Relativistic effect of J/ψ hadroproduction in large p_T region

Rong Li,^a An-Ping Chen,^{b,c} Jing-Kai Huang^b and Yan-Qing Ma^{b,c,d}

^a*School of Science, Xi'an Jiaotong University,
Xi'an 710049, China*

^b*School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University,
Beijing 100871, China*

^c*Center for High Energy Physics, Peking University,
Beijing 100871, China*

^d*Collaborative Innovation Center of Quantum Matter,
Beijing 100871, China*

E-mail: rongli@xjtu.edu.cn, chenanping@pku.edu.cn,
jkhuanphysics@hotmail.com, yqma@pku.edu.cn

ABSTRACT: By combining NRQCD factorization and collinear factorization, we compute a series of relativistic corrections for J/ψ hadroproduction to all orders in v^2 at large p_T limit. The v^2 expansion converges well for all channels. We find that the ratio of relativistic correction term to the corresponding leading term is independent of kinematic variables for any channel, which generalizes the proportional relations found in previous works to all orders.

KEYWORDS: Perturbative QCD, Resummation

ARXIV EPRINT: [1909.03554](https://arxiv.org/abs/1909.03554)

Contents

1	Introduction	1
2	Relativistic corrections within NRQCD factorization	2
3	Collinear factorization for J/ψ production	5
4	Relativistic effect of color-octet J/ψ production	7
4.1	Relativistic effect in $^3S_1^{[8]}$ channel	7
4.2	Relativistic effect in $^1S_0^{[8]}$ channel	9
4.3	Relativistic effect in $^3P^{[8]}$ channel	11
4.3.1	Axial-vector channel	11
4.3.2	Vector channel	12
4.3.3	Summation of the two channels	13
5	Summary and outlook	14
A	Average over direction of relative momentum	15
A.1	S-wave	15
A.2	P-wave	17

1 Introduction

The study of heavy quarkonium production is important to understand hadronization physics of QCD. Due to the successful performance of the Tevatron and the LHC, the J/ψ hadroproduction receives especial attention recently. The most widely used theory to describe heavy quarkonium production in the past two decades is the non-relativistic QCD (NRQCD) factorization [1]. However, based on it, prediction of prompt J/ψ hadroproduction at leading order (LO) in α_s is significantly transversely polarized [2] which contradicts with the measurement by the CDF Collaboration [3, 4], where one found that the produced J/ψ is almost unpolarized and even slightly longitudinally polarized as p_T increases.

To solve the polarization puzzle, the complete next-to-leading order (NLO) in α_s corrections for all important channels ($^3S_1^{[1]}$, $^1S_0^{[8]}$, $^3S_1^{[8]}$ and $^3P_J^{[8]}$) have been calculated [5–23]. Full NLO prediction can indeed describes both yield and polarization of prompt J/ψ hadroproduction consistently [14, 24], but the corresponding color-octet (CO) long-distance matrix elements (LDMEs) are much larger than that needed in J/ψ production at B factories [25–27], which challenges the universality of LDMEs. It seems like that still some other important contributions have not been included yet in J/ψ hadroproduction. A possible source may be relativistic effects, as the relative momentum between $c\bar{c}$ pair is not very small in J/ψ system ($v^2 \approx 0.3$). Along this line, a soft gluon factorization was proposed

to resum relativistic corrections to all orders [28]. In this paper, we will concentrate on relativistic corrections within NRQCD factorization. For J/ψ hadroproduction, NLO relativistic corrections are calculated in refs. [29–32], where it was found that, although corrections for $^3S_1^{[1]}$ channel are small [29], corrections for CO channels are non-ignorable [30–32]. As NLO relativistic corrections are important, it is also needed to study the importance of even higher order relativistic corrections to explore all important contributions for J/ψ hadroproduction and to test the convergence of relativistic expansion. Another interesting finding in refs. [29, 30] is that, in large transverse momentum p_T limit, the ratio of relativistic correction term to the corresponding leading term is a constant number for each of the four channels. The proportional relation for $^3S_1^{[1]}$ channel is well understood [33], but not for other three channels. An understanding of these proportional relations is helpful to illustrate the structure of relativistic effects.

In high p_T region, it is more convenient to use collinear factorization framework [34–40], which has been rigorously proven to leading power (LP) and next-to-leading power (NLP) in p_T expansion [39]. As contributions beyond NLP are surely negligible when p_T is sufficiently large, this collinear factorization method can capture the main feature in this region. For example, using the collinear factorization, a simple LO calculation can already reproduce NLO NRQCD calculations at large p_T [41]. As the proportional relations mentioned in the last paragraph are large p_T behavior, they should also be reproducible in the collinear factorization. In fact, we will see that, in large p_T limit, relativistic corrections for all the three CO channels can be easily calculated to all orders in v^2 in the framework of collinear factorization and the proportional relations hold to all orders.

The rest of the paper is organized as follows. In section 2, we review the NRQCD factorization and give the explicit large p_T relations that were found in refs. [29, 30]. We then review the collinear factorization for heavy quarkonium production in section 3. The combination of NRQCD factorization and collinear factorization is also discussed. Based on these factorizations, we study the relativistic effects of CO J/ψ production via $^3S_1^{[8]}$, $^1S_0^{[8]}$ and $^3P^{[8]}$ channels in section 4 one by one. We summarize our study in section 5. Finally, we provide formulas to project short distance coefficients to definite orbital angular momentum state in appendix A.

2 Relativistic corrections within NRQCD factorization

In NRQCD factorization, differential cross section of the J/ψ production at hadron colliders is factorized as [1]

$$d\sigma_{A+B \rightarrow J/\psi+X} = \sum_{i,j,n} \int dx_1 dx_2 f_{i/A}(x_1) f_{j/B}(x_2) d\sigma_{i+j \rightarrow c\bar{c}[n]+X_H} \frac{\langle \mathcal{O}_n^{J/\psi} \rangle}{m^{d_n}}, \quad (2.1)$$

where m is the mass of charm quark, the denominator m^{d_n} has the same mass dimension as $\langle \mathcal{O}_n^{J/\psi} \rangle$ which insures $d\sigma$ to have the same mass dimension on both sides, f is parton distribution function and $d\sigma_{i+j \rightarrow c\bar{c}[n]+X_H}$ are short-distance coefficients to produce a $c\bar{c}$ pair with quantum number n . LDMEs $\langle \mathcal{O}_n^{J/\psi} \rangle$ are defined as expectation values of four

fermions operators in vacuum, which can be linear combination of forms as

$$\mathcal{O}_n = \chi^\dagger \mathcal{K}_n'^\dagger \psi \mathcal{P}_{J/\psi} \psi^\dagger \mathcal{K}_n \chi, \quad (2.2)$$

where ψ (χ) is field of heavy (anti-)quark in NRQCD effective field theory [42], $\mathcal{K}_n$ and $\mathcal{K}_n'^\dagger$ contain a color matrix, a spin matrix and a polynomial of covariant differential operator $\mathbf{D}$, respectively and $\mathcal{P}_{J/\psi}$ is the projection operator of J/ψ with the form

$$\mathcal{P}_{J/\psi} = a_{J/\psi}^\dagger a_{J/\psi} = \sum_{X_S} |J/\psi + X_S\rangle \langle J/\psi + X_S|, \quad (2.3)$$

where X_S includes all soft hadrons. In principle, one needs infinite number of LDMEs to reproduce QCD result. However, as each LDME has a definite power counting in v , not all of them are relevant to finite accuracy [1]. The most relevant LDMEs for J/ψ production are $\langle \mathcal{O}^{J/\psi}(^3S_1^{[1]}) \rangle$, $\langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle$, $\langle \mathcal{O}^{J/\psi}(^3S_1^{[8]}) \rangle$ and $\langle \mathcal{O}^{J/\psi}(^3P_J^{[8]}) \rangle$ ($J = 0, 1, 2$), with

$$\begin{aligned} \langle \mathcal{O}^{J/\psi}(^3S_1^{[1]}) \rangle &= \langle 0 | \chi^\dagger \sigma^i \psi \mathcal{P}_{J/\psi} \psi^\dagger \sigma^i \chi | 0 \rangle, \\ \langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle &= \langle 0 | \chi^\dagger T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger T^a \chi | 0 \rangle, \\ \langle \mathcal{O}^{J/\psi}(^3S_1^{[8]}) \rangle &= \langle 0 | \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger \sigma^i T^a \chi | 0 \rangle, \\ \langle \mathcal{O}^{J/\psi}(^3P_0^{[8]}) \rangle &= \frac{1}{3} \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma} \right) T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma} \right) T^a \chi | 0 \rangle, \end{aligned} \quad (2.4)$$

where the operator $\overleftrightarrow{\mathbf{D}}$ is defined as $\chi^\dagger \overleftrightarrow{\mathbf{D}} \psi = \chi^\dagger (\mathbf{D} \psi) - (\mathbf{D} \chi^\dagger) \psi$ and the approximate heavy quark spin symmetry ensures $\langle \mathcal{O}^{J/\psi}(^3P_J^{[8]}) \rangle = (2J+1) \langle \mathcal{O}^{J/\psi}(^3P_0^{[8]}) \rangle (1 + O(v^2))$. We will also use the following definition in this paper

$$\langle \mathcal{O}^{J/\psi}(^3P^{[8]}) \rangle = \sum_{J=0,1,2} \langle \mathcal{O}^{J/\psi}(^3P_J^{[8]}) \rangle = \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}^j \sigma^i \right) T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}^j \sigma^i \right) T^a \chi | 0 \rangle. \quad (2.5)$$

To consider the first order relativistic corrections, one needs the following relatively v^2 suppressed LDMEs [29, 30],

$$\begin{aligned} \langle \mathcal{P}^{J/\psi}(^3S_1^{[1]}) \rangle &= \frac{1}{2} \left[\langle 0 | \chi^\dagger \sigma^i \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi \mathcal{P}_{J/\psi} \psi^\dagger \sigma^i \chi | 0 \rangle + h.c. \right], \\ \langle \mathcal{P}^{J/\psi}(^1S_0^{[8]}) \rangle &= \frac{1}{2} \left[\langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger T^a \chi | 0 \rangle + h.c. \right], \\ \langle \mathcal{P}^{J/\psi}(^3S_1^{[8]}) \rangle &= \frac{1}{2} \left[\langle 0 | \chi^\dagger \sigma^i \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger \sigma^i T^a \chi | 0 \rangle + h.c. \right], \\ \langle \mathcal{P}^{J/\psi}(^3P^{[8]}) \rangle &= \frac{1}{2} \left[\langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}^j \sigma^i \right) T^a \psi \mathcal{P}_{J/\psi} \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}^j \sigma^i \right) T^a \chi | 0 \rangle + h.c. \right], \end{aligned} \quad (2.6)$$

which have two more $\mathbf{D}$'s comparing with the corresponding LDMEs in eqs. (2.4) and (2.5). At even higher order in v , there can be a lot of LDMEs.

To use NRQCD factorization, one needs to calculate short-distance coefficients $d\sigma_{i+j \rightarrow c\bar{c}[n]+X_H}$ in eq. (2.1). We denote the short-distance coefficient as $F(n)$ and $G(n)$ if the corresponding LDME is $\langle \mathcal{O}^{J/\psi}(n) \rangle$ and $\langle \mathcal{P}^{J/\psi}(n) \rangle$, respectively.

In the rest frame of the $c\bar{c}$ pair,¹ we denote their momenta in the amplitude as

$$\begin{aligned} p_c &= (E, \vec{\mathbf{q}}), \\ p_{\bar{c}} &= (E, -\vec{\mathbf{q}}), \end{aligned} \quad (2.7)$$

where $\vec{\mathbf{q}}$ is half of the relative momentum between $c\bar{c}$ pair, $E = \sqrt{m^2 + |\vec{\mathbf{q}}|^2}$ is half of the invariant mass of $c\bar{c}$ pair. In the complex conjugated amplitude, the relative momentum $\vec{\mathbf{q}}'$ can be in principle different from $\vec{\mathbf{q}}$. Momentum conservation gives $p_c + p_{\bar{c}} = p'_c + p'_{\bar{c}}$, which results in $E' = E$ and $|\vec{\mathbf{q}}'| = |\vec{\mathbf{q}}|$. Therefore, we do not distinguish E' and $|\vec{\mathbf{q}}'|$ from E and $|\vec{\mathbf{q}}|$ in the following. It is convenient to define a ratio

$$\beta = \frac{|\vec{\mathbf{q}}|}{E}, \quad (2.8)$$

which is also the same in amplitude and complex conjugated amplitude. In arbitrary frame,

$$\begin{aligned} p_c &= \frac{1}{2}P + q, \\ p_{\bar{c}} &= \frac{1}{2}P - q, \end{aligned} \quad (2.9)$$

with P is the total momentum of $c\bar{c}$ pair and q is boosted from $(0, \vec{\mathbf{q}})$ in the rest frame. To produce a $c\bar{c}[n]$ state, one should project both color and spin of $c\bar{c}$ pair to this definite state. There are two kinds of color states with projection operators

$$\mathcal{C}_1 = \frac{\delta_{ij}}{\sqrt{N_c}}, \quad (2.10a)$$

$$\mathcal{C}_8^a = \sqrt{2}T_{ij}^a, \quad (2.10b)$$

for singlet and octet respectively. For spin singlet or triplet, one needs the following projection operators

$$\Pi_1 = \frac{1}{\sqrt{2E}(E+m)} (\not{p}_{\bar{c}} - m) \frac{2E - \not{P}}{4E} \gamma^5 \frac{2E + \not{P}}{4E} (\not{p}_c + m), \quad (2.11a)$$

$$\Pi_3^\alpha = \frac{1}{\sqrt{2E}(E+m)} (\not{p}_{\bar{c}} - m) \frac{2E - \not{P}}{4E} \gamma^\alpha \frac{2E + \not{P}}{4E} (\not{p}_c + m). \quad (2.11b)$$

Denoting momenta of light partons in initial state and final state as k_1 , k_2 and k_3 respectively, one has Lorentz invariant Mandelstam variables

$$\begin{aligned} s &= (k_1 + k_2)^2 = (P + k_3)^2, \\ t &= (k_2 - k_3)^2 = (P - k_1)^2, \\ u &= (k_1 - k_3)^2 = (P - k_2)^2, \end{aligned} \quad (2.12)$$

¹We call it rest frame in the rest of this paper for simplicity.

with $s + t + u = P^2 = 4E^2$. In principle, both $F(n)$ and $G(n)$ are complicated functions of s , t , u , and m . However, in refs. [29, 30], it was found that there are very simple relations in large p_T limit (i.e. in the limit that $p_T^2 \sim s, t, u \gg m^2$),

$$R^{(1)}(^3S_1^{[1]}) = \left. \frac{G(^3S_1^{[1]})}{F(^3S_1^{[1]})} \right|_{p_T \gg m} = \frac{1}{6}, \quad (2.13a)$$

$$R^{(1)}(^1S_0^{[8]}) = \left. \frac{G(^1S_0^{[8]})}{F(^1S_0^{[8]})} \right|_{p_T \gg m} = -\frac{5}{6}, \quad (2.13b)$$

$$R^{(1)}(^3S_1^{[8]}) = \left. \frac{G(^3S_1^{[8]})}{F(^3S_1^{[8]})} \right|_{p_T \gg m} = -\frac{11}{6}, \quad (2.13c)$$

$$R^{(1)}(^3P^{[8]}) = \left. \frac{G(^3P^{[8]})}{F(^3P^{[8]})} \right|_{p_T \gg m} = -\frac{31}{30}, \quad (2.13d)$$

where the superscript “(1)” means the ratio of the first order of relativistic corrections to the lowest order results. Among them, relation for $^3S_1^{[1]}$ channel is understood in ref. [33]. In the rest of this paper, we will be devoted to understand the proportional relations for CO channels and generalize them to higher orders.

3 Collinear factorization for J/ψ production

Production cross section of the J/ψ in the collinear factorization is given by [39, 40]

$$\begin{aligned} d\sigma_{A+B \rightarrow J/\psi+X}(p) \approx \sum_{i,j} f_{i/A}(x_1) f_{j/B}(x_2) \left\{ \sum_f d\sigma_{i+j \rightarrow f+X}(p_f = \hat{P}/z) \otimes D_{\psi/f}(z, m) \right. \\ \left. + \sum_{[c\bar{c}(\kappa)]} d\sigma_{i+j \rightarrow [c\bar{c}(\kappa)]+X}(\hat{P}(1 \pm \zeta)/2z, \hat{P}(1 \pm \zeta')/2z) \otimes \mathcal{D}_{\psi/[c\bar{c}(\kappa)]}(z, \zeta, \zeta', m) \right\}, \end{aligned} \quad (3.1)$$

where p is the momentum of the observed J/ψ in the final states. The first (second) term on the right-hand side gives the contribution of LP (NLP) in m^2/p_T^2 expansion. $D_{\psi/f}(z, m)$ is the fragmentation function (FF) for J/ψ from a single parton f with momentum fraction z and $\mathcal{D}_{\psi/[c\bar{c}(\kappa)]}(z, \zeta, \zeta', m)$ is the charm quark-pair FF. Operator definition of single parton FF can be found in refs. [34, 35, 43], and that of double parton FF can be found in refs. [39, 40]. The hard-scattering function $d\sigma_{i+j \rightarrow f+X}(p_f = \hat{P}/z)$ ($d\sigma_{i+j \rightarrow [c\bar{c}(\kappa)]+X}(\hat{P}(1 \pm \zeta)/2z, \hat{P}(1 \pm \zeta')/2z)$) describes the production of an on-shell light parton (collinear $c\bar{c}$ pair).

The pair $[c\bar{c}(\kappa)]$ in fragmentation function is massive and, in general, is off shell, thus gauge link is needed to ensure gauge invariance. The charm quark pair $[c\bar{c}(\kappa)]$ in hard part are massless and moving in the “+ z ” direction with light-cone momentum components $\hat{P}^\mu/z = (\hat{P}^+/z, 0, \mathbf{0}_\perp)$, where $\hat{P}^\mu$ is a light like momentum whose plus component equals to the J/ψ ’s plus component $\hat{P}^+ = P_{J/\psi}^+ = n \cdot P_{J/\psi}$. κ represents the pair’s color and spin, which has two kinds of color states with the projection operators defined in eq. (2.10) for

both fragmentation and hard part, and three kinds of spin states, for effective axial vector (a), vector (v) and tensor (t) “currents”, described by relativistic Dirac spin projection operators [39, 40]

$$\tilde{\mathcal{P}}_a = \frac{1}{4p^+} \gamma^+ \gamma_5, \quad (3.2a)$$

$$\tilde{\mathcal{P}}_v = \frac{1}{4p^+} \gamma^+, \quad (3.2b)$$

$$\tilde{\mathcal{P}}_t^\mu = \frac{1}{4p^+} \gamma^+ \gamma_\perp^\mu, \quad (3.2c)$$

for FFs and

$$\mathcal{P}_a = \not{p} \gamma^5, \quad (3.3a)$$

$$\mathcal{P}_v = \not{p}, \quad (3.3b)$$

$$\mathcal{P}_t^\mu = \not{p} \gamma_\perp^\mu, \quad (3.3c)$$

for hard parts. The momentum fractions z , ζ and ζ' are defined as

$$\begin{aligned} \hat{p}_c^\mu &= \frac{1+\zeta}{2z} \hat{P}^\mu, & \hat{p}_{\bar{c}}^\mu &= \frac{1-\zeta}{2z} \hat{P}^\mu, \\ \hat{p}_c'^\mu &= \frac{1+\zeta'}{2z} \hat{P}^\mu, & \hat{p}_{\bar{c}}'^\mu &= \frac{1-\zeta'}{2z} \hat{P}^\mu. \end{aligned} \quad (3.4)$$

z measures the fractional momentum of the collinear $c\bar{c}$ pair carried by J/ψ in this leading region, which is the same on both amplitude side and its complex conjugate side.

Both single parton and double parton FFs can be in principle determined by fitting experimental data. However, it will be much powerful if we combine this collinear factorization with NRQCD factorization. Applying NRQCD factorization to J/ψ FFs, all FFs can be expressed in terms of some unknown LDMEs

$$D_{\psi/f}(z, m) = \sum_n D_{c\bar{c}[n]/f}(z, m) \frac{\langle \mathcal{O}_n^{J/\psi} \rangle}{m^{d_n}}, \quad (3.5)$$

$$\mathcal{D}_{\psi/[c\bar{c}(\kappa)]}(z, \zeta, \zeta', m) = \sum_n \mathcal{D}_{c\bar{c}[n]/[c\bar{c}(\kappa)]}(z, \zeta, \zeta', m) \frac{\langle \mathcal{O}_n^{J/\psi} \rangle}{m^{d_n}}. \quad (3.6)$$

Note the difference that $[c\bar{c}(\kappa)]$ is a perturbative QCD state, while $c\bar{c}[n]$ is a NRQCD state.

What was shown above is a two steps factorization: first, using collinear factorization to express the cross section in terms of hard parts and FFs of a heavy quarkonium; then using NRQCD factorization to express these FFs in terms of LDMEs. However, it is more convenient for us to reorganize the factorization as: first, using NRQCD factorization to express the cross section in terms of short-distance coefficients and LDMEs; then using collinear factorization to express the short-distance coefficients in terms of multiplication of hard parts with FFs of $c\bar{c}[n]$. Based on the later factorization steps, we can study relations between short-distance coefficients of J/ψ production at large p_T limit for each channel directly. Single parton fragmentation functions have been studied extensively [44–68]. Double parton fragmentation functions have also been calculated to order $\alpha_s v^0$ [69–71].

Recall that we are interested in J/ψ production in $^3S_1^{[1]}$, $^3S_1^{[8]}$, $^1S_0^{[8]}$ and $^3P^{[8]}$ channels at LO in α_s . $^3S_1^{[8]}$ channel is dominated by gluon fragmentation and $d\sigma/dp_T^2$ behaves as p_T^{-4} in large p_T limit. $^1S_0^{[8]}$ and $^3P^{[8]}$ channels do not have single particle fragmentation contributions at this order, and their leading contributions come from $c\bar{c}$ pair fragmentation, which behaves as p_T^{-6} . $^3S_1^{[1]}$ channel behaves as p_T^{-8} [33], thus it cannot be interpreted in terms of the above formula. In the following sections, we will discuss the three CO channels one by one.

4 Relativistic effect of color-octet J/ψ production

4.1 Relativistic effect in $^3S_1^{[8]}$ channel

The short-distance coefficient for producing a $c\bar{c}[^3S_1^{[8]}]$ in large p_T limit can be factorized as

$$d\sigma_{i+j \rightarrow c\bar{c}[^3S_1^{[8]}]+k} = \int_0^1 dz \, d\sigma_{i+j \rightarrow g+k}(z) D_{c\bar{c}[^3S_1^{[8]}/g]}(z, m), \quad (4.1)$$

where i, j and k denote various light partons. At LO in α_s , we can denote $D_{c\bar{c}[^3S_1^{[8]}/g]}(z, m) = \overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}(m)\delta(1-z)$, which results in

$$d\sigma_{i+j \rightarrow c\bar{c}[^3S_1^{[8]}]+k} = d\sigma_{i+j \rightarrow g+k}(1) \overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}(m). \quad (4.2)$$

Then we get the ratio

$$R(^3S_1^{[8]}) = \frac{d\sigma_{i+j \rightarrow c\bar{c}[^3S_1^{[8]}]+k}}{d\sigma_{i+j \rightarrow c\bar{c}[^3S_1^{[8]}]+k}^{(0)}} = \frac{\overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}(m)}{\overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}^{(0)}(m)}, \quad (4.3)$$

where the superscript “(0)” denotes keeping the relative momentum to the lowest order. Therefore, to calculate $R(^3S_1^{[8]})$, one needs only calculate $D_{c\bar{c}[^3S_1^{[8]}/g]}(z, m)$. Expanding the above equation to NLO of relative momentum, we get the expression of $R^{(1)}(^3S_1^{[8]})$ in eq. (2.13c)

$$R^{(1)}(^3S_1^{[8]}) = \frac{\overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}^{(1)}(m)}{\overline{D}_{c\bar{c}[^3S_1^{[8]}/g]}^{(0)}(m)}. \quad (4.4)$$

The cut diagram of gluon fragmentation to $c\bar{c}[^3S_1^{[8]}]$ is shown in figure 1. The amplitude is

$$A^{\alpha\mu} = \text{Tr} \left[\mathcal{C}_8^h \Pi_3^\alpha (-ig_s T^c \gamma^\nu) \right] \frac{-ig^{\mu\nu}}{P^2}, \quad (4.5)$$

where α (h) is the polarization (color) index of $c\bar{c}[^3S_1^{[8]}]$. Considering that P^μ and P^α can be set to zero here, we have

$$A^{\alpha\mu} = \frac{g_s \delta^{hc}}{2\sqrt{E}E^2(E+m)} [E(E+m)g^{\alpha\mu} + q^\alpha q^\mu]. \quad (4.6)$$

Figure 1. The cut diagram of gluon fragmentation to $c\bar{c}[^3S_1^{[8]}]$.

Then using eq. (A.11) to project the above amplitude to S-wave by the replacement $q^\alpha q^\mu \rightarrow \Pi^{\alpha\mu}$, we get

$$A^{\alpha\mu} = \frac{g_s \delta^{hc} (2E + m)}{6\sqrt{E}E^2} g^{\alpha\mu}. \quad (4.7)$$

FF is achieved by squaring the amplitude and summing/averaging over quantum numbers

$$D_{c\bar{c}[^3S_1^{[8]}]_g}(z, m) = \frac{1}{2} \frac{1}{3} \frac{1}{(N_c^2 - 1)^2} P_{\alpha\alpha'} d_{\mu\mu'} A^{\alpha\mu} A^{\dagger\alpha'\mu'} \delta(1-z) = \frac{g_s^2}{96} \frac{\left(\frac{1+2\Delta}{3}\right)^2}{\Delta^5 m^3} \delta(1-z), \quad (4.8)$$

with $\Delta = E/m$, $P^{\alpha\beta} = -g^{\alpha\beta} + \frac{P^\alpha P^\beta}{P^2}$ and $d^{\alpha\beta} = -g^{\alpha\beta} + \frac{P^\alpha n^\beta + n^\alpha P^\beta}{P \cdot n} - \frac{P^2 n^\alpha n^\beta}{(P \cdot n)^2}$. This result agrees with the calculation in ref. [28]. Therefore, we get

$$R(^3S_1^{[8]}) = \frac{\left(\frac{1+2\Delta}{3}\right)^2}{\Delta^5}. \quad (4.9)$$

The expansion of $R(^3S_1^{[8]})$ as powers of $\delta = |\vec{q}|^2/m^2$ is

$$R(^3S_1^{[8]}) = 1 - \frac{11}{6}\delta + \frac{191}{72}\delta^2 - \frac{167}{48}\delta^3 + \dots, \quad (4.10)$$

where the second term indeed reproduces the result in eq. (2.13c) of NLO relativistic correction in NRQCD framework. This result tells us that the proportional relation for $^3S_1^{[8]}$ channel holds to all orders in velocity expansion. We will see that this property holds also for the other two channels.

To give an estimation of the convergence of velocity expansion in this part, we choose an average value for δ in eq. (4.10). The expansions of $R(^3S_1^{[8]})$ are listed in table 1, where δ is chosen as 0.3 for J/ψ and 0.1 for Υ . From table 1, we find that after a few corrections, the expansion can converge to its exact result soon. Obviously, the convergence of Υ system is much faster than that of the J/ψ system.

n	0	1	2	3	4	$\dots$	∞
$\sum_{i=0}^n \delta^i R^{(i)}(^3S_1^{[8]}) \Big _{\delta=0.3}$	1	0.450	0.689	0.595	0.630	$\dots$	0.620
$\sum_{i=0}^n \delta^i R^{(i)}(^3S_1^{[8]}) \Big _{\delta=0.1}$	1	0.817	0.843	0.840	0.840	$\dots$	0.840

Table 1. Perturbative expansion of $R(^3S_1^{[8]})$. $\delta = |\vec{q}|^2/m^2$ is chosen to be 0.3 and 0.1 in the second and third row, respectively.

Figure 2. The cut diagram of a $c\bar{c}$ pair in perturbative QCD fragmentate to a $c\bar{c}$ pair in NRQCD.

4.2 Relativistic effect in $^1S_0^{[8]}$ channel

A $c\bar{c}$ pair which fragments to $c\bar{c}[^1S_0^{[8]}]$ at LO in α_s must be in CO. Then, there are three possible channels, $a^{[8]}$, $v^{[8]}$ and $t^{[8]}$. The hard part $i + j \rightarrow [c\bar{c}(t^{[8]})] + k$ at LO in α_s are helicity suppressed, whose contributions are at higher power in m^2/p_T^2 , thus $t^{[8]}$ channel can be neglected. $\mathcal{D}_{c\bar{c}[^1S_0^{[8]}/[c\bar{c}(v^{[8]})]}$ vanishes at LO in α_s because there are only three independent vectors but a γ^5 exists in the trace of gamma matrices. All in all, the short-distance coefficient for producing a $c\bar{c}[^1S_0^{[8]}]$ in large p_T limit can be factorized as

$$d\sigma_{i+j \rightarrow c\bar{c}[^1S_0^{[8]}] + k} = \int dz d\zeta d\zeta' d\sigma_{i+j \rightarrow [c\bar{c}(a^{[8]})] + k}(z, \zeta, \zeta') \mathcal{D}_{c\bar{c}[^1S_0^{[8]}/[c\bar{c}(a^{[8]})]}(z, \zeta, \zeta', m). \quad (4.11)$$

Let's first calculate the fragmentation function $\mathcal{D}_{c\bar{c}[^1S_0^{[8]}/[c\bar{c}(a^{[8]})]}$. The cut diagram for a perturbative QCD $c\bar{c}$ pair fragments to a NRQCD $c\bar{c}$ pair is shown in figure 2. The amplitude for $\mathcal{D}_{c\bar{c}[^1S_0^{[8]}/[c\bar{c}(a^{[8]})]}$ is

$$\begin{aligned} A &= \int \text{Tr} \left[C_8^h \Pi_1 C_8^{h_c} \tilde{\mathcal{P}}_a \right] (2\pi)^4 \delta^4 \left(\frac{P}{2} + q - \frac{P_c}{2} - q_c \right) 2\delta \left(\zeta - \frac{2q_c^+}{P_c^+} \right) \frac{d^4 q_c}{(2\pi)^4} \\ &= \text{Tr} \left[C_8^h \Pi_1 C_8^{h_c} \tilde{\mathcal{P}}_a \right] 2\delta \left(\zeta - \frac{2q^+}{P^+} \right), \end{aligned} \quad (4.12)$$

where the relation $P_c = P$ guaranteed by momentum conservation has been used. It is straightforward to get

$$A = \frac{-2m}{(2E)^{3/2}} \delta^{hh_c} \delta \left(\zeta - \frac{2q^+}{P^+} \right). \quad (4.13)$$

Squaring the amplitude and summing/averaging over quantum numbers, we get

$$\mathcal{D}_{c\bar{c}[^1S_0^{[8]}]/[c\bar{c}(a^{[8]})]}(z, \zeta, \zeta', m) = \frac{4}{(N_c^2 - 1)^2} \frac{8m^2}{(2E)^3} \delta\left(\zeta - \frac{2q^+}{P^+}\right) \delta\left(\zeta' - \frac{2q'^+}{P^+}\right) \delta(1-z). \quad (4.14)$$

It was found in ref. [40] that hard parts have the behavior $d\sigma_{i+j \rightarrow [c\bar{c}(a^{[8]})] + k}(z, \zeta, \zeta') = C_a^{ijk} + (\text{terms odd in } \zeta \text{ or } \zeta')$, where C_a^{ijk} is independent of ζ and ζ' . Thus the eq. (4.11) becomes

$$\begin{aligned} d\sigma_{i+j \rightarrow c\bar{c}[^1S_0^{[8]}] + k} &= \int dz d\zeta d\zeta' (C_a^{ijk} + \text{terms odd in } \zeta \text{ or } \zeta') \\ &\quad \times \frac{m^2}{2(1-\zeta^2)(1-\zeta'^2)(2E)^3} \delta\left(\zeta - \frac{2q^+}{P^+}\right) \delta\left(\zeta' - \frac{2q'^+}{P^+}\right) \delta(1-z) \\ &= (C_a^{ijk} + \text{terms odd in } \zeta \text{ or } \zeta') \frac{m^2}{2(1-\zeta^2)(1-\zeta'^2)(2E)^3} \Big|_{\zeta=\frac{2q^+}{P^+}, \zeta'=\frac{2q'^+}{P^+}} \\ &= C_a^{ijk} \frac{m^2}{2(1-\zeta^2)(1-\zeta'^2)(2E)^3} \Big|_{\zeta=\frac{2q^+}{P^+}, \zeta'=\frac{2q'^+}{P^+}}, \end{aligned} \quad (4.15)$$

where in the last equation we suppress terms odd in ζ (or ζ') which do not contribute to S-wave. Keep to lowest order, one gets

$$d\sigma_{i+j \rightarrow c\bar{c}[^1S_0^{[8]}] + k}^{(0)} = C_a^{ijk} \frac{m^2}{2(2m)^3}, \quad (4.16)$$

and

$$R(^1S_0^{[8]}) = \frac{m^3}{(1-\zeta^2)(1-\zeta'^2)E^3}. \quad (4.17)$$

Note that we have still not projected the result to S-wave in the above expression. This projection can be done using eq. (A.15) by replacing $(1-\zeta'^2)^{-1}$ and $(1-\zeta^2)^{-1} \rightarrow \beta^{-1} \text{arctanh}(\beta)$. The final result is

$$\begin{aligned} R(^1S_0^{[8]}) &= \left(\frac{m}{E}\right)^3 \left(\frac{\text{arctanh}(\beta)}{\beta}\right)^2 \\ &= \frac{\left(\text{arctanh}\left(\sqrt{\frac{\delta}{1+\delta}}\right)\right)^2}{\delta\sqrt{1+\delta}} \\ &= 1 - \frac{5}{6}\delta + \frac{259}{360}\delta^2 - \frac{3229}{5040}\delta^3 + \dots \end{aligned} \quad (4.18)$$

where the second term in the last line indeed reproduces the result in eq. (2.13b) of NLO relativistic correction for $^1S_0^{[8]}$ channel in NRQCD framework, while higher order terms here are new. As an estimate of convergence of perturbative expansion, choosing $\delta = 0.3$ and $\delta = 0.1$, the expansions of $R(^1S_0^{[8]})$ are listed in table 2, where a good convergence similar as $R(^3S_1^{[8]})$ is found.

n	0	1	2	3	$\dots$	∞
$\sum_{i=0}^n \delta^i R^{(i)}(^1S_0^{[8]}) \Big _{\delta=0.3}$	1	0.750	0.815	0.797	$\dots$	0.801
$\sum_{i=0}^n \delta^i R^{(i)}(^1S_0^{[8]}) \Big _{\delta=0.1}$	1	0.917	0.924	0.923	$\dots$	0.923

Table 2. Perturbative expansion of $R(^1S_0^{[8]})$. $\delta = |\vec{q}|^2/m^2$ is chosen to be 0.3 and 0.1 in second and third row, respectively.

4.3 Relativistic effect in $^3P^{[8]}$ channel

Similar analysis as that for $^1S_0^{[8]}$ channel gives that the short-distance coefficient for producing a $c\bar{c}[^3P^{[8]}]$ in large p_T limit can be factorized as

$$\begin{aligned}
 d\sigma_{i+j \rightarrow c\bar{c}[^3P^{[8]}] + k} &= \int dz d\zeta d\zeta' d\sigma_{i+j \rightarrow [c\bar{c}(a^{[8]})] + k}(z, \zeta, \zeta') \mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(a^{[8]})]}(z, \zeta, \zeta', m) \\
 &\quad + d\sigma_{i+j \rightarrow [c\bar{c}(v^{[8]})] + k}(z, \zeta, \zeta') \mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(v^{[8]})]}(z, \zeta, \zeta', m) \\
 &= d\sigma_a + d\sigma_v.
 \end{aligned} \tag{4.19}$$

4.3.1 Axial-vector channel

The amplitude for $\mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(a^{[8]})]}$ is

$$\begin{aligned}
 A_a^\alpha &= \int \text{Tr} \left[\mathcal{C}_8^h \Pi_3^\alpha \mathcal{C}_8^{h_c} \tilde{\mathcal{P}}_a \right] (2\pi)^4 \delta^4 \left(\frac{P}{2} + q - \frac{P_c}{2} - q_c \right) 2\delta \left(\zeta - \frac{2q_c^+}{P_c^+} \right) \frac{d^4 q_c}{(2\pi)^4} \\
 &= \text{Tr} \left[\mathcal{C}_8^h \Pi_3^\alpha \mathcal{C}_8^{h_c} \tilde{\mathcal{P}}_a \right] 2\delta \left(\zeta - \frac{2q^+}{P^+} \right) \\
 &= \frac{2i\epsilon^{Pnq\alpha}}{(2E)^{3/2} P^+} \delta^{hh_c} \delta \left(\zeta - \frac{2q^+}{P^+} \right).
 \end{aligned} \tag{4.20}$$

Squaring the amplitude and summing/averaging over quantum numbers, we get

$$\begin{aligned}
 \mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(a^{[8]})]}(z, \zeta, \zeta', m) &= \frac{1}{9} \frac{1}{(N_c^2 - 1)^2} P_{\alpha\alpha'} A_a^\alpha A_a^{\dagger\alpha'} \delta(1-z) \\
 &= \frac{-q \cdot q' - E^2 \zeta \zeta'}{18(2E)^3} \delta \left(\zeta - \frac{2q^+}{P^+} \right) \delta \left(\zeta' - \frac{2q'^+}{P^+} \right) \delta(1-z),
 \end{aligned} \tag{4.21}$$

which is odd in q and q' if we do not consider delta functions. As only terms with odd number of q and q' contribute to P-wave and hard parts have the behavior $d\sigma_{i+j \rightarrow [c\bar{c}(a^{[8]})] + k}(z, \zeta, \zeta') = C_a^{ijk} + (\text{terms odd in } \zeta \text{ or } \zeta')$, we have

$$\begin{aligned}
 d\sigma_a &= \int dz d\zeta d\zeta' C_a^{ijk} \frac{-q \cdot q' - E^2 \zeta \zeta'}{18(1-\zeta^2)(1-\zeta'^2)(2E)^3} \delta \left(\zeta - \frac{2q^+}{P^+} \right) \delta \left(\zeta' - \frac{2q'^+}{P^+} \right) \delta(1-z) \\
 &= C_a^{ijk} \frac{-q \cdot q' - E^2 \zeta \zeta'}{18(1-\zeta^2)(1-\zeta'^2)(2E)^3} \Big|_{\zeta=\frac{2q^+}{P^+}, \zeta'=\frac{2q'^+}{P^+}}.
 \end{aligned} \tag{4.22}$$

Using eq. (A.32) to project the above expression to P-wave, we get

$$d\sigma_a = C_a^{ijk} \frac{|\vec{q}|^2}{18(2E)^3} \{ (3\Delta_1^2 + \Delta_2\Delta_3) - (\Delta_1^2 + \Delta_2\Delta_3) \} = C_a^{ijk} \frac{|\vec{q}|^2}{9(2E)^3} \Delta_1^2, \quad (4.23)$$

and

$$d\sigma_a^{(0)} = C_a^{ijk} \frac{|\vec{q}|^2}{9(2m)^3}. \quad (4.24)$$

Therefore,

$$\begin{aligned} R_a(^3P^{[8]}) &= \left(\frac{m}{E}\right)^3 \Delta_1^2 \\ &= \frac{9}{4} \frac{\sqrt{1+\delta}}{\delta^2} \left[-\frac{1}{\sqrt{\delta(1+\delta)}} \operatorname{arctanh} \left(\sqrt{\frac{\delta}{1+\delta}} \right) + 1 \right]^2 \\ &= 1 - \frac{11}{10}\delta + \frac{1521}{1400}\delta^2 - \frac{8803}{8400}\delta^3 + \dots \end{aligned} \quad (4.25)$$

4.3.2 Vector channel

The amplitude for $\mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(v^{[8]})]}$ is

$$\begin{aligned} A_v^\alpha &= \int \operatorname{Tr} \left[\mathcal{C}_8^h \Pi_3^\alpha \mathcal{C}_8^{h_c} \tilde{\mathcal{P}}_v \right] (2\pi)^4 \delta^4 \left(\frac{P}{2} + q - \frac{P_c}{2} - q_c \right) 2\delta \left(\zeta - \frac{2q_c^+}{P_c^+} \right) \frac{d^4 q_c}{(2\pi)^4} \\ &= \operatorname{Tr} \left[\mathcal{C}_8^h \Pi_3^\alpha \mathcal{C}_8^{h_c} \tilde{\mathcal{P}}_v \right] 2\delta \left(\zeta - \frac{2q^+}{P^+} \right) \\ &= \frac{-2}{P^+ \sqrt{2E}} \left(E n^\alpha + \frac{q^+ q^\alpha}{E+m} \right) \delta^{hh_c} \delta \left(\zeta - \frac{2q^+}{P^+} \right). \end{aligned} \quad (4.26)$$

Squaring the amplitude and summing/averaging over quantum numbers, we get

$$\begin{aligned} &\mathcal{D}_{c\bar{c}[^3P^{[8]}]/[c\bar{c}(v^{[8]})]}(z, \zeta, \zeta', m) \\ &= \frac{1}{9} \frac{1}{(N_c^2 - 1)^2} P_{\alpha\alpha'} A_v^\alpha A_v^{\dagger\alpha'} \delta(1-z) \\ &= \frac{1 - \frac{E}{E+m}(\zeta^2 + \zeta'^2) - \frac{q \cdot q'}{(E+m)^2} \zeta \zeta'}{18(2E)^3} E^2 \delta \left(\zeta - \frac{2q^+}{P^+} \right) \delta \left(\zeta' - \frac{2q'^+}{P^+} \right) \delta(1-z), \end{aligned} \quad (4.27)$$

which is even in q and q' if we do not consider delta functions. As only terms with odd number of q and q' contribute to P-wave and hard parts have behaviors $d\sigma_{i+j \rightarrow [c\bar{c}(v^{[8]})] + k}(z, \zeta, \zeta') = C_v^{ijk} \zeta \zeta' + (\text{terms even in } \zeta \text{ or } \zeta')$ [40], where C_v^{ijk} is indepen-

dent of ζ and ζ' , we have

$$\begin{aligned}
 d\sigma_v &= \int dz d\zeta d\zeta' C_v^{ijk} \zeta \zeta' \frac{1 - \frac{E}{E+m}(\zeta^2 + \zeta'^2) - \frac{q \cdot q'}{(E+m)^2} \zeta \zeta'}{18(1-\zeta^2)(1-\zeta'^2)(2E)^3} E^2 \delta\left(\zeta - \frac{2q^+}{P^+}\right) \delta\left(\zeta' - \frac{2q'^+}{P^+}\right) \delta(1-z) \\
 &= C_v^{ijk} \zeta \zeta' \frac{1 - \frac{E}{E+m}(\zeta^2 + \zeta'^2) - \frac{q \cdot q'}{(E+m)^2} \zeta \zeta'}{18(1-\zeta^2)(1-\zeta'^2)(2E)^3} E^2 \Bigg|_{\zeta=\frac{2q^+}{P^+}, \zeta'=\frac{2q'^+}{P^+}} \\
 &= \frac{C_v^{ijk}}{144E} \left\{ \frac{m-E}{m+E} \frac{\zeta \zeta'}{(1-\zeta^2)(1-\zeta'^2)} + \frac{E}{m+E} \zeta \zeta' \left(\frac{1}{1-\zeta^2} + \frac{1}{1-\zeta'^2} \right) \right. \\
 &\quad \left. - \frac{q \cdot q'}{(m+E)^2} \left[\frac{1}{(1-\zeta^2)(1-\zeta'^2)} - \frac{1}{1-\zeta^2} - \frac{1}{1-\zeta'^2} + 1 \right] \right\} \Bigg|_{\zeta=\frac{2q^+}{P^+}, \zeta'=\frac{2q'^+}{P^+}}. \tag{4.28}
 \end{aligned}$$

Using eq. (A.32) to project the above expression to P-wave, we get

$$\begin{aligned}
 d\sigma_v &= \frac{|\vec{\mathbf{q}}|^2}{18} \frac{C_v^{ijk}}{8E} \left\{ \frac{m-E}{E^2(m+E)} (\Delta_1^2 + \Delta_2 \Delta_3) + \frac{2}{E(m+E)} (\Delta_1 + \Delta_2) \right. \\
 &\quad \left. + \frac{1}{(m+E)^2} [(3\Delta_1^2 + \Delta_2 \Delta_3) - 2(3\Delta_1 + \Delta_2) + 3] \right\}. \tag{4.29}
 \end{aligned}$$

Considering that

$$\begin{cases} \Delta_1 = 1 + O(\delta), \\ \Delta_2 = \frac{2\delta}{5} + O(\delta^2), \\ \Delta_3 = 2 + O(\delta), \end{cases} \tag{4.30}$$

we get

$$d\sigma_v^{(0)} = C_v^{ijk} \frac{|\vec{\mathbf{q}}|^2}{18(2m)^3}. \tag{4.31}$$

Therefore,

$$\begin{aligned}
 R_v(3P^{[8]}) &= \frac{m^3}{E} \left\{ \frac{m-E}{E^2(m+E)} (\Delta_1^2 + \Delta_2 \Delta_3) + \frac{2}{E(m+E)} (\Delta_1 + \Delta_2) \right. \\
 &\quad \left. + \frac{1}{(m+E)^2} [(3\Delta_1^2 + \Delta_2 \Delta_3) - 2(3\Delta_1 + \Delta_2) + 3] \right\} \\
 &= 1 - \frac{9}{10} \delta + \frac{1069}{1400} \delta^2 - \frac{5549}{8400} \delta^3 + \dots. \tag{4.32}
 \end{aligned}$$

4.3.3 Summation of the two channels

Finally, we get

$$R(3P^{[8]}) = \frac{d\sigma_a + d\sigma_v}{d\sigma_a^{(0)} + d\sigma_v^{(0)}} = \frac{R_a(3P^{[8]}) + \frac{C_v^{ijk}}{2C_a^{ijk}} R_v(3P^{[8]})}{1 + \frac{C_v^{ijk}}{2C_a^{ijk}}}. \tag{4.33}$$

n	0	1	2	3	$\dots$	∞
$\sum_{i=0}^n \delta^i R^{(i)}(^3P^{[8]}) _{\delta=0.3}$	1	0.690	0.778	0.753	$\dots$	0.759
$\sum_{i=0}^n \delta^i R^{(i)}(^3P^{[8]}) _{\delta=0.1}$	1	0.897	0.906	0.906	$\dots$	0.906

Table 3. Perturbative expansion of $R(^3P^{[8]})$. $\delta = |\vec{q}|^2/m^2$ is chosen to be 0.3 and 0.1 in second and third row, respectively.

In ref. [40], it is interesting to find that $C_a^{ijk} = C_v^{ijk}$, which results in

$$\begin{aligned}
 R(^3P^{[8]}) &= \frac{2R_a(^3P^{[8]}) + R_v(^3P^{[8]})}{3} \\
 &= 1 - \frac{31}{30}\delta + \frac{4111}{4200}\delta^2 - \frac{4631}{5040}\delta^3 + \dots,
 \end{aligned}
 \tag{4.34}$$

where the second term in the last line indeed reproduces the result in eq. (2.13d) of NLO relativistic correction for $^3P^{[8]}$ channel in NRQCD framework, while higher order terms here are new. As an estimate of convergence of perturbation expansion, choosing $\delta = 0.3$ and $\delta = 0.1$, the expansions of $R(^3P^{[8]})$ are listed in table 3, where the convergence is good.

5 Summary and outlook

By combining NRQCD factorization and collinear factorization for heavy quarkonium production, we calculate the relativistic correction for J/ψ hadron production to all orders in v^2 at large p_T limit. As large p_T data are very important to determine CO LDMEs in NRQCD, our calculation should be useful for this purpose. It is interesting to find that the ratio of relativistic correction contribution to the leading contribution $R(n)$ is independent of kinematic variables for all production channels, which generalizes the finding in ref. [30] to all orders in v^2 expansion. Specifically, for $^3S_1^{[8]}$ channel, relative momentum does not flow into hard part after the factorization in eq. (4.1), thus kinematics dependence in $R(^3S_1^{[8]})$ cancels between denominator and numerator. For $^1S_0^{[8]}$ channel, relative momentum still flows into hard part in terms of ζ and ζ' even after the factorization in eq. (4.11). However, to produce a S-wave $c\bar{c}$ pair, only a relative momentum independent kinematic configuration C_a^{ijk} in hard part contributes, which cancels between denominator and numerator of $R(^1S_0^{[8]})$. The $^3P^{[8]}$ channel is even complicated because two terms contribute in the factorization formula in eq. (4.19). Fortunately, kinematic configurations for the two terms (C_a^{ijk} and C_v^{ijk}) are identical, thus they can be canceled in $R(^3P^{[8]})$.

By expanding $R(n)$ according to powers of $\delta = |\vec{q}|^2/m^2$, we find $O(\delta)$ terms reproduce large p_T results in ref. [30]. By setting $\delta = 0.3$ or $\delta = 0.1$, we find a good convergence for $R(n)$ expansion.

Note that, the calculation in this paper only covers one series of relativistic correction terms. Another series of relativistic correction terms caused by soft gluons emission are also important for J/ψ hadroproduction, which can be taken into account by the soft gluon factorization [28]. Our calculation is also a start point for the use of soft gluon factorization framework.

Acknowledgments

We thank J.W. Qiu and H. Zhang for useful discussions. This work is supported in part by the National Natural Science Foundation of China (Grant No. 11875071) and the China Postdoctoral Science Foundation under Grant No. 2018M631234.

A Average over direction of relative momentum

In this appendix, we give explicit expressions that project short distance coefficients to definite orbital angular momentum state. Define projection operator F_{lm} , which projects amplitude to state with orbital angular momentum l and polarization m . In the rest frame of the intermediate $c\bar{c}$ pair and parameterizing the relative momentum as $q^\mu = (0, \vec{q}) = |\vec{q}|(0, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$, F_{lm} acts on amplitude $A(\vec{q})$ gives

$$F_{lm}(A(\vec{q})) = \sqrt{\frac{(2l+1)!!}{4\pi(l!)}} \int d\Omega Y_l^m(\theta, \phi) A(\vec{q}), \quad (\text{A.1})$$

where $Y_l^m(\theta, \phi)$ are spherical harmonics functions. If there is no L-S coupling, we can use the following projection operator to get unpolarized differential cross section

$$F_l(M(\vec{q}, \vec{q}')) = \sum_{m=\pm l, \pm(l-1), \dots, 0} \frac{(2l+1)!!}{4\pi(l!)} \int d\Omega d\Omega' Y_l^m(\theta, \phi) (Y_l^m(\theta', \phi'))^* M(\vec{q}, \vec{q}'), \quad (\text{A.2})$$

which acts on the squared amplitudes. In this work we consider only S-wave ($l=0$) and P-wave ($l=1$), explicit spherical harmonics functions of which are

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}.$$

Once the projected results in the rest frame are calculated, results in arbitrary frame can be easily obtained using Lorentz covariance. Specifically, one needs to do the following replacement

$$|\vec{q}|^2 \rightarrow -q^2, \quad (\text{A.3})$$

$$k^0 \rightarrow \frac{P \cdot k}{P^2} P^\mu, \quad (\text{A.4})$$

$$\delta^{ij} \rightarrow \Pi^{\mu\nu} = -g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2}, \quad (\text{A.5})$$

where k^0 is the “0”-component of a momentum k .

A.1 S-wave

For S-wave, projection operator gives

$$F_0(A(\vec{q})) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y_0^0(\theta, \phi) A(\vec{q}) = \frac{1}{4\pi} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi A(\vec{q}), \quad (\text{A.6})$$

which satisfies

$$F_0(1) = 1. \quad (\text{A.7})$$

Because of parity symmetry, F_0 acts on product of odd number of $\vec{\mathbf{q}}$ vanishes

$$F_0 \left(\prod_{j=1}^{2N+1} q^{i_j} \right) = 0. \quad (\text{A.8})$$

F_0 acts on two $\vec{\mathbf{q}}$ gives

$$\begin{aligned} & F_0(\vec{\mathbf{q}} \cdot \vec{\mathbf{k}}_1 \vec{\mathbf{q}} \cdot \vec{\mathbf{k}}_2) \\ &= \frac{|\vec{\mathbf{q}}|^2}{4\pi} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi (k_{1x} \sin\theta \cos\phi + k_{1y} \sin\theta \sin\phi + k_{1z} \cos\theta) \\ & \quad \times (k_{2x} \sin\theta \cos\phi + k_{2y} \sin\theta \sin\phi + k_{2z} \cos\theta) \\ &= \frac{|\vec{\mathbf{q}}|^2}{4\pi} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi (k_{1x} k_{2x} \sin^2\theta \cos^2\phi + k_{1y} k_{2y} \sin^2\theta \sin^2\phi + k_{1z} k_{2z} \cos^2\theta) \\ &= \frac{|\vec{\mathbf{q}}|^2}{3} (k_{1x} k_{2x} + k_{1y} k_{2y} + k_{1z} k_{2z}) \\ &= \frac{|\vec{\mathbf{q}}|^2}{3} \vec{\mathbf{k}}_1 \cdot \vec{\mathbf{k}}_2, \end{aligned} \quad (\text{A.9})$$

which means

$$F_0(q^i q^j) = \frac{|\vec{\mathbf{q}}|^2}{3} \delta^{ij}. \quad (\text{A.10})$$

In arbitrary frame, it gives

$$F_0(q^\mu q^\nu) = \frac{-q^2}{3} \Pi^{\mu\nu}. \quad (\text{A.11})$$

Expression in eqs. (A.10) and (A.11) can be generalized to any even number of $\vec{\mathbf{q}}$. Notice that there is no vector in $F_0(\prod_{j=1}^{2N} q^{i_j})$ after integration, thus its result must be a symmetric tensor containing only terms like δ^{ij} . Therefore we have

$$F_0 \left(\prod_{j=1}^{2N} q^{i_j} \right) \propto \{[\delta]^N\}^{i_1 \dots i_{2N}}, \quad (\text{A.12})$$

where $\{[\delta]^N\}^{i_1 \dots i_{2N}}$ means a tensor combination symmetric under $i_1, \dots, i_{2N}$, and each term in the combination has N delta functions. For example, $\{[\delta]^2\}^{i_1 i_2 i_3 i_4} = \delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3}$. It is easy to find that $\{[\delta]^N\}^{i_1 \dots i_{2N}}$ contains $(2N-1)!!$ terms. Contract both sides of eq. (A.12) with $\delta^{i_1 i_2} \delta^{i_3 i_4} \dots \delta^{i_{2N-1} i_{2N}}$, one can determine its proportionality factor to be $\frac{|\vec{\mathbf{q}}|^{2N}}{(2N+1)!!}$. Thus we get

$$F_0 \left(\prod_{j=1}^{2N} q^{i_j} \right) = \frac{|\vec{\mathbf{q}}|^{2N}}{(2N+1)!!} \{[\delta]^N\}^{i_1 \dots i_{2N}}, \quad (\text{A.13})$$

and

$$F_0 \left(\prod_{j=1}^{2N} q^{\mu_j} \right) = \frac{(-q^2)^N}{(2N+1)!!} \{[\Pi]^N\}^{\mu_1 \cdots \mu_{2N}}. \quad (\text{A.14})$$

In this work, we also need to calculate $F_0(\frac{1}{1-\zeta^2})$ with $\zeta = \frac{2q \cdot n}{P \cdot n}$. It is convenient to calculate it in the rest frame and choose n to only have t and z directions. We get

$$F_0\left(\frac{1}{1-\zeta^2}\right) = \frac{1}{4\pi} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi \frac{1}{1 - \left(\frac{2|\vec{q}|\cos\theta}{2E}\right)^2} = \beta^{-1} \text{arctanh}(\beta), \quad (\text{A.15})$$

where

$$\beta = \frac{|\vec{q}|}{E}. \quad (\text{A.16})$$

A.2 P-wave

For P-wave, projection operator gives

$$F_{1m}(A(\vec{q})) = \sqrt{\frac{3}{4\pi}} \int d\Omega Y_1^m(\theta, \phi) A(\vec{q}) = \sqrt{\frac{3}{4\pi}} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi Y_1^m(\theta, \phi) A(\vec{q}). \quad (\text{A.17})$$

Parity symmetry in this case results in that F_{1m} acts on product of even number of $\vec{q}$ vanishes

$$F_{1m} \left(\prod_{j=1}^{2N} q^{i_j} \right) = 0, \quad (\text{A.18})$$

which can be checked explicitly in eq. (A.17). F_{1m} acts on one $\vec{q}$ gives

$$\begin{aligned} F_{1m}(\vec{q} \cdot \vec{k}) &= \sqrt{\frac{3}{4\pi}} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi Y_1^m(\theta, \phi) |\vec{q}| (k_x \sin\theta \cos\phi + k_y \sin\theta \sin\phi + k_z \cos\theta) \\ &= |\vec{q}| \left(\delta_{m,0} k_z + \delta_{m,1} \frac{-k_x - ik_y}{\sqrt{2}} + \delta_{m,-1} \frac{k_x - ik_y}{\sqrt{2}} \right) \\ &\equiv |\vec{q}| \vec{\epsilon}_m \cdot \vec{k}, \end{aligned} \quad (\text{A.19})$$

where $\vec{\epsilon}_m$ are polarization vectors

$$\vec{\epsilon}_0 = (0, 0, 1), \quad \vec{\epsilon}_{\pm 1} = \frac{1}{\sqrt{2}} (\mp 1, -i, 0), \quad (\text{A.20})$$

which satisfy the orthonormality

$$\sum_{i=1}^3 \epsilon_m^i (\epsilon_{m'}^*)^i = \delta_{mm'}, \quad \text{and} \quad \sum_{m=0,\pm 1} \epsilon_m^i (\epsilon_m^*)^{i'} = \delta^{ii'}. \quad (\text{A.21})$$

Therefore, by summing over m for the squared amplitude, we get

$$\sum_{m=0,\pm 1} F_{1m}(\vec{q} \cdot \vec{k}) \left[F_{1m}(\vec{q}' \cdot \vec{k}') \right]^* = |\vec{q}|^2 \vec{k} \cdot \vec{k}', \quad (\text{A.22})$$

where the fact that $|\vec{q}'| = |\vec{q}|$ is used. Then we have

$$F_1(q^i q'^j) = \sum_{m=0,\pm 1} F_{1m}(q^i) [F_{1m}(q'^j)]^* = |\vec{q}|^2 \delta^{ij}. \quad (\text{A.23})$$

This method also applies when there are more $\vec{q}$ or $\vec{q}'$. For example, one can easily get

$$F_1(q^{i_1} q^{i_2} q^{i_3} q^{i_4}) = \frac{|\vec{q}|^4}{5} (\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3}). \quad (\text{A.24})$$

We then calculate $F_{1m}(\frac{q^i}{1-\zeta^2})$. Choosing the same convention as for the S-wave case, we have

$$\begin{aligned} & F_{1m} \left(\frac{\vec{q} \cdot \vec{k}}{1-\zeta^2} \right) \\ &= \sqrt{\frac{3}{4\pi}} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi Y_1^m(\theta, \phi) \frac{|\vec{q}|}{1 - \left(\frac{2|\vec{q}|\cos\theta}{2E} \right)^2} (k_x \sin\theta \cos\phi + k_y \sin\theta \sin\phi + k_z \cos\theta) \\ &= |\vec{q}| \left\{ \frac{3}{\beta^2} [\beta^{-1} \text{arctanh}(\beta) - 1] \delta_{m,0} k_z + \frac{3}{2\beta^2} [(\beta - \beta^{-1}) \text{arctanh}(\beta) + 1] \right. \\ &\quad \left. \times \left(\delta_{m,1} \frac{-k_x - ik_y}{\sqrt{2}} + \delta_{m,-1} \frac{k_x - ik_y}{\sqrt{2}} \right) \right\} \\ &= |\vec{q}| \left\{ \Delta_1 \vec{\epsilon}_m \cdot \vec{k} + \Delta_2 \delta_{m,0} k_z \right\}, \end{aligned} \quad (\text{A.25})$$

where Δ_1 and Δ_2 are defined as

$$\Delta_1 = \frac{3}{2\beta^2} [(\beta - \beta^{-1}) \text{arctanh}(\beta) + 1], \quad (\text{A.26})$$

$$\Delta_2 = \frac{3}{2\beta^2} [(3\beta^{-1} - \beta) \text{arctanh}(\beta) - 3]. \quad (\text{A.27})$$

Summing over m , we get

$$\begin{aligned} & \sum_{m=0,\pm 1} F_{1m} \left(\frac{\vec{q} \cdot \vec{k}}{1-\zeta^2} \right) \left[F_{1m} \left(\frac{\vec{q}' \cdot \vec{k}'}{1-\zeta'^2} \right) \right]^* \\ &= |\vec{q}|^2 \left\{ \Delta_1^2 \vec{k} \cdot \vec{k}' + \Delta_2 (2\Delta_1 + \Delta_2) k_z k'_z \right\} \\ &= |\vec{q}|^2 \left\{ \Delta_1^2 \vec{k} \cdot \vec{k}' + \Delta_2 \Delta_3 \frac{P^2 \vec{k} \cdot \vec{n} \vec{k}' \cdot \vec{n}}{(P \cdot n)^2} \right\}, \end{aligned} \quad (\text{A.28})$$

where

$$\Delta_3 = 2\Delta_1 + \Delta_2 = \frac{3}{2\beta^2} [(\beta + \beta^{-1}) \text{arctanh}(\beta) - 1]. \quad (\text{A.29})$$

Therefore

$$F_1 \left(\frac{q^i q'^j}{(1-\zeta^2)(1-\zeta'^2)} \right) = |\vec{q}|^2 \left\{ \Delta_1^2 \delta^{ij} + \Delta_2 \Delta_3 \frac{P^2 n^i n^j}{(P \cdot n)^2} \right\}. \quad (\text{A.30})$$

Similarly, we have

$$F_1\left(\frac{q^i q'^j}{1-\zeta^2}\right) = |\vec{q}|^2 \left\{ \Delta_1 \delta^{ij} + \Delta_2 \frac{P^2 n^i n^j}{(P \cdot n)^2} \right\}. \quad (\text{A.31})$$

Generalizing eqs. (A.23), (A.30) and (A.31) to arbitrary frame and contracting indexes, we get the following explicit expressions that are used in this work,

$$F_1(-q \cdot q') = |\vec{q}|^2 3, \quad (\text{A.32a})$$

$$F_1\left(\frac{-q \cdot q'}{1-\zeta^2}\right) = |\vec{q}|^2 (3\Delta_1 + \Delta_2), \quad (\text{A.32b})$$

$$F_1\left(\frac{-q \cdot q'}{(1-\zeta^2)(1-\zeta'^2)}\right) = |\vec{q}|^2 (3\Delta_1^2 + \Delta_2 \Delta_3), \quad (\text{A.32c})$$

$$F_1(\zeta \zeta') = |\vec{q}|^2 \frac{1}{E^2}, \quad (\text{A.32d})$$

$$F_1\left(\frac{\zeta \zeta'}{1-\zeta^2}\right) = |\vec{q}|^2 \frac{1}{E^2} (\Delta_1 + \Delta_2), \quad (\text{A.32e})$$

$$F_1\left(\frac{\zeta \zeta'}{(1-\zeta^2)(1-\zeta'^2)}\right) = |\vec{q}|^2 \frac{1}{E^2} (\Delta_1^2 + \Delta_2 \Delta_3). \quad (\text{A.32f})$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] G.T. Bodwin, E. Braaten and G.P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, *Phys. Rev. D* **51** (1995) 1125 [Erratum *ibid.* **D 55** (1997) 5853] [[hep-ph/9407339](#)] [[INSPIRE](#)].
- [2] E. Braaten, B.A. Kniehl and J. Lee, *Polarization of prompt J/ψ at the Tevatron*, *Phys. Rev. D* **62** (2000) 094005 [[hep-ph/9911436](#)] [[INSPIRE](#)].
- [3] CDF collaboration, *Measurement of J/ψ and $\psi(2S)$ polarization in $p\bar{p}$ collisions at $\sqrt{s} = 1.8$ TeV*, *Phys. Rev. Lett.* **85** (2000) 2886 [[hep-ex/0004027](#)] [[INSPIRE](#)].
- [4] CDF collaboration, *Polarization of J/ψ and ψ_{2S} Mesons Produced in $p\bar{p}$ Collisions at $\sqrt{s} = 1.96$ TeV*, *Phys. Rev. Lett.* **99** (2007) 132001 [[arXiv:0704.0638](#)] [[INSPIRE](#)].
- [5] J.M. Campbell, F. Maltoni and F. Tramontano, *QCD corrections to J/ψ and Υ production at hadron colliders*, *Phys. Rev. Lett.* **98** (2007) 252002 [[hep-ph/0703113](#)] [[INSPIRE](#)].
- [6] P. Artoisenet, J.P. Lansberg and F. Maltoni, *Hadroproduction of J/ψ and Υ in association with a heavy-quark pair*, *Phys. Lett. B* **653** (2007) 60 [[hep-ph/0703129](#)] [[INSPIRE](#)].
- [7] B. Gong and J.-X. Wang, *Next-to-leading-order QCD corrections to J/ψ polarization at Tevatron and Large-Hadron-Collider energies*, *Phys. Rev. Lett.* **100** (2008) 232001 [[arXiv:0802.3727](#)] [[INSPIRE](#)].
- [8] B. Gong and J.-X. Wang, *QCD corrections to polarization of J/ψ and Υ at Tevatron and LHC*, *Phys. Rev. D* **78** (2008) 074011 [[arXiv:0805.2469](#)] [[INSPIRE](#)].
- [9] Y.-Q. Ma, K. Wang and K.-T. Chao, *QCD radiative corrections to χ_{cJ} production at hadron colliders*, *Phys. Rev. D* **83** (2011) 111503 [[arXiv:1002.3987](#)] [[INSPIRE](#)].

- [10] Y.-Q. Ma, K. Wang and K.-T. Chao, $J/\psi(\psi')$ production at the Tevatron and LHC at $\mathcal{O}(\alpha_s^4 v^4)$ in nonrelativistic QCD, *Phys. Rev. Lett.* **106** (2011) 042002 [[arXiv:1009.3655](#)] [[INSPIRE](#)].
- [11] M. Butenschoen and B.A. Kniehl, *Reconciling J/ψ production at HERA, RHIC, Tevatron and LHC with NRQCD factorization at next-to-leading order*, *Phys. Rev. Lett.* **106** (2011) 022003 [[arXiv:1009.5662](#)] [[INSPIRE](#)].
- [12] D. Li, Y.-Q. Ma and K.-T. Chao, χ_{cJ} production associated with a $c\bar{c}$ pair at hadron colliders, *Phys. Rev. D* **83** (2011) 114037 [[arXiv:1106.4262](#)] [[INSPIRE](#)].
- [13] M. Butenschoen and B.A. Kniehl, J/ψ polarization at Tevatron and LHC: Nonrelativistic-QCD factorization at the crossroads, *Phys. Rev. Lett.* **108** (2012) 172002 [[arXiv:1201.1872](#)] [[INSPIRE](#)].
- [14] K.-T. Chao, Y.-Q. Ma, H.-S. Shao, K. Wang and Y.-J. Zhang, J/ψ Polarization at Hadron Colliders in Nonrelativistic QCD, *Phys. Rev. Lett.* **108** (2012) 242004 [[arXiv:1201.2675](#)] [[INSPIRE](#)].
- [15] B. Gong, L.-P. Wan, J.-X. Wang and H.-F. Zhang, Polarization for Prompt J/ψ , $\psi(2S)$ Production at the Tevatron and LHC, *Phys. Rev. Lett.* **110** (2013) 042002 [[arXiv:1205.6682](#)] [[INSPIRE](#)].
- [16] K. Wang, Y.-Q. Ma and K.-T. Chao, $\Upsilon(1S)$ prompt production at the Tevatron and LHC in nonrelativistic QCD, *Phys. Rev. D* **85** (2012) 114003 [[arXiv:1202.6012](#)] [[INSPIRE](#)].
- [17] B. Gong, L.-P. Wan, J.-X. Wang and H.-F. Zhang, Complete next-to-leading-order study on the yield and polarization of $\Upsilon(1S, 2S, 3S)$ at the Tevatron and LHC, *Phys. Rev. Lett.* **112** (2014) 032001 [[arXiv:1305.0748](#)] [[INSPIRE](#)].
- [18] H. Han, Y.-Q. Ma, C. Meng, H.-S. Shao, Y.-J. Zhang and K.-T. Chao, $\Upsilon(nS)$ and $\chi_b(nP)$ production at hadron colliders in nonrelativistic QCD, *Phys. Rev. D* **94** (2016) 014028 [[arXiv:1410.8537](#)] [[INSPIRE](#)].
- [19] M. Butenschoen, Z.-G. He and B.A. Kniehl, η_c production at the LHC challenges nonrelativistic-QCD factorization, *Phys. Rev. Lett.* **114** (2015) 092004 [[arXiv:1411.5287](#)] [[INSPIRE](#)].
- [20] H. Han, Y.-Q. Ma, C. Meng, H.-S. Shao and K.-T. Chao, η_c production at LHC and indications on the understanding of J/ψ production, *Phys. Rev. Lett.* **114** (2015) 092005 [[arXiv:1411.7350](#)] [[INSPIRE](#)].
- [21] H.-F. Zhang, Z. Sun, W.-L. Sang and R. Li, Impact of η_c hadroproduction data on charmonium production and polarization within NRQCD framework, *Phys. Rev. Lett.* **114** (2015) 092006 [[arXiv:1412.0508](#)] [[INSPIRE](#)].
- [22] G.T. Bodwin, K.-T. Chao, H.S. Chung, U.-R. Kim, J. Lee and Y.-Q. Ma, Fragmentation contributions to hadroproduction of prompt J/ψ , χ_{cJ} and $\psi(2S)$ states, *Phys. Rev. D* **93** (2016) 034041 [[arXiv:1509.07904](#)] [[INSPIRE](#)].
- [23] Y. Feng, B. Gong, C.-H. Chang and J.-X. Wang, Remaining parts of the long-standing J/ψ polarization puzzle, *Phys. Rev. D* **99** (2019) 014044 [[arXiv:1810.08989](#)] [[INSPIRE](#)].
- [24] G.T. Bodwin, H.S. Chung, U.-R. Kim and J. Lee, Fragmentation contributions to J/ψ production at the Tevatron and the LHC, *Phys. Rev. Lett.* **113** (2014) 022001 [[arXiv:1403.3612](#)] [[INSPIRE](#)].
- [25] Y.-Q. Ma, Y.-J. Zhang and K.-T. Chao, QCD correction to $e^+e^- \rightarrow J/\psi + gg$ at B Factories, *Phys. Rev. Lett.* **102** (2009) 162002 [[arXiv:0812.5106](#)] [[INSPIRE](#)].

- [26] B. Gong and J.-X. Wang, *Next-to-Leading-Order QCD Corrections to $e^+e^- \rightarrow J/\psi gg$ at the B Factories*, *Phys. Rev. Lett.* **102** (2009) 162003 [[arXiv:0901.0117](#)] [[INSPIRE](#)].
- [27] Y.-J. Zhang, Y.-Q. Ma, K. Wang and K.-T. Chao, *QCD radiative correction to color-octet J/ψ inclusive production at B Factories*, *Phys. Rev. D* **81** (2010) 034015 [[arXiv:0911.2166](#)] [[INSPIRE](#)].
- [28] Y.-Q. Ma and K.-T. Chao, *A new factorization theory for heavy quarkonium production and decay*, [arXiv:1703.08402](#) [[INSPIRE](#)].
- [29] Y. Fan, Y.-Q. Ma and K.-T. Chao, *Relativistic Correction to J/ψ Production at Hadron Colliders*, *Phys. Rev. D* **79** (2009) 114009 [[arXiv:0904.4025](#)] [[INSPIRE](#)].
- [30] G.-Z. Xu, Y.-J. Li, K.-Y. Liu and Y.-J. Zhang, *Relativistic Correction to Color Octet J/ψ Production at Hadron Colliders*, *Phys. Rev. D* **86** (2012) 094017 [[arXiv:1203.0207](#)] [[INSPIRE](#)].
- [31] Z.-G. He and B.A. Kniehl, *Relativistic corrections to prompt J/ψ photo- and hadroproduction*, *Phys. Rev. D* **90** (2014) 014045 [Erratum *ibid.* **D 94** (2016) 079903] [[arXiv:1507.03882](#)] [[INSPIRE](#)].
- [32] Z.-G. He and B.A. Kniehl, *Relativistic corrections to J/ψ polarization in photo- and hadroproduction*, *Phys. Rev. D* **92** (2015) 014009 [[arXiv:1507.03883](#)] [[INSPIRE](#)].
- [33] Y.-Q. Ma, *General form of s , t and u symmetric polynomials and heavy quarkonium physics*, *Phys. Rev. D* **87** (2013) 034001 [[arXiv:1207.3073](#)] [[INSPIRE](#)].
- [34] G.C. Nayak, J.-W. Qiu and G.F. Sterman, *Fragmentation, NRQCD and NNLO factorization analysis in heavy quarkonium production*, *Phys. Rev. D* **72** (2005) 114012 [[hep-ph/0509021](#)] [[INSPIRE](#)].
- [35] G.C. Nayak, J.-W. Qiu and G.F. Sterman, *Fragmentation, factorization and infrared poles in heavy quarkonium production*, *Phys. Lett. B* **613** (2005) 45 [[hep-ph/0501235](#)] [[INSPIRE](#)].
- [36] Z.-B. Kang, J.-W. Qiu and G. Sterman, *Heavy quarkonium production and polarization*, *Phys. Rev. Lett.* **108** (2012) 102002 [[arXiv:1109.1520](#)] [[INSPIRE](#)].
- [37] S. Fleming, A.K. Leibovich, T. Mehen and I.Z. Rothstein, *The Systematics of Quarkonium Production at the LHC and Double Parton Fragmentation*, *Phys. Rev. D* **86** (2012) 094012 [[arXiv:1207.2578](#)] [[INSPIRE](#)].
- [38] S. Fleming, A.K. Leibovich, T. Mehen and I.Z. Rothstein, *Anomalous dimensions of the double parton fragmentation functions*, *Phys. Rev. D* **87** (2013) 074022 [[arXiv:1301.3822](#)] [[INSPIRE](#)].
- [39] Z.-B. Kang, Y.-Q. Ma, J.-W. Qiu and G. Sterman, *Heavy Quarkonium Production at Collider Energies: Factorization and Evolution*, *Phys. Rev. D* **90** (2014) 034006 [[arXiv:1401.0923](#)] [[INSPIRE](#)].
- [40] Z.-B. Kang, Y.-Q. Ma, J.-W. Qiu and G. Sterman, *Heavy Quarkonium Production at Collider Energies: Partonic Cross Section and Polarization*, *Phys. Rev. D* **91** (2015) 014030 [[arXiv:1411.2456](#)] [[INSPIRE](#)].
- [41] Y.-Q. Ma, J.-W. Qiu, G. Sterman and H. Zhang, *Factorized power expansion for high- p_T heavy quarkonium production*, *Phys. Rev. Lett.* **113** (2014) 142002 [[arXiv:1407.0383](#)] [[INSPIRE](#)].
- [42] G.P. Lepage and B.A. Thacker, *Effective Lagrangians for Simulating Heavy Quark Systems*, *Nucl. Phys. Proc. Suppl.* **4** (1988) 199 [[INSPIRE](#)].

- [43] E. Braaten and T.C. Yuan, *Gluon fragmentation into heavy quarkonium*, *Phys. Rev. Lett.* **71** (1993) 1673 [[hep-ph/9303205](#)] [[INSPIRE](#)].
- [44] E. Braaten, K.-m. Cheung and T.C. Yuan, *Z^0 decay into charmonium via charm quark fragmentation*, *Phys. Rev. D* **48** (1993) 4230 [[hep-ph/9302307](#)] [[INSPIRE](#)].
- [45] E. Braaten and T.C. Yuan, *Gluon fragmentation into P wave heavy quarkonium*, *Phys. Rev. D* **50** (1994) 3176 [[hep-ph/9403401](#)] [[INSPIRE](#)].
- [46] E. Braaten and T.C. Yuan, *Gluon fragmentation into spin triplet S wave quarkonium*, *Phys. Rev. D* **52** (1995) 6627 [[hep-ph/9507398](#)] [[INSPIRE](#)].
- [47] J.P. Ma, *Quark fragmentation into p wave triplet quarkonium*, *Phys. Rev. D* **53** (1996) 1185 [[hep-ph/9504263](#)] [[INSPIRE](#)].
- [48] W. Qi, C.F. Qiao and J.X. Wang, *On the Application of Gluon to Heavy Quarkonium Fragmentation Functions*, *Phys. Rev. D* **75** (2007) 074012 [[hep-ph/0701264](#)] [[INSPIRE](#)].
- [49] G. Hao, Y. Zuo and C.-F. Qiao, *The Fragmentation Function of Gluon Splitting into P -wave Spin-singlet Heavy Quarkonium*, [arXiv:0911.5539](#) [[INSPIRE](#)].
- [50] G.T. Bodwin, H.S. Chung, U.-R. Kim and J. Lee, *Quark fragmentation into spin-triplet S -wave quarkonium*, *Phys. Rev. D* **91** (2015) 074013 [[arXiv:1412.7106](#)] [[INSPIRE](#)].
- [51] S.M. Moosavi Nejad and D. Mahdi, *Gluon fragmentation into S -wave heavy quarkonium*, [arXiv:1401.5223](#) [[INSPIRE](#)].
- [52] S.M. Moosavi Nejad, *Fragmentation functions of $g \rightarrow \eta_c(^1S_0)$ and $g \rightarrow J/\psi(^3S_1)$ considering the role of heavy quarkonium spin*, *Eur. Phys. J. Plus* **130** (2015) 136 [[arXiv:1506.08861](#)] [[INSPIRE](#)].
- [53] S.M. Moosavi Nejad and M. Delpasand, *Spin-dependent fragmentation functions of Gluon splitting into heavy quarkonia considering three different scenarios*, *Int. J. Mod. Phys. A* **30** (2015) 1550179 [[arXiv:1512.01342](#)] [[INSPIRE](#)].
- [54] P. Zhang, Y.-Q. Ma, Q. Chen and K.-T. Chao, *Analytical calculation for the gluon fragmentation into spin-triplet S -wave quarkonium*, *Phys. Rev. D* **96** (2017) 094016 [[arXiv:1708.01129](#)] [[INSPIRE](#)].
- [55] F. Feng, S. Ishaq, Y. Jia and J.-Y. Zhang, *Fragmentation function of gluon into spin-singlet P -wave quarkonium*, [arXiv:1712.09986](#) [[INSPIRE](#)].
- [56] S.M. Moosavi Nejad, *The impact of Fermi motion on the heavy quarkonia fragmentation using the light-cone wave function*, *Eur. Phys. J. Plus* **133** (2018) 25 [[INSPIRE](#)].
- [57] E. Braaten and J. Lee, *Next-to-leading order calculation of the color octet 3S_1 gluon fragmentation function for heavy quarkonium*, *Nucl. Phys. B* **586** (2000) 427 [[hep-ph/0004228](#)] [[INSPIRE](#)].
- [58] J. Lee, *Next-to-leading order calculation of a fragmentation function in a light-cone gauge*, *Phys. Rev. D* **71** (2005) 094007 [[hep-ph/0504285](#)] [[INSPIRE](#)].
- [59] G.T. Bodwin and J. Lee, *Relativistic corrections to gluon fragmentation into spin triplet S wave quarkonium*, *Phys. Rev. D* **69** (2004) 054003 [[hep-ph/0308016](#)] [[INSPIRE](#)].
- [60] W.-l. Sang, L.-f. Yang and Y.-q. Chen, *Relativistic corrections to heavy quark fragmentation to S -wave heavy mesons*, *Phys. Rev. D* **80** (2009) 014013 [[INSPIRE](#)].
- [61] G.T. Bodwin, U.-R. Kim and J. Lee, *Higher-order relativistic corrections to gluon fragmentation into spin-triplet S -wave quarkonium*, *JHEP* **11** (2012) 020 [[arXiv:1208.5301](#)] [[INSPIRE](#)].

- [62] X. Gao, Y. Jia, L. Li and X. Xiong, *Relativistic correction to gluon fragmentation function into pseudoscalar quarkonium*, *Chin. Phys. C* **41** (2017) 023103 [[arXiv:1606.07455](#)] [[INSPIRE](#)].
- [63] S.M. Moosavi Nejad and P. Sartipi Yarahmadi, *Heavy quark fragmentation functions at next-to-leading perturbative QCD*, *Eur. Phys. J. A* **52** (2016) 315 [[arXiv:1609.07422](#)] [[INSPIRE](#)].
- [64] R. Sepahvand and S. Dadfar, *NLO corrections to c- and b-quark fragmentation into j/ψ and γ* , *Phys. Rev. D* **95** (2017) 034012 [[INSPIRE](#)].
- [65] P. Artoisenet and E. Braaten, *Gluon fragmentation into quarkonium at next-to-leading order*, *JHEP* **04** (2015) 121 [[arXiv:1412.3834](#)] [[INSPIRE](#)].
- [66] P. Artoisenet and E. Braaten, *Gluon fragmentation into quarkonium at next-to-leading order using FKS subtraction*, *JHEP* **01** (2019) 227 [[arXiv:1810.02448](#)] [[INSPIRE](#)].
- [67] F. Feng and Y. Jia, *Next-to-leading-order QCD corrections to gluon fragmentation into $^1S_0^{(1,8)}$ quarkonia*, [arXiv:1810.04138](#) [[INSPIRE](#)].
- [68] P. Zhang, C.-Y. Wang, X. Liu, Y.-Q. Ma, C. Meng and K.-T. Chao, *Semi-analytical calculation of gluon fragmentation into $^1S_0^{[1,8]}$ quarkonia at next-to-leading order*, *JHEP* **04** (2019) 116 [[arXiv:1810.07656](#)] [[INSPIRE](#)].
- [69] Y.-Q. Ma, J.-W. Qiu and H. Zhang, *Heavy quarkonium fragmentation functions from a heavy quark pair. I. S wave*, *Phys. Rev. D* **89** (2014) 094029 [[arXiv:1311.7078](#)] [[INSPIRE](#)].
- [70] Y.-Q. Ma, J.-W. Qiu and H. Zhang, *Heavy quarkonium fragmentation functions from a heavy quark pair. II. P wave*, *Phys. Rev. D* **89** (2014) 094030 [[arXiv:1401.0524](#)] [[INSPIRE](#)].
- [71] Y.-Q. Ma, J.-W. Qiu and H. Zhang, *Fragmentation functions of polarized heavy quarkonium*, *JHEP* **06** (2015) 021 [[arXiv:1501.04556](#)] [[INSPIRE](#)].