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Excitations of the Myers-Perry black holes

Oleg Lunin

*Department of Physics, University at Albany (SUNY),
Albany, NY 12222, U.S.A.*

E-mail: olunin@albany.edu

ABSTRACT: We demonstrate separability of the dynamical equations for all p -form fluxes in the Myers-Perry-(A)dS geometry, extending the earlier results for electromagnetic field. In the physically important cases of $p = (1, 2, 3, 4)$, we explicitly write the ODEs governing the dynamics of separable solutions.

KEYWORDS: Black Holes, Black Holes in String Theory

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1 Introduction

Black holes are important laboratories for studying classical and quantum gravity. We are entering a new era in exploration of these objects that was launched by the first direct observations of electromagnetic [1] and gravitational [2–6] waves produced by them. Reconstruction of images from these measurements required detailed understanding of classical fields in the vicinity of black holes, and the relevant calculations are usually done numerically [7–15]. Nevertheless, in the idealized case of an isolated black hole, all excitations

can be studied analytically [16, 17], and these classic results have played an important role in our understanding of classical and quantum properties of the holes. In four dimensions, light excitations of black holes contain four types of fields (scalars, spinors, vectors, and gravitons), and all of them are covered by the Teukolsky's equations [16, 17]. In higher dimensions, one encounters additional excitations, and exploration of their dynamics is the main goal of this article. The motivation for our investigation comes from the desire to understand classical scattering from black holes, as well as quantum effects encoded in the Hawking radiation [18].

Hawking radiation is one of the most important quantum phenomena in gravitational physics. While the essence of this process is the same for all fields [18], the detailed wavefunctions of the emitted particles depend on the properties of the underlying dynamical equations. Furthermore, since black holes radiate all light fields at comparable rates, quantitative understanding of quantum dynamics requires analysis of all available excitations. In four dimensions, wavefunctions for all relevant fields have been computed in [16, 17]. Although eventually we are interested in explaining four-dimensional phenomena, in the last three decades great insights into physics of black holes have been gained from analyzing these objects in higher dimensions within the framework of string theory [19–24]. In this case light bosonic excitations are not exhausted by scalars, vectors, and symmetric rank-two tensors, but they also contain higher forms predicted by string theory. Detailed study of such fields, which would shed new light on classical and quantum properties of black holes, is the main objective of this article.

Hawking radiation originates from dynamics of quantum fields on nontrivial gravitational backgrounds. For the static black holes, the original calculations for radiation rates and scattering amplitudes have been extended to various fields in higher dimensions [25–31], and important lessons extracted from such investigations inspired formulation of the AdS/CFT correspondence [32–34]. Success of these studies relied on rich $SO(d-1) \times U(1)$ isometries of static black holes in d dimensions, which give rise to $(d-1)$ commuting conserved charges and guarantee full separation of variables in equations for all dynamical fields. Unfortunately, rotating black holes, described by the Myers-Perry geometry [35], have only $[U(1)]^{[(d+1)/2]}$ isometries, which give rise to $[\frac{d+1}{2}]$ conserved quantities, and they are not sufficient for ensuring full integrability of the dynamical equations. Nevertheless, equations of motion for probe particles and scalar fields in the Myers-Perry geometry turn out to be fully separable due to additional conserved quantities associated with rank-two Killing tensors. The hidden symmetries responsible for this separation were first discovered for the Kerr black hole in [36–38], and in the last two decades they have been extended to rotating black holes in arbitrary dimensions in [39–54], including some geometries carrying charges [55–60]. Remarkably, separation of variables in the wave equation persists for the GLPP geometries [61],¹ which generalize the Myers-Perry black holes to solutions with non-zero values of the cosmological constant, so we will briefly discuss excitations of these backgrounds in section 7.

¹See also [62–65] for a general discussion of the GLPP black holes and their properties.

In contrast to the static black holes, where decomposition into spherical harmonics persists for fields of arbitrary spin,² for rotating spacetimes separation of variables for various fields has been worked out only on a case-by-case basis. In the four-dimensional Kerr geometry, separability of equations for electromagnetic and gravitational waves was proven in the classic work by Teukolsky [16, 17]. Recently separability of Maxwell’s equations has been demonstrated for rotating Myers-Perry and GLPP black holes in all dimensions [73],³ and the goal of our article is to extend these results to dynamics of higher forms⁴ which play an important role in string theory. A better understanding of the black hole excitations will give new handles on probing these fascinating objects.

This paper has the following organization. In section 2 we review the known results pertaining to the Myers-Perry geometry and its excitations. Specifically, we recall the special frames [42–53], which have played crucial role in separating Maxwell’s equations in [73], and review the procedure for separating variables in the equations for the scalar and vector fields. The notation established in section 2 is used throughout this article.

In section 3 we demonstrate separability of the dynamical equation for the two-form, which can be used to describe either NS-NS or Ramond-Ramond fluxes in string theory. We derive the most general separable ansatz in the Myers-Perry geometry and verify that the resulting ODEs contain the expected number of separation constants. In section 4 these results are extended to the three-form potentials. While equations for the higher forms follow the same pattern, the resulting ODEs become rather complicated, so we just present the general structure in section 6 and write the explicit equation for the four-form in the physically interesting ten-dimensional case in section 5. All our results are extended to the GLPP geometry in section 7. Some technical details are presented in the appendices.

2 Summary of the known results

This article is dedicated to analyzing dynamics of various fields in the backgrounds of the Myers-Perry black holes [35], so we begin with reviewing some general properties of these geometries. In particular, section 2.1 establishes the notation used throughout this paper. The main conclusion of this article is that the procedures for solving dynamical equations for all p -forms,

$$d \star dA^{(p)} = 0, \tag{2.1}$$

follow similar patterns, so in sections 2.2 and 2.3 we review the known results for the scalar fields [39–54] and for the one-form potentials [73]. These constructions will be extended to higher forms in the remaining parts of this article. Readers familiar with the earlier work can go directly to section 3.

²While the analysis of spherical harmonics for all p -form fields is rather straightforward [66], study of gravitational waves is quite challenging even in the backgrounds of static black holes. We refer to [67–72] for the detailed discussion.

³Various extensions of this work, including incorporation of massive vector fields, have been reported in [74–79].

⁴To our knowledge, in the past, the dynamics of p -forms in black holes geometries has been studied only in static backgrounds [66].

2.1 Myers-Perry black hole and its symmetries

Let us review some well-known properties of the d -dimensional Myers-Perry geometry [35]. The form of metric differs between even and odd values of d , so we begin with quoting the solution in $d = 2n + 2$ dimensions [35, 80]:

$$ds^2 = -dt^2 + \frac{Mr}{FR} \left(dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 + \frac{FRdr^2}{R - Mr} + \sum_{i=1}^n (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 d\alpha^2. \quad (2.2)$$

Here variables (μ_i, α) are subject to a constraint

$$\alpha^2 + \sum_{i=1}^n \mu_i^2 = 1, \quad (2.3)$$

and functions F, R are defined by

$$F = 1 - \sum_{k=1}^n \frac{a_k^2 \mu_k^2}{r^2 + a_k^2}, \quad R = \prod_{k=1}^n (r^2 + a_k^2). \quad (2.4)$$

To study equations (2.1) for various fields in the background (2.2), we need to review the symmetries of this geometry, which are encoded the Killing-Yano tensors associated with it [42–54, 60].

Metric (2.2) has $(n + 1)$ Killing vectors corresponding to constant shifts of t and ϕ_i , but these isometries are not sufficient to ensure full separation of the wave equation or dynamical equations for tensor fields. For the wave equation such separation is guaranteed by a family of Killing tensors [42–47, 60], while already for the Maxwell field separation requires a more rigid structure associated with Killing-Yano tensors (KYT) [73]. The general equation for an anti-symmetric KYT [81–83] is

$$\nabla_\mu Y_{\nu_1 \dots \nu_p} + \nabla_{\nu_1} Y_{\mu \nu_2 \dots \nu_p} = 0, \quad (2.5)$$

and for the Myers-Perry black holes (2.2), all Killing-Yano tensors are given by a very elegant formula [42–53]

$$Y^{2(n-k)} = \star \left[\wedge h^k \right]. \quad (2.6)$$

Here h is a special two-form that will be written below. As demonstrated in [73], the eigenvectors of the antisymmetric tensor h play the central role in separation of Maxwell's equations, and we expect them to be important in the study of higher forms as well. Using the eigenvectors of a symmetric tensor $t_{\mu\nu} = h_{\mu\alpha} h_\nu^\alpha$ as frames, one finds very simple expressions for the two-form h and for the metric [42–53]:

$$h = r e^r \wedge e^t + \sum_i x_i e^{x_i} \wedge e^i, \quad ds^2 = -(e^t)^2 + (e^r)^2 + \sum_k [(e^{x_k})^2 + (e^k)^2]. \quad (2.7)$$

The frames were constructed in [42–53], and we will write only the components with upper indices, e_A^μ , using the notation introduced in [60]:

$$\begin{aligned} e_t &= -\sqrt{\frac{R^2}{FR\Delta}} \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right], & e_r &= \sqrt{\frac{\Delta}{FR}} \partial_r, \\ e_i &= -\sqrt{\frac{H_i}{d_i}} \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_i^2} \partial_{\phi_k} \right], & e_{x_i} &= \sqrt{\frac{H_i}{d_i}} \partial_{x_i}. \end{aligned} \quad (2.8)$$

Here we defined convenient expressions

$$d_i = (r^2 + x_i^2) \prod_{k \neq i} (x_k^2 - x_i^2), \quad H_i = \prod_k (a_k^2 - x_i^2), \quad \Delta = R - Mr. \quad (2.9)$$

The map between coordinates $\{\mu_i\}$ and $\{x_k\}$ can be found in [60], we just recall that in the variables (r, x_i) , functions (2.4) become

$$R = \prod_k (r^2 + a_k^2), \quad FR = \prod_k (r^2 + x_k^2). \quad (2.10)$$

Note that, apart from the common overall factors, the components (e_t^μ, e_r^μ) of the frames depend only on r , while the components $(e_i^\mu, e_{x_i}^\mu)$ depend only on x_i . This crucial fact is responsible for separation of variables in all equations discussed in this article.

As found in [73] and reviewed in section 2.3, separation of the Maxwell's equations occurs not in components $(A_t, A_r, A_{\phi_i}, A_{x_i})$, but in projections of the gauge fields to particular combinations of the frames (2.8). Specifically, it is convenient to define $m_\pm^{(I)}$ by

$$\begin{aligned} m_\pm^{(0)} &\equiv \sqrt{FR} (e_r \mp e_t) = \frac{R}{\sqrt{\Delta}} \left\{ \frac{\Delta}{R} \partial_r \pm \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right] \right\}, \\ m_\pm^{(j)} &\equiv \sqrt{d_i} (e_{x_i} \mp i e_i) = \sqrt{H_j} \left\{ \partial_{x_j} \pm i \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_j^2} \partial_{\phi_k} \right] \right\}. \end{aligned} \quad (2.11)$$

Then, as demonstrated in [73], it is the set of projections $[m_\pm^{(I)}]^\mu A_\mu$ that separates. Note that the components $m_\pm^{(j)\mu}$ depend only on x_j , and $m_\pm^{(0)\mu}$ depend only on r . The metric and the generator h of the Killing-Yano tensors (2.6)–(2.7) can be written as

$$\begin{aligned} ds^2 &= \frac{1}{FR} \tilde{m}_+^{(0)} \tilde{m}_-^{(0)} + \sum_k \frac{1}{d_k} \tilde{m}_+^{(k)} \tilde{m}_-^{(k)}, \quad \tilde{m}_\pm^{(I)} \equiv [m_\pm^{(I)}]_\mu dx^\mu \\ h &= \frac{r}{2FR} \tilde{m}_+^{(0)} \wedge \tilde{m}_-^{(0)} + \sum_{k=1}^n \frac{x_k}{2id_k} \tilde{m}_+^{(k)} \wedge \tilde{m}_-^{(k)}. \end{aligned} \quad (2.12)$$

The frames (2.11) will play an important role throughout this article.

Let us now discuss the Myers-Perry black holes in odd dimensions $(d = 2n + 1)$. The metric is given by [35, 80]

$$ds^2 = -dt^2 + \frac{Mr^2}{FR} \left(dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 + \frac{FRdr^2}{R - Mr^2} + \sum_{i=1}^n (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2), \quad (2.13)$$

and coordinates μ_i are subject to a constraint

$$\sum_{i=1}^n \mu_i^2 = 1.$$

The counterparts of the special frames (2.7) are given by [42–47, 60]

$$\begin{aligned} e_t &= -\sqrt{\frac{R^2}{FR\Delta}} \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right], & e_r &= \sqrt{\frac{\Delta}{FR}} \partial_r, & e_{x_i} &= \sqrt{\frac{H_i}{x_i^2 d_i}} \partial_{x_i} \\ e_i &= -\sqrt{\frac{H_i}{x_i^2 d_i}} \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_i^2} \partial_{\phi_k} \right], & e_\psi &= -\frac{\prod a_i}{r \prod x_k} \left[\partial_t - \sum_k \frac{1}{a_k} \partial_{\phi_k} \right]. \end{aligned} \quad (2.14)$$

Most relations (2.9), (2.10) still hold, with two exceptions:

$$FR = r^2 \prod_k (r^2 + x_k^2), \quad \Delta = R - Mr^2. \quad (2.15)$$

The Killing-Yano tensors still have the form (2.6)–(2.7), although the metric acquires an extra term $(e_\psi)^2$, and we refer to [60] for the detailed discussion. Finally, the light-cone frames $m_\pm^{(I)}$ are given by counterparts of (2.11), and there is an additional frame n^μ :

$$\begin{aligned} [m_\pm^{(0)}]^\mu \partial_\mu &= \frac{R}{\sqrt{\Delta}} \left\{ \frac{\Delta}{R} \partial_r \pm \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right] \right\}, & \Delta &= R - Mr^2, \\ [m_\pm^{(j)}]^\mu \partial_\mu &= \sqrt{H_j} \left\{ \partial_{x_j} \pm i \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_j^2} \partial_{\phi_k} \right] \right\}, & n^\mu \partial_\mu &= \partial_t - \sum_k \frac{1}{a_k} \partial_{\phi_k}. \end{aligned} \quad (2.16)$$

To study dynamical equations, we will need the expression for the inverse metric in terms of these frames:

$$\begin{aligned} g^{\mu\nu} \partial_\mu \partial_\nu &= \frac{1}{FR} [m_+^{(0)}]^\mu [m_-^{(0)}]^\nu \partial_\mu \partial_\nu + \sum_{j=1}^n \frac{1}{x_j^2 d_j} [m_+^{(j)}]^\mu [m_-^{(j)}]^\nu \partial_\mu \partial_\nu \\ &\quad + \left[\frac{\prod a_i}{r \prod x_k} \right]^2 n^\mu n^\nu \partial_\mu \partial_\nu. \end{aligned} \quad (2.17)$$

Note that the components $[m_\pm^{(j)}]^\mu$ depend only on x_j , $[m_\pm^{(0)}]^\mu$ depend only on r , and n^μ are constants. This feature is crucial for ensuring separation of variables in the Klein-Gordon equation and in dynamical equations for p -form potentials. In this article we are focusing on massless fields, so we begin with reviewing separation of the wave equation.

2.2 Separation of the wave equation

As we will see in the later sections, dynamics of a p -form in the Myers-Perry geometry is governed by a separable scalar function that satisfies a system of ordinary differential equations. To get an inspiration for the resulting ODEs, it is useful to review separation of variables in the wave equation. Our discussion of the scalar field in the Myers-Perry geometry will be very brief, and we refer to [73] for the details.

Let us consider the wave equation

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi] = 0 \quad (2.18)$$

in the background (2.2). Imposing a separable ansatz

$$\Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}, \quad (2.19)$$

we conclude that the functions (Φ, X_i) satisfy a system of differential equations [73]:

$$\begin{aligned} \frac{d}{dr} \left[\Delta \frac{d\Phi}{dr} \right] + \frac{R^2}{\Delta} \left[\omega - \sum_k \frac{a_k n_k}{r^2 + a_k^2} \right]^2 \Phi - P_{n-1}[-r^2] \Phi &= 0, \\ \frac{d}{dx_i} \left[H_i \frac{dX_i}{dx_i} \right] - H_i \left[\omega - \sum_k \frac{a_k n_k}{a_k^2 - x_i^2} \right]^2 X_i + P_{n-1}[x_i^2] X_i &= 0. \end{aligned} \quad (2.20)$$

Here $P_{n-1}[y]$ is an arbitrary polynomial of degree $(n-1)$, and it is crucial that all equations (2.20) contain *the same* function P_{n-1} .

Note that the separable ansatz (2.19) reduces one PDE (2.18) to a system of $d = 2(n+1)$ ordinary differential equations for function Ψ : $(n+1)$ of them come from (2.20) upon replacement $(\Phi, X_i) \rightarrow \Psi$, and the remaining $(n+1)$ relations are

$$\partial_t \Psi = i\omega \Psi, \quad \partial_{\phi_k} \Psi = i n_k \Psi. \quad (2.21)$$

Thus equations (2.19), (2.20) guarantee full separability of the wave equation, and as expected, the resulting solution depends on $d-1 = 2n+1$ integration constants:

- (i) $(n+1)$ parameters (ω, n_i) ;
- (ii) n free coefficients of the polynomial P_{n-1} .

The separation (2.19) also extends to rotating black holes with cosmological constant [42–53], and we refer to [73] for the counterparts of equations (2.20) for that case. Remarkably, at least in some special situations, solutions of the resulting ODEs can be written in terms of Painleve transcendental functions [84]. It would be interesting to see whether a similar reduction to Painleve transcendentals occurs for the ODEs encountered in sections 3–6 of our article.⁵

We conclude this short subsection by writing the counterpart of equations (2.20) for the geometry (2.13) in $d = 2n+1$ dimensions [73]:

$$\begin{aligned} r \frac{d}{dr} \left[\frac{\Delta}{r} \frac{d\Phi}{dr} \right] + \frac{R^2}{\Delta} \left[\omega - \sum_k \frac{a_k n_k}{r^2 + a_k^2} \right]^2 \Phi - P_{n-2}[-r^2] \Phi &= 0, \\ x_i \frac{d}{dx_i} \left[\frac{H_i}{x_i} \frac{dX_i}{dx_i} \right] - H_i \left[\omega - \sum_k \frac{a_k n_k}{a_k^2 - x_i^2} \right]^2 X_i + P_{n-2}[-x_i^2] X_i &= 0. \end{aligned} \quad (2.22)$$

⁵A recent paper [85] found such a reduction for the Teukolsky's equations [16, 17] describing electromagnetic and gravitational waves in the background of the Kerr black hole.

As before, the relations (2.19), (2.22) guarantee full separability of the wave equation, but the counting of $d - 1 = 2n$ free parameters is slightly different:

- (i) $(n + 1)$ parameters (ω, n_i) ;
- (ii) $(n - 1)$ free coefficients of the polynomial P_{n-2} .

In the remainder of this article we will extend the separation (2.19), (2.20), (2.22) to the dynamical equations for all p -form potentials. We begin with reviewing the $p = 1$ case that has been solved in [73].

2.3 Maxwell's equations in the Myers-Perry geometry

The first extension of (2.19), (2.20), (2.22) beyond the scalar field has been accomplished in [73], which demonstrated separability of the Maxwell's equations,

$$d \star dA^{(1)} = 0 : \quad \frac{1}{\sqrt{-g}} \partial_\alpha \left[\sqrt{-g} g^{\mu\nu} g^{\alpha\beta} (\partial_\nu A_\beta - \partial_\beta A_\nu) \right] = 0, \quad (2.23)$$

in the Myers-Perry geometry. The ansatz found in [73] was inspired by the classic analysis of the Maxwell field in the four-dimensional Kerr geometry [16, 17], and we refer to [73] for the detailed discussion of the relation between the higher-dimensional ansatz for the gauge field and the four-dimensional variables introduced by Teukolsky [16, 17]. Here we just mention that, in contrast to a gauge-invariant formulation used in [16, 17], separation of variables in higher dimensions occurred in certain projections of the *gauge potential*, so it worked only in a specific gauge. As we will see in later sections, similar preferred gauges persist for all p -forms, so it is useful to recall the construction of [73]. It is convenient to separate the discussions of odd and even dimensions.

Even dimensions. As demonstrated in [73], the most general separable ansatz for the Maxwell field in even dimensions has the form

$$[m_\pm^{(I)}]^\nu A_\nu = \mp \frac{i}{x_I \pm \mu} [m_\pm^{(I)}]^\nu \partial_\nu \Psi, \quad x_0 = -ir, \quad (2.24)$$

where the scalar function Ψ is given by (2.19). Relations (2.24) have a free parameter μ that will play a role of one of the separation constants. Equations (2.24) can be easily solved for the gauge potential,

$$A_\mu = K_\mu{}^\nu \partial_\nu \Psi, \quad K_\mu{}^\nu = - \sum_I \sum_{\alpha=\pm} \frac{i\alpha}{x_I + \alpha\mu} \frac{1}{d_I} [m_{-\alpha}^{(I)}]_\mu [m_\alpha^{(I)}]^\nu, \quad (2.25)$$

but we find the expression (2.24) to be more compact and more useful for extending to higher forms. Substitution of the ansatz (2.24) into Maxwell's equations leads to a system of ODEs [73]:

$$\begin{aligned} D_j \frac{d}{dx} \left[\frac{H_j}{D_j} X_j' \right] + \left\{ \frac{2\Lambda}{D_j} - H_j W_j^2 - \Lambda + P_{n-2}[x_j^2] D_j \right\} X_j &= 0, \\ D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \dot{\Phi} \right] - \left\{ \frac{2\Lambda}{D_r} - \frac{R^2 W_r^2}{\Delta} - \Lambda + P_{n-2}[-r^2] D_r \right\} \Phi &= 0. \end{aligned} \quad (2.26)$$

Various functions appearing in these equations are defined by

$$\begin{aligned} \Omega &= \omega - \sum \frac{n_k a_k}{\Lambda_k}, & W_j &= \omega - \sum \frac{n_k a_k}{a_k^2 - x_j^2}, & W_r &= \omega - \sum \frac{n_k a_k}{a_k^2 + r^2}, \\ D_j &= 1 - \frac{x_j^2}{\mu^2}, & D_r &= 1 + \frac{r^2}{\mu^2}, & \Lambda &= \frac{\Omega}{\mu} \prod \Lambda_k, & \Lambda_i &= (a_i^2 - \mu^2). \end{aligned} \quad (2.27)$$

Note that the wave equation (2.20) is also covered by a minor modification of the system (2.26):

$$\text{scalar :} \quad D_r = D_j = 1, \quad \forall \Lambda, \quad P_{n-2} \rightarrow P_{n-1}. \quad (2.28)$$

This implies that in the process of going from scalar to vector systems (i.e., from (2.20) to (2.26)), one coefficient in the polynomial P_{n-1} is traded for a separation constant μ . Thus the vector counterpart of the counting presented on page 7 is

- (i) $(n+1)$ parameters (ω, n_i) ;
- (ii) $(n-1)$ free coefficients of the polynomial P_{n-2} ;
- (iii) constant μ .

As expected, the total number of free parameters is still $d-1 = 2n+1$. Furthermore, as demonstrated in [73], the ansatz (2.24) covers all $d-2$ polarizations of the electromagnetic field: they correspond to different ranges of μ .

Note that the ansatz (2.24) fully fixes the gauge since it is not preserved under a gauge transformation. However, after equations (2.24), (2.26) were *derived* in [73] as the system describing the most general separable solution of the Maxwell's equations, it was *observed* in [74, 75] that the resulting potential happened to be in the Lorenz gauge:

$$\nabla_\mu A^\mu = 0. \quad (2.29)$$

This observation allowed the authors of [74, 75] to develop an elegant shortcut in deriving the system (2.26) by imposing the gauge condition (2.29) from the beginning. Unfortunately a similar strategy seems to be failing for higher forms,⁶ so the original derivation presented in [73] appears to be more suitable for the extension to p -forms.

Odd dimensions. The counterpart of the ansatz (2.24) for odd dimensions was also derived in [73], where it was shown that the most general separable solution of the Maxwell's equations has the form

$$[m_\pm^{(I)}]^\nu A_\nu = \mp \frac{i}{x_I \pm \mu} [m_\pm^{(I)}]^\nu \partial_\nu \Psi, \quad n^\nu A_\nu = -\frac{i}{\mu} n^\nu \partial_\nu \Psi, \quad (2.30)$$

with Ψ given by (2.19). The resulting ordinary differential equations are [73]

$$\begin{aligned} \frac{D_j}{x_j} \frac{d}{dx_j} \left[\frac{H_j}{x_j D_j} X_j' \right] + \left\{ \frac{2\Lambda}{D_j} - \frac{H_j W_j^2}{x_j^2} + \frac{\mathcal{A} D_j}{x_j^2} \Omega^2 + P_{n-3}[-x_j^2] D_j \right\} &= 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \dot{\Phi} \right] + \left\{ \frac{2\Lambda}{D_r} + \frac{R^2 W_r^2}{r^2 \Delta} - \frac{\mathcal{A} D_r}{r^2} \Omega^2 + P_{n-3}[r^2] D_r \right\} &\Phi = 0. \end{aligned} \quad (2.31)$$

⁶See appendix B for a detailed discussion.

Here functions (W_j, W_r) are still given by (2.27), while the constants $(\mathcal{A}, \Omega)$ are defined as

$$\mathcal{A} = \left[\prod_k a_k \right]^2, \quad \Omega = \omega - \sum_k \frac{n_k}{a_k}. \quad (2.32)$$

As in even dimensions, one can show that the system (2.30)–(2.31) describes the most general separable solution of the Maxwell’s equations, and it depends on $d - 1$ separation parameters and covers $d - 2$ independent polarizations [73]. In the rest of this article we will extend the systems (2.24), (2.26) and (2.30)–(2.31) to equations governing the dynamics of p -form potentials.

3 Two-form potential in the Myers-Perry geometry

In the last section we reviewed separation of variables in equations for scalar and vector fields in the Myers-Perry geometry. While in four dimensions such excitations, along with spinors and gravitons, exhaust all interesting modes, for $d > 4$ one also encounters higher forms, such as Ramond-Ramond potentials predicted by string theory. Since the Myers-Perry black hole is a solution of the type II supergravity, it is interesting to study SUGRA excitations of this geometry, and they include higher forms corresponding to either Ramond-Ramond potentials or the Kalb-Ramond field. In this section we will extend the success in separating equations for the scalar and vector perturbations to equations of motion for a two-form field, and dynamics of higher forms will be discussed in sections 4–6. To emphasize the analogy with the Maxwell field and with higher forms, we will denote the dynamical variables by A , and our analysis is equally applicable to the Kalb-Ramond field B and to the Ramond-Ramond potential $C^{(2)}$. As in the case of electromagnetism, the discussion naturally splits into even- and odd-dimensional cases, which are covered in subsections 3.1, 3.2.

3.1 Even dimensions

Let us consider a two-form potential A that obeys a system of linear differential equations

$$d \star dA = 0. \quad (3.1)$$

Here Hodge dual is taken with respect to the Myers-Perry geometry (2.2). Since the metric has $(n + 1)$ isometries corresponding to translations along the (t, ϕ_i) coordinates, we can look for solutions in the form

$$A = \frac{1}{2} e^{i\omega t + \sum n_i \phi_i} \tilde{A}_{\mu\nu}(r, x_i) dx^\mu \wedge dx^\nu, \quad x^\mu = \{r, x_i, t, \phi_i\}. \quad (3.2)$$

As in the case of electromagnetism, we expect that separable solutions are governed by one “master function” Ψ , and now we will determine the relation between the gauge potential and such function.

We begin with focusing on a special case

$$\omega = n_i = 0. \quad (3.3)$$

Since the metric has a block form

$$g_{\mu\nu} = \begin{bmatrix} g_{ab} & 0 \\ 0 & g_{ij} \end{bmatrix}, \quad \{a, b\} = \{t, \phi_i\}, \quad \{i, j\} = \{r, x_i\}, \quad x_0 = -ir, \quad (3.4)$$

and there is no dependence on $\phi^a = \{t, \phi_i\}$, we conclude that the field equations for configurations (3.3) can be divided into three decoupled groups that involve $A^{(1)}$, $A^{(2)}$, or $A^{(3)}$:

$$A^{(1)} = \frac{1}{2} A_{ab} d\phi^a \wedge d\phi^b, \quad A^{(2)} = \frac{1}{2} A_{ij} dx^i \wedge dx^j, \quad A^{(3)} = A_{ai} d\phi^a \wedge dx^i. \quad (3.5)$$

Thus in the special case (3.3) one would have three independent “master functions” ($\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}$), but if a configuration (3.5) is viewed as a limit of a general solution with nontrivial (ω, n_i) , then all $\Psi^{(a)}$ must be the same, so the dynamics is governed by a single scalar Ψ :

$$\Psi = \Psi^{(1)} = \Psi^{(2)} = \Psi^{(3)}. \quad (3.6)$$

It is clear that we are interested only in excitations of this type. For separable solutions of equations (3.1), function Ψ must have the form

$$\Psi = \Phi(r) \prod_i^n X_j(x_j). \quad (3.7)$$

All three cases (3.5) are analyzed in the appendix A.2, where it is shown that the most general separable solutions in each class are given by⁷

$$\begin{aligned} A_{ij}^{(1)} &= (x_i^2 - x_j^2) \frac{px_i^2 + q}{Q_i^{(1)}} \frac{px_j^2 + q}{Q_j^{(1)}} \partial_i \partial_j \Psi^{(1)}, \\ \sum_{\mu, \nu} [m_\alpha^{(I)}]^\mu [m_\beta^{(J)}]^\nu A_{\mu\nu}^{(2)} &= \alpha\beta \frac{(x_I^2 - x_J^2) x_I x_J}{Q_I^{(2)} Q_J^{(2)}} \sum_{i, j} [m_+^{(I)}]^i [m_+^{(J)}]^j \partial_i \partial_j \Psi^{(2)}, \\ \sum_\mu [m_\alpha^{(I)}]^\mu A_{\mu j}^{(3)} &= \alpha \frac{(x_I^2 - x_j^2) x_I (\tilde{p}x_i^2 + \tilde{q})}{Q_I^{(3)} Q_j^{(3)}} \sum_i [m_+^{(I)}]^i \partial_i \partial_j \Psi^{(3)}. \end{aligned} \quad (3.8)$$

Polynomials $(Q_j^{(1)}, Q_j^{(2)}, Q_j^{(3)})$ introduced here have the form

$$Q_j^{(a)} = a^{(a)} x_j^4 + b^{(a)} x_j^2 + c^{(a)}, \quad x_0 = -ir, \quad (3.9)$$

and a priori there are independent sets of coefficients $(a^{(a)}, b^{(a)}, c^{(a)})$ for three values of the superscript. Configurations (3.8) solve the field equations (3.1) if and only if each of the functions $(\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)})$ obeys a system of ODEs

$$\partial_j \left[\frac{H_j}{Q_j^{(a)}} \partial_j \Psi^{(a)} \right] + P_{n-3}^{(a)} [x_j^2] \Psi^{(a)} = 0, \quad \partial_r \left[\frac{\Delta}{Q_0^{(a)}} \partial_r \Psi^{(a)} \right] - P_{n-3}^{(a)} [-r^2] \Psi^{(a)} = 0, \quad (3.10)$$

⁷Labels (α, β) take values $\pm$, and the relevant frames are given by (2.11).

where $P_{n-3}^{(a)}[x_j^2]$ is a polynomial of degree $(n-3)$ in its argument. Imposing the restriction (3.6), we conclude that functions $Q_j^{(a)}$ must be the same for all three values of a , and $P_{n-3}^{(a)}$ must have the same property.

Combining relations (3.8), we find the expression for various projections of the most general gauge potential with a separable function Ψ :

$$\sum_{\mu,\nu} [m_\alpha^{(I)}]^\mu [m_\beta^{(J)}]^\nu A_{\mu\nu} = \frac{(x_I^2 - x_J^2)}{Q_I Q_J} F(x_I, x_J) \sum_{i,j} [m_+^{(I)}]^i [m_+^{(J)}]^j \partial_i \partial_j \Psi. \quad (3.11)$$

Here F is a polynomial that is at most quadratic in each of its arguments, and motivated by the form of (2.24) and (3.8), we require function F to be separable. This leads to identification of (p, q) with $(\tilde{p}, \tilde{q})$ in (3.8) and to the final expression for the most general separable ansatz in the special case (3.3):⁸

$$\sum_{\mu,\nu} [m_\alpha^{(I)}]^\mu [m_\beta^{(J)}]^\nu A_{\mu\nu} = (x_I^2 - x_J^2) \frac{V_{I,\alpha}}{Q_I} \frac{V_{J,\beta}}{Q_J} \sum_{i,j} [m_\alpha^{(I)}]^i [m_\beta^{(J)}]^j \partial_i \partial_j \Psi, \quad (3.12)$$

$$Q_I = ax_I^4 + bx_I^2 + c, \quad V_{I,\alpha} = px_I^2 + q + \alpha x_I.$$

Here Ψ is a function with factorized dependence on the coordinates (3.7), that satisfies ordinary differential equations

$$\partial_j \left[\frac{H_j}{Q_j} \partial_j \Psi \right] + P_{n-3}[x_j^2] \Psi = 0, \quad \partial_r \left[\frac{\Delta}{Q_0} \partial_r \Psi \right] - P_{n-3}[-r^2] \Psi = 0, \quad (3.13)$$

with *the same* polynomial P_{n-3} for all values of j . Let us now extend the solution (3.12)–(3.13) beyond the special case (3.3).

Using the separable ansatz (2.24) for the Maxwell's equations as a guide, we assume that addition of nontrivial angular dependence does not modify relations (3.12), but results in a new expression for function Ψ :

$$\Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}. \quad (3.14)$$

In other words, we assume that coefficients (a, b, c, p, q) in (3.12) define a separable ansatz for all values of (ω, n_i) , and the quantum numbers (ω, n_i) affect only the final ODEs. Specifically, using the scalar and vector cases as guides (see equations (2.20), (2.26)) we impose differential equations

$$\partial_j \left[\frac{H_j}{Q_j} \partial_j X_j \right] + f_j[x_j^2] X_j = 0, \quad \partial_r \left[\frac{\Delta}{Q_0} \partial_r \Phi \right] - f_r[-r^2] \Phi = 0, \quad (3.15)$$

and assume that (ω, m_i) dependence appears only in functions $f_j[x_j^2]$. For nontrivial angular dependence, such restrictive ansatz leads to inconsistency of the field equations (3.1) unless coefficients (a, b, c, p, q) are constrained by relations:⁹

$$\frac{a}{c} = \frac{p^2}{q^2}, \quad \frac{b}{c} = \frac{2p}{q} - \frac{1}{q^2}. \quad (3.16)$$

⁸We used the relation $[m_-^{(I)}]^i = [m_+^{(I)}]^i$ to replace $[m_+^{(I)}]^i$ by $[m_-^{(I)}]^i$ and $[m_+^{(J)}]^j$ by $[m_-^{(J)}]^j$.

⁹The easiest way to see this is to look at (ij) components of (3.1), but since the relevant expressions are rather unwieldy, we do not quote them here. A detailed derivation of the counterpart of (3.16) for Maxwell's equations was presented in [73], and extension to higher forms follows the same logic.

These restrictions imply factorization of the quartic polynomials Q_I :

$$Q_I = \frac{a}{p^2} V_{I,+} V_{I,-}. \quad (3.17)$$

Once the ansatz (3.12), (3.14) with factorization (3.17) is imposed, all field equations (3.1) reduce to a system of ODEs, just as in the case of electromagnetism.

To present the final answer in a compact form, we begin with simplifying the ansatz (3.12) in the presence of the constraint (3.17):¹⁰

$$m_\alpha^{I\mu} m_\beta^{J\nu} A_{\mu\nu} = (x_I^2 - x_J^2) h_\alpha^I h_\beta^J m_\alpha^{I\mu} m_\beta^{J\nu} \partial_\mu \partial_\nu \Psi, \quad h_\pm^I = \frac{1}{e_1 + e_2 x_I^2 \pm i e_3 x_I}. \quad (3.18)$$

Substitution of this configuration with separable function Ψ (3.14) into the field equations (3.1) in $d = 2(n + 1)$ dimensions leads to a system of ODEs:

$$D_i \frac{d}{dx_i} \left[\frac{H_i}{D_i} X_i' \right] + \left[-W_i^2 H_i - (e_1 - e_2 x_i^2) \left(G_i - \frac{G_*}{2} \right) + D_i P_{n-3}[x_i^2] \right] X_i + \frac{i e_3 x_i^2}{e_2} X_i \frac{d}{dx_i} \left[\frac{x_i}{x_*^2 - x_i^2} \left\{ \frac{H_i W_i}{x_i^2} - \frac{H_* W_*}{x_*^2} \right\} \right] = 0, \quad (3.19)$$

$$D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] + \left[\frac{W_r^2 H_r^2}{\Delta} + (e_1 + e_2 r^2) \left(G_r - \frac{G_*}{2} \right) - D_r P_{n-3}[-r^2] \right] \Phi - \frac{i e_3 r^2}{e_2} \Phi \frac{d}{dr} \left[\frac{r}{r^2 + x_*^2} \left\{ \frac{H_r W_r}{r^2} + \frac{H_* W_*}{r_*^2} \right\} \right] = 0. \quad (3.20)$$

Here we introduced convenient notation that will be used throughout this article:

$$G_i = \frac{2i e_3 H_i}{D_i} W_i, \quad W_i = \left[\omega - \sum_k \frac{n_k a_k}{d_k^2 - x_i^2} \right], \quad x_* = i \sqrt{\frac{e_1}{e_2}}, \quad (3.21)$$

$$(G_r, H_r, W_r) = (G_i, H_i, W_i) \Big|_{x_i = ir}, \quad (G_*, H_*, W_*) = (G_i, H_i, W_i) \Big|_{x_i = x_*}.$$

We also defined functions D_j and D_r that are specific to the two-form:

$$D_j = \frac{1}{h_+^j h_-^j}, \quad D_r = \frac{1}{h_+^r h_-^r}. \quad (3.22)$$

Equations (3.18), (3.14), (3.19), (3.20) summarize our main result for the two-form gauge potential in the even number of dimensions. As expected, there are $d - 1 = 2n + 1$ free separation constants:

- (i) $(n + 1)$ parameters (ω, n_i) ;
- (ii) $(n - 2)$ coefficients of the polynomial P_{n-3} ;
- (iii) two ratios $(e_2/e_1, e_3/e_1)$ appearing in the expressions (3.14) for the functions $h_\pm^I$.

Although one can set $e_1 = 1$ by rescaling the gauge potential, we kept all three parameters (e_1, e_2, e_3) to make the expressions more symmetric.

¹⁰Recall that $x_0 = -ir$.

The counting presented above works for $n \geq 3$, but it clearly fails in four dimensions. Formally, for $n = 2$ equations (3.19)–(3.20) contain a polynomial P_{-2} with (-1) degree of freedom,¹¹ and this clearly indicates the failure of the description (3.19)–(3.20). It turns out that in the degenerate four-dimensional case, the ratio e_2/e_1 is not an independent parameter, but rather it is determined in terms of (ω, n) . A direct calculation for the Kerr black holes gives

$$h_{\pm}^x = \frac{1}{e_2(x^2 - a^2)W_x \pm i\omega e_3 x}, \quad h_{\pm}^r = \frac{1}{-e_2(r^2 + a^2)W_r \pm \omega e_3 r}, \quad D_x = \frac{1}{h_+^x h_-^x}. \quad (3.23)$$

The differential equations are

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] + \left[-W_x^2 H_x + \frac{ie_3 \omega}{e_2} + \frac{4ie_2 e_3 H_x W_x (\omega x)^2}{D_x} - \frac{2i(e_3 \omega)^3 x^2}{e_2 D_x} \right] X &= 0, \\ D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] + \left[\frac{(H_r W_r)^2}{\Delta} - \frac{ie_3 \omega}{e_2} + \frac{4ie_2 e_3 H_r W_r (\omega r)^2}{D_r} - \frac{2i(e_3 \omega)^3 r^2}{e_2 D_r} \right] \Phi &= 0. \end{aligned} \quad (3.24)$$

Four-dimensional system has another peculiarity: a harmonic two-form A can be dualized into a scalar $\tilde{\Psi}$ using the standard relation

$$d\tilde{\Psi} = \star dA. \quad (3.25)$$

A direct calculation shows that a two-form satisfying the ansatz (3.18), (3.14), (3.23) and differential equations (3.24) maps into a separable scalar:

$$\tilde{\Psi} = \frac{4e_3 \omega}{e_2} E \tilde{\Phi}(r) \tilde{X}(x), \quad (3.26)$$

where

$$\tilde{X} \equiv \frac{1}{D_x} [e_2(a^2 - x^2)\partial_x + ie_3 \omega x] X, \quad \tilde{\Phi} \equiv \frac{1}{D_r} [e_2(r^2 + a^2 - Mr)\partial_r + e_3 \omega r] \Phi \quad (3.27)$$

As a consistency check, one can verify that functions $(\tilde{X}, \tilde{\Phi})$ indeed satisfy the ordinary differential equations (2.22), which follow from the wave equation for the separable scalar (3.26). This concludes our discussion of a two-form field in even dimensions, and in the next subsection our results will be extended to the odd-dimensional case.

3.2 Odd dimensions

The analysis presented in the last subsection can also be carried out for the two-form potential in the odd-dimensional Myers-Perry geometry (2.13). Instead of repeating the algebraic manipulations, we just outline the logical steps:

- (1) As in section 3.1 we begin with a special case (3.3) of the general two-form (3.2). Then manipulations presented in the appendix A.2 lead to the most general separable ansatz (3.8), but now there is a new set of projections: $n^\mu A_{\mu\nu} = 0$. Recall that the frames for the metric (2.13) were introduced in (2.16).

¹¹Note that P_{-1} with zero degrees of freedom should be interpreted as absence of a polynomial. Thus in six dimensions, the separation constants come only from (ω, n_i) and the ratios $(e_2/e_1, e_3/e_1)$.

- (2) Once the condition (3.3) is relaxed, three types of components (3.5) become mixed in the equations of motion (3.1), so consistency of the separable ansatz (3.8) requires the identification (3.6). Furthermore, by combining various components into counterparts of the relation (3.11), and by requiring separability in the resulting projections, we conclude that the *limit* (3.3) of the general separable solution must have the form (3.12):

$$\sum_{\mu,\nu} [m_\alpha^{(I)}]^\mu [m_\beta^{(J)}]^\nu A_{\mu\nu} = (x_I^2 - x_J^2) \frac{V_{I,\alpha}}{Q_I} \frac{V_{J,\beta}}{Q_J} \sum_{i,j} [m_\alpha^{(I)}]^i [m_\beta^{(J)}]^j \partial_i \partial_j \Psi. \quad (3.28)$$

$$n^\mu A_{\mu\nu} = 0, \quad Q_I = ax_I^4 + bx_I^2 + c, \quad V_{I,\alpha} = px_I^2 + q + \alpha x_I.$$

This ansatz leads to an odd-dimensional counterpart of equations (3.13):

$$\frac{1}{x_j} \partial_j \left[\frac{H_j}{x_i Q_j} \partial_j \Psi \right] + P_{n-4}[x_j^2] \Psi = 0, \quad \frac{1}{r} \partial_r \left[\frac{\Delta}{r Q_0} \partial_r \Psi \right] + P_{n-4}[-r^2] \Psi = 0. \quad (3.29)$$

- (3) Extension of the solution (3.28) beyond the special case (3.3) imposes consistency conditions (3.16) that lead to factorization (3.17). In the odd-dimensional case, introduction of angular dependence also leads to specific nontrivial expressions for $n^\mu A_{\mu\nu}$. The final results for the unique separable ansatz and for the ODEs governing the dynamics of the two-form are written below.

The three steps outlined above lead to the full set of components of the gauge potential

$$\begin{aligned} m_\alpha^{I\mu} m_\beta^{J\nu} A_{\mu\nu} &= (x_I^2 - x_J^2) h_\alpha^I h_\beta^J m_\alpha^{I\mu} m_\beta^{J\nu} \partial_\mu \partial_\nu \Psi, \\ m_\alpha^{I\mu} n^\nu A_{\mu\nu} &= x_I^2 h_\alpha^I h_\beta^J m_\alpha^{I\mu} n^\nu \partial_\mu \partial_\nu \Psi, \end{aligned} \quad (3.30)$$

where

$$h_\pm^I = \frac{1}{e_1 + e_2 x_I^2 \pm i e_3 x_I}, \quad \Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}. \quad (3.31)$$

Various ingredients of function Ψ satisfy the system of ordinary differential equations

$$\begin{aligned} \frac{D_i}{x_i} \frac{d}{dx_i} \left[\frac{H_i}{x_i D_i} X_i' \right] + \left[-\frac{W_i^2 H_i}{x_i^2} - 2 \frac{e_1 - e_2 x_i^2}{x_i^2} G_i - 2e_2 G_* + P_{n-4}[x_i^2] D_i \right] X_i \\ + \frac{\mathcal{A} \Omega^2 D_i}{e_1^2 x_i^2} X_i - \frac{i e_3 x_i}{e_2} X_i \frac{d}{dx_i} \left[\frac{1}{x_i^2 - x_*^2} \left\{ \frac{H_i W_i}{x_i^2} - \frac{H_* W_*}{x_*^2} \right\} \right] = 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \Phi' \right] + \left[\frac{W_r^2 H_r^2}{r^2 \Delta} + 2 \frac{e_1 + e_2 r^2}{r^2} G_r - 2e_2 G_* + P_{n-4}[-r^2] D_r \right] \Phi \\ - \frac{\mathcal{A} \Omega^2 D_r}{e_1^2 r^2} \Phi + \frac{i e_3 r}{e_2} \Phi \frac{d}{dr} \left[\frac{1}{r^2 + x_*^2} \left\{ -\frac{H_r W_r}{r^2} - \frac{H_* W_*}{x_*^2} \right\} \right] = 0. \end{aligned} \quad (3.32)$$

Functions appearing in these expressions are given by (3.21)–(3.22), and the constants $(\mathcal{A}, \Omega)$ are defined in (2.32). As expected, solutions of (3.32) contain $d - 1 = 2n$ free separation constants:

- (i) $(n + 1)$ parameters (ω, n_i) ;
- (ii) $(n - 3)$ coefficients of the polynomial P_{n-4} ;
- (iii) two ratios $(e_2/e_1, e_3/e_1)$ appearing in the expressions (3.14) for the functions $h_\pm^I$.

Furthermore, as in the case of the Maxwell's equations, different ranges of the ratios $(e_2/e_1, e_3/e_1)$ lead to the correct number of independent polarizations of the two-form.

The counting presented above fails for $d = 5$, which corresponds to $n = 2$. In the last subsection we have encountered a similar situation in the four-dimensional case, where the ratios $(e_2/e_1, e_3/e_1)$ were not independent, but rather they were determined by (ω, n) . For the five-dimensional black hole there are two options:

- (a) The general ansatz (3.30)–(3.31) can be kept, as long as three parameters (ω, n_1, n_2) are constrained by

$$\Omega = \omega - \sum_k \frac{n_k}{a_k} = 0. \quad (3.33)$$

Then the differential equations are given by a simpler version of (3.32):

$$\begin{aligned} \frac{D_i}{x_i} \frac{d}{dx_i} \left[\frac{H_i}{x_i D_i} X'_i \right] + \left[-\frac{W_i^2 H_i}{x_i^2} - 2 \frac{e_1 - e_2 x_i^2}{x_i^2} G_i - 2e_2 G_* \right] X_i &= 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \Phi' \right] + \left[\frac{W_r^2 H_r^2}{r^2 \Delta} + 2 \frac{e_1 + e_2 r^2}{r^2} G_r - 2e_2 G_* \right] \Phi &= 0. \end{aligned}$$

The second lines in both equations (3.32) disappear due to the constraint (3.33) and the explicit form of the product $H_i W_i$ in five dimensions. As expected, there are four separation constants: two ratios $(e_2/e_1, e_3/e_1)$ and two angular momenta (n_1, n_2) . The frequency ω is fixed by the constraint (3.33).

- (b) If the constraint (3.33) is not satisfied, then the ansatz (3.31) becomes inconsistent unless $e_2 = 0$:

$$h_{\pm}^I = \frac{1}{e_1 \pm i e_3 x_I}. \quad (3.34)$$

The differential equations (3.32) are replaced by

$$\begin{aligned} \frac{D_i}{x_i} \frac{d}{dx_i} \left[\frac{H_i}{x_i D_i} X'_i \right] + \left[-\frac{W_i^2 H_i}{x_i^2} + \frac{\mathcal{A} \Omega^2 D_i}{e_1^2 x_i^2} + \frac{2i e_3^3 W_{\#} H_{\#}}{e_1 D_i} \right] X_i &= 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \Phi' \right] + \left[\frac{W_r^2 H_r^2}{r^2 \Delta} - \frac{\mathcal{A} \Omega^2 D_r}{e_1^2 r^2} + \frac{2i e_3^3 W_{\#} H_{\#}}{e_1 D_r} \right] \Phi &= 0. \end{aligned} \quad (3.35)$$

Here we defined

$$(W_{\#}, H_{\#}) = (W_x, H_x)|_{x=ie_1/e_3}.$$

Again, there are four separation constants: three unconstrained parameters (ω, n_1, n_2) and one ratio e_1/e_3 .

In five dimensions, a two-form $A^{(2)}$ can be dualized into a vector $\tilde{A}^{(1)}$ using a standard relation

$$d\tilde{A}^{(1)} = \star dA^{(2)}. \quad (3.36)$$

The resulting $\tilde{A}^{(1)}$ turns out to have a separable form (2.30), and the expression for its “master function” $\tilde{\Psi}$ in terms of Ψ is similar to (3.26)–(3.27).

To summarize, in this section we have demonstrated separability of equations for the two-form potential in all dimensions and found the resulting systems of ODEs that govern the dynamics. In even dimensions, the ansatz and the ODEs are given by (3.18) and (3.19), while the odd-dimensional results are (3.30) and (3.32). In the degenerate cases of four and five dimensions, the separable configurations are given by (3.23)–(3.24) and (3.34)–(3.35). The differential equations contain the correct number of separation constants,¹² and all polarizations are covered by various ranges of the these constants.

4 3-form potential in the Myers-Perry geometry

In this section we will analyze a three-form potential

$$A = \frac{1}{6} A_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda \quad (4.1)$$

and construct the most general separable solution of the equations of motion

$$d \star dA = 0. \quad (4.2)$$

Since the procedure is very similar to the one implemented in the last section, we will only outline the logic and present the results for the most crucial steps. As before, we will have separate discussions for even and odd dimensions.

4.1 Even dimensions

In this subsection we construct the most general separable solution of equations (4.2) in the Myers-Perry geometry (2.2). Using the results for the Maxwell field and for the two-form as an inspiration, we impose an ansatz

$$m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\lambda} A_{\mu\nu\lambda} = F[x_I, x_J, x_K] h_\alpha^I h_\beta^J h_\gamma^K m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\lambda} \partial_\mu \partial_\nu \partial_\lambda \Psi, \quad (4.3)$$

with some unknown functions F and $h_\pm^I(x_I)$. Indices (α, β, γ) in (4.3) are not summed over, and they take values $(+, -)$. The frames $m_\alpha^{I\mu}$ are given by (2.11). For separable solutions, function Ψ must have the form

$$\Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}. \quad (4.4)$$

As in the case of the two-form, we begin with the special case

$$\omega = n_i = 0 \quad (4.5)$$

to determine the functions $F[x_I, x_J, x_K]$ and h_α^I , then we switch on the angular dependence to find the differential equations for (Φ, X_j) . Specifically, we perform the following steps:

¹²To demonstrate this explicitly, one would need to apply the analysis presented in the appendix F of [73] to the two-form potential. While the logic of such proof is straightforward, the intermediate formulas are rather complicated, so we are not presenting the details.

- (1) In the special case (4.5), the ansatz (4.3)–(4.4) and its counterparts for higher forms are analyzed in appendix A.3, and the result for the three-form potential reads

$$F[x_I, x_J, x_K] = (x_I^2 - x_J^2)(x_I^2 - x_K^2)(x_J^2 - x_K^2), \quad h_{\pm}^I = \frac{p_1 x_I^2 + p_2 \pm p_3 x_I}{Q_I}. \quad (4.6)$$

Here polynomials Q_I are defined by

$$Q_I = b_3 x_I^6 + b_2 x_I^4 + b_1 x_I^2 + b_0, \quad (4.7)$$

and coefficients (p_k, b_k) are the same for all values of I . Function Ψ satisfies a system of ordinary differential equations (A.72)

$$\partial_j \left[\frac{H_j}{Q_j} \partial_j \right] \Psi + P_{n-4}[x_j^2] \Psi = 0, \quad \partial_r \left[\frac{\Delta}{Q_0} \partial_r \right] \Psi - P_{n-4}[-r^2] \Psi = 0, \quad (4.8)$$

where $P_{n-4}[y]$ is an arbitrary polynomial of degree $(n-4)$ in y . The system (4.6)–(4.8) gives the *most general* separable solution (4.3) of equations (4.2) in the special case (4.5).

- (2) Introduction of angular dependence (i.e., relaxation of the requirements (4.5)) leads to severe constraints on the coefficients (p_k, b_k) . In particular, the roots of two equations

$$p_1 x_I^2 + p_2 + p_3 x_I = 0 \quad \text{and} \quad p_1 x_I^2 + p_2 - p_3 x_I = 0$$

must also be the roots of $Q_I = 0$. A similar situation has already been encountered for the electromagnetic field and for the two-form potential, and in the present case the constraints lead to the final expression for $h_{\pm}^I$:

$$h_{\pm}^I = \frac{1}{[1 + e_2 x_I^2 \pm i e_3 x_I][1 + q x_I^2]}. \quad (4.9)$$

This relation is the counterpart of the expression for $h_{\pm}^I$ from (3.18), although now we have rescaled the gauge potential to set $e_1 = 1$.

- (3) In the case of the three-form potential, there are further constraints on parameter q , and the separable solution (4.3)–(4.4) exists if and only if q satisfies an algebraic equation

$$\left[\prod_i (1 + q a_i^2) \right] \left[\omega - \sum_i \frac{q a_i n_i}{1 + q a_i^2} \right] = 0. \quad (4.10)$$

This constraint is automatically satisfied in the special case (4.5), but any nontrivial angular dependence leads to n solutions¹³ for the parameter q . Once the constraint (4.10) is imposed, equations (4.2) reduce to a system of ODEs (4.12) for the function Ψ . Note that the constraint (4.10) is a new feature of the three-forms that has not been encountered in $p = 1, 2$ cases, and in sections 5, 6 we will show that the counterparts of the restriction (4.10) have to be imposed for the p -forms with $p > 3$ as well.

¹³Recall that n is related to the dimensions d of the spacetime by $d = 2(n+1)$.

To summarize, we have found that the most general separable solution of equations (4.2) has the form

$$m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\sigma} A_{\mu\nu\sigma} = X_{IJ} X_{IK} X_{JK} h_\alpha^I h_\beta^J h_\gamma^K m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\sigma} \partial_\mu \partial_\nu \partial_\sigma \Psi, \quad X_{IJ} \equiv x_I^2 - x_J^2, \quad (4.11)$$

where functions $h_\pm^I$ are given by (4.9), and parameter q satisfies the constraint (4.10). The differential equations for functions (X_j, Φ) are¹⁴

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] - W_x^2 H_x X - (1+qx^2)(1-e_2 x^2) \left[G_x - \frac{G_\star}{2} \right] X + D_x P_{n-4}[x^2] X \\ + \frac{ie_3 x^2 (1+qx^2)}{e_2 - q} X \frac{d}{dx} \left[\frac{x(W_x \hat{H}_x - W_\star \hat{H}_\star)}{1+e_2 x^2} - \frac{x(W_x \hat{H}_x - W_q \hat{H}_q)}{1+qx^2} \right] = 0, \\ -D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] - \frac{W_r^2 H_r^2}{\Delta} \Phi - (1-qr^2)(1+e_2 r^2) \left[G_r - \frac{G_\star}{2} \right] \Phi + D_r P_{n-4}[-r^2] \Phi \\ - \frac{ie_3 r^2 (1-qr^2)}{e_2 - q} \Phi \frac{d}{dr} \left[\frac{r(W_r \hat{H}_r - W_\star \hat{H}_\star)}{1-e_2 r^2} - \frac{r(W_r \hat{H}_r - W_q \hat{H}_q)}{1-qr^2} \right] = 0. \end{aligned} \quad (4.12)$$

Here we used various functions defined in (3.21), as well as some new ingredients:

$$\begin{aligned} D_x &= \frac{1}{h_+ h_- [1+qx^2]}, & \hat{H}_x &= \frac{1}{x^4} \prod (a_i^2 - x^2), & (D_r, \hat{H}_r) &= (D_x, \hat{H}_x)_{x=ir} \\ \hat{H}_\star &= e_2^2 \prod \left(a_i^2 + \frac{1}{e_2} \right), & \hat{H}_q &= q^2 \prod \left(a_i^2 + \frac{1}{q} \right). \end{aligned} \quad (4.13)$$

This concludes our discussion of three-form potentials in even dimensions.

4.2 Odd dimensions

In odd dimensions, the metric and the frames are given by (2.13), (2.16). Repeating the analysis presented in the last subsection, we arrive at the most general separable ansatz for the three-form satisfying equations (4.2):

$$\begin{aligned} m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\sigma} A_{\mu\nu\sigma} &= X_{IJ} X_{IK} X_{JK} h_\alpha^I h_\beta^J h_\gamma^K m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\sigma} \partial_\mu \partial_\nu \partial_\sigma \Psi, \quad X_{IJ} \equiv x_I^2 - x_J^2, \\ m_\alpha^{I\mu} m_\beta^{J\nu} n^\sigma A_{\mu\nu\sigma} &= X_{IJ} x_I^2 x_J^2 h_\alpha^I h_\beta^J m_\alpha^{I\mu} m_\beta^{J\nu} n^\sigma \partial_\mu \partial_\nu \partial_\sigma \Psi. \end{aligned} \quad (4.14)$$

Functions h_α^I are still given by (4.9), where parameter q satisfies an algebraic equation (4.10). For function Ψ that has the form (4.4), the dynamical equations (4.2) are equivalent to a system of ODEs:

$$\begin{aligned} \frac{D_x}{x} \frac{d}{dx} \left[\frac{H_x}{x D_x} X' \right] - \left[\frac{W_x^2 H_x}{x^2} + N_x \left[G_x - \frac{G_\star}{2} \right] - D_x P_{n-5}[x^2] - \frac{\mathcal{A} \Omega^2 D_x}{x^2} \right] X \\ + \left\{ - \frac{ie_2^4 H_\star W_\star x^2 (1+qx^2)(1+e_2 x^2)}{e_3(e_2 - q)} - ie_3 e_2 (1+qx^2) \mathcal{F}[W_x H_x] \right\} X = 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \Phi' \right] - \left[- \frac{W_r^2 H_r^2}{r^2 \Delta} + N_r \left[G_r - \frac{G_\star}{2} \right] - D_r P_{n-5}[-r^2] + \frac{\mathcal{A} \Omega^2 D_r}{r^2} \right] \Phi \\ + \left\{ \frac{ie_2^4 H_\star W_\star r^2 (1-qr^2)(1-e_2 r^2)}{e_3(e_2 - q)} - ie_3 e_2 (1-qr^2) \left[\mathcal{F}[W_x H_x] \right]_{x=ir} \right\} \Phi = 0. \end{aligned} \quad (4.15)$$

¹⁴To avoid unnecessary clutter, here in similar equations for higher forms we write $X(x)$ instead of $X_j(x_j)$. All functions X_j satisfy the same ODE.

Here we used functions $(H_x, H_r, \Delta, W_x, W_r, G_x, G_r, W_\star, H_\star, G_\star)$ defined in previous sections, (D_x, D_r) given by (4.13), and two new factors (N_x, N_r) :

$$N_x = (1 + qx^2)(1 - e_2x^2)(e_2^2x^2 + 2e_2 + e_3^2), \quad N_r = N_x \Big|_{x=ir}. \quad (4.16)$$

Unlike the explicit equations (3.19), (3.32), (4.12) encountered earlier, the system (4.15) contains some unspecified function $\mathcal{F}$, and unfortunately we were not able to find a closed form expression for $\mathcal{F}$ that holds in all dimensions. Let us list the properties of $\mathcal{F}$ which are sufficient for writing equations (4.15) in all $d \leq 11$ that are of interest for supergravity.

(a) Function $\mathcal{F}$ maps polynomials in x into polynomials.

(b) On the space of polynomials, function $\mathcal{F}$ is linear:

$$\mathcal{F}[\alpha x^m + \beta x^n] = \alpha \mathcal{F}[x^m] + \beta \mathcal{F}[x^n]. \quad (4.17)$$

(c) Property (b) implies that to specify function $\mathcal{F}$, it is sufficient to evaluate $\mathcal{F}[x^m]$ for all values of m . Furthermore, $H_x W_x$ is a polynomial of degree $\frac{d-1}{2}$ in x^2 , so for $d \leq 11$ we need $\mathcal{F}[x^m]$ only for the even values of $m \leq 10$. The results are¹⁵

$$\begin{aligned} \mathcal{F}[1] &= \mathcal{P}_0 - q\mathcal{P}_2 + q^2\mathcal{P}_4, & \mathcal{F}[x^2] &= \mathcal{P}_2 - q\mathcal{P}_4, & \mathcal{F}[x^4] &= \mathcal{P}_4, \\ \mathcal{F}[x^6] &= 0, & \mathcal{F}[x^8] &= \frac{1}{q}\mathcal{P}_6, & \mathcal{F}[x^{10}] &= \frac{1}{q}\left[\mathcal{P}_8 - \frac{1}{q}\mathcal{P}_6\right], \\ \mathcal{F}[x^{12}] &= \frac{1}{q}\left[\mathcal{P}_{10} - \frac{1}{q}\mathcal{P}_8 + \frac{1}{q^2}\mathcal{P}_6\right]. \end{aligned} \quad (4.18)$$

Here we defined convenient building blocks $\mathcal{P}_k$:

$$\begin{aligned} \mathcal{P}_0 &= e_2x^2 + 1, & \mathcal{P}_2 &= x^2 - \frac{1}{e_2}, & \mathcal{P}_4 &= 2x^4 - \frac{x^2}{e_2} - \frac{1}{e_2^2}, \\ \mathcal{P}_6 &= 2x^6 - \frac{2x^4}{e_2} - \frac{3x^2}{e_2^2} + \frac{1}{e_2^3}, & \mathcal{P}_8 &= 2y^8 - \frac{2x^6}{e_2} - \frac{4x^4}{e_2^2} + \frac{3x^2}{e_2^3} - \frac{1}{e_2^4}, \\ \mathcal{P}_{10} &= 2x^{10} - \frac{2x^8}{e_2} - \frac{6x^6}{e_2^2} + \frac{4x^4}{e_2^3} - \frac{3x^2}{e_2^4} + \frac{1}{e_2^5}. \end{aligned} \quad (4.19)$$

Note that analogues of function $\mathcal{F}$ have appeared in equations (3.19), (3.32), (4.12), but we found closed-form expressions for them. For example, the first equation in (4.12) can be rewritten as

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] - W_x^2 H_x X - (1 + qx^2)(1 - e_2x^2) \left[G_x - \frac{G_\star}{2} \right] X + D_x P_{n-3}[x^2] X \\ + \frac{ie_3x^2(1 + qx^2)}{e_2 - q} \tilde{\mathcal{F}}[x] X = 0, \end{aligned}$$

¹⁵We also write the result for $m = 12$, just to stress that function $\mathcal{F}$ can be determined even beyond eleven dimensions.

where function $\tilde{\mathcal{F}}$ is defined by

$$\tilde{\mathcal{F}}[x] = \frac{d}{dx} \left[\frac{x(W_x \hat{H}_x - W_\star \hat{H}_\star)}{1 + e_2 x^2} - \frac{x(W_x \hat{H}_x - W_q \hat{H}_q)}{1 + q x^2} \right]. \quad (4.20)$$

This $\tilde{\mathcal{F}}$ satisfies the properties (a)–(b) listed above. Unfortunately we were not able to find a simple counterpart of (4.20) for the function $\mathcal{F}$ appearing in (4.15), so we have to rely on the case-by-case construction listed in the item (c). Relations (4.18)–(4.19) are sufficient for writing the ODEs (4.15) in $d \leq 13$ dimensions.

Although equations (4.14) give the most compact form of the ODEs corresponding to a separable three-form in odd dimensions, it is convenient to rearrange some terms in (4.14) to inspire the extension to higher forms discussed in section 6. It is sufficient to look only at the alternative form of the equation for X :

$$\begin{aligned} \frac{D_x}{x} \frac{d}{dx} \left[\frac{H_x}{x D_x} X' \right] + \left[-\frac{W_x^2 H_x}{x^2} + (1 + q x^2) \left[e_2 G_\star + \frac{1 - e_2 x^2}{x^2} G_x \right] + D_x P_{n-3}[x^2] \right] X \\ + \left[\frac{\mathcal{A} \Omega^2 D_x}{x^2} - \frac{2i e_3 H_x W_x (1 - e_2 x^2)}{x^2} \right] X - i e_3 (1 + q x^2) \mathcal{G}[W_x H_x] X = 0 \end{aligned} \quad (4.21)$$

Here $\mathcal{G}$ is defined a linear map between polynomials of x :

$$\mathcal{G}[W_x H_x] \equiv \left\{ \frac{e_2^2 H_\star W_\star x^2 (1 - e_2 x^2)}{(e_2 - q)} + e_2 \mathcal{F}[W_x H_x] \right\},$$

and properties of $\mathcal{G}$ are similar to those of $\mathcal{F}$. In section 6 we will propose a conjecture for an extension of equation (4.21) to all p -forms in odd dimensions.

To summarize, in this section we have demonstrated separability of equations for the three-form potential. In even dimensions, the most general separable ansatz is given by (4.11), and it leads to the system of ODEs (4.12). In odd dimensions the results are (4.14) and (4.15).

5 Four-form potential in ten dimensions

The main motivation for studying p -form potentials comes from string theory, which contains dynamical fields with $p \leq 4$. In previous sections we have analyzed equations of motion for $p = (1, 2, 3)$, and we will now study the last important case: the four-form potential. Unfortunately, differential equations describing separable configurations of this field in arbitrary dimensions are rather complicated, so in this section we will focus only on the physically relevant situation of ten dimensions and find the full answer for that case. The structure of equations for the four-potential in arbitrary dimensions will be discussed in the next section, which will also cover all p -forms with $p > 4$.

A four-form potential appears in the ten-dimensional type IIB supergravity [86–88], so our analysis focuses on space-times of the form

$$\text{MP}_d \times T^{10-d}, \quad (5.1)$$

where MP_d is a d -dimensional Myers-Perry geometry, and T^{10-d} is a torus. The Ramond-Ramond potential $C^{(4)}$ has four types of components,

$$C_{\mu\nu\lambda\sigma}^{(4)}, \quad C_{\mu\nu\lambda a}^{(4)}, \quad C_{\mu\nu ab}^{(4)}, \quad C_{\mu abc}^{(4)}, \quad C_{abcd}^{(4)}, \quad (5.2)$$

where Greek indices correspond to the directions on MP_d , and Latin indices cover the torus. Only the first ingredient in (5.2) describes a genuine 4-form on MP_d , while other sectors reduce to lower forms. Furthermore, unless $d = 10$, the field $C_{\mu\nu\lambda\sigma}^{(4)}$ can be dualized to a four-potential that has at least one index on the torus, i.e., to one of the last three sectors in (5.2). Therefore, equations for the four-potential become interesting only for $d = 10$, and here we will focus only on that case.

In the ten-dimensional type IIB supergravity [86–88], the five-form field strength F_5 satisfies the self-duality relation and the Bianchi identity¹⁶

$$F_5 = \star F_5, \quad dF_5 = 0. \quad (5.3)$$

To construct separable solutions of these equations using the techniques developed in the previous sections, we begin with the Maxwell-type equation for $C^{(4)} \equiv A^{(4)}$ and build the five-form by adding an exact form $dA^{(4)}$ and its dual:¹⁷

$$d \star dA^{(4)} = 0, \quad (5.4)$$

$$F_5 = dA^{(4)} + \star dA^{(4)}. \quad (5.5)$$

In this section we will focus on solving the first equation.

Using the constructions (2.24), (3.18), (4.11) as inspirations, we impose a separable ansatz

$$m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\lambda} m_\delta^{L\sigma} A_{\mu\nu\lambda\sigma} = F_{IJKL} (h_\alpha^I h_\beta^J h_\gamma^K h_\delta^L) m_\alpha^{I\mu} m_\beta^{J\nu} m_\gamma^{K\lambda} m_\delta^{L\sigma} \partial_\mu \partial_\nu \partial_\lambda \partial_\sigma \Psi, \quad (5.6)$$

where function Ψ has the form

$$\Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}. \quad (5.7)$$

In the special case (4.5), the resulting equations (5.4) are analyzed in the appendix A.3, where it is shown that, for consistency, function F_{IJKL} must have the form

$$F_{IJKL} = X_{IJ} X_{IK} X_{IL} X_{JK} X_{JL} X_{KL}, \quad X_{IJ} \equiv x_I^2 - x_J^2, \quad (5.8)$$

and that factors $h_\pm^I$ must be given by

$$h_\pm^I = \frac{p_1 x_I^2 + p_2 \pm p_3 x_I}{Q_I}, \quad Q_I = b_4 x_I^8 + b_3 x_I^6 + b_2 x_I^4 + b_1 x_I^2 + b_0. \quad (5.9)$$

¹⁶We assume that all three-form fluxes are turned off.

¹⁷Here and below, the Ramond-Ramond four-form will be denoted by $A^{(4)}$ rather than $C^{(4)}$ to agree with notation adopted throughout this article.

Then the Maxwell-type equations (5.4) reduce to a system of ODEs:

$$\frac{d}{dx_j} \left[\frac{H_j}{Q_j} X'_j \right] = 0, \quad \frac{d}{dr} \left[\frac{\Delta}{Q_r} \Phi' \right] = 0. \quad (5.10)$$

As in the case of the lower forms, extension of the ansatz (5.6)–(5.9) beyond the special configurations (4.5) leads to certain restrictions on coefficients b_k in (5.9). Specifically, we find that factors $h_{\pm}^I$ must have the form¹⁸

$$h_{\pm}^I = \frac{1}{[1 + e_2 x_I^2 \pm i e_3 x_I][1 + q_1 x_I^2][1 + q_2 x_I^2]}, \quad (5.11)$$

where (q_1, q_2) are two different roots of the algebraic equation

$$\left[\prod_i (1 + q a_i^2) \right] \left[\omega - \sum_i \frac{q a_i n_i}{1 + q a_i^2} \right] = 0. \quad (5.12)$$

Note that the same equation (4.10) has been encountered in our discussion of three-forms. Once the ansatz (5.6) with functions (5.8) and (5.11) is imposed, equation (5.4) reduces to a system of ODEs¹⁹

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] - W_x^2 H_x X - F_x (1 - e_2 x^2) \left[G_x - \frac{G_{\star}}{2} \right] X + \frac{i e_3 \omega}{e_2 q_1 q_2} x^2 F_x X &= 0, \\ -D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] - \frac{W_r^2 H_r^2}{\Delta} \Phi - F_r (1 + e_2 r^2) \left[G_r - \frac{G_{\star}}{2} \right] \Phi - \frac{i e_3 \omega}{e_2 q_1 q_2} r^2 F_r \Phi &= 0. \end{aligned} \quad (5.13)$$

Here functions $(G_x, G_r, G_{\star})$ are given by (3.21), and functions (F_x, D_x, F_r, D_r) are defined by

$$F_x = (1 + q_1 x^2)(1 + q_2 x^2), \quad D_x = \frac{1}{h_+ h_- F_x}, \quad (F_r, D_r) = (F_x, D_x)|_{x=ir}. \quad (5.14)$$

Equations (5.6)–(5.8), (5.11), (5.13) give the full separable solution of the equation (5.4) for the four-form in ten dimensions.

Although solving equation $d \star dA^{(4)} = 0$ in arbitrary dimensions is beyond the scope of this paper, the experience with lower forms gained in previous sections suggests that the ansatz (5.6), (5.8), (5.11) should hold for all $d \geq 10$. Furthermore, by comparing equations (5.13) with the structure of the systems (2.26), (3.19), and (4.12), we arrive at a reasonable guess for the extension of (5.13) to all even dimensions $d = 2(n + 1)$:

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] - W_x^2 H_x X - F_x \left\{ (1 - e_2 x^2) \left[G_x - \frac{G_{\star}}{2} \right] + i e_3 x^2 \mathcal{G}[W_x H_x] \right\} X \\ + D_x P_{n-5}[x^2] X &= 0, \\ -D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] - \frac{W_r^2 H_r^2}{\Delta} \Phi - F_r \left\{ (1 + e_2 r^2) \left[G_r - \frac{G_{\star}}{2} \right] - i e_3 r^2 \mathcal{G}[W_x H_x]_{x=ir} \right\} \Phi \\ + D_x P_{n-5}[-r^2] \Phi &= 0. \end{aligned} \quad (5.15)$$

¹⁸To simplify the formulas below, we used the freedom in rescaling $A^{(4)}$ to require $h_{\pm}^I[0] = 1$. An analogous choice in (3.18) would have set $e_1 = 1$.

¹⁹As in previous sections, we write $X(x)$ instead of $X_j(x_j)$ to avoid unnecessary clutter.

Function $\mathcal{G}$ is a linear operator on the space of polynomials²⁰ of x , and it will be discussed in more detail in the next section. Equations (5.13) clearly fit the pattern (5.15) with a specific function $\mathcal{G}$, and finding the expression for $\mathcal{G}$ in arbitrary dimensions is an interesting open problem.

To summarize, in this section we have demonstrated separability of the equation for the four-form potential and derived the most general separable ansatz (5.6), (5.8), (5.11). In the ten-dimensional case, we also found the system of ODEs (5.13) that governs the dynamics, and we conjectured an extension of these equations to all dimensions (5.15). In the next section we will discuss additional conjectures for the ODEs describing p -forms with $p > 4$.

6 Separability of equations for higher forms

Although supergravity contains only p -forms with $p \leq 4$, the construction presented in this article can be extended to cover dynamics of all p -forms in all dimensions. In the appendix A.3 we demonstrate that equations for such fields in the Myers-Perry geometry are separable, and in this section we summarize the resulting ansatz. We also propose a conjecture for the system of ordinary differential equations governing the dynamics of p -form in all dimensions.

Let us consider a dynamical equation for a p -form field $A^{(p)}$,

$$d \star dA^{(p)} = 0, \quad (6.1)$$

in the even-dimensional Myers-Perry geometry (2.2). Inspired by the results of the previous sections, we impose an ansatz

$$m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_p}^{I_p \mu_p} A_{\mu_1 \dots \mu_p} = F_{I_1 \dots I_p} \left[\prod h_{\alpha_k}^{I_k} \right] m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_p}^{I_p \mu_p} \partial_{\mu_1} \dots \partial_{\mu_p} \Psi, \quad (6.2)$$

where

$$\Psi = E \Phi(r) \left[\prod X_i(x_i) \right], \quad E = e^{i\omega t + i \sum n_i \phi_i}. \quad (6.3)$$

In the appendix A.3 we analyze the resulting equations (6.1) in the special case

$$\omega = n_i = 0, \quad (6.4)$$

and demonstrate that the ansatz (6.2)–(6.3) is consistent if and only if²¹

$$F_{I_1 \dots I_p} = \left[\prod_{k < l}^p [x_{I_k}^2 - x_{I_l}^2] \right], \quad h_{\pm}^J = \frac{g_0 + g_1 x_J^2 \pm g_2 x_J}{Q_J}, \quad Q_J = \sum_{k=0}^p b_k x_J^{2k}. \quad (6.5)$$

With this choice, the dynamical equations (6.1) reduce to a system of ODEs

$$\frac{d}{dx_j} \left[\frac{H_j}{Q_j} X_j' \right] + P_{n-p-1}[x_j^2] X_j = 0, \quad \frac{d}{dr} \left[\frac{\Delta}{Q_r} \Phi' \right] - P_{n-p-1}[-r^2] \Phi = 0. \quad (6.6)$$

²⁰In other words, $\mathcal{G}$ maps polynomials into polynomials and $\mathcal{G}[\alpha x^m + \beta x^n] = \alpha \mathcal{G}[x^m] + \beta \mathcal{G}[x^n]$.

²¹We focus on $d \geq 2(p+1)$, but lower dimensions admit additional degenerate solutions. Equations (6.5) summarize the final result (A.92)–(A.93) of the appendix A.3.

Here $P_{n-p-1}[y]$ is an arbitrary polynomial of degree $(n - p - 1)$ in its argument, and it must be the same in all equations (6.6).

Relaxation of the condition (6.4) leads to a complicated set of equations, and a full analysis of this system is beyond the scope of this article. However, by comparing the answers (2.24), (3.18), (4.9), (5.11) for $p = 1, 2, 3, 4$, we conjecture that the ansatz (6.2)–(6.3) still works, but relations (6.5) must be replaced by more restrictive requirements²²

$$F_{I_1 \dots I_p} = \left[\prod_{k < l}^p [x_{I_k}^2 - x_{I_l}^2] \right], \quad h_{\pm}^I = \frac{1}{[1 + e_2 x_I^2 \pm i e_3 x_I] F_I^{(p)}}, \quad F_I^{(p)} = \prod_{k=1}^{p-2} [1 + q_k x_I^2]. \quad (6.7)$$

Here $\{q_k\}$ are $(p - 2)$ different roots of the algebraic equation

$$\left[\prod_i (1 + q a_i^2) \right] \left[\omega - \sum_i \frac{q a_i n_i}{1 + q a_i^2} \right] = 0. \quad (6.8)$$

Comparison of the systems (2.26), (3.19), (4.12) leads to the conjecture for the ODEs governing the dynamics (6.1) of the p -form:

$$\begin{aligned} D_x \frac{d}{dx} \left[\frac{H_x}{D_x} X' \right] - W_x^2 H_x X - F_x^{(p)} \left\{ (1 - e_2 x^2) \left[G_x - \frac{G_{\star}}{2} \right] + i e_3 x^2 \mathcal{G}_p [W_x H_x] \right\} X \\ + D_x P_{n-p-1}[x^2] X = 0, \\ - D_r \frac{d}{dr} \left[\frac{\Delta}{D_r} \Phi' \right] - \frac{W_r^2 H_r^2}{\Delta} \Phi - F_r^{(p)} \left\{ (1 + e_2 r^2) \left[G_r - \frac{G_{\star}}{2} \right] - i e_3 r^2 \mathcal{G}_p [W_x H_x]_{x=ir} \right\} \Phi \\ + D_x P_{n-p-1}[-r^2] \Phi = 0. \end{aligned} \quad (6.9)$$

Here we used the expressions (3.21) and introduced factors (D_x, D_r) :

$$D_x = \frac{1}{h_+^x h_-^x F_x^{(p)}}, \quad D_r = \frac{1}{h_+^r h_-^r F_r^{(p)}} = D_x|_{x=ir}. \quad (6.10)$$

Equations (6.9) contain a function $\mathcal{G}_p$, which depends on p , but *not* on n , and has the following properties:

- (a) Function $\mathcal{G}_p$ maps polynomials in x into polynomials.
- (b) On the space of polynomials, function $\mathcal{G}$ is linear:

$$\mathcal{G}_p[\alpha x^m + \beta x^n] = \alpha \mathcal{G}_p[x^m] + \beta \mathcal{G}_p[x^n].$$

- (c) Coefficients of the polynomial $\mathcal{G}_p[x^m]$ depend only on parameters $(e_2, q_1, \dots, q_{p-2})$ and integers (p, m) .

²²As in (4.9) and (5.11), we normalized the p -form potential to ensure that $h_{\pm}^I[0] = 1$. This avoids unnecessary complications in differential equations (6.9).

In previous sections we have found explicit expressions for $\mathcal{G}_p$ in all dimensions for $p = (1, 2, 3)$ (see equations (2.26), (3.19), (4.12)). For the four-form, we looked only at the ten-dimensional case (5.13) that gave limited information about $\mathcal{G}_4$:

$$\mathcal{G}_4[x^8] = -\frac{1}{e_2 q_1 q_2}, \quad \mathcal{G}_4[x^k] = 0 \text{ for } k < 8. \quad (6.11)$$

Evaluation of $\mathcal{G}_p[x^m]$ for all values of parameters (p, m) , which would amount to finding the ordinary differential equations (6.9) for all forms in all even dimensions, is beyond the scope of this paper.

We conclude this section with a brief discussion of the odd-dimensional case. Our results for $p = (1, 2, 3)$ forms suggest that the ansatz (6.2) should be modified as

$$\begin{aligned} m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_p}^{I_p \mu_p} A_{\mu_1 \dots \mu_p} &= F_{I_1 \dots I_p} \left[\prod_{k=1}^p h_{\alpha_k}^{I_k} \right] m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_p}^{I_p \mu_p} \partial_{\mu_1} \dots \partial_{\mu_p} \Psi, \\ m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_{p-1}}^{I_{p-1} \mu_{p-1}} n^{\mu_p} A_{\mu_1 \dots \mu_p} &= \hat{F}_{I_1 \dots I_{p-1}} \left[\prod_{k=1}^{p-1} h_{\alpha_k}^{I_k} \right] m_{\alpha_1}^{I_1 \mu_1} \dots m_{\alpha_{p-1}}^{I_{p-1} \mu_{p-1}} n^{\mu_p} \partial_{\mu_1} \dots \partial_{\mu_p} \Psi. \end{aligned} \quad (6.12)$$

Functions $F_{I_1 \dots I_p}$, $h_{\pm}^I$, and Ψ are still given by (6.7), (6.3), and $\hat{F}_{I_1 \dots I_{p-1}}$ is defined as

$$\hat{F}_{I_1 \dots I_{p-1}} = \left[\prod_{k < l}^{p-1} [x_{I_k}^2 - x_{I_l}^2] \right] \left[\prod_k^{p-1} x_{I_k}^2 \right]. \quad (6.13)$$

Substitution of the ansatz (6.12) into the dynamical equation (6.1) leads to a system of ODEs that generalizes (2.31), (3.32), (4.15), (4.21):

$$\begin{aligned} \frac{D_x}{x} \frac{d}{dx} \left[\frac{H_x}{x D_x} X' \right] - \frac{W_x^2 H_x X}{x^2} + F_x^{(p)} \left\{ e_2 G_{\star} + \frac{1 - e_2 x^2}{x^2} G_x + i e_3 \mathcal{G}_p [W_x H_x] \right\} X \\ + \left\{ \frac{\mathcal{A} \Omega^2 D_x}{x^2} - \frac{2 i e_3 H_x W_x (1 - e_2 x^2)}{x^2} + D_x P_{n-p-2}[x^2] \right\} X = 0, \\ \frac{D_r}{r} \frac{d}{dr} \left[\frac{\Delta}{r D_r} \Phi' \right] + \frac{W_r^2 H_r^2}{r^2 \Delta} \Phi + F_r^{(p)} \left\{ e_2 G_{\star} - \frac{1 + e_2 r^2}{r^2} G_r + i e_3 \mathcal{G}_p [W_x H_x]_{x=ir} \right\} \Phi \\ - \left\{ \frac{\mathcal{A} \Omega^2 D_r}{r^2} - \frac{2 i e_3 H_r W_r (1 + e_2 r^2)}{r^2} - D_r P_{n-p-2}[-r^2] \right\} \Phi = 0. \end{aligned} \quad (6.14)$$

Functions $\mathcal{G}$ have the properties (a)–(c) listed on the previous page, and the explicit expressions for $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ can be inferred from sections 2.3, 3.2, 4.2.

To summarize, in this section we proposed the separable ansatz (6.2), (6.12) for arbitrary p -forms and conjectured the structures (6.9), (6.14) of the resulting ordinary differential equations. For the special configurations (4.5), the derivation of these results is presented in the appendix A.3, but full justification of equations (6.9), (6.14) in the general case for $p > 4$ is an interesting open problem.

7 Extension to the Kerr-(A)dS geometry

The main motivation for this article comes from the desire to understand the dynamics of p -form fluxes on nontrivial stringy backgrounds. The Myers-Perry geometry, which solves equations of motion of supergravity, is a natural example of such a background, so until now we have focused on excitations of this space. The equations for scalar and vector fields on this geometry have been solved in the past, and interestingly, success in separation of variables for such excitations extends to rotating black holes with cosmological constant [42–47, 73]. This turns out to be true for the higher forms as well, but so far we have focused only on vanishing cosmological constant since, as we will explain below, this seems to be the only physically interesting case. For completeness, in this section all results are extended to black holes with cosmological constant, even though physical applications of equation (6.1) on such spaces are not clear.

In certain limits, string theory reduces to supergravities in ten and eleven dimensions, and light excitations of such theories contain various p -forms. In particular, the $D = 10, 11$ supergravities admit solutions of the form

$$\text{MP}_d \times T^{D-d}, \quad (7.1)$$

and in this article we have studied the p -form excitations of such spaces. Since the geometry (7.1) is a pure metric, linearized equations for all fluxes have the form (6.1), so all results obtained in previous sections are directly applicable to string theoretic excitations of the geometry (7.1).

In addition to backgrounds (7.1), supergravities also admit solutions of the form²³

$$\text{AdS}_d \times S^q \times T^{D-d-q} \quad \text{supported by fluxes.} \quad (7.2)$$

One can try to combine the structures appearing in (7.1) and (7.2) by replacing the AdS factor in the last equation by a black hole with a negative cosmological constant. Such “Myers-Perry-AdS” solutions were constructed in the article [61], and we will refer to them as GLPP geometries, whose explicit metrics will be written below. Therefore in string theory it is very natural to study excitations of spacetimes

$$\text{GLPP}_d \times S^q \times T^{D-d-q} \quad \text{supported by fluxes.} \quad (7.3)$$

Note that, unlike the pure metric (7.1), the geometry (7.3) necessarily contains fluxes, which modify the equation (6.1) for the p -forms, making separation of (6.1) with $p > 1$ irrelevant for studying supergravity excitations of (6.1). Let us explain this in more detail.

The dynamics of scalars and vectors on the GLPP geometry was analyzed in [42–47, 73], where it was shown that the relevant equations separate, just as in the case of the Myers-Perry black hole. Furthermore, a wave equation for a scalar on the full space (7.3) reduces to a Klein-Gordon equation on GLPP, and similar reduction occurs for Maxwell’s equations, so the results of [42–47, 73] lead to full integrability of dynamical equations for scalar and vector fields on (7.3). Unfortunately, equations of motion for higher forms depend on the

²³The best-known examples are $(d, q) = (5, 5), (3, 3), (2, 2)$ for $D = 10$ and $(d, q) = (4, 7), (7, 4)$ for $D = 11$.

structure of background fluxes, and they do not have the universal form (6.1). For example, there are two two-form excitations ($B^{(2)}, C^{(2)}$) of the space

$$\text{GLPP}_5 \times S^5 \quad \text{supported by} \quad F_5 = L [\text{vol}_{S^5} + \text{vol}_{\text{GLPP}_5}] , \quad (7.4)$$

and linearized equations of supergravity mix them [86–88, 93]:

$$d \star dB^{(2)} = -F_5 \wedge dC^{(2)}, \quad d \star dC^{(2)} = F_5 \wedge dB^{(2)} . \quad (7.5)$$

This system is more complicated than a single equation $d \star dA^{(2)} = 0$, which appears to be irrelevant for analyzing supergravity excitations of (7.4). Nevertheless in this section we will study a *formal equation* $d \star dA^{(2)} = 0$, as well as its generalization (6.1), in the GLPP geometry and demonstrate full separation of variables, extending the results from the previous section. The analysis of the *physical equations*, such as (7.5) on (7.4), is an interesting open problem.

To demonstrate separability of equations (6.1) in the GLPP geometry, we should begin with giving the explicit form of the metrics. As in the Myers-Perry case, the odd and even dimensions must be treated separately. The geometries were constructed in [61], and the special frames that generalize (2.8) and (2.14) were found in [42–53]. In even dimension one gets

$$\begin{aligned} e_t &= -\frac{1}{Q_r} \sqrt{\frac{R^2}{FR\Delta}} \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right], & e_r &= Q_r \sqrt{\frac{\Delta}{FR}} \partial_r, \\ e_i &= -\frac{1}{Q_i} \sqrt{\frac{H_i}{d_i}} \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_i^2} \partial_{\phi_k} \right], & e_{x_i} &= Q_i \sqrt{\frac{H_i}{d_i}} \partial_{x_i}. \end{aligned} \quad (7.6)$$

The notation is the same as in (2.8), with two new ingredients:

$$Q_r = \sqrt{1 - Lr^2 \frac{R}{\Delta}}, \quad Q_i = \sqrt{1 + Lx_i^2}. \quad (7.7)$$

Parameter L introduced in [61] is related to the value of the cosmological constant, and expressions (2.8) are recovered for $L = 0$. The counterparts of the null vectors (2.11) are defined by

$$\begin{aligned} m_{\pm}^{(0)} &\equiv \sqrt{FR} (e_r \mp e_t) = \frac{R}{\sqrt{\Delta}} \left\{ \frac{Q_r \Delta}{R} \partial_r \pm \frac{1}{Q_r} \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right] \right\}, \\ m_{\pm}^{(j)} &\equiv \sqrt{d_i} (e_{x_i} \mp e_i) = \sqrt{H_j} \left\{ Q_j \partial_{x_j} \pm \frac{i}{Q_j} \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_j^2} \partial_{\phi_k} \right] \right\}. \end{aligned} \quad (7.8)$$

To solve the equation (6.1) in the even-dimensional GLPP geometry, we introduce the

ansatz (6.2), (6.3), (6.7)–(6.8). This results in the generalization of the ODEs (6.9),

$$\begin{aligned}
 & D_x \frac{d}{dx} \left[\frac{Q_x^2 H_x}{D_x} X' \right] - \frac{W_x^2 H_x}{Q_x^2} X - F_x^{(p)} \left\{ (1 - e_2 x^2) \left[G_x - \frac{G_0}{2} \right] + i e_3 x^2 \mathcal{G}_p[W_x H_x] \right\} X \\
 & + D_x P_{n-p-1}[x^2] X = 0, \\
 & - D_r \frac{d}{dr} \left[\frac{Q_r^2 \Delta}{D_r} \Phi' \right] - \frac{W_r^2 H_r^2}{Q_r^2 \Delta} \Phi - F_r^{(p)} \left\{ (1 + e_2 r^2) \left[G_r - \frac{G_0}{2} \right] - i e_3 r^2 \mathcal{G}_p[W_x H_x]_{x=ir} \right\} \Phi \\
 & + D_x P_{n-p-1}[-r^2] \Phi = 0.
 \end{aligned} \tag{7.9}$$

Note that the cosmological parameter L appears only in the terms

$$D_x \frac{d}{dx} \left[\frac{Q_x^2 H_x}{D_x} X' \right], \quad \frac{W_x^2 H_x}{Q_x^2} X \quad \text{and} \quad D_r \frac{d}{dr} \left[\frac{Q_r^2 \Delta}{D_r} \Phi' \right], \quad \frac{W_r^2 H_r^2}{Q_r^2 \Delta} \Phi, \tag{7.10}$$

while all other ingredients of equations (6.9) remain the same as in the Myers-Perry case. Modifications (7.10) can also be easily implemented in the more explicit equations (2.26), (3.19), (4.12), (5.13) for $p = (1, 2, 3, 4)$.

In odd dimensions, the GLPP metric [61] can be written in terms of separable frames [42–53, 73]

$$\begin{aligned}
 e_t &= -\frac{1}{Q_r} \sqrt{\frac{R^2}{FR\Delta}} \left[\partial_t - \sum_k \frac{a_k}{r^2 + a_k^2} \partial_{\phi_k} \right], \quad e_r = Q_r \sqrt{\frac{\Delta}{FR}} \partial_r, \quad e_{x_i} = Q_i \sqrt{\frac{H_i}{x_i^2 d_i}} \partial_{x_i} \\
 e_i &= -\frac{1}{Q_i} \sqrt{\frac{H_i}{x_i^2 d_i}} \left[\partial_t - \sum_k \frac{a_k}{a_k^2 - x_i^2} \partial_{\phi_k} \right], \quad e_\psi = -\frac{\prod a_i}{r \prod x_k} \left[\partial_t - \sum_k \frac{1}{a_k} \partial_{\phi_k} \right],
 \end{aligned} \tag{7.11}$$

which are obtained by inserting the factors (Q_r, Q_i) given by (7.7) into (2.14). The separable ansatz for the p -forms ((6.12), (6.13), (6.7), (6.3)) is built using the null vectors (7.8),²⁴ as well as $n = e_\psi$. Substitution of the resulting form into the dynamical equation (6.1) leads to a system of ODEs

$$\begin{aligned}
 & \frac{D_x}{x} \frac{d}{dx} \left[\frac{Q_x^2 H_x}{x D_x} X' \right] - \frac{W_x^2 H_x X}{x^2 Q_x^2} + F_x^{(p)} \left\{ e_2 G_\star + \frac{1 - e_2 x^2}{x^2} G_x + i e_3 \mathcal{G}_{n,p}[W_x H_x] \right\} X \\
 & + \left\{ \frac{\mathcal{A} \Omega^2 D_x}{x^2} - \frac{2i e_3 H_x W_x (1 - e_2 x^2)}{x^2} + D_x P_{n-p-2}[x^2] \right\} X = 0, \\
 & \frac{D_r}{r} \frac{d}{dr} \left[\frac{Q_r^2 \Delta}{r D_r} \Phi' \right] + \frac{W_r^2 H_r^2}{r^2 Q_r^2 \Delta} \Phi + F_r^{(p)} \left\{ e_2 G_\star - \frac{1 + e_2 r^2}{r^2} G_r + i e_3 \mathcal{G}_p[W_x H_x]_{x=ir} \right\} \Phi \\
 & - \left\{ \frac{\mathcal{A} \Omega^2 D_r}{r^2} - \frac{2i e_3 H_r W_r (1 + e_2 r^2)}{r^2} - D_r P_{n-p-2}[-r^2] \right\} \Phi = 0.
 \end{aligned} \tag{7.12}$$

This is a very simple modification of equations (6.14), and similar insertions of the Q -factors should be made in the special cases (2.31), (3.32), and (4.15).

To summarize, in this section all results obtained in our article have been extended to the GLPP geometries, which generalize the Myers-Perry solutions to black holes in the

²⁴Recall that the expressions for Δ are different in even and odd dimensions: see equations (2.9) and (2.15).

presence of cosmological constant. It would be very interesting to find further extensions to equations like (7.5), which describe the dynamics of p -forms in the spaces with *effective* cosmological constants generated by fluxes. Such extensions will provide valuable tools for studying excitations of black holes in string theory.

8 Discussion

In this article we have demonstrated separability of the dynamical equations describing all p -forms in the backgrounds of the Myers-Perry and GLPP black holes, generalizing the results known for the scalar and vector fields. The most general separable ansatze are given by ((6.2), (6.7)) in even and by (6.12)–(6.13) in odd dimensions. The structures of the resulting ordinary differential equations are summarized in (6.9) and (6.14), and in the special cases of $p = (1, 2, 3, 4)$, the explicit ODEs are given by (2.26), (2.31), (3.19), (3.32), (4.12), (4.15), (5.13).

Our procedure relied on existence of the Killing-Yano tensors, and very restrictive uniqueness theorems for such objects [89–92] imply that it would be hard to extend the methods presented here beyond the Myers-Perry and the GLPP geometries, if one focuses only on vacuum solutions of Einstein’s equations. However, one should be able to incorporate charged black holes since they also admit Killing-Yano tensors [60]. Unfortunately, equations for interesting excitations of such spacetimes mix various fluxes (see the discussion around formula (7.5)), so we leave exploration of these modes to future work. Another interesting direction involves the study of integrable excitations which are not covered by separable ansatze. Although separability of dynamical equations guarantees integrability, the inverse relationship is not true. In particular, there are several examples of gravitational backgrounds where the equation for the scalar field is fully integrable, while there is no separation of variables [94–96]. It would be interesting to see whether a similar phenomenon occurs for the higher forms. Finally, it would also be very interesting to extend the framework introduced this article to gravitational waves.

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A Derivation of the separable ansatz

This appendix is dedicated to justifying the ansatze (3.18), (4.11), (6.2), (6.7). Specifically, we focus on configurations which do not depend on the cyclic coordinates and determine the relations between various components of the gauge potential and the “master” scalar function. The extensions of these relations to configurations with general angular dependence are straightforward, but the intermediate formulas in the derivations are rather cumbersome, so we just write the final answers in the appropriate sections.²⁵

²⁵For the Maxwell’s equations, the detailed derivations were presented in [73], and similar analysis is applicable to all forms.

We begin with reviewing the results for electromagnetism by presenting a modification of the analysis performed in [73] that can be easily extended to higher forms. Then in section A.2 we derive the ansatz (3.18) for the two-form, and in section A.3 the results are extended to all p -form potentials.

A.1 Maxwell's equations

In this subsection we will justify the ansatz (2.24) for special configurations with

$$\omega = 0, \quad n_i = 0. \quad (\text{A.1})$$

Although the result (2.24) has been already derived in [73] using a slightly different method, it is the logic presented here that can be easily extended to higher forms, so it is instructive to begin with applying the new method to the Maxwell field before extending it to higher forms in sections A.2 and A.3.

We begin with studying Maxwell's equations in even dimensions, and some peculiarities of the odd-dimensional case will be discussed after equation (A.39). Since the frame $[m_{\pm}^{(I)}]^{\mu}$ defined by (2.11) is associated with coordinate x_I , a reasonable requirement for separation of variables is²⁶

$$[m_{\alpha}^{(I)}]^{\mu} A_{\mu} = S_{I,\alpha}(x_I) [m_{\alpha}^{(I)}]^{\mu} \partial_{\mu} \Psi, \quad \alpha = \pm \quad (\text{A.2})$$

with some unknown functions $(S_{r,\pm}, S_{i,\pm})$. For Ψ that does not depend on (t, ϕ_k) , the relation above simplifies

$$[m_{\alpha}^{(I)}]^{\mu} A_{\mu} = S_{I,\alpha}(x_I) [m_{\alpha}^{(I)}]^I \partial_I \Psi. \quad (\text{A.3})$$

In the absence of (t, ϕ_k) -dependence the Maxwell's equations for (A_r, A_{x_i}) decouple from the ones for (A_t, A_{ϕ_i}) , so it is convenient to analyze these sectors separately. Similar decoupling will take place for the higher forms discussed in sections A.2, A.3. For the Maxwell field, we will analyze the (A_r, A_{x_i}) components in subsection A.1.1 and focus on (A_t, A_{ϕ_i}) components in section A.1.2.

A.1.1 Non-cyclic components of the gauge field

Let us consider the Maxwell's equations involving

$$A_r \quad \text{and} \quad A_a \equiv A_{x_a}, \quad a = 1 \dots n.$$

Recalling the frames (2.11), we find

$$A_r = [S_{r,+} + S_{r,-}] \partial_r \Psi \equiv S_r \partial_r \Psi, \quad A_a = [S_{a,+} + S_{a,-}] \partial_a \Psi \equiv S_a \partial_a \Psi. \quad (\text{A.4})$$

The cyclic components of the gauge potential, (A_t, A_{ϕ_k}) , are more complicated, but they do not mix with (A.4) in the Maxwell's equations. In four dimensions, $n = 1$ and configurations (A.4) describe a pure gauge (see [73] for details), therefore here we focus on $n \geq 2$.

²⁶In (A.2) and (A.3) summation is performed only over index μ .

Using indices $(i, j, \dots)$ to cover r and x_a and defining a function

$$H_r \equiv \Delta = R - Mr, \quad (\text{A.5})$$

we find the expressions for the gauge potential and the field strength in the sector (A.4)

$$A_i = S_i \partial_i \Psi, \quad F_{ij} = (S_j - S_i) \partial_i \partial_j \Psi, \quad F^{ij} = \frac{H_i H_j}{d_i d_j} (S_j - S_i) \partial_i \partial_j \Psi. \quad (\text{A.6})$$

Furthermore, evaluating the determinant of the Myers-Perry metric (2.2) in coordinates (r, x_a, t, ϕ_a) ,

$$\sqrt{-g} = \frac{1}{\prod a_k} \left[FR \prod \frac{d_k}{c_k^2} \right]^{1/2}, \quad (\text{A.7})$$

we write the i -th component of Maxwell's equations as

$$0 = \frac{1}{\sqrt{\prod d_k}} \sum_{j \neq i} \partial_j \left[\sqrt{\prod d_k} \frac{H_i H_j}{d_i d_j} (S_j - S_i) \partial_i \partial_j \Psi \right] = \sum_{j \neq i} \frac{H_i}{d_j} \partial_j \left[\frac{H_j}{d_i} (S_j - S_i) \partial_j \partial_i \Psi \right]. \quad (\text{A.8})$$

Simplifications lead to the final system of equations

$$\sum_{j \neq i} \frac{x_i^2 - x_j^2}{d_j} \partial_j \left[\frac{H_j (S_j - S_i)}{x_i^2 - x_j^2} \partial_j \partial_i \Psi \right] = 0, \quad i = (r, x_a). \quad (\text{A.9})$$

Separable ansatz in $d = 2(n+1)$ dimensions implies that function Ψ can be written as

$$\Psi = X_r(r) \prod X_a(x_a), \quad (\text{A.10})$$

then the set of equations (A.9) reduces to a system²⁷

$$\partial_j \left[\frac{H_j (S_j - S_i)}{x_i^2 - x_j^2} \partial_j X_j \right] + P_{n-2}^{(j,i)}(x_j^2; x_i) X_j = 0, \quad j \neq i. \quad (\text{A.11})$$

Here $P_{n-2}^{(j,i)}(x_j^2; x_i)$ is a polynomial of degree $(n-2)$ in x_j^2 , and x_i -dependence of this function is still undetermined. Substitution of (A.11) back into (A.9), leads to a system of algebraic relations for $P_{n-2}^{(i,j)}$:

$$\sum_{j \neq i} \frac{x_i^2 - x_j^2}{d_j} P_{n-2}^{(j,i)}(x_j^2; x_i) = 0. \quad (\text{A.12})$$

The left-hand side of the last relation is a meromorphic function of all x_j for $j \neq i$, then analysis of the asymptotic behavior and residues at various poles demonstrates that polynomials $P_{n-2}^{(j,i)}$ must be the same for all values of j :

$$P_{n-2}^{(j,i)}(x_j^2; x_i) = P_{n-2}^{(i)}(x_j^2; x_i). \quad (\text{A.13})$$

Once the relation (A.13) is satisfied, equation (A.12) becomes an identity.

²⁷In four dimensions, $n = 1$, and there is no polynomial $P_{n-2}^{(i)}$ in equation (A.11). However, as we already mentioned, in this case configuration (A.4) describes a pure gauge, so we are focusing on $n \geq 2$.

To summarize, for the separable ansatz (A.10), the Maxwell's equations (A.9) are equivalent to the system of ODEs (A.11) with restrictions (A.13). Furthermore, as in the scalar case (2.20), the polynomials $P_{n-2}^{(i)}$ contain some number of separation constants, in particular, setting these constants to zero, one can get a special case of the system (A.11) with $P_{n-2}^{(i)} = 0$:

$$\partial_j \left[\frac{H_j(S_j - S_i)}{x_i^2 - x_j^2} \partial_j X_j \right] = 0, \quad j \neq i. \quad (\text{A.14})$$

The current discussion is applicable to $n > 2$, then taking $j = 0$ and combining equations (A.14) with $i = 1, 2$, we find

$$\frac{x_1^2 - x_0^2}{H_0(S_0 - S_1)} \partial_0 \left[\frac{H_0(S_0 - S_1)}{x_1^2 - x_0^2} \right] - \frac{x_2^2 - x_0^2}{H_0(S_0 - S_2)} \partial_0 \left[\frac{H_0(S_0 - S_2)}{x_2^2 - x_0^2} \right] = 0.$$

In particular, this implies that

$$\partial_2 \partial_0 \ln \left[\frac{H_0(S_0 - S_2)}{x_2^2 - x_0^2} \right] = 0. \quad (\text{A.15})$$

This relation and its counterparts with other values of (i, j) in (A.14) can be summarized as a system of equations

$$\frac{S_i - S_j}{x_i^2 - x_j^2} = g(x_i)g(x_j), \quad (\text{A.16})$$

with some function $g(x)$. The analogues of the functional equations (A.16) will play the crucial role in derivations of separable ansatze for higher forms, which will be presented in sections A.2, A.3.

Explicit evaluation of the derivatives in (A.15) gives

$$\frac{S'_0 S'_2}{(S_0 - S_2)^2} = \frac{4x_0 x_2}{(x_2^2 - x_0^2)^2}. \quad (\text{A.17})$$

The most general solution of this differential equation for function S is

$$S(x) = \frac{\alpha x^2 + \beta}{\gamma x^2 + \delta} \quad (\text{A.18})$$

with arbitrary constants $(\alpha, \beta, \gamma, \delta)$. Although the result (A.18) was derived for the special choice of separation constants corresponding to reduction of (A.11) to (A.14), it is clear that function S cannot depend on the separation constants, so relations (A.18) also hold in the general case (A.11). Substitution of (A.18) into (A.11) with (A.13) gives

$$\frac{\alpha\delta - \beta\gamma}{\gamma x_i^2 + \delta} \partial_j \left[\frac{H_j}{\gamma x_j^2 + \delta} \partial_j X_j \right] + P_{n-2}^{(i)}(x_j^2; x_i) X_j = 0, \quad j \neq i. \quad (\text{A.19})$$

This equation implies that

$$P_{n-2}^{(i)}(x_j^2; x_i) = \frac{\alpha\delta - \beta\gamma}{\gamma x_i^2 + \delta} P_{n-2}(x_j^2), \quad (\text{A.20})$$

where $P_{n-2}(x_j^2)$ is a polynomial with *constant* coefficients.

To summarize, we have demonstrated that the polarization (A.4) is consistent with the separable ansatz (A.10) if and only if (S_r, S_a) are given by (A.18), and functions X_j satisfy ordinary differential equations

$$\partial_j \left[\frac{H_j}{\gamma x_j^2 + \delta} \partial_j X_j \right] + P_{n-2}(x_j^2) X_j = 0. \quad (\text{A.21})$$

Let us now discuss the equations for (A_t, A_{ϕ_i}) .

A.1.2 Angular components of the gauge field

The angular components of the gauge field are determined from combinations of (A.3), which are complementary to (A.4):

$$([m_+^{(I)}]^\mu - [m_-^{(I)}]^\mu) A_\mu = [S_{I,+} - S_{I,-}] [m_+^{(I)}]^I \partial_I \tilde{\Psi}. \quad (\text{A.22})$$

For configuration without angular dependence, (A.1), functions Ψ and $\tilde{\Psi}$ are independent, but for nontrivial m_i or ω we expect to have only one “master function” Ψ . Thus if one views polarizations (A.4) and (A.22) as *limits* (A.1) of the general solution, then Ψ and $\tilde{\Psi}$ must be the same. In particular, function $\tilde{\Psi}$ must have the form (A.10) with equations (A.21) for various ingredients. Rewriting the angular polarizations (A.22) as

$$[m_\pm^{(I)}]^\mu A_\mu = \pm \tilde{S}_I [m_\pm^{(I)}]^I \partial_I \Psi, \quad (\text{A.23})$$

and substituting the ansatz (A.22) into Maxwell’s equations, we find $(n+1)$ independent equations:

$$\mathcal{M}^\mu = 0, \quad \text{where} \quad \mathcal{M}^\mu \equiv \frac{1}{\sqrt{-g}} \partial_i [\sqrt{-g} F^{i\mu}], \quad \mu = (t, \phi_i). \quad (\text{A.24})$$

It is convenient to look at projections of $\mathcal{M}^\mu$ on various basis vectors, and straightforward calculations give

$$\begin{aligned} [m_+^{(C)}]_\mu \mathcal{M}^\mu = & -\frac{2}{x_C} \partial_C \left[\frac{x_C^2}{d_C} \partial_C \left[\frac{H_C}{x_C} \tilde{S}_C \partial_C \Psi \right] \right] + \sum_i \frac{4x_C H_i \tilde{S}_i}{(x_C^2 - x_i^2) d_i} \partial_C \partial_i \Psi \\ & + \sum_i \frac{4x_i^2}{(x_C^2 - x_i^2) d_i} \partial_i \left[\frac{H_i}{x_i} \tilde{S}_i \partial_i \Psi \right] + \sum_i \frac{2\tilde{S}_C (x_C^2 - x_i^2)}{d_i} \partial_i \left[\frac{H_i}{x_i^2 - x_C^2} \partial_i \partial_C \Psi \right]. \end{aligned} \quad (\text{A.25})$$

We will focus on n scalar equations,

$$[m_+^{(C)}]_\mu \mathcal{M}^\mu = 0, \quad C = 1 \dots n,$$

and the projection $[m_+^{(0)}]_\mu \mathcal{M}^\mu$ can be analyzed in the same way. To use the differential equation (A.21), it is convenient to rewrite the derivatives appearing in the last term of (A.25) as

$$\partial_i \left[\frac{H_i}{x_i^2 - x_C^2} \partial_i \partial_C \Psi \right] = \left\{ \partial_i \frac{x_i}{\tilde{S}_i (x_i^2 - x_C^2)} \right\} \frac{\tilde{S}_i H_i}{x_i} \partial_C \partial_i \Psi + \frac{x_i}{\tilde{S}_i (x_i^2 - x_C^2)} \partial_i \left[\frac{\tilde{S}_i H_i}{x_i} \partial_i \partial_C \Psi \right].$$

Then equation (A.25) becomes

$$[m_+^{(C)}]_\mu \mathcal{M}^\mu = -\frac{2}{x_C} \partial_C \left[\frac{x_C^2}{d_C} \partial_C \left[\frac{H_C}{x_C} \tilde{S}_C \partial_C \Psi \right] \right] + \sum_i \frac{H_i \tilde{S}_i W_{C,i}}{d_i} \partial_C \partial_i \Psi \\ + \sum_i \frac{4x_i^2}{(x_C^2 - x_i^2) d_i} \partial_i \left[\frac{H_i}{x_i} \tilde{S}_i \partial_i \Psi \right] - \sum_i \frac{2\tilde{S}_C x_i}{d_i \tilde{S}_i} \partial_i \left[\frac{\tilde{S}_i H_i}{x_i} \partial_i \partial_C \Psi \right]. \quad (\text{A.26})$$

Here we defined

$$W_{C,i} = \frac{4x_C}{x_C^2 - x_i^2} + \frac{2\tilde{S}_C(x_C^2 - x_i^2)}{x_i} \left\{ \partial_i \frac{x_i}{\tilde{S}_i(x_i^2 - x_C^2)} \right\}. \quad (\text{A.27})$$

Substituting the separable ansatz (A.10) into equation (A.26), one arrives at several consistency conditions, in particular, all $W_{C,i}$ must vanish, and all ∂_i derivatives must disappear from the last line of (A.25). The latter condition leads to differential equations for functions X_i :

$$\partial_i \left[\frac{H_i}{x_i} \tilde{S}_i \partial_i \right] \Psi + P^{(i)}[x_i] \Psi = 0, \quad (\text{A.28})$$

while equations $W_{C,i} = 0$ allow one to determine the functions $\tilde{S}_i$. Specifically, defining $Q_i = \frac{1}{\tilde{S}_i}$, we find

$$W_{1,2} = \frac{2}{Q_1 x_2 (x_2^2 - x_1^2)} [-2x_1 x_2 Q_1 + (x_1^2 + x_2^2) Q_2 + x_2 (x_1^2 - x_2^2) Q_2'] = 0 \quad (\text{A.29})$$

Treating this relation as an algebraic equation for Q_1 , we find that

$$Q_i = c_1 x_i + \frac{c_2}{x_i} \quad (\text{A.30})$$

for $i = 1$. Equation $W_{2,1} = 0$ leads to a similar relation for Q_2 , and substituting the result into (A.29), we conclude that coefficients (c_1, c_2) can be arbitrary, but they must be the same for all values of i . This leads to the expressions for $\tilde{S}_i$ and to a more explicit form of equation (A.28):

$$\tilde{S}_i = \frac{x_i}{c_1 x_i^2 + c_2}, \quad \partial_i \left[\frac{H_i}{c_1 x_i^2 + c_2} \partial_i \right] \Psi + P^{(i)}[x_i] \Psi = 0. \quad (\text{A.31})$$

As expected, for an appropriate choice of coefficients (c_1, c_2) the last relation reproduces the ODE (A.21) describing the “radial” polarization. This is a consistency check for our construction since (A.31) and (A.21) emerge in the limit of the same equation (2.26) when one takes $\omega = n_k = 0$.

Let us now determine the properties of function $P^{(i)}[x_i]$ appearing in (A.31). Substitution of relations (A.31) into expression (A.25) gives

$$[m_+^{(C)}]_\mu \mathcal{M}^\mu = \frac{2}{x_C} \partial_C \left[\frac{x_C^2}{d_C} P^{(C)} \Psi \right] - \sum_i \frac{4x_i^2 P^{(i)}}{(x_C^2 - x_i^2) d_i} \Psi + \sum_i \frac{2\tilde{S}_C x_i P^{(i)}}{d_i \tilde{S}_i} \partial_C \Psi. \quad (\text{A.32})$$

The coefficient in front of $\partial_C \Psi$ vanishes if and only if

$$\frac{2x_C}{d_C} \frac{P^{(C)}}{\tilde{S}_C} + \sum_{i \neq C} \frac{2x_i P^{(i)}}{d_i \tilde{S}_i} = 0. \quad (\text{A.33})$$

Let us define a new set of functions $\tilde{P}_{n-1}^{(k)}$ by

$$\tilde{P}_{n-1}^{(k)}[x_k^2] \equiv \frac{x_k P^{(k)}}{\tilde{S}_k}. \quad (\text{A.34})$$

Rewriting equation (A.33) in terms of these functions and recalling the expression (2.9) for d_k , we conclude that $\tilde{P}_{n-1}^{(k)}$ must be a polynomial of degree $(n-1)$ in its argument. Furthermore, taking the limit $x_i \rightarrow x_C$ in (A.33), we find that $\tilde{P}_{n-1}^{(C)}[x^2] = \tilde{P}_{n-1}^{(i)}[x^2]$. Therefore, functions $\tilde{P}_{n-1}^{(k)}$ must be the same for all values of k , then

$$P^{(k)} = \frac{\tilde{S}_k}{x_k} \tilde{P}_{n-1}[x_k^2] \quad (\text{A.35})$$

Substitution of the last formula into equation (A.33) leads to a relation

$$\frac{\tilde{P}_{n-1}[x_C^2]}{d_C} + \sum_{i \neq C} \frac{\tilde{P}_{n-1}[x_i^2]}{d_i} = 0, \quad (\text{A.36})$$

which is identically satisfied for an arbitrary polynomial²⁸ $\tilde{P}_{n-1}$. Thus functions $P^{(k)}$ must have the form (A.35), and relation (A.32) reduces to

$$[m_+^{(C)}]_\mu \mathcal{M}^\mu = \frac{2}{x_C} \partial_C \left[\frac{x_C \tilde{S}_C}{d_C} \tilde{P}_{n-1}[x_C^2] \right] \Psi - \sum_i \frac{4x_i \tilde{S}_i \tilde{P}_{n-1}[x_i^2]}{(x_C^2 - x_i^2) d_i} \Psi. \quad (\text{A.37})$$

To avoid pole at $x_C^2 = -c_2/c_1$, we must require that $\tilde{P}_{n-1}[-c_2/c_1] = 0$, then relation (A.35) can be rewritten as

$$P^{(k)} = P_{n-2}[x_k^2], \quad (\text{A.38})$$

where $\tilde{P}_{n-2}$ is an arbitrary polynomial of degree $(n-2)$.

To summarize, we have demonstrated that the ansatz (A.23), (A.10) solves Maxwell's equations if and only if

$$\tilde{S}_i = \frac{x_i}{c_1 x_i^2 + c_2}, \quad \partial_i \left[\frac{H_i}{c_1 x_i^2 + c_2} \partial_i X_i \right] + P_{n-2}[x_k^2] X_i = 0. \quad (\text{A.39})$$

As expected, this “angular” polarization and its “radial” counterpart (A.21) lead to the same system of ODEs for $X_k = (R, X_a)$. Although the derivation of (A.39) turned out to be somewhat tedious, we note that the main results pertaining to the Maxwell field, equations (A.21) and (A.18), were obtained using very transparent calculations, and it is

²⁸To prove (A.36), one observes the left-hand side is a meromorphic function of all x_k and shows that residues at all poles, including infinity, vanish. More details can be found in the appendix E of [73].

this new derivation that will be easily generalized to higher forms in the remaining part of this appendix.

We conclude the discussion of the Maxwell field by making comments about odd dimensions. Substituting the ansatz (A.4) into Maxwell's equations, we arrive at a counterpart of (A.9):

$$\sum_{j \neq i} \frac{x_j(x_i^2 - x_j^2)}{d_j} \partial_j \left[\frac{H_j(S_j - S_i)}{x_j(x_i^2 - x_j^2)} \partial_j \partial_i \Psi \right] = 0, \quad i = (r, x_a). \quad (\text{A.40})$$

Additional factors of x_j are caused by different expressions for the frames and the determinant of the metric. The rest of the derivation proceeds as before, leading to (A.18) and the counterpart of (A.21):

$$x_j \partial_j \left[\frac{H_j}{x_j(\gamma x_j^2 + \delta)} \partial_j X_j \right] + P_{n-2}(x_j^2) X_j = 0. \quad (\text{A.41})$$

For the angular polarization we again find (A.22) and the expression for $\tilde{S}_i$ from (A.39). Furthermore, the static configurations (A.1) must have $n^\mu A_\mu = 0$.

Although all results of this subsection have already been obtained in [73], the new derivation presented here can be easily extended to higher forms, and such generalizations are carried out in the next two subsections.

A.2 Two-form potential

Let us now extend the construction presented in the last subsection to the two-form potential. As in the case of Maxwell's field, we will look for separable solutions of linear equations $d \star dA = 0$ in the absence of angular dependence. We will mostly focus on the x -components of the gauge field, but as we explicitly saw in the last subsection, the resulting equations would apply to other components as well. The verification of this fact is an straightforward extension of the arguments presented in the appendix A.1.2, although the intermediate formulas become rather complicated.

We begin with looking at x -components of a two-form field in even dimensions and imposing a natural generalization of the ansatz (A.4)²⁹

$$A_{ij} = S_{ij}(x_i, x_j) \partial_i \partial_j \Psi. \quad (\text{A.42})$$

Evaluating the field strength,

$$F_{ijk} = (S_{jk} - S_{ik} + S_{ij}) \partial_i \partial_j \partial_k \Psi \equiv \mathcal{S}_{ijk} \partial_i \partial_j \partial_k \Psi, \quad F^{ijk} = \frac{H_i H_j H_k}{d_i d_j d_k} \mathcal{S}_{ijk} \partial_i \partial_j \partial_k \Psi, \quad (\text{A.43})$$

and substituting the result into equation $d \star F = 0$, we arrive at a counterpart of equation (A.8):

$$0 = \frac{1}{\sqrt{\prod d_p}} \sum_j \partial_j \left[\sqrt{\prod d_p} \frac{H_i H_j H_k}{d_i d_j d_k} \mathcal{S}_{ijk} \partial_i \partial_j \partial_k \Psi \right] = \sum_j \frac{H_i H_k}{d_j} \partial_j \left[\frac{H_j}{d_i d_k} \mathcal{S}_{ijk} \partial_j \partial_i \partial_k \Psi \right].$$

²⁹Recall that in this appendix indices (i, j, k) cover n coordinates x_a , as well as $x_0 = -ir$.

Simplifications of the last relation leads to a system of equations labeled by indices i and k :

$$\sum_{j \neq i, k} \frac{(x_i^2 - x_j^2)(x_k^2 - x_j^2)}{d_j} \partial_j \left[\frac{H_j \mathcal{S}_{ijk}}{(x_i^2 - x_j^2)(x_k^2 - x_j^2)} \partial_j \partial_i \partial_k \Psi \right] = 0 \quad (\text{A.44})$$

Imposing a separable ansatz for function Ψ ,

$$\Psi = X_r(r) \prod X_a(x_a), \quad (\text{A.45})$$

one arrives at a system of differential equations

$$\partial_j \left[\frac{H_j \mathcal{S}_{ijk}}{(x_i^2 - x_j^2)(x_k^2 - x_j^2)} \partial_j X_j \right] + f^{(j;ik)}(x_j; x_i, x_k) X_j = 0.$$

The last relation should become a second-order ODE for $X_j(x_j)$, which means that, upon multiplication by an overall function, the coefficients in front of $(\partial_j^2 X_j, \partial_j X_j, X_j)$ must become functions of x_j only. In particular, this implies the following separation of variables

$$\frac{\mathcal{S}_{ijk}}{(x_i^2 - x_j^2)(x_k^2 - x_j^2)} = g(x_j) G(x_i, x_k).$$

Recalling the definition of $\mathcal{S}_{ijk}$ given by (A.43), the last relation can be written in a more symmetric form

$$\frac{S_{jk} - S_{ik} + S_{ij}}{(x_i^2 - x_j^2)(x_k^2 - x_i^2)(x_k^2 - x_j^2)} = g(x_i)g(x_j)g(x_k). \quad (\text{A.46})$$

Although a priori we should have used $g^{(i,j,k)}(x)$ instead of $g(x)$ in the last equation, the final ODE (A.48) ensures that all $g^{(i,j,k)}(x)$ must be the same, so we avoided unnecessary complications already in (A.46). The functional equation (A.46), a counterpart of (A.16), is the main consistency condition of the separable ansatz for the two-form potential.

Substitution of (A.46) into the field equations (A.44) gives

$$\sum_{j \neq i, k} \frac{(x_i^2 - x_j^2)(x_k^2 - x_j^2)}{d_j} \partial_j [H_j g(x_j) \partial_j \partial_i \partial_k \Psi] = 0. \quad (\text{A.47})$$

Imposing the separable ansatz (A.45) and recalling that d_j is a polynomial of degree $2n$, we arrive at the system of ordinary differential equations

$$\partial_j [H_j g(x_j) \partial_j] X_j + P_{n-3}[x_j^2] X_j = 0, \quad (\text{A.48})$$

where P_{n-3} is a polynomial of degree $(n-3)$.³⁰ In particular, for $n = 2$ equation (A.48) does not contain the last term. Furthermore, for $n = 1$, which corresponds to four dimensions, the logic outlined here breaks down, and this degenerate case is discussed in the end of section 3.1.

³⁰To see this, we observe that, upon multiplication of (A.47) by $\frac{d_m}{(x_i^2 - x_m^2)(x_k^2 - x_m^2)}$ the terms that do not contain ∂_m become polynomials of $(n-3)$ in x_m^2 .

Let us now solve the functional equations (A.46) to determine functions $g(x)$ and $S_{ij}(x_i, x_j)$. Introducing new variables $y_k = x_k^2$ and defining $\tilde{g}_k = \tilde{g}(y_k) \equiv g(\sqrt{y_k})$, we can write some integrability conditions of (A.46) as (no summations)

$$\partial_{y_i} \partial_{y_j} \partial_{y_k} \left[(y_i - y_j)(y_i - y_k)(y_j - y_k) \tilde{g}_i \tilde{g}_j \tilde{g}_k \right] = 0. \quad (\text{A.49})$$

Treating this relation as a first-order differential equation for $\tilde{g}_i$, we find

$$\tilde{g}(y) = \frac{1}{ay^2 + by + c} \quad (\text{A.50})$$

with some constants (a, b, c) . Substitution of this relation back into (A.49) yields an identity. Let us now look at another integrability condition of (A.46):

$$\begin{aligned} \partial_{y_i} \partial_{y_j} S_{ij} &= \partial_{y_i} \partial_{y_j} \left[(y_i - y_j)(y_i - y_k)(y_j - y_k) \tilde{g}_i \tilde{g}_j \tilde{g}_k \right] \\ &= \frac{(y_i - y_j)[2ay_i y_j + b(y_i + y_j) + 2c]}{[ay_i^2 + by_i + c]^2 [ay_j^2 + by_j + c]^2} \end{aligned} \quad (\text{A.51})$$

Integration of this relation gives

$$S_{ij} = \frac{y_i - y_j}{a[ay_i^2 + by_i + c][ay_j^2 + by_j + c]} + A_{ij}(y_i) + B_{ij}(y_j) + C_{ij}.$$

Substitution of this formula into equation (A.46) produces restrictions on (A_{ij}, B_{ij}) :

$$B_{ij}(y) = -A_{ij}(y), \quad \partial_{y_i} [A_{ij}(y_i) - A_{ik}(y_i)] = 0 \quad \Rightarrow \quad A_{ij}(y) = A(y) + \tilde{C}_{ij}. \quad (\text{A.52})$$

This leads to the final formula for the most general S_{ij} :

$$S_{ij} = \frac{y_i - y_j}{a[ay_i^2 + by_i + c][ay_j^2 + by_j + c]} + A(y_i) - A(y_j) + C_{ij}, \quad (\text{A.53})$$

where constants C_{ij} satisfy a system of homogeneous equations

$$C_{jk} - C_{ik} + C_{ij} = 0. \quad (\text{A.54})$$

With this assignment the functional equation (A.46) becomes an identity.

To summarize, we have found the most general solution of the functional equation (A.46):

$$g(x_i) = \frac{1}{Q_i}, \quad S_{ij} = \frac{x_i^2 - x_j^2}{aQ_i Q_j} + A(x_i^2) - A(x_j^2) + C_{ij}, \quad Q_j = ax_j^4 + bx_j^2 + c. \quad (\text{A.55})$$

Here the set of constants $\{C_{ij}\}$ satisfies the relations (A.54). Note that these constants, as well as $A(y)$, don't contribute to S_{ijk} and to the field strength, so the physical degrees of freedom are described by a simpler version of equation (A.55):

$$S_{ij} = \frac{x_i^2 - x_j^2}{aQ_i Q_j}. \quad (\text{A.56})$$

To turn on (ω, n_i) , we would need a slight generalization of the last relation. Using (A.56) as an inspiration, we require a multiplicative separation in S_{ij} :

$$S_{ij} = (x_i^2 - x_j^2)f(x_i)f(x_j). \quad (\text{A.57})$$

Then consistency with (A.55),

$$\partial_i \partial_j \left[S_{ij} - \frac{x_i^2 - x_j^2}{aQ_i Q_j} \right] = 0, \quad (\text{A.58})$$

leads to differential equations for function f , and the most general solution is³¹

$$f(x_j) = \frac{px_j^2 + q}{ax_j^4 + bx_j^2 + c}. \quad (\text{A.59})$$

We conclude that the most general *separable* expression for (g, S_{ij}) is a special case of (A.55):

$$g(x_i) = \frac{1}{Q_i}, \quad S_{ij} = (x_i^2 - x_j^2)f(x_i)f(x_j), \quad Q_j = ax_j^4 + bx_j^2 + c, \quad f(x_j) = \frac{px_j^2 + q}{Q_j}. \quad (\text{A.60})$$

In section 3 we demonstrate that this ansatz works for nontrivial (ω, n_i) as well.

To summarize, we have shown that the most general “radial” polarization (A.42) with separable function Ψ leads to a system of ODEs (A.48). Furthermore, S_{ij} and g must be given by (A.60).

One can also analyze the “angular” polarization by modifying the relevant parts of section A.1.2. This logic of such extension from the Maxwell field to the two-form potential is rather straightforward, but the intermediate formulas are somewhat messy, so here we just quote the result. Starting from an ansatz for the cyclic components³² of the gauge field A_{ab} ,

$$\sum_{a,b} [m_\alpha^{(I)}]^a [m_\beta^{(J)}]^b A_{ab} = \alpha\beta \tilde{S}_{IJ} \sum_{i,j} [m_+^{(I)}]^i [m_+^{(J)}]^j \partial_i \partial_j \Psi, \quad (\text{A.61})$$

with a separable function Ψ , (A.45), and substituting this A_{ab} into the field equations, we find the system of ODEs (A.48) with

$$g(x_i) = \frac{1}{Q_i^{(2)}}, \quad \tilde{S}_{IJ} = \frac{(x_I^2 - x_J^2)x_I x_J}{Q_I^{(2)} Q_J^{(2)}}, \quad Q_j^{(2)} = ax_j^4 + bx_j^2 + c. \quad (\text{A.62})$$

Although a priori polynomials Q_j and $Q^{(2)}$ in (A.62) and (A.60) have different coefficients, we are interested in obtaining polarization (A.42) and (A.61) as degenerate cases of solutions with nonzero (ω, m_i) . Then these two configurations must be described by the

³¹Strictly speaking, relation (A.58) imposes an additional linear constraint on (p, q) . However, this constraint can be eliminated by a rescaling of function g and polynomial P_{n-3} that introduces a multiplicative constant in the right-hand side of equation (A.46) without affecting the ODE (A.48). Thus we can keep four free parameters (a, b, c, p, q) in (A.59).

³²Recall that for cyclic components A_{ab} , indices (a, b) take values (t, ϕ_i) .

same function Ψ , and differential equation (A.48) ensures that relations (A.62) and (A.60) contain the same parameters (a, b, c) .

Finally, for the mixed polarization, we start with the ansatz

$$\sum_{\mu} [m_{\alpha}^{(I)}]^a A_{a\mu} = \alpha \hat{S}_{IJ} \sum_i [m_+^{(I)}]^i \partial_i \partial_j \Psi, \quad (\text{A.63})$$

and determine functions $\hat{S}_{IJ}$ by applying the arguments of section A.1.2. Once again, the calculations are straightforward but tedious, so we just present the final result:

$$\sum_{\mu} [m_{\alpha}^{(I)}]^a A_{a\mu}^{(3)} = \alpha \frac{(x_I^2 - x_J^2) x_I (\tilde{p} x_i^2 + \tilde{q})}{Q_I^{(3)} Q_J^{(3)}} \sum_i [m_+^{(I)}]^i \partial_i \partial_j \Psi. \quad (\text{A.64})$$

Function Ψ is given by (A.45), and its ingredients satisfy a system of differential equations (A.48) with

$$g(x_i) = \frac{1}{Q_i^{(3)}}, \quad Q_j^{(3)} = ax_j^4 + bx_j^2 + c. \quad (\text{A.65})$$

A priori the coefficients (a, b, c) appearing in the last relation are not the same as parameters in (A.62) and (A.60), but if all three polarizations originate from the same “master function” Ψ , then the differential equation (A.48) ensures that all three Q -factors, (A.62), (A.60), (A.65), are built from the same set (a, b, c) . Combining polarizations (A.42), (A.61), and (A.64), and requiring the result to be separable, we find

$$\begin{aligned} \sum_{\mu, \nu} [m_{\alpha}^{(I)}]^{\mu} [m_{\beta}^{(J)}]^{\nu} A_{\mu\nu} &= (x_I^2 - x_J^2) h_{\alpha}^I h_{\beta}^J \sum_{i,j} [m_+^{(I)}]^i [m_+^{(J)}]^j \partial_i \partial_j \Psi, \\ h_{\pm}^I &= \frac{px_I^2 + q \pm ex_I}{ax_I^4 + bx_I^2 + c}. \end{aligned} \quad (\text{A.66})$$

This ansatz is the starting point for our discussion of separable two-form potentials in section 3. The analysis of odd dimensions follows the same route, and the final result is presented in section 3.2.

A.3 Higher forms

Let us now extend the results obtained in the last subsection to p -form potentials with $p > 2$. We will mostly focus on the “radial polarization” (A.42) in even dimensions, and comment on the counterparts of (A.61) and (A.64) in the end of this subsection. The extension to the odd-dimensional case is straightforward, and it is discussed in sections 4.2 and 6.

The generalization of the ansatz (A.42), (A.45) is

$$A_{i_1 \dots i_p} = S_{i_1 \dots i_p}(x_{i_1}, \dots, x_{i_p}) \partial_{i_1} \dots \partial_{i_p} \Psi, \quad \Psi = X_r(r) \prod X_a(x_a). \quad (\text{A.67})$$

Here $S_{i_1 \dots i_p}$ is antisymmetric its indices, and we can also introduce an antisymmetric tensor $S_{i_1 \dots i_{p+1}}$ by generalizing the definition (A.43):

$$S_{i_1 \dots i_{p+1}} = S_{i_2 \dots i_{p+1}} - S_{i_1 i_3 \dots i_{p+1}} + S_{i_1 i_2 i_4 \dots i_{p+1}} - \dots \quad (\text{A.68})$$

Then the field equations, $d \star dA = 0$, lead to a counterpart of the system (A.44):

$$\sum_{j \neq i_1, \dots, i_p} \frac{\prod_k (x_{i_k}^2 - x_j^2)}{d_j} \partial_j \left[\frac{H_j \mathcal{S}_{j i_1 \dots i_p}}{\prod_k (x_{i_k}^2 - x_j^2)} \partial_j \partial_{i_1} \dots \partial_{i_p} \Psi \right] = 0 \quad (\text{A.69})$$

This gives one equation for every set of indices $(i_1, \dots, i_p)$. Imposing the separable ansatz (A.67) for Ψ , and repeating manipulations that led to (A.46) and (A.47), we can rewrite the system (A.69) as

$$\sum_{j \neq i_1, \dots, i_p} \frac{\prod_k (x_{i_k}^2 - x_j^2)}{d_j} \partial_j [H_j g(x_j) \partial_j \partial_{i_1} \dots \partial_{i_p} \Psi] = 0, \quad (\text{A.70})$$

where tensor $\mathcal{S}_{i_1 \dots i_{p+1}}$ and function g are subject to constraints

$$\mathcal{S}_{i_1 \dots i_{p+1}} \prod_{k < l} \frac{1}{(x_{i_k}^2 - x_{i_l}^2)} = \prod_{k=1}^{p+1} g(x_{i_k}). \quad (\text{A.71})$$

This functional equation for $\mathcal{S}_{i_1 \dots i_p}$ and g is a generalization of the constraints (A.16) and (A.46) encountered for the one- and two-forms. As before, equations (A.70) with separable function Ψ lead to a system of ODEs

$$\partial_j [H_j g(x_j) \partial_j] X_j + P_{n-p-1}[x_j^2] X_j = 0, \quad (\text{A.72})$$

and all these equations contain *the same* P_{n-p-1} , a polynomial of degree $(n-p-1)$ in its argument. Equations (A.72) apply only for $n \geq (p+1)$, and degenerate cases of small n can be analyzed separately.

Function $g(x)$ should be determined by solving the functional equation (A.71). The definition (A.68) implies that

$$\partial_{i_1} \dots \partial_{i_{p+1}} \mathcal{S}_{i_1 \dots i_{p+1}} = 0 \quad (\text{no summations}), \quad (\text{A.73})$$

leading to an integrability condition for the system (A.71):

$$\partial_{i_1} \dots \partial_{i_{p+1}} \left\{ \left[\prod_{k < l} (x_{i_k}^2 - x_{i_l}^2) \right] \prod_{k=1}^{p+1} g(x_{i_k}) \right\} = 0 \quad (\text{no summations}). \quad (\text{A.74})$$

Treating the last relation as a first order ODE for the function $g(x_{i_1})$, we find the most general solution of (A.74):

$$g(x_j) = \frac{1}{Q_j}, \quad Q_j = \sum_{s=0}^p e_s x_j^{2s}. \quad (\text{A.75})$$

Substituting (A.75) into the functional equation (A.71) and solving for $\mathcal{S}_{i_1 \dots i_p}$, we find³³

$$\mathcal{S}_{i_1 \dots i_p} = \frac{1}{e_p} \left[\prod_k \frac{1}{Q_{i_k}} \right] \prod_{k < l} [x_{i_k}^2 - x_{i_l}^2] + \hat{\mathcal{S}}_{i_1 \dots i_p}, \quad (\text{A.76})$$

³³To derive this, one can follow the steps that led from (A.50) to (A.53).

where $\hat{S}_{i_1 \dots i_p}$ satisfies the homogeneous equation

$$\hat{S}_{i_1 \dots i_p} - \hat{S}_{i_1 \dots i_{p-1} i_{p+1}} + \dots = 0. \quad (\text{A.77})$$

Taking various derivatives of this relation and using symmetries of the tensor $\hat{S}$, we find the most general solution of (A.77):

$$\hat{S}_{i_1 \dots i_p}(x_{i_1}, \dots, x_{i_p}) = [A(x_{i_1}, \dots, x_{i_{p-1}}) + \text{permutations}] + C_{i_1 \dots i_p}. \quad (\text{A.78})$$

Here $C_{i_1 \dots i_p}$ are arbitrary constants satisfying the homogeneous equation

$$C_{i_1 \dots i_p} - C_{i_1 \dots i_{p-1} i_{p+1}} + \dots = 0. \quad (\text{A.79})$$

To summarize, we have shown that the most general solution of the functional equation (A.71) is given by (A.75), (A.76), (A.78), (A.79), and that the system of ODEs governing the dynamics of p -forms is given by (A.72).

Using the separable ansatz (A.57) as an inspiration, we focus on a special case of (A.76):

$$S_{i_1 \dots i_p} = \frac{1}{e_p} \left[\prod_k^p f(x_{i_k}) \right] \prod_{k < l}^p [x_{i_k}^2 - x_{i_l}^2]. \quad (\text{A.80})$$

We have already found an example that corresponds to $\hat{S}_{i_1 \dots i_p} = 0$:

$$f(x_{i_k}) = \frac{1}{Q_{i_k}}, \quad (\text{A.81})$$

and now we will determine the most general function f that solves equation (A.71). Since expressions $S_{i_1 \dots i_p}$ must have the form (A.76) with (A.78), we conclude that the ansatz (A.80) solving equation (A.71) must satisfy an integrability condition

$$\partial_{i_1} \dots \partial_{i_p} \left\{ \left[\prod_k^p f(x_{i_k}) - \prod_k^p \frac{1}{Q_{i_k}} \right] \prod_{k < l}^p [x_{i_k}^2 - x_{i_l}^2] \right\} = 0. \quad (\text{A.82})$$

Viewing the left-hand side of this relation as a holomorphic function of x_{i_1} , we conclude that the poles of $f(x_{i_1})$ must coincide with zeroes of Q_{i_1} . Furthermore, function f must have the form

$$f(x_i) = \frac{T[x_i^2]}{Q_i}, \quad (\text{A.83})$$

where $T[x_i^2]$ is a polynomial in $y_i \equiv (x_i)^2$. Rewriting equation (A.82) in terms of variables y_i , we find

$$\partial_{y_{i_1}} \dots \partial_{y_{i_p}} \left\{ \left[\prod_k^p T[y_{i_k}] - 1 \right] \left[\prod_k^p \frac{1}{Q_{i_k}} \right] \prod_{k < l}^p [y_{i_k} - y_{i_l}] \right\} = 0. \quad (\text{A.84})$$

Let us now demonstrate that $T[y_{i_k}]$ must be either a linear polynomial or a constants.

We begin with factorizing the polynomial Q :

$$Q[y] = e_p \prod_{s=1}^p (y - c_s). \quad (\text{A.85})$$

To simplify the discussion, we assume that all roots of Q are distinct, but the extension to degenerate roots is straightforward. In the vicinity of $y_{i_p} = c_p$, the leading contribution to the left-hand side of (A.84) is proportional to $(y_{i_p} - c_p)^{-2}$, and equation (A.84) can hold only if the residue at this pole vanishes:

$$\partial_{y_{i_1}} \dots \partial_{y_{i_{p-1}}} \left\{ \left[T[c_p] \prod_k^{p-1} T[y_{i_k}] - 1 \right] \left[\prod_k^{p-1} \frac{y_{i_k} - c_p}{Q_{i_k}} \right] \prod_{k < l}^{p-1} [y_{i_k} - y_{i_l}] \right\} = 0. \quad (\text{A.86})$$

Rescaling the coefficients of the polynomial $T[y]$ and recalling the factorization (A.85), we can rewrite the relation (A.86) as

$$\partial_{y_{i_1}} \dots \partial_{y_{i_{p-1}}} \left\{ \left[\prod_k^{p-1} \tilde{T}[y_{i_k}] - 1 \right] \left[\prod_k^{p-1} \frac{1}{\tilde{Q}_{i_k}} \right] \prod_{k < l}^{p-1} [y_{i_k} - y_{i_l}] \right\} = 0, \quad (\text{A.87})$$

where

$$\tilde{Q}_{i_k} = \prod_{s=1}^{p-1} (y_{i_k} - c_s). \quad (\text{A.88})$$

Relation (A.87) is a *necessary condition* for (A.84) to hold, but it has the same form as (A.84) with p replaced by $(p-1)$. Thus polynomial T must have the same degree for p - and $(p-1)$ -forms, and by induction, we conclude that the degree of T is the same for all forms. In section A.2 we demonstrated that for $p=2$, the polynomial T must be linear (see equation (A.59)), so for arbitrary p the degree of T cannot exceed one. It turns out that there are no additional restrictions on T : a direct check shows that the functional equation (A.71) is identically satisfied for any $S_{i_1 \dots i_p}$ given by (A.80) and (A.83) with an arbitrary linear polynomial³⁴ T . To summarize, we have found that the most general separable solution for $S_{i_1 \dots i_p}$ is a counterpart of (A.60)

$$g(x_i) = \frac{1}{Q_i}, \quad S_{i_1 \dots i_p} = \left[\prod_j^p \frac{g_0 + g_1 x_j^2}{Q_j} \right] \prod_{k < l}^p [x_{i_k}^2 - x_{i_l}^2], \quad Q_j = \sum_{k=0}^p b_k x_j^{2k}, \quad (\text{A.89})$$

This concludes our discussion of the “non-cyclic polarization” (A.67) for the p -form fields.

In the absence of angular dependence (A.1), field equations $d \star dA = 0$ split into sectors with a fixed number of cyclic indices. So far we have focused on the non-cyclic components of the gauge field, and we derived the most general separable form of the solution (A.89).

³⁴To keep arbitrary coefficients in T , one may have to rescale function $g(x)$, as discussed in the footnote on page 40.

Other sectors can be analyzed in a similar fashion, for example, the result for the sector with one cyclic index reads

$$\sum_a e_l^a A_{i_1 \dots i_{p-1} a} = \sum_{i_p} \left[\prod_{k < l}^p [x_{i_k}^2 - x_{i_l}^2] \right] \left[\prod_k^{p-1} \frac{T[x_{i_k}^2]}{Q_{i_k}} \right] \frac{x_{i_p}}{Q_{i_p}} e_{x_l}^{i_p} \partial_{i_1} \dots \partial_{i_p} \Psi. \quad (\text{A.90})$$

The frames $(e_j^a, e_{x_j}^{i_p})$ appearing here are given by (2.8). Note that for a given value of j , the summation in the right-hand side of (A.90) contains only one term. Function Ψ still has the form

$$\Psi = X_r(r) \prod X_a(x_a), \quad (\text{A.91})$$

and the dynamical equations $d \star dA = 0$ reduce to the system of ODEs (A.72). Analyzing other sectors in the same fashion and combining the results, we arrive at a counterpart of the ansatz (A.66):

$$\begin{aligned} & \sum_{\mu_i} [m_{\alpha_1}^{(I_1)}]^{\mu_1} \dots [m_{\alpha_p}^{(I_p)}]^{\mu_p} A_{\mu_1 \dots \mu_p} \\ &= \left[\prod_{k < l}^p [x_{i_k}^2 - x_{i_l}^2] \right] \left[\prod_k^p h_{\alpha_k}^{I_k} \right] \sum_{i_k} [m_+^{(I_1)}]^{i_1} \dots [m_+^{(I_p)}]^{i_p} \partial_{i_1} \dots \partial_{i_p} \Psi, \end{aligned} \quad (\text{A.92})$$

Coefficients $h_{\alpha_k}^{I_k}$ are rational functions of x_J :

$$h_{\pm}^J = \frac{g_0 + g_1 x_J^2 \pm g_2 x_J}{Q_J}, \quad Q_J = \sum_{k=0}^p a_k x_J^{2k}. \quad (\text{A.93})$$

As demonstrated in sections 4, 5 and 6, the ansatz (A.92) can be easily extended to configurations with nontrivial (ω, n_i) , although we managed to derive the resulting ODEs only on a case-by-case basis.

To summarize, in this appendix we have derived the most general separable ansatz for arbitrary p -form fields in the Myers-Perry geometry. Our final result is summarized by (A.92)–(A.93), and differential equations produced by this ansatz are discussed in the main body of this article.

B Divergence of the two-form potential

In the appendix A we derived the separable ansatz for the p -forms and outlined the derivation of the resulting ordinary differential equations for the “master field” Ψ . In the presence of angular dependence, the analysis of the equation

$$d \star dA^{(p)} = 0 \quad (\text{B.1})$$

was somewhat messy even in the simplest case of the Maxwell field, i.e., for $p = 1$. After such a derivation had been performed in [73], the authors of [74, 75] observed that the resulting field $A^{(1)}$ satisfied the Lorenz condition

$$\nabla_\mu [A^{(1)}]^\mu = 0. \quad (\text{B.2})$$

Then by imposing the ansatz (2.24) introduced in [73] along with the Lorenz gauge (B.2), the authors of [74, 75] found an elegant shortcut for arriving at (2.26), (2.31) which simplified the original derivation of [73]. In this appendix we explain the origin of this shortcut and demonstrate that, unfortunately, it cannot be directly extended to higher forms.

Let us begin with the discussion of the one-form $A \equiv A^{(1)}$ in even dimensions. The ansatz (2.24),

$$[m_{\pm}^{(I)}]^{\mu} A_{\mu} = \mp \frac{i}{x_I \pm \mu} [m_{\pm}^{(I)}]^{\mu} \partial_{\mu} \Psi, \quad x_0 = -ir, \quad (\text{B.3})$$

fully fixes the gauge, so one cannot impose the Lorenz condition (B.2) *in addition* to (B.3). However, as we will now demonstrate, any separable solution (B.3) of the Maxwell's equations $d \star dA = 0$ *automatically* satisfies the condition (B.3). We will later see that this property does not extend to higher forms, so the Lorenz gauge

$$\nabla_{\mu} [A^{(p)}]^{\mu\nu_1 \dots \nu_{p-1}} = 0 \quad (\text{B.4})$$

is inconsistent with the dynamical equations (B.1) for the separable configurations (6.2)–(6.3) with $p > 1$. To avoid unnecessary complications, we will focus on a special class of solutions

$$\Psi = E\Phi(r) \left[\prod X_i(x_i) \right], \quad E = 1, \quad (\text{B.5})$$

although the conclusion about satisfaction or failure of the Lorenz gauge holds in the presence of (t, ϕ_i) dependence as well.

In the special case (B.5), the non-cyclic components of the Maxwell field (B.3) become³⁵

$$A_j = \frac{1}{2} \left[-\frac{i}{x_j + \mu} + \frac{i}{x_j - \mu} \right] \partial_j \Psi = \frac{i\mu}{x_j^2 - \mu^2} \partial_j \Psi. \quad (\text{B.6})$$

Substitution into the left-hand side of (B.2) gives

$$\nabla_{\mu} A^{\mu} = \frac{1}{\sqrt{-g}} \partial_j \left[\sqrt{-g} g^{jk} A_k \right] = \frac{1}{FR} \partial_r \left[FR \frac{\Delta}{FR} A_r \right] + \sum_{j=1}^n \frac{1}{d_j} \partial_j \left[d_j \frac{H_j}{d_j} A_j \right]. \quad (\text{B.7})$$

Here we used the expressions for g^{ij} from (2.12), as well as equation (A.7) for the determinant of the metric. As we have shown in the appendix A, the Maxwell's equations for the separable configurations (B.3) are equivalent to the system of ODEs (2.26), then in the present case $\omega = n_i = 0$ we find (no summation)

$$\begin{aligned} \partial_j \left[d_j \frac{H_j}{d_j} A_j \right] &= \frac{\partial}{\partial x_j} \left[\frac{i\mu H_j}{x_j^2 - \mu^2} \frac{\partial \Psi}{\partial x_j} \right] = -\frac{i\mu}{\mu^2} \frac{\partial}{\partial x_j} \left[\frac{H_j}{D_j} \frac{\partial \Psi}{\partial x_j} \right] = \frac{i}{\mu} P_{n-2}[x_j^2] \Psi, \quad j = 1, \dots, n; \\ \partial_r \left[FR \frac{\Delta}{FR} A_r \right] &= \frac{\partial}{\partial r} \left[\frac{i\mu \Delta}{r^2 + \mu^2} \frac{\partial \Psi}{\partial r} \right] = \frac{i\mu}{\mu^2} \frac{\partial}{\partial r} \left[\frac{\Delta}{D_r} \frac{\partial \Psi}{\partial r} \right] = \frac{i}{\mu} P_{n-2}[-r^2] \Psi. \end{aligned}$$

³⁵As in the appendix A, here we use the notation $A_j = (A_r, A_{x_a})$.

Substitution into (B.7) gives

$$\nabla_\mu A^\mu = \frac{i}{\mu} \left[\frac{P_{n-2}[-r^2]}{FR} + \sum_{j=1}^n \frac{P_{n-2}[x_j^2]}{d_j} \right].$$

As we saw in the appendix A.1.1, the right-hand side of the last equation vanishes identically, so we have shown that any separable solution (B.3), (B.5) of the Maxwell's equations satisfies the Lorenz gauge

$$\nabla_\mu A^\mu = 0.$$

The conclusion holds even for a more general function E given in (2.19), but the proof requires longer algebraic manipulations, and we will not present them here.

Let us now look at the separable two-form potential (3.18):

$$m_\alpha^{I\mu} m_\beta^{J\nu} A_{\mu\nu} = (x_I^2 - x_J^2) h_\alpha^I h_\beta^J m_\alpha^{I\mu} m_\beta^{J\nu} \partial_\mu \partial_\nu \Psi, \quad h_\pm^I = \frac{1}{e_1 + e_2 x_I^2 \pm i e_3 x_I}. \quad (\text{B.8})$$

Once again we will focus on the special configurations whose “master function” has the form (B.5). The non-cyclic components of the gauge field are given by

$$A_{jk} = (x_j^2 - x_k^2) f_j f_k \partial_j \partial_k \Psi, \quad f_j = \frac{e_1 + e_2 x_j^2}{(e_1 + e_2 x_j^2)^2 + e_3^2 x_j^2}. \quad (\text{B.9})$$

Following the steps used to evaluate $\nabla_\mu A^\mu$, we find the non-cyclic components of the divergence of the two-form:

$$\begin{aligned} \frac{1}{\sqrt{-g}} \partial_j \left[\sqrt{-g} g^{jj'} g^{kk'} A_{j'k'} \right] &= \frac{1}{FR} \partial_r \left[FR \frac{\Delta}{FR} \frac{H_k}{d_k} A_{rk} \right] + \sum_{j=1}^n \frac{1}{d_j} \partial_j \left[d_j \frac{H_j}{d_j} \frac{H_k}{d_k} A_{jk} \right] \\ &= \frac{(-r^2 - x_k^2) f_k H_k}{d_k FR} \partial_r [f_r \Delta \partial_r] \partial_k \Psi + \sum_{j=1}^n \frac{(x_j^2 - x_k^2) f_k H_k}{d_j d_k} \partial_j [H_j f_j \partial_j] \partial_k \Psi \\ &= \frac{2e_2 r(r^2 + x_k^2) f_k H_k \Delta}{d_k FR D_r} \partial_r \partial_k \Psi + \sum_{j=1}^n \frac{2e_2 x_j (x_j^2 - x_k^2) f_k H_k H_j}{d_j d_k D_k} \partial_j \partial_k \Psi. \end{aligned}$$

Here we used the ODEs (3.19) for function Ψ , as well as identities (A.9)–(A.11) for the polynomial P_{n-2} . The last equation can be summarized as

$$\nabla_j A^{jk} = \frac{2e_2 r(r^2 + x_k^2) f_k H_k \Delta}{d_k FR D_r} \partial_r \partial_k \Psi + \sum_{j=1}^n \frac{2e_2 x_j (x_j^2 - x_k^2) f_k H_k H_j}{d_j d_k D_k} \partial_j \partial_k \Psi. \quad (\text{B.10})$$

Although this expression is very compact, it is clear that the two-form does not satisfy the Lorenz condition (B.4). The cyclic components of the divergence can be computed in a similar fashion, and the result reads

$$\nabla_j A^{jb} = \sum_{j < k} \frac{2e_2 e_3 a_b x_j x_k (x_j^2 - x_k^2)^2 H_j H_k}{(a_b^2 - x_j^2)(a_b^2 - x_k^2) D_j D_k d_j d_k} \partial_j \partial_k \Psi \quad (\text{B.11})$$

Once again, this expression does not vanish.

The expressions (B.10), (B.11) can be extended to configurations with nontrivial (t, ϕ_i) dependence and generalized to higher forms, and although the explicit formulas are more complicated, the conclusion remains the same: the separable configurations (6.2) satisfying the equations of motion (6.1) do not belong to the Lorenz gauge (B.4) for any $p > 1$. Thus existence of an elegant shortcut for deriving the ODEs (2.26) proposed in [74, 75] seems to be a peculiarity associated with Maxwell's equations, and for higher forms one has to follow the longer route discussed in the appendix A.

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