RECEIVED: July 2, 2019

REVISED: September 5, 2019

ACCEPTED: September 16, 2019

PUBLISHED: October 7, 2019

Condensates of interacting non-Abelian $SO(5)_{N_f}$ anyons

Daniel Borcharding and Holger Frahm

*Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstraße 2, 30167 Hannover, Germany*

E-mail: frahm@itp.uni-hannover.de

ABSTRACT: Starting from a one-dimensional model of relativistic fermions with $SO(5)$ spin and $U(N_f)$ flavour degrees of freedom we study the condensation of $SO(5)_{N_f}$ anyons. In the low-energy limit the quasi-particles in the spin sector of this model are found to be massive solitons forming multiplets in the $SO(5)$ vector or spinor representations. The solitons carry internal degrees of freedom which are identified as $SO(5)_{N_f}$ anyons. By controlling the external magnetic fields the transitions from a dilute gas of free anyons to various collective states of interacting ones are observed. We identify the generalized parafermionic cosets describing these collective states and propose a low temperature phase diagram for the anyonic modes.

KEYWORDS: Bethe Ansatz, Conformal Field Theory, Integrable Field Theories, Anyons

ARXIV EPRINT: [1906.09929](https://arxiv.org/abs/1906.09929)

Contents

1	Introduction	1
2	Bethe ansatz for a perturbed $\text{SO}(5)_{N_f}$ WZNW model	3
2.1	Low-temperature thermodynamics	6
2.2	Non-interacting solitons	9
2.3	Condensate of $[1, 0]$ -solitons	11
2.4	Condensate of $[1, 1]$ -solitons	14
2.5	Condensate of $[1, 0]$ - and $[1, 1]$ -solitons	16
3	Summary and conclusion	18
A	Representation theory of $\text{SO}(5)$	20
B	TBA of the perturbed $\text{SO}(5)_{N_f}$ WZNW model	21

1 Introduction

The fractionalized quasi-particle excitations of topological states of matter have attracted a lot of attention in the recent years. A particularly interesting class of these quasi-particles are so-called non-Abelian anyons. Their remarkable exchange statistics makes them a resource for decoherence-free quantum computing [1] which has further driven the search for physical realizations. Candidate systems are the topologically ordered phases of two-dimensional quantum matter such as the fractional quantum Hall states or $p + ip$ superconductors where non-Abelian anyons may appear as zero-energy degrees of freedom of gapped excitations in the bulk [2–4].

Mathematically, anyons are objects in a braided tensor category. In this description they are characterized by their braiding and fusion properties. These completely determine the physics of a dilute anyon gas and quantum computing operations can be realized based on the braiding of the anyonic quasi-particles [5]. The fusion rules on the other hand determine the Hilbert space of a many-anyon system as well as the possible local interactions between pairs of anyons [6]. The presence of the latter lifts the degeneracy of the zero-energy modes and leads to the anomalous collective behaviour of systems with a finite density of anyons, e.g. when they condense at the boundaries between phases of different topological order. This can be exploited to stabilize topological quantum memories [7].

The properties of interacting anyons forming a high density condensate on the edge of the topologically ordered phase of a two dimensional quantum system have been studied in various effective lattice models [6, 8–13]. Combining numerical methods with insights from

exactly solvable models and conformal field theory important insights into the collective behaviour of different types of non-Abelian anyons have been obtained. Unfortunately, these lattice models do not allow to tune the anyon density. To study the transition between the low density phase of ‘bare’ anyons and the collective state realized at high anyon densities one can follow the approach of [14–16]: at sufficiently low temperatures the Hamiltonian of an integrable one-dimensional model of fermions carrying $SU(k)$ spin and $U(N_f)$ flavour indices with a particular current-current interaction can be separated into commuting parts describing the fractionalized charge, spin and flavour degrees of freedom separately. Concentrating on the spin sector the elementary excitations are found to be massive solitons forming multiplets in fundamental representations of $SU(k)$ and bound states thereof. Residing on these solitons are $SU(k)_{N_f}$ anyons. Therefore the density of the anyonic degrees of freedom can be controlled together with that of the solitons by the variation of the magnetic fields. This allows, by solving the thermodynamic Bethe ansatz equations for different external fields, to study the condensation of $SU(k)_{N_f}$ anyons in detail.

In the present paper we extend this approach to fermions with an $SO(5)$ spin degree of freedom.¹ Specifically we consider a model defined by Hamiltonian densities describing relativistic chiral fermions in external magnetic fields h_i ($i = 1, 2$) perturbed by an anisotropic spin-spin interaction

$$\begin{aligned}\mathcal{H} &= -i\bar{\psi}_f\gamma_\mu\partial^\mu\psi_f - \sum_{f=1}^{N_f} h_i \bar{\psi}_f\gamma_0 H_c^i \psi_f + \mathcal{H}_{\text{int}}, \\ \mathcal{H}_{\text{int}} &= \lambda_{\parallel} \sum_{i=1}^2 (\bar{\psi}_f\gamma_\mu H_c^i \psi_f)^2 + \lambda_{\perp} \sum_{\alpha>0} \frac{|\alpha|^2}{2} (\bar{\psi}_f\gamma_\mu E^\alpha \psi_f)(\bar{\psi}_{f'}\gamma_\mu E^{-\alpha} \psi_{f'}),\end{aligned}\tag{1.1}$$

where ψ_{fa} are Dirac spinors with $U(N_f)$ ‘flavour’ indices $f, f' = 1, \dots, N_f$ and $SO(5)$ ‘spin’ indices $a = 1, \dots, 5$ (the latter are suppressed in (1.1)). H_c^i ($i = 1, 2$) are the generators of the $SO(5)$ Cartan subalgebra while the $SO(5)$ ladder operator for a root α in the Cartan-Weyl basis is denoted by E^α . Moreover, the γ_μ ($\mu = 1, 2$) are Dirac matrices and $\bar{\psi}_f = \gamma_0\psi_f^\dagger$.

Similar as in the models for fermions carrying $SU(k)$ spin mentioned above we find that the excitations in the spin sector of (1.1) are massive solitons. Here they form multiplets in the $SO(5)$ vector and spinor representations and, in addition, carry an internal degree of freedom which we identify as non-Abelian $SO(5)_{N_f}$ anyons. The density of solitons can be controlled by the external fields coupling to the $SO(5)$ Cartan generators. For sufficient large magnetic fields solitons form a condensate described by a $U(1)$ Gaussian model. The condensation of the solitons is accompanied by the formation of collective states for the anyon degrees of freedom which are found to be described by generalized parafermionic conformal field theories. We note that this complements previous results obtained for the high density collective states of interacting $SO(5)_2$ anyons in lattice models [10, 13].

¹ $SO(5)$ symmetric electron models have been constructed e.g. in refs. [17, 18]. In the present paper, however, we do not discuss the origin of this and the additional flavour degrees of freedom but rather concentrate on the possible existence of $SO(5)_{N_f}$ anyons in such models and their signature in the thermodynamical properties.

2 Bethe ansatz for a perturbed $\text{SO}(5)_{N_f}$ WZNW model

In the models of fermions with $\text{SU}(k)$ spin studied previously [14–16] conformal embedding has been used to isolate the part of the Hamiltonian describing the collective excitations in the spin sector [19]. Here, we rely instead on the spectrum of the model obtained from the exact solution of (1.1): the isotropic model, $\lambda_{\parallel} = \lambda_{\perp}$ has been solved using the Bethe ansatz [20–22]. In the limit $N_f \rightarrow \infty$ the fermionic model is equivalent to the $\text{SO}(5) \times \text{SO}(5)$ chiral model and its spectrum and magnetic properties have been studied in ref. [23]. For generic anisotropic choice of the coupling constants the integrability of the Hamiltonian is based on a deformation of the corresponding factorized scattering matrices [24–27]. The low-energy excitations of (1.1) carry charge, flavour and spin degrees of freedom (cf. [22] for the isotropic case). Here we are interested solely in the $\text{SO}(5)$ spin degrees of freedom. By placing $\mathcal{N}$ fermions into a box of length L with periodic boundary conditions the energy contribution of the spin excitations specified by quantum numbers N_1, N_2 is

$$E = \frac{\mathcal{N}}{L} \sum_{\alpha=1}^{N_1} \sum_{\tau=\pm 1} \frac{\tau}{2i} \ln \left(\frac{\sinh(\frac{\pi}{2p_0}(\lambda_{\alpha}^{(1)} + \tau/g - N_f i))}{\sinh(\frac{\pi}{2p_0}(\lambda_{\alpha}^{(1)} + \tau/g + N_f i))} \right) + N_1 H_1 + N_2 H_2 - \mathcal{N}(H_1 + H_2), \quad (2.1)$$

where g, p_0 are functions of the coupling constants $\lambda_{\parallel}$ and $\lambda_{\perp}$ in (1.1). H_1 and H_2 are linear combinations of the magnetic fields introduced above, i.e. $H_1 \equiv \alpha^1 \cdot \vec{h}$, $H_2 \equiv \alpha^2 \cdot \vec{h}$ with the simple roots α^j , $j = 1, 2$, of $\text{SO}(5)$, see appendix A. Also notice that the relativistic invariance of the fermion model is broken by the choice of boundary conditions but will be restored later by considering observables in the scaling limit $L, \mathcal{N} \rightarrow \infty$ and $g \ll 1$ such that the mass of the elementary excitations is small compared to the particle density $\mathcal{N}/L$. The complex parameters $\lambda_{\alpha}^{(m)}$ with $\alpha = 1, \dots, N_m$ ($m = 1, 2$) appearing in (2.1) are so called Bethe roots solving the hierarchy of Bethe equations (cf. refs. [23, 28] for the isotropic case)

$$\begin{aligned} \prod_{\tau=\pm 1} e_{N_f}(\lambda_{\alpha}^{(1)} + \tau/g)^{\mathcal{N}/2} &= \prod_{\beta \neq \alpha}^{N_1} e_2(\lambda_{\alpha}^{(1)} - \lambda_{\beta}^{(1)}) \prod_{\beta=1}^{N_2} e_{-1}(\lambda_{\alpha}^{(1)} - \lambda_{\beta}^{(2)}), \quad \alpha = 1, \dots, N_1, \\ \prod_{\beta=1}^{N_1} e_1(\lambda_{\alpha}^{(2)} - \lambda_{\beta}^{(1)}) &= \prod_{\beta \neq \alpha}^{N_2} e_1(\lambda_{\alpha}^{(2)} - \lambda_{\beta}^{(2)}), \quad \alpha = 1, \dots, N_2, \end{aligned} \quad (2.2)$$

where $e_k(x) = \sinh\left(\frac{\pi}{2p_0}(x + ik)\right) / \sinh\left(\frac{\pi}{2p_0}(x - ik)\right)$.

Based on equations (2.2), (2.1) the thermodynamics of the model can be studied provided that the solutions to the Bethe equations describing the eigenstates in the limit $\mathcal{N} \rightarrow \infty$ are known. Here we argue that the root configurations corresponding to the ground state and excitations relevant for the low-temperature behavior of (1.1) can be built based on a generalized string hypothesis, see e.g. refs. [29, 30]: in the thermodynamic limit the Bethe roots $\lambda_{\alpha}^{(m)}$ are grouped into j -strings of length n_j and with parity $v_{n_j} \in \{\pm 1\}$

$$\begin{aligned} \lambda_{j,\alpha,\ell}^{(1)} &= \lambda_{j,\alpha}^{(1)} + i(n_j + 1 - 2\ell) + \frac{p_0}{2}(1 - v_{N_f}^{(1)} v_{n_j}^{(1)}), \quad \ell = 1, \dots, n_j, \\ \lambda_{j,\alpha,\ell}^{(2)} &= \lambda_{j,\alpha}^{(2)} + \frac{i}{2}(n_j + 1 - 2\ell) + \frac{p_0}{2}(1 - v_{2N_f}^{(2)} v_{n_j}^{(2)}), \quad \ell = 1, \dots, n_j \end{aligned} \quad (2.3)$$

with real centers $\lambda_{j,\alpha}^{(m)} \in \mathbb{R}$. The allowed lengths and parities depend on the parameter p_0 . To simplify the discussion below we assume that $p_0 = N_f + 1/\nu$ with integer $\nu > 2$ where only a few string configurations are relevant for the low-temperature thermodynamics

$$\begin{aligned}
 n_{j_{2,1}}^{(1)} &= j_{2,1}, & v_{j_{2,1}}^{(1)} &= 1, & 1 \leq j_{2,1} &\leq N_f - 1, \\
 n_{j_{2,2}}^{(2)} &= j_{2,2}, & v_{j_{2,2}}^{(2)} &= 1, & 1 \leq j_{2,2} &\leq 2N_f - 1, \\
 n_{\tilde{j}_{0,1}}^{(1)} &= N_f(\nu - 1) + 1, & v_{\tilde{j}_{0,1}}^{(1)} &= (-1)^\nu, & \tilde{j}_{0,1} &= N_f, \\
 n_{j_{0,1}}^{(1)} &= N_f, & v_{j_{0,1}}^{(1)} &= 1, & j_{0,1} &= N_f + 1, \\
 n_{j_{0,2}}^{(2)} &= 2N_f, & v_{j_{0,2}}^{(2)} &= 1, & j_{0,2} &= 2N_f + 1,
 \end{aligned}$$

together with the string configuration for even ν

$$n_{\tilde{j}_{0,2}}^{(2)} = 2N_f(\nu/2 - 1) + 1, \quad v_{\tilde{j}_{0,2}}^{(2)} = (-1)^{\nu/2}, \quad \tilde{j}_{0,2} = 2N_f.$$

Within the root density approach the Bethe equations are rewritten as coupled integral equations for the densities of these strings [31]. For vanishing external fields one finds that the Bethe root configuration corresponding to the lowest energy state is described by finite densities of $j_{0,m}$ -strings on the levels $m = 1, 2$. The elementary excitations above this ground state are of three types: similar as for the discussion of the perturbed $SU(3)_{N_f}$ WZNW model the excitations corresponding to holes in the distributions of $j_{0,m}$ -strings on level $m = 1, 2$ are solitons. From their coupling to the fields it is found that they carry quantum numbers of the five-dimensional vector representation of $SO(5)$ with Young diagram $[1, 0]$ and the four-dimensional spinor representation $[1, 1]$, respectively. Hence, we refer to solitons of the first level as $[1, 0]$ -solitons and to solitons of the second level as $[1, 1]$ -solitons. The excitations corresponding to $j_{2,m}$ -strings are denoted by auxiliary modes, while the contributions of breather excitations are assumed to be negligible for low temperatures. The densities $\rho_j^{(m)}(\lambda)$ of these excitations (and $\rho_j^{h(m)}(\lambda)$ for the corresponding holes) satisfy the integral equations

$$\rho_k^{h(m)}(\lambda) = \rho_{0,k}^{(m)}(\lambda) - \sum_{l=1}^2 \sum_j B_{kj}^{(m,l)} * \rho_j^{(l)} \quad m = 1, 2, \quad (2.4)$$

see appendix B. As mentioned above relativistic invariance is restored in the scaling limit $g \ll 1$ where the solitons are massive particles with bare densities $\rho_{0,j_{0,m}}^{(m)}$ and bare energies $\epsilon_{0,j_{0,m}}^{(m)}$

$$\rho_{0,j_{0,m}}^{(m)}(\lambda) \stackrel{g \ll 1}{=} \begin{cases} \frac{2\sqrt{3}M_0}{6} \cosh(\pi\lambda/3) & \text{if } m = 1, \\ \frac{2M_0}{6} \cosh(\pi\lambda/3) & \text{if } m = 2, \end{cases} \quad (2.5)$$

$$\epsilon_{0,j_{0,m}}^{(m)}(\lambda) \stackrel{g \ll 1}{=} \begin{cases} 2\sqrt{3}M_0 \cosh(\pi\lambda/3) - zH_1 - zH_2 & \text{if } m = 1, \\ 2M_0 \cosh(\pi\lambda/3) - \frac{z}{2}H_1 - zH_2 & \text{if } m = 2, \end{cases} \quad (2.6)$$

where $z = (1 + N_f\nu)$. The prefactors $2\sqrt{3}M_0$ and $2M_0$ with $M_0 \equiv e^{-\pi/3g}$ are the masses of the $[1, 0]$ - and $[1, 1]$ -solitons, respectively. Furthermore, the corresponding charges can be

Figure 1. Relationship between the weights (a) of the $[1, 0]$ (Dynkin labels $(1, 0)$) and $[1, 1]$ (Dynkin labels $(0, 1)$) representation of $\text{SO}(5)$ and the charges (b) determined by (2.7) with the projection of the magnetic fields $\vec{h} = (h_1, h_2)$ in (1.1) on the directions of the simple roots, $H_1 = \alpha^1 \cdot \vec{h}$, $H_2 = \alpha^2 \cdot \vec{h}$.

read off from (2.6): for a general excitation with mass M and bare energy $\epsilon_0(\lambda)$ its charges (q_1, q_2) are defined by

$$\epsilon_0(\lambda) = M \cosh\left(\frac{\pi\lambda}{3}\right) - z(\omega_1 h_1 + \omega_2 h_2) = M \cosh\left(\frac{\pi\lambda}{3}\right) - z(q_1 H_1 + q_2 H_2), \quad (2.7)$$

where ω_1, ω_2 are the components of a weight in a $\text{SO}(5)$ representation. Consequently, $[1, 0]$ -solitons of type $j_{0,1}$ carry the charge $(q_1, q_2) = (1, 1)$, while $[1, 1]$ -solitons of type $j_{0,2}$ carry the charge $(q_1, q_2) = (1/2, 1)$. These charges correspond to the highest weight states of the $[1, 0]$ (vector) and $[1, 1]$ (spinor) representation of $\text{SO}(5)$. All possible charges of $[1, 0]$ - and $[1, 1]$ -solitons are shown in figure 1.

Similarly, the bare densities and energies of the $\tilde{j}_{0,m}$ -strings are

$$\rho_{0,\tilde{j}_{0,m}}^{(m)}(\lambda) \stackrel{g \ll 1}{=} \begin{cases} \frac{2\sqrt{3}M_0}{6} \cosh(\pi\lambda/3) & \text{if } m = 1, \\ \frac{2M_0}{6} \cosh(\pi\lambda/3) & \text{if } m = 2, \end{cases} \quad (2.8)$$

$$\epsilon_{0,\tilde{j}_{0,m}}^{(m)}(\lambda) \stackrel{g \ll 1}{=} \begin{cases} 2\sqrt{3}M_0 \cosh(\pi\lambda/3) - zH_2 & \text{if } m = 1, \\ 2M_0 \cosh(\pi\lambda/3) - \frac{z}{2}H_1 & \text{if } m = 2. \end{cases} \quad (2.9)$$

The corresponding masses of these excitations coincide with the masses of the $[1, 0]$ - and $[1, 1]$ -solitons, respectively. However, they couple to these modes in a different way, i.e. $\tilde{j}_{0,1}$ -strings carry the charge $(0, 1)$ and $\tilde{j}_{0,2}$ -strings the charge $(1/2, 0)$. Therefore, the excitations of type $\tilde{j}_{0,1}$ and $\tilde{j}_{0,2}$ are descendant states of the highest weight states in the $[1, 0]$ and $[1, 1]$ representation. From now on excitations of type $\{j_{0,1}, \tilde{j}_{0,1}\}$ are labeled as $[1, 0]$ -solitons while excitations of type $\{j_{0,2}, \tilde{j}_{0,2}\}$ are labeled as $[1, 1]$ -solitons. The masses and $\text{SO}(5)$ charges of the auxiliary modes vanish, i.e. $\rho_{0,j_2}^{(m)}(\lambda) = 0 = \epsilon_{0,j_2}^{(m)}(\lambda)$.

The energy density of a macro-state with densities given by (2.4) is

$$\Delta\mathcal{E} = \sum_{m=1}^2 \sum_j \int_{-\infty}^{\infty} d\lambda \epsilon_{0,j}^{(m)}(\lambda) \rho_j^{(m)}(\lambda). \quad (2.10)$$

Furthermore, it is convenient to define the masses $M_k^{(m)}$ of the different solitons as

$$M_k^{(m)} \equiv \begin{cases} 2\sqrt{3}M_0 & \text{if } m=1, k \in \{j_{0,1}, \tilde{j}_{0,1}\}, \\ 2M_0 & \text{if } m=2, k \in \{j_{0,2}, \tilde{j}_{0,2}\} \end{cases}.$$

2.1 Low-temperature thermodynamics

To derive the physical properties of the different quasi-particles appearing in the Bethe ansatz solution of the model (1.1) its low-temperature thermodynamics is studied. The equilibrium state at finite temperature is obtained by minimizing the free energy, $F/N = \mathcal{E} - T\mathcal{S}$, with the combinatorial entropy [32]

$$\mathcal{S} = \sum_{j \geq 1} \int_{-\infty}^{+\infty} d\lambda \left[(\rho_j + \rho_j^h) \ln(\rho_j + \rho_j^h) - \rho_j \ln \rho_j - \rho_j^h \ln \rho_j^h \right]. \quad (2.11)$$

The resulting thermodynamic Bethe ansatz (TBA) equations read

$$T \ln(1 + e^{\epsilon_k^{(m)}/T}) = \epsilon_{0,k}^{(m)}(\lambda) + \sum_{l=1}^2 \sum_{j \geq 1} B_{jk}^{(l,m)} * T \ln(1 + e^{-\epsilon_j^{(l)}/T}), \quad (2.12)$$

where the dressed energies $\epsilon_j^{(m)}(\lambda)$ have been introduced through $e^{-\epsilon_j^{(m)}/T} = \rho_j^{(m)}/\rho_j^{h(m)}$. To study the properties of free and interacting solitons it is convenient to rewrite the integral equations of the auxiliary modes: the auxiliary modes become independent of λ for temperatures small compared to the soliton gaps, $T \ll \epsilon_{j_{0,m}}^{(m)}(0)$. Similarly, they take constant values for finite values of λ in the condensed phases when $T \ll |\epsilon_{j_{0,m}}^{(m)}(0)|$. In these cases the effective equations describing the auxiliary modes are

$$\begin{aligned} \epsilon_{j_{2,1}}^{(1)} &= \delta_{j_{2,1}, N_f-1} T \ln \left(1 + e^{-\epsilon_{j_{0,1}}^{(1)}/T} \right)^{\frac{1}{2}} + T \ln \left(1 + e^{\epsilon_{j_{2,1}-1}^{(1)}/T} \right)^{\frac{1}{2}} \left(1 + e^{\epsilon_{j_{2,1}+1}^{(1)}/T} \right)^{\frac{1}{2}} \\ &\quad - T \ln \left(1 + e^{-\epsilon_{2j_{2,1}-1}^{(2)}/T} \right)^{\frac{1}{2}} \left(1 + e^{-\epsilon_{2j_{2,1}}^{(2)}/T} \right) \left(1 + e^{-\epsilon_{2j_{2,1}+1}^{(2)}/T} \right)^{\frac{1}{2}}, \\ \epsilon_{j_{2,2}}^{(2)} &= \delta_{j_{2,2}, 2N_f-1} T \ln \left(1 + e^{-\epsilon_{j_{0,2}}^{(2)}/T} \right)^{\frac{1}{2}} + T \ln \left(1 + e^{\epsilon_{j_{2,2}-1}^{(2)}/T} \right)^{\frac{1}{2}} \left(1 + e^{\epsilon_{j_{2,2}+1}^{(2)}/T} \right)^{\frac{1}{2}} \\ &\quad - T \ln \left(1 + e^{-\epsilon_{j_{2,2}/2}^{(1)}/T} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.13)$$

Figure 2. Zero temperature phase diagram of the model (1.1): as the fields $H_{1,2}$ are increased solitons carrying quantum numbers of the highest weight states in the $[1,0]$ vector and the $[1,1]$ spinor representation of $SO(5)$ condense (the actual location of the phase boundaries is obtained from the numerical solution of the TBA equations (2.12) for $T = 0$ and $p_0 = 2 + 1/3$).

where $\epsilon_0^{(1)} = \epsilon_0^{(2)} = -\infty$ and $\epsilon_{j_{2,2}/2}^{(1)} = \infty$ if $j_{2,2}$ is odd. In terms of the dressed energies the free energy per particle is

$$\begin{aligned} \frac{F}{\mathcal{N}} &= -T \sum_{m=1}^2 \sum_{j \notin \{j_{2,m}\}} \int_{-\infty}^{\infty} d\lambda \rho_{0,j}^{(m)}(\lambda) \ln(1 + e^{-\epsilon_j^{(m)}(\lambda)/T}) \\ &\stackrel{g \ll 1}{\cong} -\frac{T}{6} \sum_{m=1}^2 \sum_{j \notin \{j_{2,m}\}} M_j^{(m)} \int_{-\infty}^{\infty} d\lambda \cosh(\pi\lambda/3) \ln(1 + e^{-\epsilon_j^{(m)}(\lambda)/T}). \end{aligned} \quad (2.14)$$

Solving the equations (2.12) the spectrum of the model (1.1) for a given temperature T and fields H_1, H_2 is obtained. From the expressions (2.6) and (2.9) for the bare energies of the elementary excitations the qualitative behavior of these modes at low temperatures can be deduced, see figure 2 for $T \rightarrow 0$: as long as $zH_2 < \min(2\sqrt{3}M_0 - zH_1, 2M_0 - zH_1/2)$ solitons remain gapped. By increasing the field H_1 (above $zH_1 \geq 2\sqrt{3}M_0 - zH_2$) for sufficiently small H_2 ($zH_2 < (4 - 2\sqrt{3})M_0$) the gap of the $[1,0]$ -solitons closes and they condense into a phase with finite density. In this collective state the degeneracy of the auxiliary modes is lifted while the gap of the $[1,1]$ -solitons remains open until $zH_1 \gg M_0$, see figure 3(a) for the $T = 0$ spectrum with $H_2 \equiv 0$. Similarly, for sufficiently small H_1 ($zH_1 < 4(\sqrt{3} - 1)M_0$) increasing the field H_2 (above $zH_2 \geq 2M_0 - zH_1/2$) closes the gap of the $[1,1]$ -solitons, while the gap of the $[1,0]$ -solitons remains open until $zH_2 \gg M_0$, see

Figure 3. The zero temperature spectrum of elementary excitations (and Fermi energy of solitons in the condensed phases) $\epsilon_j^{(m)}(0)$ obtained from the numerical solution of (2.12) for $p_0 = 2 + 1/4$: (a) for $H_2 = 0$ as a function of the field H_1 , (b) for $H_1 = 0$ as function of H_2 , and (c) for $zH_2/M_0 = -0.06 + 0.21 zH_1/M_0$ as function of H_1 . Once the gap of $[1, 0]$ -solitons in (a) or $[1, 1]$ -solitons in (b) closes the system forms a collective state of these objects. For sufficiently large fields both $[1, 0]$ - and $[1, 1]$ -solitons condense as in (c). In these phases the degeneracy of the auxiliary modes is lifted.

figure 3(b) for the $T = 0$ spectrum with $H_1 \equiv 0$. In figure 3(c) we display the spectrum of elementary excitations for a combination of magnetic fields, where the gaps of $[1, 0]$ - and $[1, 1]$ -solitons close simultaneously.

Notice that the string hypothesis (2.3) does not capture all solitons of the $[1, 0]$ and $[1, 1]$ multiplet that may occur. However, from the coupling of their charges to the fields the energy gaps of all $[1, 0]$ - and $[1, 1]$ -solitons can be predicted in the non-interacting regime, see figures 3. In the following we will choose the temperatures to be sufficiently small such that only the solitons with charges corresponding to the highest weight states of the $[1, 0]$ and $[1, 1]$ multiplet, i.e. excitations of type $j_{0,m}$, contribute to the thermodynamics.

N_f	$\left\{ \exp \left(-\epsilon_{j_{2,1}}^{(1)}/T \right) \right\}_{j_{2,1}=1}^{N_f-1}$	$\left\{ \exp \left(-\epsilon_{j_{2,2}}^{(2)}/T \right) \right\}_{j_{2,2}=1}^{2N_f-1}$
2	3	$\frac{2}{3}, \frac{4}{5}, \frac{2}{3}$
3	$\sqrt{3}, \sqrt{3}$	$\frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{\sqrt{3}}$
4	1.34601, 0.91557, 1.34601	0.53679, 0.40477, 0.24991, 0.27673, 0.24991, 0.40477, 0.53679
5	1.16591, 0.66818, 0.66818, 1.16591	0.51415, 0.35935, 0.21297, 0.20711, 0.17157, 0.20711, 0.21297, 0.35935, 0.51415

Table 1. Asymptotic solution ($|\lambda| \rightarrow \infty$) of auxiliary modes ($\epsilon_{j_{2,m}}^{(m)}/T \equiv \epsilon_{j_{2,m}}^{(m)}(\lambda \rightarrow \infty)/T$) for $2 \leq N_f \leq 5$ derived numerically from eqs. (2.13) with $\epsilon_{j_{0,m}}^{(m)}/T = \infty$.

2.2 Non-interacting solitons

For fields $zH_2 < \min(2\sqrt{3}M_0 - zH_1, 2M_0 - zH_1/2)$ temperatures below the gaps of the solitons are considered, i.e. $T \ll \min(\epsilon_{0,j_{0,1}}^{(1)}(0), \epsilon_{0,j_{0,2}}^{(2)}(0))$. Analogously to [14–16] the nonlinear integral equations (2.12) can be solved iteratively in this regime: the energies $\epsilon_k^{(m)}$ of solitons are well described by their first order approximation while those of the auxiliary modes can be replaced by the asymptotic solution for $|\lambda| \rightarrow \infty$, see table 1 for $2 \leq N_f \leq 5$. For the other modes

$$\epsilon_j^{(m)}(\lambda) = \epsilon_{0,j}^{(m)}(\lambda) - T \ln Q_j^{(m)}$$

is obtained for $m = 1, j \in \{j_{0,1}, \tilde{j}_{0,1}\}$ and $m = 2, j \in \{j_{0,2}, \tilde{j}_{0,2}\}$ resulting in the free energy

$$\frac{F}{\mathcal{N}} = - \sum_{m=1}^2 \sum_{j \notin \{j_{2,m}\}} T Q_j^{(m)} \int \frac{dp}{2\pi} e^{-\epsilon_{0,j}^{(m)}(0)/T - p^2/2M_j^{(m)}T}, \quad (2.15)$$

where $Q_k^{(m)}$ ($k \neq j_{2,m}$) depends on the asymptotic solution of the auxiliary modes

$$Q_k^{(m)} = \prod_{i=1}^2 \prod_{j_{2,i}} \left(1 + e^{-\epsilon_{j_{2,i}}^{(i)}/T} \right)^{-B_{j_{2,m}k}^{(i,m)}(0)}. \quad (2.16)$$

See table 2 for explicit values of $Q_k^{(m)}$ for $2 \leq N_f \leq 5$. Following [14, 15] each of the terms appearing in eq. (2.15) is the free energy of an ideal gas of particles with the corresponding mass carrying an internal degree of freedom with possibly non-integer quantum dimension $Q_k^{(m)}$ for the solitons. It is found that solitons of the same multiplet carry the same quantum dimension, i.e. $Q^{(m)} \equiv Q_{j_{0,m}}^{(m)} = Q_{\tilde{j}_{0,m}}^{(m)}$. The densities of the solitons

$$n_j^{(m)} = Q^{(m)} \sqrt{\frac{M_j^{(m)}T}{2\pi}} e^{-\epsilon_{0,j}^{(m)}(0)/T}, \quad (2.17)$$

N_f	$Q^{(1)}$	$Q^{(2)}$
2	2	$\sqrt{5}$
3	$1 + \sqrt{3}$	$1 + \sqrt{3}$
4	$2 + \sin\left(\frac{3\pi}{14}\right)$	$\frac{1}{2\sin\left(\frac{3\pi}{14}\right) - 1}$
5	$1 + \sqrt{4 + 2\sqrt{2}}$	$\sqrt{2} + \sqrt{2 + \sqrt{2}}$

Table 2. Quantum dimensions of the internal degrees of freedom of $[1, 0]$ -solitons ($Q^{(1)} \equiv Q_{j_0,1}^{(1)}$) and $[1, 1]$ -solitons ($Q^{(2)} \equiv Q_{j_0,2}^{(2)}$) derived from (2.16) using the asymptotic solutions of the auxiliary modes (see table 1).

derived from the free energy (2.15) for $j \in \{j_{0,m}, \tilde{j}_{0,m}\}$, can be controlled by variation of the temperature and the fields.

In order to identify the quantum dimensions $Q_k^{(m)}$ with the quantum dimensions of $\text{SO}(5)_{N_f}$ anyons the topological charges are written in terms of Young diagrams: according to [19] the admissible weights Λ of the affine Lie algebra $\text{SO}(5)_{N_f}$ have to satisfy

$$(\Lambda, \theta) \leq N_f, \quad (2.18)$$

where θ is the highest root. In terms of the Dynkin labels (m_1, m_2) , the condition (2.18) results in

$$m_1 + m_2 \leq N_f. \quad (2.19)$$

Hence, $\text{SO}(5)_{N_f}$ anyons may be labeled by Dynkin labels (m_1, m_2) satisfying (2.19). Equivalently, they can be expressed using Young diagrams. For $N_f = 2$ the admissible topological charges in terms of Young diagrams are

$$[0, 0], [1, 1], [1, 0], [2, 1], [2, 0], [2, 2].$$

The corresponding fusion rules can be found in [10] using the identification

$$\psi_1 = [0, 0], \psi_2 = [1, 1], \psi_3 = [1, 0], \psi_4 = [2, 1], \psi_5 = [2, 0], \psi_6 = [2, 2].$$

Notice that these fusion rules are consistent with the tensor product reductions of $\text{SO}(5)$ irreducible representations with reasonable modifications due to the level $N_f = 2$.² The quantum dimensions extracted from the fusion rules for $N_f = 2$ are given by

$$d([0, 0]) = d([2, 2]) = 1, d([1, 1]) = d([2, 0]) = 2, d([1, 0]) = d([2, 1]) = \sqrt{5}.$$

Therefore, the appearance of the internal degrees of freedom, $Q^{(1)}$ and $Q^{(2)}$, can be interpreted as $[1, 1]$ or $[2, 0]$ anyons being bound to the $[1, 0]$ -solitons and $[1, 0]$ or $[2, 1]$ anyons being bound to the $[1, 1]$ -solitons.

²An elegant graphical method for deriving tensor product reductions for Lie algebras with rank $r \leq 2$ can be found in [33].

For $N_f > 2$ this identification cannot be done, since the fusion rules and quantum dimensions of $SO(5)_{N_f > 2}$ anyons have not yet been derived. However, following the results from the perturbed $SU(3)_{N_f}$ WZNW model it is conjectured that the internal degrees of freedom, $Q^{(1)}$ and $Q^{(2)}$, coincide with the quantum dimensions of $[1, 1]$ and $[1, 0]$ anyons for arbitrary $N_f \geq 2$, respectively.

The densities of $[1, 1]$ and $[1, 0]$ anyons appearing in the one-dimensional model are determined by the densities of the corresponding solitons (2.17). For fields satisfying

$$zH_1 > (4\sqrt{3} - 4)M_0 - 2T \log \left(3^{1/4} \frac{Q^{(1)}}{Q^{(2)}} \right)$$

the dominant contribution to the free energy is that of the $[1, 0]$ -solitons with $[1, 1]$ anyons being bound to them. In the remaining region of non-interacting solitons the $[1, 1]$ -solitons with $[1, 0]$ anyons bound to them are the dominant excitations.

2.3 Condensate of $[1, 0]$ -solitons

For fields $zH_2 < (4 - 2\sqrt{3})M_0$, $zH_1 > 2\sqrt{3}M_0 - zH_2$ and temperatures $T \ll zH_1 + zH_2 - 2\sqrt{3}M_0$ the $[1, 0]$ -solitons (of type $j_{0,1}$) form a condensate, while the contribution to the free energy of the other quasi-particles can be neglected. Following [34] we observe that the dressed energies and densities can be related as

$$\begin{aligned} \rho_j^{(m)}(\lambda) &= (-1)^{\delta_{j \in \{j_2, m\}}} \frac{1}{2\pi} \frac{d\epsilon_j^{(m)}(\lambda)}{d\lambda} f\left(\frac{\epsilon_j^{(m)}(\lambda)}{T}\right), \\ \rho_j^{h(m)}(\lambda) &= (-1)^{\delta_{j \in \{j_2, m\}}} \frac{1}{2\pi} \frac{d\epsilon_j^{(m)}(\lambda)}{d\lambda} \left(1 - f\left(\frac{\epsilon_j^{(m)}(\lambda)}{T}\right)\right), \end{aligned} \quad (2.20)$$

for $\lambda > \lambda_\delta$ with $\exp(\pi\lambda_\delta/3) \gg 1$, where $f(\epsilon) = (1 + e^\epsilon)^{-1}$ is the Fermi function. Inserting this into (2.11) we get $(\phi_j^{(m)} = \epsilon_j^{(m)}/T)$

$$\begin{aligned} \mathcal{S} &= -\frac{T}{\pi} \sum_{m,j} (-1)^{\delta_{j \in \{j_2, m\}}} \int_{\phi_j^{(m)}(\lambda_\delta)}^{\phi_j^{(m)}(\infty)} d\phi_j^{(m)} \\ &\quad \times \left[f(\phi_j^{(m)}) \ln f(\phi_j^{(m)}) + (1 - f(\phi_j^{(m)})) \ln(1 - f(\phi_j^{(m)})) \right] \\ &\quad + \sum_{m,j} \mathcal{S}_j^{(m)}(\lambda_\delta), \end{aligned} \quad (2.21)$$

$$\mathcal{S}_j^{(m)}(\lambda_\delta) \equiv \int_{-\lambda_\delta}^{\lambda_\delta} d\lambda \left[(\rho_j^{(m)} + \rho_j^{h(m)}) \ln(\rho_j^{(m)} + \rho_j^{h(m)}) - \rho_j^{(m)} \ln \rho_j^{(m)} - \rho_j^{h(m)} \ln \rho_j^{h(m)} \right].$$

The integrals over $\phi_j^{(m)}$ can be performed giving

$$\mathcal{S} = \sum_{m,j} \mathcal{S}_j^{(m)}(\lambda_\delta) - \frac{2T}{\pi} \sum_{m,j} (-1)^{\delta_{j \in \{j_2, m\}}} [L(f(\phi_j^{(m)}(\infty))) - L(f(\phi_j^{(m)}(\lambda_\delta)))] \quad (2.22)$$

in terms of the Rogers dilogarithm $L(x)$

$$L(x) = -\frac{1}{2} \int_0^x dy \left(\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y} \right).$$

For large fields $zH_1 \gg 2\sqrt{3}M_0 - zH_2$ we have $\log((zH_1 + zH_2)/2\sqrt{3}M_0) > \lambda_\delta \gg 1$. Using eqs. (2.12) and (2.13) this implies

$$\begin{aligned} f(\phi_{j_0,m}^{(m)}(\lambda_\delta)) &= \begin{cases} 1 & \text{for } m = 1 \\ 0 & \text{for } m = 2 \end{cases}, & f(\phi_{j_0,m}^{(m)}(\infty)) &= 0, \\ f(\phi_{j_0,m}^{(m)}(\lambda_\delta)) &= 0, & f(\phi_{j_0,m}^{(m)}(\infty)) &= 0, \\ f(\phi_{j_2,m}^{(m)}(\lambda_\delta)) &= \begin{cases} 0 & \text{for } m = 1 \\ \left(\frac{\sin\left(\frac{\pi}{2N_f+2}\right)}{\sin\left(\frac{\pi(j_2,2+1)}{2N_f+2}\right)} \right)^2 & \text{for } m = 2 \end{cases}. \end{aligned} \quad (2.23)$$

For the remaining term, $f(\phi_{j_2,m}^{(m)}(\infty))$, an analytical expression is not known. However, it can be computed numerically using the results for the asymptotic behavior of the auxiliary modes from table 1. From (2.23) one can further conclude that the densities for $|\lambda| < \lambda_\delta$ are given by

$$\rho_{j_0,1}^{h(1)}(\lambda) = \rho_{j_0,2}^{(2)}(\lambda) = \rho_{j_0,m}^{(m)}(\lambda) = \rho_{j_2,1}^{(1)} = 0, \quad \rho_{j_2,2}^{(2)}(\lambda) = e^{-\epsilon_{j_2,2}^{(2)}/T} \rho_{j_2,2}^{h(2)}(\lambda),$$

where $e^{-\epsilon_{j_2,2}^{(2)}/T} = \text{const.}$ for $|\lambda| < \lambda_\delta$. Since the integral equations (2.4) for $\rho_{j_2,2}^{h(2)}$ simplify in this regime to

$$\rho_{j_2,2}^{h(2)} = - \sum_{k_{2,2}} B_{j_2,2k_{2,2}}^{(2,2)} * e^{-\epsilon_{k_{2,2}}^{(2)}/T} \rho_{k_{2,2}}^{h(2)} \quad \text{for } |\lambda| < \lambda_\delta,$$

one can conclude that $\rho_{j_2,2}^{h(2)} \rightarrow 0$, $\rho_{j_2,2}^{(2)} \rightarrow 0$ such that $\rho_{j_2,2}^{(2)}/\rho_{j_2,2}^{h(2)} = e^{-\epsilon_{j_2,2}^{(2)}/T} = \text{const.}$ Consequently, $\mathcal{S}_j^{(m)}(\lambda_\delta) = 0$ for all j, m is obtained. Using the Rogers dilogarithm identity

$$\sum_{k=2}^{n-2} L\left(\frac{\sin^2(\pi/n)}{\sin^2(\pi k/n)}\right) = \frac{\pi^2}{6} \frac{2(n-3)}{n} \quad (2.24)$$

it is found that

$$\sum_{j_2,2} L\left(f(\phi_{j_2,2}^{(2)}(\lambda_\delta))\right) = \frac{\pi^2}{6} \left(\frac{3N_f}{N_f+1} - 1 \right).$$

In general Rogers dilogarithm identities giving the relationship between Lie algebras and central charges of parafermion conformal field theories have only been proven for the simply laced case [35]. However, for the non-simply laced Lie algebra $\text{SO}(5)$ similar relations can be verified numerically

$$\sum_{m=1}^2 \sum_{j_2,m} L\left(f(\phi_{j_2,m}^{(m)}(\infty))\right) = \frac{\pi^2}{6} \left(\frac{10N_f}{N_f+3} - 2 \right). \quad (2.25)$$

Hence, we obtain the following low-temperature behavior of the entropy

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{10N_f}{N_f + 3} - \frac{3N_f}{N_f + 1} \right) T \quad (2.26)$$

which is consistent with a conformal field theory describing the collective modes given by the coset $\text{SO}(5)_{N_f}/\text{SO}(3)_{N_f}$ with central charge

$$c = \frac{10N_f}{N_f + 3} - \frac{3N_f}{N_f + 1}.$$

Using the conformal embedding

$$\frac{\text{SO}(5)_{N_f}}{\text{SO}(3)_{N_f}} = \text{U}(1) + \frac{Z_{\text{SO}(5)_{N_f}}}{Z_{\text{SO}(3)_{N_f}}} \quad (2.27)$$

where Z_G denotes generalized parafermions given as the quotient $G/\text{U}(1)^{\text{rank}(G)}$ involving the group G [36], the collective modes can equivalently be described by a product of a free $\text{U}(1)$ boson and a parafermion coset $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SO}(3)_{N_f}}$ contributing $c = 1$ and

$$c = \frac{8N_f - 6}{N_f + 3} - \frac{2N_f - 1}{N_f + 1},$$

respectively. Notice that the central charge of the coset $Z_{\text{SO}(5)_2}/Z_{\text{SO}(3)_2}$ is $c = 1$, which is consistent with the results for interacting chains of $[1, 1]$ $\text{SO}(5)_2$ anyons [10].

Following [15] the entropy $\mathcal{S} = -\frac{d}{dT} \frac{F}{N}$ is computed numerically to study the transition from free anyons to a condensate of anyons. In the region $2\sqrt{3}M_0 - zH_2 \lesssim zH_1$ the entropy deviates from the asymptotic expression (2.26): in this range of H_1 the auxiliary modes of the first level propagate with a velocity (independent of $j_{2,1}$) differing from that of the $[1, 0]$ -solitons, $v_{[1,0]}$, namely

$$v_{[1,0]} = \left. \frac{\partial_\lambda \epsilon_{j_{0,1}}^{(1)}(\lambda)}{2\pi \rho_{j_{0,1}}^{(1)}(\lambda)} \right|_{\Lambda_1}, \quad v_{pf}^{(1)} = - \left. \frac{\partial_\lambda \epsilon_{j_{2,1}}^{(1)}(\lambda)}{2\pi \rho_{j_{2,1}}^{h(1)}(\lambda)} \right|_{\lambda \rightarrow \infty},$$

where Λ_1 denotes the Fermi point of $[1, 0]$ -solitons defined by $\epsilon_{j_{0,1}}^{(1)}(\pm\Lambda_1) = 0$. Also notice that Fermi velocities of the second level do not exist in this regime. As a consequence the bosonic (spinon) and parafermionic degrees of freedom in the first level separate and the low-temperature entropy is

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{1}{v_{[1,0]}} + \frac{1}{v_{pf}^{(1)}} \left(\frac{8N_f - 6}{N_f + 3} - \frac{2N_f - 1}{N_f + 1} \right) \right) T. \quad (2.28)$$

This behavior can be explained by the conformal embedding (2.27). Note that both Fermi velocities depend on the field H_1 and approach 1 as $H_1 \gtrsim H_{1,\delta}$ such that $\Lambda_1(H_{1,\delta}) > \lambda_\delta$, see figure 4 (a), giving the entropy (2.26) of the coset $\text{SO}(5)_{N_f}/\text{SO}(3)_{N_f}$. In figure 4 the computed entropy is shown for $T = 0.02 M_0$ as a function of the field H_1 together with the $T \rightarrow 0$ behavior (2.28) expected from conformal field theory.³

³Actually, this behavior can only be seen for temperatures $T < 0.02 M_0$, which was not accessible by available numerical methods. To overcome this problem the entropy for $T = 0.02 M_0$ was computed, while already neglecting the contribution of $\epsilon_{j_{0,2}}^{(2)}$ in the integral equations (2.12).

Figure 4. (a) Fermi velocities of the $[1,0]$ -solitons and first level parafermion modes as a function of the field zH_1/M_0 for $p_0 = 2 + 1/3$, $H_2 \equiv 0$ at zero temperature. For large field, $H_1 > H_{1,\delta}$, both Fermi velocities approach 1 leading to the asymptotic result for the low-temperature entropy (2.26). (b) Entropy obtained from numerical solution of the TBA equations (2.12) for $p_0 = 2 + 1/3$ and $H_2 \equiv 0$ as a function of the field zH_1/M_0 for $T = 0.02M_0$. For fields large compared to the $[1,0]$ -soliton mass, $zH_1 \gg 2\sqrt{3}M_0$, the entropy approaches the expected analytical value (2.26) for a field theory with a free bosonic sector and a $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SO}(3)_{N_f}}$ parafermion sector propagating with velocities $v_{[1,0]}$ and $v_{pf}^{(1)}$, respectively (full red line). For magnetic fields $zH_1 < 2\sqrt{3}M_0$ and temperature $T \ll 2\sqrt{3}M_0$ the entropy is that of a dilute gas of non-interacting quasi-particles with degenerate internal degree of freedom due to the anyons.

2.4 Condensate of $[1,1]$ -solitons

For fields $zH_1 < (4\sqrt{3} - 4)M_0$, $zH_2 > 2M_0 - zH_1/2$ and temperatures $T \ll zH_1/2 + zH_2 - 2M_0$ the $[1,1]$ -solitons (of type $j_{0,2}$) form a condensate, while the contribution to the free energy of the other quasi-particles can be neglected. For large fields $zH_2 \gg 2M_0 - zH_1/2$ such that $\log((zH_1/2 + zH_2)/2M_0) > \lambda_\delta \gg 1$, eq. (2.12) implies

$$\begin{aligned}
 f(\phi_{j_{0,m}}^{(m)}(\lambda_\delta)) &= \begin{cases} 0 & \text{for } m = 1 \\ 1 & \text{for } m = 2 \end{cases}, & f(\phi_{j_{0,m}}^{(m)}(\infty)) &= 0, \\
 f(\phi_{j_{0,m}}^{(m)}(\lambda_\delta)) &= 0, & f(\phi_{j_{0,m}}^{(m)}(\infty)) &= 0, \\
 f(\phi_{j_{2,m}}^{(m)}(\lambda_\delta)) &= \begin{cases} \left(\frac{\sin\left(\frac{\pi}{N_f+2}\right)}{\sin\left(\frac{\pi(j_{2,1}+1)}{N_f+2}\right)} \right)^2 & \text{for } m = 1 \\ 0 & \text{for } m = 2 \end{cases}
 \end{aligned} \tag{2.29}$$

together with the numerical expressions for $f(\phi_{j_{2,m}}^{(m)}(\infty))$ obtained from the asymptotic behavior of the auxiliary modes shown in table 1. The densities for $|\lambda| < \lambda_\delta$ following from (2.29) are

$$\rho_{j_{0,2}}^{h(2)}(\lambda) = \rho_{j_{0,1}}^{(1)}(\lambda) = \rho_{j_{0,m}}^{(m)}(\lambda) = \rho_{j_{2,2}}^{(2)} = 0, \quad \rho_{j_{2,1}}^{(1)}(\lambda) = e^{-\epsilon_{j_{2,1}}^{(1)}/T} \rho_{j_{2,1}}^{h(1)}(\lambda),$$

where $e^{-\epsilon_{j_{2,1}}^{(1)}/T} = \text{const.}$ for $|\lambda| < \lambda_\delta$. Since the integral equations (2.4) for $\rho_{j_{2,1}}^{h(1)}$ simplify in this regime to

$$\rho_{j_{2,1}}^{h(1)} = - \sum_{k_{2,1}} B_{j_{2,1}k_{2,1}}^{(1,1)} * e^{-\epsilon_{k_{2,1}}^{(1)}/T} \rho_{k_{2,1}}^{h(1)} \quad \text{for } |\lambda| < \lambda_\delta,$$

one can conclude that $\rho_{j_{2,1}}^{h(1)} \rightarrow 0$, $\rho_{j_{2,1}}^{(1)} \rightarrow 0$ such that $\rho_{j_{2,1}}^{(1)}/\rho_{j_{2,1}}^{h(1)} = e^{-\epsilon_{j_{2,1}}^{(1)}/T} = \text{const.}$ Consequently, $\mathcal{S}_j^{(m)}(\lambda_\delta) = 0$ for all j, m is obtained. Using the Rogers dilogarithm identity (2.24) the relation for $Z_{\text{SU}(2)_{N_f}}$ parafermions is found:

$$\sum_{j_{2,1}} L\left(f(\phi_{j_{2,1}}^{(1)}(\lambda_\delta))\right) = \frac{\pi^2}{6} \left(\frac{3N_f}{N_f + 2} - 1 \right)$$

Hence, the following low-temperature behavior of the entropy is obtained using (2.25)

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{10N_f}{N_f + 3} - \frac{3N_f}{N_f + 2} \right) T, \quad (2.30)$$

which is consistent with a conformal field theory describing the collective modes given by the coset $\text{SO}(5)_{N_f}/\text{SU}(2)_{N_f}$ with central charge

$$c = \frac{10N_f}{N_f + 3} - \frac{3N_f}{N_f + 2}.$$

Using the conformal embedding

$$\frac{\text{SO}(5)_{N_f}}{\text{SU}(2)_{N_f}} = \text{U}(1) + \frac{Z_{\text{SO}(5)_{N_f}}}{Z_{\text{SU}(2)_{N_f}}},$$

where $Z_{\text{SU}(N)_{N_f}} = \text{SU}(N)_{N_f}/\text{U}(1)^N$ denotes generalized $\text{SU}(N)_{N_f}$ parafermions [36], the collective modes can equivalently be described by a product of a free $\text{U}(1)$ boson and a parafermion coset $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SU}(2)_{N_f}}$ contributing $c = 1$ and

$$c = \frac{8N_f - 6}{N_f + 3} - \frac{2(N_f - 1)}{N_f + 2}.$$

Notice that for $N_f = 2$ the central charge of the coset $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SU}(2)_{N_f}}$ is $c = 3/2$, which is consistent with the results for interacting chains of $[1, 0]$ $\text{SO}(5)_{N_f}$ anyons [13].

Analogously to the regime discussed in section 2.3, the entropy deviates from the asymptotic expression in the region $2M_0 - zH_1 \lesssim zH_2$, since the auxiliary modes of the second level propagate with a velocity differing from that of the $[1, 1]$ -solitons, $v_{[1,1]}$, namely

$$v_{[1,1]} = \frac{\partial_\lambda \epsilon_{j_{0,2}}^{(2)}(\lambda)}{2\pi \rho_{j_{0,2}}^{(2)}(\lambda)} \Big|_{\Lambda_2}, \quad v_{pf}^{(2)} = - \frac{\partial_\lambda \epsilon_{j_{2,2}}^{(2)}(\lambda)}{2\pi \rho_{j_{2,2}}^{h(2)}(\lambda)} \Big|_{\lambda \rightarrow \infty},$$

Figure 5. (a) Fermi velocities of the $[1, 1]$ -solitons and second level parafermion modes as a function of the field zH_2/M_0 for $p_0 = 2 + 1/3$, $H_1 \equiv 0$ at zero temperature. For large fields, $H_2 > H_{2,\delta}$, both Fermi velocities approach 1 leading to the asymptotic result for the low-temperature entropy (2.30). (b) Entropy obtained from numerical solution of the TBA equations (2.12) for $p_0 = 2 + 1/3$ and $H_1 \equiv 0$ as a function of the field zH_2/M_0 for $T = 0.02M_0$. For fields large compared to the $[1, 1]$ -soliton mass, $zH_2 \gg 2M_0$, the entropy approaches the expected analytical value (2.30) for a field theory with a free bosonic sector and a $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SU}(2)_{N_f}}$ parafermion sector propagating with velocities $v_{[1,1]}$ and $v_{pf}^{(2)}$, respectively (full red line). For magnetic fields $zH_2 < 2M_0$ and temperature $T \ll 2M_0$ the entropy is that of a dilute gas of non-interacting quasi-particles with degenerate internal degree of freedom due to the anyons.

where Λ_2 denotes the Fermi point of $[1, 1]$ -solitons defined by $\epsilon_{j_0,2}^{(2)}(\pm\Lambda_2) = 0$. Also notice that Fermi velocities of the first level do not exist in this regime. As a consequence the bosonic (spinon) and parafermionic degrees of freedom in the first level separate and the low-temperature entropy is

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{1}{v_{[1,1]}} + \frac{1}{v_{pf}^{(2)}} \left(\frac{8N_f - 6}{N_f + 3} - \frac{2(N_f - 1)}{N_f + 2} \right) \right) T. \quad (2.31)$$

Figure 5 (a) shows how both Fermi velocities depend on the field H_2 and approach 1 as $H_2 \geq H_{2,\delta}$ such that $\Lambda_2(H_{2,\delta}) > \lambda_\delta$. In figure 5 (b) the computed entropy is shown as a function of the field H_2 together with the $T \rightarrow 0$ behavior expected from conformal field theory.

2.5 Condensate of $[1, 0]$ - and $[1, 1]$ -solitons

For fields H_1, H_2 satisfying $zH_2 > \max(2\sqrt{3}M_0 - zH_1, 2M_0 - zH_1/2)$ and temperatures $T \ll -\min(\epsilon_{0,j_0,1}^{(1)}(0), \epsilon_{0,j_0,2}^{(2)}(0))$ the highest weight $[1, 0]$ - and $[1, 1]$ -solitons condense. From figure 3(c) one can further conclude that descendent $[1, 0]$ - and $[1, 1]$ -solitons are negligible in this regime of temperatures and magnetic fields.

The condensation of highest weight $[1, 0]$ - and $[1, 1]$ -solitons results in non-zero Fermi velocities $v_{[1,0]}, v_{[1,1]}, v_{pf}^{(m)}$ ($m = 1, 2$) for the solitons and the auxiliary modes. For large

fields $zH_1 \gg M_0$, $zH_2 \gg M_0$ the following relations are found using (2.12)

$$\begin{aligned} f(\phi_{j_0,m}^{(m)}(\lambda_\delta)) &= 1, & f(\phi_{j_0,m}^{(m)}(\infty)) &= 0, \\ f(\phi_{j_0,m}^{(m)}(\lambda_\delta)) &= 0, & f(\phi_{j_0,m}^{(m)}(\infty)) &= 0, \\ f(\phi_{j_2,m}^{(m)}(\lambda_\delta)) &= 0 \end{aligned}$$

and therefore

$$\rho_{j_0,m}^{h(m)}(\lambda) = \rho_{j_1,m}^{(m)}(\lambda) = \rho_{j_2,m}^{(m)} = 0, \quad \text{for } |\lambda| < \lambda_\delta,$$

giving $\mathcal{S}_j^{(m)}(\lambda_\delta) = 0$ for all j, m . Using the relation (2.25) the low-temperature behavior of the entropy becomes

$$\mathcal{S} = \frac{\pi}{3} \frac{10N_f}{N_f + 3} T \quad (2.32)$$

in the phase with finite $[1, 0]$ - and $[1, 1]$ -soliton density. The low-energy excitations near the Fermi points $\epsilon_{j_0,m}^{(m)}(\pm\Lambda_m) = 0$ of the soliton dispersion propagate with velocity $v_{[1,0]} = v_{[1,1]} \rightarrow 1$ for fields $H_m > H_{m,\delta}$ such that $\Lambda_m(H_{m,\delta}) > \lambda_\delta$. Hence, the conformal field theory describing the collective low-energy modes is the $\text{SO}(5)$ WZNW model at level N_f or, by conformal embedding [36], a product of two free $\text{U}(1)$ bosons and a $\text{SO}(5)$ parafermionic coset $\text{SO}(5)_{N_f}/\text{U}(1)^2$ contributing $c = 2$ and

$$c = \frac{10N_f}{N_f + 3} - 2 = \frac{8N_f - 6}{N_f + 3} \quad (2.33)$$

to the central charge, respectively.

For fields H_1, H_2 such that $v_{[1,0]} = v_{[1,1]} < 1$ and $v_{pf}^{(1)} = v_{pf}^{(2)} < 1$ the degeneracy between the solitons and the parafermions is lifted resulting in the low-temperature behavior of the entropy given by

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{2}{v_{[1,0]}} + \frac{1}{v_{pf}^{(m)}} \frac{8N_f - 6}{N_f + 3} \right) T.$$

Additionally, the fields can be chosen such that the remaining degeneracies are lifted, i.e. $v_{[1,0]} < v_{[1,1]}$ and $v_{pf}^{(1)} < v_{pf}^{(2)}$. In this case the entropy becomes

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{1}{v_{[1,0]}} + \frac{1}{v_{pf}^{(1)}} \frac{2(N_f - 1)}{N_f + 2} + \frac{1}{v_{[1,1]}} + \frac{1}{v_{pf}^{(2)}} \left(\frac{8N_f - 6}{N_f + 3} - \frac{2(N_f - 1)}{N_f + 2} \right) \right) T,$$

which is consistent with the conformal embedding

$$\text{SO}(5)_{N_f} = \text{U}(1) + Z_{\text{SU}(2)_{N_f}} + \text{U}(1) + \frac{Z_{\text{SO}(5)_{N_f}}}{Z_{\text{SU}(2)_{N_f}}},$$

see figure 6 (a) for the Fermi velocities and figure 6 (b) for the entropy in this regime.

Figure 6. (a) Fermi velocities as a function of the field zH_1/M_0 for $p_0 = 2 + 1/3$, $zH_2 = -0.06M_0 + 0.21zH_1$ at zero temperature. For large fields, $H_1 > H_{1,\delta}$, all Fermi velocities approach 1 leading to the asymptotic result for the low-temperature entropy (2.32). (b) Entropy obtained from numerical solution of the TBA equations (2.12) as a function of the field zH_2/M_0 for $p_0 = 2 + 1/3$, fixed $zH_2 = -0.06M_0 + 0.21zH_1$ and different temperatures. For fields large compared to the kink mass, $zH_1 \gg M_0$, the entropy approaches the expected analytical value (2.32) (full red line). For magnetic fields $zH_1 < 2(M_{j0,1}^{(1)} - M_{j0,2}^{(2)})$ and temperature $T \ll M_0$ the entropy is that of a dilute gas of non-interacting quasi-particles with degenerate internal degree of freedom due to the anyons.

At last, for Fermi velocities $v_{[1,1]} < v_{[1,0]}$ and $v_{pf}^{(2)} < v_{pf}^{(1)}$ the entropy results in

$$\mathcal{S} = \frac{\pi}{3} \left(\frac{1}{v_{[1,1]}} + \frac{1}{v_{pf}^{(2)}} \frac{2N_f - 1}{N_f + 1} + \frac{1}{v_{[1,0]}} + \frac{1}{v_{pf}^{(1)}} \left(\frac{8N_f - 6}{N_f + 3} - \frac{2N_f - 1}{N_f + 1} \right) \right) T,$$

which is consistent with the conformal embedding

$$\text{SO}(5)_{N_f} = \text{U}(1) + Z_{\text{SO}(3)_{N_f}} + \text{U}(1) + \frac{Z_{\text{SO}(5)_{N_f}}}{Z_{\text{SO}(3)_{N_f}}}.$$

3 Summary and conclusion

Our findings are summarized in a phase diagram based on the numerical analysis of the TBA equations (2.12), see figure 7. For sufficiently small fields a dilute gas of anyons with quantum dimension $Q^{(1)}$ or $Q^{(2)}$ is dominating the contribution to the free energy. By varying the magnetic fields the condensation of anyons can be driven into various collective states described by parafermionic cosets: the collective state describing the condensation of $[1, 1]$ $\text{SO}(5)_{N_f}$ anyons is identified as the $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SO}(3)_{N_f}}$ parafermion coset, while the condensation of $[1, 0]$ $\text{SO}(5)_{N_f}$ anyons results in the $Z_{\text{SO}(5)_{N_f}}/Z_{\text{SU}(2)_{N_f}}$ parafermionic theory. Moreover, the condensation of a mixture of $[1, 0]$ and $[1, 1]$ anyons is studied resulting in the $Z_{\text{SO}(5)_{N_f}}$ parafermion theory describing the collective state. Other theories describing the condensation of $\text{SO}(5)_{N_f}$ anyons are based on conformal embeddings, see figure 7.

Figure 7. Contribution of the $SO(5)_{N_f}$ anyons to the low-temperature properties of the model (1.1), see figure 2 for the phases of the solitonic quasi-particles: using the criteria described in the main text the parameter regions are identified using analytical arguments for $T \rightarrow 0$ (the actual location of the boundaries is based on numerical data for $p_0 = 2 + 1/3$ and $T = 0.035 M_0$). For small fields (regions $Q^{(1)}$, $Q^{(2)}$) a dilute gas of non-interacting quasi-particles with an internal anyonic (zero-energy) degree of freedom with quantum dimension $Q^{(1)}$ or $Q^{(2)}$ is realized. In the shaded region the degeneracy of the zero modes is lifted by the presence of thermally activated solitons with a small but finite density. All the other phases are labelled by the CFT describing the collective state formed by the condensed degrees of freedom.

In summary we can conclude that the effective model describing the $SO(5)$ spin excitations is the $SO(5)_{N_f}$ WZNW model with an anisotropic current-current perturbation. In contrast to the previous application of this approach to $SU(k)_{N_f}$ anyons in [15, 16] this was not clear from the beginning, since corresponding non-Abelian bosonization results of free fermions with $SO(5)$ spin and $U(N_f)$ flavour degrees of freedom are missing.

Acknowledgments

Funding for this work has been provided by the *School for Contacts in Nanosystems*. Additional support by the research unit *Correlations in Integrable Quantum Many-Body Systems* (FOR2316) is gratefully acknowledged.

A Representation theory of SO(5)

The algebra SO(5) has dimension 10 and rank two. In terms of the self adjoint generators $M_{ab} = -M_{ba}$, $a, b = 1, 2, \dots, 5$, the commutation relations of the algebra read

$$[M_{ab}, M_{cd}] = -i(\delta_{bc} M_{ad} - \delta_{ac} M_{bd} - \delta_{bd} M_{ac} + \delta_{ad} M_{bc}) \quad (\text{A.1})$$

We choose the generators of the Cartan subalgebra to be $H_c^1 = M_{1,2}$ and $H_c^2 = M_{3,4}$. The SO(5) root diagram is

The corresponding ladder operators E_α used in the construction of the Hamiltonian (1.1) are linear combinations of the other generators, e.g. $E_{(\pm 1,0)} = (M_{1,5} \pm iM_{2,5})$. The two simple roots $\alpha^1 = (1, -1)$ and $\alpha^2 = (0, 1)$ have different length and the Cartan matrix is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}. \quad (\text{A.2})$$

The fundamental weights are

$$\omega^1 = e^1, \quad \omega^2 = \frac{1}{2}(e^1 + e^2). \quad (\text{A.3})$$

Equivalently, these representations can be labelled by their Dynkin labels $(1, 0)$ and $(0, 1)$ or Young diagrams $[1, 0]$ and $[1, 1]$, respectively (the diagram $[x_1, x_2]$ consists of x_i nodes in the i -th row). The generators in the five dimensional vector representation corresponding to ω^1 are

$$[M_{ab}]_{xy} = -i(\delta_{ax}\delta_{by} - \delta_{bx}\delta_{ay}), \quad (\text{A.4})$$

while the four dimensional spinor representation ω^2 is built from tensor products of Pauli matrices

$$\begin{aligned} M_{1,5} &= \frac{1}{2}\sigma^x \otimes \mathbf{1}, & M_{2,5} &= \frac{1}{2}\sigma^y \otimes \mathbf{1}, \\ M_{3,5} &= \frac{1}{2}\sigma^z \otimes \sigma^x, & M_{4,5} &= \frac{1}{2}\sigma^z \otimes \sigma^y. \end{aligned} \quad (\text{A.5})$$

The other generators can be obtained from the commutation relations (A.1) giving, e.g., the Cartan generators $H_c^1 = \frac{1}{2}\sigma^z \otimes \mathbf{1}$ and $H_c^2 = \frac{1}{2}\mathbf{1} \otimes \sigma^z$.

B TBA of the perturbed $SO(5)_{N_f}$ WZNW model

In order to obtain the integral equations (2.4) a root configuration consisting of $\nu_j^{(m)}$ strings of type $(n_j^{(m)}, v_{n_j}^{(m)})$ on the m -th level is considered and the Bethe equations (2.2) are rewritten in terms of the real string-centers $\lambda_\alpha^{(m,j)} \equiv \lambda_\alpha^{(m)n_j}$ using (2.3). In their logarithmic form they read

$$\begin{aligned} \sum_{\tau=\pm 1} \frac{N}{2} t_{k,N_f}(\lambda_\alpha^{(1,k)} + \tau/g) &= 2\pi I_\alpha^{(1,k)} + \sum_{m=1}^2 \sum_j \sum_{\beta=1}^{\nu_j^{(m)}} (-1)^{m+1} \theta_{kj}^{(1,m)}(\lambda_\alpha^{(1,k)} - \lambda_\beta^{(m,j)}), \\ 0 &= 2\pi I_\alpha^{(2,k)} + \sum_{m=1}^2 \sum_j \sum_{\beta=1}^{\nu_j^{(m)}} (-1)^m \theta_{kj}^{(2,m)}(\lambda_\alpha^{(2,k)} - \lambda_\beta^{(m,j)}), \end{aligned} \quad (\text{B.1})$$

where $I_\alpha^{(m,k)}$ are integers (or half-integers) and the functions

$$\begin{aligned} t_{k,N_f}(\lambda) &= \sum_{l=1}^{\min(n_k, N_f)} f(\lambda, |n_k - N_f| + 2l - 1, v_k v_{N_f}), \\ \theta_{kj}^{(m,m)}(\lambda) &= f(\lambda, |n_k^{(m)} - n_j^{(m)}|, v_k^{(m)} v_j^{(m)}) + f(\lambda, n_k^{(m)} + n_j^{(m)}, v_k^{(m)} v_j^{(m)}) \\ &\quad + 2 \sum_{\ell=1}^{\min(n_k^{(m)}, n_j^{(m)})-1} f(\lambda, |n_k^{(m)} - n_j^{(m)}| + 2\ell, v_k^{(m)} v_j^{(m)}) \quad \text{with } m = 1, 2, \\ \theta_{kj}^{(1,2)}(\lambda) &= \sum_{l=1}^{\min(2n_k^{(1)}, n_j^{(2)})} f(\lambda, |n_j^{(2)}/2 - n_k^{(1)}| + 2l - 1, v_k^{(1)} v_j^{(2)}), \\ \theta_{kj}^{(2,1)}(\lambda) &\equiv \theta_{jk}^{(1,2)}(\lambda) \end{aligned} \quad (\text{B.2})$$

were introduced with

$$f(\lambda, n, v) = \begin{cases} 2 \arctan \left(\tan\left(\left(\frac{1+v}{4} - \frac{n}{2p_0}\right)\pi\right) \tanh\left(\frac{\pi\lambda}{2p_0}\right) \right) & \text{if } \frac{n}{p_0} \neq \text{integer} \\ 0 & \text{if } \frac{n}{p_0} = \text{integer} \end{cases}.$$

In the thermodynamic limit, $N_m, \mathcal{N} \rightarrow \infty$ with $N_m/\mathcal{N}$ fixed, the centers $\lambda_\alpha^{(m,k)}$ are distributed continuously with densities $\rho_k^{(m)}(\lambda)$ and hole densities $\rho_k^{h(m)}(\lambda)$. Following [37] the densities are defined through the following integral equations

$$\begin{aligned} \tilde{\rho}_{0,k}^{(1)}(\lambda) &= (-1)^{r^{(1)}(k)} \rho_k^{h(1)}(\lambda) + \sum_{m=1}^2 \sum_j (-1)^{m+1} A_{kj}^{(1,m)} * \rho_j^{(m)}(\lambda) \quad \text{with } k = 1, \dots, j_{0,1}, \\ 0 &= (-1)^{r^{(2)}(k)} \rho_k^{h(2)}(\lambda) + \sum_{m=1}^2 \sum_j (-1)^m A_{kj}^{(2,m)} * \rho_j^{(m)}(\lambda) \quad \text{with } k = 1, \dots, j_{0,2}, \end{aligned} \quad (\text{B.3})$$

where $a * b$ denotes a convolution and $r^{(m)}(j)$ is given by

$$r^{(m)}(j_{2,m}) = 0, \quad r^{(m)}(\tilde{j}_{0,m}) = 1, \quad r^{(m)}(j_{0,m}) = 2.$$

The bare densities $\tilde{\rho}_{0,j}^{(1)}(\lambda)$ and the kernels $A_{jk}^{(m)}(\lambda)$ of the integral equations are defined by

$$\begin{aligned}\tilde{\rho}_{0,j}^{(1)}(\lambda) &= \frac{1}{2} (a_{j,N_f}(\lambda + 1/g) + a_{j,N_f}(\lambda - 1/g)) , \quad a_{j,N_f}(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} t_{j,N_f}(\lambda) , \\ A_{kj}^{(m,l)}(\lambda) &= \frac{1}{2\pi} \frac{d}{d\lambda} \theta_{kj}^{(m,l)}(\lambda) + (-1)^{r^{(m)}(k)} \delta_{m,l} \delta_{jk} \delta(\lambda) .\end{aligned}\tag{B.4}$$

Using (2.1) and the solutions $\rho_k^{(m)}$ of (B.3) the energy density $\mathcal{E} = E/\mathcal{N}$ is rewritten as

$$\begin{aligned}\mathcal{E} &= \frac{1}{\mathcal{N}} \sum_j \sum_{\alpha=1}^{\nu_j^{(1)}} \left(\sum_{\tau=\pm 1} \frac{\tau}{2} t_{j,N_f}(\lambda_{\alpha}^{(j,1)} + \tau/g) + n_j H_1 \right) + \frac{1}{\mathcal{N}} \sum_j \sum_{\alpha=1}^{\nu_j^{(2)}} n_j H_2 - H_1 - H_2 \\ &\stackrel{\mathcal{N} \rightarrow \infty}{=} \sum_{m=1}^2 \sum_{j \geq 1} \int_{-\infty}^{+\infty} d\lambda \tilde{\epsilon}_{0,j}^{(m)}(\lambda) \rho_j^{(m)}(\lambda) - H_1 - H_2 ,\end{aligned}\tag{B.5}$$

where the bare energies

$$\tilde{\epsilon}_{0,j}^{(1)}(\lambda) = \sum_{\tau=\pm 1} \frac{\tau}{2} t_{j,N_f}(\lambda + \tau/g) + n_j H_1, \quad \tilde{\epsilon}_{0,j}^{(2)}(\lambda) = n_j H_2$$

were introduced. It turns out that the energy (B.5) is minimized by a configuration, where only the strings of length N_f on the first level and strings of length $2N_f$ on the second level have a finite density (cf. ref. [23] for the isotropic case). After inverting the kernels $A_{j_{0,1}j_{0,1}}^{(1,1)}$ and $A_{j_{0,2}j_{0,2}}^{(2,2)}$ in equation (B.3) and inserting the resulting expression for $\rho_{j_{0,1}}^{(1)}(\lambda)$ and $\rho_{j_{0,2}}^{(2)}(\lambda)$ into the other equations for $k \neq j_{0,1}$ on the first level and $k \neq j_{0,2}$ on the second level the integral equations (2.4) are found, where the densities $\rho_{j_{0,m}}^{h(m)} \leftrightarrow \rho_{j_{0,m}}^{(m)}$ were redefined and the Fourier-transformed kernels $B_{kj}^{(l,m)}(\omega)$ were introduced:

$$\begin{aligned}B_{j_{0,m}j_{0,m}}^{(m,m)} &= (-1)^{r_{j_{0,m}}^{(m)}} A_{j_{0,\tilde{m}}j_{0,\tilde{m}}}^{(\tilde{m},\tilde{m})} / K , \\ B_{j_{0,m}j_{0,l}}^{(m,l)} &= (-1)^{r_{j_{0,l}}^{(l)}} A_{j_{0,m}j_{0,l}}^{(m,l)} / K & m \neq l , \\ B_{j_{0,l}j}^{(l,m)} &= (-1)^{l+m} \left(A_{j_{0,l}j}^{(l,m)} A_{j_{0,\tilde{l}}j_{0,\tilde{l}}}^{(\tilde{l},\tilde{l})} - A_{j_{0,l}j_{0,\tilde{l}}}^{(l,\tilde{l})} A_{j_{0,\tilde{l}}j}^{(\tilde{l},m)} \right) / K & j \neq j_{0,l} \text{ if } l = m , \\ B_{kj_{0,m}}^{(l,m)} &= (-1)^{r_k^{(l)}} \left(A_{kj_{0,\tilde{l}}}^{(l,\tilde{l})} B_{j_{0,\tilde{l}}j_{0,m}}^{(\tilde{l},m)} - A_{kj_{0,l}}^{(l,l)} B_{j_{0,l}j_{0,m}}^{(l,m)} \right) & k \neq j_{0,m} \text{ if } l = m , \\ B_{kj}^{(l,m)} &= (-1)^{r_k^{(l)}} \left(A_{kj_{0,\tilde{l}}}^{(l,\tilde{l})} B_{j_{0,\tilde{l}}j}^{(\tilde{l},m)} - A_{kj_{0,l}}^{(l,l)} B_{j_{0,l}j}^{(l,m)} + (-1)^{l+m} A_{kj}^{(l,m)} \right) & k \neq j_{0,l} , j \neq j_{0,m} ,\end{aligned}$$

where $\tilde{m} = m \bmod 2 + 1$, $\tilde{l} = l \bmod 2 + 1$ and

$$K(\omega) = A_{j_{0,1}j_{0,1}}^{(1,1)}(\omega) A_{j_{0,2}j_{0,2}}^{(2,2)}(\omega) - A_{j_{0,2}j_{0,1}}^{(2,1)}(\omega) A_{j_{0,1}j_{0,2}}^{(1,2)}(\omega) .$$

The Fourier-transformed kernels $A_{jk}^{(m,l)}(\omega)$ ($m, l \in \{1, 2\}$) can be derived using (B.2), (B.4), while $A_{jk}^{(1,1)}$ corresponds to A_{jk} in [15].

The expressions determining the bare densities $\rho_{0,k}^{(m)}(\lambda)$ of (2.5), (2.8) and the bare energies $\epsilon_{0,k}^{(m)}(\lambda)$ of (2.6), (2.9) are

$$\begin{aligned}\rho_{0,k}^{(m)}(\lambda) &= B_{kj_{0,1}}^{(m,1)} * \tilde{\rho}_{0,j_{0,1}}^{(1)}(\lambda) && \text{for } k = j_{0,m}, \\ \rho_{0,k}^{(m)}(\lambda) &= \delta_{m,1} \tilde{\rho}_{0,k}^{(1)}(\lambda) + B_{kj_{0,1}}^{(m,1)} * \tilde{\rho}_{0,j_{0,1}}^{(1)}(\lambda) && \text{for } k \neq j_{0,m}, \\ \epsilon_{0,k}^{(m)}(\lambda) &= -B_{j_{0,1}k}^{(1,m)} * \tilde{\epsilon}_{0,j_{0,1}}^{(1)}(\lambda) - B_{j_{0,2}k}^{(2,m)} * \tilde{\epsilon}_{0,j_{0,2}}^{(2)}(\lambda) && \text{for } k = j_{0,m}, \\ \epsilon_{0,k}^{(m)}(\lambda) &= \tilde{\epsilon}_{0,k}^{(m)} - B_{j_{0,1}k}^{(1,m)} * \tilde{\epsilon}_{0,j_{0,1}}^{(1)}(\lambda) - B_{j_{0,2}k}^{(2,m)} * \tilde{\epsilon}_{0,j_{0,2}}^{(2)}(\lambda) && \text{for } k \neq j_{0,m}.\end{aligned}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] A. Yu. Kitaev, *Fault tolerant quantum computation by anyons*, *Annals Phys.* **303** (2003) 2 [[quant-ph/9707021](#)] [[INSPIRE](#)].
- [2] G.W. Moore and N. Read, *Nonabelions in the fractional quantum Hall effect*, *Nucl. Phys. B* **360** (1991) 362 [[INSPIRE](#)].
- [3] N. Read and E. Rezayi, *Beyond paired quantum Hall states: parafermions and incompressible states in the first excited Landau level*, *Phys. Rev. B* **59** (1999) 8084 [[cond-mat/9809384](#)] [[INSPIRE](#)].
- [4] N. Read and D. Green, *Paired states of fermions in two-dimensions with breaking of parity and time reversal symmetries and the fractional quantum Hall effect*, *Phys. Rev. B* **61** (2000) 10267 [[cond-mat/9906453](#)] [[INSPIRE](#)].
- [5] C. Nayak, S.H. Simon, A. Stern, M. Freedman and S. Das Sarma, *Non-Abelian anyons and topological quantum computation*, *Rev. Mod. Phys.* **80** (2008) 1083 [[arXiv:0707.1889](#)] [[INSPIRE](#)].
- [6] A. Feiguin et al., *Interacting anyons in topological quantum liquids: the golden chain*, *Phys. Rev. Lett.* **98** (2007) 160409 [[cond-mat/0612341](#)] [[INSPIRE](#)].
- [7] B.J. Brown, D. Loss, J.K. Pachos, C.N. Self and J.R. Wootton, *Quantum memories at finite temperature*, *Rev. Mod. Phys.* **88** (2016) 045005 [[arXiv:1411.6643](#)] [[INSPIRE](#)].
- [8] C. Gils et al., *Anyonic quantum spin chains: spin-1 generalizations and topological stability*, *Phys. Rev. B* **87** (2013) 235120 [[arXiv:1303.4290](#)].
- [9] P.E. Finch, H. Frahm, M. Lewerenz, A. Milsted and T.J. Osborne, *Quantum phases of a chain of strongly interacting anyons*, *Phys. Rev. B* **90** (2014) 081111 [[arXiv:1404.2439](#)] [[INSPIRE](#)].
- [10] P.E. Finch, M. Flohr and H. Frahm, *Integrable anyon chains: from fusion rules to face models to effective field theories*, *Nucl. Phys. B* **889** (2014) 299 [[arXiv:1408.1282](#)] [[INSPIRE](#)].
- [11] N. Braylovskaya, P.E. Finch and H. Frahm, *Exact solution of the D_3 non-Abelian anyon chain*, *Phys. Rev. B* **94** (2016) 085138 [[arXiv:1606.00793](#)] [[INSPIRE](#)].
- [12] E. Vernier, J. Jacobsen and H. Saleur, *Elaborating the phase diagram of spin-1 anyonic chains*, *SciPost Phys.* **2** (2017) 004 [[arXiv:1611.02236](#)].

- [13] P.E. Finch, M. Flohr and H. Frahm, Z_n clock models and chains of $SO(N)_2$ non-Abelian anyons: symmetries, integrable points and low energy properties, *J. Stat. Mech.* **1802** (2018) 023103 [[arXiv:1710.09620](#)] [[INSPIRE](#)].
- [14] A.M. Tsvelik, Integrable model with parafermion zero energy modes, *Phys. Rev. Lett.* **113** (2014) 066401 [[arXiv:1404.2840](#)] [[INSPIRE](#)].
- [15] D. Borcharding and H. Frahm, Signatures of non-Abelian anyons in the thermodynamics of an interacting fermion model, *J. Phys. A* **51** (2018) 195001 [[arXiv:1706.09822](#)] [[INSPIRE](#)].
- [16] D. Borcharding and H. Frahm, Condensation of non-Abelian $SU(3)_{N_f}$ anyons in a one-dimensional fermion model, *J. Phys. A* **51** (2018) 495002 [[arXiv:1808.05808](#)] [[INSPIRE](#)].
- [17] D. Scalapino, S.-C. Zhang and W. Hanke, $SO(5)$ symmetric ladder, *Phys. Rev. B* **58** (1998) 443 [[cond-mat/9711117](#)].
- [18] H. Frahm and M. Stahlsmeier, Electronic ladders with $SO(5)$ symmetry: phase diagrams and correlations at half filling, *Phys. Rev. B* **63** (2001) 125109 [[cond-mat/0009443](#)].
- [19] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Springer, New York, NY, U.S.A. (1997) [[INSPIRE](#)].
- [20] A.M. Polyakov and P.B. Wiegmann, Theory of non-Abelian Goldstone bosons, *Phys. Lett. B* **131** (1983) 121 [[INSPIRE](#)].
- [21] E. Ogievetsky and P. Wiegmann, Factorized S matrix and the Bethe ansatz for simple Lie groups, *Phys. Lett. B* **168** (1986) 360 [[INSPIRE](#)].
- [22] E. Ogievetsky, P. Wiegmann and N. Reshetikhin, The principal chiral field in two-dimensions on classical Lie algebras: the Bethe ansatz solution and factorized theory of scattering, *Nucl. Phys. B* **280** (1987) 45 [[INSPIRE](#)].
- [23] M. Nakagawara, Magnetic properties of two-dimensional $SO(5) \times SO(5)$ chiral model, *Prog. Theor. Phys.* **76** (1986) 216 [[INSPIRE](#)].
- [24] V.V. Bazhanov, Trigonometric solution of triangle equations and classical Lie algebras, *Phys. Lett. B* **159** (1985) 321 [[INSPIRE](#)].
- [25] M. Jimbo, Quantum R matrix for the generalized Toda system, *Commun. Math. Phys.* **102** (1986) 537 [[INSPIRE](#)].
- [26] V.V. Bazhanov, Integrable quantum systems and classical Lie algebras, *Commun. Math. Phys.* **113** (1987) 471 [[INSPIRE](#)].
- [27] N. Yu. Reshetikhin and P.B. Wiegmann, Towards the classification of completely integrable quantum field theories, *Phys. Lett. B* **189** (1987) 125 [[INSPIRE](#)].
- [28] N. Yu. Reshetikhin, Integrable models of quantum one-dimensional magnets with $O(N)$ and $Sp(2k)$ symmetry, *Theor. Math. Phys.* **63** (1985) 555 [*Teor. Mat. Fiz.* **63** (1985) 347] [[INSPIRE](#)].
- [29] M. Takahashi and M. Suzuki, One-dimensional anisotropic Heisenberg model at finite temperatures, *Prog. Theor. Phys.* **48** (1972) 2187 [[INSPIRE](#)].
- [30] M.J. Martins, Fractional strings hypothesis and nonsimple laced integrable models, *J. Phys. A* **24** (1991) L159 [[INSPIRE](#)].
- [31] C.N. Yang and C.P. Yang, One-dimensional chain of anisotropic spin spin interactions. 2. Properties of the ground state energy per lattice site for an infinite system, *Phys. Rev.* **150** (1966) 327 [[INSPIRE](#)].

- [32] C.-N. Yang and C.P. Yang, *Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction*, *J. Math. Phys.* **10** (1969) 1115 [[INSPIRE](#)].
- [33] N.D. Vlasii, F. von Rütte and U.J. Wiese, *Graphical tensor product reduction scheme for the Lie algebras $so(5) = sp(2)$, $su(3)$ and G_2* , *Annals Phys.* **371** (2016) 199 [[arXiv:1511.02015](#)] [[INSPIRE](#)].
- [34] A.N. Kirillov and N.Y. Reshetikhin, *Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. II. Thermodynamics of the system*, *J. Phys. A* **20** (1987) 1587.
- [35] T. Nakanishi, *Dilogarithm identities for conformal field theories and cluster algebras: simply laced case*, [arXiv:0909.5480](#) [[INSPIRE](#)].
- [36] D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, *Nucl. Phys. B* **290** (1987) 10 [[INSPIRE](#)].
- [37] C.-N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system*, *Phys. Rev.* **150** (1966) 321 [[INSPIRE](#)].