB^{\log}(\hat{x}')\right). \tag{7.1}$$

Next we need to evaluate the action of above operators on a Fock state. From (5.12), we get the expression of $A^{0,1}_{\bar{z}}$.

$$\begin{aligned}A^{0,1}_{\bar{z}}(\hat{x}) = &\frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi}\int_{-\infty}^{\infty}du'\int d^2z'\frac{q^\mu\epsilon^\sigma_-}{q.q'}q'_{[\sigma}\partial'_{q^\mu]}\left[-\frac{1}{2}h^1_{rr}(\hat{x}')j^2_u(\hat{x}') + 2e^2A^1_r(\hat{x}')|\phi^1(\hat{x}')^2\right]\\ &-\frac{\sqrt{\gamma_{z\bar{z}}}}{8\pi}h^1_{rr}(\hat{x})\lim_{\omega\to 0}\omega a_-(\omega,\hat{x}).\end{aligned} \tag{7.2}$$

It is interesting to note that the first line resembles tree level subleading soft operator acting on $h^1_{rr} + A^1_r$. Similarly the second line is h^1_{rr} times the leading soft operator. The

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action of (7.2) on an outgoing Fock state can be easily evaluated.

$$
\begin{aligned}
< \text{out}|Q_-^{\text{hard}} =& < \text{out}|4\pi \sum_{i\epsilon out} U^{\sigma\mu}(q_i) q_{i[\sigma}\partial_{q_i^{\mu}]} \left[\frac{e_i}{2} h^1_{rr}(z_i) + e_i^2 \frac{A^1_r(z_i)}{\omega_i}\right] \\
& - < \text{out}| \int d^2z' D_{\bar{z}}^{\prime 2} V^z(z') \frac{\sqrt{\gamma'_{z\bar{z}}}}{8\pi} \sum_i \frac{e_i \epsilon^- . p_i}{q'.p_i} h^1_{rr}(z'),
\end{aligned}
\tag{7.3}
$$

where we have defined

$$
U^{\sigma\mu}(q_i) = \int d^2z' D_z^{\prime 2} V^z(z,z') \frac{\sqrt{\gamma'^{z\bar{z}}}}{16\pi^2} \frac{\epsilon_-^{\sigma} q'^{\mu}}{q'.q_i},
\tag{7.4}
$$

to make the expressions compact. Similarly we can use (5.14) to get the action of the past hard charge on an incoming state. Then we need to substitute for h^1_{rr} and A^1_r.

Classical part. Let us substitute the classical part of h^1_{rr} and A^1_r. As noted from (6.1), the classical part of A^1_r is trivial and does not contribute. Using (6.2) for h^1_{rr}, we get:

$$
\begin{aligned}
& < \text{out}| \left[Q^{\text{hard}}, S\right]_{\text{class}} |\text{in} > \\
& = - \sum_{i,j;\eta_i\eta_j=1} e_i U^{\sigma\mu}(q_i) q_{i[\sigma}\partial_{q_i^{\mu}]}(p_j.q_i)\mathcal{M}_n + \sum_i e_i q_{i\sigma} U^{\sigma\mu} \sum_{j;\eta_j=1} p_{j\mu}\mathcal{M}_n
\end{aligned}
\tag{7.5}
$$

Here, we have $\mathcal{M}_n =< \text{out}|S|\text{in} >$.

Quantum part. Next we will use the quantum part of h^1_{rr} and A^1_r from (6.14) and (6.15) to get:

$$
\begin{aligned}
& < \text{out}| \left[Q^{\text{hard}}, S\right]_{\text{quan}} |\text{in} > \\
& = -\frac{i}{\pi} \sum_{i,j;i\neq j} e_i U^{\sigma\mu}(q_i) q_{i[\sigma}\partial_{q_i^{\mu}]} \left[p_j.q_i \log\left[\frac{2(p_j.q_i)}{m_j^2}\right] + \frac{e_ie_j}{\omega_i}\log\left[\frac{2(p_j.q_i)}{m_j^2}\right]\right]\mathcal{M}_n \\
& \quad + i\int d^2z' D_{\bar{z}}^{\prime 2} V^z(z') \frac{\sqrt{\gamma'_{z\bar{z}}}}{16\pi^3} \sum_i \frac{e_i\epsilon^-.p_i}{q'.p_i} \sum_j q'.p_j \log\left[\frac{2(p_j.q')}{m_j^2}\right]\mathcal{M}_n
\end{aligned}
\tag{7.6}
$$

The terms that depend on m_j are the divergent pieces. The second divergent term in the first line is killed by the derivative operator $q_{i[\sigma}\partial_{q_i^{\mu}]}$. The other two divergent terms cancel each other. Thus, we get a finite action of the charge:

$$
\begin{aligned}
& < \text{out}| \left[Q^{\text{hard}}, S\right]_{\text{quan}} |\text{in}> \\
& = -\frac{i}{\pi} \sum_{i,j;i\neq j} e_i U^{\sigma\mu}(q_i) q_{i[\sigma}\partial_{q_i^{\mu}]} \left[p_j.q_i \log(p_j.q_i) + \frac{e_ie_j}{\omega_i}\log(p_j.q_i)\right] + \frac{i}{\pi}\sum_i e_i q_{i\sigma} \sum_j \tilde{U}_j^{\sigma\mu} p_{j\mu}\mathcal{M}_n
\end{aligned}
\tag{7.7}
$$

here we have defined

$$
\tilde{U}_j^{\sigma\mu}(q_i) = \int d^2z' D_z^{\prime 2} V^z(z,z') \frac{\sqrt{\gamma'^{z\bar{z}}}}{16\pi^2} \frac{\epsilon_-^{\sigma} q'^{\mu}}{q'.q_i} \log(q'.p_j).
\tag{7.8}
$$

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Collecting together (7.5) and (7.7) we get the complete action of the hard charge and we can write down the Ward identity. Thus, the S-matrix needs to satisfy following Ward identity for a generic V^z that lives on a sphere:

$$\Big[Q^{\rm soft}(V^z)\,, S\Big] = -C_{\rm hard}(V^z)S. \tag{7.9}$$

$Q^{\rm soft}(V)$ defined in (5.7) inserts soft modes of photon. We have:

$$\begin{aligned}C_{\rm hard}(V^z) &= \sum_i e_i q_{i\sigma} U^{\sigma\mu} \sum_{j;\eta_j=1} p_{j\mu} - \sum_{i,j;\eta_i\eta_j=1} e_i U^{\sigma\mu}(q_i)(p_{j\mu}q_{i\sigma} - p_{j\sigma}q_{i\mu})\\ &\quad - \frac{i}{\pi}\sum_{i,j;i\neq j} e_i U^{\sigma\mu}(q_i)\left[\frac{e_i e_j}{q_j.q_i}(q_{j\mu}q_{i\sigma} - q_{j\sigma}q_{i\mu}) + (p_{j\mu}q_{i\sigma} - p_{j\sigma}q_{i\mu})\log(-p_j.p_i)\right]^{10}\\ &\quad + \frac{i}{\pi}\sum_i e_i q_{i\sigma}\sum_j \tilde{U}_j^{\sigma\mu} p_{j\mu}\,.\end{aligned} \tag{7.10}$$

Dependence on V^z is via the U's defined in (7.4) and (7.8). Ward identity involving $V^{\bar z}$ can be written down similarly.

7.1 The Sahoo-Sen soft theorem

Let us derive the Sahoo-Sen soft theorem from above Ward identity. To derive negative helicity soft theorem we choose [1]:

$$V^z(z,z') = \sqrt{2}(1+z'\bar{z}')\frac{z-z'}{\bar{z}-\bar{z}'}, V^{\bar z} = 0. \tag{7.11}$$

Performing the sphere $(z', \bar{z}')$ integral in (5.7), we get:

$$Q_+^{\rm soft} = -i\lim_{\omega\to 0}\omega\partial_\omega^2 \omega a_-(\omega,\hat{x}). \tag{7.12}$$

Next we will use (7.11) in the expression for hard charge (7.10). The sphere integral in the expression for U (7.4) and for $\tilde{U}$ in (7.8) can be done easily. We get:

$$\begin{aligned}C_{\rm hard} &= \frac{1}{4\pi}\sum_i e_i \frac{\epsilon.p_i}{p_i.k}\sum_{j;\eta_j=1} k.p_j - \frac{1}{4\pi}\sum_{i,j;i\neq j\ \eta_i\eta_j=1} e_i\frac{\epsilon_\mu k_\rho}{p_i.k}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu)\\ &\quad - \frac{i}{4\pi^2}\sum_{i,j;i\neq j} e_i\frac{\epsilon_\mu k_\rho}{p_i.k}\left[\frac{e_i e_j}{p_i.p_j}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) + (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu)\log[p_i.p_j]\right]\\ &\quad + \frac{i}{4\pi^2}\sum_i e_i\frac{\epsilon.p_i}{p_i.k}\sum_j k.p_j \log p_j.q\,.\end{aligned} \tag{7.13}$$

[10] The first term in (7.5) produces a term that vanishes due to conservation of momenta. This is the term discussed in footnote 2.

So, the Ward identity can be recast as:

$$
\begin{aligned}
&\lim_{\omega\to 0}\omega\partial_\omega^2\omega\mathcal{M}_{n+1}\\
&=\Bigg[-\frac{i}{4\pi}\sum_i e_i\frac{\epsilon.p_i}{p_i.k}\sum_{j;\eta_j=1}k.p_j+\frac{i}{4\pi}\sum_{i,j;i\neq j\ \eta_i\eta_j=1}e_i\frac{\epsilon_\mu k_\rho}{p_i.k}(p_j^\rho p_i^\mu-p_i^\rho p_j^\mu)\\
&\quad-\frac{1}{4\pi^2}\sum_{i,j;i\neq j}e_i\frac{\epsilon_\mu k_\rho}{p_i.k}\left[\frac{e_ie_j}{p_i.p_j}(p_j^\rho p_i^\mu-p_i^\rho p_j^\mu)+(p_j^\rho p_i^\mu-p_i^\rho p_j^\mu)\log[p_i.p_j]\right]\\
&\quad+\frac{1}{4\pi^2}\sum_i e_i\frac{\epsilon.p_i}{p_i.k}\sum_j k.p_j\log p_j.q\Bigg]\mathcal{M}_n.
\end{aligned}\tag{7.14}
$$

This is exactly the Sahoo-Sen soft theorem. So, we have derived the soft theorem from the Ward identity. The Ward identity (with $V^{\bar{z}}=0$) can be derived from the soft theorem by multiplying both sides of the statement of soft theorem with $\int d^2zD_{\bar{z}}^2V^z(z)\frac{\sqrt{\gamma_{z\bar{z}}}}{16\pi^2}$. Thus, we can conclude that the Ward identity (7.9) is exactly equivalent to the Sahoo-Sen soft photon theorem (1.1).

Soft theorems are expected to be related to asymptotic symmetries. It is well known that QED amplitudes exhibit leading soft theorem that is equivalent to Ward identity of large U(1) gauge transformations. In this paper we studied this equivalence for loop level subleading soft theorem. This study was initiated in [1] for the case of massive scalar QED. In this paper, we showed that the Sahoo-Sen soft photon theorem for massless scalar QED coupled to gravity is equivalent to the conservation law in (5.2). It would be interesting to understand the symmetry underlying this conservation law.

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A Calculating the $1/u$ mode in A_A^0

Dressing of scalars under long range forces lead to logarithmic modes in the current:

$$j_A=j_A^{\log}\frac{\log r}{r^2}+\frac{j_A^2}{r^2}+\dots.$$

We also have:

$$j_r=j_r^{\log}\frac{\log r}{r^4}+\frac{j_r^4}{r^4}+\dots,\qquad j_u=\frac{j_u^2}{r^2}+j_u^{\log}\frac{\log r}{r^3}+\dots.$$

For the Cartesian components of the U(1) current we have:

$$j_\mu=\frac{j_\mu^2}{r^2}+j_\mu^{\log}\frac{\log r}{r^3}+\dots.\tag{A.1}$$

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We will substitute above current source in:

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x'\delta((x-x')^2)\Theta(t-t')j_\sigma(x'). \tag{A.2}$$

Let us take the limit $r\to\infty$ keeping u finite:

$$\begin{aligned}
&A_\sigma(u,r,\hat{x})\\
&=\frac{1}{4\pi r}\int_{-\infty}^{\infty}du'\int_0^\infty dr'\int_{S^2}\frac{d^2z'}{-q.q'}\delta\left(r'+\frac{u-u'}{q.q'}\right)\left[j^2_\sigma(u',z')+j^{\log}_\sigma(u',z')\frac{\log r'}{r'}+j^3_\sigma(u',z')\frac{1}{r'}+\dots\right],\\
&=\frac{1}{4\pi r}\int_{-\infty}^{\infty}du'\int_{S^2}d^2z'\left[\frac{j^2_\sigma(u',z')}{-q.q'}+j^{\log}_\sigma(u',z')\frac{\log(u-u')}{u-u'}+\frac{[j^3_\sigma(u',z')-j^{\log}_\sigma(u',z')\log(-q.q')]}{u-u'}\right].
\end{aligned} \tag{A.3}$$

We are interested in studying the u-behaviour in $u\to\infty$ limit. In (A.3), the j^2_σ term contributes to u^0 term as $u\to\infty$. The next dominant fall off in $u\to\infty$ limit is $\frac{\log u}{u}$. It comes from the region $u'\ll u$. Thus, we have:

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r}\int_{-\infty}^{\infty}du'\int_{S^2}d^2z'\left[\frac{j^2_\sigma(u',z')}{-q.q'}+j^{\log}_\sigma(u',z')\frac{\log u}{u}+\dots\right]. \tag{A.4}$$

First we rewrite the coefficient in retarded co-ordinates (Recalling that $q^\mu=(1,\hat{x})$.):

$$j^{\log}_\sigma = -q_\sigma j^{\log}_u + \gamma^{AB}\partial_B q_\sigma j^{\log}_A. \tag{A.5}$$

$j^{\log}_u$ can be eliminated using the conservation equation of current:

$$j^{\log}_u = \partial_u j^{\log}_r - D^A j^{\log}_A. \tag{A.6}$$

Substituting in the expression for $j^{\log}_\sigma$:

$$j^{\log}_\sigma = -q_\sigma\partial_u j^{\log}_r + D^A[q_\sigma j^{\log}_A]. \tag{A.7}$$

Thus, $j^{\log}_\sigma$ is a total derivative. When (A.7) is substituted in (A.4), the $D^A[q_\sigma j^{\log}_A]$ term vanishes trivially due to sphere integral. Using the logarithmic fall off of the gauge field: $A_r=\frac{A^1_r}{r}+A^{\log}_r\frac{\log r}{r^2}+\dots$ in the expression of U(1) current we get

$$j^{\log}_r = -2e^2A^{\log}_r|\phi^1|^2. \tag{A.8}$$

Using (A.8) let us study the behaviour of $j^{\log}_r$ as $|u|\to\infty$. Following the logic of [14], we know that $\phi\sim\frac{1}{u^{1+\epsilon}}$ as $|u|\to\infty$. Now, let us find the u-fall off of $A^{\log}_r$. Using the gauge condition we have: $\partial_u A^{\log}_r = -A^{\log}_u$. Then $A^{\log}_u$ can be related to the current by Maxwell's equation: $2\partial_u A^{\log}_u = j^2_u$. Hence, $A^{\log}_r$ can have a $\mathcal{O}(u)$ term as $|u|\to\infty$. Using these u-fall offs in the expression (A.8) we get $j^{\log}_r\to 0$ as $|u|\to\infty$. Thus, the first term in (A.7) also gives a vanishing contribution. Hence the coefficient of $\frac{\log u}{u}$ vanishes.

The next term falls off as $1/u$ and this is the term that is relevant for loop level charge. Let us rewrite the $1/u$-term in a nice form. To start with, we have:

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r u}\int_{-\infty}^{\infty}du'\int_{S^2}d^2z'\Big[j^3_\sigma(z')-j^{\log}_\sigma(z')\log(-q.q')\Big].$$

We manipulate j^3_σ in similar fashion:

$$j^3_\sigma = -q_\sigma j^3_u + \gamma^{AB}\partial_B q_\sigma j^2_A = -q_\sigma \partial_u j^4_r - q_\sigma j^{\log}_u + D^A[q_\sigma j^2_A], \tag{A.9}$$

and we get:

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi ru}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \Big[-q'_\sigma[j^{\log}_u + \partial'_u j^4_r] + [q'_\sigma \partial_u j^{\log}_r - D'^A[q'_\sigma j^{\log}_A]]\log(-q.q')\Big].$$

We again substitute for $j^{\log}_u$ using (A.6). Upto total sphere derivative terms, above expression can be rewritten as:

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi ru}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \Big[q'_\sigma[D'^A j^{\log}_A + j^{\log}_A D'^A \log(-q.q')] - q'_\sigma \partial'_u[j^{\log}_r + j^4_r - j^{\log}_r \log(-q.q')]\Big].$$

The last term drops out as $j^4_r \to 0$ as $|u| \to \infty$ (proved in [14]) and we have already checked that $j^{\log}_r \to 0$ as $|u| \to \infty$. We can rewrite the first term as

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi ru}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' q^\mu \frac{q'_{[\sigma} D'^A q'_{\mu]}}{q.q'} j^{\log}_A, \tag{A.10}$$

where, $q_{[\mu} D^A q_{\nu]} = q_\mu D^A q_\nu - q_\nu D^A q_\mu$. Finally we perform a co-ordinate transformation (using (2.2)) to get:

$$A^0_{\bar{z}}(u,\hat{x}) = \frac{1}{4\pi u}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \frac{\epsilon^\sigma_- q^\mu}{q.q'} q'_{[\sigma} D'^A q'_{\mu]} j^{\log}_A. \tag{A.11}$$

At past null infinity. Next let us repeat above calculation at past null infinity. We have:

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x' \delta((x-x')^2)\Theta(t-t')\left[\frac{j^2_\mu}{r^2} + j^{\log}_\mu \frac{\log r}{r^3} + \dots\right].$$

Let us take the limit $r \to \infty$ keeping $v = t + r$ finite:

$$\begin{aligned} &A_\sigma(v,r,\hat{x}) \\ &= \frac{1}{4\pi r}\int_{-\infty}^{\infty} dv' \int_0^{\infty} dr' \int_{S^2} \frac{d^2z'}{(1+\hat{x}.\hat{x}')}\delta(r' + 2r(1+\hat{x}.\hat{x}'))\left[j^2_\sigma(v',z') + j^{\log}_\sigma(v',z')\frac{\log r'}{r'} + \dots\right] \end{aligned} \tag{A.12}$$

In (A.12), the j^2_σ term contributes to $\frac{v^0}{r}$ term as $v \to \infty$. The next dominant fall off in $v \to \infty$ limit is $v^0 \frac{\log r}{r^2}$. From above equation, we get:

$$A_\sigma(x) \sim -\frac{\log r}{8\pi r^2}\int_{-\infty}^{\infty} dv' \int_{S^2} \frac{d^2z'}{(q.q')^2} j^{\log}_\sigma(v',\hat{x}').$$

In above expression we have used '$\sim$' as we have written down only the $\frac{\log r}{r^2}$ mode in the field, ignoring other modes which are not relevant to us. Next we repeat the steps similar to futute null infinity calculations and with a co-ordinate transformation we get:

$$F_{rz}|_{\mathcal{I}^-_+} = \frac{1}{8\pi}\frac{\log r}{r^2} q^\nu \partial_z q^\mu \int_{S^2} \frac{d^2z'}{(q.q')^3} q'_{[\mu} D^A q'_{\nu]} j^{\log}_A(v', -\hat{x}') + \dots . \tag{A.13}$$

'$\dots$' denote $\mathcal{O}(\frac{1}{r^2})$ terms. Above expression can be rewritten as:

$$F_{rz}(\hat{x})|_{\mathcal{I}^-_+} = \frac{1}{4\pi}\frac{\log r}{r^2}\int_{-\infty}^{\infty} dv' \int_{S^2} d^2z' \gamma^{z\bar{z}} D^2_z \left[\frac{q^\nu \partial_z q^\mu}{q.q'} q'_{[\mu} D^A q'_{\nu]} j^{\log}_A(v', -\hat{x}')\right]. \tag{A.14}$$

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B Quantum modes in h^1_{rr} and A^1_r

Let us start with the expression for $\overset{\text{quan}}{h^1_{rr}}$ given in (6.13):

$$\overset{\text{quan}}{h^1_{rr}}(x) = \frac{1}{4\pi}\int d^2z'(1+z'\bar{z}')^2 \frac{1}{q'.q}\left[\epsilon^-.q\epsilon^-.q\overset{\log}{C_{zz}}(\hat{x}') + \epsilon^+.q\epsilon^+.q\overset{\log}{C_{\bar{z}\bar{z}}}(\hat{x}')\right]. \tag{B.1}$$

We will use the leading soft theorem to evaluate action of (6.11) and (6.12). So, action of h^1_{rr} on a generic state is given by:

$$\begin{aligned} &< \text{out}|\overset{\text{quan}}{h^1_{rr}}(x)S|\text{in}> \\ &= i < \text{out}|\int \frac{d^2z'}{16\pi^3}\left[\frac{\epsilon^-\cdot q\epsilon^-\cdot q}{q'.q}\sum_j \frac{\epsilon^+\cdot p_j\epsilon^+\cdot p_j}{q'.p_j} + \frac{\epsilon^+\cdot q\epsilon^+\cdot q}{q'.q}\sum_j \frac{\epsilon^-\cdot p_j\epsilon^-\cdot p_j}{q'.p_j}\right] S|\text{in}> . \end{aligned} \tag{B.2}$$

Using completeness relations for polarisation tensors:

$$\frac{\epsilon^-\cdot q\epsilon^-\cdot q}{q'.q}\sum_j \frac{\epsilon^+\cdot p_j\epsilon^+\cdot p_j}{q'.p_j} + \frac{\epsilon^+\cdot q\epsilon^+\cdot q}{q'.q}\sum_j \frac{\epsilon^-\cdot p_j\epsilon^-\cdot p_j}{q'.p_j} = \sum_j \frac{2(q.p_j)^2}{q.q'\ q'.p_j}. \tag{B.3}$$

Thus, in (B.2), we need to do following integral:

$$\begin{aligned} I = \int d^2z' \frac{1}{q.q'\ q'.p_j} &= \int d^2z' \int_0^1 dx \frac{1}{[\hat{q}'.(x\hat{q} + (1-x)\omega_j\hat{q}_j) - x - (1-x)\omega_j]^2}, \\ &= 2\pi\int_0^1 dx \frac{1}{[x(1-x)\omega_j(1-\hat{q}_j.\hat{q})]}. \end{aligned} \tag{B.4}$$

But I is divergent. These are collinear divergences that appear as we are dealing with massless particles. We will see that the diverging terms cancel and the charge is finite. Let us regulate the integral by introducing a regulator m_j by making p_j massive. Repeating previous steps for a massive p_j, we get:

$$I = \frac{4\pi}{q.p_j}\int_0^1 dx \frac{1}{[2x(1-x) + \frac{m_j^2}{q.p_j}(1-x)^2]}.$$

Thus, I still has divergence coming from $x=1$. But we will see that $x=1$ term vanishes due to conservation of momentum.

$$I = \frac{4\pi}{q.p_j}\frac{m_j^2 - 2q.p_j}{(-2q.p_j)}\left[\lim_{x\to 1}\log(1-x) + \log\left[\frac{m_j^2}{m_j^2 - 2q.p_j}\right]\right] \tag{B.5}$$

Let us study above expression in the limit when the regulator is taken to 0:

$$\lim_{m_j\to 0} I = \frac{4\pi}{q.p_j}\left[\lim_{x\to 1}\log(1-x) + \log[m_j^2] - \log[2q.p_j] + \dots\right]. \tag{B.6}$$

Here, '...' denote terms that vanish when regulator is set to 0. The infinite piece is as follows:

$$< \text{out}|\overset{\text{quan}}{h^1_{rr}}(x)S|\text{in} > |_{inf} = \frac{i}{2\pi^2}\sum_j (q.p_j)\left[\lim_{x\to 1}\log(1-x) + \log\left[-\frac{m_j^2}{2}\right]\right]$$
$$= \frac{i}{2\pi^2}\sum_j (q.p_j)\log\left[\frac{m_j^2}{2}\right]. \tag{B.7}$$

Here, the first piece vanishes due to conservation of momenta. We could have regulated the $x = 1$ divergence right from the beginning by introducing a mass for the null vector q^μ,[11] and gotten the same result for I. The finite piece is:

$$< \text{out}|\overset{\text{quan}}{h^1_{rr}}(x)S|\text{in} > = -\frac{i}{2\pi^2}\sum_j (q.p_j)\log(q.p_j). \tag{B.8}$$

Next we will repeat the calculation for gauge field. To start with, we have [1]:

$$\overset{\text{quan}}{A_\mu}(x) = \frac{1}{2\pi}\int d^2z' \frac{(1+|z'|^2)}{\sqrt{2}}\frac{1}{q'.x}\left[\varepsilon^-_\mu \overset{\log}{A_z} + \varepsilon^+_\mu \overset{\log}{A_{\bar{z}}}\right], \tag{B.9}$$

where,

$$\overset{\log}{A_{\bar{z}}}(\hat{x}') = \frac{i}{8\pi^2}\frac{\sqrt{2}}{(1+|z'|^2)}\lim_{\omega\to 0}\omega\left[a_-(\omega,\hat{x}') + a_+^\dagger(-\omega,\hat{x}')\right],$$
$$\overset{\log}{A_z}(\hat{x}') = \frac{i}{8\pi^2}\frac{\sqrt{2}}{(1+|z'|^2)}\lim_{\omega\to 0}\omega\left[a_+(\omega,\hat{x}') + a_-^\dagger(-\omega,\hat{x}')\right]. \tag{B.10}$$

We extract out the $1/r$-term:

$$\overset{\text{quan}}{A^1_r}(\hat{x}) = \frac{1}{2\pi}q^\mu \int d^2z' \frac{(1+|z'|^2)}{\sqrt{2}}\frac{1}{q'.q}\left[\varepsilon^-_\mu \overset{\log}{A_z} + \varepsilon^+_\mu \overset{\log}{A_{\bar{z}}}\right]. \tag{B.11}$$

The action of A^1_r can be evaluated on a generic out state:

$$< \text{out}|\overset{\text{quan}}{A^1_r}(\hat{x})S|\text{in} > = i < \text{out}|\int \frac{d^2z'}{16\pi^3}\left[\frac{\varepsilon^-\cdot q}{q'.q}\sum_j e_j\frac{\varepsilon^+\cdot p_j}{q'.p_j} + \frac{\varepsilon^+\cdot q}{q'.q}\sum_j e_j\frac{\varepsilon^-\cdot p_j}{q'.p_j}\right]S|\text{in} >,$$
$$= \frac{i}{16\pi^3} < \text{out}|\int d^2z' \sum_j e_j \frac{p_j\cdot q}{q'.q\ q'.p_j}S|\text{in} > . \tag{B.12}$$

Above integral can be calculated similar to the earlier one. The infinite piece is a constant:

$$< \text{out}|\overset{\text{quan}}{A^1_r}(x)S|\text{in} > |_{inf} = \frac{i}{4\pi^2}\log\left[\frac{m_j^2}{2}\right]. \tag{B.13}$$

We have for the finite part:

$$< \text{out}|\overset{\text{quan}}{A^1_r}(\hat{x})S|\text{in} > = -\frac{i}{4\pi^2}\sum_j e_j\log(q.p_j). \tag{B.14}$$

[11]The corresponding integral is same as the integral in equation (5.27) of [2] and can be evaluated accordingly.

C Maxwell's equations in presence of gravity

In this section we write down Maxwell's equations in presence of gravitational fluctuations given by (2.9).

Let us study the $\nabla^\mu F_{u\mu} = j_u$ equation. Expanding the equation around $r \to \infty$, at $\mathcal{O}(\frac{1}{r^2})$ we get:

$$\partial_u \overset{2}{F}_{ru} + \partial_u D^B A^0_B - j^2_u = -\gamma^{CB} h^0_{Cr} \partial_u F^0_{uB}. \tag{C.1}$$

In the equation $\nabla^\mu F_{A\mu} = 0$, there appears a gravity correction even at $\mathcal{O}(\frac{1}{r})$:

$$\partial_u \overset{1}{F}_{rA} - h^1_{rr} \partial_u \overset{0}{F}_{Au} = 0. \tag{C.2}$$

This implies $\log r$ dressing of A_A that has also been derived in (4.7). $\nabla^\mu F_{A\mu} = 0$ at $\mathcal{O}(\frac{1}{r^2})$ gives:

$$\begin{aligned} &2\partial_u \overset{2}{F}_{rA} - \partial_A \overset{2}{F}_{ru} + D^B F^0_{AB} - j^2_A \\ &= h^1_{rr} \partial_u \overset{1}{F}_{Au} + h^2_{rr} \partial_u \overset{0}{F}_{Au} - \gamma^{CB} h^0_{Cr} \partial_u F^0_{AB} - \gamma^{CB} h^0_{Cr} D_B F^0_{Au} - \gamma^{BC} h^{-1}_{AB} F^0_{uC} \\ &\quad + \partial_u h^2_{rr} \overset{0}{F}_{Au} + \frac{1}{2} h^1_{rr} \overset{0}{F}_{Au} - \frac{1}{2} \gamma^{BC} h^{-1}_{BC} F^0_{Au} - \gamma^{BC} D_B h^0_{Ar} F^0_{uC} - D^B h^0_{Br} F^0_{Au} - 2h^{1ur} F^0_{Au}. \end{aligned} \tag{C.3}$$

We use above equation to substitute for $\partial_u \overset{2}{F}_{rA}$ in (5.3) i.e. in

$$Q^{\text{soft}}_+ = -\int dud^2z' V^A \partial_u \left[u^2 \partial_u^2 \overset{2}{F}_{rA} \right], \tag{C.4}$$

and we get (5.4) i.e.

$$Q^{\text{soft}}_+ = -\frac{1}{2} \int dud^2z' V^A \partial_u \left[u^2 \partial_u \left[\partial_A \overset{2}{F}_{ru} - D^B F^0_{AB} + j^2_A \right] \right] + \ldots, \tag{C.5}$$

where "..." refers to the gravity corrections that come from r.h.s. of (C.3) and (C.1). We will analyse them one by one. Out of the metric components appearing in Maxwell's equations only h^2_{rr} and h^{-1}_{AB} depend on u, rest of them are u-independent. This simplifies the analysis for most of the terms.

Term $h^1_{rr}\partial_u F^1_{Au}$.

$$Q^{\text{cor}}_1 = -\frac{1}{2} \int dud^2z V^z \partial_u \left[u^2 \partial_u \left[h^1_{rr} \partial_u \overset{1}{F}_{uz} \right] \right] + z \leftrightarrow \bar{z}.$$

Using Bianchi identities we can simplify above expression to:

$$Q^{\text{cor}}_1 = -\frac{1}{2} \int d^2z V^z \int du h^1_{rr} \partial_u \left[u^2 \partial_u^2 D^2_z A^0_{\bar{z}} \right] + z \leftrightarrow \bar{z}. \tag{C.6}$$

The operator picks out difference between boundary values of $\log u$ piece of A_A which is 0.

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Term $h^2_{rr}\partial_u F^0_{uA}$.

$$Q_2^{\rm cor} = -\frac{1}{2}\int dud^2z V^A \partial_u \left[u^2\partial_u\left[h^2_{rr}\partial_u \overset{0}{F}_{uA}\right]\right]. \tag{C.7}$$

h^2_{rr} has atmost a $\mathcal{O}(u)$ term. Using the u-behaviour of $\overset{0}{F}_{uA}$ we see that this term is also 0.

Term $\gamma^{BC}h^{-1}_{BA}F^0_{uC}$.

$$Q_3^{\rm cor} = \frac{1}{2}\int dud^2z V^A \partial_u \left[u^2\partial_u\left[\gamma^{BC}h^{-1}_{BA}F^0_{uC}\right]\right].$$

This term vanishes trivially for classical fall offs of F^0_{uC}. For the quantum $\log u$ fall offs we get 2 terms for A=z (the analysis is similar for $A=\bar{z}$):

$$Q_3^{\rm cor} = -\frac{1}{2}\int dud^2z V^z\gamma^{\bar{z}z}\partial_u h^{-1}_{zz}A^{0,\log}_{\bar{z}} + \frac{1}{2}\int dud^2z V^z\gamma^{\bar{z}z}\partial_u A^0_{\bar{z}}h^{-1,\log}_{zz}. \tag{C.8}$$

Upto unimportant overall factors that are common to both terms, the first integrand is: $\lim_{\omega\to 0}\omega[c_+(\omega)+c_-^\dagger(\omega)]\lim_{\omega\to 0}\omega[a_-(\omega)-a_+^\dagger(\omega)]$. Similarly the second integrand is: $\lim_{\omega\to 0}\omega[c_+(\omega)-c_-^\dagger(\omega)]\lim_{\omega\to 0}\omega[a_-(\omega)+a_+^\dagger(\omega)]$. Thus, $Q_3^{\rm cor}=0$.

Term $h^0_{rC}\partial_u F^0_{AB}$.

$$\begin{aligned} Q_4^{\rm cor} &= \frac{1}{2}\int dud^2z V^A \partial_u \left[u^2\partial_u\left[\gamma^{CB}h^0_{Cr}\partial_u \overset{0}{F}_{AB}\right]\right], \\ &= \frac{1}{2}\int dud^2z \gamma^{CB}h^0_{Cr}V^A\partial_u\left[u^2\partial_u^2\partial_{(B}A^0_{A)}\right]. \end{aligned} \tag{C.9}$$

This is similar to (C.6) and vanishes by same logic. The analysis for rest of the terms is exactly similar.

D Herdegen like representation for graviton

The usual momentum space expression for free metric field is:

$$h_{\mu\nu}(x) = \sum_{r=+,-}\frac{1}{2(2\pi)^3}\int_0^\infty \omega d\omega d^2q\left[e^{-i\omega(u+r-\hat{q}.\vec{x})}c^r_{\mu\nu}(\omega,q) - e^{-i\omega(u+r-\hat{q}.\vec{x})}c^{\dagger r}_{\mu\nu}(-\omega,q)\right]. \tag{D.1}$$

The angular integral can be performed using stationary phase approximation at large r, we can obtain following well known expressions [5]:

$$h_{zz}(u,q) = \frac{r}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega u}\tilde{C}_{zz}(\omega,q). \tag{D.2}$$

where:

$$\tilde{C}_{zz}(\omega,q) = \frac{c_+(\omega,q)}{2\pi i(1+|z|^2)^2}\ldots\omega>0, \quad \tilde{C}_{zz}(\omega,q) = \frac{-\,c_-^\dagger(-\omega,q)}{2\pi i(1+|z|^2)^2}\ldots\omega<0. \tag{D.3}$$

And

$$h_{\bar{z}\bar{z}}(u,q) = \frac{r}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega u}\tilde{C}_{\bar{z}\bar{z}}(\omega,q). \tag{D.4}$$

$$\tilde{C}_{\bar{z}\bar{z}}(\omega) = \frac{c_-(\omega,q)}{2\pi i(1+|z|^2)^2}\dots\omega>0, \quad \tilde{C}_{\bar{z}\bar{z}}(\omega) = \frac{-c_+^\dagger(-\omega,q)}{2\pi i(1+|z|^2)^2}\dots\omega<0. \tag{D.5}$$

Thus (D.1) can be rewritten as:

$$\begin{aligned} h_{\mu\nu}(x) &= \frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}\omega d\omega d^2q(1+|z|^2)^2[\varepsilon^-_{\mu\nu}\tilde{C}_{zz}+\varepsilon^+_{\mu\nu}\tilde{C}_{\bar{z}\bar{z}}]e^{-i\omega(u+r-\hat{q}.\vec{x})}, \\ &= \frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}\omega d\omega d^2q(1+|z|^2)^2\varepsilon^-_{\mu\nu}\left[\int_{-\infty}^{\infty}du' e^{i\omega u'}C_{zz}\right]e^{-i\omega(u+r-\hat{q}.\vec{x})} \\ &\quad+\frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}\omega d\omega d^2q(1+|z|^2)^2\varepsilon^+_{\mu\nu}\left[\int_{-\infty}^{\infty}du' e^{i\omega u'}C_{\bar{z}\bar{z}}\right]e^{-i\omega(u+r-\hat{q}.\vec{x})}, \\ &= -\frac{1}{(4\pi)}\int d^2q(1+|z|^2)^2\Big[\varepsilon^-_\mu\varepsilon^-_\nu\dot{C}_{zz}(u=-x\cdot q,\hat{q})+\varepsilon^+_\mu\varepsilon^+_\nu\dot{C}_{\bar{z}\bar{z}}(u=-x\cdot q,\hat{q})\Big], \end{aligned} \tag{D.6}$$

here, $C_{AB} = \lim_{r\to\infty}\frac{1}{r}h_{AB}$. Above expression is analogous to the expression for gauge field obtained by Herdegen [28].

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