Boundary scattering in the ϕ^6 model

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ABSTRACT: We study the non-integrable ϕ^6 model on the half-line. The model has two topological sectors. We chose solutions from just one topological sector to fix the initial conditions. The scalar field satisfies a Neumann boundary condition $\phi_x(0, t) = H$. We study the scattering of a kink (antikink) with all possible regular and stable boundaries. For $H = 0$ the results are the same observed for scattering for the same model in the full line. For $H \neq 0$, sensible modifications appear in the dynamics with several possibilities for the output depending on the initial velocity and the boundary. Our results are confronted with the topological structure and linear stability analysis of kink, antikink and boundary solutions.

KEYWORDS: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Nonperturbative Effects

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1 Introduction

Some solutions in nonlinear field theories have the very special property of localized energy density that can propagate freely without loosing form [1–4]. The kink (antikink) is the simplest of such solutions in (1,1) dimensions with solitary character, with interesting aspects of scattering that depends strongly on the integrability of the model. This work deals with kink/antikink scattering in the half-line with a boundary condition.

The kink can be applied in scenarios with large symmetry in high energy physics. In the string landscape, it is of interest the physics of bubble collisions. In a model with false vacuum, it is known that bubbles accelerate when grow in size. Then a high speed collision means a collision with large bubbles. In this limit one can treat this problem as the scattering of two planar walls, that can be reduced to a (1,1) dimensional problem [5–7]. The ϕ^6 model under a generic perturbation was studied in the ref. [8], motivated by the undetection of topological defects expected to be produced at a large rate via the Kibble mechanism [9] in the early universe. In condensed matter, the kink in an nonintegrable model was proposed theoretically in buckled graphene nanoribbon [10, 11]. The possibility of existence of negative radiation pressure effects in buckled graphene was considered in the ref. [12]. This was based in the observation, using molecular dynamics simulations, that such system can reveal some properties of the ϕ^4 model [13]. The transition to a metallic state in heavily doped *trans*- polyacetylene was explained by Kivelson and Heeger [14] as a crossover from a soliton lattice to a polaron lattice. Recently the interaction between the

kink-antikink pair in photoexcited *trans*- polyacetylene was studied using ab-initio excited state dynamics [15]. For $T = 0$ a soliton/antisoliton nucleates and pass through each other, as expected by a sine-Gordon model. For $T \neq 0$ a surprising result appears, with the solitons scattering after bouncing twice, a characteristic of a nonintegrable model [15]. Border effects in solitons have also being studied in polyacetylene chains [16].

To better understand the consequences of a boundary condition in the kink/antikink dynamics, it is important to review some aspects of kink-antikink collisions in the full line. In an integrable model such as the sine-Gordon one, the kink-antikink pair gains after scattering at most a phase shift upon collisions [17]. On the other hand, kink/antikink scattering in a nonintegrable system has several surprising effects. Several classes of non-integrable models have already been studied. One can cite the polynomials ϕ^4 [18–24], ϕ^6 [25, 26], ϕ^8 [27–30], general high-order polynomials [31, 32], polynomials with extensions [33] or with an external perturbation [34] and nonpolynomials [35–41]. Interesting scattering structures were observed with multikinks [42–45] and with models with two scalar fields [46–49]. Despite the existence of analytical results for some classes of potentials [50], the study of kink scattering is mostly numerical.

The well-studied ϕ^4 model presents a rich structure, which depends crucially on their initial velocity v_i [23]. For large initial velocity, above a critical value v_c , the pair $K\bar{K}$ recedes from each other, i.e., the kinks always escape to infinity after one collision. On the other hand, for initial velocities bellow v_c , the structure of the collision is far more complex. In this region, the kink and antikink capture one another, forming the bion state. For $v_i \lesssim v_c$ there is the possibility of scattering after a two-bouncing process, where the pair is again able to escape to infinity after colliding twice [18, 19, 21–24]. Each two-bounce process can be identified as corresponding to a particular two-bounce window. Such windows in velocity are characterized by widths that are reduced with the grow of v_i and accumulate around $v_i = v_c$. Near to each two-bounce window usually there is a sequence of three-bounce windows. This structure is reproduced in a fractal pattern [23].

In the half-line, the study of nonlinear field theories with boundary have a long tradition in integrable systems. In such models, one aspect of investigation is to find suitable boundary conditions compatible with integrability [51–57]. Supersymmetric theories on the half-line keeping integrability were also considered [58–61]. One interesting aspect that deserves to be more investigated is the consequence of nonintegrability in the interaction of a kink or antikink with a boundary. In this line there are few examples of previous investigation for the ϕ^4 [62] and sine-Gordon [63] models.

In this paper, we will discuss the process of collision of antikinks and kinks in the ϕ^6 model with a Neumann boundary condition. Since the antikink (kink) ϕ^6 has no associated vibrational state, our main motivation was to investigate if the boundary condition is able to produce bounce windows. Our numerical analysis shows that this is the case for antikink-boundary scattering, with a very intricate structure emerging from the scattering process. The kink-boundary scattering leads to different results depending on the type of boundary to be scattered, with the most notable aspect the possibility of he changing of the topological sector and the nature of the boundary due to the scattering. Another motivation was to investigate if the boundary has a repulsive character of the scattering

defect. We found that this is true in most cases, but for some configurations the defect can be “trapped” around or even annihilated by the boundary.

In the next section we review the first-order formalism for obtaining static solutions and stability analysis. The section 3 reviews some known results of solutions and stability analysis for the ϕ^6 model in the full line. In the section 4 we present solutions and stability analysis in the half line. The numerical analysis of antikink boundary scattering is presented in the section 5. Corresponding results for kink boundary scattering is presented in the section 6. We conclude and indicate some perspectives for future work in the section 7.

2 BPS states

We consider the following action with standard dynamics:

$$S = \int dx dt \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right). \quad (2.1)$$

The equation of motion is given by

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + V_\phi = 0, \quad (2.2)$$

where $V_\phi = dV/d\phi$. If the potential is

$$V(\phi) = \frac{1}{2} W_\phi^2, \quad (2.3)$$

the solutions of the first-order equation are

$$\frac{d\phi}{dx} = \pm W_\phi, \quad (2.4)$$

and are also static solutions of the second-order equation of motion. The defects formed with this prescription minimize energy, connect adjacent minima of the potential and are known as BPS defects [64, 65]. Stability analysis around a static solution $\phi_S(x)$ considers $\phi(x, t) = \phi_S(x) + \eta(x)e^{i\omega t} + c.c.$ and the first-order corrections in the equation of motion. This results in a Schrödinger-like eigenvalue equation

$$-\frac{d^2 \eta(x)}{dx^2} + U(x)\eta(x) = \omega^2 \eta(x), \quad (2.5)$$

where the potential is

$$U(x) = \frac{d^2 V(\phi)}{d\phi^2} \Big|_{\phi=\phi_S}. \quad (2.6)$$

The Schrödinger-like equation can be rewritten as $\mathcal{H}\eta(x) = \omega^2 \eta(x)$. That is, linear stability is assured if the Hamiltonian $\mathcal{H}$ is positive definite.

Now we consider the model on the half-line $-\infty < x < 0$. For this we consider the action

$$S = \int_{-\infty}^0 dx \int dt \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \delta(x) H\phi \right\}. \quad (2.7)$$

This action is very similar to that which describes an interaction of kink with a delta-functional spatial defect (impurity) on the full line [66–69]. Here, however, we work on the half-line and we shall be interested in border effects. Varying the action with respect to ϕ leads to an equation of motion for $x \neq 0$ identical to eq. (2.2) obtained previously for the full-line. In addition, when extended and considered around $x = 0$, the action gives the Neumann condition

$$\left. \frac{d\phi}{dx} \right|_{x=0} = H. \quad (2.8)$$

Static solutions have energies given by

$$E = \int_{-\infty}^0 \left[\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + V(\phi) \right] dx - H\phi(x=0), \quad (2.9)$$

or, by eqs. (2.3) and (2.4),

$$E[\phi] = \pm W \Big|_{\phi(-\infty)}^{\phi(0)} - H\phi(0, t). \quad (2.10)$$

3 The ϕ^6 model on the full line

We consider the model given by

$$V(\phi) = \frac{1}{2} \phi^2 (1 - \phi^2)^2. \quad (3.1)$$

The associated superpotential is $W(\phi) = \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4$. The first-order equations are then

$$\frac{d\phi}{dx} = \phi - \phi^3, \quad (3.2)$$

$$\frac{d\phi}{dx} = -(\phi - \phi^3). \quad (3.3)$$

Note that the potential has vacua given by $\{-1, 0, 1\}$. The kink connecting the minima $\{0, 1\}$ is given by [25]

$$\Phi_{1+}(x) = \sqrt{\frac{1 + \tanh(x)}{2}}. \quad (3.4)$$

The antikink connecting the minima $\{1, 0\}$ is given by [25]

$$\Phi_{2+}(x) = \sqrt{\frac{1 - \tanh(x)}{2}}. \quad (3.5)$$

The antikink and kink connecting the minima $\{0, -1\}$ and $\{-1, 0\}$ are given, respectively, by $\Phi_{1-}(x) = -\Phi_{1+}(x)$ and $\Phi_{2-}(x) = -\Phi_{2+}(x)$. Each of these solutions has energy $E = 1/4$, and their profiles can be seen in the figure 1a. In addition we have four other solutions of the second-order equation of motion, divergent at $x = 0$:

$$\Phi_{3\pm}(x) = \pm \sqrt{\frac{1 + \coth(x)}{2}} \quad (3.6)$$

$$\Phi_{4\pm}(x) = \pm \sqrt{\frac{1 - \coth(x)}{2}}. \quad (3.7)$$

The profile of these solutions are presented in figure 1b.

Figure 1. The ϕ^6 model on the full line: a) (left) Static BPS kinks and antikinks on the full line, centered at $x = 0$. Plots are for $\Phi_{1+}(x)$ (red dashed), $\Phi_{2+}(x)$ (blue dash dotted), $\Phi_{1-}(x)$ (black dotted), $\Phi_{2-}(x)$ (green solid line). b) (right) Irregular solutions $\Phi_{3+}(x)$ (red dashed), $\Phi_{4+}(x)$ (blue dash dotted), $\Phi_{3-}(x)$ (black dotted), $\Phi_{4-}(x)$ (green solid line).

Figure 2. The ϕ^6 model on the full line: Schrödinger-like potential (solid line) and scalar field profile (traced line) for a) (left) kink Φ_{1+} and b) (right) antikink Φ_{2+} . See ref. [25].

Stability analysis for $\Phi_{1\pm}$ leads to the following potential of perturbations:

$$U_1(x) = \frac{5}{2} + \frac{3}{2} \tanh(x) - \frac{15}{4} \text{sech}^2(x). \quad (3.8)$$

Similarly, for $\Phi_{2\pm}$ we have

$$U_2(x) = \frac{5}{2} - \frac{3}{2} \tanh(x) - \frac{15}{4} \text{sech}^2(x). \quad (3.9)$$

Figures 2a-b show the plots of the potentials. There one can see three characteristic regions: i) $0 < \omega^2 < 1$ where one can look for bound states, ii) $1 < \omega^2 < 4$ where there is a continuum of reflecting states and iii) $\omega^2 > 4$ where there is a continuum of free states. In particular, the analysis of these states for $\Phi_{1\pm}$ was presented in detail in the ref. [70] (see also ref. [71]). The potentials have just one bound state, the zero-mode connected with the translational symmetry of the solutions. According to the Campbell, Schonfeld and Wingate (CSW) mechanism [21], the absence of a vibrational state would result in the absence of two-bounce windows in collisions involving kinks (or antikinks). However, an interesting counterexample was presented in ref. [25]. In their paper the authors showed the presence of bounce windows in antikink-kink collisions. The explanation was that the energy after the initial impact was trapped in the composited antikink-kink configuration. For example, the composite antikink-kink system given by $\phi(x) = \Phi_{2+}(x+a) + \Phi_{1+}(x-a)$ has

Figure 3. The ϕ^6 model on the full line: Schrödinger-like potential (solid line) and scalar field profile (traced line) for a) (left) antikink-kink (composite system $\Phi_{1+}(x-a) + \Phi_{2+}(x+a)$) and b) (right) kink-antikink (composite system $\Phi_{1+}(x-a) + \Phi_{2+}(x+a) - 1$). Here we considered $a = 5$.

a potential of perturbations represented in figure 3a. This potential has a tower of bound states. On the contrary, the kink-antikink system given by $\phi(x) = \Phi_{1+}(x+a) + \Phi_{2+}(x-a) - 1$ has the Schrödinger-like potential showed in figure 3b. There one has only two quasi-zero modes. That is the reason why bounce windows are absent in kink-antikink collisions.

4 The ϕ^6 model on the half-line

Now we consider the ϕ^6 model on the half-line $-\infty < x < 0$. Due to the Z_2 symmetry of the potential $V(\phi)$, and without loosing generality, we consider only solutions from the topological sector $\{0, 1\}$. Firstly we will consider the solutions given by

$$\varphi_{1+}(x) = \sqrt{\frac{1 + \tanh[x - \chi(H)]}{2}}, \quad (4.1)$$

$$\varphi_{4+}(x) = \sqrt{\frac{1 - \coth[x - X_1(H)]}{2}}, \quad (4.2)$$

which correspond respectively to $\Phi_{1+}(x)$ and $\Phi_{4+}(x)$ on the full line. The Neumann condition (eq. (2.8)), impose that

$$\frac{\text{sech}^2 \chi}{\sqrt{8 - 8 \tanh \chi}} = H, \quad (4.3)$$

$$\frac{\coth X_1^2 - 1}{\sqrt{8 + 8 \coth X_1}} = H. \quad (4.4)$$

The eqs. (4.3) and (4.4) have no real solutions for $H < 0$. The eq. (4.3) has two solutions for each value of H in the interval $0 \leq H \leq H_m$, where $H_m = 2/\sqrt{27}$. We will name these solutions as $X_0(H)$ and $\hat{X}_0(H)$ with $\hat{X}_0 \leq X_0$. For $H = H_m$ we have $\hat{X}_0 = X_0$. As a comparison, for the ϕ^4 model on the half-line we have [62] $|\hat{X}_0| = |X_0|$ and $H_m = 1$. In addition the eq. (4.4) has a unique real solution for $X_1(H)$ in the interval $0 < H < \infty$. Similar reasoning can be done with the solutions

$$\varphi_{2+}(x) = \sqrt{\frac{1 - \tanh[x + \chi(H)]}{2}}, \quad (4.5)$$

$$\varphi_{3+}(x) = \sqrt{\frac{1 + \coth[x + X_1(H)]}{2}}. \quad (4.6)$$

Figure 4. a) The two real solutions of eq. (4.7) for $\chi(H)$: X_0 (red) and $\hat{X}_0$ (green); b) the unique real solution $X_1(H)$ of eq. (4.8).

In this case the Neumann condition (eq. (2.8)) gives (compare with the eqs. (4.3) and (4.4))

$$\frac{\text{sech}^2 \chi}{\sqrt{8 - 8 \tanh \chi}} = -H, \quad (4.7)$$

$$\frac{\coth X_1^2 - 1}{\sqrt{8 + 8 \coth X_1}} = -H. \quad (4.8)$$

The eqs. (4.7) and (4.8) have no real solutions for $H > 0$. The eq. (4.7) has two solutions, $X_0(H)$ and $\hat{X}_0(H)$ in the interval $-H_m \leq H \leq 0$ whereas eq. (4.8) has one solution for each value of $H < 0$. The figure 4a presents the behavior the parameters X_0 and $\hat{X}_0$ as function of $|H|/H_m$. The figure 4b presents the parameter X_1 as function of $|H|$. In the figures the signal of H depends on the particular φ_{i+} , $i = 1 \dots 4$.

We can classify the solutions depending on the BPS equations they satisfy. Then we have

$$\phi_1(x) = \sqrt{\frac{1 + \tanh(x - X_0)}{2}} \quad (4.9)$$

$$\phi_2(x) = \sqrt{\frac{1 + \tanh(x - \hat{X}_0)}{2}} \quad (4.10)$$

$$\phi_3(x) = \sqrt{\frac{1 + \coth(x + X_1)}{2}}. \quad (4.11)$$

Figure 5. Static solutions for $|H| = 0.90H_m$.

which satisfy BPS eq. (3.2). Also we have

$$\tilde{\phi}_1(x) = \sqrt{\frac{1 - \tanh(x + \hat{X}_0)}{2}} \quad (4.12)$$

$$\tilde{\phi}_2(x) = \sqrt{\frac{1 - \tanh(x + X_0)}{2}}, \quad (4.13)$$

$$\tilde{\phi}_3(x) = \sqrt{\frac{1 - \coth(x - X_1)}{2}}, \quad (4.14)$$

which satisfy BPS eq. (3.3). Note that ϕ_1, ϕ_2 and $\tilde{\phi}_3$ are solutions for $H > 0$ whereas $\tilde{\phi}_1, \tilde{\phi}_2$ and ϕ_3 are solutions for $H < 0$.

The profiles of the solutions in the region $x < 0$, for a particular value of H , are depicted in the figure 5. Solution $\phi_3(x)$ is not regular for $x < 0$ and will not be considered here. The energy of static solutions can be found using eq. (2.10). Then, the energies for solutions (4.9)–(4.10) are given by

$$E[\phi_i(x)] = \frac{3}{4}\phi_i(0)^4 - \frac{1}{2}\phi_i(0)^2, \quad (4.15)$$

with $i = 1 \dots 2$ whereas those for solutions (4.12)–(4.14) are given by

$$E[\tilde{\phi}_i(x)] = \frac{1}{4} + \frac{1}{2}\tilde{\phi}_i(0)^2 - \frac{3}{4}\tilde{\phi}_i(0)^4 \quad (4.16)$$

with $i = 1 \dots 3$. The eqs. (4.15) and (4.16) shows that the energies presents an implicit dependence with H through of $\phi_i(0)$ and $\tilde{\phi}_i(0)$.

Figure 6. Schrödinger-like potential for static configurations: a) $\Phi_{2+}(x - a) + \phi_1(x)$ b) $\Phi_{2+}(x - a) + \phi_2(x)$ c) $\Phi_{1+}(x - a) + \tilde{\phi}_1(x) - 1$ d) $\Phi_{1+}(x - a) + \tilde{\phi}_3(x) - 1$. The lines are for $|H| = 0.05H_m$ (red), $|H| = 0.50H_m$ (green) and $|H| = 0.95H_m$ (blue). We fix $a = -12.50$.

The figure 6 shows the Schrödinger-like potential for some static configurations of antikink-boundary and kink-boundary for several values of H . Note from the figure that the potential is composed of two parts: one, with a hole centered at x_0 (chosen as -12.5 in the figure), characterizes the kink/antikink defect (known to have only the translational mode $\omega^2 = 0$); the second one, close to $x = 0$, represents the influence of the boundary. Note also that for $\omega^2 > 4$ one has a continuum of states. The possibility of vibrational states depends on the boundary. The figures 6a-b shows that, for the boundaries ϕ_1 and ϕ_2 , vibrational states must be searched for $\omega^2 \leq 4$. On the other hand, the potential of the figure 6c shows that, for boundary $\tilde{\phi}_1$, true vibrational states can occur for $\omega^2 < 1$; for $1 < \omega^2 < 4$ we have metastable states, similar to the reflecting states in the full line of section 3. The potential for $\Phi_{1+}(x - a) + \tilde{\phi}_2(x) - 1$ is similar to figure 6c, leading to the same possibilities for occurrence of vibrational and metastable states for the boundary $\tilde{\phi}_2$. The potential of the figures 6d shows no possibility of vibrational or metastable states for the boundary $\tilde{\phi}_3$.

In order to study the occurrence of localized boundary modes, we solved numerically the Schrödinger-like equation when the defect is too far from the boundary. The figure 7 shows the squared frequency of these modes as a function of H . For $0 < H < H_m$, we analyzed the solutions $\phi_1(x)$, $\phi_2(x)$ and $\tilde{\phi}_3(x)$. For $\phi_1(x)$ we have one vibrational state with $0 < \omega^2 < 1$ and the solution is stable. For $\phi_2(x)$ we have one mode with $\omega^2 < 0$, meaning

Figure 7. The squared frequencies ω^2 of localized boundary modes as a function of H . The curves are for solutions $\phi_1(x)$ (black solid line) and $\phi_2(x)$ (green dashed line), $\tilde{\phi}_1(x)$ (blue solid line) and $\tilde{\phi}_2(x)$ (red dashed and solid lines).

instability. The solution $\tilde{\phi}_3(x)$ has neither vibrational states, nor signal of instability. For $-H_m < H < 0$ we analyzed the solutions $\tilde{\phi}_1(x)$ and $\tilde{\phi}_2(x)$. For $\tilde{\phi}_1(x)$ there is one state with $\omega^2 > 0$ and the solution is stable. The classification of this state depends on the value of H . The figure 7 shows that for $-0.90 \lesssim H/H_m < 0$ we have $1 < \omega^2 < 4$. When we consider the whole system antikink-boundary, this is in fact a metastable state, as already discussed. For $-1 < H/H_m \lesssim -0.90$ we have $0 < \omega^2 < 1$ and the solution is a true vibrational state of the boundary. The solution $\tilde{\phi}_2(x)$ has three bound states, one of them with $\omega^2 < 0$, meaning instability.

The stability of the numerical method was analyzed, considering only each boundary isolated, without any scattering process. Set $\xi(x)$ one of the boundaries $\phi_1(x)$, $\tilde{\phi}_1(x)$, ϕ_2 and $\tilde{\phi}_3(x)$. The initial conditions are

$$\phi(x, 0) = \xi(x), \quad (4.17)$$

$$\dot{\phi}(x, 0) = 0. \quad (4.18)$$

For the numerical solutions we used a 4th-order finite-difference method on a grid $N = 2048$ nodes with the “infinity” at $x_{\min} = -100$ and a spatial step $\delta x \approx 0.05$. For the time dependence we used a 6th order symplectic integrator method, with a time step $\delta t = 0.02$. To demonstrate that the method does not develop significant instabilities we show in the

Figure 8. Stability analysis of the numerical method, considering the $\phi_1(x)$ configuration, compatible with stability. a) the evolution of the field on the boundary, b) the field (radiation) away from the boundary at $x = -50$. For both figures $H = 0.8H_m$.

figure 8a the evolution of the field on the stable boundary ϕ_1 and in the figure 8b the field (radiation) away from the boundary at some fixed position. The figures show that the scalar field varies less than one part in 10^7 in the former, and less than one part in 10^8 in the latter, measuring the accuracy of the symplectic method for this problem.

Now the same procedure was used for this and the remaining boundaries to build up multidimensional plots to better see significant emitted radiation. We found no significant modification of the boundaries $\phi_1(x)$, $\tilde{\phi}_1(x)$ and $\tilde{\phi}_3(x)$ with the increasing of H even for large times, which is compatible with stability (see the figures 9a-b for the boundaries $\phi_1(x)$ and $\tilde{\phi}_3(x)$). The instability of the $\phi_2(x)$ shows up in the simulations. One sees, for $H \lesssim 0.72H_m$, the emission of radiation by the boundary, as shown in the figure 9c. For $H \gtrsim 0.72H_m$ one has the production of a kink by the boundary, as shown in the figure 9d. For such high values of H there is no emission of radiation before or after the production of the kink. In this case the effect of instability shows up abruptly, with no signal of instability thereafter. This is compatible with a decaying of the unstable ϕ_2 boundary to the stable ϕ_1 .

In the remaining of this work we will study boundary scattering in the ϕ^6 model. For this purpose only the regular and stable solutions $\phi_1(x)$, $\tilde{\phi}_1(x)$ and $\tilde{\phi}_3(x)$ will be considered. We will see that even these stable boundaries can emit radiation, produce a kink/antikink or even change their nature due to the scattering. Clearly this is a consequence of the interaction defect-boundary, not a signal of instability.

Figure 9. Stability analysis of the boundaries: a) unchanging of $\phi_1(x)$ configuration, compatible with stability; b) unchanging of $\tilde{\phi}_3(x)$ configuration, compatible with stability; c) emission of radiation by the unstable ϕ_2 boundary; d) emission of a kink by the unstable ϕ_2 boundary. The values of H are shown in the figures.

5 Antikink-boundary scattering

Here we will study the scattering of an antikink from the sector $(0, 1)$ with the boundary. The figure 5 shows that an antikink in this sector can be connected with solutions $\phi_1(x)$ and $\phi_2(x)$. However, the discussion from the last section showed that only $\phi_1(x)$ is regular and stable. The antikink is the static solution $\Phi_{2+}(x)$ given by the eq. (3.5), now boosted with a velocity v_i with $x(t=0) = a$ and given by:

$$\Phi_{2+}(v_i, x, t) = \sqrt{\frac{1 - \tanh[\gamma(x - v_i t - a)]}{2}} \quad (5.1)$$

with $\gamma = 1/\sqrt{1 - v_i^2}$. The initial conditions are

$$\phi(x, 0) = \Phi_{2+}(v_i, x, 0) \pm \phi_1(x), \quad (5.2)$$

$$\dot{\phi}(x, 0) = \dot{\Phi}_{2+}(v_i, x, t)|_{t=0}, \quad (5.3)$$

Figure 10. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 boundary, showing a) bion state at the boundary, b) inelastic scattering (one-bounce) with the boundary, c) two-bounce collision with the boundary, d) production of a $\Phi_{1+}(x)$ kink from the boundary, forming a composed oscillating state, e) production of a $\Phi_{1+}(x)$ kink from the boundary, showing the antikink colliding once with the formed kink, f) production of a $\Phi_{1+}(x)$ kink from the boundary, showing the antikink colliding twice with the formed kink. One can localize the structure of the output field profile at the boundary in the next figure 11.

where in the eq. (5.2) the plus (minus) signal is for $H > 0$ ($H < 0$). For the spatial dependence we used $a = -12.5$ and a 4th-order finite-difference method. For the time dependence we used a 6th order symplectic integrator method. We will consider separately the two ranges of H .

Figure 11. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 ($-\phi_1$) boundary for $H > 0$ ($H < 0$): final state of the scalar field $\phi(0, t_f)$, with $t_f = |a|/v_i + 100$.

5.1 $0 < H < H_m$

The figure 10 shows the richness of possibilities of the scattering products for the range $0 < H < H_m$. The realization of each scenario depends on the parameters (v_i, H) . The figure 10a shows a bion state at the boundary. The figure 10b shows an inelastic scattering (one-bounce) with the boundary. The figure 10c shows a two-bounce collision with the boundary. The figure 10d shows the production of a $\Phi_{1+}(x)$ kink by the boundary together with an inelastic scattering of the antikink. Note that the antikink-kink pair do not separate completely, forming a composed oscillating state. The figures 10e (10f) show cases where the produced $\Phi_{1+}(x)$ kink scatter once (twice) with the reflected antikink. After this collision, the separation between antikink and kink grows slowly with time, favoring the exchanging of radiation between the antikink-kink pair.

The contribution of the bound states to bion and two-bounce windows can be seen in the figures 10a and 10c. There we can see that between bounces there is emission of radiation whose frequency decreases with H . This agrees with the figure 7 (black curve). The same effect is observed in the pattern of the oscillations at the boundary after the scattering (compare the figures 10c and 10b).

Figure 12. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 boundary: critical velocity v_c as a function of H . For $v_i > v_c$ there is no possibility of formation of a travelling kink at the boundary (see also figure 11).

The structure of our results on the boundary is presented in the figure 11. There one can see, in the bidimensional (H, v_i) phase space, the final state of the scalar field at $x = 0$. Let us analyze here the range $H > 0$, postponing the discussion of $H < 0$ for the next subsection. In the blue region the antikink is inelastically scattered by the boundary after one bounce. The red region characterizes the production of a $\Phi_{1+}(x)$ kink by the boundary. In the figure we can observe the dependence with H of the structure of bounce windows. There the bounce windows are characterized by the finite intervals in blue ($\phi(0, t_f) \sim 0$) in the region $0 < H \lesssim 0.6H_m$, $v_i \lesssim 0.55$. The frontier red-blue characterizes the critical velocity v_c , above which the antikink collides only once with the boundary, thereafter retreating to $x \rightarrow -\infty$ (one can better visualize this in the figure 12). This is the analogous situation of a one-bounce solution for antikink-kink scattering in the full line. For velocities $v_i < v_c$ there are several possibilities: i) bion states with bounce windows for $v_i \lesssim v_c$. This is mostly observed for small values of H (middle part of figure 11). ii) production of kink by the boundary for large values of H (far right side of figure 11).

The figure 12 shows the behavior of the critical velocity as a function of the parameter H . Note that the larger is H , the larger is the critical velocity. In particular, in the limit $H \rightarrow H_m$ we have $v_c \rightarrow 1$ and the formation of a travelling kink at the boundary. Note also that the curve v_c versus H presents an inflection point around $H \simeq 0.6H_m$, which coincides with the maximum value where two bounce windows are observed. For $H \gtrsim 0.6H_m$, v_c grows with H at a higher rate, favoring the occurrence of kink emission by the boundary.

The effect of H on the structure of two-bounce windows is described in the figures 13a-13d, where we see the times for the first three bounces as a function of the initial velocity. Each two-bounce window is delimited by the divergences in the curves of the time t for

Figure 13. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 boundary: time of the first (black), second (red) and third (blue) bounce *versus* initial velocity for (a) $H = 0$, (b) $H = 0.2H_m$, (c) $H = 0.4H_m$, and (d) $H = 0.5H_m$.

the third bounce. First of all, we observe that when $H = 0$ we recover the structure of two-bounce windows presented in antikink-kink scattering for the ϕ^6 model in the full line, as described in the ref. [25]. As H grows from zero, also grows the critical velocity v_c that separated one-bounce inelastic scattering from bion states. We note also that with the growth of H the two-bounce windows accumulate around $v_i = v_c$. For $H \simeq 0.6H_m$ they disappear, remaining just the phenomena of kink production on the boundary.

One remarkable effect of the variation of H is indicated by the integer m , the number of oscillations between bounces of $\phi(0, t)$ for the first visible two-bounce windows. This can be seen in the figures 14 and 15. For small values of H ($H \leq 0.1H_m$), the structure of bounce windows is slightly altered, as can be seen in figure 14a. The first two-bounce windows in this figure is labeled $m = 7$. This corresponds to the number of oscillations between bounces shown in figure 14a. This is smaller than the value $m = 11$ obtained for $H = 0$ (our m corresponds to $n - 1$ in the ref. [25]). Growing more the parameter H , we see that m decreases — compare figure 14b ($H = 0.2H_m$, $m = 4$) with figure 14c ($H = 0.4H_m$, $m = 2$). A quantitative view of this decay of m with H is presented in the figure 15. There one can see that m decreases until the value $m = 1$ of the first window for $H = 0.5H_m$. This shows that the

Figure 14. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 boundary: scalar field at the boundary $\phi(0, t)$ as a function of t for collisions from the first visible two-bounce window. a) $H = 0.1H_m$, b) $H = 0.2H_m$, c) $H = 0.4H_m$. The number m of oscillations between bounces for the first visible two-bounce windows is also indicated. In each figure the time t was rescaled as $t - \tilde{t}$ such that $t = 0$ corresponds to the first peak.

Figure 15. $\Phi_{2+}(x)$ antikink scattering with ϕ_1 boundary: order of first two-bounce windows as a function of H/H_m .

analogue of the magnetic field contributes to reduce the order of the first two-bounce windows, approaching to those expected from CSW mechanism. However, this occurs at the price of an increasing distortion around the bounces of $\phi(0, t)$, as can be seen in figures 14b and 14c. For $H > H_c \simeq 0.6H_m$ two-bounce windows disappear and the vibrational mode of the boundary now contributes to the production of kink during the scattering.

We have also checked that even in the regime of H connected with an irregular pattern of oscillations between bounces (see for instance figure 14c for $H = 0.4H_m$), the classification of m used here is in accord to the CSW mechanism. This was confirmed by the expected linear behavior between the time interval between bounces and the parameter m for both regimes $H = 0.1H_m$ (where the fluctuations are regular) and $H = 0.4H_m$ (where the fluctuations are irregular).

5.2 $-H_m < H < 0$

In this case the boosted antikink has a repulsive interaction with the boundary. Following ref. [62] one can find the value of the H -dependent initial velocity, v_* for which the energy of the initial configuration (boosted antikink and boundary) is equal to energy of the final boundary state. The initial velocity $v_i = v_*$ is the equilibrium point where the final state becomes as the unstable solution $\tilde{\phi}_2(x)$. Then, when $v_i = v_*$ and from the eqs. (4.15) and (4.16) we get $E_i = \frac{\gamma(v_*)}{4} + E[-\phi_1]$ and $E_f = E[\tilde{\phi}_2]$. The equality $E_i = E_f$ then gives

$$v_*(H) = \sqrt{1 - \left[2 \left(\phi_1(0)^2 + \tilde{\phi}_2(0)^2 \right) - 3 \left(\phi_1(0)^4 + \tilde{\phi}_2(0)^4 \right) + 1 \right]^{-2}} \quad (5.4)$$

The curve $v_*(H)$ matches with the curve in the left side of the figure 11. Below v_* the boosted antikink does not reach the boundary. For $v_* < v_i < v_c$ there is an extremely narrow intermediate region where one can observe resonance phenomena. For $v_i > v_c$ we always observe one-bounce collision. An exception to this scenario occurs for $H < -0.25H_m$ and large initial velocity, where the final state is the antikink $\Phi_{1-}(x)$ (green region from figure 11).

The figure 16 shows some representative results from antikink scattering for $H < 0$: a) the antikink does not even reach the boundary due to the repulsive force, b) two-bounce scattering, c) one-bounce scattering, d) the scalar field accesses the other topological sector.

The figure 17 shows the modulus of the final velocity of the antikink. We note that the existence of a vibrational mode for $H < 0$ results in some resonance windows. We report an interesting feature: the windows occurs in a much narrow interval of initial velocity, hardly seen on the left side of the figure 11. In the figure 17 the dashed line is the curve $v_f = v_i$ (always in absolute value) and the dash-dotted line is the value of v_* for the respective H value. In all cases if $v_i < v_*$ then the antikink does not reach the boundary and returns with the same velocity. Increasing v_i from v_* we find the much narrow region where can happen resonance phenomena. The middle region in plots gives final velocity only for two-bounce windows as the figure 17b. Increasing further v_i we arrive in one-bounce region, where the antikink always collides once with boundary. Similarly to what happens in [62], the final velocity in a escape window is always greater than v_* . Moreover when v_i overcomes the upper critical escape velocity, $v_i > v_c$, the final velocity v_f is slightly higher than v_* contrary to what happens for the ϕ^4 model (see the ref. [62]).

For $H < 0$, it is instructive to note that, for the ϕ^4 model [62], the boundary does not have a vibrational mode able to storage the energy of the translational mode, whereas the traveling antikink presents one. On the other hand, for the ϕ^6 model, the traveling antikink does not have a vibrational mode whereas the boundary presents a non-zero mode described by black curve in the figure 7. In both ϕ^4 and ϕ^6 models the boundary decays to an unstable configuration $\tilde{\phi}_2$ when v_i is equal or slightly higher than v_* .

6 Kink-boundary

Here we will consider scattering with a kink in the sector $(0, 1)$ with the boundary. The kink is the static solution $\Phi_{1+}(x)$ given by the eq. (3.4), now boosted with a velocity v_i

Figure 16. Antikink scattering for $H < 0$ a) for $v_i < v_c$ the antikink does not even reach the boundary due to the repulsive force, b) two-bounce scattering for $v_* < v_i < v_c$, c) one-bounce scattering for $v_i > v_c$ then the antikink collides once and return, d) For $v_i > v_c$ and large values of H the scalar field accesses the other topological sector. Compare with figure 11.

with $x(t = 0) = a$ and given by:

$$\Phi_{1+}(v_i, x, t) = \sqrt{\frac{1 + \tanh(\gamma(x - v_i t - a))}{2}}. \quad (6.1)$$

The figure 5 shows that this kink can be connected only with solutions $\tilde{\phi}_1(x)$ and $\tilde{\phi}_3(x)$. In the following we consider separately the two possibilities.

6.1 Kink- $\tilde{\phi}_1$ boundary

For this solution we consider $-H_m < H < 0$. The initial conditions are

$$\phi(x, 0) = \Phi_{1+}(v_i, x, 0) + \tilde{\phi}_1(x) - 1, \quad (6.2)$$

$$\dot{\phi}(x, 0) = \dot{\Phi}_{1+}(v_i, x, t)|_{t=0}. \quad (6.3)$$

with $a = -12.50$.

The figure 18 shows different aspects of the scattering process. We found that, as a result of the scattering there is a changing of the topological sector with the production a Φ_{1-}

Figure 17. Final velocity of the antikink Φ_{2+} . a) $H = -0.05H_m$ b) $H = -0.20H_m$ c) $H = -0.30H_m$ and d) $H = -0.40H_m$. Dashed lines represent $v_f = v_i$ and dotted-dashed lines represent the value of $v_*(H)$ according to the eq. (5.4).

antikink. Note also that the nature of the boundary $\tilde{\phi}_1$ has changed due to the scattering to another boundary $-\tilde{\phi}_3$. In this way we have the process $\Phi_{1+} + \tilde{\phi}_1 \rightarrow \Phi_{1-} + (-\tilde{\phi}_3)$. The final $-\tilde{\phi}_3$ boundary has no bound state, only a continuum of states. The modulus v_f of the velocity of the produced antikink has an intricate dependence with v_i and H , as shown for instance in the figure 19. Another interesting effect is the appearing of a travelling oscillon, as shown in the figure 18c. This effect starts to appear for $|H| \approx 0.70H_m$ but it is more evident for $|H| > 0.90H_m$. The oscillon has the scalar field around the vacuum $\phi = 0$ when its velocity is larger than v_f , as in the example of the figure 18c. If its velocity is lower than v_f , the scalar field oscillating around the vacuum $\phi = -1$. Note also that, contrarily to the rich pattern observed for antikink-boundary scattering in the figure 11, here we have that for all phase space (v_i, H) the final state of the boundary is close to the vacuum $\phi = -1$.

6.2 Kink- $\tilde{\phi}_3$ boundary

For this solution we have $0 \leq H < \infty$. However, to ease the comparison with the other cases, in our simulations we will consider the specific range $0 \leq H \leq H_m$. The initial conditions are

$$\phi(x, 0) = \Phi_{1+}(v_i, x, 0) + \tilde{\phi}_3(x) - 1, \quad (6.4)$$

$$\dot{\phi}(x, 0) = \dot{\Phi}_{1+}(v_i, x, t)|_{t=0}. \quad (6.5)$$

with $a = -12.50$.

In this case the boundary presents an initially-repulsive force. The figure 20 shows the pattern of scattering for different values of v_i and H . The figure 21 shows the value of the scalar field at the boundary for a large scale of time. For small initial velocities,

Figure 18. Φ_{1+} kink scattering with $\tilde{\phi}_1$ boundary for different values of v_i and H , showing: in a) and b), the changing of the topological sector with a produced Φ_{1-} antikink; in c) and d), a produced oscillon and Φ_{1-} antikink.

the kink does not even reach the boundary at $x = 0$, and the scattering is almost elastic, with a small emission of radiation (see the figure 20a). Increasing the value of v_i until a certain velocity v_* , the kink becomes stuck to the boundary, with very low rate of emitted radiation, as shown in the figure 20b. This final state is the unstable configuration ϕ_2 . This is supported by energy considerations. Indeed, since there is a small amount of emitted radiation, we can obtain v_* considering the conservation of energy between initial and final states. For the initial state we have $E_i = \frac{1}{4}\gamma(v_*) + E[\tilde{\phi}_3]$ whereas for the final state we have $E_f = E[\phi_2]$. The equality $E_i = E_f$ gives (see a similar reasoning for the ϕ^4 model in the ref. [62])

$$v_*(H) = \sqrt{1 - \left[3\phi_2(0)^4 + 3\tilde{\phi}_3(0)^4 - 2\phi_2(0)^2 - 2\tilde{\phi}_3(0)^2 - 1 \right]^{-2}}. \quad (6.6)$$

The curve $v_*(H)$ is included in the figure 21 (green traced line). Note the agreement between this figure, which is constructed with the results of numerical solutions of the kink-boundary scattering and the expression given by the eq. (6.6). Increasing the initial velocity above v_* , there is an intermediate region where we observe that the kink approaches, overcome the potential barrier and is absorbed by the boundary, that now radiates in a

Figure 19. Final velocity modulus v_f of the produced Φ_{1-} antikink as a function of $|H|/H_m$ with a) $v_i = 0.05$ b) $v_i = 0.20$ c) $v_i = 0.50$ d) $v_i = 0.80$.

typical pattern of the unstable solution ϕ_2 (see the figure 20c and the discussion of the end of section 4). Here one has a process like $\Phi_{1+} + \tilde{\phi}_3 \rightarrow \phi_2$. This type of scattering corresponds to the central blue region of the figure 21. Keeping H fixed and increasing v_i , there occurs a scenario like that of the figure 20d where the nature of the $\tilde{\phi}_3$ boundary does not change due to the scattering. There is the emission of a Φ_{1+} kink, followed by the emission of radiation from the continuum states of the boundary. This type of scattering corresponds to the upper orange region of the figure 21. Increasing further the velocity, one has the scenario of the figure 20e, which shows the changing of the topological sector and the production of an Φ_{1-} antikink. This corresponds to the green region of the figure 21. The possibility of changing the topological sector decreases for large values of H . Larger values of H in the same orange region of figure 21 we have the effect that during the collision the scalar field at the center of mass oscillates around $\phi = 0$, as in the figure 20f.

7 Conclusions

In this work we have considered the ϕ^6 model with a Neumann boundary condition characterized by the parameter H . We found several solutions for kink, antikink and boundary. Stability analysis for the antikink and kink lead to a translational mode. A similar procedure for the boundary solutions gives the behavior of the squared frequency of discrete

Figure 20. Φ_{1+} kink scattering with $\tilde{\phi}_3$ boundary, showing a) quasi-elastic return b) kink with energy to reach the energy equilibrium point c) emission of radiation by the unstable ϕ_2 boundary d) Typical collision on top orange region of figure 21 e) Inelastic scattering f) Final state for large H .

modes as a function of H . This resulted in three regular and stable solutions for the boundaries. The study of the unstable boundary was important since the nature of a boundary can be changed to an unstable one due to the scattering process. The changing of the nature of a regular and stable boundary due to the scattering was verified in some examples in two ways: i) a topological argument — the changing of the topological sector, and ii) energy conservation leading to an accurate description of $v_*(H)$. After a period of time without any sensible alteration, and depending on the parameter H , there are two possibilities for the time evolution of an isolated unstable boundary: i) radiation being emitted continuously, or ii) creation of a kink and decaying to a stable boundary. As initial conditions, we considered kink/antikink scattering with regular and stable boundaries.

Figure 21. $\Phi_{1+}(x)$ kink scattering with $\tilde{\phi}_3$ boundary for $H > 0$: final state of the scalar field $\phi(0, t_f)$, with $t_f = \frac{|a|}{v_i} + 100$. The green traced line is given by the eq. (6.6).

Our numerical analysis showed that the antikink-boundary scattering model has a rich pattern. We report the following phenomena, depending on the initial velocity and the parameter $H > 0$: i) the production of a new kink from the boundary at $x = 0$ that travels with the reflected antikink. Such reflected antikink-kink pair can have a finite number of mutual collisions separating afterwards. Another possibility is the pair travelling together and colliding an infinite number of times, in a bion-like state. Such effect was observed previously in the ϕ^4 model (see the ref. [62]). ii) the presence of a critical escape velocity, above which the antikink always recedes to $-\infty$ without the production of a new travelling kink at the boundary. iii) the formation of two-bounce windows. For $H < 0$ we report: i) the antikink has a repulsive interaction with boundary. Due to this we can find the velocity v_* , such that for $v_i < v_*$ the antikink does not reach the boundary, ii) there is a narrow range $v_* < v_i < v_c$ where one can find bounce windows and iii) for $v_i > v_c$ we always observe one-bounce collisions.

For the kink-boundary scattering here are two possibilities for a regular and stable boundary. For one boundary the scalar field always changes to the other topological sector. As a result, the new boundary emits an antikink with or without an oscillon. Another possibility for the initial boundary has a richer pattern. In this case the interaction with the

kink is repulsive. This means that there is a critical velocity where the kink become trapped by the boundary. Depending on the parameters (v_i, H) one can also see the changing of the boundary with the emission of radiation or the changing of the topological sector.

When compared with the results for the ϕ^4 model [62], despite its higher nonlinearity, the structure revealed by the antikink-boundary collisions for the ϕ^6 model is simpler. Indeed, as we saw in figure 12, v_c grows monotonically with H for $0 < H < H_m$, favoring the emission of a kink from the boundary. On the contrary, for the ϕ^4 model and $0 < H \lesssim 0.5H_m$, v_c initially decreases with H , favoring the 1-bounce region. Figure 11 shows that the region close to $H \sim H_m$ for the ϕ^6 model has a clear separation between two regions: 1-bounce (blue) and kink formation from the boundary (red). The corresponding region for the ϕ^4 model (see the figure 3 from ref. [62]) is intricate, and the v_c has no functional form. The complexity of the ϕ^6 model comes from the existence of two topological sectors. Even in one topological sector there are more solutions for kink, antikink and boundaries. This results, for the ϕ^6 model, in different structures for antikink-boundary and kink-boundary scattering and in the possibility of changing the topological sector.

Despite our investigation being mostly numerical, we connected some of the reported effects to the presence of bound states and continuum states in the system antikink-boundary or kink-boundary. Also, our results extended, for $H \neq 0$, the known results for scattering in the full-line.

The further study of border effects in a nonintegrable field theory has a broad range of possibilities. The interesting connections with one-dimensional finite systems such as nanotubes and polymers could be further explored. Another direction could be to investigate a relation with multidimensional physics in the brane context. Also, despite studying other class of nonlinearities, one could also study of defects with two or more scalar fields. In any case, we agree with the literature about the importance of stability analysis and the possible occurrence of vibrational modes. This is a crucial step to gain a deeper comprehension in the complexity of nonlinear dynamics.

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