Quiver indices and Abelianization from Jeffrey-Kirwan residues

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ABSTRACT: In quiver quantum mechanics with 4 supercharges, supersymmetric ground states are known to be in one-to-one correspondence with Dolbeault cohomology classes on the moduli space of stable quiver representations. Using supersymmetric localization, the refined Witten index can be expressed as a residue integral with a specific contour prescription, originally due to Jeffrey and Kirwan, depending on the stability parameters. On the other hand, the physical picture of quiver quantum mechanics describing interactions of BPS black holes predicts that the refined Witten index of a non-Abelian quiver can be expressed as a sum of indices for Abelian quivers, weighted by ‘single-centered invariants’. In the case of quivers without oriented loops, we show that this decomposition naturally arises from the residue formula, as a consequence of applying the Cauchy-Bose identity to the vector multiplet contributions. For quivers with loops, the same procedure produces a natural decomposition of the single-centered invariants, which remains to be elucidated. In the process, we clarify some under-appreciated aspects of the localization formula. Part of the results reported herein have been obtained by implementing the Jeffrey-Kirwan residue formula in a public Mathematica code.

KEYWORDS: Supersymmetric Gauge Theory, Black Holes in String Theory, M(atric) Theories

ARXIV EPRINT: [1907.01354](https://arxiv.org/abs/1907.01354)

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1 Introduction

In four-dimensional quantum field theories or string vacua with 8 supercharges, BPS states with mutually non-local electromagnetic charges γ_i ($i = 1 \dots n$), can generically form supersymmetric bound states, or BPS molecules, which are stable in some domain in moduli space. These bound states can be understood as multi-centered black hole solutions to $\mathcal{N} = 2$ supergravity in the string theory context [1, 2], or as multi-centered Dirac-Julia-Zee dyons in the low energy description of $\mathcal{N} = 2$ gauge theories on the Coulomb branch [3, 4].

In the non-relativistic limit, the interactions between the centers can be described by a supersymmetric quantum mechanics of n particles in $\mathbb{R}^3$, interacting through electromagnetic, scalar exchange and possibly gravitational forces [5–9].

Alternatively, in a regime where the BPS constituents can be represented by D-branes, their dynamics is well described by a supersymmetric gauge theory in 0+1 dimension [7, 10–15] with a unitary gauge group $U(N_a)$ for each species of charge α_a occurring among the γ_i 's with multiplicity N_a , and with κ_{ab} chiral multiplets $\phi_{ab,A}$, $A = 1, \dots, \kappa_{ab}$ in the bifundamental representation $(N_a, \bar{N}_b)$ whenever the charges α_a, α_b have non-negative Dirac-Schwinger-Zwanziger product $\kappa_{ab} = \langle \alpha_a, \alpha_b \rangle$. The field content is graphically represented by a quiver Q with vertices $V_a \in \mathcal{V}_Q$ associated to the gauge groups $U(N_a)$ and with arrows $e_{ab,A} \in \mathcal{A}_Q$ associated to the chiral multiplets. We shall refer to the corresponding quantum mechanics with 4 supercharges as quiver quantum mechanics. The dimension vector $(N_1, \dots, N_K)$ is identified with the total electromagnetic charge $\gamma = \sum_{a=1}^K N_a \alpha_a = \sum_{i=1}^n \gamma_i$ of the BPS molecule.

The dependence of the dynamics on the gauge or string theory moduli, which we denote collectively by z , is encoded through Fayet-Iliopoulos parameters $\{\zeta_a(z)\}$ associated to each factor in the gauge group, as well as through a gauge invariant superpotential $\mathcal{W}(\phi, z)$ whenever the quiver has oriented closed loops. BPS bound states correspond to supersymmetric ground states of this quantum mechanics [7]. In the simplest case where the ranks N_a are coprime and the superpotential is generic, these BPS bound states are in one-to-one correspondence with Dolbeault cohomology classes on the moduli space $\mathcal{M}_Q \equiv \mathcal{M}_Q(\{N_a, \zeta_a\})$ of stable representations, an algebraic variety of central interest in representation theory (see e.g. [16]). In particular, the refined Witten index $\Omega(\gamma, z, y) = \text{Tr}(-1)^{2J_3} y^{2(I_3+J_3)}$ in the supersymmetric quantum mechanics (where J_3, I_3 are Cartan generators of the R-symmetry group $SU(2)_L \times SU_2(R)$) coincides (up to a simple prefactor) with the χ -genus of the moduli space $\mathcal{M}_Q$ [17, 18],

$$\Omega(\gamma, z, y) := \chi_Q(\{N_a, \zeta_a\}, y) := \sum_{p,q=0}^d (-1)^{p+q-d} y^{2p-d} h_{p,q}(\mathcal{M}_Q) \quad (1.1)$$

which in turn is the specialization at $t = y$ of the Hodge polynomial

$$\Omega(\gamma, z, y, t) := \sum_{p,q=0}^d (-y)^{p+q-d} t^{p-q} h_{p,q}(\mathcal{M}_Q) \quad (1.2)$$

Here d is the complex dimension of $\mathcal{M}_Q$ and $h_{p,q}$ are the Hodge numbers, such that χ_Q coincides with $(-1)^d$ times the Euler number $\chi(\mathcal{M}_Q)$ when $y = 1$. While the Hodge numbers are insensitive to the choice of superpotential away from complex codimension-one loci, they depend crucially on the stability parameters ζ_a , and may jump across real codimension-one known as ‘walls of marginal stability’, where $\sum_{a=1}^K n_a \zeta_a = 0$ for some positive integers n_a . The jump is given by a universal wall-crossing formula [19–21] which has a transparent physical interpretation in terms of the (dis)appearance of multi-centered black hole solutions [22–24].

In general, the direct computation of the Hodge numbers $h_{p,q}(\mathcal{M}_Q)$ is a difficult task. In the absence of oriented loops and for primitive dimension vector, $\mathcal{M}_Q$ is a pure projective variety, whose Poincaré polynomial, which can be computed by counting points over finite fields [25], and moreover the cohomology is supported in degree (p, p) [26], so the χ -genus coincides with the Poincaré polynomial. Recently, building on previous work on two-dimensional gauge linear sigma models [27, 28],¹ the χ -genus (1.1) for any quiver was computed using the method of supersymmetric localization [31–33], which is closely related to the Atiyah-Bott Lefschetz fixed point theorem. The result is expressed as a suitable combination of residues of a rational function $Z_Q(\{u_{a,i}\})$ of the (complexified) adjoint scalars $u_{a,i}, i = 1 \dots N_a$ for each gauge group $U(N_a)$, restricted to the Cartan torus. For a fixed value of the stability parameters $\{\zeta_a\}$, the contributing poles and the order of integration around each of them are determined according to a prescription originally due to Jeffrey and Kirwan [34]. Different integration contours arise in different chambers, such that discontinuities across walls of marginal stability are consistent with the wall-crossing formula [31]. Despite being completely algorithmic, the residue formula becomes quickly unwieldy even for moderate ranks, due to a proliferation of possible poles and orders of integration.

Based on the interpretation in terms of a multi-centered configuration of n BPS black holes with charges γ_i , it was suggested in [36], and further elaborated in [17, 37, 38], that the Witten index (or χ -genus) of a non-Abelian quiver Q with gauge group $\prod_{a=1}^K U(N_a)$ could be decomposed as a sum of Witten indices $\chi_{Q(\{\gamma_i\})}$ for a family of Abelian quivers $Q(\{\gamma_i\})$ associated to all decompositions of the dimension vector $\gamma = \sum_{a=1}^K N_a \alpha_a$ into a sum $\gamma = \sum_{i=1}^m \gamma_i$ where each $\gamma_i = \sum_{a=1}^K n_{i,a} \alpha_a$ is itself a linear combination of the basis vectors associated to the nodes of the original quiver, with positive integer coefficients $n_{a,i}$. The general formula is known as the ‘Coulomb branch formula’ from [17, 36] (see section 2 below for a precise statement) and the coefficients appearing in front of each $\chi_{Q(\{\gamma_i\})}$ involve a new set of invariants $\bar{\Omega}_S(\gamma_i)$ known as ‘intrinsic Higgs invariants’, ‘quiver invariants’ or ‘single-centered invariants’ [17, 18, 39, 40], which are independent of the stability conditions, and conjecturally depend only on the variable t conjugate to $p - q$ in (1.2), but not on y [17]. These invariants are currently defined in an indirect, recursive way (see [41] for a concise explanation of the Coulomb branch formula). This conjecture, if true, gives a powerful way of obtaining the the full Hodge polynomial from the knowledge of χ -genus of Q and of its subquivers.

For quivers without oriented loops and primitive dimension vector however, the single-centered invariants are known to have support only on multiples of the basic vectors $\gamma_i = \ell \alpha_a$, in which case they are simply given by $\bar{\Omega}_S(\ell \alpha_a) = (y - 1/y)/[\ell(y^\ell - y^{-\ell})]$ for any vertex $V_a \in \mathcal{V}_Q$. The Coulomb branch formula (sometimes known as the MPS formula in this restricted setting) then becomes completely explicit, and reduces to a sum of the χ -genera for the Abelian quivers $Q(\{\gamma_i\})$, with simple combinatorics coefficients. This formula was first derived in [24, appendix D] based on the Reineke formula [25] for quivers without loops, and put on a rigorous mathematical ground in [42].

¹This type of computation was pioneered in [29, 30].

Our main goal in this paper will be to derive the Abelianization formula for quivers without oriented loops (where the MPS formula is already known to hold) by manipulating the residue formula of [31]. The key idea is to use the Cauchy-Bose formula

$$\det \frac{1}{\sinh(\mu_i - \nu_j)} = \frac{\prod_{i < j} \sinh(\mu_i - \mu_j) \sinh(\nu_j - \nu_i)}{\prod_{i,j} \sinh(\mu_i - \nu_j)} \quad (1.3)$$

to decompose the vector multiplet determinant for a given $U(N_a)$ gauge group into a sum over conjugacy classes in the permutation group S_{N_a} , which are labelled by partitions $\lambda = \sum_{\ell \geq 1} \ell N_{a,\ell}$. Applying (1.3) to each $U(N_a)$ factor, leads to a sum over all decompositions of the dimension vector $\gamma = \sum_i \gamma_i$ where the γ_i 's are positive linear combinations of the vectors $\ell \alpha_a$. We shall see that the contribution χ_Q^λ of permutations² in the conjugacy class λ associated to a decomposition $\gamma = \sum_{\ell,a} n_{\ell,a}(\ell \alpha_a)$ reproduce the contribution of the Abelian quiver $Q(\{\gamma_i\})$ to the MPS formula for χ_Q . We note that the formula (1.3) (or rather its rational limit) was used in a similar context in the computation of the index of $\mathcal{N} = 4$ SYM in [30], and as a tool to construct grand canonical partition functions for matrix models in [30, 43, 44]. An elliptic version (3.13) of this equation also allows to implement a similar Abelianization process for computing the elliptic genus two-dimensional gauge linear sigma models [28]. Finally, we note that the Abelianization formula for quivers was also investigated using toric geometry in [45], where a similar but less manipulation of the vector multiplet determinant was used.

The outline of this work is as follows. In section 2, we recall basic definitions and facts about quiver moduli spaces, and briefly review the Coulomb branch formula and the MPS formula. In section 3, we review the residue prescription of [28, 31] for computing the χ -genus $\chi_Q(\{N_a, \zeta_a\})$ of the quiver moduli space $\mathcal{M}_Q$, recall the relation with the Atiyah-Bott fixed point theorem, examine its value in the attractor chamber, and introduce the Cauchy-Bose formula as a useful way of decomposing the index into a sum over partitions of the dimension vector. In section 4, we apply the residue prescription to various examples of Abelian quivers with or without oriented loop, clarifying the origin of the stability condition for flags contributing to the residue formula. In section 5, we consider various examples of non-Abelian quivers, and explain the origin of the various terms in the Coulomb branch or MPS formula in terms of conjugacy classes in the Cauchy-Bose formula. In appendix A, we indicate how to reproduce some of our results by using the mathematica package `CoulombHiggs.m` developed by the last-named author, which was originally released along with [37], and has been extended to include the Jeffrey-Kirwan residue formula and other functionalities.

2 A brief review of the Coulomb branch formula for quivers

Let Q be a quiver with vertices $V_a \in \mathcal{V}_Q$ ($a = 1 \dots K$), arrows $e_{ab,A} \in \mathcal{A}_Q$. We assume that there are no arrows from one vertex to itself, and that all arrows $e_{ab,A}$ between vertices V_a and V_b point in the same direction. We denote by κ_{ab} the number of arrows from V_a to V_b ,

²In fact, all permutations in the same conjugacy class turn out to contribute equally, naturally leading to the Boltzmann symmetry factor in the MPS formula.

and by $-\kappa_{ab}$ the number of arrows from V_b to V_a , so that κ_{ab} is an antisymmetric matrix with integer entries. Let $\gamma = (N_1, \dots, N_K)$ be a vector of non-negative integers, known as the dimension vector. The quiver quantum mechanics associated to (Q, γ) is a 0+1 dimensional gauge theory with four supercharges [7]. It includes vector multiplets for the gauge group $G = \prod_{a=1}^K \mathrm{U}(N_a)$ and $|\alpha_{ab}|$ chiral multiplets transforming in the bifundamental representation $(N_a, \bar{N}_b)$ if $\alpha_{ab} > 0$, or its complex conjugate $(\bar{N}_a, N_b)$ if $\alpha_{ab} < 0$. We shall denote the bosonic component of these chiral multiplets by $\phi_{ab,A,ss'}$, where $1 \leq A \leq |\alpha_{ab}|$, $1 \leq s \leq N_a$, $1 \leq s' \leq N_b$. When the quiver has oriented loops, the Lagrangian depends on a superpotential $\mathcal{W}(\phi)$, which is a sum of G -invariant monomials in the chiral multiplets $\phi_{ab,A,ss'}$. In addition, the Lagrangian depends on a real vector $\zeta = (\zeta_1, \dots, \zeta_K)$, known as the stability vector, whose entries are Fayet-Iliopoulos (FI) parameters for the $\mathrm{U}(1)$ center in each gauge group $\mathrm{U}(N_a)$. Since the diagonal $\mathrm{U}(1)$ action leaves all fields invariants, we may assume that $\sum_{a=1}^K N_a \zeta_a = 0$. For the purpose of counting BPS states, the overall scale of the ζ_a 's is also irrelevant, so this vector can be viewed as a point in real projective space $\mathbb{RP}^K$.

Semi-classically, the quiver quantum mechanics admits two branches of supersymmetric vacua. On the Higgs branch, the gauge symmetry is broken to the $\mathrm{U}(1)$ center by the vevs of the chiral multiplet scalars $\phi_{ab,A,ss'}$, which are subject to the D and F-term relations

$$\sum_{b: \kappa_{ab} > 0} \sum_{\substack{s'=1 \dots b \\ A=1 \dots \kappa_{ab}}} \phi_{ab,A,ss'}^* \phi_{ab,A,ts'} - \sum_{b: \kappa_{ab} < 0} \sum_{\substack{s'=1 \dots b \\ A=1 \dots |\kappa_{ab}|}} \phi_{ba,A,s's}^* \phi_{ba,A,s't} = \zeta_a \delta_{st} \quad \forall a, s, t \quad (2.1)$$

$$\frac{\partial \mathcal{W}}{\partial \phi_{ab,A,ss'}} = 0 \quad \forall a, b, A, s, s' \quad (2.2)$$

where $1 \leq a \leq K$, $1 \leq s, t \leq N_a$ in the first equation while $1 \leq a, b \leq K$, $1 \leq s \leq N_a$, $1 \leq s' \leq N_b$, $1 \leq A \leq \kappa_{ab}$ in the second equation, whenever $\kappa_{ab} > 0$. Classical supersymmetric vacua are in one-to-one correspondence with orbits of solutions to (2.1), (2.2) under the compact gauge group $G = \prod_{a=1}^K \mathrm{U}(N_a)$. Equivalently, they are in one-to-one correspondence with stable orbits of solutions of (2.2) under the action of the complexified gauge group $G_{\mathbb{C}} = \prod_{a=1}^K \mathrm{GL}(N_a, \mathbb{C})$, where the stability condition is determined by the vector ζ . The set $\mathcal{M}_Q$ of supersymmetric vacua thus coincides with the moduli space of stable quiver representations widely studied in mathematics (see e.g. [16, 46] for entry points in the vast literature on this subject). Quantum mechanically, BPS states on the Higgs branch are harmonic forms on $\mathcal{M}_Q$, or equivalently Dolbeault cohomology classes [7]. The group $\mathrm{SO}(3)$ associated to physical rotations in $\mathbb{R}^3$ acts on the cohomology of the Higgs branch via the Lefschetz action generated by contraction and wedge product with the natural Kähler form on $\mathcal{M}_Q$, induced from the flat Kähler form on the ambient space $\oplus_{e_{ab,A} \in \mathcal{A}_Q} \mathbb{C}^{N_a} \otimes \mathbb{C}^{N_b}$.

On the Coulomb branch, the gauge symmetry is broken to the diagonal subgroup $\mathrm{U}(1)^{\sum_{a=1}^K N_a}$ and all chiral multiplets as well as off-diagonal vector multiplets are massive. After integrating out these degrees of freedom, the diagonal part $\vec{r}_i$ of the scalars in the

vector multiplets must be solutions to Denef's equations [7]

$$\forall i = 1 \dots n, \quad \sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i. \quad (2.3)$$

Here, the index i runs over all $n = \sum_a N_a$ pairs (a, s) with $s = 1 \dots N_a$, and the corresponding γ_{ij} and c_i are equal to α_{ab} and ζ_a , in such a way that $\sum_{i=1}^n c_i = \sum_{a=1}^K N_a \zeta_a = 0$. The same equations (2.3) govern the positions of n centers with charges γ_i such that $\langle \gamma_i, \gamma_j \rangle = \gamma_{ij}$ in $\mathcal{N} = 2$ supergravity [1, 2]. The space of solutions modulo common translations is a phase space $\mathcal{M}_n(\{\gamma_{ij}, c_i\})$ of dimension $2n - 2$, equipped with a natural symplectic form (inherited from the symplectic form on the full non-BPS phase space) such that the moment map for spatial rotations is given by the total angular momentum $\vec{J} = \frac{1}{2} \sum_{i < j} \gamma_{ij} (\vec{r}_i - \vec{r}_j) / |\vec{r}_i - \vec{r}_j|$ [47]. BPS states on the Coulomb branch are harmonic spinors for the natural Dirac operator on $\mathcal{M}_n(\{\gamma_{ij}, c_i\})$ [9, 47].

The Witten index $\Omega(\gamma, z, y)$ defined in (1.1) counts BPS states on the Higgs branch. In the case where the dimension vector γ is primitive and the superpotential $\mathcal{W}$ is generic, $\mathcal{M}_Q$ is compact, so $\Omega(\gamma, z, y)$ is a symmetric Laurent polynomial in y , which can be viewed as the character of the Lefschetz action of $\text{SO}(3)$ on the cohomology of the quiver moduli space $\mathcal{M}_Q$. When γ is not primitive, $\mathcal{M}_Q$ is no longer compact, but one can still define the χ -genus using intersection cohomology. It is useful to introduce the rational invariant [36]

$$\bar{\Omega}(\gamma, z, y) := \bar{\chi}_Q(\{N_a, \zeta_a\}, y) := \sum_{d|N_a} \frac{y - 1/y}{d(y^d - y^{-d})} \chi_Q(\{N_a/d, \zeta_a\}, y^d), \quad (2.4)$$

which coincides with $\chi_Q(\{N_a, \zeta_a\}, y)$ when the dimension vector $\gamma = (N_1, \dots, N_K)$ is primitive, but is in general a rational function of y whenever γ is not primitive. The advantage is that $\bar{\chi}_Q(\{N_a, \zeta_a\}, y)$ satisfies a much simpler wall-crossing formula than $\chi_Q(\{N_a, \zeta_a\}, y)$ [20, 21, 24].

2.1 The Coulomb branch formula

The Coulomb branch formula conjecturally expresses the rational index $\bar{\Omega}(\gamma, z, y)$ in terms of single-centered indices $\bar{\Omega}_S(\gamma_i)$ as follows [36, 37, 41]:

$$\bar{\chi}_Q(\{N_a, \zeta_a\}, y) = \sum_{\gamma = \sum_{i=1}^n \gamma_i} \frac{g_C(\{\gamma_i, c_i\}, y)}{|\text{Aut}\{\gamma_i\}|} \prod_{i=1}^n \bar{\Omega}_T(\gamma_i, y) \quad (2.5)$$

where $\bar{\Omega}_T(\gamma_i, y)$ is constructed in terms of $\Omega_T(\gamma_i, y)$ by a relation similar to (2.4). The ‘total’ invariant $\Omega_T(\gamma_i, y)$ is in turn determined in terms of the single-centered invariants $\Omega_S(\gamma_i, y)$ via

$$\Omega_T(\gamma, y) = \Omega_S(\gamma, y) + \sum_{\gamma = \sum_{i=1}^m m_i \beta_i} H(\{\beta_i, m_i\}, y) \prod_{i=1}^m \Omega_S(\beta_i, y^{m_i}). \quad (2.6)$$

In (2.5), the sum runs over unordered decompositions of γ into a sum of vectors $\gamma_i = \sum_a n_{i,a} \alpha_a$ which are linear combinations of the basis vectors α_a with positive integer coefficients. Similarly, in (2.6) the sums run over unordered decompositions of γ into sums of

vectors $m_i \beta_i$ with $m_i \geq 1$ and β_i a linear combination of the α_a 's with positive integer coefficients.³ The functions $H(\{\beta_i, m_i\}, y)$ are determined recursively by the so called “minimal modification hypothesis” (see [37, 41] for details) and their role is to ensure that the full refined index $\bar{\Omega}(\gamma, z)$ is a symmetric Laurent polynomial in y . The function $g_C(\{\gamma_i, c_i\}, y)$, known as the Coulomb index, is the only quantity on the r.h.s. of (2.5) which depends on the stability parameters ζ_a . It is defined as the equivariant index of the Dirac operator on the phase space $\mathcal{M}_n(\{\gamma_i, c_i\})$, computed by localization with respect to rotations around a fixed axis [24, 36, 37]. The fixed points of the action of J_3 on $\mathcal{M}_n(\{\gamma_i, c_i\})$ are collinear black hole solutions, which are classified by permutations σ of $\{1, 2, \dots, n\}$,

$$g_C(\{\gamma_i, c_i\}, y) = \frac{(-1)^{n-1+\sum_{i<j} \gamma_{ij}}}{(y - y^{-1})^{n-1}} \sum_{\sigma \in S_n} F_{C,n}(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\}) y^{\sum_{i<j} \gamma_{\sigma(i)\sigma(j)}}, \quad (2.7)$$

where the ‘partial Coulomb index’ $F_{C,n}(\{\gamma_i, c_i\}) \in \mathbb{Z}$ counts (with sign) collinear solutions for a fixed ordering $x_1 < x_2 < \dots < x_n$ along the axis. A recursive procedure for computing it was given in [37] and has been implemented in a Mathematica package (see section A).

When the Abelian quiver $Q(\{\gamma_i\})$ constructed from the adjacency matrix $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle = \sum_{a,b} n_{i,a} n_{j,b} \kappa_{ab}$ has no oriented loop, the Coulomb index (2.7) coincides with the χ -index $\chi_{Q(\{\gamma_i\})}$ with stability parameters $c_i = \sum_a n_{i,a} \zeta_a$, in particular it is a symmetric Laurent polynomial in y . When $Q(\{\gamma_i\})$ has oriented closed loops, this relation is lost, and the Coulomb index (2.7) is in general a rational function. The functions $H(\{\beta_i, m_i\}, y)$ are then adjusted in such a way that the full index $\bar{\Omega}(\gamma, z)$ obtained via (2.5) is a symmetric Laurent polynomial in y , provided the single-centered indices $\Omega_S(\gamma_i, y)$ are. The minimal modification hypothesis of [37] gives a unique prescription for computing H , based on the assumption that the missing contributions from the boundary of $\mathcal{M}_n$ carry the minimal possible angular momentum. Note that this prescription does not take into account the condition of absence of closed timelike curves, which is irrelevant in the context of quiver quantum mechanics, but needs to be checked by hand for general supergravity bound states (see e.g. [36, §3.2] for an example where this condition makes an important difference).

In the special case where the dimension vector γ is primitive and such that all charge vectors γ_i appearing in each decomposition $\gamma = \sum \gamma_i$ are distinct and primitive (which in particular applies when the original quiver Q is Abelian), the Coulomb branch formula (2.5) simplifies to

$$\Omega(\gamma, z, y) = \sum_{\gamma = \sum_{i=1}^n \gamma_i} g_C(\{\gamma_i, c_i\}, y) \prod_{i=1}^n \left\{ \Omega_S(\gamma_i, y) + \sum_{\sum_{j=1}^{m_i} \beta_j = \gamma_i} H_{m_i}(\{\beta_j\}, y) \prod_{j=1}^{m_i} \Omega_S(\beta_j, y) \right\}. \quad (2.8)$$

In this case, the rational functions $H(\{\beta_j\}, y)$ are fixed by demanding that the coefficient of the monomial $\prod_{j=1}^{m_i} \Omega_S(\beta_j, y)$ in $\Omega(\gamma, z, y)$ be a Laurent polynomial in y . Requiring that $H(\{\beta_j\}, y)$ are invariant under $y \rightarrow 1/y$ and vanish at $y = \infty$ fixes them uniquely [36]. In [17] it is conjectured that the Hodge polynomial (1.2) of $\mathcal{M}_Q$ can be obtained from the

³If one of the constituents β_i is not primitive, all choices $(dm_i, \beta_i/d)$ are counted as distinct contributions.

χ -genus given by (2.5) or (2.8) by replacing the argument y in $\Omega_S(\gamma_i, y)$ by the fugacity t conjugate to $p - q$.

2.2 The MPS formula

For quiver without oriented loops, the single-centered invariants $\bar{\Omega}_S(\gamma_i)$ vanish unless γ_i is a multiple of a basis vector α_a , in which case $\bar{\Omega}_S(\ell\alpha_a) = (y - 1/y)/[\ell(y^\ell - y^{-\ell})]$. The Coulomb branch formula therefore reduces to

$$\bar{\chi}_Q(\{N_a, \zeta_a\}, y) = \sum_{\substack{N_a = \sum_{\ell=1}^{\infty} \ell n_{\ell,a} \\ a=1 \dots K}} \chi_{Q(\{n_{\ell,a}\})}(\{c_i\}, y) \prod_{a=1}^K \prod_{\ell} \frac{1}{n_{\ell,a}!} \left[\frac{y - 1/y}{\ell(y^\ell - y^{-\ell})} \right]^{n_{\ell,a}} \quad (2.9)$$

The factor $\chi_{Q(\{n_{\ell,a}\})}(\{c_i\}, y)$ is the χ -genus of an Abelian quiver with $n = \sum_{\ell,a} n_{\ell,a}$ vertices $\{W_{a,\ell,m}, 1 \leq a \leq K, 1 \leq m \leq n_{\ell,a}\}$, which we denote by $Q(\{n_{\ell,a}\})$ (see figure 1). Denoting the nodes of the ‘blown up quiver’ by $\{W_i, 1 \leq i \leq n\}$ using an arbitrary bijection $i \mapsto (a(i), \ell(i), m(i))$, the adjacency matrix of $Q(\{n_{\ell,a}\})$ is given by $\gamma_{ij} = \ell(i)\ell(j)\kappa_{a(i)a(j)}$, the dimension vector has entries $N_i = 1$ while the stability vector is $c_i = \ell(i)\zeta_{a(i)}$, in such a way that $\sum_{i=1}^n c_i = \sum_{\ell,a} \ell n_{\ell,a} \zeta_a = 0$. The positive integer $\ell(i)$ is sometimes called the *level* of the vertex W_i . Physically, the quiver $Q(\{n_{\ell,a}\})$ describes the interactions of $n = \sum_{\ell} n_{\ell,a}$ black holes with charges $\gamma_i = \ell(i)\alpha_{a(i)}$, treated as distinguishable particles, so we use the notation the equivalent notation $Q(\{\gamma_i\})$ for the quiver $Q(\{n_{\ell,a}\})$.

The formula (2.9) is a special case $S = \{1, \dots, K\}$ of a more general relation operating only on a subset $S \subset \{1, \dots, K\}$ of the nodes:

$$\bar{\chi}_Q(\{N_a, \zeta_a\}, y) = \sum_{\substack{N_a = \sum_{\ell} \ell n_{\ell,a} \\ a \in S}} \chi_{Q(\{n_{\ell,a}\})}(\{c_i\}, y) \prod_{a \in S} \prod_{\ell} \frac{1}{n_{\ell,a}!} \left[\frac{y - 1/y}{\ell(y^\ell - y^{-\ell})} \right]^{n_{\ell,a}} \quad (2.10)$$

where the quiver $Q(\{n_{\ell,a}\})$ is obtained from Q by replacing every vertex $V_a \in \mathcal{V}_Q \cap S$ with vertices $W_{a,\ell,m}$, $1 \leq m \leq n_{\ell,a}$, carrying level ℓ while the vertices $V_a \in \mathcal{V}_Q \setminus S$ are unaffected. Again, we denote by $\{W_i\}$ the set of all vertices, and by $a(i), \ell(i), N(i)$ their type, level and dimension, equal to $(a, \ell, 1)$ for $W_{a,\ell,m}$ and to $(a, 1, N_a)$ for $V_a \in \mathcal{V}_Q \setminus S$. The adjacency matrix is given as before by $\gamma_{ij} = \ell(i)\ell(j)\kappa_{a(i)a(j)}$, the stability vector is $c_i = \ell(i)\zeta_{a(i)}$ and the dimension vector has entries $N_i = N(i)$, in such a way that $\sum_{i=1}^n N_i c_i = 0$. Clearly, this relation is only non-trivial if the entries in S correspond to non-Abelian nodes, i.e. $N_a > 1$ for $a \in S$. Moreover, the relation (2.10) follows iteratively from the special case where S contains a single element. In the case where the dimension vector is primitive, this relation was established in [24, appendix D] based on Reineke’s formula [25] for quivers without oriented loops and with primitive dimension vector. It is worth noting that the proof also works for quivers with oriented loops but with vanishing superpotential, since Reineke’s formula also holds in that case [17].

3 Indices from residues

Using supersymmetric localization techniques, the authors of [31, 33] have found a general prescription for computing the Witten index in any gauged quantum mechanics with

Figure 1. Blowing up a non-Abelian node of rank $N_a = \sum_\ell \ell n_{a,\ell}$ (on the left) into $\sum_\ell n_{a,\ell}$ Abelian nodes (on the right). The number of arrows from $W_{a,\ell,m}$ to V_b is ℓ times the number of arrows from V_a to V_b , where ℓ is the level of the node $W_{a,\ell,m}$. This operation may be performed on any number of non-Abelian nodes.

two supercharges. In the case of quiver quantum mechanics with four supercharges, the prescription reads:

$$\bar{\chi}_Q(\{N_a, \zeta_a\}, y) = \frac{1}{\prod_{a=1}^K N_a!} \sum_p \text{JK-Res}_p(Z_Q(\{u_{a,s}\}, y), \zeta) \quad (3.1)$$

Here, the variables $u_{a,s}$ parametrize the complexified Cartan torus $\prod_{a=1}^K (\mathbb{C}^\times)^{N_a}$ of the gauge group $[\prod_a \text{U}(N_a)]/\text{U}(1)$, the factor $Z_Q(\{u_{a,s}\}, y)$ is a holomorphic top form on this space and the sum runs over all poles of $Z_Q(\{u_{a,s}\}, y)$. The symbol JK-Res denotes a specific prescription, originally due to Jeffrey and Kirwan [34],⁴ for extracting the residue at the corresponding pole. The prescription crucially depends on the stability vector ζ , and ensures that the result is consistent with the wall-crossing formula. The function

⁴The specific prescription used in [28, 31] and in this paper can be found in [35].

$Z_Q(\{u_{a,s}\}, y)$ originates from the one-loop fluctuation determinant of all fields in the vector and chiral multiplets, and is given by

$$\begin{aligned}
 Z_Q(\{u_{a,s}\}, y) = & \left[\frac{\pi \hbar}{\sin(\pi \hbar)} \right]^{\sum_{a=1}^K N_a - 1} \prod_{\substack{a=1 \dots K \\ s, s'=1 \dots N_a \\ s \neq s'}} \frac{\sin[\pi \hbar(u_{a,s'} - u_{a,s})]}{\sin[\pi \hbar(u_{a,s} - u_{a,s'} - 1)]} \\
 & \times \prod_{\substack{a,j=1 \dots K \\ \kappa_{ab} > 0}} \prod_{\substack{s=1 \dots N_a \\ s'=1 \dots N_b}} \prod_{A=1 \dots \kappa_{ab}} \frac{\sin[\pi \hbar(u_{b,s'} - u_{a,s} + 1 - \frac{R_{ab}}{2} - \theta_{ab,A})]}{\sin[\pi \hbar(u_{a,s} - u_{b,s'} + \frac{R_{ab}}{2} + \theta_{ab,A})]} \\
 & \times \widetilde{\prod_{\substack{a=1 \dots K \\ s=1 \dots N_a}} du_{a,s}}
 \end{aligned} \tag{3.2}$$

where $y = e^{i\pi \hbar}$, and $\widetilde{\prod}$ denotes the omission of any one of the $du_{a,s}$'s in the measure (due to the decoupling of the diagonal $U(1)$ action). The quantity R_{ab} denotes the R -charge of the chiral fields $\phi_{ab,A,ss'}$, while the parameters $\theta_{ab,A}$ are chemical potentials⁵ for the $U(\kappa_{ab})$ flavor symmetry permuting the chiral fields $\phi_{ab,A,ss'}$ with $A = 1 \dots \kappa_{ab}$. In the absence of oriented loops, the assignment of R -charges is irrelevant, but if oriented loops are present, it constrains the possible superpotential, which must be a gauge invariant polynomial in the $\phi_{ab,A,ss'}$ with R -charge equal to 2, invariant under flavor symmetries with non-zero potential $\theta_{ab,A}$. In particular, in order to allow for a generic superpotential, one should set all flavor potentials to zero and ensure that the total R -charge for any oriented loop is equal to 2. Clearly, (3.2) is invariant under $\hbar \rightarrow -\hbar$, or equivalently $y \rightarrow 1/y$.

In the limit $y \rightarrow 1$ (or $\hbar \rightarrow 0$), and for vanishing chemical potentials $\theta_{ab,A} = 0$, the χ -genus reduces to the Euler number, given by the sum of residues of

$$\begin{aligned}
 Z_Q(\{u_{a,s}\}) = & \prod_{\substack{a=1 \dots K \\ s, s'=1 \dots N_a \\ s \neq s'}} \frac{u_{a,s'} - u_{a,s}}{u_{a,s} - u_{a,s'} - 1} \prod_{\substack{a,b=1 \dots K \\ \kappa_{ab} > 0}} \prod_{s=1 \dots N_a} \prod_{s'=1 \dots N_b} \left[\frac{u_{b,s'} - u_{a,s} + 1 - \frac{R_{ab}}{2}}{u_{a,s} - u_{b,s'} + \frac{R_{ab}}{2}} \right]^{\kappa_{ab}} \\
 & \times \widetilde{\prod_{\substack{a=1 \dots K \\ s=1 \dots N_a}} du_{a,s}}
 \end{aligned} \tag{3.3}$$

When $\theta_{ab,A} \neq 0$, the limit $\hbar \rightarrow 0$ instead produces the equivariant Euler number for the action of the Cartan torus of the flavor symmetry. It is worth noting that (3.2) arises as the limit $\tau \rightarrow i\infty$ from the elliptic genus of a two-dimensional gauged linear sigma model with the same matter content, given by a sum of residues of

$$\begin{aligned}
 Z_Q(\{u_{a,s}\}, y, \tau) = & \left[\frac{2\pi\eta^3\hbar}{\theta_1(\hbar)} \right]^{\sum_{a=1}^K N_a - 1} \prod_{\substack{a=1 \dots K \\ s, s'=1 \dots N_a \\ s \neq s'}} \frac{\vartheta_1[\pi \hbar(u_{a,s'} - u_{a,s})]}{\vartheta_1[\pi \hbar(u_{a,s} - u_{a,s'} - 1)]} \\
 & \times \prod_{\substack{a,j=1 \dots K \\ \kappa_{ab} > 0}} \prod_{\substack{s=1 \dots N_a \\ s'=1 \dots N_b}} \left[\frac{\vartheta_1[\pi \hbar(u_{b,s'} - u_{a,s} + 1 - \frac{R_{ab}}{2})]}{\vartheta_1[\pi \hbar(u_{a,s} - u_{b,s'} + \frac{R_{ab}}{2})]} \right]^{\kappa_{ab}} \widetilde{\prod_{\substack{a=1 \dots K \\ s=1 \dots N_a}} du_{a,s}}
 \end{aligned} \tag{3.4}$$

⁵Since the R -charge is defined only up to the addition of flavor symmetries, one may decide to absorb $\theta_{ab,A}$ into $R_{ab,A} = R_{ab} + 2\theta_{ab,A}$. However we find it convenient to separate $R_{ab,A}$ into a flavor invariant part R_{ab} and chemical potentials $\theta_{ab,A}$ which may break part of the flavor symmetry.

where $\theta_1(v, \tau) = \theta_1(v) = 2q^{1/8} \sin(\pi v) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i v})(1 - q^n e^{-2\pi i v})$. Note however that in 0+1 dimensions, there is no need to cancel the R-symmetry anomaly, unlike for the elliptic genus of two-dimensional sigma models.

In order to compute the residue in (3.1), it is more efficient to change variables to $v_{a,s} = e^{2\pi i h u_{a,s}}$, such that (3.2) may be replaced by

$$\begin{aligned}
 Z_Q(\{v_{a,s}\}, y) = & \prod_{\substack{a=1\dots K \\ s,s'=1\dots N_a \\ s \neq s'}} \frac{v_{a,s'} - v_{a,s}}{v_{i,s}/y - y v_{a,s'}} \\
 & \times \prod_{\substack{a,b=1\dots K \\ \kappa_{ab} > 0}} \prod_{\substack{s=1\dots N_a \\ s'=1\dots N_b}} \prod_{A=1\dots \kappa_{ab}} \frac{y^{1-R_{ab}} \nu_{ab,A} v_{b,s'} - v_{a,s}/y}{v_{a,s} - y^{-R_{ab}} \nu_{ab,A} v_{b,s'}} \widetilde{\prod_{\substack{a=1\dots K \\ s=1\dots N_a}}} \frac{dv_{a,s}}{v_{a,s}(y - 1/y)}
 \end{aligned} \tag{3.5}$$

where $\nu_{ab,A} = e^{-i\pi \theta_{ab,A}}$ are flavor fugacities. In this representation, the symmetry under $y \rightarrow 1/y$ requires inverting $v_{a,s} \rightarrow 1/v_{a,s}$, $\alpha_{ab,A} \rightarrow 1/\alpha_{ab,A}$.

3.1 Review of the JK residue prescription

To explain the prescription for identifying and extracting the relevant residues, let us introduce the weight lattice $\Lambda = \mathbb{Z} \sum N_a$, with basis $e_{a,s}$, $a = 1 \dots K, s = 1 \dots N_a$. To each root of $U(N_a)$ we associate the vector $e_{a,s} - e_{a,s'}$ dual to the hyperplane $H_{a,s,s'} = u_{a,s} - u_{a,s'} - 1$. To each bifundamental chiral field $\phi_{ab,A,ss'}$ with $\kappa_{ab} > 0$ we associate the vector $e_{a,s} - e_{b,s'}$ dual to the hyperplane $H_{ab,A,ss'} = u_{a,s} - u_{b,s'} + \frac{R_{ab}}{2} + \theta_{ab,A}$. If the flavor chemical potentials $\theta_{ab,A}$ vanish for all A , we may as well consider a single hyperplane $H_{ab,ss'}$ with multiplicity $|\kappa_{ab}|$. Finally, since all fields are neutral under the diagonal $U(1)$ action, we set one Cartan parameter to zero, e.g. $u_{1,1} = 0$, and disregard the corresponding entry of all charge vectors. We denote by $r = \sum_{a=1}^K N_a - 1$ the rank of the reduced lattice $\Lambda/\mathbb{Z}$.

Now, for each isolated intersection p of hyperplanes, let $\mathcal{H}_p$ be the list of hyperplanes intersecting at p ; to construct this list, consider all r -plets of hyperplanes, determine their intersection,⁶ collect all non-degenerate intersection points, and finally for each point in this list, collect all hyperplanes which meet at that point. Let $\mathcal{Q}_p$ be the corresponding list of charge vectors. We further assume that the intersection is projective, i.e. that all vectors in $\mathcal{Q}_p$ lies in a positive half-space of $\mathbb{R}^r$. If that is not the case, it is usually possible to perturb the charges R_{ab} so as to resolve a non-projective intersection into multiple projective intersections.

Let $\mathcal{F}_p$ be the list of flags $F = (F_1, \dots, F_r)$ made out of vectors in $\mathcal{Q}_p$; equivalently, the list of ordered r -plets $Q_F = (Q_1, \dots, Q_r)$ of vectors in $\mathcal{Q}_p$, subject to the equivalence relation $Q_F \sim Q'_F$ if $Q_F Q'^{-1}_F$ is a lower triangular matrix, if the rows of Q_F denote the charge vectors. The space spanned by Q_j with $1 \leq i \leq j$ defines the i -th graded space F_i

⁶For quiver theories, each non-degenerate r -plet of hyperplanes appears to have only one intersection point on the Cartan torus, but this is not true for more general matter content.

in the flag F . Let $\kappa_{F,i}$ be the sum of all charge vectors⁷ in $\mathcal{Q}_p$ which belongs to F_i , for all $i = 1, \dots, r$. Note that $\kappa_{F,i}$ includes contributions from the vectors $Q_1, \dots, Q_i$, but it may also include charge vectors which do not belong to the list $(Q_1, \dots, Q_r)$ if more than r hyperplanes intersect at p , i.e. if the intersection is degenerate. In particular, $\kappa_{F,r}$ is the sum of all vectors in $\mathcal{Q}_p$, irrespective of the flag F . Note also that the matrix κ_F is independent of the choice of representative Q_F for the flag F .

Now, let us promote the stability condition $\zeta \in \mathbb{R}^K$ to a vector $\eta \in \mathbb{R}^r$, by using the diagonal embedding $\eta_{a,s} = \zeta_a$, and perturbing slightly away from this point (the perturbation must be chosen once for all, and be the same for all flags). A flag $F \in \mathcal{F}_p$ is said to be stable if η belongs to the positive cone spanned by the vectors $\kappa_{F,i}$ with $i = 1 \dots r$, i.e.

$$\eta = \sum_{i=1}^r \lambda_i \kappa_{F,i}, \quad \lambda_i > 0 \quad (3.6)$$

In the case where the vectors $\kappa_i(F)$ are linearly dependent, we discard the corresponding flag, since they will not be stable for generic values of η . Let $\mathcal{F}_p(\eta)$ be the set of stable flags constructed from the list of hyperplanes $\mathcal{H}_p$ meeting at p . The Jeffrey-Kirwan residue at p is then the sum of iterated residues

$$\text{JK-Res}_p(Z(u), \eta) = \sum_{F \in \mathcal{F}_p(\eta)} \text{sign}(\det \kappa) \text{Res}_{\tilde{u}_r=0} \dots \text{Res}_{\tilde{u}_1=0} \tilde{Z}(\tilde{u}) \quad (3.7)$$

where $\tilde{u}_i = H_{Q_i}(u)$ (or in matrix notation, $\tilde{u} = u \cdot Q_F$) and $\tilde{Z}(\tilde{u}) = Z(u) d\tilde{u}/du$. The residue is most efficiently computed from the representation (3.5), but one must of course keep track of the Jacobians $\partial v/\partial u$ and $\partial u/\partial \tilde{u}$.

It is worth stressing that while the condition (3.6) restricts the possible flags contributing to the index, it frequently happens that some of the allowed flags give a vanishing residue. In particular, note that in the absence of flavor fugacities and upon treating the hyperplanes $H_{ab,A,ss'}$, $A = 1 \dots \kappa_{ab}$ as a single hyperplane of degree κ_{ab} , the stability condition for flags depends only on the sign of κ_{ab} , not on its absolute value, but the complex dimension d of the moduli space $\mathcal{M}_Q$ does depend on $|\kappa_{ab}|$, and the χ -genus must vanish if $d < 0$. It would be interesting to find a criterium predicting when a given stable flag will produce a vanishing contribution.

3.2 Residues and fixed points

As explained in [31, 32], the residue formula (3.1) arises by applying supersymmetric localization to the functional integral defining the gauge theory on an Euclidean circle of radius β . For this, one constructs a linear combination Q_h of the 4 supercharges which squares to a combination $Q_h^2 = J$ of a $U(1)_R$ rotation and global flavor symmetries, such that the action becomes Q_h -exact. Fixed points of Q_h are such that the complexified scalars $u_{a,s}$ in the Cartan part of vector multiplets become constant (independent of the coordinate τ along the thermal circle), while the chiral multiplets must vanish for generic values of the

⁷Here it matters whether we treat hyperplanes with $|\kappa_{ab}| > 1$ as multiple copies with unit multiplicity, or a single copy with multiplicity $|\kappa_{ab}|$. We find in examples that the two prescriptions lead to the same result.

$u_{a,s}$'s. The one-loop determinant for the off-diagonal components of the vector multiplets and for the chiral multiplets gives a (r, r) form ω on the space of the $u_{a,s}$'s, which has the topology of a cylinder $(\mathbb{C}^\times)^r = (\mathbb{R} \times S_1)^r$. By integration by parts, the integral can be rewritten as a sum of contour integrals around the poles at finite and infinite distance. At poles at finite distance in the u -plane, corresponding to an intersection of hyperplanes $H_1, \dots, H_r$, the chiral multiplets (in case $H_i = H_{ab,A,ss'}$) or the off-diagonal scalars in the vector multiplets (in case $H_i = H_{a,s,s'}$) may acquire a non-zero vev. These correspond to fixed points of the one-parameter subgroup of the flavor symmetry $\prod_{a,b,\kappa_{ab}>0} U(\kappa_{ab})$ acting on the Higgs branch. The condition (3.6) ensures that fixed points with $\phi_{ab,A,ss'} \neq 0$ for a, b, s, s' associated to the hyperplanes H_i are allowed by the D-term constraints (2.1). Moreover, the contribution of each fixed point agrees with the Atiyah-Bott Lefschetz fixed point theorem [48] for a Kähler manifold X with a holomorphic action of $f : X \rightarrow X$, which lifts to an action f^* on its cohomology $H^*(X)$,

$$\sum_{p,q \geq 0} (-1)^{p+q-d} y^{2p-d} \text{Tr}_{H^{p,q}} f^* = \sum_{p \geq 0} (-1)^{p-d} y^{2p-d} \sum_{f(P)=P} \frac{\text{Tr}(\lambda^p df_P)}{\det_{\mathbb{C}}(1 - df_P)} \quad (3.8)$$

where df_P is the linear action of f on the holomorphic tangent space at the fixed point P , and $\lambda^p df_P$ is the p -th antisymmetric power of this map. In our context, f is the $U(1)$ flavor symmetry determined by the chemical potentials $\theta_{ab,A}$. In section 4 below we discuss the relation between poles and fixed points in more detail in the case of Abelian quivers.

3.3 Index in attractor chamber

For any quiver, there is a special choice of stability conditions known as the attractor chamber [49]

$$\zeta_a^* = - \sum_b \kappa_{ab} N_b. \quad (3.9)$$

This chamber precludes the existence of two-centered bound states, since for any splitting $\gamma = \gamma_L + \gamma_R$ of the dimension vector, or equivalently $N_a = N_a^L + N_a^R$, the DSZ product $\langle \gamma_L, \gamma_R \rangle = \kappa_{ab} N_L^a N_R^b$ and the effective stability parameter $\zeta_L = N_L^a \zeta_a^* = -N_R^a \zeta_a^*$ have opposite sign. For a quiver without oriented loops, the index in this chamber automatically vanishes. This is because such a quiver always admits (possibly more than) one sink, where all arrows are incoming, and (possibly more than) one source, where all arrows are outgoing. The D-term conditions (2.1) have no solutions unless $\zeta_a < 0$ for a sink, and $\zeta_a > 0$ for a source. In contrast, at the attractor point $\zeta_a^* > 0$ for sink, and $\zeta_a^* < 0$ for a source. By the same token, the index vanishes at the attractor point for any quiver which admits a sink or source, whether or not it contains an oriented loop; more generally the index vanishes for any stability condition of the form $\zeta_a = -\mu_a \sum_b \kappa_{ab} N_b$ such that $\mu_a > 0$ provided $\sum_a N_a \zeta_a = 0$.

It is easy to see that the JK residue prescription is consistent with this vanishing property. Indeed, summing up the relation (3.6) over the indices $s = 1 \dots N_a$ for each a , we get

$$\zeta = \sum_{i=1}^r \lambda_i \tilde{\kappa}_i \quad (3.10)$$

where $\tilde{\kappa}_i \in \mathbb{R}^K$ gets contributions only from chiral multiplet charge vectors in $\mathcal{Q}_p$. If a is a sink, all the contributions to the a -th component of $\tilde{\kappa}_i$ are strictly negative, so there are no stable flags when $\zeta_a > 0$, in particular at the attractor point ζ^* . Similarly, if a is a source, sink, all the contributions to the a -th component of $\tilde{\kappa}_i$ are strictly positive, so there are again no stable flags when $\zeta_a < 0$, in particular at the attractor point ζ^* .

In the absence of sources or sinks, in particular for quivers with oriented loops, the index at the attractor point does not necessarily vanishes, but rather gets contributions from single-centered invariants and from scaling solutions thereof, as we discuss further in section 4.3.

3.4 Cauchy-Bose formula

One of the bottlenecks in the practical evaluation of computation of (3.1) is the enumeration of intersections points and flags. Each non-Abelian group contributes $\mathcal{O}(N_a^2)$ singular hyperplanes, while the rank grows only linearly in N_a , so the number of possible ℓ -plets grows exponentially. One remedy is to use the Cauchy-Bose identity (1.3) to rewrite the vector multiplet determinant in (3.2) as

$$\prod_{\substack{s,s'=1\dots N \\ s \neq s'}} \frac{\sin[\pi\hbar(u_{s'} - u_s)]}{\sin[\pi\hbar(u_s - u_{s'} - 1)]} = \sum_{\sigma \in S_N} \frac{\epsilon(\sigma) \sin(\pi\hbar)^N}{\prod_{s=1}^N \sin[\hbar(u_s - u_{\sigma(s)} + 1)]} \quad (3.11)$$

Equivalently, in terms of the exponentiated variables $v_s = e^{2\pi i \hbar u_s}$,

$$\frac{1}{\prod_{s=1}^N v_s} \prod_{\substack{s,s'=1\dots N \\ s \neq s'}} \frac{(v_{s'} - v_s)}{v_s/y - y v_{s'}} = (1/y - y)^N \sum_{\sigma \in S_N} \frac{\epsilon(\sigma)}{\prod_{s=1}^N (v_s/y - y v_{\sigma(s)})} \quad (3.12)$$

Using the elliptic generalization of (1.3) due to Frobenius [50]

$$\det \frac{1}{\theta_1(\mu_i - \nu_j)} = \frac{\prod_{i < j} \theta_1(\mu_i - \mu_j) \theta_1(\nu_j - \nu_i)}{\prod_{i,j} \theta_1(\mu_i - \nu_j)}, \quad (3.13)$$

we may similarly write the vector multiplet determinant for the elliptic genus in (3.2) as

$$\prod_{\substack{s,s'=1\dots N \\ s \neq s'}} \frac{\vartheta_1[\pi\hbar(u_{s'} - u_s)]}{\vartheta_1[\pi\hbar(u_s - u_{s'} - 1)]} = \sum_{\sigma \in S_N} \frac{\epsilon(\sigma) \vartheta_1(\pi\hbar)^N}{\prod_{s=1}^N \vartheta_1[\hbar(u_s - u_{\sigma(s)} + 1)]} \quad (3.14)$$

In each of these formulae, the denominator now involves N hyperplanes rather than $N(N-1)/2$, which drastically simplifies the classification of intersections flags, though it requires sifting through $N!$ permutations rather than a single product of many hyperplanes. Fortunately, all permutations in the same conjugacy class (labelled by the partition $\lambda = \sum_{\ell} \ell n_{\ell}$) turn out to produce the same contribution,⁸ so it suffices to pick one particular permutation for each partition, and multiply the result by $\prod_{\ell} \ell^{n_{\ell}} n_{\ell}!$, corresponding

⁸This fact is non-trivial since the generalized stability vector $\eta \in \mathbb{R}^r$ breaks the S_N symmetry. The integrand coming from different permutations with the same cycle shape can be mapped to the same integrand by relabelling the integration variables $u_{s,a}$, at the expense of permuting the entries in the flag $F = (Q_1, \dots, Q_r)$ used for integrating each of them. For non-degenerate intersections, these permutations do not affect the residue, so the result is the same for all permutations with the same cycle shape. For degenerate intersections, the change of variables typically permutes the flags as well (not only the entries Q_i in a given flag) and a more detailed analysis is required.

to the number of permutations in the same conjugacy class. As we shall see in section 5, if one applies this trick for each of the gauge groups $U(N_a)$, the resulting sum of multi-partitions coincides with the MPS formula, at least in the case of non-Abelian quivers without oriented loops.

4 Abelian quivers

In this section, we apply the residue formula in the context of Abelian quivers with or without loops. In the absence of oriented loops, we show that the stability condition (3.6) on flags coincides with the condition for existence of fixed points satisfying the D-term equations.

4.1 Abelian quivers without loops

To demonstrate this in a simple example, let us focus on Abelian quivers without oriented loops (but possibly with non-oriented ones). The D-term equations (2.1) can be written as

$$\zeta = \sum_{a,b;\kappa_{ab}>0} |\phi_{ab}|^2 (e_a - e_b), \quad |\phi_{ab}|^2 := \sum_{A=1}^{\kappa_{ab}} |\phi_{ab,A}|^2 \quad (4.1)$$

For a quiver without (oriented nor unoriented) loop, the rank $r = K - 1$ is equal to the total number of edges $E = \#\{(a,b \in \mathcal{A}_Q, \kappa_{ab} > 0)\}$ (counted *without* multiplicity), so each hyperplane H_i originates from a set of chiral fields $\phi_{i,A} \equiv \phi_{a(i)b(i),A}$, and vice-versa, and (4.1) becomes

$$\zeta = \sum_{i=1}^r |\phi_i|^2 Q_i. \quad (4.2)$$

For the flag $F = (Q_1, \dots, Q_r)$, one has $\kappa_{F,i} = \sum_{k=1}^i Q_k$ and it is easy to check that the solution to $\zeta = \sum_{i=1}^r \lambda_i \kappa_{F,i}$ is given by

$$\lambda_i = |\phi_i|^2 - |\phi_{i+1}|^2 \quad \text{for } i < r, \quad \lambda_r = |\phi_r|^2 \quad (4.3)$$

This gives a transparent interpretation of the parameters λ_i in (3.6) as difference of vevs of chiral fields. In particular, the flag F is stable if the stability parameters allow for solutions of the D-term equations where the only non-vanishing chiral fields ϕ_i satisfy $|\phi_1|^2 > |\phi_2|^2 > \dots > |\phi_r|^2 > 0$. Different flags correspond to different ordering of these vevs. The associated residue then computes the contribution of the fixed points satisfying these constraints.

As a simple example, consider the Abelian ‘star’ quiver with K vertices $V_0, V_1, \dots, V_{K-1}$, with a_i arrows from V_i to V_0 , with a_i a non-zero integer of arbitrary sign. We denote this

quiver by $Q = S_{a_1, \dots, a_K}$:

The D-term conditions

$$\text{sgn}(a_i) \sum_{A=1}^{|a_i|} |\phi_{i,A}|^2 = \zeta_i, \quad i = 1 \dots K-1 \quad (4.4)$$

admit solutions only when $\text{sgn}(a_i) = \text{sgn}(\zeta_i)$, in which case the quiver moduli space reduces to a product of projective spaces $\mathcal{M}_Q = \prod_i \mathbb{P}^{|a_i|-1}$. This agrees with the result from the residue formula

$$\chi_Q(\{1, \zeta_a\}, y) = \sum_p \text{JK-Res}_p \left[\prod_{i=1}^{K-1} \left(\frac{y^{\text{sgn} a_i} u_0 - y^{-\text{sgn} a_i} u_i}{u_i - u_0} \right)^{|a_i|} \frac{du_i}{(y - 1/y) u_i} \right]. \quad (4.5)$$

In this case, the singular hyperplanes $u_i = u_0$ have a non-degenerate intersection and a single stable flag contributes, which depends on the ordering of the ζ_i 's. The result however is independent of that ordering, since the integral factorizes into a product of residues in each variable u_i :

$$\chi_Q(\{1, \zeta_a\}, y) = \prod_{i=1}^{K-1} \left[(-1)^{a_i-1} \frac{y^{|a_i|} - y^{-|a_i|}}{y - 1/y} \right] \quad (4.6)$$

In the presence of chemical potentials $\theta_{i,A}$ for the global symmetry group $\prod_i \text{U}(|a_i|)$, the intersection at the origin splits into $\prod_i |a_i|$ intersection points, corresponding to a choice of $A_i \in \{1, \dots, |a_i|\}$ for each arrow, producing

$$\chi_Q(\{1, \zeta_a\}, y, \theta) = \prod_{i=1}^{K-1} \left[\sum_{A_i=1}^{|a_i|} \prod_{A \neq A_i} \frac{\nu_{i,A}/y - \nu_{i,A_i} y}{\nu_{i,A_i} - \nu_{i,A}} \right] \quad (4.7)$$

Each of these contributions correspond to one fixed point with $\phi_{i,A} = \sqrt{|\zeta_i|} \delta_{A,A_i}$, up to $\text{U}(1)$ gauge rotations of the phase of $\phi_{i,A}$. The determinant $\det_{\mathbb{C}}(1 - \text{d}f_P)$ appearing in the Lefschetz fixed point formula (3.8) evaluates to $\prod_i \prod_{A \neq A_i} (1 - \nu_{i,A}/\nu_{i,A_i})$, in agreement with the denominator in (4.7). After summing over $A \neq A_i$, each bracket in (4.7) reduces to the corresponding bracket in (4.6), as expected since each factor $\mathbb{P}^{|a_i|-1}$ is compact. Moreover, as $y \rightarrow 1$, each of these non-degenerate intersections contributes ± 1 , so that $\chi_Q(\{1, \zeta_a\}, y)$ counts the number of fixed points, up to an overall sign.

4.2 Abelian quivers with unoriented loops

For Abelian quivers with h unoriented loops, the rank $r = K - 1$ is equal to $E - h$, where E is the total of number of edges (not counting multiplicity). Choosing R-charges R_{ab} and flavor fugacities $\theta_{ab,A}$ such that only non-degenerate intersections of r hyperplanes $(Q_1, \dots, Q_r)$ contribute, then only the chiral fields corresponding to $Q_1, \dots, Q_r$ can get non-trivial expectation values at the intersection, while the remaining $E - r = h$ must be set to zero. Removing these arrows from the original quiver Q defines a ‘reduced’ quiver $\tilde{Q}$. In order for the matrix κ_F to be non-degenerate, it is easy to see that $\tilde{Q}$ must be a tree, which is therefore a spanning tree of the original quiver Q . The same argument as in the previous section shows that the stable flags are again in one-to-one correspondence with fixed points satisfying the D-term equations, for any ordering of the vevs $|\phi_i|^2$ along the spanning tree.

4.2.1 Three node quiver

As an example with one unoriented loop, let us consider a quiver with three nodes V_1, V_2, V_3 and $a = \gamma_{12} < 0, b = \gamma_{23} > 0, c = \gamma_{31} < 0$, which we denote by $Q = C_{a,b,c}$:

The D-term equations

$$|\phi_{13}|^2 - |\phi_{21}|^2 = \zeta_1, \quad |\phi_{23}|^2 + |\phi_{21}|^2 = \zeta_2, \quad -|\phi_{13}|^2 - |\phi_{23}|^2 = \zeta_3 \quad (4.8)$$

admit solutions only when $\zeta_2 > 0$ and $\zeta_3 < 0$. If $\zeta_1 > 0$, then $\phi_{13} \neq 0$ defines a point in $\mathbb{P}^{|c|-1}$, fibered over $\mathbb{P}^{|a|+b-1}$ parametrized by $(\phi_{21}, \phi_{23}) \neq (0, 0)$, with index

$$\chi_Q^+(y) = (-1)^{a+b+c} \frac{(y^{|c|} - y^{-|c|})(y^{|a|+b} - y^{-|a|-b})}{(y - 1/y)^2} \quad (4.9)$$

If instead $\zeta_1 < 0$, then $\phi_{21} \neq 0$ defines a point in $\mathbb{P}^{|a|-1}$, fibered over $\mathbb{P}^{|c|+b-1}$ parametrized by $(\phi_{13}, \phi_{23}) \neq (0, 0)$, with index

$$\chi_Q^-(y) = (-1)^{a+b+c} \frac{(y^{|a|} - y^{-|a|})(y^{|c|+b} - y^{-|c|-b})}{(y - 1/y)^2} \quad (4.10)$$

The difference

$$\chi_Q^+(y) - \chi_Q^-(y) = (-1)^{a+c-1} \frac{y^{|c|-a} - y^{-|c|+a}}{y - 1/y} \times (-1)^{b-1} \frac{y^b - y^{-b}}{y - 1/y} \quad (4.11)$$

is interpreted as the contribution of a two-particle bound state with charges $\{\gamma_{1+2}, \gamma_3\}$ [22].

Let us see how these results can be recovered from the localization formula (3.1). For vanishing R-charges or flavor fugacities, the index is given by a residue of

$$Z_Q = \left(\frac{yu_1 - u_2/y}{u_2 - u_1} \right)^{|a|} \left(\frac{yu_3 - u_2/y}{u_2 - u_3} \right)^{|b|} \left(\frac{yu_3 - u_1/y}{u_1 - u_3} \right)^c \widetilde{\prod_{i=1\dots 3} \frac{du_i}{u_i(y - 1/y)}} \quad (4.12)$$

at the degenerate intersection $u_1 = u_2 = u_3$. The flag (Q_{23}, Q_{12}) , or equivalently (Q_{23}, Q_{13}) , has a degenerate matrix κ_F so does not contribute for generic values of the ζ_i 's. For $\zeta_1 < 0, \zeta_2 > 0, \zeta_3 < 0$, the only stable flag is $(Q_{12}, Q_{23}) \sim (Q_{12}, Q_{13})$, reproducing (4.10). For $\zeta_1 > 0, \zeta_2 > 0, \zeta_3 < 0$, the only stable flag is instead $(Q_{13}, Q_{12}) \sim (Q_{13}, Q_{23})$, reproducing (4.9). For generic R-charges R_{21}, R_{23}, R_{13} , the degenerate intersection at $(0, 0, 0)$ splits into three non-degenerate intersections $p_1 = H_{31} \cap H_{12}, p_2 = H_{12} \cap H_{23}, p_3 = H_{23} \cap H_{31}$. We denote by $p_2^+ = (Q_{12}, Q_{23})$ and $p_2^- = (Q_{23}, Q_{12})$ the two possible flags at p_2 , which are no longer equivalent, and similarly for $p_3^\pm, p_1^\pm$. Depending on ζ_1 , there are now two stable flags, given in the following table,

$\zeta_1 < \zeta_3 < 0$	$\zeta_3 < \zeta_1 < 0$	$0 < \zeta_1 < \zeta_2$	$0 < \zeta_2 < \zeta_1$
$\{p_1^-, p_2^+\}$	$\{p_1^-, p_2^-\}$	$\{p_1^+, p_3^+\}$	$\{p_1^+, p_3^-\}$

(4.13)

In each interval, the contribution of each flag is a complicated rational function of $y^{\pm R_{ij}}$, but they combine in the same result (4.9) or (4.10), depending on the sign of ζ_1 . For generic flavor fugacities, the intersections become non-degenerate, and are in one-to-one correspondence with the possible fixed points. For those, only 2 out of the 3 set of arrows can be non-vanishing, leading to a total of either $|c|(|a| + b)$ or $|a|(b + |c|)$ fixed points, depending on the sign of ζ_1 . This agrees with (4.9) and (4.10) in the limit $y \rightarrow 1$.

4.2.2 Four node quiver

Next, we consider an example will become relevant in section 5.2: an Abelian quiver with 4 nodes $V_{1,2,3,4}$ and arrows $\kappa_{12} = a > 0, \kappa_{23} = b < 0, \kappa_{34} = c > 0, \kappa_{41} = d < 0$, which we denote by $Q = C_{a,b,c,d}$:

The D-term equations

$$\begin{aligned} |\phi_{12}|^2 + |\phi_{14}|^2 &= \zeta_1, & -|\phi_{12}|^2 - |\phi_{32}|^2 &= \zeta_2, \\ |\phi_{32}|^2 + |\phi_{34}|^2 &= \zeta_3, & -|\phi_{14}|^2 - |\phi_{34}|^2 &= \zeta_4 \end{aligned} \quad (4.14)$$

admit solutions only when $\zeta_1, \zeta_3 > 0$ and $\zeta_2, \zeta_4 < 0$. Using the symmetries exchanging (V_1, V_3) and (V_2, V_4) , there is no loss of generality in assuming that $c_1 < c_3$, $c_2 < c_4$. For vanishing R-charges and flavor fugacities, the index is given by a residue of

$$Z_Q = \left(\frac{yu_2 - u_1/y}{u_1 - u_2} \right)^a \left(\frac{yu_2 - u_3/y}{u_3 - u_2} \right)^{|b|} \left(\frac{yu_4 - u_3/y}{u_3 - u_4} \right)^c \left(\frac{yu_4 - u_1/y}{u_1 - u_4} \right)^{|d|} \times \widetilde{\prod_{i=1\dots 4} \frac{du_i}{u_i(y-1/y)}} \quad (4.15)$$

at the degenerate intersection $u_1 = u_2 = u_3 = u_4$. For $\zeta_2 + \zeta_3 > 0$, we find a single stable flag (Q_{32}, Q_{34}, Q_{12}) contributing

$$\chi_Q^+ = (-1)^{a+b+c+d+1} \frac{(y^{|b|} - y^{-|b|})(y^c - y^{-c})(y^{a+|d|} - y^{-a-|d|})}{(y-1/y)^3} \quad (4.16)$$

If instead $\zeta_2 + \zeta_3 < 0$, there is a single stable flag (Q_{32}, Q_{12}, Q_{34}) contributing

$$\chi_Q^- = (-1)^{a+b+c+d+1} \frac{(y^a - y^{-a})(y^{|b|} - y^{-|b|})(y^{c+|d|} - y^{-c-|d|})}{(y-1/y)^3} \quad (4.17)$$

The difference between the two contributions

$$\chi_Q^+ - \chi_Q^- = (-1)^{a+b+c+d} \frac{(y^{a-c} - y^{c-a})(y^{|b|} - y^{-|b|})(y^{|d|} - y^{-|d|})}{(y-1/y)^3} \quad (4.18)$$

is recognized as the index of a bound state of $\gamma_L = \gamma_1 + \gamma_4$ and $\gamma_R = \gamma_2 + \gamma_3$. In the presence of generic R-charges, the degenerate intersection splits into 4 non-degenerate intersections, with 2 stable flags contributing in any chamber of the (ζ_2, ζ_3) plane (for fixed values of ζ_1, ζ_4 satisfying the previous assumptions). For generic flavor fugacities (and vanishing R-charges), the degenerate intersection splits into $a|b|c+a|bd|+ac|d|+c|bd|$ non-degenerate intersections, grouped in 4 subsets corresponding to the 4 possible spanning trees. Depending on the stability conditions, only two of these subsets support a stable flag contributing ± 1 in the limit $y \rightarrow 1$, in agreement with the previous answers (4.16), (4.18).

4.3 Abelian quivers with oriented loops

In the presence of oriented loops and absence of a superpotential, the the quiver moduli space $\mathcal{M}_Q$ is in general non-compact. If one naively tries to apply the residue formula (3.1) for vanishing R-charges and flavor fugacities, the intersections are typically degenerate and non-projective. One way to resolve this problem is to switch on generic flavor fugacities θ_A for the chiral fields; the resulting equivariant index is then a rational function of y and $\nu_A = e^{i\hbar\theta_A}$, with no natural way of taking the limit $\theta_A \rightarrow 0$. Another way out is to allow for a generic superpotential, by tuning the R-charges R_{ab} such that the gauge invariant product of chiral fields around any oriented loops carries R-charge two. Additionally, one may switch on some flavor fugacities so as to restrict the form of the superpotential, but possibly at the cost of opening non-compact directions.

Let us demonstrate this in the case of an Abelian quiver $Q = C_{a,b,c}$ with 3 nodes V_1, V_2, V_3 with an oriented loop, $a = \gamma_{12} > 0, b = \gamma_{23} > 0, c = \gamma_{31} > 0$:

Choosing $R_{12} = R_{23} = R_{31} = 2/3$ so that the total R-charge of the loop is 2, the index is given by a sum of residues of

$$Z_Q = \left(\frac{y^{1/3}u_2 - u_1/y}{u_1 - y^{-2/3}u_2} \right)^a \left(\frac{y^{1/3}u_3 - u_2/y}{u_2 - y^{-2/3}u_3} \right)^b \left(\frac{y^{1/3}u_1 - u_3/y}{u_3 - y^{-2/3}u_1} \right)^c \widetilde{\prod_{i=1\dots 3} \frac{du_i}{u_i(y - 1/y)}} \quad (4.19)$$

Denoting $H_{12} = u_1 - y^{-2/3}u_2$, $H_{23} = u_2 - y^{-2/3}u_3$, $H_{31} = u_3 - y^{-2/3}u_1$ the singular hyperplanes, and Q_{12}, Q_{23}, Q_{31} the corresponding charge vectors, we find three non-degenerate intersections $p_2 = H_{12} \cap H_{23}, p_3 = H_{23} \cap H_{31}, p_1 = H_{31} \cap H_{12}$. We denote by p_2^+ the flag (Q_{12}, Q_{23}) , and by p_2^- the flag (Q_{23}, Q_{12}) , and similarly for p_3^+, p_3^- . A single flag contributes for any signs of $\zeta_1, \zeta_2, \zeta_3$, given in the following table:

ζ_1	ζ_2	ζ_3	p
−	−	+	p_1^+
+	−	+	p_1^-
+	−	−	p_2^+
+	+	−	p_2^-
−	+	−	p_3^+
−	+	+	p_3^-

(4.20)

The corresponding residue of course gives the same result as the computation based on the identification of the quiver moduli space as a complete intersection in a product of projective spaces [17, 39], since both rely on the same index theorem.

Across the wall at $\zeta_3 = 0$ (say), assuming $\zeta_1 > 0, \zeta_2 < 0$, the flag p_1^- contributes when $\zeta_3 > 0$, while p_2^+ contributes when $\zeta_3 < 0$. Defining $Z_1(u_2, u_3) = \text{Res}_{u_1=y^{-2/3}u_2} Z_Q$, the difference of indices across the wall gives

$$\Delta\chi = \text{Res}_{u_2=y^{4/3}u_3} Z_1 + \text{Res}_{u_2=y^{-2/3}u_3} Z_1 = -(\text{Res}_{u_2=0} + \text{Res}_{u_2=\infty}) Z_1. \quad (4.21)$$

The integrand $Z_1(u_2, u_3)$ is of the form $f(u_2, u_3) \frac{du_2}{u_2}$, where $f(u_2, u_3)$ is a rational function of degree 0. By homogeneity, the residues at $u_2 = 0$ and $u_2 = \infty$ can be traded for residues

at $u_3 = \infty$ and $u_3 = 0$, respectively. The latter are easily computed from the limits

$$\begin{aligned} f(u_2, u_3) \xrightarrow{u_3 \rightarrow 0} \frac{(-y)^{b-c}}{(y-1/y)^2} \oint_{y^{-2/3}u_2} \frac{du_1}{u_1} \left(\frac{y^{1/3}u_2 - u_1/y}{u_1 - y^{-2/3}u_2} \right)^a &= (-1)^{a+b+c} \frac{y^{b-c}(y^a - y^{-a})}{(y-1/y)^2} \\ f(u_2, u_3) \xrightarrow{u_3 \rightarrow \infty} \frac{(-y)^{c-b}}{(y-1/y)^2} \oint_{y^{-2/3}u_2} \frac{du_1}{u_1} \left(\frac{y^{1/3}u_2 - u_1/y}{u_1 - y^{-2/3}u_2} \right)^a &= (-1)^{a+b+c} \frac{y^{c-b}(y^a - y^{-a})}{(y-1/y)^2} \end{aligned} \quad (4.22)$$

leading to

$$\Delta\chi = (-1)^{b+c-1} \frac{y^{c-b} - y^{b-c}}{y-1/y} \times (-1)^{a-1} \frac{y^a - y^{-a}}{y-1/y} \quad (4.23)$$

which is recognized as the contribution from the bound state with charges $\{\gamma_1 + \gamma_2, \gamma_3\}$. This computation illustrates how discontinuities across the wall come from poles at infinity [31].

Let us now consider the attractor point (3.9) where $\zeta_1^* = c-a$, $\zeta_2^* = a-b$, $\zeta_3^* = b-c$. For $a, b < c$, the contribution comes from the flag p_2^+ if $b > a$, or from the flag p_2^- if $a < b$. The residue associated to p_i^+ and p_i^- coincide, since there are only two hyperplanes intersecting at p_i . The result should be compared with the Coulomb branch formula evaluated at the attractor point,

$$\Omega_*(1, 1, 1) = \Omega_S(1, 1, 1) + g_C(\{\gamma_1, \gamma_2, \gamma_3\}) + H(\{\gamma_1, \gamma_2, \gamma_3\}, \{1, 1, 1\}) \quad (4.24)$$

where $\Omega_S(1, 1, 1)$ is the single-centered invariant, and the modified Coulomb index is given by [17, §3.3]

$$g_C(\{\gamma_1, \gamma_2, \gamma_3\}) = (-1)^k \frac{y^k + y^{-k}}{(y-1/y)^2} \quad (4.25)$$

$$H(\{\gamma_1, \gamma_2, \gamma_3\}, \{1, 1, 1\}) = \begin{cases} -2(y-1/y)^{-2} & k \text{ even} \\ (y+1/y)(y-1/y)^{-2} & k \text{ odd} \end{cases} \quad (4.26)$$

with $k = c - a - b$ (more generally, k is equal to the largest of a, b, c minus the sum of the other two). Evaluating the l.h.s. of (4.24) via the residue formula allows to read off the single-centered invariant $\Omega_S(1, 1, 1)$. Note that both $\Omega_S(1, 1, 1)$ and $g_C + H$ vanish unless $c \leq a + b - 2$, although g_C by itself does not vanish for $a + b - 2 < c \leq a + b - c$. This is consistent with the fact that the quiver moduli space, when non-empty, has complex dimension $d = a + b - c - 2$.

It is instructive to compare the index in the presence of a generic superpotential, computed using an assignment of R-charges such that the oriented loop carries charge 2, to the index for vanishing superpotential. As explained in [17, §2.5], the latter can be computed using the Harder-Narasimhan recursion, or equivalently Reineke's formula. Either way, one finds, in the chamber $\zeta_1 > \zeta_2 > 0, \zeta_3 < 0$,

$$\chi_Q^{W=0} = (-1)^{a+b+c} y^{2-a-b-c} (1-y^{2a})(1-y^{2b})/(1-y^2)^2 \quad (4.27)$$

which in particular is not invariant under $y \rightarrow 1/y$. This is consistent with the fact that the quiver moduli space with vanishing superpotential is a $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$ bundle over

the non-compact base $\mathbb{C}^c$ [17, §3.3]. The effect of the superpotential is to restrict to the point at the origin in $\mathbb{C}^c$, and to a complete intersection of c hypersurfaces in the fiber over that point. The first operation removes a factor y^{-c} in the index, and the second, by virtue of the Lefschetz hyperplane theorem, multiplies by an additional factor of y^c , as far as only negative powers of y are concerned. The result of these two operations, $(-1)^{a+b+c} y^{2+c-a-b} (1-y^{2a})(1-y^{2b})/(1-y^2)^2$ agrees with the Coulomb index (4.25) up to positive powers of y , which therefore correctly captures the non-middle part of the cohomology, while the middle part is captured by the single-centered invariant $\Omega_S(1, 1, 1)$.

Here we note that the same result (4.27) can be obtained by applying the formula (3.1) for vanishing R-charge but generic flavor potentials ν_{1A} ($A = 1, \dots, a$), ν_{2B} ($B = 1, \dots, b$), ν_{3C} ($C = 1, \dots, c$), which requires extracting a suitable residue of

$$Z_Q = \prod_{A=1}^a \left(\frac{u_2 y \nu_{1A} - u_1/y}{u_1 - u_2 \nu_{1A}} \right) \prod_{B=1}^b \left(\frac{u_3 y \nu_{2B} - u_2/y}{u_2 - u_3 \nu_{2B}} \right) \prod_{C=1}^c \left(\frac{u_1 y \nu_{3C} - u_3/y}{u_3 - u_1 \nu_{3C}} \right) \\ \times \widetilde{\prod_{i=1 \dots 3} \frac{du_i}{u_i(y - 1/y)}} \quad (4.28)$$

Denoting $H_{12,A} = u_1 - u_2 \nu_{1A}$, etc the singular hyperplanes, there are $a \times b$ non-degenerate intersections $H_{12,A_0} \cap H_{23,B_0}$, $b \times c$ intersections $H_{2B_0} \cap H_{31,C_0}$ and $a \times c$ intersections $H_{12,A_0} \cap H_{31,C_0}$. In the chamber $c_1 > 0, c_2 < 0, c_3 < 0$, only the flags (Q_{23,B_0}, Q_{12,A_0}) contribute, each of them giving

$$\prod_{A \neq A_0}^a \left(\frac{y \nu_{1A} - \nu_{1A_0}/y}{\nu_{1A_0} - \nu_{1i}} \right) \prod_{B \neq B_0}^b \left(\frac{y \nu_{2B} - \nu_{2B_0}/y}{\nu_{2B_0} - \nu_{2j}} \right) \\ \times \prod_{C=1}^c \left(\frac{y \nu_{1A_0} \nu_{2B_0} \nu_{3C} - 1/y}{1 - \nu_{1A_0} \nu_{2B_0} \nu_{3C}} \right) \xrightarrow{y \rightarrow 1} (-1)^{a+b+c} \quad (4.29)$$

Summing over all choices of $A_0 B_0$ and taking the limit $y \rightarrow 1$, we obtain $(-1)^{a+b+c} ab$ in agreement with (4.27). Keeping $y \neq 1$, we recognize in (4.29) the contribution of the fixed point under the flavor rotation $\varphi_{i,A} \mapsto e^{i\hbar\theta_{i,A}}$ given by $P : \phi_{12,A} = \delta_{A,A_0}, \phi_{23,B} = \delta_{B,B_0}, \phi_{31,C} = 0$. The flavor rotation act on the tangent space by a compensating gauge rotation,

$$\Phi_{1A} \mapsto e^{i(\theta_{1A} + \phi_1 - \phi_2)} \Phi_{1A}, \quad \Phi_{2B} \mapsto e^{i(\theta_{2B} + \phi_2 - \phi_3)} \Phi_{2B}, \quad \Phi_{3C} \mapsto e^{i(\theta_{3C} + \phi_3 - \phi_1)} \Phi_{3C}, \quad (4.30)$$

with $\theta_{1A_0} + \phi_1 - \phi_2 = 0$ and $\theta_{2B_0} + \phi_2 - \phi_3 = 0$, so that its determinant reproduces the denominator in (4.29), in agreement with (3.8),

$$\det_{\mathbb{C}}(1 - df_P) = \prod_{A \neq A_0} (1 - \nu_{1A}/\nu_{1A_0}) \prod_{B \neq B_0} (1 - \nu_{2B}/\nu_{2B_0}) \prod_C (1 - \nu_{3C} \nu_{1A_0} \nu_{2B_0}). \quad (4.31)$$

Rescaling the fugacities as $\nu_{1,A} \rightarrow t^A \nu_{1,A}$, $\nu_{2,B} \rightarrow t^B \nu_{1,B}$, $\nu_{3,C} \rightarrow t^C \nu_{3,C}$ and then take the limit $t \rightarrow \infty$, then each of the ab contributions (4.29) produces a single power of y , which sum up to the same result (4.27) as obtained from Reineke's formula.

5 Non-Abelian quivers

In this section, we finally turn to non-Abelian quivers, and demonstrate how the residue formula (3.1) combined with the Cauchy-Bose identity (3.12) gives a natural decomposition of the index in terms according to partitions of the total dimension vector, in agreement with the Coulomb branch formula.

5.1 Kronecker quiver with rank $(N, 1)$

We consider a quiver $Q = K_m$ with two nodes V_1, V_2 and $m > 0$ arrows from V_1 to V_2 :

We choose the dimension vector $(N_1, N_2) = (N, 1)$ and stability parameters $\zeta_1 < 0$, $\zeta_2 > 0$ such that $N\zeta_1 + \zeta_2 = 0$. For $m \geq N$, the quiver moduli space $\mathcal{M}_Q$ is known to be the Grassmannian $G(N, m)$ of N -dimensional planes inside $\mathbb{C}^m$, of dimension $d = N(m - N)$, with χ -genus given by

$$\chi_Q(y) = \frac{(-y)^{-N(m-N)} [m, y]!}{[N, y]! [m - N, y]!} \quad (5.1)$$

where $[m, y]! = \prod_{k=1}^m (1 - y^{2k}/(1 - y^2))$ is the deformed factorial. For $m < N$, the moduli space is empty. Our aim is to rederive this well-known result using localization, and explain how the decomposition predicted by the Coulomb branch formula naturally emerges in this context.

In order to apply the localization formula (3.1), we upgrade the stability vector $\zeta = (\zeta_1, \zeta_2)$ to $\eta = (\eta_1, \dots, \eta_N, \zeta_2)$ where $\eta_1 < \eta_2 < \dots < \eta_N$ and $\eta_s \simeq \zeta_1$ for $s = 1 \dots N$. We denote by $(v_1, \dots, v_N)$ and u the exponentiated Cartan variables associated to the vertices V_1 and V_2 . The equivariant χ -genus is then a sum of residues of

$$Z_Q = \left(\prod_{s \neq s'} \frac{u_{s'} - u_s}{u_s/y - y u_{s'}} \right) \prod_{s=1}^N \left(\prod_{A=1}^m \frac{y \nu_A u_s - u/y}{u - \nu_A u_s} \frac{du_s}{u_s(y - 1/y)} \right) \quad (5.2)$$

As argued in [51, 52], residues involving vector multiplet poles $u_s/y - y u_{s'}$ always vanish, so the only contribution comes from the intersection of the hyperplanes $(u - \nu_{A_1} u_1, u - \nu_{A_2} u_2, \dots, u - \nu_{A_N} u_N)$, for any subset of N distinct elements $\{A_1, \dots, A_N\} \subset \{1, \dots, m\}$, taken in this particular order for the above choice of η . In this way one arrives at⁹

$$\chi_Q(y, \nu_A) = \frac{1}{N!} \oint Z_Q = \sum_{\mathcal{I} \in C(N, m)} \prod_{A \in \mathcal{I}} \prod_{B \notin \mathcal{I}} \frac{\nu_A/y - \nu_B y}{\nu_A - \nu_B} \quad (5.3)$$

where $\mathcal{I}$ runs over all subsets of N distinct elements $\{A_1, \dots, A_N\} \subset \{1, \dots, m\}$. Each term in this sum originates from a $U(1)$ fixed point on $G(N, m)$, where the $N \times m$ matrix ϕ_{sA}

⁹This agrees with [28, §4.6], upon taking the limit $\tau \rightarrow i\infty$ in their formula.

has N non-zero elements in positions (s, A_s) . The result can be shown to be independent¹⁰ of the fugacities ν_A , and equal to the deformed binomial coefficient (5.1).

In order to explain the connection with the Coulomb branch formula, we set to one the flavor fugacities ν_A and fix the gauge $u = 1$, obtaining

$$\chi_Q(y) = \frac{1}{(y - 1/y)^N N!} \int \prod_{s=1}^N \frac{du_s}{2\pi i u_s} \prod_{s \neq s'} \frac{u_{s'} - u_s}{u_s/y - y u_{s'}} \prod_{s=1}^N \left(\frac{y u_s - 1/y}{1 - u_s} \right)^m \quad (5.4)$$

where the integral runs over a product of small circles around $u_s = 1$. Using the Cauchy-Bose identity (3.12), we can rewrite (5.4) as a sum over permutations,

$$\chi_Q = \frac{(-1)^N}{N!} \int \prod_{s=1}^N \frac{du_s}{2\pi i} \sum_{\sigma \in S_N} \frac{\epsilon(\sigma)}{\prod_{s=1}^N (u_s/y - y u_{\sigma(s)})} \prod_{s=1}^N \left(\frac{y u_s - 1/y}{1 - u_s} \right)^m \quad (5.5)$$

Decomposing each permutation σ into a product of cycles $\prod_{\ell} (\mathcal{C}_{\ell})^{n_{\ell}}$, the integral factorizes into a product of factors associated to each cycle $\mathcal{C}_{\ell}$ of length ℓ ,

$$\chi_Q = \frac{(-1)^N}{N!} \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{\ell} (\mathcal{I}_{\ell})^{n_{\ell}} \quad (5.6)$$

where $\mathcal{I}_{\ell}$ denotes the integral (with $x_i \equiv x_{i+L}$),

$$\mathcal{I}_{\ell} = \prod_{i=1}^{\ell} \oint_1 \frac{dx_i}{2\pi i} \frac{1}{\prod_{i=1}^{\ell} (y x_i - x_{i+1}/y)} \left(\frac{y x_i - 1/y}{x_i - 1} \right)^m \quad (5.7)$$

Moreover, the signature of the permutation is $\epsilon(\sigma) = \prod_{\ell} (-1)^{n_{\ell}(\ell-1)}$. By successively integrating over each u_i , one can establish that

$$\mathcal{I}_{\ell} = \frac{y^{m\ell} - y^{-m\ell}}{y^{\ell} - y^{-\ell}} \quad (5.8)$$

For this, one may deform the contour around $x_i = 1$ to a sum of contours around $x_i = x_{i+1}/y^2$ and $x_i = y^2 x_{i-1}$. The residue at $x_i = x_{i+1}/y^2$ is regular at $x_{i+1} = 1$ (due to a cancellation of factors of $(x_{i+1} - 1)^m$ in the nominator and denominator), so can be dropped. The residue at $x_i = y^2 x_{i-1}$ produces a factor $[(y^2 x_{i-1} - 1/y^2)/(y x_{i-1} - 1/y)]^m$, which cancels partially against $[(y x_{i-1} - 1/y)/(x_{i-1} - 1)]^m$, etc. Alternatively, one may reinstate the flavor fugacities and evaluate

$$\mathcal{I}_{\ell} = \int \prod_{i=1}^{\ell} \frac{dx_i}{2\pi i} \frac{1}{\prod_{i=1}^{\ell} (x_i/y - y x_{i+1})} \prod_{A=1}^m \frac{y \nu_A x_i - 1/y}{1 - \nu_A x_i} \quad (5.9)$$

where the integral circles around each pole at $x_i = \nu_A(i)$ for all maps $A : [1, \ell] \rightarrow [1, m]$. In the limit $y \rightarrow 1$ it is easy to check that constant maps contribute +1 while non-constant maps contribute 0, leading to $i_{\ell} \rightarrow m$. For $y \neq 1$ but assuming $\nu_1 \ll \dots \ll \nu_m$, one

¹⁰For this it suffices to show that the residue at the potential singularities $\nu_A = \nu_B$ vanish.

also finds that the only non-vanishing contribution come from constant maps $A(i) = A$, contributing $y^{\ell(2A-m-1)}$, in agreement with (5.8).

Using (5.8), and noting that the number of permutations with cycle shape $\lambda = \sum \ell n_\ell$ is $N! / \prod_\ell \ell^{n_\ell} n_\ell!$, one may rewrite (5.6) as a sum over partitions of N ,

$$\chi_Q = (-1)^{N(m+1)} \sum_{N=\sum_{\ell=1}^N \ell n_\ell} \prod_{\ell=1}^N \frac{[(-1)^{\ell+1} \mathcal{I}_\ell]^{n_\ell}}{\ell^{n_\ell} n_\ell!} = \sum_{N=\sum_{\ell=1}^N \ell n_\ell} \prod_{\ell=1}^N \frac{[(-1)^{m\ell+1} \mathcal{I}_\ell]^{n_\ell}}{\ell^{n_\ell} n_\ell!} \quad (5.10)$$

Note that this coincides with the cycle index $Z_N(\{t_\ell\})$ for the permutation group S_N , evaluated at $t_\ell = (-1)^{m\ell+1} \mathcal{I}_\ell$. In order to evaluate the sum over partitions, it is expedient to construct the generating function,

$$G = \sum_{N=0}^{\infty} Z_N(\{t_\ell\}) t_0^N = \exp \left(- \sum_{\ell=1}^{\infty} \frac{(-y)^{m\ell} - (-y)^{-m\ell}}{\ell(y^\ell - y^{-\ell})} t_0^\ell \right). \quad (5.11)$$

The result agrees with Z_{halo} in [24, (4.72)], up to a change $m \rightarrow -m$ since we have $m > 0$ while the result of loc.cit. assumed $m < 0$.

We can now compare (5.10) with the Coulomb branch formula (2.5) for this system (or equivalently (2.9)),

$$\chi_Q = \sum_{N=\sum_{\ell=1}^N \ell n_\ell} \frac{g_C(\{n_1 \times \gamma_1, n_2 \times 2\gamma_1, \dots, n_\ell \times \ell\gamma_1, \gamma_2\})}{\prod_{\ell=1}^N n_\ell!} \prod_{\ell=1}^N [\bar{\Omega}_S(\ell\gamma_1)]^{n_\ell} \bar{\Omega}_S(\gamma_2) \quad (5.12)$$

where $n \times \ell\gamma$ denotes n copies of the vector $\ell\gamma$. Since the only non-vanishing DSZ products are $\langle \ell\gamma_1, \gamma_2 \rangle = m\ell$, the Coulomb index factorizes into

$$g_C(n_1 \times \gamma_1, n_2 \times 2\gamma_1, \dots, n_\ell \times \ell\gamma_1, \gamma_2) = \prod_{\ell=1}^N \left(\frac{y^{m\ell} - y^{-m\ell}}{y - 1/y} \right)^{n_\ell} \quad (5.13)$$

which is also the χ -genus of the Abelian quiver $Q(\{\gamma_i\})$ described below (2.9). Moreover, in the absence of loops, the only non-vanishing single centered invariants are $\Omega_S(\gamma_1) = \Omega_S(\gamma_2) = 1$, hence

$$\bar{\Omega}_S(\ell\gamma_1) = \frac{y - 1/y}{\ell(y^\ell - y^{-\ell})}, \quad \bar{\Omega}_S(\ell\gamma_2) = 1 \quad (5.14)$$

Combining these relations, we conclude that (5.10) is in perfect agreement with the Coulomb branch formula. Moreover, the sum over partitions of $\gamma = N\gamma_1 + \gamma_2$ clearly originates from the sum over conjugacy classes in the permutation group S_N .

5.2 Star quivers

We now turn to a generalization of both the Abelian star quiver $S_{\{a_i\}}$ considered in section 4.1, and the Kronecker quiver K_m of rank $(N, 1)$ in the previous subsection. Namely, we consider a quiver with $K + 1$ vertices $V_0, V_1, \dots, V_K$, with $a_i > 0$ arrows from V_i to V_0 , and with dimension vector $(N, 1, 1, \dots)$. Our aim is to evaluate the index using the Cauchy-Bose formula for the contribution of the $U(N)$ vector multiplets, and show that

the resulting decomposition agrees with the Coulomb branch formula (2.5), or equivalently the MPS formula (2.9).

Clearly, the moduli space is trivial unless $\zeta_0 < 0$ and $\zeta_i > 0$ for $i = 1 \dots K$. Up to relabelling the nodes, we can assume that $\zeta_1 > \zeta_2 > \dots > \zeta_K$. We shall further assume that $\zeta_1 \gg \zeta_2 \gg \dots \gg \zeta_K$, and upgrade the stability vector to $\eta = (\eta_1, \dots, \eta_N, \zeta_1, \dots, \zeta_K)$ with $\eta_1 < \eta_2 < \dots < \eta_N$ and $\eta_s \simeq \zeta_0$ for $s = 1 \dots N$. We denote by $(v_1, \dots, v_N)$ and $(u_1, \dots, u_K)$ the exponentiated Cartan variables associated to the vertices V_0 and V_i . The χ -genus is then a sum of residues of

$$Z_Q = \frac{1}{(y-1/y)^{N+K-1}} \left(\prod_{s \neq s'} \frac{v_{s'} - v_s}{v_s/y - y v_{s'}} \right) \left(\prod_{\substack{s=1 \dots N \\ i=1 \dots K}} \left[\frac{y v_s - u_i/y}{u_i - v_s} \right]^{a_i} \right) \prod_{i=1}^K \frac{du_i}{u_i} \widetilde{\prod_{s=1 \dots N} \frac{dv_s}{v_s}} \quad (5.15)$$

As for the Grassmannian in the previous subsection, residues involving vector multiplet poles $v_s/y - y v_{s'}$ always vanish [51, 52], so the only contribution comes from the intersection of the NK hyperplanes $u_i - v_s$, which is degenerate if $K \geq 2$. For the above choice of η , this intersection carries a single stable flag $(u_1 - v_1, u_1 - v_2, \dots, u_1 - v_N, u_2 - v_N, u_K - v_N)$, corresponding to the following integration prescription

$$\chi_Q = \frac{1}{N!} \oint Z_Q, \quad \oint := (-1)^{N-1} \oint_{v_N} \frac{du_K}{u_K} \dots \oint_{v_N} \frac{du_1}{u_1} \oint_{u_1} \frac{dv_{N-1}}{v_{N-1}} \dots \oint_{u_1} \frac{dv_1}{v_1} \quad (5.16)$$

We now apply the Cauchy-Bose formula (3.12) to the vector multiplet product in (5.15), and collect contributions according to the cycle shape of the permutation $\sigma \in S_N$, corresponding to a partition $\lambda = \ell_1 + \dots + \ell_p = \sum \ell n_\ell$, where n_ℓ is the number of n_i 's equal to ℓ . For convenience we relabel the Cartan variables for the non-Abelian group accordingly,

$$\{v_1, \dots, v_N\} \rightarrow \{v_1^{(1)}, \dots, v_{\ell_1}^{(1)}; v_1^{(2)}, \dots, v_{\ell_2}^{(2)}; \dots; v_1^{(p)}, \dots, v_{\ell_p}^{(p)}\}. \quad (5.17)$$

All permutations with the same cycle shape λ give the same contribution to the χ -genus,

$$\begin{aligned} \chi_\lambda &= (-1)^{N-p} (y - y^{-1})^{-N-K+1} (-1)^{N-1} (1 - y^2)^N \\ &\times \oint_{v_N} \frac{du_K}{u_K} \dots \oint_{v_N} \frac{du_1}{u_1} \oint_{u_1} \frac{dv_{N-1}}{v_{N-1}} \dots \oint_{u_1} \frac{dv_1}{v_1} \\ &\times \left[\prod_{\beta=1}^p \frac{v_1^{(\beta)} \dots v_{\ell_\beta}^{(\beta)}}{(v_1^{(\beta)} - y^2 v_2^{(\beta)}) \dots (v_{\ell_\beta}^{(\beta)} - y^2 v_1^{(\beta)})} \right] \prod_{s=1}^N \prod_{\alpha=1}^K \left(\frac{y v_s - y^{-1} u_\alpha}{u_\alpha - v_s} \right)^{a_\alpha}. \end{aligned} \quad (5.18)$$

where v_N can be gauged fixed to any value. The sum over all permutations with the same cycle shape turn the prefactor $1/N!$ into a factor $1/(\prod_\ell n_\ell! \ell^{n_\ell})$. According to the MPS formula (2.9), (5.18) should coincide with

$$\tilde{\chi}_\lambda = \frac{(y - y^{-1})^p}{\prod_{\beta=1}^p (y^{\ell_\beta} - y^{-\ell_\beta})} \chi(Q_\lambda) \quad (5.19)$$

where Q_λ is the Abelian quiver

with $\ell_\beta \times a_i$ arrows going from the node 1_i on the top row to the node 1_β on the bottom row. The χ -index for Q_λ follows from the residue formula (3.1),

$$\tilde{\chi}_\lambda = \frac{(-1)^{p-1} (y - y^{-1})^{-K+1}}{\prod_{\beta=1}^p (y^{\ell_\beta} - y^{-\ell_\beta})} \times \oint_{v_p} \frac{du_K}{u_K} \dots \oint_{v_p} \frac{du_1}{u_1} \oint_{u_1} \frac{dv_{p-1}}{v_{p-1}} \dots \oint_{u_1} \frac{dv_1}{v_1} \prod_{r=1}^p \left[\prod_{\alpha=1}^K \left(\frac{yv_r - y^{-1}u_\alpha}{u_\alpha - v_r} \right)^{a_\alpha} \right]^{\ell_r}. \quad (5.20)$$

where v_p can be gauge fixed to any value. We shall now prove, by induction on K , that (5.18) and (5.20) coincide. For $K = 2$, this is the Kronecker quiver with rank $(N, 1)$ discussed in the previous subsection. We shall assume that the equality $\chi_\lambda = \tilde{\chi}_\lambda$ holds for star quivers with $K - 1$ nodes, and show that it continues to hold for K nodes.

To show that (5.18) and (5.20) coincide, we focus on the last contour integral over u_K around v_N in (5.18), or around v_p in (5.20), which we free to identify by a choice of gauge. Denoting by $\chi_\Lambda(u_K, v)$ and $\tilde{\chi}_\Lambda(u_K, v)$ the two integrands, we need to prove

$$\oint_v \frac{du_K}{u_K} [\chi_\Lambda(u_K, v) - \tilde{\chi}_\Lambda(u_K, v)] = 0, \quad (5.21)$$

By construction, $\chi_\Lambda(u_K, v)$ and $\tilde{\chi}_\Lambda(u_K, v)$ are homogenous rational functions of degree 0. The recursion shows that their difference can be put to the form

$$\chi_\Lambda(u_K, v) - \tilde{\chi}_\Lambda(u_K, v) = \frac{N_m(u_K, v)}{(u_K - v)^m}, \quad (5.22)$$

where m is a non-negative integer and $N_m(u_K, v)$ is a homogeneous polynomial of degree m . In particular, the denominator does not have any factor of u_K or v . The equality (5.21) will follow if we can show that the numerator factorizes as

$$N_m(u_K, v) = v^{m_1} u_K^{m_2} \tilde{N}_{m-m_1-m_2} \quad (5.23)$$

with $m_1, m_2 \geq 1$. Indeed, setting $v = 1$ by homogeneity, it follows from (5.23) that

$$\oint_1 \frac{du_K}{u_K} [\chi_\Lambda(u_K, 1) - \tilde{\chi}_\Lambda(u_K, 1)] = \oint_1 du_K \frac{u_K^{m_2-1} \tilde{N}_{m-m_1-m_2}(u_K, 1)}{(u_K - 1)^m} = 0 \quad (5.24)$$

since the numerator has monomials of degree less than $m - 1$ are annihilated by the $(m - 1)$ differentiations needed to extract the residue. To show that $N_m(u_K, v)$ factorizes as in (5.23), it suffices to check that $I_K(u_K, v)$ vanishes both when $u_K \rightarrow 0$ and when $u_K \rightarrow \infty$ (which is equivalent to $v \rightarrow 0$ by homogeneity). In either of those limits, the integrands $\chi_\lambda(u_K, v)$ and $\tilde{\chi}_\lambda(u_K, v)$ reduce to the indices for the star quiver obtained by removing the node v_K . By induction, the equality $\chi_\lambda = \tilde{\chi}_\lambda$ therefore holds for star quivers with arbitrary number of Abelian nodes.

5.3 Kronecker quiver with rank (N_1, N_2)

We now consider the Kronecker quiver K_m introduced in section 5.1, now for general dimension vector. The χ -genus of the moduli space $\mathcal{M}_Q$ has been computed in the mathematics literature [42, 53, 54] using the Atiyah-Bott Lefschetz fixed point theorem, and in the physics literature [32, 51, 52, 55] using supersymmetric localization. Our goal here is to clarify the relation between the two approaches, and explain how the Coulomb branch formula arises from applying the Cauchy-Bose formula to both nodes.

The quiver moduli space $\mathcal{M}_Q$ is non-trivial in the chamber $\zeta_1 < 0 < \zeta_2$ with $N_1\zeta_0 + N_2\zeta_1 = 0$, in which case its complex dimension is $d = mN_1N_2 - N_1^2 - N_2^2 + 1$. In order to apply the residue formula (3.1), we upgrade the stability vector $\zeta = (\zeta_0, \zeta_1)$ to $\eta = (\eta_1, \dots, \eta_{N_1}, \tilde{\eta}_1, \dots, \eta_{N_2})$ where $\eta_1 < \dots < \eta_{N_1} < 0 < \tilde{\eta}_1 < \dots < \eta_{N_2}$ and $\eta_s \simeq \zeta_1, \tilde{\eta}_{s'} \simeq \zeta_2$ for $s = 1 \dots N_1, s' = 1 \dots N_2$. The equivariant χ -genus is then given by a suitable residue of

$$\begin{aligned}
 Z_Q = & \frac{1}{(y - 1/y)^{N_1+N_2-1}} \prod_{\substack{s,s'=1\dots N_1 \\ s \neq s'}} \frac{u_{s'} - u_s}{u_s/y - y u_{s'}} \prod_{\substack{s,s'=1\dots N_2 \\ s \neq s'}} \frac{v_{s'} - v_s}{v_s/y - y v_{s'}} \\
 & \times \prod_{\substack{s=1\dots N_1 \\ s'=1\dots N_2}} \prod_{A=1}^m \frac{y \nu_A v_{s'} - u_s/y}{u_s - \nu_A v_{s'}} \prod_{s=1\dots N_1} \frac{du_s}{u_s} \widetilde{\prod_{s'=1\dots N_2} \frac{dv_{s'}}{v_{s'}}}
 \end{aligned} \tag{5.25}$$

For (N_1, N_2) coprime, it turns out that residues involving the vector multiplet hyperplanes $u_s/y - y u_{s'}, v_s/y - y v_{s'}$ vanish. For generic fugacities, the intersections of the chiral multiplet hyperplanes are non-degenerate, and in one-to-one correspondence with bipartite trees linking N_1 ‘black’ vertices $e_1, \dots, e_{N_1}$ to N_2 ‘white’ vertices $f_1, \dots, f_{N_2}$, with edges $e_s - f_{s'}$ colored by an integer $A \in \{1, \dots, m\}$. Each edge represents one hyperplane $u_s - \nu_A v_{s'}$, and each tree corresponds to a particular non-degenerate intersection, which may or may not carry a stable flag. As usual, stable flags correspond to fixed points of the toric action $\phi_{Ass'} \mapsto \nu_A \phi_{Ass'}$ on the space of solutions to the D-term equations (2.1), up to gauge transformations. Since the quiver moduli space is compact, the dependence on θ_A must cancel after summing over all fixed points. In the limit $y \rightarrow 1$, each stable flag contributes a rational number $\pm 1/(N_1!N_2!)$, which must add up to the Euler characteristic which is integer. Alternatively, by assuming that the θ_A ’s are hierarchically ordered, e.g. $\theta_1 \ll \theta_2 \ll \dots \ll \theta_m$, one may associate a given power $\pm y^{2J_3}/(N_1!N_2!)$ to each fixed point, which must add up to the χ -genus.

5.3.1 Rank (2, 3)

For illustration, we consider the rank (2, 3) case in some detail. For generic values of the fugacities, one finds contributions from two types of graphs, depicted in figure 2:

- $m(m-1)(m-2)$ flags of type F_{ABCC} with A, B, C all distinct,

$$F_1 = (-u_1 + \nu_A v_3, -u_1 + \nu_B v_2, -u_1 + \nu_C v_1, -u_2 + \nu_A v_3) \tag{5.26}$$

contributing $-\frac{1}{12}$ as $y \rightarrow 1$.

Figure 2. Graphs associated to the flags contributing to the equivariant χ -genus of the Kronecker quiver $K_m(2, 3)$ (up to permutations of $\{u_s\}$ and $\{v_{s'}\}$)

- $3 \times m(m-1)(m-2)$ flags of type F_{ABCC} with A, B, C all distinct,

$$\begin{aligned} F_2 &= (-u_2 + \nu_A v_3, -u_1 + \nu_B v_2, -u_1 + \nu_C v_1, -u_1 + \nu_A v_3) \\ F_3 &= (-u_1 + \nu_C v_3, -u_2 + \nu_A v_2, -u_1 + \nu_B v_1, -u_1 + \nu_A v_2) \\ F_4 &= (-u_1 + \nu_C v_3, -u_1 + \nu_B v_2, -u_2 + \nu_A v_1, -u_1 + \nu_A v_1) \end{aligned} \quad (5.27)$$

contributing $+\frac{1}{12}$ as $y \rightarrow 1$; note that F_1 and F_2 only differ by the order of the hyperplanes, while $F_{2,3,4}$ differ by a permutation of $\{v_1, v_2, v_3\}$.

- $6 \times m(m-1)^3$ flags of type W_{ABCD} with $A \neq B, B \neq C, C \neq D$,

$$\begin{aligned} W_1 &= (-u_2 + \nu_A v_3, -u_1 + \nu_D v_1, -u_1 + \nu_C v_2, -u_2 + \nu_B v_2) \\ W_2 &= (-u_2 + \nu_A v_3, -u_1 + \nu_D v_2, -u_1 + \nu_C v_1, -u_2 + \nu_B v_1) \\ W_3 &= (-u_2 + \nu_A v_2, -u_1 + \nu_D v_1, -u_1 + \nu_C v_3, -u_2 + \nu_B v_3) \\ W_4 &= (-u_1 + \nu_D v_3, -u_2 + \nu_A v_1, -u_1 + \nu_C v_2, -u_2 + \nu_B v_2) \\ W_5 &= (-u_1 + \nu_D v_2, -u_2 + \nu_A v_1, -u_1 + \nu_C v_3, -u_2 + \nu_B v_3) \\ W_6 &= (-u_1 + \nu_D v_3, -u_2 + \nu_A v_2, -u_1 + \nu_C v_1, -u_2 + \nu_B v_1) \end{aligned} \quad (5.28)$$

contributing $+\frac{1}{12}$ as $y \rightarrow 1$. These flags differ by permutations of $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$.

In total, we thus find $2m(3m^3 - 7m^2 + 3m + 1)$ stable flags, contributing as $y \rightarrow 1$

$$\begin{aligned} m(m-1)(m-2) \times \frac{-1}{12} + 3 \times m(m-1)(m-2) \times \frac{1}{12} \\ + 6 \times m(m-1)^3 \times \frac{1}{12} = \frac{m^4}{2} - \frac{4m^3}{3} + m^2 - \frac{m}{6} \end{aligned} \quad (5.29)$$

More generally, for $y \neq 1$ we find

$$\chi_Q = \frac{y^{6m-4} + y^{4-6m} + (y + 1/y)^4 - (y^6 + 3y^4 + 3y^2 + 2)y^{2m-4} - (2y^6 + 3y^4 + 3y^2 + 1)y^{-2-2m}}{(y - 1/y)^4 (y + 1/y)^2 (y^2 + 1 + y^{-2})} \quad (5.30)$$

in agreement with Reineke's formula.

In the absence of flavor fugacities, the non-degenerate intersections collide into a single degenerate intersection involving all chiral multiplet hyperplanes. We find that 4 stable flags contribute, corresponding to F_1, F_2, F_3, F_4 above with $\nu_A = 1$, with the flags $F_{2,3,4}$ producing equal contributions. As an example, for $m = 3$ we find

$$\begin{aligned}
 & -\frac{1}{2} \left(\frac{7}{y^6} + \frac{27}{y^4} + \frac{55}{y^2} + 69 + 55y^2 + 27y^4 + 7y^6 \right) \\
 & \quad + \frac{3}{6} \left(\frac{9}{y^6} + \frac{29}{y^4} + \frac{61}{y^2} + 75 + 61y^2 + 29y^4 + 9y^6 \right) \quad (5.31) \\
 & = \frac{1}{y^6} + \frac{1}{y^4} + \frac{3}{y^2} + 3 + 3y^2 + y^4 + y^6 \xrightarrow{y \rightarrow 1} -\frac{247}{2} + 3 \times \frac{91}{2} = 13
 \end{aligned}$$

again in agreement with (5.30).

Let us now apply the Cauchy-Bose identity for the $U(2)$ vector multiplets, keeping the fugacities generic. In this case, the same non-degenerate intersections contribute separately to each of the two partitions $(1 + 1, 2)$:

- Each of the $m(m-1)(m-2)$ flags of type F_1 contribute $(\frac{1}{4}, -\frac{1}{3})$
- Each of the $3 \times m(m-1)(m-2)$ flags of type $F_{2,3,4}$ contribute $(0, \frac{1}{12})$
- Each of the $6 \times m(m-1)^3$ flags of type $W_{1,\dots,6}$ contribute $(\frac{1}{12}, 0)$
- In addition, there are $3 \times 2 \times m(m-1)$ flags of type F_{AABB} with $A \neq B$ which contribute $(\frac{1}{12}, -\frac{1}{12})$

After summing these contributions, we arrive at the result predicted by the MPS formula (2.10) with $S = \{1\}$,

$$\chi_{K_m(2,3)} = \frac{y - y^{-1}}{2(y^2 - y^{-2})} \chi_{K_{2m}(1,3)} + \frac{1}{2} \chi_{S_{m,m}(3,1,1)} \quad (5.32)$$

where K_{2m} is the Kronecker quiver with $2m$ arrows, while $S_{m,m}$ is the star quiver considered in section 5.2 for $a_1 = a_2 = m$. Similarly, applying the Cauchy-Bose identity for the $U(3)$ vector multiplets, one finds

$$\chi_{K_m(2,3)} = \frac{y - y^{-1}}{3(y^3 - y^{-3})} \chi_{K_{3m}(2,1)} + \frac{y - y^{-1}}{2(y^2 - y^{-2})} \chi_{S_{2m,m}(2,1,1)} + \frac{1}{6} \chi_{S_{m,m,m}(2,1,1,1)} \quad (5.33)$$

in agreement with the MPS formula (2.10) with $S = \{2\}$.

Finally, let us apply the Cauchy-Bose identity for both the $U(2)$ and $U(3)$ vector multiplets. We find contributions from the 6 possible partitions $\lambda = (\lambda_1, \lambda_2)$ of the dimension vector $(2, 3)$:

- For $\lambda = (2, 3)$, we find m flags of type F_{AAAA} contributing $-\frac{2}{3}$ each, and $3m$ flags of type F_{AAAA} contributing $\frac{1}{6}$, leading to $\chi_Q^\lambda = -m/6$ in the limit $y \rightarrow 1$;
- For $\lambda = (2, 2 + 1)$, we find m^2 flags of type F_{AABB} contributing $+1$, and $2m^2$ flags of type F_{AABB} contributing $-\frac{1}{4}$, leading to $\chi_Q^\lambda = m^2/2$;

- For $\lambda = (2, 1 + 1 + 1)$, we find m^3 flags of type F_{ABCC} contributing $-\frac{1}{3}$, leading to $\chi_Q^\lambda = -m^3/2$;
- For $\lambda = (1 + 1, 3)$, we find m^2 flags of type F_{AAAB} contributing $\frac{1}{2}$, leading to $\chi_Q^\lambda = m^2/2$;
- For $\lambda = (1 + 1, 2 + 1)$, we find $2m(m - 1)$ flags of type F_{AAAB} with $A \neq B$ and m flags of type F_{AAAA} , each contributing $-\frac{3}{4}$; $2m(m - 1)$ flags of type F_{AAAB} with $A \neq B$ and m flags of type F_{AAAA} , each contributing $-\frac{1}{4}$; and $2m(m - 1)^2$ flags of type W_{AABC} with $A \neq B, B \neq C$, each contributing $-\frac{1}{2}$; in total, $\chi_Q^\lambda = -m^3$;
- For $\lambda = (1 + 1, 1 + 1 + 1)$, we find m^3 flags of type F_{ABCC} contributing $\frac{1}{4}$; $m(m - 1)$ flags of type F_{AAAB} contributing $\frac{1}{4}$; $3m^3$ flags of type F_{ABCC} plus $3m(m - 1)$ flags of type F_{AAAB} with $A \neq B$, each contributing $\frac{1}{12}$; $6m(m - 1)^2$ flags of type W_{AABC} with $A \neq B, B \neq C$ contributing $\frac{1}{6}$; $6m(m - 1)^3$ flags of type W_{ABCD} with $A \neq B, B \neq C, C \neq D$ contributing $\frac{1}{12}$; in total, $\chi_Q^\lambda = m^4/2$;

These results are summarized in the table below, retaining the dependence on y :

λ	χ_Q^λ	$y \rightarrow 1$
$(2, 3)$	$-\kappa(6m) \rho_2 \rho_3$	$-m/6$
$(2, 2 + 1)$	$\kappa(2m) \kappa(4m) \rho_2^2$	$m^2/2$
$(2, 1 + 1 + 1)$	$-\frac{1}{6} \kappa(2m)^3 \rho_2$	$-m^3/3$
$(1 + 1, 3)$	$\frac{1}{2} \kappa(3m)^2 \rho_3$	$m^2/2$
$(1 + 1, 2 + 1)$	$-\frac{1}{2} \kappa(m) \kappa(2m) (\kappa(3m) + \kappa(m)) \rho_2$	$-m^3$
$(1 + 1, 1 + 1 + 1)$	$\frac{1}{12} \kappa(m)^3 (\kappa(3m) + 3\kappa(m))$	$m^4/2$

(5.34)

where $\rho_k = (y - 1/y)/k/(y^k - y^{-k})$ and $\kappa(m) = (-1)^m (y^m - y^{-m})/(y - 1/y)$. The contributions of the various partitions perfectly match the result of the Coulomb branch formula, which in this case follows from the wall-crossing formula [24, (A.4)] (changing $m \rightarrow -m$ in this equation, and setting $\bar{\Omega}(N_1, N_2) = 0$ unless $N_1 N_2 = 0$)

$$\begin{aligned}
 \chi_{K_m(2,3)} = & -\kappa(6m) \bar{\Omega}(2, 0) \bar{\Omega}(0, 3) + \frac{1}{2} [\kappa(3m)]^2 \bar{\Omega}(1, 0)^2 \bar{\Omega}(0, 3) \\
 & + \kappa(2m) \kappa(4m) \bar{\Omega}(2, 0) \bar{\Omega}(0, 2) \bar{\Omega}(0, 1) \\
 & - \frac{1}{2} [\kappa(m)^2 \kappa(2m) + \kappa(m) \kappa(2m) \kappa(3m)] \bar{\Omega}(1, 0)^2 \bar{\Omega}(0, 2) \bar{\Omega}(0, 1) \\
 & - \frac{1}{6} \kappa(2m)^3 \bar{\Omega}(2, 0) \bar{\Omega}(0, 1)^3 + \frac{1}{2} [3\kappa(m)^4 + \kappa(m)^3 \kappa(3m)] \bar{\Omega}(1, 0)^2 \bar{\Omega}(0, 1)^3
 \end{aligned}
 \tag{5.35}$$

where $\bar{\Omega}(k, 0) = \bar{\Omega}(0, k) = \rho_k$.

5.3.2 Rank (2, 2)

In the case where the dimension vector is not primitive, there can be contributions from singularities involving vector multiplet hyperplanes. As a result, the contributions are still

classified by trees but they are no longer bipartite, since they can involve edges of the form $e_s - e_{s'}$ or $f_s - f_{s'}$ corresponding to the hyperplanes $u_s/y - y u_{s'}$ or $v_s/y - y v_{s'}$. As an example, we consider the rank $(2, 2)$ case. In the absence of flavor fugacities, we find that among the 21 singularities, three intersections support stable flags with non-trivial contributions. As an example, for $m = 3$ we get

flag	residue	
$F = (v_2 - u_1, v_1 - u_1, v_2 - u_2)$	$-\frac{1}{2}(\frac{2}{y^5} + \frac{11}{y^3} + \frac{20}{y} + 20y + 11y^3 + 2y^5)$	(5.36)
$G_1 = (v_1 - u_1, v_2 - u_2, u_2 - y^2 u_1)$	$\frac{10+27y^2+37y^4+27y^6+10y^8}{4y^3(1+y^2)}$	
$G_2 = (v_2 - u_1, v_1 - u_2, u_2 - y^2 u_1)$	$\frac{10+27y^2+37y^4+27y^6+10y^8}{4y^3(1+y^2)}$	

Adding up these terms, we find

$$\bar{\chi}_{K_3(2,2)} = -\frac{(2+3y^2+2y^4)(1+y^4+y^8)}{2y^5(1+y^2)} \xrightarrow{y \rightarrow 1} -\frac{21}{4} \quad (5.37)$$

in agreement with Reineke's formula's for the rational invariant,

$$\bar{\chi}_{K_m(2,2)} = -\frac{(y^m - y^{-m})(2y^{3m-2} + 2y^{2-3m} - 2y^{m+2} - 2y^{-2-m} - y^3 - y^{-3} + y + y^{-1})}{2(y-1/y)^3(y+1/y)^2} \xrightarrow{y \rightarrow 1} \frac{-m^3}{2} + m^2 - \frac{m}{4} \quad (5.38)$$

For generic flavor fugacities, the singularities become non-degenerate. We find that the following stable flags contribute: (see figure 3)

- $2 \times m(m-1)^2$ flags of type F_{ABC} with $A \neq B, B \neq C$,

$$\begin{aligned} F_1 &= (-u_1 + \nu_B v_2, -u_1 + \nu_A v_1, -u_2 + \nu_C v_2) \\ F_2 &= (-u_1 + \nu_A v_1, -u_2 + \nu_C v_2, -u_1 + \nu_B v_2) \end{aligned} \quad (5.39)$$

contributing $-\frac{1}{4}$ as $y \rightarrow 1$;

- $2m$ flags of type G_A ,

$$\begin{aligned} G_1 &= (-u_1 + \nu_A v_1, -u_2 + \nu_A v_2, -u_1 y^2 + u_2) \\ G_2 &= (-u_1 + \nu_A v_2, -u_2 + \nu_A v_1, -u_1 y^2 + u_2) \end{aligned} \quad (5.40)$$

contributing $\frac{1}{8}$ in the limit $y \rightarrow 1$ as $y \rightarrow 1$;

In total, we get

$$\bar{\chi}_{K_m(2,2)} = 2m(m-1)^2 \times -\frac{1}{4} + 2m \times \frac{1}{8} = -\frac{m^3}{2} + m^2 - \frac{m}{4} \quad (5.41)$$

in agreement with [32, §6.3].

Applying the Cauchy-Bose formula for both nodes, this result decomposes as a sum over partitions $\lambda = (\lambda_1, \lambda_2)$ of the dimension vector $(2, 2)$:

Figure 3. Graphs associated to the flags contributing to the equivariant χ -genus of the Kronecker quiver $K_m(2, 2)$ (up to permutations of $\{u_s\}$ and $\{v_{s'}\}$)

- For $\lambda = (2, 2)$, we find m flags of type F_{AAA} contributing $-\frac{1}{2}$ each and $2m$ flags of type G_A contributing $-\frac{1}{8}$ each, leading to $\chi_Q^\lambda = -\frac{m}{4}$;
- For $\lambda = (2, 1 + 1)$ or $\lambda = (1 + 1, 2)$, we find m^2 flags of type F_{AAB} contributing $\frac{1}{2}$ each, leading to $\chi_Q^\lambda = \frac{m^2}{2}$;
- For $\lambda = (1 + 1, 1 + 1)$, we find $2m(m - 1)$ flags of type F_{AAB} with $A \neq B$ and m flags of type F_{AAA} , contributing $-\frac{1}{2}$ each; and $2m(m - 1)^2$ flags of type F_{ABC} with $A \neq B \neq C$; in total $\chi_Q^\lambda = -\frac{m^3}{2}$;

These results are summarized in the table below, retaining the dependence on y :

λ	χ_Q^λ	$y \rightarrow 1$
$(2, 2)$	$\kappa(4m) \rho_2^2$	$-m/4$
$(2, 1 + 1)$	$\frac{1}{2}[\kappa(2m)]^2 \rho_2$	$m^2/2$
$(1 + 1, 2)$	$\frac{1}{2}[\kappa(2m)]^2 \rho_2$	$m^2/2$
$(1 + 1, 1 + 1)$	$\frac{1}{4}\kappa(m)^2 \kappa(2m)$	$-m^3/2$

(5.42)

The contributions of the various partitions match the result of the Coulomb branch formula [24, (A.4)]

$$\begin{aligned}
 \bar{\chi}_{K_m(2,2)} = & -\kappa(4m) \bar{\Omega}(2, 0) \bar{\Omega}(0, 2) + \frac{1}{2}[\kappa(2m)]^2 \bar{\Omega}(1, 0)^2 \bar{\Omega}(0, 2) \\
 & + \frac{1}{2}[\kappa(2m)]^2 \bar{\Omega}(2, 0) \bar{\Omega}(0, 1)^2 - \frac{1}{4}\kappa(m)^2 \kappa(2m) \bar{\Omega}(1, 0)^2 \bar{\Omega}(0, 1)^2
 \end{aligned}
 \tag{5.43}$$

where $\bar{\Omega}(k, 0) = \bar{\Omega}(0, k) = \rho_k$.

5.4 Non-abelian quivers with oriented loops

Finally, we turn to an example of non-abelian quiver with oriented loops. In this case, applying the Cauchy-Bose formula to all nodes leads to a sum over partitions of the total dimension vector as a sum of multiples of basis vectors ℓ_{α_a} , whereas the Coulomb branch formula also includes contributions of single-centered invariants involving combinations of these basis vectors which support an oriented loop. Upon matching the two formulae, we find that the single-centered invariants naturally decompose into contributions from different partitions, whose mathematical meaning remains to elucidate.

To exhibit this phenomenon, let us consider the simple case of a 3-node cyclic quiver $C_{a,b,c}$ with $a > 0, b > 0, c > 0$ and dimension vector $(2, 1, 1)$. In order to compare with the analysis in Sec 6.1 of [17] (up to a cyclic permutation of the nodes), we assume

$$\begin{aligned} b < 2a, \quad c < a, \quad k = b + 2(c - a) > 0, \\ \zeta_2 > 0, \quad \zeta_2 + \zeta_3 > 0, \quad \zeta_3 < 0, \quad \zeta_1 \rightarrow 0^- \end{aligned} \quad (5.44)$$

which includes the attractor point $\zeta^* = (c - a, 2a - b, b - 2c)$. Simple choices of (a, b, c) satisfying these conditions are

$$(4, 7, 3), (5, 7, 4), (5, 8, 4), (6, 7, 5), (6, 8, 5), (7, 7, 6), (7, 8, 6), (8, 7, 7), (8, 8, 7), \dots \quad (5.45)$$

We upgrade the stability vector ζ into $(\eta_0, \eta_1; \eta_2, \eta_3) = (\zeta_1 - \epsilon, \zeta_1 + \epsilon, \zeta_2, \zeta_3)$ with $0 < \epsilon \ll |\zeta_1|$, and choose the R-charges to be $R_{12} = R_{23} = R_{31} = 2/3$ so as to allow for a generic superpotential. Denoting the Cartan variables by $(v_1, v_2; u_1, u_2)$, the χ -genus of the quiver moduli space is given by a sum of residues of

$$\begin{aligned} Z_Q = \frac{1}{2(y - 1/y)^3} \frac{(v_1 - v_2)(v_2 - v_1)}{(v_2/y - v_1y)(v_1/y - v_2y)} \times \left(\frac{y^{1/3}u_2 - u_1/y}{u_1 - y^{-2/3}u_2} \right)^b \\ \times \prod_{s=1,2} \left(\frac{y^{1/3}u_1 - v_s/y}{v_s - y^{-2/3}u_1} \right)^a \left(\frac{y^{1/3}v_s - u_2/y}{u_2 - y^{-2/3}v_s} \right)^c \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{dv_1}{v_1} \end{aligned} \quad (5.46)$$

Out of the 15 singular points, 12 intersections are non-degenerate and three are degenerate. Out of those three, one is projective and two are not, but have $\det \kappa = 0$ so can be safely ignored. For the stability conditions above, the only contributing stable flag is $(u_1 - y^{-2/3}u_2, u_2 - y^{-2/3}v_1, u_2 - y^{-2/3}v_2)$, producing a symmetric Laurent polynomial of degree $k - 5$, in agreement with the complex dimension of $\mathcal{M}_Q$. On the other hand, the Coulomb branch formula in this chamber predicts

$$\begin{aligned} \chi_Q = \Omega_S(2, 1, 1) + \left(\frac{y - 1/y}{2(y^2 - 1/y^2)} g_C(\{2\gamma_1, \gamma_2, \gamma_3\}) + H(\{\gamma_1, \gamma_2, \gamma_3\}, \{2, 1, 1\}) \right) \\ + \left(\frac{1}{2} g_C(\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}) + H(\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}, \{1, 1, 1, 1\}) \right) \end{aligned} \quad (5.47)$$

where $\Omega_S(2, 1, 1)$ is the single-centered invariant, and the modified Coulomb indices are given by [17, §6.1]

$$\begin{aligned} g_C(\{2\gamma_1, \gamma_2, \gamma_3\}) &= \frac{(-1)^k (y^k + y^{-k})}{(y - 1/y)^2} \\ H(\{\gamma_1, \gamma_2, \gamma_3\}, \{1, 1, 2\}) &= \begin{cases} \frac{1}{4} (y - y^{-1})^{-2} (y + y^{-1})^{-1} \\ \quad \times \{ -(y + y^{-1})^2 + (-1)^{k/2} (y - y^{-1})^2 \} \\ \frac{1}{2} (y - y^{-1})^{-2} \end{cases} \\ g_C(\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}) &= \frac{(-1)^{k+1} (y^k - y^{-k})}{(y - 1/y)^3} \\ H(\{\gamma_1, \gamma_2, \gamma_3, \gamma_3\}, \{1, 1, 1, 1\}) &= \begin{cases} \frac{1}{4} k (y - y^{-1})^{-2} (y + y^{-1}) \\ -\frac{1}{2} k (y - y^{-1})^{-2} \end{cases}, \end{aligned} \quad (5.48)$$

Figure 4. Abelian quivers rising in the MPS formula for the 3-node quiver with rank $(2, 1, 1)$.

where the two options correspond to $k = b + 2(c - a)$ even and odd. respectively. By comparing the result from the residue formula with (5.47), we can read off the single-centered invariant $\Omega_S(2, 1, 1)$, e.g. for $(a, b, c) = (7, 8, 6)$ we get

$$\Omega(2, 1, 1) = 1862(y + 1/y) \quad \rightarrow \quad \Omega_S(2, 1, 1) = 1863(y + 1/y) \quad (5.49)$$

Now, applying the Cauchy-Bose formula for the $U(2)$ vector multiplets, the χ -index naturally splits into two terms $\chi_Q^{(2)}, \chi_Q^{(1+1)}$ and associated to the partitions of $N = 2$. From the Coulomb branch picture, these two partitions correspond to bound states of charges $\{2\gamma_1, \gamma_2, \gamma_3\}$ and $\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}$, corresponding to the two terms in bracket in (5.47). Thus, it is natural to define ‘partial’ single-centered invariants $\Omega_S^{(2)}(2, 1, 1)$ and $\Omega_S^{(1+1)}(2, 1, 1)$ via

$$\begin{aligned} \chi_Q^{(2)} &= \Omega_S^{(2)}(2, 1, 1) + \frac{y - 1/y}{2(y^2 - 1/y^2)} g_C(\{2\gamma_1, \gamma_2, \gamma_3\}) + H(\{\gamma_1, \gamma_2, \gamma_3\}, \{2, 1, 1\}) \\ \chi_Q^{(1+1)} &= \Omega_S^{(1+1)}(2, 1, 1) + \frac{1}{2} g_C(\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}) + H(\{\gamma_1, \gamma_1, \gamma_2, \gamma_3\}, \{1, 1, 1, 1\}) \end{aligned} \quad (5.50)$$

such that $\Omega_S(2, 1, 1) = \Omega_S^{(2)}(2, 1, 1) + \Omega_S^{(1+1)}(2, 1, 1)$. E.g. for $(a, b, c) = (7, 8, 6)$,

$$\begin{aligned} \Omega_S^{(2)}(2, 1, 1) &= -3(y + 1/y)(1903y^2 + 10976 + 1903y^{-2}) \\ \Omega_S^{(1+1)}(2, 1, 1) &= 3(y + 1/y)(1903y^2 + 11597 + 1903y^{-2}) \end{aligned} \quad (5.51)$$

which correctly add up to $1863(y + 1/y)$. One might expect these two contributions to be related to the single-centered invariants of the 3-node and 4-node Abelian quivers obtained by applying the MPS formula (2.10) to the node V_1 (see figure 4). This expectation is indeed borne out for $\Omega_S^{(1,1)}(2, 1, 1)$, which is equal to half of the single-centered invariant $\Omega_S(1, 1, 1, 1)$ of the 4-node Abelian quiver, but not for $\Omega_S^{(2)}(2, 1, 1)$ which appears to be unrelated to the single-centered invariant $\Omega_S(1, 1, 1)$ associated to the 3-node Abelian quiver $C_{2a,b,2c}$. It would be interesting to understand the mathematical significance of $\Omega_S^{(2)}(2, 1, 1)$, or similar ‘partial’ single-centered invariants arising at higher rank. More generally, it would be of great interest to find a residue prescription (or otherwise) for computing the single-centered invariants $\Omega_S(\gamma, y)$ directly.

Acknowledgments

The authors are grateful to M. Reineke and T. Weist for helpful correspondence. S. M. acknowledges the hospitality of LPTHE during part of this project, and the J C Bose Fellowship of Rajesh Gopakumar, from the SERB, Govt. of India. The research of B.P. is supported in part by French state funds managed by the Agence Nationale de la Recherche (ANR) in the context of the LABEX ILP (ANR-11-IDEX-0004-02, ANR-10- LABX-63).

A Implementation of the residue formula in CoulombHiggs.m

The computations reported in this paper were in part carried out using the Mathematica package `CoulombHiggs.m` by the last-named author, which was first released along with [37] and has since then been extended to include an implementation of the residue formula (3.1) for general non-Abelian quivers. This package can be freely downloaded from the last author's home page.¹¹ Below we briefly outline how to use this package to reproduce some of the computations in this paper, referring to the documentation for more complete information.

Assuming that the file `CoulombHiggs.m` is present in the user's MATHEMATICA Application directory, the package is loaded by entering

```
In[1]:= <<CoulombHiggs`

Out[1]:= CoulombHiggs 5.0 - A package for evaluating quiver
invariants.
```

To specify to the Kronecker quiver $K_3(2,3)$ considered in section 5.3.1, we first feed in the adjacency matrix, R-charge matrix, stability parameter, dimension vector to the routine `JKInitialize`:

```
In[2]:= Mat={{0, -3},{3,0}; RMat=0*Mat; Cvec={-1/2,1/3};Nvec={2,3};
JKInitialize[Mat, RMat, Cvec, Nvec]; JKChargeMatrix //
MatrixForm

Out[2]:= 
$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & -1 & 0 & 0 & 1 & 0 & 3 \end{pmatrix}$$

```

Among other things, this routine constructs the matrix `JKChargeMatrix`, whose rows contain the charges of the chiral multiplets under the Cartan algebra $U(1)^2 \times U(1)^3$, as well as their R-charge (here, 0) and multiplicity (here, 3); as well as the vector `JKEta`, which is a small perturbation of $(\zeta_1, \zeta_1, \zeta_2, \zeta_2, \zeta_2)$ with increasing entries for each node. The integrand

¹¹<http://www.lpthe.jussieu.fr/~pioline/computing.html>

Z_Q in (3.2) can be displayed by calling `ZTrig[JKChargeMatrix, Nvec]`, or `ZEuler[]`, `ZRational[]`, `ZElliptic[]` for (3.3), (3.5) or (3.4), respectively. For computing the index of non-quiver type systems, the charge matrix can be specified by hand.

Having constructed the charge matrix and generalized stability vector, we can then feed these data into the routine `JKIndex`,

```
In[3]:= JKIndex[JKChargeMatrix, Nvec, JKEta]; JKChiGenus
66 stable flags in total; from computing the Euler number,
4 stable flags appear to contribute

Out[3]:= { - $\frac{7y^{12}+27y^{10}+55y^8+69y^6+55y^4+27y^2+7}{2y^6}$ , - $\frac{-9y^{12}-29y^{10}-61y^8-75y^6-61y^4-29y^2-9}{6y^6}$ ,
- $\frac{-9y^{12}-29y^{10}-61y^8-75y^6-61y^4-29y^2-9}{6y^6}$ , - $\frac{-9y^{12}-29y^{10}-61y^8-75y^6-61y^4-29y^2-9}{6y^6}$  }
```

Each of the entries in the result, stored in the global variable `JKChiGenus`, gives the contribution to the χ -genus of those stable flags which contribute non-trivially to the Euler number. In this case, out of 66 stable flags, 4 of those give non-trivial contributions, reproducing the result in (5.31). The relevant flags are stored in the global variable `JKRelevantStableFlags`,

```
In[4]:= JKListuDisplay = {u1, u2, v1, v2, v3};
DisplayFlagList[JKRelevantStableFlags]

Out[4]:= ( {0,0,0,0} {v3-u1,v2-u1,v1-u1,v3-u2} -1 )
{0,0,0,0} {v3-u2,v2-u1,v1-u1,v3-u1} -1
{0,0,0,0} {v3-u1,v2-u2,v1-u1,v2-u1} -1
{0,0,0,0} {v3-u1,v2-u1,v1-u2,v1-u1} -1 )
```

where the first entry in each row corresponds to the intersection point (dropping the coordinate along the frozen coordinate u_1 , specified by the variable `JKFrozenCartan` defined by `JKInitialize`), the second entry to the ordered list of hyperplanes defining each flag F , and the last entry to $\text{sgn}(\det \kappa_F)$. The covering tree associated to each flag can also be displayed by using `DisplayFlagTree`,

```
In[5]:= JKVertexLabels = {1 -> u1, 2 -> u2, 3 -> v1, 4 -> v2, 5 -> v3};
Table[ DisplayFlagTree[JKRelevantStableFlags[[i]]], i,
Length[JKRelevantStableFlags]]
```

The red zeros on each arrow indicate half the R-charge, which vanishes in this example. Flavor fugacities can be switched on by defining

```
In[6]:= RMat = 2 FlavoredRMatrix[Mat]; JKInitialize[Mat,RMat, Cvec,
        Nvec];

Out[6]:=
```

The global variable *JKChargeMatrix* now includes $2 \times 3 \times 3$ rows, each of which with R-charge $2\text{th}[A]$ where $\text{th}[A]$ are generic fugacity parameters. Running *JKIndex* will now produce 1410 stable flags, out of which 168 contribute a non-zero residue as in (5.29).

Finally, the same computation can be carried out using the Cauchy-Bose formula for either, or both factors of the gauge group $U(2) \times U(3)$. This is done using *JKIndexSplit*, which takes the same arguments as *JKIndex* plus the subset $S \subset \{1, 2\}$ as in (2.10). E.g. the results reported above (5.32) can be verified by using

```
In[7]:= JKIndexSplit[JKChargeMatrix, Nvec, JKEta, {1}]; JKChiGenus

Out[7]:=
```

and studying the structure of the relevant flags in *JKRelevantStableFlags*.

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