Analytic results for the planar double box integral relevant to top-pair production with a closed top loop

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ABSTRACT: In this article we give the details on the analytic calculation of the master integrals for the planar double box integral relevant to top-pair production with a closed top loop. We show that these integrals can be computed systematically to all order in the dimensional regularisation parameter ε. This is done by transforming the system of differential equations into a form linear in ε, where the ε^0-part is a strictly lower triangular matrix. Explicit results in terms of iterated integrals are presented for the terms relevant to NNLO calculations.

Contents

1   Introduction

Precision particle physics relies on our ability to compute the relevant quantum corrections. We are now entering an era of precision physics, where two-loop corrections to processes with massive particles are required. It is well known that starting from two loops not all Feynman integrals may be expressed in terms of multiple polylogarithms. The simplest Feynman integral which cannot be expressed in terms of multiple polylogarithms is given by the two-loop equal-mass sunrise integral [1–20]. This integral is related to an elliptic curve and can be expressed to all orders in the dimensional regularisation parameter ε in terms of iterated integrals of modular forms [21]. Integrals, which do not evaluate to multiple polylogarithms are now an active field of research in particle physics [21–40] and string theory [41–46]. The equal mass sunrise integral is an integral which depends on a single scale p^2/m^2.

In realistic scattering processes we face in general Feynman integrals, which depend on multiple scales. A prominent example is given by the planar double box integral for $t\bar{t}$-production with a closed top loop. This integral enters the next-to-next-to-leading order (NNLO) contribution for the process $pp \to t\bar{t}$. Until quite recently, it has not been known analytically. The existing NNLO calculation for the process $pp \to t\bar{t}$ treats this integral numerically [47–51]. This integral is clearly a cornerstone and should be understood for further progress on the analytical side. In a recent letter we reported how this integral can be treated analytically [52]. With this longer article we would like to give all the technical details.

Our starting point is the method of differential equations [53–63] for the master integrals. In the modern incarnation of this method one tries to find a basis of master integrals $\vec{J}$, such that the system of differential equations is in ε-form [59]:

$$d\vec{J} = \varepsilon A \vec{J}, \tag{1.1}$$

where A does not depend on ε. If such a form is achieved, a solution in terms of iterated integrals is immediate, supplemented by appropriate boundary conditions. This strategy has successfully been applied to many Feynman integrals evaluating to multiple polylogarithms. The difficulty of Feynman integral calculations is therefore reduced to finding

the right basis of master integrals. In the case of multiple polylogarithms the transformation from a Laporta basis $\vec{I}$ to the basis $\vec{J}$ is algebraic in the kinematic variables. If the transformation is rational in the kinematic variables, several algorithms exist to find such a transformation [62, 64–72]. Less is known in the genuine algebraic case (i.e. involving roots) [71]. In ref. [26] we showed that an ε-form can even be achieved for the equal-mass sunrise / kite system, essential steps leading to an ε-form have been discussed in [17, 23]. This is made possible by enlarging the set of transformations from the Laporta basis I to the basis J from algebraic functions in the kinematic variables towards algebraic functions in the kinematic variables, the periods of the elliptic curve and their derivatives.

We may slightly relax the form of the differential equation and consider

$$d\vec{J} = \left(A^{(0)} + \varepsilon A^{(1)}\right) \vec{J}, \tag{1.2}$$

where $A^{(0)}$ is strictly lower-triangular and $A^{(0)}$ and $A^{(1)}$ are independent of ε. This does not spoil the property, that the system of differential equations is easily solved in terms of iterated integrals. Since $A^{(0)}$ is strictly lower triangular, one can easily transform the system to an ε-form by introducing primitives for the terms occurring in the ε^0-part. In this paper we give for the system of the double box integral a transformation matrix, which transforms the system from a pre-canonical form to a form linear in ε, as in eq. (1.2). The transformation is rational in the kinematic variables, the periods of the elliptic curves and their derivatives. A subsequent transformation, which brings the system into ε-form is possible, however this would introduce additional transcendental functions.

We choose the basis of master integrals $\vec{J}$ such that $A^{(0)}$ vanishes if either $t = m^2$ or $s = \infty$. In the former case the solution reduces to multiple polylogarithms, in the latter case the solution reduces to iterated integrals of modular forms already encountered in the sunrise / kite system.

The attentive reader might have noticed that we put "elliptic curves" into the plural. The system of differential equations for the double box integral is not governed by a single elliptic curve. We find that there are three elliptic curves involved, originating from different sub-topologies. We show how the elliptic curves can be extracted from the maximal cuts. This fact has consequences: elliptic multiple polylogarithms are defined as iterated integrals on a punctured elliptic curve [14–17, 22, 24, 35–38, 73–78]. Inherent to this definition is the notion of a single elliptic curve. Since in our problem there are three distinct elliptic curves involved, we do not expect our results to be expressible in terms of elliptic multiple polylogarithms (which by definition are tied to one specific elliptic curve). We express our results as iterated integrals of the integration kernels appearing in eq. (1.2). Let us however also stress the other side of the medal: our analysis also shows that nothing worse than elliptic curves appears in the calculation.

This paper is organised as follows: in section 2 we introduce our notation and review a few basic facts about Feynman integrals. In section 3 we discuss iterated integrals. There are two special cases of iterated integrals, which we briefly review: multiple polylogarithms and iterated integrals of modular forms. In section 4 we discuss in detail the kinematic variables related to the subset of Feynman integrals, which evaluate to multiple polylogarithms. Section 5 is devoted to elliptic curves. After a discussion of the generic quartic
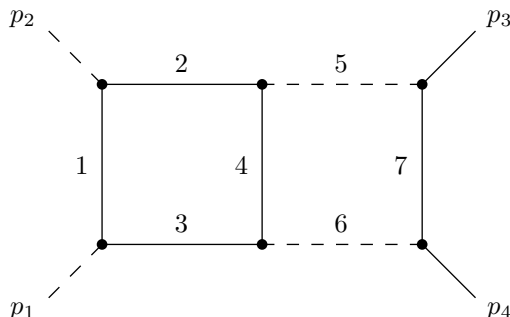
Figure 1. The planar double box. Solid lines correspond to massive propagators of mass m, dashed lines correspond to massless propagators. All external momenta are out-going and on-shell: $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m^2$.

case, we show how to extract the elliptic curves from the maximal cuts. We discuss the three occurring elliptic curves in detail. In section 6 we define the basis of master integrals $\vec{J}$. The system of differential equations for this set of master integrals is given in section 7. In section 8 we show how the boundary values are obtained. The results for the master integrals up to order ε^4 are given in section 9. Finally, our conclusions are given in section 10. The article is supplemented by an appendix. In appendix A we show Feynman graphs for all master topologies. In appendix B we give an extra relation, which reduces the number of master integrals in the sector 93 from five to four. In appendix C we collected useful information on the modular forms occurring in the $s \to \infty$ limit. Appendix D lists the full set of bondary constants. Appendix E describes the supplementary electronic file attached to this article, which gives the definition of the master integrals, the differential equation and the results in electronic form.

2 Notation, definitions and review of basics facts

We consider the planar double box integral shown in figure 1. This integral is relevant to the NNLO corrections for $t\bar{t}$-production at the LHC. In figure 1 the solid lines correspond to propagators with a mass m, while dashed lines correspond to massless propagators. All external momenta are out-going and on-shell. Thus we have

$$p_1 + p_2 + p_3 + p_4 = 0, \quad p_1^2 = p_2^2 = 0, \quad p_3^2 = p_4^2 = m^2. \tag{2.1}$$

We further set

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2. \tag{2.2}$$

Since there are two independent loop momenta and three independent external momenta we have nine independent scalar products involving the loop momenta. We therefore consider an auxiliary topology with nine propagators, shown in figure 2. In D-dimensional Minkowski space the integral family for this auxiliary topology is given by

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9}\left(D, s, t, m^2, \mu^2\right) = e^{2\gamma_E\varepsilon}\left(\mu^2\right)^{\nu-D}\int \frac{d^Dk_1}{i\pi^{\frac{D}{2}}}\frac{d^Dk_2}{i\pi^{\frac{D}{2}}}\prod_{j=1}^{9}\frac{1}{P_j^{\nu_j}}, \tag{2.3}$$
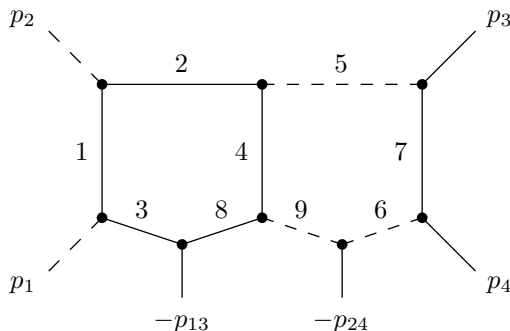
Figure 2. The auxiliary topology with nine propagators.

where γ_E denotes the Euler-Mascheroni constant, μ is an arbitrary scale introduced to render the Feynman integral dimensionless, the quantity ν is given by

$$\nu = \sum_{j=1}^{9} \nu_j \tag{2.4}$$

and

$$
\begin{aligned}
&P_1 = -(k_1+p_2)^2+m^2, &&P_2 = -k_1^2+m^2, &&P_3 = -(k_1+p_1+p_2)^2+m^2,\\
&P_4 = -(k_1+k_2)^2+m^2, &&P_5 = -k_2^2, &&P_6 = -(k_2+p_3+p_4)^2,\\
&P_7 = -(k_2+p_3)^2+m^2, &&P_8 = -(k_1+p_2-p_3)^2+m^2, &&P_9 = -(k_2-p_2+p_3)^2.
\end{aligned}
\tag{2.5}
$$

The original double box integral corresponds to $\nu_8 = \nu_9 = 0$. It will be convenient to introduce a short-hand notation: if $\nu_8 = \nu_9 = 0$, we may suppress these indices. Furthermore we will not always write explicitly the dependency on the variables s, t, m^2 and μ^2. Thus

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_700}\left(D,s,t,m^2,\mu^2\right). \tag{2.6}$$

We are interested in the Laurent expansion of these integrals in ε, where $\varepsilon = (4-D)/2$ denotes the dimensional regularisation parameter. Thus we write

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(4-2\varepsilon) = \sum_{j=j_{\min}}^{\infty} \varepsilon^j \, I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}^{(j)}. \tag{2.7}$$

A sector (or topology) is defined by the set of propagators with positive exponents. We define a sector id (or topology id) by

$$\mathrm{id} = \sum_{j=1}^{9} 2^{j-1}\Theta(\nu_j). \tag{2.8}$$

Most parts of our paper are valid for arbitrary values of s and t. In detail, there are no restrictions on s and t in section 2 to section 8. In particular, the system of differential equations is valid for all values of s and t. The results in sections 9.1–9.4 are given in terms of iterated integrals. These are valid for all values of s and t, if a proper analytic

continuation around branch cuts according to Feynman's $i\varepsilon$-prescription is understood. In a neighbourhood of the boundary point (which we take as $s = \infty$ and $t = m^2$) no analytic continuation is needed. For the analytic continuation we have to choose the integration path such that it avoids the singularities of the integration kernels according to Feynman's $i\varepsilon$-prescription. At the same time we have to ensure that the integration kernels are continuous along the integration path. The integration kernels will involve the periods of the elliptic curves. We have to ensure that the periods vary continuously along the integration path. We express the periods in a neighbourhood of the boundary point in terms of complete elliptic integrals. The complete elliptic integral $K(k)$, when viewed as a function of k^2 has a branch cut along $[1, \infty[$. It may happen that the image of the integration path in k^2-space crosses this cut. If this happens, we have to compensate for the discontinuity of $K(k)$ by taking the monodromy around $k^2 = 1$ into account. This has been discussed in [25]. In section 9.5 we perform a numerical check. We expand all integrands in power series and integrate term by term. This is limited to the region of convergence of the power series expansions.

2.1 Chains and cycles

It will be useful to group the internal propagators P_j into chains [79]. Two propagators belong to the same chain, if their momenta differ only by a linear combination of the external momenta. Obviously, each internal line can only belong to one chain. In figure 2 we have three chains:

$$C^{(1)} = \{P_8, P_3, P_1, P_2\}, \quad C^{(2)} = \{P_9, P_6, P_7, P_5\}, \quad C^{(3)} = \{P_4\}. \tag{2.9}$$

We define a cycle to be a closed circuit in the diagram. We can denote a cycle by specifying the chains which belong to the cycle. In the two-loop diagram of figure 2 there are three different cycles, given by

$$C^{(13)}, \quad C^{(23)}, \quad C^{(12)}. \tag{2.10}$$

Here we used the notation that $C^{(ij)}$ denotes the cycle consisting of the chains $C^{(i)}$ and $C^{(j)}$. A cycle corresponds to a one-loop sub-graph. The discussion carries over to sub-topologies of eq. (2.3) by deleting the appropriate propagators from the chains $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$. If a chain is empty, the two-loop integral factorises into two one-loop integrals. For non-empty chains $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ we are interested in a cycle with a minimum number of propagators. A cycle with a minimum number of propagators simplifies the calculation of the maximal cut of a Feynman integral within the loop-by-loop approach. The choice of such a cycle may not be unique, for example for the sunrise topology all three cycles have two propagators. For the auxiliary topology shown in figure 2 and all non-trivial sub-topologies (i.e. sub-topologies which are not products of one-loop integrals) it is always possible to choose either

$$C_1 = C^{(13)} \qquad \text{or} \qquad C_2 = C^{(23)} \tag{2.11}$$

as a cycle with a minimal number of propagators. This is due to the fact that the chain $C^{(3)}$ contains already the minimal number of propagators for a non-trivial chain.

2.2 The Feynman parameter representation

It is useful to discuss the Feynman integral representation for the double box integral. The Feynman parameter integral for $\nu_8 = \nu_9 = 0$ reads

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = e^{2\gamma_E\varepsilon} \frac{\Gamma(\nu-D)}{\prod\limits_{j=1}^{7}\Gamma(\nu_j)} \int\limits_{\sigma} \left(\prod_{j=1}^{7} x_j^{\nu_j-1}\right) \frac{\mathcal{U}^{\nu-\frac{3}{2}D}}{\mathcal{F}^{\nu-D}}\omega, \tag{2.12}$$

where the integration is over

$$\sigma = \left\{[x_1 : \ldots : x_7] \in \mathbb{RP}^6 | x_i \geq 0\right\}. \tag{2.13}$$

The differential form ω is given by

$$\omega = \sum_{j=1}^{7} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_7, \tag{2.14}$$

where the hat indicates that the corresponding term is omitted. The graph polynomials are given by

$$\mathcal{U} = (x_1 + x_2 + x_3)(x_5 + x_6 + x_7) + x_4(x_1 + x_2 + x_3 + x_5 + x_6 + x_7),$$

$$\mathcal{F} = [x_2 x_3 (x_4 + x_5 + x_6 + x_7) + x_5 x_6 (x_1 + x_2 + x_3 + x_4) + x_2 x_4 x_6 + x_3 x_4 x_5] \left(\frac{-s}{\mu^2}\right)$$

$$+ x_1 x_4 x_7 \left(\frac{-t}{\mu^2}\right) + x_7 [(x_2 + x_3) x_4 + (x_5 + x_6)(x_1 + x_2 + x_3 + x_4)] \left(\frac{-m^2}{\mu^2}\right)$$

$$+ (x_1 + x_2 + x_3 + x_4 + x_7)\mathcal{U}\frac{m^2}{\mu^2}. \tag{2.15}$$

The graph polynomial $\mathcal{U}$ reads in expanded form

$$\mathcal{U} = x_1 x_5 + x_1 x_6 + x_1 x_7 + x_2 x_5 + x_2 x_6 + x_2 x_7 + x_3 x_5 + x_3 x_6 + x_3 x_7$$

$$+ x_1 x_4 + x_2 x_4 + x_3 x_4 + x_4 x_5 + x_4 x_6 + x_4 x_7. \tag{2.16}$$

Let us further define the derivatives of the graph polynomial $\mathcal{F}$ with respect to s and t by

$$\mathcal{F}'_s = -\mu^2 \frac{d}{ds}\mathcal{F} = x_2 x_3 (x_4 + x_5 + x_6 + x_7) + x_5 x_6 (x_1 + x_2 + x_3 + x_4) + x_2 x_4 x_6 + x_3 x_4 x_5,$$

$$\mathcal{F}'_t = -\mu^2 \frac{d}{dt}\mathcal{F} = x_1 x_4 x_7. \tag{2.17}$$

The graph polynomials for sub-topologies are obtained from $\mathcal{U}$ and $\mathcal{F}$ by setting the Feynman parameters to zero which correspond to propagators not present in the sub-topology.

2.3 Dimensional shift relations and differential equations

Let us introduce an operator $\mathbf{i}^+$, which raises the power of the propagator i by one, e.g.

$$\mathbf{1}^+ I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = I_{(\nu_1+1)\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D). \tag{2.18}$$

In addition we define two operators $\mathbf{D}^\pm$, which shift the dimension of space-time by two through

$$\mathbf{D}^\pm I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D \pm 2). \tag{2.19}$$

The dimensional shift relations for $\nu_1,\ldots,\nu_7 \geq 0$ read [80, 81]

$$\mathbf{D}^- I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = \mathcal{U}\left(\nu_1\mathbf{1}^+, \nu_2\mathbf{2}^+, \nu_3\mathbf{3}^+, \nu_4\mathbf{4}^+, \nu_5\mathbf{5}^+, \nu_6\mathbf{6}^+, \nu_7\mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D). \tag{2.20}$$

The dimensional shift relations for integrals with irreducible numerators (i.e. $\nu_8 < 0$ or $\nu_9 < 0$) can be obtained as follows: one first converts to a basis of master integrals with $\nu_8 = \nu_9 = 0$ (and raised propagators), applies the dimensional shift relations to the latter and converts back to the original basis.

The differential equations for $\nu_1,\ldots,\nu_7 \geq 0$ can be obtained from

$$\mu^2 \frac{d}{ds} I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = \mathbf{D}^+ \mathcal{F}_s'\left(\nu_1\mathbf{1}^+, \ldots, \nu_7\mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D),$$
$$\mu^2 \frac{d}{dt} I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = \mathbf{D}^+ \mathcal{F}_t'\left(\nu_1\mathbf{1}^+, \ldots, \nu_7\mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D). \tag{2.21}$$

The right-hand side is given by integrals in $(D+2)$ dimensions with three propagators raised by an additional unit. Reducing these integrals to the basis in D dimensions gives the differential equation. Let us mention that eq. (2.21) allows us to write the differential equations in a compact way. However, from a computational point of view it is not the most advantageous representation, since it requires reduction of integrals with large ν.

We use the programs Reduze [82], Kira [83], Fire [84] and LiteRed [85, 86] to reduce the integrals to master integrals. These programs are based on integration-by-parts identities [87, 88] and implement the Laporta algorithm [89]. There are 44 master integrals. We may choose a basis of master integrals for which the auxiliary propagators P_8 and P_9 are not present, at the expense of having higher powers of the propagators for the propagators P_1–P_7. It is therefore possible to label these master integrals by $I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}$. In some sectors we have more than one master integral. The list of master integrals is shown in table 1. The master topologies are shown in appendix A.

We may change the basis of master integrals. Let us denote by $\vec{I}$ the vector of the pre-canonical master integrals. The differential equation for $\vec{I}$ reads for $\mu = m$

$$d\vec{I} = A\vec{I}, \quad A = A_s \frac{ds}{m^2} + A_t \frac{dt}{m^2}. \tag{2.22}$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A = 0. \tag{2.23}$$

Under a change of basis [59]

$$\vec{J} = U\vec{I}, \tag{2.24}$$

number of propagators	block	sector	master integrals basis $\vec{I}$	master integrals basis $\vec{J}$	kinematic dependence
2	1	9	$I_{1001000}$	J_1	$-$
3	2	14	$I_{0111000}$	J_2	s
	3	28	$I_{0011100}, I_{0021100}$	J_3, J_4	s
	4	49	$I_{1000110}$	J_5	s
	5	73	$I_{1001001}, I_{2001001}$	J_6, J_7	t
	6	74	$I_{0101001}$	J_8	$-$
4	7	15	$I_{1111000}$	J_9	s
	8	29	$I_{1011100}$	J_{10}	s
	9	54	$I_{0110110}$	J_{11}	s
	10	57	$I_{1001110}, I_{2001110}$	J_{12}, J_{13}	s
	11	75	$I_{1101001}$	J_{14}	t
	12	78	$I_{0111001}, I_{0211001}$	J_{15}, J_{16}	s
	13	89	$I_{1001101}$	J_{17}	t
	14	92	$I_{0011101}, I_{0021101}$	J_{18}, J_{19}	s
	15	113	$I_{1000111}$	J_{20}	s
5	16	55	$I_{1110110}$	J_{21}	s
	17	59	$I_{1101110}$	J_{22}	s
	18	62	$I_{0111110}$	J_{23}	s
	19	79	$I_{1111001}, I_{2111001}, I_{1211001}$	J_{24}, J_{25}, J_{26}	s, t
	20	93	$I_{1011101}, I_{2011101}, I_{1021101},$ $I_{1012101}$	$J_{27}, J_{28}, J_{29},$ J_{30}	s, t
	21	118	$I_{0110111}$	J_{32}	s
	22	121	$I_{1001111}, I_{2001111}, I_{1001112}$	J_{33}, J_{34}, J_{35}	s, t
6	23	63	$I_{1111110}$	J_{36}	s
	24	119	$I_{1110111}$	J_{37}	s
	25	123	$I_{1101111}, I_{1101211}$	J_{38}, J_{39}	s, t
	26	126	$I_{0111111}$	J_{40}	s
7	27	127	$I_{1111111}, I_{2111111}, I_{1211111},$ $I_{1111112}, I_{3111111}$	$J_{41}, J_{42}, J_{43},$ J_{44}, J_{45}	s, t

Table 1. Overview of the set of master integrals. The first column denotes the number of propagators, the second column labels consecutively the sectors or topologies, the third column gives the sector id (defined in eq. (2.8)), the fourth column lists the master integrals in the basis $\vec{I}$, the fifth column the corresponding ones in the basis $\vec{J}$. The last column denotes the kinematic dependence.

one obtains

$$d\vec{J} = A'\vec{J}, \tag{2.25}$$

where the matrix A' is related to A by

$$A' = UAU^{-1} - UdU^{-1}. \tag{2.26}$$

A comment is in order: the statement that there are 44 master integrals (and not 45) is already a non-trivial statement. We first run `Reduze`, `Kira` and `Fire` (the latter in combination with `LiteRed` [85, 86] and without). Taking trivial symmetry relations into account, all programs give with standard settings 45 master integrals. However, the reductions disagree for the three most complicated topologies and at first sight the results of two of the three programs seem to violate the integrability condition eq. (2.23). All these symptoms are resolved once an additional relations is taken into account. The relation is given in appendix B. This relation reduces the number of master integrals in sector 93 from 5 to 4 (and in turn the total number of master integrals from 45 to 44). Imposing this relation, the results from `Reduze`, `Kira` and `Fire` agree and the integrability condition is satisfied. We have found this relation by comparing the output of the three programs above. All differences are proportional to a single equation. In addition, we verified numerically the first few terms in the ε-expansion of this relation. This extra relation comes from a higher sector (i.e. sector 123). We would like to mention that `Reduze` is able to find the relation and can be forced to use this relation with the command `distribute_external`.[1] We also would like to mention that the new version 1.1 of `Kira` gives 44 master integrals.[2] Let us also mention that `MINT` [90] and `AZURITE` [91] are programs, which can be used to count the number of master integrals. `MINT` analyses critical points, `AZURITE` is based on syzygy relations. Applied to our problem, `MINT` reports 4 master integrals for sector 93, `AZURITE` gives 5.

3 Iterated integrals

Let us first review Chen's definition of iterated integrals [92]: let M be a n-dimensional manifold and

$$\gamma : \quad [0,1] \to M \tag{3.1}$$

a path with start point $x_i = \gamma(0)$ and end point $x_f = \gamma(1)$. Suppose further that $\omega_1, \ldots, \omega_k$ are differential 1-forms on M. Let us write

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j \tag{3.2}$$

for the pull-backs to the interval $[0,1]$. For $\lambda \in [0,1]$ the k-fold iterated integral of $\omega_1, \ldots, \omega_k$ along the path γ is defined by

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k). \tag{3.3}$$

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[1]We thank L. Tancredi for pointing this out.

[2]We thank P. Maierhoefer and J. Usovitsch.

We define the 0-fold iterated integral to be

$$I_\gamma\left(;\lambda\right) = 1. \tag{3.4}$$

We have

$$\frac{d}{d\lambda}I_\gamma\left(\omega_1,\omega_2,\ldots,\omega_k;\lambda\right) = f_1\left(\lambda\right)\,I_\gamma\left(\omega_2,\ldots,\omega_k;\lambda\right). \tag{3.5}$$

Let us now specialise to our case of interest: without loss of generality we may set $\mu = m$ in eq. (2.3). Then the Feynman integrals $I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}^{(j)}$ appearing in the Laurent expansion of eq. (2.7) depend only on two dimensionless ratios, which may be taken as

$$\frac{s}{m^2},\quad \frac{t}{m^2}. \tag{3.6}$$

In other words, we may view the integrals $I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}^{(j)}$ as functions on $M = \mathbb{P}^2(\mathbb{C})$, where

$$\left[s:t:m^2\right] \tag{3.7}$$

denote the homogeneous coordinates. We will express $I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}^{(j)}$ as iterated integrals on $\mathbb{P}^2(\mathbb{C})$.

Note that we are free to choose any convenient coordinates on M. One possibility is given by eq. (3.6). We will refer to this choice as (s,t)-coordinates.

A second possibility is given by the set (x,y), where x and y are related to s and t by

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x},\quad \frac{t}{m^2} = y. \tag{3.8}$$

We will refer to this choice as (x,y)-coordinates.

The (s,t)-coordinates and the (x,y)-coordinates will be our main coordinate systems, although not the only ones. The (s,t)-coordinates are closest to physics, however in these coordinates we encounter square roots already in very simple sub-topologies. The (x,y)-coordinates rationalise the most prominent square root $\sqrt{-s(4m^2-s)}$. However, this is not the only occurring square root. For example, we also encounter the square root $\sqrt{-s(-4m^2-s)}$. In order to simultaneously rationalise both square roots we use the coordinates $(\tilde{x},y)$, where $\tilde{x}$ is defined by

$$\frac{s}{m^2} = -\frac{\left(1+\tilde{x}^2\right)^2}{\tilde{x}\left(1-\tilde{x}^2\right)}. \tag{3.9}$$

However, there is a price to pay: rationalising square roots will increase the degree of the polynomials in intermediate stages of the calculation. For this reason we work bottom-up and treat each sub-topology in a coordinate system adapted to this sub-topology.

On top of this there are several elliptic topologies. Here we use the modular parameter τ of the associated elliptic curve as one variable.

3.1 Multiple polylogarithms

Multiple polylogarithms are a special case of iterated integrals. For $z_k \neq 0$ they are defined by [93–96]

$$G(z_1, \ldots, z_k; y) = \int\limits_0^y \frac{dy_1}{y_1 - z_1} \int\limits_0^{y_1} \frac{dy_2}{y_2 - z_2} \cdots \int\limits_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}. \tag{3.10}$$

The number k is referred to as the depth of the integral representation or the weight of the multiple polylogarithm. Let us introduce the short-hand notation

$$G_{m_1,\ldots,m_k}(z_1, \ldots, z_k; y) = G(\underbrace{0, \ldots, 0}_{m_1-1}, z_1, \ldots, z_{k-1}, \underbrace{0 \ldots, 0}_{m_k-1}, z_k; y), \tag{3.11}$$

where all z_j for $j = 1, \ldots, k$ are assumed to be non-zero. This allows us to relate the integral representation of the multiple polylogarithms to the sum representation of the multiple polylogarithms. The sum representation is defined by

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = \sum_{n_1 > n_2 > \ldots > n_k > 0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdots \frac{x_k^{n_k}}{n_k^{m_k}}. \tag{3.12}$$

The number k is referred to as the depth of the sum representation of the multiple polylogarithm, the weight is now given by $m_1 + m_2 + \ldots m_k$. The relations between the two representations are given by

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \ldots, \frac{1}{x_1 \ldots x_k}; 1\right),$$

$$G_{m_1,\ldots,m_k}(z_1, \ldots, z_k; y) = (-1)^k \, \mathrm{Li}_{m_1,\ldots,m_k}\left(\frac{y}{z_1}, \frac{z_1}{z_2}, \ldots, \frac{z_{k-1}}{z_k}\right). \tag{3.13}$$

Note that in the integral representation one variable is redundant due to the following scaling relation (recall $z_k \neq 0$):

$$G(z_1, \ldots, z_k; y) = G(xz_1, \ldots, xz_k; xy). \tag{3.14}$$

If one further sets $g(z; y) = 1/(y - z)$, then one has

$$\frac{d}{dy} G(z_1, \ldots, z_k; y) = g(z_1; y) G(z_2, \ldots, z_k; y) \tag{3.15}$$

and

$$G(z_1, z_2, \ldots, z_k; y) = \int\limits_0^y dy_1 \, g(z_1; y_1) G(z_2, \ldots, z_k; y_1). \tag{3.16}$$

One can slightly enlarge the set of multiple polylogarithms and define $G(0, \ldots, 0; y)$ with k zeros for z_1 to z_k to be

$$G(0, \ldots, 0; y) = \frac{1}{k!} \left(\ln y\right)^k. \tag{3.17}$$

This permits us to allow trailing zeros in the sequence $(z_1, \ldots, z_k)$ by defining the function G with trailing zeros via eq. (3.16) and eq. (3.17). Please note that the scaling relation eq. (3.14) does not hold for multiple polylogarithms with trailing zeros. Using the shuffle product it is possible to remove trailing zeros and to express any multiple polylogarithms as a linear combination of terms which involve multiple polylogarithms without trailing zeros and powers of $\ln y$.

It will be convenient to introduce the following notation: for differential one-forms

$$\omega_j = \sum_{r=1}^{r_j} c_{j,r} \, \frac{dy}{y - z_{j,r}} \tag{3.18}$$

we define $G(\omega_1, \ldots, \omega_k; y)$ recursively through

$$G(\omega_1, \omega_2, \ldots, \omega_k; y) = \sum_{r=1}^{r_1} c_{1,r} \int_0^y dy_1 \, g(z_{1,r}, y_1) \, G(\omega_2, \ldots, \omega_k; y_1). \tag{3.19}$$

Methods for the numerical evaluation of multiple polylogarithms can be found in [97].

3.2 Iterated integrals of modular forms

A second special case of iterated integrals are iterated integrals of modular forms. Let $f_1(\tau)$, $f_2(\tau)$, $\ldots$, $f_k(\tau)$ be modular forms of a congruence subgroup.

The (full) modular group $\mathrm{SL}_2(\mathbb{Z})$ is the group of (2×2)-matrices over the integers with unit determinant:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \, \middle| \, a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\}. \tag{3.20}$$

The standard congruence subgroups of the modular group $\mathrm{SL}_2(\mathbb{Z})$ are defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; c \equiv 0 \bmod N \right\},$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; b, c \equiv 0 \bmod N \right\}. \tag{3.21}$$

The most prominent example in our application will be the congruence subgroup $\Gamma_1(6)$. Let us further assume that $f_k(\tau)$ vanishes at the cusp $\tau = i\infty$. We define the k-fold iterated integral by

$$F(f_1, f_2, \ldots, f_k; q) = (2\pi i)^k \int_{i\infty}^{\tau} d\tau_1 \, f_1(\tau_1) \int_{i\infty}^{\tau_1} d\tau_2 \, f_2(\tau_2) \ldots \int_{i\infty}^{\tau_{k-1}} d\tau_k \, f_k(\tau_k), \quad q = e^{2\pi i \tau}. \tag{3.22}$$

The case where $f_k(\tau)$ does not vanishes at the cusp $\tau = i\infty$ is discussed in [21, 98] and is similar to trailing zeros in the case of multiple polylogarithms. This is easily seen by changing the variable from τ to q:

$$2\pi i \int_{i\infty}^{\tau} d\tau_1 \, f(\tau_1) = \int_0^q \frac{dq_1}{q_1} \, f(\tau_1(q_1)), \quad \tau_1(q_1) = \frac{1}{2\pi i} \ln q_1 \tag{3.23}$$

Modular forms have a Fourier expansion around the cusp $\tau = i\infty$:

$$f_j(\tau) = \sum_{n=0}^{\infty} a_{j,n} q^n. \tag{3.24}$$

$f_j(\tau)$ vanishes at $\tau = i\infty$ if $a_{j,0} = 0$. Using the Fourier expansion and integrating term-by-term one obtains the q-series of the iterated integral of modular forms corresponding to eq. (3.22):

$$F(f_1, f_2, \ldots, f_k; q) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \frac{a_{1,n_1}}{n_1 + \ldots + n_k} \cdots \frac{a_{k-1,n_{k-1}}}{n_{k-1} + n_k} \frac{a_{k,n_k}}{n_k} q^{n_1+\ldots+n_k}. \tag{3.25}$$

4 The kinematic variables for the multiple polylogarithms

A large fraction of the integrals under consideration will only depend on the kinematic variable s, but not on t. These integrals are expressible in terms of multiple polylogarithms. Furthermore one finds that all integrals under consideration are expressible in terms of multiple polylogarithms in the special kinematic configuration $t = m^2$. Let us now introduce several variables related to the multiple polylogarithms. They all replace the kinematic variable s and rationalise one or several square roots. The difference lies in the square roots they rationalise.

Let us start with the simplest case relevant to integrals with a singular point at $s = 4m^2$. The variable x replaces s and is defined by [99–102]

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}, \quad x = \frac{1}{2}\left(\frac{-s}{m^2} + 2 - \sqrt{\frac{-s}{m^2}}\sqrt{4 - \frac{s}{m^2}}\right). \tag{4.1}$$

The interval $s \in]-\infty, 0]$ is mapped to $x \in [0, 1]$, with the point $s = -\infty$ being mapped to $x = 0$ and the point $s = 0$ being mapped to $x = 1$. This change of variables rationalises the square root $\sqrt{-s(4m^2 - s)}$. In more detail we have

$$\frac{ds}{s} = \frac{2dx}{x-1} - \frac{dx}{x}, \quad \frac{ds}{s-4m^2} = \frac{2dx}{x+1} - \frac{dx}{x}, \quad \frac{ds}{\sqrt{-s(4m^2-s)}} = \frac{dx}{x}. \tag{4.2}$$

We will encounter sub-topologies, which have a singular point at $s = -4m^2$. The simplest example is given by sector 57. For these integrals we change from the variable s to a variable x' defined through

$$\frac{s}{m^2} = -\frac{(1+x')^2}{x'}, \quad x' = \frac{1}{2}\left(\frac{-s}{m^2} - 2 - \sqrt{-\frac{s}{m^2}}\sqrt{-4 - \frac{s}{m^2}}\right). \tag{4.3}$$

The interval $s \in]-\infty, -4m^2]$ is mapped to $x' \in [0,1]$. The point $s = -\infty$ is mapped to $x' = 0$, the point $s = 0$ is mapped to $x' = -1$. This change of variables rationalises the square root $\sqrt{-s(-4m^2 - s)}$, but not the square root $\sqrt{-s(4m^2 - s)}$. We have

$$\frac{ds}{s} = \frac{2dx'}{x'+1} - \frac{dx'}{x'}, \quad \frac{ds}{s+4m^2} = \frac{2dx'}{x'-1} - \frac{dx'}{x'}, \quad \frac{ds}{\sqrt{-s(-4m^2 - s)}} = \frac{dx'}{x'}. \tag{4.4}$$

We further have

$$x' = -1 + \frac{(1-x)^2}{2x} - \frac{1-x}{2x}\sqrt{x^2 - 6x + 1},$$

$$x = 1 + \frac{(1+x')^2}{2x'} - \frac{1+x'}{2x'}\sqrt{x'^2 + 6x' + 1}. \tag{4.5}$$

In order to rationalise simultaneously the two square roots $\sqrt{-s(4m^2 - s)}$ and $\sqrt{-s(-4m^2 - s)}$ we introduce a variable $\tilde{x}$ through

$$x = \tilde{x}\frac{(1-\tilde{x})}{(1+\tilde{x})}, \quad \tilde{x} = \frac{1}{2}\left(1 - x - \sqrt{x^2 - 6x + 1}\right). \tag{4.6}$$

Expressing x' in terms of $\tilde{x}$ yields

$$x' = \tilde{x}\frac{(1+\tilde{x})}{(1-\tilde{x})}. \tag{4.7}$$

We have

$$\begin{aligned}
\omega_0 &= \frac{ds}{s} &&= \frac{2(2\tilde{x})d\tilde{x}}{\tilde{x}^2 + 1} - \frac{d\tilde{x}}{\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} + 1} - \frac{d\tilde{x}}{\tilde{x}}, \\
\omega_4 &= \frac{ds}{s - 4m^2} &&= \frac{2(2\tilde{x} - 2)d\tilde{x}}{\tilde{x}^2 - 2\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} + 1} - \frac{d\tilde{x}}{\tilde{x}}, \\
\omega_{-4} &= \frac{ds}{s + 4m^2} &&= \frac{2(2\tilde{x} + 2)d\tilde{x}}{\tilde{x}^2 + 2\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} + 1} - \frac{d\tilde{x}}{\tilde{x}}, \\
\omega_{0,4} &= \frac{ds}{\sqrt{-s(4m^2 - s)}} &&= \frac{d\tilde{x}}{\tilde{x} - 1} - \frac{d\tilde{x}}{\tilde{x} + 1} + \frac{d\tilde{x}}{\tilde{x}}, \\
\omega_{-4,0} &= \frac{ds}{\sqrt{-s(-4m^2 - s)}} &&= -\frac{d\tilde{x}}{\tilde{x} - 1} + \frac{d\tilde{x}}{\tilde{x} + 1} + \frac{d\tilde{x}}{\tilde{x}}.
\end{aligned} \tag{4.8}$$

In eq. (4.8) we defined for later convenience the differential forms ω_0, ω_4, ω_{-4}, $\omega_{0,4}$ and $\omega_{-4,0}$. Note that

$$\begin{aligned}
\frac{2\tilde{x}d\tilde{x}}{\tilde{x}^2 + 1} &= \frac{d\tilde{x}}{\tilde{x} - i} + \frac{d\tilde{x}}{\tilde{x} + i}, \\
\frac{(2\tilde{x} - 2)d\tilde{x}}{\tilde{x}^2 - 2\tilde{x} - 1} &= \frac{d\tilde{x}}{\tilde{x} - (1 + \sqrt{2})} + \frac{d\tilde{x}}{\tilde{x} - (1 - \sqrt{2})}, \\
\frac{(2\tilde{x} + 2)d\tilde{x}}{\tilde{x}^2 + 2\tilde{x} - 1} &= \frac{d\tilde{x}}{\tilde{x} - (-1 + \sqrt{2})} + \frac{d\tilde{x}}{\tilde{x} - (-1 - \sqrt{2})}.
\end{aligned} \tag{4.9}$$

From eq. (4.2), eq. (4.4) and eq. (4.8) we may read off the alphabets $\mathcal{A}$, $\mathcal{A}'$ and $\tilde{\mathcal{A}}$ for the variables x, x' and $\tilde{x}$, respectively. We have

$$
\begin{aligned}
\mathcal{A} &= \{-1, 0, 1\}, \\
\mathcal{A}' &= \{-1, 0, 1\}, \\
\tilde{\mathcal{A}} &= \left\{-1, 0, 1, i, -i, 1 + \sqrt{2}, 1 - \sqrt{2}, -1 + \sqrt{2}, -1 - \sqrt{2}\right\}.
\end{aligned}
\tag{4.10}
$$

Thus, iterated integrals in the variable x involving the differential forms of eq. (4.2) may be expressed in terms of the smaller class of harmonic polylogarithms [103, 104]. The same holds true for iterated integrals in the variable x' involving the differential forms of eq. (4.4). On the other hand, iterated integrals in the variable $\tilde{x}$ involving the differential forms of eq. (4.8) have a larger alphabet and are expressed in terms of multiple polylogarithms [93–96].

In practice, the results for the more complicated integrals are most compactly expressed by introducing the notation of eq. (3.19). All sub-topologies, which depend only on s, can be expressed as iterated integrals with integration kernels given by the five differential one-forms

$$
\{\omega_0, \omega_4, \omega_{-4}, \omega_{0,4}, \omega_{-4,0}\}.
\tag{4.11}
$$

In addition, for $t = m^2$ (or equivalently $y = 1$) all master integrals can be expressed as iterated integrals with these integration kernels. From eq. (4.8) and eq. (4.9) it is clear that all iterated integrals in these integration kernels are expressible in terms of multiple polylogarithms.

5 Elliptic curves

In this section we discuss elliptic curves. We start with a review of the general quartic case in section 5.1. The relevant elliptic curves are extracted from the maximal cuts. This is done in section 5.2. We find three different elliptic curves, which we label $E^{(a)}$, $E^{(b)}$ and $E^{(c)}$. These curves are discussed individually in sections 5.3–5.5.

5.1 The general quartic case

Let us consider the elliptic curve

$$
E \; : \; w^2 - (z - z_1)(z - z_2)(z - z_3)(z - z_4) = 0,
\tag{5.1}
$$

where the roots z_j may depend on variables $x = (x_1, \ldots, x_n)$:

$$
z_j = z_j(x), \quad j \in \{1, 2, 3, 4\}.
\tag{5.2}
$$

We use the notation

$$
df(x) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right) dx_i.
\tag{5.3}
$$

We set

$$Z_1 = (z_2 - z_1)(z_4 - z_3), \quad Z_2 = (z_3 - z_2)(z_4 - z_1), \quad Z_3 = (z_3 - z_1)(z_4 - z_2). \quad (5.4)$$

Note that we have

$$Z_1 + Z_2 = Z_3. \quad (5.5)$$

We define the modulus and the complementary modulus of the elliptic curve E by

$$k^2 = \frac{Z_1}{Z_3}, \quad \bar{k}^2 = 1 - k^2 = \frac{Z_2}{Z_3}. \quad (5.6)$$

Note that there are six possibilities of defining k^2. Our standard choice for the periods and quasi-periods is

$$\psi_1 = \frac{4K(k)}{Z_3^{\frac{1}{2}}}, \qquad \psi_2 = \frac{4iK(\bar{k})}{Z_3^{\frac{1}{2}}},$$
$$\phi_1 = \frac{4[K(k) - E(k)]}{Z_3^{\frac{1}{2}}}, \qquad \phi_2 = \frac{4iE(\bar{k})}{Z_3^{\frac{1}{2}}}. \quad (5.7)$$

These periods satisfy the first-order system of differential equations

$$d \begin{pmatrix} \psi_i \\ \phi_i \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}d\ln Z_2 & \frac{1}{2}d\ln\frac{Z_2}{Z_1} \\ -\frac{1}{2}d\ln\frac{Z_2}{Z_3} & \frac{1}{2}d\ln\frac{Z_2^2}{Z_3^2} \end{pmatrix} \begin{pmatrix} \psi_i \\ \phi_i \end{pmatrix}, \quad i \in \{1, 2\}, \quad (5.8)$$

and the Legendre relation

$$\psi_1\phi_2 - \psi_2\phi_1 = \frac{8\pi i}{Z_3}. \quad (5.9)$$

The parameter τ and the nome squared q are defined by

$$\tau = \frac{\psi_2}{\psi_1}, \quad q = e^{2i\pi\tau}. \quad (5.10)$$

We have

$$2\pi i \, d\tau = d\ln q = \frac{2\pi i}{\psi_1^2} \frac{4\pi i}{Z_3} d\ln\frac{Z_2}{Z_1}. \quad (5.11)$$

Let us now consider a path $\gamma : [0,1] \to \mathbb{C}^n$ such that $x_i = x_i(\lambda)$, where the variable λ parametrises the path. A specific example is the path $\gamma_\alpha : [0,1] \to \mathbb{C}^n$, indexed by $\alpha = [\alpha_1 : \ldots : \alpha_n] \in \mathbb{CP}^{n-1}$ and given explicitly by

$$x_i(\lambda) = x_i(0) + \alpha_i\lambda, \quad 1 \le i \le n. \quad (5.12)$$

For a path γ we may view the periods ψ_1 and ψ_2 as functions of the variable λ. We then have

$$\left[\frac{d^2}{d\lambda^2} + p_{1,\gamma}\frac{d}{d\lambda} + p_{0,\gamma} \right]\psi_i = 0, \quad i \in \{1, 2\}, \quad (5.13)$$

where

$$p_{1,\gamma} = \frac{d}{d\lambda}\ln Z_3 - \frac{d}{d\lambda}\ln\left(\frac{d}{d\lambda}\ln\frac{Z_2}{Z_1}\right), \tag{5.14}$$

$$p_{0,\gamma} = \frac{1}{2}\left(\frac{d}{d\lambda}\ln Z_1\right)\left(\frac{d}{d\lambda}\ln Z_2\right) - \frac{1}{2}\frac{\left(\frac{d}{d\lambda}Z_1\right)\left(\frac{d^2}{d\lambda^2}Z_2\right) - \left(\frac{d^2}{d\lambda^2}Z_1\right)\left(\frac{d}{d\lambda}Z_2\right)}{Z_1\left(\frac{d}{d\lambda}Z_2\right) - Z_2\left(\frac{d}{d\lambda}Z_1\right)}$$

$$+ \frac{1}{4Z_3}\left[\frac{1}{Z_1}\left(\frac{d}{d\lambda}Z_1\right)^2 + \frac{1}{Z_2}\left(\frac{d}{d\lambda}Z_2\right)^2\right].$$

This defines the Picard-Fuchs operator along the path γ:

$$L_\gamma = \frac{d^2}{d\lambda^2} + p_{1,\gamma}\frac{d}{d\lambda} + p_{0,\gamma}. \tag{5.15}$$

The Wronskian is defined by

$$W_\gamma = \psi_1\frac{d}{d\lambda}\psi_2 - \psi_2\frac{d}{d\lambda}\psi_1 = \frac{4\pi i}{Z_3}\frac{d}{d\lambda}\ln\frac{Z_2}{Z_1}. \tag{5.16}$$

We have

$$\frac{d}{d\lambda}W_\gamma = -p_{1,\gamma}W_\gamma,$$

$$2\pi i d\tau = \frac{2\pi i\, W_\gamma}{\psi_1^2}d\lambda. \tag{5.17}$$

Let us now specify to the case, where the base space is given by the two variables (x,y). Eq. (5.14) and eq. (5.16) allows us to obtain the Picard-Fuchs operator and the Wronskian from the roots z_1, z_2, z_3 and z_4 for the variation of the elliptic curve along the paths

$$\gamma_\alpha : [0,1] \to \mathbb{C}^2,$$

$$x(\lambda) = x + \alpha_1\lambda, \quad y(\lambda) = y + \alpha_2\lambda. \tag{5.18}$$

We define the Wronskians W_x and W_y at the point (x,y) as the derivatives in the directions x and y, respectively. Thus

$$W_x = \psi_1\frac{d}{dx}\psi_2 - \psi_2\frac{d}{dx}\psi_1, \quad W_y = \psi_1\frac{d}{dy}\psi_2 - \psi_2\frac{d}{dy}\psi_1. \tag{5.19}$$

We have

$$W_{\gamma_\alpha} = \alpha_1 W_x + \alpha_2 W_y,$$

$$2\pi i d\tau = \frac{2\pi i W_{\gamma_\alpha}}{\psi_1^2}d\lambda = \frac{2\pi i}{\psi_1^2}\left(W_x dx + W_y dy\right). \tag{5.20}$$

We may use the Picard-Fuchs operator to eliminate second derivatives:

$$\frac{d^2}{dx^2}\psi_1 = -p_{1,x}\frac{d}{dx}\psi_1 - p_{0,x}\psi_1, \tag{5.21}$$

$$\frac{d^2}{dy^2}\psi_1 = -p_{1,y}\frac{d}{dy}\psi_1 - p_{0,y}\psi_1,$$

$$2\frac{d^2}{dxdy}\psi_1 = -\left(p_{1,x+y} - p_{1,x}\right)\frac{d}{dx}\psi_1 - \left(p_{1,x+y} - p_{1,y}\right)\frac{d}{dy}\psi_1 - \left(p_{0,x+y} - p_{0,x} - p_{0,y}\right)\psi_1,$$

where the subscript $x + y$ refers to the path with $(\alpha_1, \alpha_2) = (1, 1)$. This leaves us with

$$\psi_1, \quad \frac{d}{dx}\psi_1, \quad \frac{d}{dy}\psi_1. \tag{5.22}$$

There is a further relation, since we may exchange any derivative of ψ_1 in favour of ϕ_1:

$$\frac{\frac{1}{2}\left(\frac{d}{dx}\ln Z_2\right)\psi_1 + \frac{d}{dx}\psi_1}{\frac{1}{2}\frac{d}{dx}\ln\frac{Z_2}{Z_1}} = \phi_1 = \frac{\frac{1}{2}\left(\frac{d}{dy}\ln Z_2\right)\psi_1 + \frac{d}{dy}\psi_1}{\frac{1}{2}\frac{d}{dy}\ln\frac{Z_2}{Z_1}}. \tag{5.23}$$

This yields

$$\frac{1}{2}\left(\frac{d}{dy}\ln\frac{Z_2}{Z_1}\right)\frac{d}{dx}\psi_1 - \frac{1}{2}\left(\frac{d}{dx}\ln\frac{Z_2}{Z_1}\right)\frac{d}{dy}\psi_1 =$$
$$\frac{1}{4}\left[\left(\frac{d}{dx}\ln Z_2\right)\left(\frac{d}{dy}\ln Z_1\right) - \left(\frac{d}{dy}\ln Z_2\right)\left(\frac{d}{dx}\ln Z_1\right)\right]\psi_1. \tag{5.24}$$

Using eq. (5.24) we may eliminate one derivative, say $\frac{d}{dx}\psi_1$. This leaves us with

$$\psi_1, \qquad \frac{d}{dy}\psi_1, \tag{5.25}$$

as expected, since the first cohomology group of an elliptic curve is two dimensional.

5.2 Maximal cuts

In order to identify the elliptic curves associated to a Feynman integrals we use maximal cuts in the Baikov representation [105–111]. For the maximal cut of an integral we use the loop-by-loop approach [109]. Let us first assume that all propagators occur to the power one, although we allow irreducible numerators. We first consider a one-loop sub-graph with a minimal number of propagators. This is equivalent to the statement that the dimension of the sub-space spanned by the external momenta for this sub-graph is minimal. Let us assume that this sub-graph contains $(e + 1)$ propagators $P_1, \ldots, P_{e+1}$, therefore the sub-space spanned by the external momenta for this sub-graph has dimension e. We change the integration variables for this sub-graph according to

$$\frac{d^D k}{i\pi^{\frac{D}{2}}} = u \frac{2^{-e}\pi^{-\frac{e}{2}}}{\Gamma\left(\frac{D-e}{2}\right)} G(p_1, \ldots, p_e)^{\frac{1+e-D}{2}} G(k, p_1, \ldots, p_e)^{\frac{D-e-2}{2}} \prod_{j=1}^{e+1} dP_j, \tag{5.26}$$

where the momenta $p_1, \ldots, p_e$ denote the linearly independent external momenta for this sub-graph, the Gram determinant (in Minkowski space) is defined by

$$G(p_1, \ldots, p_e) = \det\left(-p_i \cdot p_j\right)_{1 \le i, j \le e}, \tag{5.27}$$

and u denotes an (irrelevant) phase ($|u| = 1$). The maximal cuts are solutions of the homogeneous differential equations [112]. Any such solution remains a solution upon multiplication with a non-zero constant. Therefore the phase u is not relevant.

We then repeat this procedure for the second loop, replacing $p_1, \ldots, p_e$ by the set of independent external momenta for the full graph. For an integral of the form

$$I = e^{2\gamma_E \varepsilon} \left(\mu^2\right)^{n-D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \, N\left(k_1, k_2\right) \prod_{j=1}^{n} \frac{1}{P_j}, \tag{5.28}$$

where $N(k_1, k_2)$ is a polynomial in k_1 and k_2, a maximal cut is given by

$$\mathrm{MaxCut}_{\mathcal{C}} \, I = e^{2\gamma_E \varepsilon} \left(\mu^2\right)^{n-D} \int_{\mathcal{C}} \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \, N\left(k_1, k_2\right) \prod_{j=1}^{n} \delta\left(P_j\right), \tag{5.29}$$

where the integration measure is re-written according to eq. (5.26) and the integration is over a (yet to be) specified contour in the variables P_j not eliminated by the delta distributions. The exact definition of the integration contour is not relevant for the extraction of the elliptic curve from the maximal cut. We aim for a one-dimensional integral representation for the maximal cut with a constant in the numerator and a square root of a quartic polynomial in the denominator. This defines the elliptic curve. The possible choices for the integration contour are then given by an integration between any pair of roots of the quartic polynomial. This integration gives a period of the elliptic curve. The result for any choice of integration contour may be expressed as a linear combination of two independent periods. In practice we label/order the roots and define the periods by eq. (5.7).

For integrals with $\nu_j > 1$ we may compute a maximal cut by first converting to a basis with $\nu_j = 1$ and possibly irreducible numerators and then computing the maximal cut in this basis. Alternatively, we may interpret the delta distribution $\delta(P_j)$ as a contour integration along a small circle around $P_j = 0$. We may therefore compute the maximal cut from residues.

Let us look at a few examples. We start with the equal mass sunrise integral (sector 73) in two space-time dimensions. Starting with the sub-loop C_1 first we obtain

$$\mathrm{MaxCut}_{\mathcal{C}} \, I_{1001001} \left(2 - 2\varepsilon\right) = \tag{5.30}$$

$$\frac{u\mu^2}{\pi^2} \int_{\mathcal{C}} \frac{dP'}{\left(P' - t + 2m^2\right)^{\frac{1}{2}} \left(P' - t + 6m^2\right)^{\frac{1}{2}} \left(P'^2 + 6m^2 P' - 4m^2 t + 9m^4\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

For the sunrise integral we could equally well start with the sub-loop C_2. Doing so we find

$$\mathrm{MaxCut}_{\mathcal{C}} \, I_{1001001} \left(2 - 2\varepsilon\right) = \tag{5.31}$$

$$\frac{u\mu^2}{\pi^2} \int_{\mathcal{C}} \frac{dP}{\left(P - t\right)^{\frac{1}{2}} \left(P - t + 4m^2\right)^{\frac{1}{2}} \left(P^2 + 2m^2 P - 4m^2 t + m^4\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

The two representations are related by $P' = P - 2m^2$.

Let us now look at the maximal cut of the double box integral (sector 127), this time in four space-time dimensions. We have

$$\mathrm{MaxCut}_{\mathcal{C}} \, I_{1111111} \left(4 - 2\varepsilon\right) = \tag{5.32}$$

$$\frac{u\mu^6}{4\pi^4 s^2} \int_{\mathcal{C}} \frac{dP}{\left(P - t\right)^{\frac{1}{2}} \left(P - t + 4m^2\right)^{\frac{1}{2}} \left(P^2 + 2m^2 P - 4m^2 t + m^4 - \frac{4m^2(m^2-t)^2}{s}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

One recognises in eq. (5.30), eq. (5.31) and eq. (5.32) the typical period integrals of an elliptic curve. We note that the integrand of eq. (5.32) differs from the one of eq. (5.31). The difference is given by the additional term

$$-\frac{4m^2\left(m^2-t\right)^2}{s}.$$ (5.33)

This term vanishes in the limit $s \to \infty$.

Let us now look at the maximal cut in the sector 79. We find with $P = P_8$

$$\text{MaxCut}_{\mathcal{C}}\, I_{1112001}\left(4-2\varepsilon\right) =$$ (5.34)

$$\frac{u\mu^4}{4\pi^3 s} \int_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2(m^2-t)^2}{s}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

Up to the prefactor, this is the same maximal cut integral as in eq. (5.32). Therefore the sectors 79 and 127 are associated to the same elliptic curve.

Our next example is the maximal cut in the sector 121. Here we find with $P = P_9 + 2m^2$

$$\text{MaxCut}_{\mathcal{C}}\, I_{2001111}\left(4-2\varepsilon\right) = \frac{u\mu^4}{4\pi^3\left(-s\right)^{\frac{1}{2}}\left(4m^2-s\right)^{\frac{1}{2}}}$$ (5.35)

$$\times \int_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2\frac{(s+4t)}{(s-4m^2)}P+m^2\left(m^2-4t\right)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

This corresponds to an elliptic curve different from the one found in sectors 79 and 127. In the limit $s \to \infty$ the maximal cut integral reduces again up to a prefactor to the one of eq. (5.31).

The most complicated example is the maximal cut in sector 93. For this sector we find first within the loop-by-loop approach a two-fold integral representation in P_2 and P_8 for the maximal cut. The integrand has a single pole at $P_2 = 0$. Choosing as a contour for the P_2-integration a small circle around this pole leads (with $P = P_8$) to

$$\frac{1}{\varepsilon}\,\text{MaxCut}_{\mathcal{C}}\, I_{1012101}\left(4-2\varepsilon\right) =$$ (5.36)

$$\frac{u\mu^4}{\pi^2 s} \int_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2(m^2-t)^2}{s}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

We recognise again the elliptic curve of sector 79 and 127.

Our last example is the maximal cut in the sector 123. Here we find with $P = P_9 + 2m^2$

$$\text{MaxCut}_{\mathcal{C}}\, I_{1101111}\left(4-2\varepsilon\right) = \frac{u\mu^4}{4\pi^3\left(-s\right)^{\frac{1}{2}}\left(4m^2-s\right)^{\frac{1}{2}}}$$ (5.37)

$$\times \int_{\mathcal{C}} \frac{dP}{\left(P-t\right)\left(P^2+2m^2\frac{(s+4t)}{(s-4m^2)}P+m^2\left(m^2-4t\right)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

The denominator may be viewed as a square root of a quartic polynomial, where two roots coincide. This does not involve an elliptic curve and corresponds to genus zero.

5.3 The elliptic curve associated to sector 73

From eq. (5.31) we may read off the elliptic curve for the sunrise integral:

$$E^{(a)}: \qquad w^2 - \left(z - \frac{t}{\mu^2}\right)\left(z - \frac{t - 4m^2}{\mu^2}\right)\left(z^2 + \frac{2m^2}{\mu^2}z + \frac{m^4 - 4m^2 t}{\mu^4}\right) = 0. \qquad (5.38)$$

The roots of the quartic polynomial are

$$z_1^{(a)} = \frac{t - 4m^2}{\mu^2}, \quad z_2^{(a)} = \frac{-m^2 - 2m\sqrt{t}}{\mu^2}, \quad z_3^{(a)} = \frac{-m^2 + 2m\sqrt{t}}{\mu^2}, \quad z_4^{(a)} = \frac{t}{\mu^2}. \qquad (5.39)$$

This curve has the j-invariant

$$j\left(E^{(a)}\right) = \frac{\left(3m^2 + t\right)^3 \left(3m^6 + 75m^4 t - 15m^2 t^2 + t^3\right)^3}{m^6 t \left(m^2 - t\right)^6 \left(9m^2 - t\right)^2}. \qquad (5.40)$$

Two elliptic curves over $\mathbb{C}$ are isomorphic, if and only if they have the same j-invariant. Let us now consider a path γ_β in (s, t)-space parametrised by

$$s = s_0 + \beta_1 \lambda \mu^2, \quad t = t_0 + \beta_2 \lambda \mu^2. \qquad (5.41)$$

For the Wronskian and the Picard-Fuchs operator $\frac{d}{d\lambda^2} + p_{1,\gamma_\beta}^{(a)} \frac{d}{d\lambda} + p_{0,\gamma_\beta}^{(a)}$ we find

$$W_{\gamma_\beta}^{(a)} = 2\pi i \mu^6 \frac{3\beta_2}{t\left(t - m^2\right)\left(t - 9m^2\right)},$$

$$p_{1,\gamma_\beta}^{(a)} = -\mu^2 \left(\beta_1 \frac{d}{ds} + \beta_2 \frac{d}{dt}\right) \ln W_{\gamma_\beta}^{(a)},$$

$$p_{0,\gamma_\beta}^{(a)} = \mu^{10} \frac{2\pi i}{W_{\gamma_\beta}^{(a)}} \frac{3\beta_2^3 \left(t - 3m^2\right)}{t^2 \left(t - m^2\right)^2 \left(t - 9m^2\right)^2}. \qquad (5.42)$$

Eq. (5.24) reduces to the trivial equation

$$0 = 0. \qquad (5.43)$$

We have

$$16 \frac{\eta\left(\frac{\tau^{(a)}}{2}\right)^{24} \eta\left(2\tau^{(a)}\right)^{24}}{\eta\left(\tau^{(a)}\right)^{48}} = \left(k^{(a)} \bar{k}^{(a)}\right)^2 = 16 \frac{m^3 \sqrt{t} \left(m - \sqrt{t}\right)^3 \left(3m + \sqrt{t}\right)}{\left(m + \sqrt{t}\right)^6 \left(3m - \sqrt{t}\right)^2}. \qquad (5.44)$$

Dedekind's eta function is defined by

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i \tau}. \qquad (5.45)$$

For a path γ_α in (x, y)-space

$$x = \alpha_1 \lambda, \quad y = 1 + \alpha_2 \lambda \qquad (5.46)$$

we may use eq. (5.44) to express λ as a power series in $q^{(a)}$ and vice versa. The point $(x, y) = (0, 1)$ corresponds to $\tau^{(a)} = i\infty$.

For $y = 1$ we have

$$\psi_1^{(a)}\Big|_{y=1} = \frac{\pi}{2}, \qquad \frac{d}{dy}\psi_1^{(a)}\Big|_{y=1} = -\frac{\pi}{8}. \qquad (5.47)$$

5.4 The elliptic curve associated to sector 79, sector 93 and sector 127

From eq. (5.32) we obtain the elliptic curve associated to the double box integral:

$$E^{(b)}: \quad w^2 - \left(z - \frac{t}{\mu^2}\right)\left(z - \frac{t - 4m^2}{\mu^2}\right)\left(z^2 + \frac{2m^2}{\mu^2}z + \frac{m^4 - 4m^2 t}{\mu^4} - \frac{4m^2\left(m^2 - t\right)^2}{\mu^4 s}\right) = 0. \tag{5.48}$$

The roots of the quartic polynomial are now

$$z_1^{(b)} = \frac{t - 4m^2}{\mu^2}, \quad z_2^{(b)} = \frac{-m^2 - 2m\sqrt{t + \frac{(m^2 - t)^2}{s}}}{\mu^2}, \quad z_3^{(b)} = \frac{-m^2 + 2m\sqrt{t + \frac{(m^2 - t)^2}{s}}}{\mu^2},$$
$$z_4^{(b)} = \frac{t}{\mu^2}. \tag{5.49}$$

The j-invariant is given by

$$j\left(E^{(b)}\right) = \tag{5.50}$$

$$\frac{\left\{s\left(3m^2 + t\right)\left[s\left(3m^6 + 75m^4 t - 15m^2 t + t^3\right) + 8m^2\left(m^2 - t\right)^2\left(9m^2 - t\right)\right] + 16m^4\left(m^2 - t\right)^4\right\}^3}{sm^6\left(s - 4m^2\right)^2\left[st + \left(m^2 - t\right)^2\right]\left(m^2 - t\right)^6\left[s\left(9m^2 - t\right) - 4m^2\left(m^2 - t\right)\right]^2}.$$

For the Wronskian and the Picard-Fuchs operator $\frac{d}{d\lambda^2} + p_{1,\gamma_\beta}^{(b)}\frac{d}{d\lambda} + p_{0,\gamma_\beta}^{(b)}$ we find for the path γ_β defined in eq. (5.41)

$$W_{\gamma_\beta}^{(b)} = 2\pi i \mu^6 \frac{\beta_1\left(m^2 - t\right)\left[s\left(t + 3m^2\right) - 4m^2\left(m^2 - t\right)\right] + \beta_2 s\left(s - 4m^2\right)\left(2m^2 - 3s - 2t\right)}{\left(s - 4m^2\right)\left(t - m^2\right)\left[st + \left(m^2 - t\right)^2\right]\left[s\left(9m^2 - t\right) - 4m^2\left(m^2 - t\right)\right]},$$

$$p_{1,\gamma_\beta}^{(b)} = -\mu^2\left(\beta_1\frac{d}{ds} + \beta_2\frac{d}{dt}\right)\ln W_{\gamma_\beta}^{(b)}, \tag{5.51}$$

$$p_{0,\gamma_\beta}^{(b)} = \mu^{10}\frac{2\pi i}{W_{\gamma_\beta}^{(b)}}\frac{N^{(b)}}{s^2\left(s - 4m^2\right)^2\left(t - m^2\right)^2\left[st + \left(m^2 - t\right)^2\right]^2\left[s\left(9m^2 - t\right) - 4m^2\left(m^2 - t\right)\right]^2},$$

with

$$\begin{aligned}
N^{(b)} = {}&2\beta_1^3\left(m^2 - t\right)^4 m^2\left(8m^8 - 10m^6 s - 16m^6 t + 9m^4 s^2 + 4m^4 st + 8m^4 t^2 + 8m^2 s^2 t\right.\\
&\left. + 6m^2 st^2 - s^2 t^2\right)\\
&+ \beta_1^2\beta_2 s\left(m^2 - t\right)^2\left(96m^{12} - 248m^{10}s - 288m^{10}t + 276m^8 s^2 + 504m^8 st + 288m^8 t^2\right.\\
&\left. - 63m^6 s^3 - 360m^6 s^2 t - 264m^6 st^2 - 96m^6 t^3 + 119m^4 s^3 t + 84m^4 s^2 t^2 + 8m^4 st^3\right.\\
&\left. - 18m^2 s^4 t - 25m^2 s^3 t^2 + 2s^4 t^2 + s^3 t^3\right)\\
&+ 2\beta_1\beta_2^2 s^2\left(4m^2 - s\right)\left(m^2 - t\right)^2\left(24m^8 - 78m^6 s - 48m^6 t + 88m^4 s^2 + 84m^4 st\right.\\
&\left. + 24m^4 t^2 - 18m^2 s^3 - 24m^2 s^2 t - 6m^2 st^2 + s^3 t\right)\\
&+ \beta_2^3 s^3\left(4m^2 - s\right)^2\left(8m^8 - 30m^6 s - 24m^6 t + 36m^4 s^2 + 62m^4 st + 24m^4 t^2 - 9m^2 s^3\right.\\
&\left. - 42m^2 s^2 t - 34m^2 st^2 - 8m^2 t^3 + 3s^3 t + 6s^2 t^2 + 2st^3\right).
\end{aligned} \tag{5.52}$$

Eq. (5.24) yields

$$\psi_1^{(b)} = \frac{\left(s - 4m^2\right)\left(3s + 2t - 2m^2\right)}{t - m^2}\frac{d}{ds}\psi_1^{(b)} - \frac{s\left(t + 3m^2\right) - 4m^2\left(m^2 - t\right)}{s}\frac{d}{dt}\psi_1^{(b)}. \quad (5.53)$$

In (x, y)-space this translates to

$$\psi_1^{(b)} = -\frac{(x+1)\left(3x^2 - 2xy - 4x + 3\right)}{(x-1)(y-1)}\frac{d}{dx}\psi_1^{(b)} - \frac{x^2y + 3x^2 - 6xy - 2x + y + 3}{(x-1)^2}\frac{d}{dy}\psi_1^{(b)}. \quad (5.54)$$

We have with

$$\chi^{(b)} = \sqrt{t + \frac{(m^2 - t)^2}{s}} \quad (5.55)$$

the relation

$$16\frac{\eta\left(\frac{\tau^{(b)}}{2}\right)^{24}\eta\left(2\tau^{(b)}\right)^{24}}{\eta\left(\tau^{(b)}\right)^{48}} = \left(k^{(b)}\bar{k}^{(b)}\right)^2 \quad (5.56)$$

$$= 16\frac{m^3\chi^{(b)}\left(m^2 + t - 2m\chi^{(b)}\right)\left(3m^2 - t - 2m\chi^{(b)}\right)}{\left(m^2 + t + 2m\chi^{(b)}\right)^2\left(3m^2 - t + 2m\chi^{(b)}\right)^2}.$$

For a path γ_α in (x, y)-space

$$x = \alpha_1\lambda, \quad y = 1 + \alpha_2\lambda \quad (5.57)$$

we may use eq. (5.56) to express λ as a power series in $q^{(b)}$ and vice versa. The point $(x, y) = (0, 1)$ corresponds to $\tau^{(b)} = i\infty$.

For $y = 1$ we have

$$\psi_1^{(b)}\Big|_{y=1} = \frac{\pi}{2}, \quad \frac{d}{dy}\psi_1^{(b)}\Big|_{y=1} = -\frac{\pi}{8}. \quad (5.58)$$

5.5 The elliptic curve associated to sector 121

From eq. (5.35) we obtain the elliptic curve associated to sector 121:

$$E^{(c)}: \quad w^2 - \left(z - \frac{t}{\mu^2}\right)\left(z - \frac{t - 4m^2}{\mu^2}\right)\left(z^2 + \frac{2m^2\left(s + 4t\right)}{\mu^2\left(s - 4m^2\right)}z + \frac{sm^2\left(m^2 - 4t\right) - 4m^2t^2}{\mu^4\left(s - 4m^2\right)}\right) = 0. \quad (5.59)$$

The roots of the quartic polynomial are now

$$z_1^{(c)} = \frac{t - 4m^2}{\mu^2},$$

$$z_2^{(c)} = \frac{1}{\mu^2}\left(-m^2\frac{(s + 4t)}{(s - 4m^2)} - \frac{2}{4m^2 - s}\sqrt{sm^2\left(st + (m^2 - t)^2\right)}\right),$$

$$z_3^{(c)} = \frac{1}{\mu^2}\left(-m^2\frac{(s + 4t)}{(s - 4m^2)} + \frac{2}{4m^2 - s}\sqrt{sm^2\left(st + (m^2 - t)^2\right)}\right),$$

$$z_4^{(c)} = \frac{t}{\mu^2}. \quad (5.60)$$

The j-invariant is given by

$$j\left(E^{(c)}\right) = \frac{\left\{s\left(3m^2+t\right)\left(3m^6+75m^4t-15m^2t+t^3\right)+192m^6\left(m^2-t\right)^2\right\}^3}{m^6\left[st+\left(m^2-t\right)^2\right]\left(m^2-t\right)^4\left[s\left(m^2-t\right)\left(9m^2-t\right)-64m^6\right]^2}.$$

For the Wronskian and the Picard-Fuchs operator $\frac{d}{d\lambda^2}+p_{1,\gamma_\beta}^{(c)}\frac{d}{d\lambda}+p_{0,\gamma_\beta}^{(c)}$ we find for the path γ_β defined in eq. (5.41)

$$W_{\gamma_\beta}^{(c)} = 2\pi i \mu^6$$

$$\frac{\left(s-4m^2\right)\left\{\beta_1\left(t-m^2\right)\left(t^2-6m^2t-3m^4\right)+\beta_2 s\left(3s\left(t-m^2\right)+2\left(t-m^2\right)^2+16m^4\right)\right\}}{s\left(t-m^2\right)\left[st+\left(m^2-t\right)^2\right]\left[s\left(m^2-t\right)\left(9m^2-t\right)-64m^6\right]},$$

$$p_{1,\gamma_\beta}^{(c)} = -\mu^2\left(\beta_1\frac{d}{ds}+\beta_2\frac{d}{dt}\right)\ln W_{\gamma_\beta}^{(c)},$$

$$p_{0,\gamma_\beta}^{(c)} = \mu^{10}\frac{2\pi i}{W_{\gamma_\beta}^{(c)}}\frac{N^{(c)}}{s^3\left(s-4m^2\right)\left(t-m^2\right)^2\left[st+\left(m^2-t\right)^2\right]^2\left[s\left(m^2-t\right)\left(9m^2-t\right)-64m^6\right]^2},$$

(5.61)

with

$$N^{(c)} =$$

$$-2\beta_1^3 m^2\left(m^2-t\right)^2\left(3m^4+6m^2t-t^2\right)\left(32m^{12}-62m^{10}s-64m^{10}t+3m^8s^2-24m^8st\right.$$
$$\left.+32m^8t^2-26m^6s^2t-20m^6st^2-36m^4s^2t^2-24m^4st^3-6m^2s^2t^3+2m^2st^4+s^2t^4\right)$$
$$-\beta_1^2\beta_2 s\left(m^2-t\right)\left(9600m^{18}-8232m^{16}s-17920m^{16}t+1656m^{14}s^2+18736m^{14}st\right.$$
$$+7424m^{14}t^2-81m^{12}s^3-8724m^{12}s^2t-7512m^{12}st^2+512m^{12}t^3+1356m^{10}s^3t$$
$$+2972m^{10}s^2t^2-416m^{10}st^3+384m^{10}t^4-54m^8s^4t-1045m^8s^3t^2+632m^8s^2t^3+1896m^8st^4$$
$$+96m^6s^4t^2-256m^6s^3t^3-720m^6s^2t^4-400m^6st^5-28m^4s^4t^3+37m^4s^3t^4+92m^4s^2t^5$$
$$+24m^4st^6-16m^2s^4t^4-12m^2s^3t^5-4m^2s^2t^6+2s^4t^5+s^3t^6\big)$$
$$-2\beta_1\beta_2^2 s^2\left(4m^2-s\right)\left(544m^{16}-518m^{14}s-768m^{14}t+84m^{12}s^2+1252m^{12}st-192m^{12}t^2\right.$$
$$-420m^{10}s^2t-58m^{10}st^2+512m^{10}t^3+39m^8s^3t+416m^8s^2t^2+536m^8st^3-96m^8t^4$$
$$-76m^6s^3t^2-176m^6s^2t^3-250m^6st^4+34m^4s^3t^3+108m^4s^2t^4+68m^4st^5+4m^2s^3t^4$$
$$-12m^2s^2t^5-6m^2st^6-s^3t^5\big)$$
$$+\beta_2^3 s^3\left(4m^2-s\right)^2\left(736m^{12}-542m^{10}s-672m^{10}t+120m^8s^2+562m^8st-96m^8t^2-9m^6s^3\right.$$
$$-206m^6s^2t-340m^6st^2+32m^6t^3+21m^4s^3t+122m^4s^2t^2+76m^4st^3-15m^2s^3t^2$$
$$-42m^2s^2t^3-14m^2st^4+3s^3t^3+6s^2t^4+2st^5\big).$$

(5.62)

Eq. (5.24) yields

$$\psi_1^{(c)} = \frac{s\left(s-4m^2\right)\left[3s\left(t-m^2\right)+2\left(t-m^2\right)^2+16m^4\right]}{\left(t+3m^2\right)\left[s\left(t-m^2\right)+8m^4\right]}\frac{d}{ds}\psi_1^{(c)}$$

$$-\frac{\left(s-4m^2\right)\left(t-m^2\right)\left(t^2-6m^2t-3m^4\right)}{\left(t+3m^2\right)\left[s\left(t-m^2\right)+8m^4\right]}\frac{d}{dt}\psi_1^{(c)}.$$

(5.63)

In (x, y)-space this translates to

$$\psi_1^{(c)} = -\frac{(x+1)(x-1)\left(3x^2y - 2xy^2 - 3x^2 - 2xy - 12x + 3y - 3\right)}{(y+3)\left(x^2y - x^2 - 2xy - 6x + y - 1\right)}\frac{d}{dx}\psi_1^{(c)}$$
$$-\frac{(x+1)^2(y-1)\left(y^2 - 6y - 3\right)}{(y+3)\left(x^2y - x^2 - 2xy - 6x + y - 1\right)}\frac{d}{dy}\psi_1^{(c)}. \tag{5.64}$$

We have with

$$\chi^{(c)} = \sqrt{sm^2\left(st + (m^2 - t)^2\right)} \tag{5.65}$$

the relation

$$16\frac{\eta\left(\frac{\tau^{(c)}}{2}\right)^{24}\eta\left(2\tau^{(c)}\right)^{24}}{\eta\left(\tau^{(c)}\right)^{48}} = \left(k^{(c)}\bar{k}^{(c)}\right)^2 \tag{5.66}$$

$$= 16\frac{m^2\left(4m^2 - s\right)\chi^{(c)}\left(sm^2 + st + 2\chi^{(c)}\right)\left(3sm^2 - st - 16m^4 + 2\chi^{(c)}\right)}{\left(sm^2 + st - 2\chi^{(c)}\right)^2\left(3sm^2 - st - 16m^4 - 2\chi^{(c)}\right)^2}.$$

For a path γ_α in (x, y)-space

$$x = \alpha_1\lambda, \quad y = 1 + \alpha_2\lambda \tag{5.67}$$

we may use eq. (5.66) to express λ as a power series in $q^{(c)}$ and vice versa. The point $(x, y) = (0, 1)$ corresponds to $\tau^{(c)} = i\infty$.

For $y = 1$ we have

$$\psi_1^{(c)}\Big|_{y=1} = \frac{\pi(1+x)}{2(1-x)}, \quad \frac{d}{dy}\psi_1^{(c)}\Big|_{y=1} = -\frac{\pi(1+x)}{8(1-x)}. \tag{5.68}$$

5.6 Modular forms

The integrals which only depend on t, but not on s, are all related to the elliptic curve $E^{(a)}$. Associated to the curve $E^{(a)}$ are modular forms of $\Gamma_1(6)$. The differential one-forms relevant to the integrals dependent on t but not on s are of the form

$$f\,(2\pi i)\,d\tau_6^{(a)}, \tag{5.69}$$

where

$$\tau_6^{(a)} = \frac{1}{6}\frac{\psi_2^{(a)}}{\psi_1^{(a)}}, \tag{5.70}$$

which we substitute for y (or t). Furthermore, f is a modular form of $\Gamma_1(6)$ from the set

$$\{1, f_2, f_3, f_4, g_{2,1}\}. \tag{5.71}$$

The modular weights are given by 0, 2, 3, 4 and 2, respectively. The non-trivial modular forms are given by

$$f_2 = -\frac{1}{4}\left(3y^2 - 10y - 9\right)\left(\frac{\psi_1^{(a)}}{\pi}\right)^2,$$

$$f_3 = -\frac{3}{2}y\left(y - 1\right)\left(y - 9\right)\left(\frac{\psi_1^{(a)}}{\pi}\right)^3,$$

$$f_4 = \frac{1}{16}\left(y + 3\right)^4\left(\frac{\psi_1^{(a)}}{\pi}\right)^4,$$

$$g_{2,1} = -\frac{1}{2}y\left(y - 9\right)\left(\frac{\psi_1^{(a)}}{\pi}\right)^2. \tag{5.72}$$

In appendix C we collected useful information on the occurring modular forms of $\Gamma_1(6)$ and give alternative representations.

5.7 The high-energy limit

In the high-energy limit $s \to \infty$ (or equivalently $x = 0$) the elliptic curves $E^{(b)}$ and $E^{(c)}$ degenerate to the elliptic curve $E^{(a)}$. In this limit we therefore have only one elliptic curve $E^{(a)}$. In this limit we may express all master integrals in terms of iterated integrals of modular forms. The resulting set of modular forms is slightly larger than eq. (5.71). In order to present this set, we first set

$$g_{n,r} = -\frac{1}{2}\frac{y\left(y - 1\right)\left(y - 9\right)}{y - r}\left(\frac{\psi_1^{(a)}}{\pi}\right)^n,$$

$$h_{n,s} = -\frac{1}{2}y\left(y - 1\right)^{1+s}\left(y - 9\right)\left(\frac{\psi_1^{(a)}}{\pi}\right)^n. \tag{5.73}$$

Relevant to the high-energy limit is the set

$$\{1, g_{2,0}, g_{2,1}, g_{2,9}, g_{3,1}, h_{3,0}, g_{4,0}, g_{4,1}, g_{4,9}, h_{4,0}, h_{4,1}\}. \tag{5.74}$$

These are again modular forms of $\Gamma_1(6)$ in the variable $\tau_6^{(a)}$. Additional details are given in appendix C. All integration kernels reduce in the high-energy limit to $\mathbb{Q}$-linear combinations of elements of this set (times $(2\pi i)\, d\tau_6^{(a)}$).

6 Master integrals

In this section we define the 44 master integrals in the basis $\vec{J}$. They are labelled $J_1 - J_{30}$ and $J_{32} - J_{45}$. An integral J_{31} is missing due to the extra relation given in eq. (B.1). In the following we set $\mu = m$ and $D = 4 - 2\varepsilon$.

The basis $\vec{J}$ is constructed as follows: the master integrals, which only depend on s do not pose any problems and can in principal be constructed with the algorithms of [66–71].

The master integrals, which only depend on t are similar to the kite / sunrise system and can be constructed along the lines of [21, 26]. This leaves the master integrals, which depend on s and t. Here we first analyse the diagonal blocks. Combining the information from the maximal cuts with the technique based on the factorisation properties of Picard-Fuchs operators [62] we first obtain an intermediate basis with the desired properties up to sub-topologies. We then use a slightly modified version of the algorithm of Meyer [69, 70] for the non-diagonal blocks and fix the sub-topologies.

Sector 9: $J_1 = \varepsilon^2\, \mathbf{D}^- I_{1001000},$

Sector 14: $J_2 = \varepsilon^2 \dfrac{(1-x)(1+x)}{2x}\, \mathbf{D}^- I_{0111000},$

Sector 28: $J_3 = \varepsilon^2 \dfrac{(1-x^2)}{x} \left[I_{0021200} + \dfrac{1}{2} I_{0022100} \right],$

$\qquad\qquad J_4 = \varepsilon^2 \dfrac{(1-x)^2}{x} I_{0022100},$

Sector 49: $J_5 = -\dfrac{(1-x)^2}{2x} \varepsilon^2\, \mathbf{D}^- I_{1000110},$

Sector 73: $J_6 = \varepsilon^2 \dfrac{\pi}{\psi_1^{(a)}}\, \mathbf{D}^- I_{1001001},$

$\qquad\qquad J_7 = \dfrac{6}{\varepsilon} \dfrac{\left(\psi_1^{(a)}\right)^2}{2\pi i W_y^{(a)}} \dfrac{d}{dy} J_6 - \dfrac{1}{4}\left(3y^2 - 10y - 9\right)\left(\dfrac{\psi_1^{(a)}}{\pi}\right)^2 J_6,$

Sector 74: $J_8 = 4\varepsilon^2\, \mathbf{D}^- I_{0101001},$

Sector 15: $J_9 = -\varepsilon^3(1-\varepsilon)\dfrac{(1-x)^2}{x} I_{1111000},$

Sector 29: $J_{10} = -\varepsilon^3 \dfrac{(1-x)^2}{x} I_{1012100},$

Sector 54: $J_{11} = -\varepsilon^2 \dfrac{(1+x)(1-x)^3}{4x^2}\, \mathbf{D}^- I_{0110110},$

Sector 57: $J_{12} = \varepsilon^2 \dfrac{(1-x'^2)}{x'} \left[-\dfrac{(1+x')^2}{x'} I_{2001210} - \dfrac{\varepsilon}{2(1+2\varepsilon)} I_{2002000} \right],$

$\qquad\qquad J_{13} = -\varepsilon^3 \dfrac{(1+x')^2}{x'} I_{2001110},$

Sector 75: $J_{14} = \varepsilon^3(1-y) I_{1102001},$

Sector 78: $J_{15} = \varepsilon^2 \dfrac{(1-x^2)^2}{x^2} I_{0212001} - \dfrac{3\varepsilon^2(1-x)^2}{2x} I_{0202001},$

$\qquad\qquad J_{16} = \varepsilon^3 \dfrac{(1-x^2)}{x} I_{0112001},$

Sector 89: $J_{17} = \varepsilon^3(1-y) I_{2001101},$

Sector 92: $J_{18} = \varepsilon^2 \dfrac{(1+x)^2}{x} I_{0021102} - \varepsilon^2 \dfrac{(1+x)}{x}\left(I_{0021200} + \dfrac{1}{2} I_{0022100} \right),$

$\qquad\qquad J_{19} = \varepsilon^3 \dfrac{(1-x^2)}{x} I_{0021101},$

Sector 113: $J_{20} = \varepsilon^3 (1-\varepsilon) \dfrac{(1-x^2)}{x} I_{1000111}$,

Sector 55: $J_{21} = \varepsilon^3 (1-2\varepsilon) \dfrac{(1-x)^2}{x} I_{1110110}$,

Sector 59: $J_{22} = \varepsilon^4 \dfrac{(1-x)^2}{x} I_{1101110}$,

Sector 62: $J_{23} = \varepsilon^3 (1-2\varepsilon) \dfrac{(1-x)^2}{x} I_{0111110}$,

Sector 79: $J_{24} = \varepsilon^3 \dfrac{(1-x)^2}{x} \dfrac{\pi}{\psi_1^{(b)}} I_{1112001}$,

$J_{25} = \varepsilon^3 (1-2\varepsilon) \dfrac{(1-x)^2}{x} I_{1111001} - \dfrac{1}{3} (y-9) \dfrac{\psi_1^{(b)}}{\pi} J_{24}$,

$J_{26} = \dfrac{6}{\varepsilon} \dfrac{\left(\psi_1^{(b)}\right)^2}{2\pi i W_y^{(b)}} \dfrac{d}{dy} J_{24} - \dfrac{1}{4} \left(3y^2 - 10y - 9\right) \left(\dfrac{\psi_1^{(b)}}{\pi}\right)^2 J_{24}$

$\qquad\qquad - \dfrac{1}{24} \left(y^2 - 30y - 27\right) \dfrac{\psi_1^{(b)}}{\pi} \dfrac{\psi_1^{(a)}}{\pi} J_6$,

Sector 93: $J_{27} = \varepsilon^3 \dfrac{(1-x)^2}{x} \dfrac{\pi}{\psi_1^{(b)}} I_{1012101}$,

$J_{28} = \varepsilon^3 \left[1 - y + \dfrac{(1-x)^2}{x}\right] (I_{1021101} + I_{2011101}) - \dfrac{1}{6} (y-3) \dfrac{\psi_1^{(b)}}{\pi} J_{27}$,

$J_{29} = \varepsilon^4 \left[1 - y + \dfrac{(1-x)^2}{x}\right] I_{1011101}$,

$J_{30} = \dfrac{6}{\varepsilon} \dfrac{\left(\psi_1^{(b)}\right)^2}{2\pi i W_y^{(b)}} \dfrac{d}{dy} J_{27} - \dfrac{1}{4} \left(3y^2 - 10y - 9\right) \left(\dfrac{\psi_1^{(b)}}{\pi}\right)^2 J_{27}$

$\qquad\qquad - \dfrac{1}{12} \left(y^2 - 30y - 27\right) \dfrac{\psi_1^{(b)}}{\pi} \dfrac{\psi_1^{(a)}}{\pi} J_6$,

Sector 118: $J_{32} = \varepsilon^3 (1-2\varepsilon) \dfrac{(1-x)^2}{x} I_{0110111} + 2\varepsilon^3 (1-\varepsilon) \dfrac{(1-x)}{x} I_{1000111}$,

Sector 121: $J_{33} = \varepsilon^3 \dfrac{(1-x^2)}{x} \dfrac{\pi}{\psi_1^{(c)}} I_{2001111}$,

$J_{34} = \varepsilon^3 (1-2\varepsilon) \dfrac{(1-x^2)}{x} I_{1001111} - \dfrac{1}{3} (y-9) \dfrac{(1+2x)}{(1+x)} \dfrac{\psi_1^{(c)}}{\pi} J_{33}$,

$J_{35} = \dfrac{6}{\varepsilon} \dfrac{\left(\psi_1^{(c)}\right)^2}{2\pi i W_y^{(c)}} \dfrac{d}{dy} J_{33} - \dfrac{1}{4} \left(3y^2 - 10y - 9\right) \dfrac{(1-x)^2}{(1+x)^2} \left(\dfrac{\psi_1^{(c)}}{\pi}\right)^2 J_{33}$

$\qquad\qquad + \dfrac{1}{8} \left(y^2 - 2y + 9\right) \dfrac{(1-x)}{(1+x)} \dfrac{\psi_1^{(c)}}{\pi} \dfrac{\psi_1^{(a)}}{\pi} J_6$,

Sector 63: $J_{36} = \varepsilon^4 \dfrac{(x+1)(x-1)^3}{x^2} I_{1111110}$,

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Sector 119: $J_{37} = \varepsilon^4 \dfrac{(x+1)(x-1)^3}{x^2} I_{1110111}$,

Sector 123: $J_{38} = 2\varepsilon^4 \dfrac{(x^2-1)}{x} \left[I_{11011110(-1)} - (y-2) I_{1101111} \right] - \dfrac{4}{x-1} J_{22}$,

$\qquad\qquad J_{39} = \varepsilon^4 (1-y) \dfrac{(1-x)^2}{x} I_{1101111}$,

Sector 126: $J_{40} = \varepsilon^4 \dfrac{(x+1)(x-1)^3}{x^2} I_{0111111}$,

Sector 127: $J_{41} = \varepsilon^4 \dfrac{(1-x)^4}{x^2} \dfrac{\pi}{\psi_1^{(b)}} I_{1111111}$,

$\qquad\quad J_{42} = 8\varepsilon^4 \dfrac{(1-x)^2}{x} I_{1111111(-1)(-1)} - 8\varepsilon^4 \dfrac{(y-2)(1-x)^2}{x} I_{1111111(-1)0}$

$\qquad\qquad\quad - 8\varepsilon^4 \dfrac{y(1-x)^2}{x} I_{11111110(-1)} - \dfrac{8x}{(1-x)^2} \dfrac{\psi_1^{(b)}}{\pi} J_{41}$

$\qquad\qquad\quad - 4 \dfrac{(x-1)(x^2-2xy+1)}{(x+1)^3} J_{40} - 4 \dfrac{x^2-2xy+1}{(x-1)(x+1)} J_{37}$

$\qquad\qquad\quad - 4 \dfrac{x^2-2xy-4x+1}{(x-1)(x+1)} J_{36} - \dfrac{4}{3}(y+3) \dfrac{\psi_1^{(b)}}{\pi} J_{27} + \dfrac{8}{3}(y+3) \dfrac{\psi_1^{(b)}}{\pi} J_{24}$

$\qquad\qquad\quad - \left(4 + \dfrac{32\varepsilon}{(1-2\varepsilon)} \dfrac{(x^4-yx^3-xy+1)}{(x-1)^2(x+1)^2} \right) J_{23} - 16 \dfrac{(y-1)x}{(x-1)^2} J_{22}$

$\qquad\qquad\quad - 8 \dfrac{(x-1)(x^2-xy+x+1)}{(x+1)^3} J_{19} + 4 \dfrac{(x-1)^2}{(y-1)x} J_{17}$

$\qquad\qquad\quad + 16 \dfrac{(x-1)(x^2-xy+x+1)}{(x+1)^3} J_{16} - \dfrac{4}{3} \dfrac{(6x^2-5xy-7x+6)}{(y-1)x} J_{14}$,

$\qquad\quad J_{43} = \dfrac{6}{\varepsilon} \dfrac{\left(\psi_1^{(b)}\right)^2}{2\pi i W_y^{(b)}} \dfrac{d}{dy} J_{41} - \dfrac{1}{4}(3y^2-10y-9) \left(\dfrac{\psi_1^{(b)}}{\pi} \right)^2 J_{41} + 4y \dfrac{(1-x)}{(1+x)} \dfrac{\psi_1^{(b)}}{\pi} \dfrac{\psi_1^{(c)}}{\pi} J_{33}$

$\qquad\qquad\quad + \dfrac{2}{3} y(y-9) \left(\dfrac{\psi_1^{(b)}}{\pi} \right)^2 J_{27} + \dfrac{2}{3} y(y-3) \left(\dfrac{\psi_1^{(b)}}{\pi} \right)^2 J_{24}$,

$\qquad\quad J_{44} = \varepsilon^4 \dfrac{(1-x)^4}{x^2} I_{1111111(-1)0} - \dfrac{1}{3}(2y-3) \dfrac{\psi_1^{(b)}}{\pi} J_{41} - \varepsilon^4 \dfrac{(1-x)^4}{x^2} I_{0111111}$

$\qquad\qquad\quad + \varepsilon^4 \dfrac{(1-x)^2(x^2-2xy+1)}{x^2} I_{1101111}$,

$\qquad\quad J_{45} = \varepsilon^4 \dfrac{(x-1)^2(x+1)^2}{x^2} I_{11111110(-1)} - \dfrac{(2x^2y-9x^2+8xy-6x+2y-9)}{3(x-1)^2} \dfrac{\psi_1^{(b)}}{\pi} J_{41}$

$\qquad\qquad\quad - \dfrac{2}{x-1} J_{36} + \dfrac{1}{2} \left(\dfrac{1}{x} + x - \dfrac{2}{3}y \right) \dfrac{\psi_1^{(b)}}{\pi} J_{27} - \left(\dfrac{1}{x} + x - \dfrac{2}{3}y \right) \dfrac{\psi_1^{(b)}}{\pi} J_{24}$. $\qquad$ (6.1)

7   The system of differential equations

In the basis $\vec{J}$ the system of differential equations is linear in ε, i.e. of the form

$$d\vec{J} = \left(A^{(0)} + \varepsilon A^{(1)} \right) \vec{J}. \tag{7.1}$$

The matrices $A^{(0)}$ and $A^{(1)}$ are independent of ε. Furthermore, $A^{(0)}$ is strictly lower-triangular, i.e.

$$A^{(0)}_{ij} = 0 \quad \text{for} \quad j \geq i. \tag{7.2}$$

As usual, $A^{(1)}$ is block-triangular.

The system of differential equations simplifies for $t = m^2$ (corresponding to $y = 1$) as well as for $s = \infty$ (corresponding to $x = 0$). In both limits the matrix $A^{(0)}$ vanishes. In the former case ($t = m^2$) the integration kernels are linear combinations of the one-forms given in eq. (4.8) and the solution for the master integrals can be expressed in terms of multiple polylogarithms. In the latter case ($s = \infty$) the integration kernels are of the form

$$f \ (2\pi i) \, d\tau_6^{(a)}, \tag{7.3}$$

where f is a modular form of the congruence subgroup $\Gamma_1(6)$ from the set given in eq. (5.74). In this case the solution for the master integrals can be expressed in terms of iterated integrals of modular forms. For all modular forms from the set given in eq. (5.74) the modular weight can be inferred from the scaling behaviour under a rescaling of the periods. One has

$$\text{modular weight} = \text{scaling power} + 2, \tag{7.4}$$

where the additional 2 is due to the Jacobian obtained by replacing dy by $d\tau_6^{(a)}$.

We may view the entries of

$$A = A^{(0)} + \varepsilon A^{(1)} \tag{7.5}$$

as differential one-forms rational in

$$\varepsilon, \tilde{x}, y, \psi_1^{(a)}, \psi_1^{(b)}, \psi_1^{(c)}, \partial_y \psi_1^{(a)}, \partial_y \psi_1^{(b)}, \partial_y \psi_1^{(c)}. \tag{7.6}$$

We observe that each entry of A is homogeneous under a simultaneous rescaling of all periods and their derivatives

$$\psi_1^{(r)} \to \lambda \, \psi_1^{(r)}, \quad \partial_y \psi_1^{(r)} \to \lambda \, \partial_y \psi_1^{(r)}, \qquad r \in \{a, b, c\}. \tag{7.7}$$

This allows us to group the entries of A according to the scaling behaviour under a simultaneous rescaling of all periods and their derivatives. We define a m-weight as

$$m\text{-weight} = \text{scaling power} + 2. \tag{7.8}$$

This is an ad hoc definition, which we find useful for bookkeeping. No further properties are implied. In the limit $x = 0$ the m-weight agrees with the modular weight.

Let us note that the requirements that

– A is linear in ε,

– $A^{(0)}$ is strictly lower-triangular,

– $A^{(0)}$ vanishes for $x = 0$ or $y = 1$,

– $A^{(1)}$ reduces to integration kernels for multiple polylogarithms for $y = 1$,

– $A^{(1)}$ reduces to modular forms for $x = 0$,

do not fix uniquely the set of master integrals $\vec{J}$ and the matrix A. There is still a "gauge freedom" of transformations left, leaving these conditions intact.

7.1 Integration kernels

Let us now discuss the integration kernels appearing in the matrix A. The entries of the matrix A can be written as linear combinations (with rational coefficients) of fewer basic building blocks, such that no further linear relations with rational coefficients exist among these building blocks. We call these building blocks $\mathbb{Q}$-independent integration kernels. Let us explain this concept with an example. Let us restrict to the subset of Feynman integrals which only depend on s. For this subset of Feynman integrals, all entries of A are linear combinations of

$$\frac{d\tilde{x}}{\tilde{x} - c},\tag{7.9}$$

with

$$c \in \tilde{\mathcal{A}} = \left\{ -1, 0, 1, i, -i, 1 + \sqrt{2}, 1 - \sqrt{2}, -1 + \sqrt{2}, -1 - \sqrt{2} \right\}.\tag{7.10}$$

The alphabet $\tilde{\mathcal{A}}$ has nine letters. However, in the matrix A only specific linear combinations of these one-forms appear. All entries of A can be expressed as $\mathbb{Q}$-linear combinations of

$$\{\omega_0, \omega_4, \omega_{-4}, \omega_{0,4}, \omega_{-4,0}\},\tag{7.11}$$

where the ω's have been defined in eq. (4.8). Thus the set of $\mathbb{Q}$-independent integration kernels contains for this example only five elements, given by eq. (7.11). It is clear that the results in terms of iterated integrals are shorter, if we work with a $\mathbb{Q}$-independent set of integration kernels.

Let us now return to the general case. For our choice of basis $\vec{J}$ we find 107 $\mathbb{Q}$-independent integration kernels. It is a matter of personal taste, if one considers this number to be large or small. On the one hand, it can be considered a large number as it limits our possibilities to present explicit results on paper: for iterated integrals of depth 4 we face a priori 107^4 combinations. On the other hand, it can be considered a small number, given the fact that we are dealing with a matrix of size 44×44. The matrix A can be considered to be sparse.

Let us mention explicitly that the number 107 is the number for our choice of master integrals $\vec{J}$. Other choices of master integrals (respecting the criteria given at the beginning of this section) may lead to a different number of $\mathbb{Q}$-independent integration kernels.

We group the integration kernels according to their m-weight. We have integration kernels with m-weight 0, 1, 2, 3 and 4. The complexity of the expressions increases with the m-weight. The ε^0-part $A^{(0)}$ contains only integration kernels of m-weight 3 and 4.

Our naming scheme is as follows: we keep the notation for the integration kernels, which already occurred in the special cases $t = m^2$ or $s = \infty$. This concerns

$$\{\omega_0, \omega_4, \omega_{-4}, \omega_{0,4}, \omega_{-4,0}, f_2, f_3, f_4, g_{2,1}\}. \tag{7.12}$$

Integration kernels appearing in the ε^0-part $A^{(0)}$ are denoted by

$$a_{n,j}^{(r)}, \tag{7.13}$$

where n gives the m-weight, (r) indicates the periods appearing in the integration kernel and j indexes different integration kernels with the same n and (r). We denote integration kernels appearing in the ε^1-part $A^{(1)}$ generically by

$$\eta_{n,j}^{(r)}. \tag{7.14}$$

The superscript (r) and the second subscript j are optional. The interpretation of the super- and subscripts is as above. For dlog-forms we use the notation

$$d_{2,j}. \tag{7.15}$$

These are necessarily of m-weight 2.

Let us now discuss for all m-weights typical examples, the cases of m-weight 0 and 1 are discussed completely. The full list of integration kernels is given in the supplementary electronic file attached to this article. The full list consists of the integration kernels

$$\begin{aligned}
&\Big\{\omega_0, \omega_4, \omega_{-4}, \omega_{0,4}, \omega_{-4,0}, f_2, f_3, f_4, g_{2,1}, \eta_0^{(r)}, \eta_{1,1-4}^{(b)}, \eta_{1,1-3}^{(c)}, d_{2,1-5}, \eta_{2,1-12}, \eta_2^{\left(\frac{r}{s}\right)}, \\
&\quad a_{3,1-4}^{(b)}, a_{3,1-3}^{(c)}, \eta_{3,1-3}^{(a)}, \eta_{3,1-24}^{(b)}, \eta_{3,1-11}^{(c)}, a_{4,1}^{(a,b)}, a_{4,1}^{(a,c)}, a_{4,1-5}^{(b,b)}, a_{4,1}^{(c,c)}, a_{4,1}^{(b,c)}, \\
&\quad \eta_{4,1-3}^{(a,b)}, \eta_{4,1}^{(a,c)}, \eta_{4,1-5}^{(b,b)}, \eta_{4,1}^{(c,c)}, \eta_{4,1}^{(b,c)}\Big\},
\end{aligned} \tag{7.16}$$

with $r, s \in \{a, b, c\}$ and $r \neq s$.

7.1.1 m-weight 0

At m-weight 0 we have three integration kernels. They are given by

$$\begin{aligned}
\eta_0^{(a)} &= 2\pi i \, d\tau_6^{(a)}, \\
\eta_0^{(b)} &= 2\pi i \, d\tau_6^{(b)}, \\
\eta_0^{(c)} &= 2\pi i \, d\tau_6^{(c)}.
\end{aligned} \tag{7.17}$$

These are exactly the integration kernels we expect at m-weight 0. We expressed them (compactly) in terms of the variables $\tau_6^{(a)}$, $\tau_6^{(b)}$ or $\tau_6^{(c)}$, respectively. Of course we may re-write them in terms of the variables x and y. For example, for $\eta_0^{(b)}$ one has

$$\eta_0^{(b)} = \frac{2}{3} \frac{(x-1)\left(x^2y + 3x^2 - 6xy - 2x + y + 3\right)}{\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(xy - 1)(x - y)(x + 1)} \left(\frac{\pi}{\psi_1^{(b)}}\right)^2 dx$$

$$- \frac{2}{3} \frac{(x-1)^2 \left(3x^2 - 2xy - 4x + 3\right)}{(y-1)\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(xy - 1)(x - y)} \left(\frac{\pi}{\psi_1^{(b)}}\right)^2 dy. \qquad (7.18)$$

7.1.2 m-weight 1

We find 7 integration kernels of m-weight 1, four of them are associated to the elliptic curve $E^{(b)}$, three of them to the elliptic curve $E^{(c)}$. There are no integration kernels of m-weight 1 for the elliptic curve $E^{(a)}$. The integration kernels of m-weight associated to the elliptic curve $E^{(b)}$ are

$$\eta_{1,1}^{(b)} = \frac{(x-1)}{\left(3x^2 - 2xy - 4x + 3\right)(x+1)} \frac{\pi}{\psi_1^{(b)}} dx,$$

$$\eta_{1,2}^{(b)} = \frac{(x-1)\left(x^2y^2 - 9x^2y + 6xy^2 - 2xy + y^2 + 12x - 9y\right)}{(x+1)\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(xy - 1)(x - y)} \frac{\pi}{\psi_1^{(b)}} dx$$

$$- \frac{x(x-1)^2(y-3)}{\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(xy - 1)(x - y)} \frac{\pi}{\psi_1^{(b)}} dy,$$

$$\eta_{1,3}^{(b)} = \frac{\left(x^2y - 9x^2 - 6xy + 22x + y - 9\right)}{(x+1)\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(x - 1)} \frac{\pi}{\psi_1^{(b)}} dx$$

$$+ 2 \frac{x}{\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)} \frac{\pi}{\psi_1^{(b)}} dy,$$

$$\eta_{1,4}^{(b)} = \frac{x(y-1)\left(-6xy + y + x^2y - 2x + 3 + 3x^2\right)}{(x+1)(x-1)(xy-1)(x-y)\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)} \frac{\pi}{\psi_1^{(b)}} dx$$

$$- \frac{x\left(3x^2 - 2xy - 4x + 3\right)}{\left(x^2y - 9x^2 + 2xy + 14x + y - 9\right)(xy - 1)(x - y)} \frac{\pi}{\psi_1^{(b)}} dy. \qquad (7.19)$$

Associated to the elliptic curve $E^{(c)}$ are

$$\eta_{1,1}^{(c)} = \frac{(x+1)(y+3)}{(x-1)\left(3x^2y - 2xy^2 - 3x^2 - 2xy - 12x + 3y - 3\right)} \frac{\pi}{\psi_1^{(c)}} dx,$$

$$\eta_{1,2}^{(c)} = (x+1) \frac{\pi}{\psi_1^{(c)}}$$

$$\left[\frac{\left(x^2y^3 + 3x^2y^2 - 9xy^3 - 105x^2y + 99xy^2 + 2y^3 - 27x^2 + 45xy - 12y^2 + 57x - 54y\right)}{(x-1)\left(x^2y^2 - 10x^2y - 2xy^2 + 9x^2 + 20xy + y^2 + 46x - 10y + 9\right)(xy - 1)(x - y)} dx \right.$$

$$\left. + \frac{x\left(3x^2y^2 - 4xy^3 - 30x^2y + 38xy^2 - 2y^3 + 27x^2 - 48xy + 25y^2 + 78x - 84y - 3\right)}{(y-1)\left(x^2y^2 - 10x^2y - 2xy^2 + 9x^2 + 20xy + y^2 + 46x - 10y + 9\right)(x - y)(xy - 1)} dy \right],$$

$$\eta_{1,3}^{(c)} = \frac{(x+1)^2(y-3)}{(x-1)\left(3x^2y - 2xy^2 - 3x^2 - 2xy - 12x + 3y - 3\right)\sqrt{x^2 - 6x + 1}} \frac{\pi}{\psi_1^{(c)}} dx. \qquad (7.20)$$

Note that in the last expression the square root is rationalised by using the variable $\tilde{x}$. An alternative form for $\eta_{1,3}^{(c)}$ is

$$\eta_{1,3}^{(c)} = -\frac{\pi}{\psi_1^{(c)}} d\tilde{x} \tag{7.21}$$

$$\frac{(y-3)\left(\tilde{x}^2 - 2\tilde{x} - 1\right)^2}{(\tilde{x}^2 + 1)\left(3\tilde{x}^4 y + 2\tilde{x}^3 y^2 - 3\tilde{x}^4 - 4\tilde{x}^3 y + 18\tilde{x}^3 + 6\tilde{x}^2 y - 2\tilde{x}y^2 - 6\tilde{x}^2 + 4\tilde{x}y - 18\tilde{x} + 3y - 3\right)},$$

which is manifestly rational in $\tilde{x}$, y and $\psi_1^{(c)}$.

7.1.3 m-weight 2

The integration kernels of m-weight 2 are numerous and we only list a few typical cases. The integration kernels for the multiple polylogarithms

$$\omega_0, \; \omega_4, \; \omega_{-4}, \; \omega_{0,4}, \; \omega_{-4,0}, \tag{7.22}$$

defined in eq. (4.8) belong to this class. Furthermore, the modular forms of modular weight 2 clearly belong to this class:

$$f_2 \; (2\pi i)\, d\tau_6^{(a)} = \frac{dy}{y-1} + \frac{dy}{y-9} - \frac{dy}{2y},$$

$$g_{2,1} \; (2\pi i)\, d\tau_6^{(a)} = \frac{dy}{y-1}. \tag{7.23}$$

The differential one-forms in eq. (7.22) and eq. (7.23) are all dlog-forms, depending either on x (or alternatively on $\tilde{x}$) or y, but not both. There are further dlog-forms, depending on both variables x and y. These are

$$
\begin{aligned}
d_{2,1} &= d\ln(x-y) + d\ln(xy-1), \\
d_{2,2} &= d\ln(xy-1), \\
d_{2,3} &= d\ln\left(x^2 - xy - x + 1\right), \\
d_{2,4} &= d\ln\left(3x^2 - 2xy - 4x + 3\right), \\
d_{2,5} &= d\ln\left(x^2 y - 9x^2 + 2xy + 14x + y - 9\right).
\end{aligned}
\tag{7.24}
$$

There are six differential one-forms involving ratios of periods, one for each ratio. For example

$$\eta_2^{\left(\frac{b}{a}\right)} = \frac{1}{2}\frac{(x+1)}{x\,(x-1)}\frac{\psi_1^{(b)}}{\psi_1^{(a)}} dx \tag{7.25}$$

$$-\frac{1}{12}\frac{\left(3x^2 y^2 + 2xy^3 - 90x^2 y + 52xy^2 - 81x^2 + 138xy + 3y^2 + 144x - 90y - 81\right)}{y\,(y-1)\,(y-9)\,(3x^2 - 2xy - 4x + 3)}\frac{\psi_1^{(b)}}{\psi_1^{(a)}} dy.$$

In addition there are 12 differential one-forms of m-weight 2, which do not belong to any class discussed up to now. An example is given by

$$\eta_{2,1} = \frac{(x-1)}{(x+1)\left(3x^2 - 2xy - 4x + 3\right)} dx. \tag{7.26}$$

For our choice of master integrals $\vec{J}$ we observe that in the integration kernels of m-weight 2 polynomials in denominator occur only as a single power, i.e. there are no higher poles in m-weight 2.

7.1.4 m-weight 3

At m-weight 3 we have first of all the modular form of weight 3 from the sunrise sector

$$f_3 \ (2\pi i)\, d\tau_6^{(a)} = 3\frac{\psi_1^{(a)}}{\pi} dy. \tag{7.27}$$

At m-weight 3 we have integration kernels appearing in the ε^0-part $A^{(0)}$, an example is given by

$$
\begin{aligned}
a_{3,1}^{(b)} = {} & \frac{\left(x^2 y - 3\,x^2 + 4\,xy + y - 3\right)(y-1)}{(x-1)\left(3\,x^2 - 2\,xy - 4\,x + 3\right)(x+1)} \frac{\psi_1^{(b)}}{\pi} dx \\
& + \frac{\left(x^2 y^2 - 9\,x^2 y + 6\,xy^2 - 2\,xy + y^2 + 12\,x - 9\,y\right)(y-1)}{(x-1)\left(3\,x^2 - 2\,xy - 4\,x + 3\right)(x+1)}\left(\frac{\partial_y \psi_1^{(b)}}{\pi}\right) dx \\
& - \frac{x\,(y-1)}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)}\frac{\psi_1^{(b)}}{\pi} dy - \frac{(y-3)\,x\,(y-1)}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)}\left(\frac{\partial_y \psi_1^{(b)}}{\pi}\right) dy. \tag{7.28}
\end{aligned}
$$

In addition, there are integration kernels of m-weight 3 in the ε^1-part $A^{(1)}$, an example is given by

$$\eta_{3,1}^{(b)} = 4\,\frac{(y-1)}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)}\frac{\psi_1^{(b)}}{\pi} dx - \frac{(x-1)\,(x+1)}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)}\frac{\psi_1^{(b)}}{\pi} dy. \tag{7.29}$$

7.1.5 m-weight 4

At m-weight 4 we have one modular form of weight 4 from the sunrise sector

$$f_4 \ (2\pi i)\, d\tau_6^{(a)} = -\frac{(y+3)^4}{8y\,(y-1)\,(y-9)}\left(\frac{\psi_1^{(a)}}{\pi}\right)^2 dy. \tag{7.30}$$

In addition, we encounter integration kernels appearing in the ε^0-part $A^{(0)}$. An example is given by

$$
\begin{aligned}
a_{4,3}^{(b,b)} = {} & -\frac{2}{3}\,\frac{(y-3)\,(y-1)\,N_{4,3,1}^{(b,b)}}{(x-1)\,(x+1)\left(3\,x^2 - 2\,xy - 4\,x + 3\right)^2}\left(\frac{\psi_1^{(b)}}{\pi}\right)^2 dx \\
& + \frac{4}{3}\,\frac{x\,(y-3)\,(y-1)^2\,N_{4,3,2}^{(b,b)}}{(x-1)\,(x+1)\left(3\,x^2 - 2\,xy - 4\,x + 3\right)^2}\left(\frac{\psi_1^{(b)}}{\pi}\right)\left(\frac{\partial_y \psi_1^{(b)}}{\pi}\right) dx \\
& + \frac{2}{3}\,\frac{x\,N_{4,3,3}^{(b,b)}}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)^2}\left(\frac{\psi_1^{(b)}}{\pi}\right)^2 dy \\
& + \frac{4}{3}\,\frac{x\,(y-3)\,(y-1)\,N_{4,3,4}^{(b,b)}}{\left(3\,x^2 - 2\,xy - 4\,x + 3\right)^2}\left(\frac{\psi_1^{(b)}}{\pi}\right)\left(\frac{\partial_y \psi_1^{(b)}}{\pi}\right) dy, \tag{7.31}
\end{aligned}
$$

with

$$N_{4,3,1}^{(b,b)} = 3\,x^4y - 2\,x^3y^2 + 9\,x^4 + 20\,x^3y - 20\,x^2y^2 - 18\,x^3 + 2\,x^2y - 2\,xy^2 - 6\,x^2 + 20\,xy - 18\,x$$
$$+ 3\,y + 9,$$
$$N_{4,3,2}^{(b,b)} = x^2y^2 - 24\,x^2y + 18\,xy^2 - 9\,x^2 + 16\,xy + y^2 + 30\,x - 24\,y - 9,$$
$$N_{4,3,3}^{(b,b)} = 9\,x^2y^2 - 8\,xy^3 - 6\,x^2y + 10\,xy^2 - 27\,x^2 - 20\,xy + 9\,y^2 + 66\,x - 6\,y - 27,$$
$$N_{4,3,4}^{(b,b)} = 3\,x^2y - 4\,xy^2 + 9\,x^2 - 2\,xy - 18\,x + 3\,y + 9. \tag{7.32}$$

Finally, there are integration kernels of m-weight 4 appearing in the ε^1-part $A^{(1)}$. An example is given by

$$\eta_{4,3}^{(b,b)} = \frac{1}{9} \frac{1}{(xy-1)\,(x-y)\,(3\,x^2-2\,xy-4\,x+3)^2\,(x^2y-9\,x^2+2\,xy+14\,x+y-9)}$$
$$\left[\frac{(y-1)\,P_{4,3,1}^{(b,b)}}{(x-1)\,(x+1)}\,dx - \frac{(x-1)^2\,P_{4,3,2}^{(b,b)}}{(y-1)}\,dy \right] \left(\frac{\psi_1^{(b)}}{\pi} \right)^2, \tag{7.33}$$

with

$$P_{4,3,1}^{(b,b)} = 27\,x^8y^4 - 75\,x^7y^5 + 48\,x^6y^6 - 243\,x^8y^3 + 909\,x^7y^4 - 946\,x^6y^5 + 288\,x^5y^6 + 2673\,x^8y^2$$
$$- 7182\,x^7y^3 + 5914\,x^6y^4 - 1237\,x^5y^5 - 288\,x^4y^6 - 729\,x^8y - 6966\,x^7y^2 + 17592\,x^6y^3$$
$$- 11277\,x^5y^4 + 1828\,x^4y^5 + 288\,x^3y^6 - 3159\,x^7y + 22392\,x^6y^2 - 36146\,x^5y^3$$
$$+ 15766\,x^4y^4 - 1237\,x^3y^5 + 48\,x^2y^6 + 729\,x^7 + 13770\,x^6y - 41898\,x^5y^2 + 43510\,x^4y^3$$
$$- 11277\,x^3y^4 - 946\,x^2y^5 + 1134\,x^6 - 25929\,x^5y + 52590\,x^4y^2 - 36146\,x^3y^3$$
$$+ 5914\,x^2y^4 - 75\,xy^5 - 9369\,x^5 + 30942\,x^4y - 41898\,x^3y^2 + 17592\,x^2y^3 + 909\,xy^4$$
$$+ 15012\,x^4 - 25929\,x^3y + 22392\,x^2y^2 - 7182\,xy^3 + 27\,y^4 - 9369\,x^3 + 13770\,x^2y$$
$$- 6966\,xy^2 - 243\,y^3 + 1134\,x^2 - 3159\,xy + 2673\,y^2 + 729\,x - 729\,y,$$
$$P_{4,3,2}^{(b,b)} = 27\,x^6y^4 - 45\,x^5y^5 - 30\,x^4y^6 + 48\,x^3y^7 + 243\,x^6y^3 - 603\,x^5y^4 + 828\,x^4y^5 - 460\,x^3y^6$$
$$+ 729\,x^6y^2 - 2592\,x^5y^3 + 1899\,x^4y^4 - 126\,x^3y^5 - 30\,x^2y^6 + 729\,x^6y - 4212\,x^5y^2$$
$$+ 8085\,x^4y^3 - 4686\,x^3y^4 + 828\,x^2y^5 - 2187\,x^5y + 6741\,x^4y^2 - 8864\,x^3y^3 + 1899\,x^2y^4$$
$$- 45\,xy^5 - 729\,x^5 + 7587\,x^4y - 9708\,x^3y^2 + 8085\,x^2y^3 - 603\,xy^4 + 810\,x^4 - 9954\,x^3y$$
$$+ 6741\,x^2y^2 - 2592\,xy^3 + 27\,y^4 - 810\,x^3 + 7587\,x^2y - 4212\,xy^2 + 243\,y^3 + 810\,x^2$$
$$- 2187\,xy + 729\,y^2 - 729\,x + 729\,y. \tag{7.34}$$

7.2 Singularities

As already mentioned, the integration kernels are rational in

$$\varepsilon, \tilde{x}, y, \psi_1^{(a)}, \psi_1^{(b)}, \psi_1^{(c)}, \partial_y \psi_1^{(a)}, \partial_y \psi_1^{(b)}, \partial_y \psi_1^{(c)}. \tag{7.35}$$

In the next section we will choose as boundary point the point $(x, y) = (0, 1)$ (or equivalently $(s, t) = (\infty, m^2)$). This motivates the introduction of the variable

$$\tilde{y} = 1 - y. \tag{7.36}$$

Our boundary point is then $(\tilde{x}, \tilde{y}) = (0, 0)$.

Of particular interest are the polynomials in $\tilde{x}$ and $\tilde{y}$ appearing in the denominator of the integration kernels. There aren't too many. Polynomials, which only depend on $\tilde{x}$ are (compare with eq. (4.8))

$$
\begin{aligned}
Q_1 &= \tilde{x}, \\
Q_2 &= \tilde{x} - 1, \\
Q_3 &= \tilde{x} + 1, \\
Q_4 &= \tilde{x}^2 + 1, \\
Q_5 &= \tilde{x}^2 - 2\tilde{x} - 1, \\
Q_6 &= \tilde{x}^2 + 2\tilde{x} - 1.
\end{aligned}
\tag{7.37}
$$

Polynomials, which only depend on $\tilde{y}$ are

$$
\begin{aligned}
Q_7 &= \tilde{y}, \\
Q_8 &= \tilde{y} - 1, \\
Q_9 &= \tilde{y} + 8.
\end{aligned}
\tag{7.38}
$$

Polynomials, which depend on $\tilde{x}$ and $\tilde{y}$ are

$$
\begin{aligned}
Q_{10} &= \tilde{x}^2 - \tilde{x}\,\tilde{y} - \tilde{y} + 1, \\
Q_{11} &= -\tilde{x}^2\tilde{y} + \tilde{x}^2 + \tilde{x}\,\tilde{y} + 1, \\
Q_{12} &= \tilde{x}^4 - \tilde{x}^3\tilde{y} + 2\,\tilde{x}^2 + \tilde{x}\,\tilde{y} + 1, \\
Q_{13} &= 3\,\tilde{x}^4 - 2\,\tilde{x}^3\tilde{y} + 6\,\tilde{x}^2 + 2\,\tilde{x}\,\tilde{y} + 3, \\
Q_{14} &= \tilde{x}^4\tilde{y} + 8\,\tilde{x}^4 - 4\,\tilde{x}^3\tilde{y} + 2\,\tilde{x}^2\tilde{y} + 16\,\tilde{x}^2 + 4\,\tilde{x}\,\tilde{y} + \tilde{y} + 8, \\
Q_{15} &= 3\,\tilde{x}^4\tilde{y} - 2\,\tilde{x}^3\tilde{y}^2 - 16\,\tilde{x}^3 + 6\,\tilde{x}^2\tilde{y} + 2\,\tilde{x}\,\tilde{y}^2 + 16\,\tilde{x} + 3\,\tilde{y}, \\
Q_{16} &= \tilde{x}^2\tilde{y} - 8\,\tilde{x} + \tilde{y} + 8, \\
Q_{17} &= \tilde{x}^2\tilde{y} + 8\,\tilde{x}^2 + 8\,\tilde{x} + \tilde{y}.
\end{aligned}
\tag{7.39}
$$

It is helpful to have corresponding expressions in (s,t)-space: the polynomials Q_{10} and Q_{11} appear when the expression $st + (m^2 - t)^2$ is expressed in the variables $\tilde{x}$ and $\tilde{y}$:

$$
st + \left(m^2 - t\right)^2 = \frac{Q_{10}Q_{11}}{\tilde{x}\,(\tilde{x} - 1)\,(\tilde{x} + 1)}.
\tag{7.40}
$$

The polynomial Q_{12} is related to

$$
m^2 - t - s = -\frac{Q_{12}}{\tilde{x}\,(\tilde{x} - 1)\,(\tilde{x} + 1)}.
\tag{7.41}
$$

The polynomials Q_{13} and Q_{14} are related to the elliptic curve $E^{(b)}$. We have

$$
\begin{aligned}
3s + 2t - 2m^2 &= \frac{Q_{13}}{\tilde{x}\,(\tilde{x} - 1)\,(\tilde{x} + 1)}, \\
s\left(t - 9m^2\right) + 4m^2\left(m^2 - t\right) &= -\frac{Q_{14}}{\tilde{x}\,(\tilde{x} - 1)\,(\tilde{x} + 1)}.
\end{aligned}
\tag{7.42}
$$

The polynomial Q_{13} enters through eq. (5.53) or eq. (5.54), the polynomial Q_{14} appears in the Picard-Fuchs operator for $\psi_1^{(b)}$ in eq. (5.51).

The polynomials Q_{15}, Q_{16} and Q_{17} are related to the elliptic curve $E^{(c)}$. We have

$$
3s\left(t - m^2\right) + 2\left(t - m^2\right)^2 + 16m^4 = -\frac{Q_{15}}{\tilde{x}\left(\tilde{x} - 1\right)\left(\tilde{x} + 1\right)},
$$

$$
s\left(t - 9m^2\right)\left(t - m^2\right) - 64m^6 = \frac{Q_{16}Q_{17}}{\tilde{x}\left(\tilde{x} - 1\right)\left(\tilde{x} + 1\right)}. \tag{7.43}
$$

The polynomial Q_{15} enters through eq. (5.63) or eq. (5.64), the polynomials Q_{16} and Q_{17} appears in the Picard-Fuchs operator for $\psi_1^{(c)}$ in eq. (5.61).

Note that the polynomials Q_1, Q_7, Q_{15} and Q_{17} vanish for $(\tilde{x}, \tilde{y}) = (0, 0)$.

8 Boundary conditions and boundary constants

We integrate the system of differential equations starting from the point $(x, y) = (0, 1)$ (corresponding to $s = \infty$ and $t = m^2$). In order to do so, we need the boundary constants at this point. The boundary constants may be expressed as a linear combination of transcendental constants. A basis of these transcendental constants up to weight four is given by [113]

$$
\begin{aligned}
w = 1: \quad & \ln(2), \\
w = 2: \quad & \zeta_2, \quad \ln^2(2), \\
w = 3: \quad & \zeta_3, \quad \zeta_2 \ln(2), \quad \ln^3(2), \\
w = 4: \quad & \zeta_4, \quad \mathrm{Li}_4\left(\frac{1}{2}\right), \quad \zeta_3 \ln(2), \quad \zeta_2 \ln^2(2), \quad \ln^4(2).
\end{aligned} \tag{8.1}
$$

The boundary constants for the master integrals, which are products of one-loop integrals are easily computed. For example, the master integral J_1 is a product of two tadpole integrals. The tadpole integral is given by

$$
T_\nu\left(D, m^2, \mu^2\right) = e^{\gamma_E \varepsilon}\left(\mu^2\right)^{\nu - \frac{D}{2}} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{1}{\left(-k^2 + m^2\right)^\nu} = e^{\gamma_E \varepsilon}\frac{\Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma\left(\nu\right)}\left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2} - \nu}. \tag{8.2}
$$

For $D = 2 - 2\varepsilon$, $\mu = m$ and $\nu = 1$ we have

$$
T_1\left(2 - 2\varepsilon\right) = e^{\gamma_E \varepsilon}\Gamma\left(\varepsilon\right) = \frac{1}{\varepsilon}\left[1 + \frac{1}{2}\zeta_2 \varepsilon^2 - \frac{1}{3}\zeta_3 \varepsilon^3 + \frac{9}{16}\zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right)\right]. \tag{8.3}
$$

In addition we need to calculate explicitly the boundary constants for the integrals, which neither depend on s nor on t. There are two such integrals: J_1 (which is also a product of tadpoles) and J_8 (the sunrise integral at the pseudo-threshold). For the latter we proceed as follows: we start from the Feynman parameter integral. We have

$$
J_8 = 6\varepsilon^2 e^{2\gamma_E \varepsilon}\Gamma\left(1 + 2\varepsilon\right)\int_0^1 dx_2 \int_0^1 dx_4 \left[\frac{1}{x_2 - 1} - \frac{1}{x_2 + 1}\right]\left[\frac{1}{x_4 + 1} - \frac{1}{x_4 + x_2}\right]
$$

$$
\times \left(x_2 + 1\right)^\varepsilon \left(x_4 + 1\right)^{-2\varepsilon}\left(x_4 + x_2\right)^{-2\varepsilon}\left(x_4 + \frac{x_2}{x_2 + 1}\right)^\varepsilon. \tag{8.4}
$$

At each order in ε, the x_4-integration is easily performed, resulting in multiple polylogarithms $G(z_1, \ldots, z_k; x_2)$, where the remaining variable x_2 appears in the argument list z_1, $\ldots$, z_k. With the methods of [97, 114] we convert all polylogarithms to a form, where the parameters z_1, $\ldots$, z_k do not depend on x_2, the simplest example is given by

$$G(-x_2; 1) = G(-1; x_2) - G(0; x_2). \tag{8.5}$$

We may then perform the integration over the variable x_2. The resulting expressions may be simplified with the help of the PSLQ-algorithm [115].

For all other master integrals we obtain the boundary constants from the behaviour at a specific point, where the master integral vanishes or reduces to simpler integrals. The specific points which we consider are $(x, y) = (0, 1)$, $(x, y) = (1, 1)$ and $(x, y) = (-1, 1)$. For the points $(x, y) = (1, 1)$ or $(x, y) = (-1, 1)$ we integrate the system along $y = 1$ from $x = 0$ to $x = \pm 1$. For $y = 1$ we only obtain multiple polylogarithms. We evaluate the multiple polylogarithms to high precision [97] and use the PSLQ-algorithm to extract the transcendental constants.

The complete list of boundary constants is given in appendix D.

9 Results

For all basis integrals we write

$$J_k = \sum_{j=0}^{\infty} \varepsilon^j J_k^{(j)}. \tag{9.1}$$

Obviously, the basis of master integrals involves also rather simple sub-topologies, where results for these integrals are known in the literature [116–118].

9.1 Integrals, which do not depend on s nor t

The integrals J_1 and J_8 are independent of s and t (or equivalently independent of x and y). The integrals J_1 and J_8 are given by

$$
\begin{aligned}
J_1 &= 1 + \zeta_2 \varepsilon^2 - \frac{2}{3} \zeta_3 \varepsilon^3 + \frac{7}{4} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right), \\
J_8 &= 6\zeta_2 \varepsilon^2 + \varepsilon^3 \left(21\zeta_3 - 36\zeta_2 \ln 2\right) + \varepsilon^4 \left(144 \operatorname{Li}_4\left(\frac{1}{2}\right) - 78\zeta_4 + 72\zeta_2 \ln^2(2) + 6\ln^4(2)\right) \\
&\quad + \mathcal{O}\left(\varepsilon^5\right).
\end{aligned}
\tag{9.2}
$$

9.2 Integrals, which only depend on s

The integrals $J_2 - J_5$, $J_9 - J_{13}$, $J_{15} - J_{16}$, $J_{18} - J_{23}$, J_{32}, $J_{36} - J_{37}$ and J_{40} depend only on the variable s. We may group them into three categories. The first category contains most of these integrals. More concretely, the integrals in the first category are given by $J_2 - J_5$, $J_9 - J_{11}$, $J_{15} - J_{16}$, $J_{18} - J_{21}$, J_{23}, J_{32}, J_{37} and J_{40}. These integrals are most naturally expressed in terms of harmonic polylogarithms in the variable x.

The integrals J_{12} and J_{13} belong to the second category. These integrals have a singular point at $s = -4m^2$ but not at $s = 4m^2$. These two integrals are most naturally expressed in terms of harmonic polylogarithms in the variable x'.

The integrals J_{22} and J_{36} belong to the third category. These integrals have a singular point at $s = -4m^2$ and at $s = 4m^2$. These two integrals are expressed in terms of multiple polylogarithms in the variable $\tilde{x}$.

9.2.1 Integrals, which are expressed in the variable x

The integrals $J_2 - J_5$, $J_9 - J_{11}$, $J_{15} - J_{16}$, $J_{18} - J_{21}$, J_{23}, J_{32}, J_{37} and J_{40} are most naturally expressed in terms of harmonic polylogarithms in the variable x.

$$J_2^{(0)} = 0,$$
$$J_2^{(1)} = -G\left(0;x\right),$$
$$J_2^{(2)} = 2G\left(-1,0;x\right) - G\left(0,0;x\right) + \zeta_2,$$
$$J_2^{(3)} = -4G\left(-1,-1,0;x\right) + 2G\left(-1,0,0;x\right) + 2G\left(0,-1,0;x\right) - G\left(0,0,0;x\right) - 2\zeta_2 G\left(-1;x\right)$$
$$+ 2\zeta_3,$$
$$J_2^{(4)} = 8G\left(-1,-1,-1,0;x\right) - 4G\left(-1,-1,0,0;x\right) - 4G\left(-1,0,-1,0;x\right) - 4G\left(0,-1,-1,0;x\right)$$
$$+ 2G\left(-1,0,0,0;x\right) + 2G\left(0,-1,0,0;x\right) + 2G\left(0,0,-1,0;x\right) - G\left(0,0,0,0;x\right)$$
$$+ 4\zeta_2 G\left(-1,-1;x\right) - 2\zeta_2 G\left(0,-1;x\right) - 4\zeta_3 G\left(-1;x\right) + \frac{8}{3}\zeta_3 G\left(0;x\right) + \frac{19}{4}\zeta_4,$$
$$J_3^{(0)} = 0,$$
$$J_3^{(1)} = G\left(0;x\right),$$
$$J_3^{(2)} = 4G\left(0,0;x\right) - 6G\left(-1,0;x\right) - 2G\left(1,0;x\right) - \zeta_2,$$
$$J_3^{(3)} = 10G\left(0,0,0;x\right) - 8G\left(0,1,0;x\right) - 24G\left(0,-1,0;x\right) - 24G\left(-1,0,0;x\right) - 8G\left(1,0,0;x\right)$$
$$+ 12G\left(-1,1,0;x\right) + 36G\left(-1,-1,0;x\right) + 4G\left(1,1,0;x\right) + 12G\left(1,-1,0;x\right)$$
$$+ 6\zeta_2 G\left(-1;x\right) + 2\zeta_2 G\left(1;x\right) - 3\zeta_2 G\left(0;x\right) - 11\zeta_3,$$
$$J_3^{(4)} = -20G\left(0,0,1,0;x\right) + 48G\left(0,-1,1,0;x\right) + 16G\left(0,1,1,0;x\right) + 48G\left(0,1,-1,0;x\right)$$
$$- 44G\left(0,1,0,0;x\right) - 24G\left(1,1,-1,0;x\right) + 16G\left(1,1,0,0;x\right) - 8G\left(1,1,1,0;x\right)$$
$$+ 16G\left(1,0,1,0;x\right) + 48G\left(1,-1,0,0;x\right) - 24G\left(1,-1,1,0;x\right) - 72G\left(1,-1,-1,0;x\right)$$
$$- 20G\left(1,0,0,0;x\right) + 48G\left(1,0,-1,0;x\right) - 72G\left(-1,1,-1,0;x\right) + 48G\left(-1,1,0,0;x\right)$$
$$- 24G\left(-1,1,1,0;x\right) - 72G\left(-1,-1,1,0;x\right) + 48G\left(-1,0,1,0;x\right) - 60G\left(-1,0,0,0;x\right)$$
$$+ 144G\left(-1,0,-1,0;x\right) + 144G\left(-1,-1,0,0;x\right) - 216G\left(-1,-1,-1,0;x\right)$$
$$+ 144G\left(0,-1,-1,0;x\right) + 22G\left(0,0,0,0;x\right) - 60G\left(0,0,-1,0;x\right) - 96G\left(0,-1,0,0;x\right)$$
$$+ 18\zeta_2 G\left(-1,0;x\right) - 12\zeta_2 G\left(-1,1;x\right) - 12\zeta_2 G\left(1,-1;x\right) + 22\zeta_3 G\left(1;x\right)$$
$$- 4\zeta_2 G\left(1,1;x\right) + 6\zeta_2 G\left(1,0;x\right) + 8\zeta_2 G\left(0,1;x\right) - 6\zeta_2 G\left(0,0;x\right) + 66\zeta_3 G\left(-1;x\right)$$
$$+ 24\zeta_2 G\left(0,-1;x\right) - 36\zeta_2 G\left(-1,-1;x\right) - \frac{80}{3}\zeta_3 G\left(0;x\right) - \frac{29}{2}\zeta_4,$$
$$J_4^{(0)} = 0,$$

$$J_4^{(1)} = 0,$$

$$J_4^{(2)} = 2\,G\,(0,0;x)\,,$$

$$J_4^{(3)} = 6\,G\,(0,0,0;x) - 4\,G\,(0,1,0;x) - 12\,G\,(0,-1,0;x) - 2\,\zeta_2\,G\,(0;x) + 4\,G\,(1,0,0;x) - 6\,\zeta_3,$$

$$J_4^{(4)} = 8\,G\,(1,1,0,0;x) + 8\,G\,(0,1,1,0;x) + 24\,G\,(0,1,-1,0;x) + 24\,G\,(0,-1,1,0;x)$$
$$\quad - 20\,G\,(0,1,0,0;x) - 12\,G\,(0,0,1,0;x) + 12\,G\,(1,0,0,0;x) - 24\,G\,(1,0,-1,0;x)$$
$$\quad - 8\,G\,(1,0,1,0;x) + 72\,G\,(0,-1,-1,0;x) + 14\,G\,(0,0,0,0;x) - 36\,G\,(0,0,-1,0;x)$$
$$\quad - 48\,G\,(0,-1,0,0;x) - 4\,\zeta_2\,G\,(1,0;x) + 4\,\zeta_2\,G\,(0,1;x) - 4\,\zeta_2\,G\,(0,0;x)$$
$$\quad + 12\,\zeta_2\,G\,(0,-1;x) - 16\,\zeta_3\,G\,(0;x) - 12\,\zeta_3\,G\,(1;x) - \frac{13}{2}\,\zeta_4,$$

$$J_5^{(0)} = 1,$$

$$J_5^{(1)} = -2\,G\,(1;x) + G\,(0;x),$$

$$J_5^{(2)} = 4\,G\,(1,1;x) - 2\,G\,(1;x)\,G\,(0;x) + G\,(0,0;x),$$

$$J_5^{(3)} = -8\,G\,(1,1,1;x) + 4\,G\,(1,1;x)\,G\,(0;x) - 2\,G\,(1;x)\,G\,(0,0;x) + G\,(0,0,0;x) - \frac{8}{3}\,\zeta_3,$$

$$J_5^{(4)} = 16\,G\,(1,1,1,1;x) - 8\,G\,(1,1,1;x)\,G\,(0;x) + 4\,G\,(1,1;x)\,G\,(0,0;x) - 2\,G\,(1;x)\,G\,(0,0,0;x)$$
$$\quad + G\,(0,0,0,0;x) + \frac{16}{3}\,\zeta_3\,G\,(1,x) - \frac{8}{3}\,\zeta_3\,G\,(0;x) - 3\,\zeta_4,$$

$$J_9^{(0)} = 0,$$

$$J_9^{(1)} = 0,$$

$$J_9^{(2)} = G\,(0,0;x)\,,$$

$$J_9^{(3)} = G\,(0,0,0;x) - 2\,G\,(0,-1,0;x) - \zeta_2\,G\,(0;x) - 3\,\zeta_3,$$

$$J_9^{(4)} = G\,(0,0,0,0;x) - 2\,G\,(0,0,-1,0;x) - 2\,G\,(0,-1,0,0;x) + 4\,G\,(0,-1,-1,0;x)$$
$$\quad + 2\,\zeta_2\,G\,(0,-1;x) - 2\,\zeta_3\,G\,(0;x) - \frac{5}{4}\,\zeta_4,$$

$$J_{10}^{(0)} = 0,$$

$$J_{10}^{(1)} = 0,$$

$$J_{10}^{(2)} = 0,$$

$$J_{10}^{(3)} = 2\,G\,(0,0,0;x) - 4\,G\,(1,0,0;x) - 4\,\zeta_3,$$

$$J_{10}^{(4)} = 4\,G\,(0,1,0,0;x) - 4\,G\,(0,0,1,0;x) + 6\,G\,(0,0,0,0;x) - 12\,G\,(0,0,-1,0;x)$$
$$\quad - 12\,G\,(1,0,0,0;x) + 24\,G\,(1,0,-1,0;x) + 8\,G\,(1,0,1,0;x) - 8\,G\,(1,1,0,0;x)$$
$$\quad - 2\,\zeta_2\,G\,(0,0;x) + 4\,\zeta_2\,G\,(1,0;x) + 12\,\zeta_3\,G\,(1;x) - 6\,\zeta_3\,G\,(0;x) + 4\,\zeta_4,$$

$$J_{11}^{(0)} = 0,$$

$$J_{11}^{(1)} = -G\,(0;x)\,,$$

$$J_{11}^{(2)} = 2\,G\,(-1,0;x) - 3\,G\,(0,0;x) + 2\,G\,(1;x)\,G\,(0;x) + \zeta_2,$$

$$J_{11}^{(3)} = 6\,G\,(-1,0,0;x) - 4\,G\,(-1,-1,0;x) + 4\,G\,(0,-1,0;x) - 7\,G\,(0,0,0;x)$$

$$+6\,G\,(1;x)\,G\,(0,0;x)-4\,G\,(0;x)\,G\,(1,1;x)-4\,G\,(-1,0;x)\,G\,(1;x)+2\zeta_2\,G\,(0;x)$$
$$-2\zeta_2\,G\,(-1;x)-2\zeta_2\,G\,(1;x)+2\zeta_3,$$

$$J_{11}^{(4)}=12\,G\,(0,-1,0,0;x)-15\,G\,(0,0,0,0;x)+8\,G\,(0,0,-1,0;x)-8\,G\,(0,-1,-1,0;x)$$
$$+14\,G\,(-1,0,0,0;x)-8\,G\,(-1,0,-1,0;x)-12\,G\,(-1,-1,0,0;x)$$
$$+8\,G\,(-1,-1,-1,0;x)+8\,G\,(0;x)\,G\,(1,1,1;x)-12\,G\,(1,1;x)\,G\,(0,0;x)$$
$$+14\,G\,(1;x)\,G\,(0,0,0;x)-8\,G\,(0,-1,0;x)\,G\,(1;x)-12\,G\,(-1,0,0;x)\,G\,(1;x)$$
$$+8\,G\,(-1,-1,0;x)\,G\,(1;x)+8\,G\,(1,1;x)\,G\,(-1,0;x)+4\zeta_2\,G\,(1,1;x)+4\zeta_2\,G\,(0,0;x)$$
$$+4\zeta_2\,G\,(-1,-1;x)+4\zeta_2\,G\,(-1;x)\,G\,(1;x)-4\zeta_2\,G\,(-1;x)\,G\,(0;x)-4\zeta_2\,G\,(1;x)\,G\,(0;x)$$
$$-4\zeta_3\,G\,(-1;x)-4\zeta_3\,G\,(1;x)+\frac{20}{3}\zeta_3\,G\,(0;x)+\frac{9}{4}\zeta_4,$$

$$J_{15}^{(0)}=0,$$
$$J_{15}^{(1)}=0,$$
$$J_{15}^{(2)}=-G\,(0,0;x),$$
$$J_{15}^{(3)}=4\,G\,(1,0,0;x)-4\,G\,(0,0,0;x)+2\,G\,(-1,0,0;x)+2\,G\,(0,-1,0;x)-2\zeta_2\,G\,(0;x)$$
$$+6\zeta_2\,G\,(-1;x)+\frac{11}{2}\zeta_3-6\zeta_2\ln\,(2),$$

$$J_{15}^{(4)}=8\,G\,(0,0,-1,0;x)-10\,G\,(0,0,0,0;x)-4\,G\,(0,-1,-1,0;x)+8\,G\,(0,-1,0,0;x)$$
$$-4\,G\,(-1,0,-1,0;x)+8\,G\,(-1,0,0,0;x)-4\,G\,(-1,-1,0,0;x)-8\,G\,(-1,1,0,0;x)$$
$$-16\,G\,(1,1,0,0;x)-8\,G\,(1,-1,0,0;x)+16\,G\,(1,0,0,0;x)-8\,G\,(1,0,-1,0;x)$$
$$+12\,G\,(0,1,0,0;x)-3\zeta_2\,G\,(0,0;x)-24\zeta_2\,G\,(1,-1;x)+8\zeta_2\,G\,(1,0;x)$$
$$-12\zeta_2\,G\,(-1,-1;x)+16\zeta_2\,G\,(0,-1;x)+4\zeta_2\,G\,(-1,0;x)+8\zeta_3\,G\,(0;x)$$
$$-22\zeta_3\,G\,(1;x)+10\zeta_3\,G\,(-1;x)+24\zeta_2\ln\,(2)\,G\,(1;x)-24\zeta_2\ln\,(2)\,G\,(-1;x)$$
$$-\frac{23}{2}\zeta_4+12\zeta_2\ln^2\,(2)+24\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\ln^4\,(2),$$

$$J_{16}^{(0)}=0,$$
$$J_{16}^{(1)}=0,$$
$$J_{16}^{(2)}=0,$$
$$J_{16}^{(3)}=-G\,(0,0,0;x)-\zeta_2\,G\,(0;x),$$
$$J_{16}^{(4)}=4\,G\,(0,1,0,0;x)+2\,G\,(0,0,-1,0;x)+2\,G\,(0,-1,0,0;x)-3\,G\,(0,0,0,0;x)$$
$$-2\,G\,(-1,0,0,0;x)-\zeta_2\,G\,(0,0;x)-2\zeta_2\,G\,(-1,0;x)+6\zeta_2\,G\,(0,-1;x)+2\zeta_3\,G\,(0;x)$$
$$+\frac{1}{4}\zeta_4,$$

$$J_{18}^{(0)}=0,$$
$$J_{18}^{(1)}=-\frac{1}{2}\,G\,(0;x),$$
$$J_{18}^{(2)}=G\,(1,0;x)+3\,G\,(-1,0;x)-\frac{5}{2}\,G\,(0,0;x)-\zeta_2,$$

$$J_{18}^{(3)} = -\frac{17}{2}G(0,0,0;x) + 15G(0,-1,0;x) + 14G(-1,0,0;x) - 18G(-1,-1,0;x)$$
$$+ 5G(0,1,0;x) - 6G(-1,1,0;x) + 5G(1,0,0;x) - 2G(1,1,0;x) - 6G(1,-1,0;x)$$
$$- \zeta_2 G(0;x) - \zeta_2 G(1;x) + 3\zeta_2 G(-1;x) + \frac{9}{4}\zeta_3 + 3\zeta_2 \ln(2),$$

$$J_{18}^{(4)} = 17G(1,0,0,0;x) + 12G(1,1,-1,0;x) - 30G(1,0,-1,0;x) + 12G(1,-1,1,0;x)$$
$$+ 36G(1,-1,-1,0;x) - 10G(1,0,1,0;x) - 28G(-1,1,0,0;x) + 12G(-1,1,1,0;x)$$
$$- 28G(-1,0,1,0;x) + 36G(-1,-1,1,0;x) + 36G(-1,1,-1,0;x) + 29G(0,1,0,0;x)$$
$$- 30G(0,-1,1,0;x) - 10G(0,1,1,0;x) - 30G(0,1,-1,0;x) + 17G(0,0,1,0;x)$$
$$- 80G(-1,-1,0,0;x) + 108G(-1,-1,-1,0;x) - 84G(-1,0,-1,0;x)$$
$$+ 44G(-1,0,0,0;x) - \frac{41}{2}G(0,0,0,0;x) + 51G(0,0,-1,0;x) + 66G(0,-1,0,0;x)$$
$$- 90G(0,-1,-1,0;x) - 10G(1,1,0,0;x) - 28G(1,-1,0,0;x) + 4G(1,1,1,0;x)$$
$$+ 3\zeta_2 G(0,0;x) + \zeta_2 G(-1,0;x) + 3\zeta_2 G(0,-1;x) - 6\zeta_2 G(-1,-1;x) - 5\zeta_2 G(0,1;x)$$
$$+ 2\zeta_2 G(1,0;x) - 6\zeta_2 G(1,-1;x) + 2\zeta_2 G(1,1;x) + 6\zeta_2 G(-1,1;x) - 15\zeta_3 G(1;x)$$
$$- 20\zeta_3 G(-1;x) + \frac{43}{3}\zeta_3 G(0;x) - 12\zeta_2 \ln(2)G(-1;x) + 12\zeta_2 \ln(2)G(1;x) + \frac{75}{8}\zeta_4$$
$$- 12\mathrm{Li}_4\left(\frac{1}{2}\right) - 6\zeta_2 \ln^2(2) - \frac{1}{2}\ln^4(2),$$

$$J_{19}^{(0)} = 0,$$
$$J_{19}^{(1)} = 0,$$
$$J_{19}^{(2)} = 0,$$
$$J_{19}^{(3)} = -2G(0,0,0;x) - 2\zeta_2 G(0;x),$$
$$J_{19}^{(4)} = 12G(0,0,-1,0;x) + 4G(0,-1,0,0;x) + 4G(0,0,1,0;x) - 4G(1,0,0,0;x)$$
$$- 4G(-1,0,0,0;x) - 6G(0,0,0,0;x) + 12\zeta_2 G(0,-1;x) - 4\zeta_2 G(-1,0;x)$$
$$- 4\zeta_2 G(1,0;x) - \frac{1}{2}\zeta_4,$$

$$J_{20}^{(0)} = 0,$$
$$J_{20}^{(1)} = 0,$$
$$J_{20}^{(2)} = 2G(0,1;x) - G(0,0;x) - 4\zeta_2,$$
$$J_{20}^{(3)} = 2G(0,0,1;x) + 2G(0,1,0;x) + 2G(1,0,0;x) - 4G(0,1,1;x) - G(0,0,0;x)$$
$$- 4G(1,0,1;x) - 2G(-1,0,0;x) + 4G(-1,0,1;x) + \zeta_2 G(0;x) + 8\zeta_2 G(1;x)$$
$$- 8\zeta_2 G(-1;x) - 5\zeta_3,$$
$$J_{20}^{(4)} = 2G(0,0,1,0;x) - 4G(0,0,1,1;x) + 2G(0,1,0,0;x) - 4G(0,1,0,1;x)$$
$$- 4G(0,1,1,0;x) + 8G(0,1,1,1;x) + 2G(0,0,0,1;x) - 4G(1,1,0,0;x)$$
$$- 4G(1,0,0,1;x) - 4G(1,0,1,0;x) + 8G(1,0,1,1;x) + 4G(1,-1,0,0;x)$$
$$+ 2G(1,0,0,0;x) + 8G(1,1,0,1;x) - 8G(1,-1,0,1;x) + 4G(-1,0,1,0;x)$$

$$+4G(-1,1,0,0;x)-8G(-1,1,0,1;x)+8G(-1,-1,0,1;x)+4G(-1,0,0,1;x)$$
$$-8G(-1,0,1,1;x)-G(0,0,0,0;x)-2G(-1,0,0,0;x)-4G(-1,-1,0,0;x)$$
$$+2\zeta_3 G(0;x)+10\zeta_3 G(1;x)-10\zeta_3 G(-1;x)-16\zeta_2 G(-1,-1;x)+2\zeta_2 G(-1,0;x)$$
$$+16\zeta_2 G(1,-1;x)-2\zeta_2 G(1,0;x)+16\zeta_2 G(-1,1;x)-16\zeta_2 G(1,1;x)-\frac{99}{4}\zeta_4,$$

$$J_{21}^{(0)}=0,$$
$$J_{21}^{(1)}=0,$$
$$J_{21}^{(2)}=G(0,0;x),$$
$$J_{21}^{(3)}=4G(0,0,0;x)-2G(0,0,1;x)-2G(0,1,0;x)-2G(0,-1,0;x)-2G(1,0,0;x)$$
$$-\zeta_2 G(0;x)-3\zeta_3,$$
$$J_{21}^{(4)}=4G(0,1,-1,0;x)+4G(0,-1,0,1;x)-8G(0,0,1,0;x)+4G(0,0,1,1;x)$$
$$-8G(0,1,0,0;x)+4G(0,1,0,1;x)+4G(0,1,1,0;x)+4G(0,-1,1,0;x)$$
$$-8G(0,0,0,1;x)+4G(1,1,0,0;x)+4G(1,0,0,1;x)+4G(1,0,1,0;x)$$
$$-8G(1,0,0,0;x)+4G(1,0,-1,0;x)+11G(0,0,0,0;x)-6G(0,0,-1,0;x)$$
$$-6G(0,-1,0,0;x)+4G(0,-1,-1,0;x)+2\zeta_2 G(0,-1;x)-3\zeta_2 G(0,0;x)$$
$$+2\zeta_2 G(1,0;x)+2\zeta_2 G(0,1;x)-5\zeta_3 G(0;x)+6\zeta_3 G(1;x)-\frac{5}{4}\zeta_4,$$

$$J_{23}^{(0)}=0,$$
$$J_{23}^{(1)}=0,$$
$$J_{23}^{(2)}=0,$$
$$J_{23}^{(3)}=4G(1,0,0;x)-2G(0,0,1;x)-2G(0,1,0;x)+6\zeta_3,$$
$$J_{23}^{(4)}=4G(1,0,0,1;x)+4G(0,1,0,1;x)+4G(0,0,1,1;x)-8G(0,0,0,1;x)$$
$$+10G(0,0,-1,0;x)-4G(0,-1,0,0;x)+4G(0,-1,0,1;x)+4G(0,-1,1,0;x)$$
$$-6G(0,1,0,0;x)-4G(0,0,1,0;x)+4G(0,1,1,0;x)+4G(0,1,-1,0;x)$$
$$+12G(1,0,0,0;x)-24G(1,0,-1,0;x)-4G(1,0,1,0;x)+2\zeta_2 G(0,1;x)$$
$$-4\zeta_2 G(1,0;x)+12\zeta_3 G(0;x)-24\zeta_3 G(1;x)+9\zeta_4,$$

$$J_{32}^{(0)}=0,$$
$$J_{32}^{(1)}=0,$$
$$J_{32}^{(2)}=2G(0,1;x)-G(0,0;x)-4\zeta_2,$$
$$J_{32}^{(3)}=2G(0,0,0;x)-2G(0,0,1;x)-4G(0,1,1;x)+2G(1,0,0;x)-4G(1,0,1;x)$$
$$-2G(-1,0,0;x)+4G(-1,0,1;x)+5\zeta_2 G(0;x)+8\zeta_2 G(1;x)-8\zeta_2 G(-1;x)-5\zeta_3,$$
$$J_{32}^{(4)}=4G(0,-1,1,0;x)-6G(0,1,0,0;x)+4G(0,1,0,1;x)+8G(0,1,1,1;x)$$
$$-8G(0,0,1,0;x)+4G(0,0,1,1;x)-10G(0,0,0,1;x)+4G(0,1,-1,0;x)$$
$$+4G(-1,1,0,0;x)+4G(-1,0,1,0;x)-8G(-1,1,0,1;x)+9G(0,0,0,0;x)$$
$$+8G(-1,-1,0,1;x)-2G(0,0,-1,0;x)-2G(0,-1,0,0;x)-8G(-1,0,1,1;x)$$

$$
\begin{aligned}
&+4\,G\,(-1,0,0,1;x)-4\,G\,(1,1,0,0;x)+4\,G\,(1,-1,0,0;x)-2\,G\,(-1,0,0,0;x)\\
&-4\,G\,(-1,-1,0,0;x)-8\,G\,(1,-1,0,1;x)-4\,G\,(1,0,0,0;x)+4\,G\,(1,0,0,1;x)\\
&+8\,G\,(1,0,1,1;x)+8\,G\,(1,1,0,1;x)+\zeta_2\,G\,(0,0;x)-16\,\zeta_2\,G\,(1,1;x)+16\,\zeta_2\,G\,(-1,1;x)\\
&-6\,\zeta_2\,G\,(0,1;x)-10\,\zeta_2\,G\,(1,0;x)+16\,\zeta_2\,G\,(1,-1;x)-16\,\zeta_2\,G\,(-1,-1;x)\\
&+8\,\zeta_2\,G\,(0,-1;x)+2\,\zeta_2\,G\,(-1,0;x)-10\,\zeta_3\,G\,(-1;x)+7\,\zeta_3\,G\,(0;x)+10\,\zeta_3\,G\,(1;x)\\
&-\frac{139}{4}\,\zeta_4,\\
J_{37}^{(0)}&=0,\\
J_{37}^{(1)}&=0,\\
J_{37}^{(2)}&=0,\\
J_{37}^{(3)}&=0,\\
J_{37}^{(4)}&=-6\,G\,(0,0,0,0;x)+6\,G\,(0,0,0,1;x)+4\,G\,(0,0,1,0;x)+2\,G\,(0,1,0,0;x)\\
&\quad-4\,\zeta_2\,G\,(0,0;x),\\
J_{40}^{(0)}&=0,\\
J_{40}^{(1)}&=0,\\
J_{40}^{(2)}&=0,\\
J_{40}^{(3)}&=0,\\
J_{40}^{(4)}&=-G\,(0,0,0,0;x)-8\,G\,(0,0,-1,0;x)+6\,G\,(0,0,0,1;x)+4\,G\,(0,1,0,0;x)\\
&\quad-4\,\zeta_2\,G\,(0,0;x)-2\,\zeta_3\,G\,(0;x)-3\,\zeta_4.
\end{aligned}
\tag{9.3}
$$

9.2.2 Integrals, which are expressed in the variable x'

The integrals $J_{12}-J_{13}$ are most naturally expressed in terms of harmonic polylogarithms in the variable x'.

$$
\begin{aligned}
J_{12}^{(0)}&=0,\\
J_{12}^{(1)}&=0,\\
J_{12}^{(2)}&=G\,(0,0;x')-2\,G\,(0,-1;x')+\zeta_2,\\
J_{12}^{(3)}&=4\,G\,(0,0,0;x')-8\,G\,(0,0,-1;x')-2\,G\,(0,-1,0;x')+4\,G\,(0,-1,-1;x')\\
&\quad-4\,G\,(1,0,0;x')+8\,G\,(1,0,-1;x')+4\,G\,(-1,0,-1;x')-2\,G\,(-1,0,0;x')\\
&\quad+2\,\zeta_2\,G\,(0;x')-4\,\zeta_2\,G\,(1;x')-2\,\zeta_2\,G\,(-1;x')-4\,\zeta_3,\\
J_{12}^{(4)}&=10\,G\,(0,0,0,0;x')-20\,G\,(0,0,0,-1;x')-8\,G\,(0,0,-1,0;x')+16\,G\,(0,0,-1,-1;x')\\
&\quad-32\,G\,(1,1,0,-1;x')+4\,G\,(-1,-1,0,0;x')-16\,G\,(1,0,-1,-1;x')\\
&\quad-8\,G\,(-1,-1,0,-1;x')+16\,G\,(0,-1,0,-1;x')-8\,G\,(0,-1,-1,-1;x')\\
&\quad-8\,G\,(0,-1,0,0;x')+4\,G\,(0,-1,-1,0;x')-12\,G\,(0,1,0,0;x')-16\,G\,(1,0,0,0;x')\\
&\quad-8\,G\,(-1,0,0,0;x')+32\,G\,(1,0,0,-1;x')+8\,G\,(1,0,-1,0;x')+8\,G\,(1,-1,0,0;x')\\
&\quad-16\,G\,(1,-1,0,-1;x')+24\,G\,(0,1,0,-1;x')+8\,G\,(-1,1,0,0;x')
\end{aligned}
$$

$$-16\,G\left(-1,1,0,-1;x'\right)+16\,G\left(1,1,0,0;x'\right)+16\,G\left(-1,0,0,-1;x'\right)$$
$$+4\,G\left(-1,0,-1,0;x'\right)-8\,G\left(-1,0,-1,-1;x'\right)-6\,\zeta_2\,G\left(0,-1;x'\right)+3\,\zeta_2\,G\left(0,0;x'\right)$$
$$-12\,\zeta_2\,G\left(0,1;x'\right)+16\,\zeta_2\,G\left(1,1;x'\right)+8\,\zeta_2\,G\left(1,-1;x'\right)-8\,\zeta_2\,G\left(1,0;x'\right)$$
$$-4\,\zeta_2\,G\left(-1,0;x'\right)+8\,\zeta_2\,G\left(-1,1;x'\right)+4\,\zeta_2\,G\left(-1,-1;x'\right)+16\,\zeta_3\,G\left(1;x'\right)$$
$$+8\,\zeta_3\,G\left(-1;x'\right)-8\,\zeta_3\,G\left(0;x'\right),$$
$$J_{13}^{(0)}=0,$$
$$J_{13}^{(1)}=0,$$
$$J_{13}^{(2)}=0,$$
$$J_{13}^{(3)}=G\left(0,0,0;x'\right)-2\,G\left(0,0,-1;x'\right)+\zeta_2\,G\left(0;x'\right)-2\,\zeta_3,$$
$$J_{13}^{(4)}=3\,G\left(0,0,0,0;x'\right)+8\,G\left(0,1,0,-1;x'\right)-2\,G\left(0,0,-1,0;x'\right)+4\,G\left(0,-1,0,-1;x'\right)$$
$$-4\,G\left(0,1,0,0;x'\right)+4\,G\left(0,0,-1,-1;x'\right)-2\,G\left(0,-1,0,0;x'\right)-6\,G\left(0,0,0,-1;x'\right)$$
$$+2\,G\left(-1,0,0,0;x'\right)-4\,G\left(-1,0,0,-1;x'\right)+\zeta_2\,G\left(0,0;x'\right)-4\,\zeta_2\,G\left(0,1;x'\right)$$
$$-2\,\zeta_2\,G\left(0,-1;x'\right)+2\,\zeta_2\,G\left(-1,0;x'\right)-2\,\zeta_3\,G\left(0;x'\right)-4\,\zeta_3\,G\left(-1;x'\right)+\frac{5}{4}\,\zeta_4. \tag{9.4}$$

9.2.3 Integrals, which are expressed in the variable $\tilde{x}$

The integrals J_{22} and J_{36} are expressed in terms of multiple polylogarithms in the variable $\tilde{x}$. We use the notation of eq. (3.19). The differential one-forms $\omega_0, \omega_{0,4}$ and $\omega_{-4,0}$ are defined in eq. (4.8).

$$J_{22}^{(0)}=0,$$
$$J_{22}^{(1)}=0,$$
$$J_{22}^{(2)}=0,$$
$$J_{22}^{(3)}=0,$$
$$J_{22}^{(4)}=G\left(\omega_0,\omega_{-4,0},\omega_{-4,0},\omega_0;\tilde{x}\right)-G\left(\omega_0,\omega_0,\omega_{0,4},\omega_{0,4};\tilde{x}\right)-\zeta_2\,G\left(\omega_0,\omega_{-4,0};\tilde{x}\right)+\frac{7}{4}\,\zeta_4,$$
$$J_{36}^{(0)}=0,$$
$$J_{36}^{(1)}=0,$$
$$J_{36}^{(2)}=0,$$
$$J_{36}^{(3)}=0,$$
$$J_{36}^{(4)}=-7\,G\left(\omega_{0,4},\omega_0,\omega_{0,4},\omega_{0,4};\tilde{x}\right)+2\,G\left(\omega_{0,4},\omega_{0,4},\omega_0,\omega_{0,4};\tilde{x}\right)+2\,G\left(\omega_{0,4},\omega_{0,4},\omega_{0,4},\omega_0;\tilde{x}\right)$$
$$+4\,G\left(\omega_{0,4},\omega_{-4,0},\omega_{-4,0},\omega_0;\tilde{x}\right)-4\,\zeta_2\,G\left(\omega_{0,4},\omega_{-4,0};\tilde{x}\right)-10\,\zeta_3\,G\left(\omega_{0,4};\tilde{x}\right)-\frac{39}{2}\,\zeta_4. \tag{9.5}$$

9.3 Integrals, which only depend on t

The integrals J_6-J_7, J_{14} and J_{17} depend only on the variable t (or equivalently only on the variable y). They are expressed as iterated integrals of modular forms $\{1, f_2, f_3, f_4, g_{2,1}\}$.

The modular forms have been defined in eq. (5.72).

$$J_6^{(0)} = 0,$$

$$J_6^{(1)} = 0,$$

$$J_6^{(2)} = F(1, f_3; q_6) + 3\zeta_2,$$

$$J_6^{(3)} = -F(f_2, 1, f_3; q_6) - F(1, f_2, f_3; q_6) + 3\zeta_2 F(1; q_6) - 3\zeta_2 F(f_2; q_6) + \frac{21}{2}\zeta_3 - 18\zeta_2 \ln(2),$$

$$\begin{aligned}
J_6^{(4)} &= F(f_2, f_2, 1, f_3; q_6) + F(f_2, 1, f_2, f_3; q_6) + F(1, f_2, f_2, f_3; q_6) + F(1, f_4, 1, f_3; q_6) \\
&\quad + 3\zeta_2 F(f_2, f_2; q_6) - 3\zeta_2 F(1, f_2; q_6) - 3\zeta_2 F(f_2, 1; q_6) + 3\zeta_2 F(1, f_4; q_6) + \zeta_2 F(1, f_3; q_6) \\
&\quad + \left(\frac{21}{2}\zeta_3 - 18\zeta_2 \ln(2)\right)(F(1; q_6) - F(f_2; q_6)) - 39\zeta_4 + 72\mathrm{Li}_4\left(\frac{1}{2}\right) + 36\zeta_2 \ln^2(2) \\
&\quad + 3\ln^4(2),
\end{aligned}$$

$$J_7^{(0)} = 0,$$

$$J_7^{(1)} = F(f_3; q_6),$$

$$J_7^{(2)} = -F(f_2, f_3; q_6) + 3\zeta_2,$$

$$\begin{aligned}
J_7^{(3)} &= F(f_2, f_2, f_3; q_6) + F(f_4, 1, f_3; q_6) + 3\zeta_2 F(f_4; q_6) - 3\zeta_2 F(f_2; q_6) + \zeta_2 F(f_3; q_6) + \frac{21}{2}\zeta_3 \\
&\quad - 18\zeta_2 \ln(2),
\end{aligned}$$

$$\begin{aligned}
J_7^{(4)} &= -F(f_2, f_2, f_2, f_3; q_6) - F(f_4, f_2, 1, f_3; q_6) - F(f_4, 1, f_2, f_3; q_6) - F(f_2, f_4, 1, f_3; q_6) \\
&\quad + 3\zeta_2 F(f_2, f_2; q_6) - 3\zeta_2 F(f_4, f_2; q_6) - 3\zeta_2 F(f_2, f_4; q_6) + 3\zeta_2 F(f_4, 1; q_6) - \zeta_2 F(f_2, f_3; q_6) \\
&\quad + \left(\frac{21}{2}\zeta_3 - 18\zeta_2 \ln(2)\right)(F(f_4; q_6) - F(f_2; q_6)) - \frac{2}{3}\zeta_3 F(f_3; q_6) - 39\zeta_4 + 72\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&\quad + 36\zeta_2 \ln^2(2) + 3\ln^4(2),
\end{aligned}$$

$$J_{14}^{(0)} = 0,$$

$$J_{14}^{(1)} = 0,$$

$$J_{14}^{(2)} = 0,$$

$$J_{14}^{(3)} = -\frac{1}{9}F(f_3, 1, f_3; q_6) - \frac{1}{3}\zeta_2 F(f_3; q_6),$$

$$\begin{aligned}
J_{14}^{(4)} &= \frac{1}{9}F(f_3, 1, f_2, f_3; q_6) + \frac{1}{9}F(f_3, f_2, 1, f_3; q_6) + \frac{1}{3}\zeta_2 F(f_3, f_2; q_6) - \frac{1}{3}\zeta_2 F(f_3, 1; q_6) \\
&\quad - \frac{1}{9}\left(\frac{21}{2}\zeta_3 - 18\zeta_2 \ln(2)\right)F(f_3; q_6),
\end{aligned}$$

$$J_{17}^{(0)} = 0,$$

$$J_{17}^{(1)} = 0,$$

$$J_{17}^{(2)} = 0,$$

$$J_{17}^{(3)} = -\frac{2}{9}F(f_3, 1, f_3; q_6) - \frac{2}{3}\zeta_2 F(f_3; q_6),$$

$$J_{17}^{(4)} = \frac{2}{9} F\left(f_3, 1, f_2, f_3; q_6\right) + \frac{2}{9} F\left(f_3, f_2, 1, f_3; q_6\right) - \frac{4}{9} F\left(g_{2,1}, f_3, 1, f_3; q_6\right) + \frac{2}{3} \zeta_2 F\left(f_3, f_2; q_6\right)$$

$$- \frac{2}{3} \zeta_2 F\left(f_3, 1; q_6\right) - \frac{4}{3} \zeta_2 F\left(g_{2,1}, f_3; q_6\right) - \frac{2}{9}\left(\frac{21}{2} \zeta_3 - 18\zeta_2 \ln(2)\right) F\left(f_3; q_6\right). \tag{9.6}$$

9.4 Integrals, which depend on s and t

The integrals $J_{24} - J_{30}$, $J_{33} - J_{35}$, $J_{38} - J_{39}$ and $J_{41} - J_{45}$ depend on s and t and are expressed in terms of iterated integrals with the integration kernels discussed in section 7.1. Due to the large number of integration kernels the explicit results for these integrals are rather long (at the order of $200 - 300$ terms). For this reason we list here only a few examples. We give the results for the integrals J_{24}, J_{27}, J_{33}, J_{38} and J_{39} up to order ε^3. These are rather compact. In addition we give the (not so short) result for J_{41} up to order ε^4. This integral starts at $\mathcal{O}(\varepsilon^4)$. The integral J_{41} is proportional to the double box integral with unit powers of the propagators, i.e. $I_{1111111}$, and therefore central to this article.

The results for all integrals up to order ε^4 are given in an electronic file attached to this article. More information on the electronic file accompanying this article can be found in appendix E.

The results for the integrals J_{24}, J_{27}, J_{33}, J_{38} and J_{39} up to order ε^3 read

$$J_{24}^{(0)} = 0,$$
$$J_{24}^{(1)} = 0,$$
$$J_{24}^{(2)} = 0,$$
$$J_{24}^{(3)} = I_\gamma\left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right) - \frac{3}{2} I_\gamma\left(\eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 3 I_\gamma\left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ I_\gamma\left(\eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{9}{2} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+ \frac{7}{4} \zeta_2 I_\gamma\left(\eta_0^{(b)}; \lambda\right) - 2 \zeta_2 I_\gamma\left(\eta_{1,1}^{(b)}; \lambda\right) + 3 \zeta_2 I_\gamma\left(\eta_2^{\left(\frac{a}{b}\right)}; \lambda\right) + 3 \zeta_2 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}; \lambda\right)$$

$$+ 3 \zeta_2 I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}; \lambda\right) - 3 \ln(2) \zeta_2 - \frac{7}{4} \zeta_3,$$

$$J_{27}^{(0)} = 0,$$
$$J_{27}^{(1)} = 0,$$
$$J_{27}^{(2)} = 0,$$
$$J_{27}^{(3)} = 2 I_\gamma\left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right) - 3 I_\gamma\left(\eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 6 I_\gamma\left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ 2 I_\gamma\left(\eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right) + 9 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + 2 I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+ \frac{7}{2} \zeta_2 I_\gamma\left(\eta_0^{(b)}; \lambda\right) - 4 \zeta_2 I_\gamma\left(\eta_{1,1}^{(b)}; \lambda\right) + 6 \zeta_2 I_\gamma\left(\eta_2^{\left(\frac{a}{b}\right)}; \lambda\right) + 6 \zeta_2 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}; \lambda\right)$$

$$+ 6 \zeta_2 I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}; \lambda\right) - 6 \ln(2) \zeta_2 - \frac{7}{2} \zeta_3,$$

$$J_{33}^{(0)} = 0,$$
$$J_{33}^{(1)} = 0,$$

$$J_{33}^{(2)} = 0,$$

$$J_{33}^{(3)} = I_\gamma \left(\eta_0^{(c)}, \eta_2^{\left(\frac{c}{a}\right)}, f_3; \lambda \right) + \frac{3}{2} I_\gamma \left(\eta_0^{(c)}, \eta_{3,3}^{(c)}, \omega_0; \lambda \right) + I_\gamma \left(\eta_{1,1}^{(c)}, \omega_{0,4}, \omega_0; \lambda \right)$$

$$+ 2 I_\gamma \left(\eta_{1,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda \right) + I_\gamma \left(\eta_2^{\left(\frac{a}{c}\right)}, \eta_0^{(a)}, f_3; \lambda \right) - \frac{1}{3} I_\gamma \left(\eta_0^{(c)}, a_{3,2}^{(c)}, \omega_{0,4}, \omega_0; \lambda \right)$$

$$- I_\gamma \left(\eta_0^{(c)}, a_{3,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda \right) + I_\gamma \left(\eta_0^{(c)}, a_{4,1}^{(a,c)}, \eta_0^{(a)}, f_3; \lambda \right) + \frac{3}{4} \zeta_2 I_\gamma \left(\eta_0^{(c)}; \lambda \right)$$

$$- 4 \zeta_2 I_\gamma \left(\eta_{1,1}^{(c)}; \lambda \right) - 2 \zeta_2 I_\gamma \left(\eta_{1,3}^{(c)}; \lambda \right) + 3 \zeta_2 I_\gamma \left(\eta_2^{\left(\frac{a}{c}\right)}; \lambda \right) + \frac{4}{3} \zeta_2 I_\gamma \left(\eta_0^{(c)}, a_{3,2}^{(c)}; \lambda \right)$$

$$+ \zeta_2 I_\gamma \left(\eta_0^{(c)}, a_{3,3}^{(c)}; \lambda \right) + 3 \zeta_2 I_\gamma \left(\eta_0^{(c)}, a_{4,1}^{(a,c)}; \lambda \right),$$

$$J_{38}^{(0)} = 0,$$

$$J_{38}^{(1)} = 0,$$

$$J_{38}^{(2)} = 0,$$

$$J_{38}^{(3)} = 0,$$

$$J_{39}^{(0)} = 0,$$

$$J_{39}^{(1)} = 0,$$

$$J_{39}^{(2)} = 0,$$

$$J_{39}^{(3)} = 0. \tag{9.7}$$

Finally, let us give the result the result for the integral J_{41}. We recall that this integral is proportional to the double box integral with unit powers of the propagators, e.g.

$$J_{41} = \varepsilon^4 \frac{(1-x)^4}{x^2} \frac{\pi}{\psi_1^{(b)}} I_{1111111}. \tag{9.8}$$

This integral starts at $\mathcal{O}(\varepsilon^4)$, hence

$$J_{41}^{(0)} = 0, \quad J_{41}^{(1)} = 0, \quad J_{41}^{(2)} = 0, \quad J_{41}^{(3)} = 0. \tag{9.9}$$

Due to the differential equation we may write $J_{41}^{(4)}$ as an integral over ε^3-terms $J_i^{(3)}$. Starting the integration path as usual at $(x, y) = (0, 1)$ we obtain

$$J_{41}^{(4)} = -\frac{79}{4} \zeta_4 + 8 \,\mathrm{Li}_4 \left(\frac{1}{2} \right) - 8 \zeta_2 \left(\ln(2) \right)^2 + \frac{1}{3} \left(\ln(2) \right)^4 + \int_\gamma \left(\frac{1}{9} \eta_{2,9} - \frac{1}{3} g_{2,1} - 2\omega_{0,4} \right) J_{24}^{(3)}$$

$$+ \int_\gamma \left(\frac{1}{18} \eta_{2,10} + \frac{4}{3} g_{2,1} + \frac{1}{4} \omega_0 + \frac{9}{4} \omega_{0,4} \right) J_{27}^{(3)} + \int_\gamma \eta_2^{\left(\frac{c}{b}\right)} J_{33}^{(3)}$$

$$+ \int_\gamma \eta_{1,1}^{(b)} \left[-\frac{1}{2} J_{42}^{(3)} + 2 J_{32}^{(3)} + 2 J_{28}^{(3)} + 2 J_{21}^{(3)} - 2 J_{20}^{(3)} + 4 J_{18}^{(3)} - 4 J_{15}^{(3)} + \frac{1}{3} J_{14}^{(3)} + 4 J_{13}^{(3)} \right.$$

$$\left. - 3 J_{10}^{(3)} + J_8^{(3)} - 2 J_4^{(3)} + 2 J_3^{(3)} \right] + \int_\gamma \eta_0^{(b)} J_{43}^{(3)}. \tag{9.10}$$

The explicit result for $J_{41}^{(4)}$ reads

$$J_{41}^{(4)} =$$

$$\frac{4}{3} I_\gamma \left(\eta_0^{(b)}, d_{2,4}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - 2 I_\gamma \left(\eta_0^{(b)}, d_{2,4}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) - \frac{26}{3} I_\gamma \left(\eta_0^{(b)}, \eta_{2,1}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right)$$

$$+ 13 I_\gamma \left(\eta_0^{(b)}, \eta_{2,1}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) + \frac{7}{3} I_\gamma \left(\eta_0^{(b)}, g_{2,1}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - \frac{7}{2} I_\gamma \left(\eta_0^{(b)}, g_{2,1}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right)$$

$$+ 3 I_\gamma \left(\eta_0^{(b)}, \omega_0, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) - \frac{31}{6} I_\gamma \left(\eta_0^{(b)}, \omega_4, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) + \frac{1}{6} I_\gamma \left(\eta_0^{(b)}, \eta_{2,10}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right)$$

$$- 2 I_\gamma \left(\eta_0^{(b)}, \omega_0, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) + \frac{31}{4} I_\gamma \left(\eta_0^{(b)}, \omega_4, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) - \frac{1}{9} I_\gamma \left(\eta_0^{(b)}, \eta_{2,10}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right)$$

$$- \frac{1}{9} I_\gamma \left(\eta_0^{(b)}, \eta_{2,11}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) + \frac{1}{6} I_\gamma \left(\eta_0^{(b)}, \eta_{2,11}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) + I_\gamma \left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{c}\right)}, \eta_2^{\left(\frac{c}{a}\right)}, f_3; \lambda \right)$$

$$+ \frac{3}{2} I_\gamma \left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{c}\right)}, \eta_{3,3}^{(c)}, \omega_0; \lambda \right) + \frac{1}{2} I_\gamma \left(\eta_0^{(b)}, \eta_{3,10}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right) + I_\gamma \left(\eta_0^{(b)}, \eta_{3,11}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right)$$

$$+ I_\gamma \left(\eta_0^{(b)}, \eta_{3,1}^{(b)}, \omega_{0,4}, \omega_0; \lambda \right) - I_\gamma \left(\eta_0^{(b)}, \eta_{3,24}^{(b)}, \omega_{-4,0}, \omega_0; \lambda \right) - 3 I_\gamma \left(\eta_0^{(b)}, \eta_{3,3}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right)$$

$$- 6 I_\gamma \left(\eta_0^{(b)}, \eta_{3,4}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right) + 6 I_\gamma \left(\eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_4, \omega_{0,4}; \lambda \right) - 3 I_\gamma \left(\eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}, \omega_0; \lambda \right)$$

$$- \frac{1}{3} I_\gamma \left(\eta_0^{(b)}, \eta_{3,9}^{(b)}, \omega_{0,4}, \omega_0; \lambda \right) - \frac{31}{6} I_\gamma \left(\eta_0^{(b)}, \omega_{0,4}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - \frac{1}{6} I_\gamma \left(\eta_{2,9}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right)$$

$$+ I_\gamma \left(\eta_0^{(b)}, \eta_{4,3}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda \right) + \frac{31}{4} I_\gamma \left(\eta_0^{(b)}, \omega_{0,4}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) + \frac{1}{9} I_\gamma \left(\eta_{2,9}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right)$$

$$- \frac{1}{3} I_\gamma \left(\eta_{2,9}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right) + \frac{1}{9} I_\gamma \left(\eta_{2,9}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda \right) + \frac{7}{3} I_\gamma \left(g_{2,1}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right)$$

$$- \frac{7}{2} I_\gamma \left(g_{2,1}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) - 7 I_\gamma \left(g_{2,1}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right) + \frac{7}{3} I_\gamma \left(g_{2,1}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda \right)$$

$$+ \frac{1}{2} I_\gamma \left(\omega_0, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - \frac{3}{4} I_\gamma \left(\omega_0, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) - \frac{3}{2} I_\gamma \left(\omega_0, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right)$$

$$+ \frac{1}{2} I_\gamma \left(\omega_0, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda \right) + 4 I_\gamma \left(\eta_{1,1}^{(b)}, d_{2,4}, \omega_{0,4}, \omega_{0,4}; \lambda \right) - 10 I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{2,1}, \omega_{0,4}, \omega_{0,4}; \lambda \right)$$

$$- \frac{9}{2} I_\gamma \left(\eta_{1,1}^{(b)}, \omega_0, \omega_{0,4}, \omega_{0,4}; \lambda \right) + \frac{1}{2} I_\gamma \left(\eta_{1,1}^{(b)}, \omega_4, \omega_{0,4}, \omega_{0,4}; \lambda \right) - \frac{16}{9} I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{1,2}^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right)$$

$$+ \frac{8}{3} I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{1,2}^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) + \frac{8}{3} I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{1,4}^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - 4 I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{1,4}^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right)$$

$$- \frac{2}{27} I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{3,1}^{(a)}, \eta_0^{(a)}, f_3; \lambda \right) - \frac{1}{54} I_\gamma \left(\eta_{1,1}^{(b)}, \eta_{3,2}^{(a)}, \eta_0^{(a)}, f_3; \lambda \right) + 12 I_\gamma \left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_4, \omega_{0,4}; \lambda \right)$$

$$- 8 I_\gamma \left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}, \omega_0; \lambda \right) - 4 I_\gamma \left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}, \omega_{0,4}; \lambda \right) - 4 I_\gamma \left(\eta_{1,1}^{(b)}, \omega_{-4,0}, \omega_{-4,0}, \omega_0; \lambda \right)$$

$$+ \frac{1}{9} I_\gamma \left(\eta_{2,10}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda \right) - \frac{1}{6} I_\gamma \left(\eta_{2,10}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda \right) + \frac{1}{9} I_\gamma \left(\eta_{2,10}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda \right)$$

$$- \frac{1}{3} I_\gamma \left(\eta_{2,10}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right) + I_\gamma \left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_0^{(c)}, \eta_2^{\left(\frac{c}{a}\right)}, f_3; \lambda \right) + \frac{3}{2} I_\gamma \left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_0^{(c)}, \eta_{3,3}^{(c)}, \omega_0; \lambda \right)$$

$$+ I_\gamma \left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_{1,1}^{(c)}, \omega_{0,4}, \omega_0; \lambda \right) + 2 I_\gamma \left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_{1,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda \right) - \frac{15}{2} I_\gamma \left(\omega_{0,4}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda \right)$$

$$+ I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_2^{\left(\frac{a}{c}\right)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{5}{2} I_\gamma\left(\omega_{0,4}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right) - \frac{15}{4} I_\gamma\left(\omega_{0,4}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right)$$

$$+ \frac{5}{2} I_\gamma\left(\omega_{0,4}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right) - 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, d_{2,4}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ 15 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{2,1}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + \frac{27}{4} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_0, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$- \frac{3}{4} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_4, \omega_{0,4}, \omega_{0,4}; \lambda\right) + \frac{8}{3} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,2}^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right)$$

$$- 4 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,2}^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 4 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,4}^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right)$$

$$+ 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,4}^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) + \frac{1}{9} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{3,1}^{(a)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+ \frac{1}{36} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{3,2}^{(a)}, \eta_0^{(a)}, f_3; \lambda\right) - 18 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_4, \omega_{0,4}; \lambda\right)$$

$$+ 12 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}, \omega_0; \lambda\right) + 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{-4,0}, \omega_{-4,0}, \omega_0; \lambda\right) + 6 I_\gamma\left(\eta_0^{(b)}, d_{2,4}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ \frac{4}{3} I_\gamma\left(\eta_0^{(b)}, d_{2,4}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) - 39 I_\gamma\left(\eta_0^{(b)}, \eta_{2,1}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$- \frac{26}{3} I_\gamma\left(\eta_0^{(b)}, \eta_{2,1}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{21}{2} I_\gamma\left(\eta_0^{(b)}, g_{2,1}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+ \frac{7}{3} I_\gamma\left(\eta_0^{(b)}, g_{2,1}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) - 9 I_\gamma\left(\eta_0^{(b)}, \omega_0, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$- 2 I_\gamma\left(\eta_0^{(b)}, \omega_0, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) - \frac{93}{4} I_\gamma\left(\eta_0^{(b)}, \omega_4, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$- \frac{31}{6} I_\gamma\left(\eta_0^{(b)}, \omega_4, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_0^{(c)}, \eta_2^{\left(\frac{c}{a}\right)}, f_3; \lambda\right)$$

$$+ \frac{3}{2} I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_0^{(c)}, \eta_{3,3}^{(c)}, \omega_0; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_{1,1}^{(c)}, \omega_{0,4}, \omega_0; \lambda\right)$$

$$+ 2 I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_{1,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_2^{\left(\frac{a}{c}\right)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+ I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right) - \frac{3}{2} I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right)$$

$$- 3 I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+ 2 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right) - 3 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right)$$

$$- 6 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + 2 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$- \frac{1}{2} I_\gamma\left(\eta_0^{(b)}, \eta_{2,10}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) - \frac{1}{9} I_\gamma\left(\eta_0^{(b)}, \eta_{2,10}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$- \frac{1}{2} I_\gamma\left(\eta_0^{(b)}, \eta_{2,11}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) - \frac{1}{9} I_\gamma\left(\eta_0^{(b)}, \eta_{2,11}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$- \frac{1}{3} I_\gamma\left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{c}\right)}, a_{3,2}^{(c)}, \omega_{0,4}, \omega_0; \lambda\right) - I_\gamma\left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{c}\right)}, a_{3,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda\right)$$

$$+ I_\gamma\left(\eta_0^{(b)}, \eta_2^{\left(\frac{b}{c}\right)}, a_{4,1}^{(a,c)}, \eta_0^{(a)}, f_3; \lambda\right) - \frac{93}{4} I_\gamma\left(\eta_0^{(b)}, \omega_{0,4}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$-\frac{31}{6} I_\gamma\left(\eta_0^{(b)}, \omega_{0,4}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{1}{2} I_\gamma\left(\eta_{2,9}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{1}{9} I_\gamma\left(\eta_{2,9}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{21}{2} I_\gamma\left(g_{2,1}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{7}{3} I_\gamma\left(g_{2,1}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{9}{4} I_\gamma\left(\omega_0, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{1}{2} I_\gamma\left(\omega_0, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{8}{3} I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right)$$

$$-4 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 8 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{8}{3} I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right) + 4 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right)$$

$$-6 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 12 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+4 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right) - 8 I_\gamma\left(\eta_{1,1}^{(b)}, \eta_{1,2}^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$-\frac{16}{9} I_\gamma\left(\eta_{1,1}^{(b)}, \eta_{1,2}^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + 12 I_\gamma\left(\eta_{1,1}^{(b)}, \eta_{1,4}^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{8}{3} I_\gamma\left(\eta_{1,1}^{(b)}, \eta_{1,4}^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{1}{2} I_\gamma\left(\eta_{2,10}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{1}{9} I_\gamma\left(\eta_{2,10}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) - \frac{1}{3} I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_0^{(c)}, a_{3,2}^{(c)}, \omega_{0,4}, \omega_0; \lambda\right)$$

$$-I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_0^{(c)}, a_{3,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda\right) + I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)}, \eta_0^{(c)}, a_{4,1}^{(a,c)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+\frac{45}{4} I_\gamma\left(\omega_{0,4}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + \frac{5}{2} I_\gamma\left(\omega_{0,4}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+12 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,2}^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + \frac{8}{3} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,2}^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$-18 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,4}^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) - 4 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \eta_{1,4}^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$-\frac{1}{3} I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_0^{(c)}, a_{3,2}^{(c)}, \omega_{0,4}, \omega_0; \lambda\right) - I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_0^{(c)}, a_{3,3}^{(c)}, \omega_{-4,0}, \omega_0; \lambda\right)$$

$$+I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(b,c)}, \eta_0^{(c)}, a_{4,1}^{(a,c)}, \eta_0^{(a)}, f_3; \lambda\right) + \frac{9}{2} I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + 9 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+2 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + 12 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+\frac{8}{3} I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + 18 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$+4 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) - 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, \eta_2^{\left(\frac{b}{a}\right)}, f_3; \lambda\right)$$

$$+18 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) - 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$+9 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) - 27 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right)$$

$$-6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}, \eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) + 6 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,1}^{(b)}, \eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right)$$

$$+12 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,1}^{(b)}, \eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) - 4 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,1}^{(b)}, \eta_2^{\left(\frac{a}{b}\right)}, \eta_0^{(a)}, f_3; \lambda\right)$$

$$-4\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},\eta_2^{\left(\frac{b}{a}\right)},f_3;\lambda\right)-18\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{3,2}^{(b)},\omega_{0,4},\omega_{0,4};\lambda\right)$$

$$-4\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{4,1}^{(a,b)},\eta_0^{(a)},f_3;\lambda\right)$$

$$+\left[\frac{7}{3}\,I_\gamma\left(\eta_0^{(b)},d_{2,4};\lambda\right)-\frac{91}{6}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,1};\lambda\right)+\frac{49}{12}\,I_\gamma\left(\eta_0^{(b)},g_{2,1};\lambda\right)-\frac{7}{2}\,I_\gamma\left(\eta_0^{(b)},\omega_0;\lambda\right)\right.$$

$$-\frac{217}{24}\,I_\gamma\left(\eta_0^{(b)},\omega_4;\lambda\right)-\frac{7}{36}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,10};\lambda\right)-\frac{7}{36}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,11};\lambda\right)+\frac{3}{4}\,I_\gamma\left(\eta_0^{(b)},\eta_2^{\left(\frac{b}{c}\right)};\lambda\right)$$

$$+\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{3,10}^{(b)};\lambda\right)+\frac{2}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{3,11}^{(b)};\lambda\right)-4\,I_\gamma\left(\eta_0^{(b)},\eta_{3,1}^{(b)};\lambda\right)+I_\gamma\left(\eta_0^{(b)},\eta_{3,24}^{(b)};\lambda\right)$$

$$+\frac{4}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{3,9}^{(b)};\lambda\right)+3\,I_\gamma\left(\eta_0^{(b)},\eta_{4,3}^{(a,b)};\lambda\right)-\frac{217}{24}\,I_\gamma\left(\eta_0^{(b)},\omega_{0,4};\lambda\right)+\frac{7}{36}\,I_\gamma\left(\eta_{2,9},\eta_0^{(b)};\lambda\right)$$

$$-\frac{2}{9}\,I_\gamma\left(\eta_{2,9},\eta_{1,1}^{(b)};\lambda\right)+\frac{1}{3}\,I_\gamma\left(\eta_{2,9},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)+\frac{49}{12}\,I_\gamma\left(g_{2,1},\eta_0^{(b)};\lambda\right)-\frac{14}{3}\,I_\gamma\left(g_{2,1},\eta_{1,1}^{(b)};\lambda\right)$$

$$+7\,I_\gamma\left(g_{2,1},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)+\frac{7}{8}\,I_\gamma\left(\omega_0,\eta_0^{(b)};\lambda\right)-I_\gamma\left(\omega_0,\eta_{1,1}^{(b)};\lambda\right)+\frac{3}{2}\,I_\gamma\left(\omega_0,\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)$$

$$+\frac{8}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},d_{2,4};\lambda\right)-\frac{20}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{2,1};\lambda\right)-\frac{8}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},g_{2,1};\lambda\right)+I_\gamma\left(\eta_{1,1}^{(b)},\omega_0;\lambda\right)$$

$$+\frac{1}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\omega_4;\lambda\right)-\frac{28}{9}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,2}^{(b)};\lambda\right)+\frac{14}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,4}^{(b)};\lambda\right)-\frac{2}{9}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{3,1}^{(a)};\lambda\right)$$

$$-\frac{1}{18}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{3,2}^{(a)};\lambda\right)+\frac{16}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\omega_{0,4};\lambda\right)+4\,I_\gamma\left(\eta_{1,1}^{(b)},\omega_{-4,0};\lambda\right)+\frac{7}{36}\,I_\gamma\left(\eta_{2,10},\eta_0^{(b)};\lambda\right)$$

$$-\frac{2}{9}\,I_\gamma\left(\eta_{2,10},\eta_{1,1}^{(b)};\lambda\right)+\frac{1}{3}\,I_\gamma\left(\eta_{2,10},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)+\frac{3}{4}\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_0^{(c)};\lambda\right)-4\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_{1,1}^{(c)};\lambda\right)$$

$$-2\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_{1,3}^{(c)};\lambda\right)+3\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_2^{\left(\frac{a}{c}\right)};\lambda\right)+\frac{35}{8}\,I_\gamma\left(\omega_{0,4},\eta_0^{(b)};\lambda\right)-5\,I_\gamma\left(\omega_{0,4},\eta_{1,1}^{(b)};\lambda\right)$$

$$+\frac{15}{2}\,I_\gamma\left(\omega_{0,4},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)-4\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},d_{2,4};\lambda\right)+10\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{2,1};\lambda\right)$$

$$+4\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},g_{2,1};\lambda\right)-\frac{3}{2}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\omega_0;\lambda\right)-\frac{1}{2}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\omega_4;\lambda\right)$$

$$+\frac{14}{3}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,2}^{(b)};\lambda\right)-7\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,4}^{(b)};\lambda\right)+\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{3,1}^{(a)};\lambda\right)$$

$$+\frac{1}{12}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{3,2}^{(a)};\lambda\right)-8\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\omega_{0,4};\lambda\right)-6\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\omega_{-4,0};\lambda\right)$$

$$+4\,I_\gamma\left(\eta_0^{(b)},d_{2,4},a_{3,2}^{(b)};\lambda\right)+4\,I_\gamma\left(\eta_0^{(b)},d_{2,4},a_{4,1}^{(a,b)};\lambda\right)-26\,I_\gamma\left(\eta_0^{(b)},\eta_{2,1},a_{3,2}^{(b)};\lambda\right)$$

$$-26\,I_\gamma\left(\eta_0^{(b)},\eta_{2,1},a_{4,1}^{(a,b)};\lambda\right)+7\,I_\gamma\left(\eta_0^{(b)},g_{2,1},a_{3,2}^{(b)};\lambda\right)+7\,I_\gamma\left(\eta_0^{(b)},g_{2,1},a_{4,1}^{(a,b)};\lambda\right)$$

$$-6\,I_\gamma\left(\eta_0^{(b)},\omega_0,a_{3,2}^{(b)};\lambda\right)-6\,I_\gamma\left(\eta_0^{(b)},\omega_0,a_{4,1}^{(a,b)};\lambda\right)-\frac{31}{2}\,I_\gamma\left(\eta_0^{(b)},\omega_4,a_{3,2}^{(b)};\lambda\right)$$

$$-\frac{31}{2}\,I_\gamma\left(\eta_0^{(b)},\omega_4,a_{4,1}^{(a,b)};\lambda\right)+\frac{3}{4}\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_0^{(c)};\lambda\right)-4\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_{1,1}^{(c)};\lambda\right)$$

$$-2\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_{1,3}^{(c)};\lambda\right)+3\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_2^{\left(\frac{a}{c}\right)};\lambda\right)+\frac{7}{4}\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)},\eta_0^{(b)};\lambda\right)$$

$$-2\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)},\eta_{1,1}^{(b)};\lambda\right)+3\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)+\frac{7}{2}\,I_\gamma\left(\eta_0^{(b)},a_{4,4}^{(b,b)},\eta_0^{(b)};\lambda\right)$$

$$-4\,I_\gamma\left(\eta_0^{(b)},a_{4,4}^{(b,b)},\eta_{1,1}^{(b)};\lambda\right)+6\,I_\gamma\left(\eta_0^{(b)},a_{4,4}^{(b,b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)-\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,10},a_{3,2}^{(b)};\lambda\right)$$

$$-\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,10},a_{4,1}^{(a,b)};\lambda\right)-\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,11},a_{3,2}^{(b)};\lambda\right)-\frac{1}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_{2,11},a_{4,1}^{(a,b)};\lambda\right)$$

$$+\frac{4}{3}\,I_\gamma\left(\eta_0^{(b)},\eta_2^{\left(\frac{b}{c}\right)},a_{3,2}^{(c)};\lambda\right)+I_\gamma\left(\eta_0^{(b)},\eta_2^{\left(\frac{b}{c}\right)},a_{3,3}^{(c)};\lambda\right)+3\,I_\gamma\left(\eta_0^{(b)},\eta_2^{\left(\frac{b}{c}\right)},a_{4,1}^{(a,c)};\lambda\right)$$

$$-\frac{31}{2}\,I_\gamma\left(\eta_0^{(b)},\omega_{0,4},a_{3,2}^{(b)};\lambda\right)-\frac{31}{2}\,I_\gamma\left(\eta_0^{(b)},\omega_{0,4},a_{4,1}^{(a,b)};\lambda\right)+\frac{1}{3}\,I_\gamma\left(\eta_{2,9},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+\frac{1}{3}\,I_\gamma\left(\eta_{2,9},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+7\,I_\gamma\left(g_{2,1},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)+7\,I_\gamma\left(g_{2,1},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$+\frac{3}{2}\,I_\gamma\left(\omega_0,\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)+\frac{3}{2}\,I_\gamma\left(\omega_0,\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+\frac{14}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)};\lambda\right)$$

$$-\frac{16}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,1}^{(b)},\eta_{1,1}^{(b)};\lambda\right)+8\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,1}^{(b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)+7\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)};\lambda\right)$$

$$-8\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,3}^{(b)},\eta_{1,1}^{(b)};\lambda\right)+12\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,3}^{(b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)-\frac{16}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,2}^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$-\frac{16}{3}\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,2}^{(b)},a_{4,1}^{(a,b)};\lambda\right)+8\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,4}^{(b)},a_{3,2}^{(b)};\lambda\right)+8\,I_\gamma\left(\eta_{1,1}^{(b)},\eta_{1,4}^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$+\frac{1}{3}\,I_\gamma\left(\eta_{2,10},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)+\frac{1}{3}\,I_\gamma\left(\eta_{2,10},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+\frac{4}{3}\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_0^{(c)},a_{3,2}^{(c)};\lambda\right)$$

$$+I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_0^{(c)},a_{3,3}^{(c)};\lambda\right)+3\,I_\gamma\left(\eta_2^{\left(\frac{c}{b}\right)},\eta_0^{(c)},a_{4,1}^{(a,c)};\lambda\right)+\frac{15}{2}\,I_\gamma\left(\omega_{0,4},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+\frac{15}{2}\,I_\gamma\left(\omega_{0,4},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+8\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,2}^{(b)},a_{3,2}^{(b)};\lambda\right)+8\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,2}^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$-12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,4}^{(b)},a_{3,2}^{(b)};\lambda\right)-12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},\eta_{1,4}^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$+\frac{4}{3}\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_0^{(c)},a_{3,2}^{(c)};\lambda\right)+I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_0^{(c)},a_{3,3}^{(c)};\lambda\right)$$

$$+3\,I_\gamma\left(\eta_0^{(b)},a_{4,1}^{(b,c)},\eta_0^{(c)},a_{4,1}^{(a,c)};\lambda\right)+3\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+3\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+6\,I_\gamma\left(\eta_0^{(b)},a_{4,4}^{(b,b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+6\,I_\gamma\left(\eta_0^{(b)},a_{4,4}^{(b,b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+8\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+8\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)+12\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)$$

$$+12\,I_\gamma\left(\eta_{1,1}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)-12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)$$

$$-18\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,3}^{(b)},\eta_2^{\left(\frac{a}{b}\right)};\lambda\right)-18\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$-18\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)-\frac{21}{2}\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,3}^{(b)},\eta_0^{(b)};\lambda\right)$$

$$+12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,3}^{(b)},\eta_{1,1}^{(b)};\lambda\right)-12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{4,1}^{(a,b)};\lambda\right)$$

$$-12\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)-7\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_0^{(b)};\lambda\right)$$

$$+8\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)},a_{3,1}^{(b)},\eta_{1,1}^{(b)};\lambda\right)\Bigg]\,\zeta_2$$

$$+\Bigg[\frac{49}{12}\,I_\gamma\left(\eta_0^{(b)};\lambda\right)-\frac{7}{36}\,I_\gamma\left(\eta_{2,9};\lambda\right)-\frac{49}{12}\,I_\gamma\left(g_{2,1};\lambda\right)-\frac{7}{8}\,I_\gamma\left(\omega_0;\lambda\right)+\frac{22}{3}\,I_\gamma\left(\eta_{1,1}^{(b)};\lambda\right)$$

$$-\frac{7}{36}\,I_\gamma\left(\eta_{2,10};\lambda\right)-\frac{35}{8}\,I_\gamma\left(\omega_{0,4};\lambda\right)-11\,I_\gamma\left(\eta_0^{(b)},a_{3,2}^{(b)};\lambda\right)-\frac{7}{4}\,I_\gamma\left(\eta_0^{(b)},a_{4,3}^{(b,b)};\lambda\right)$$

$$-\frac{7}{2} I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}; \lambda\right) - \frac{14}{3} I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}; \lambda\right) - 7 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}; \lambda\right) + 7 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,1}^{(b)}; \lambda\right)$$

$$+\frac{21}{2} I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}; \lambda\right) \Bigg] \zeta_3$$

$$+\Bigg[7 I_\gamma\left(\eta_0^{(b)}; \lambda\right) - \frac{1}{3} I_\gamma(\eta_{2,9}; \lambda) - 7 I_\gamma(g_{2,1}; \lambda) - \frac{3}{2} I_\gamma(\omega_0; \lambda) - 8 I_\gamma\left(\eta_{1,1}^{(b)}; \lambda\right) - \frac{1}{3} I_\gamma(\eta_{2,10}; \lambda)$$

$$-\frac{15}{2} I_\gamma(\omega_{0,4}; \lambda) + 12 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}; \lambda\right) - 3 I_\gamma\left(\eta_0^{(b)}, a_{4,3}^{(b,b)}; \lambda\right) - 6 I_\gamma\left(\eta_0^{(b)}, a_{4,4}^{(b,b)}; \lambda\right)$$

$$-8 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,1}^{(b)}; \lambda\right) - 12 I_\gamma\left(\eta_{1,1}^{(b)}, a_{3,3}^{(b)}; \lambda\right) + 18 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,3}^{(b)}; \lambda\right)$$

$$+12 I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, a_{3,1}^{(b)}; \lambda\right) \Bigg] \zeta_2 \ln(2) - \frac{79}{4} \zeta_4 + 8 \operatorname{Li}_4\left(\frac{1}{2}\right) - 8 \zeta_2 (\ln(2))^2 + \frac{1}{3} (\ln(2))^4 . \quad (9.11)$$

The large number of terms for $J_{41}^{(4)}$ can be traced back to the already large number of terms of $J_{42}^{(3)}$ and $J_{43}^{(3)}$, which are according to eq. (9.10) integrated further with $\eta_{1,1}^{(b)}$ and $\eta_0^{(b)}$, respectively. The expression in eq. (9.11) involves 47 of the 107 integration kernels.

9.5 Numerical checks

All results have been verified numerically with the help of the program `sector_decomposition` [119]. The program `sector_decomposition` allows (as `SecDec` [120–123] or `FIESTA` [124, 125]) the numerical evaluation of multi-loop integrals. On the one hand we evaluated all master integrals of the basis $\vec{I}$ at a few kinematic points numerically with the program `sector_decomposition`. On the other hand, we evaluated our results in the basis $\vec{J}$ at the same kinematic points, converted to the basis $\vec{I}$ and compared the two results. We find good agreement.

The evaluation of the iterated integrals appearing in our results is done as follows: we split the integration path into two pieces: first we integrate in $(\tilde{x}, \tilde{y})$-space from $(0,0)$ to $(\tilde{x}, 0)$, then from $(\tilde{x}, 0)$ to $(\tilde{x}, \tilde{y})$. The integration along the first part gives only multiple polylogarithms, which can be evaluated to high precision [97]. We use these results as new boundary constants for the integration along the second part. Assuming that $\tilde{y}$ is small, we may expand for the integration along the second part all integration kernels in $\tilde{y}$.

As a reference we give numerical results for the master integrals in the basis $\vec{J}$ at the kinematic point

$$s = -\frac{12769}{840} m^2, \quad t = \frac{10}{11} m^2. \quad (9.12)$$

This point corresponds to

$$x = \frac{7}{120}, \quad y = \frac{10}{11}, \quad (9.13)$$

or equivalently

$$\tilde{x} = \frac{1}{15}, \quad \tilde{y} = \frac{1}{11}. \quad (9.14)$$

The numerical results for the first five terms of ε-expansion are given in tables 2 and 3.

	ε^0	ε^1	ε^2	ε^3	ε^4
J_1	1	0	1.6449341	−0.80137127	1.8940657
J_2	0	2.8415816	−2.8295758	6.4116869	−7.7009279
J_3	0	−2.8415816	15.355894	−39.817554	97.903278
J_4	0	0	8.074586	−20.479468	55.140667
J_5	1	−2.7213737	3.7029375	−6.5645107	7.7616443
J_6	0	0	4.7951687	−7.7339091	23.583241
J_7	0	−0.13797489	5.0760627	−8.5195954	25.27333
J_8	0	0	9.8696044	−15.803336	48.383357
J_9	0	0	4.037293	−2.1975902	7.5750011
J_{10}	0	0	0	−10.577768	19.861743
J_{11}	0	2.8415816	−10.562581	19.960005	−34.628948
J_{12}	0	0	4.8094349	−23.163298	56.79741
J_{13}	0	0	0	−9.6340372	18.255071
J_{14}	0	0	0	0.074587202	−0.1198646
J_{15}	0	0	−4.037293	23.437914	−62.690651
J_{16}	0	0	0	8.4983135	−20.922966
J_{17}	0	0	0	0.1491744	0.058984085
J_{18}	0	1.4207908	−12.163995	44.930917	−88.809767
J_{19}	0	0	0	16.996627	−15.625817
J_{20}	0	0	−10.735443	−9.8004674	−37.795989
J_{21}	0	0	4.037293	−13.184573	21.864228
J_{22}	0	0	0	0	8.4599162
J_{23}	0	0	0	4.8796692	−25.793413
J_{24}	0	0	0	2.6138189	−0.23796592
J_{25}	0	0	4.037293	−9.2635254	25.950914
J_{26}	0	0	2.7276656	−1.848663	13.397014
J_{27}	0	0	0	5.2276379	8.7055971
J_{28}	0	0	0	8.9388561	12.795847
J_{29}	0	0	0	0	18.80581
J_{30}	0	0	5.4553312	−3.9355497	35.856907
J_{32}	0	0	−10.735443	−40.306104	−35.268067
J_{33}	0	0	0	7.3822471	10.116064
J_{34}	0	0	10.735443	−3.4643927	53.616756
J_{35}	0	0	0.97741243	5.1104476	15.424638

Table 2. Numerical results for the first five terms of the ε-expansion of the master integrals J_1-J_{35} at the kinematic point $s = -\frac{12769}{840}m^2$, $t = \frac{10}{11}m^2$.

	ε^0	ε^1	ε^2	ε^3	ε^4
J_{36}	0	0	0	0	-13.214347
J_{37}	0	0	0	0	-43.342128
J_{38}	0	0	0	0	44.194787
J_{39}	0	0	0	0	0.28609557
J_{40}	0	0	0	0	-26.330837
J_{41}	0	0	0	0	11.147258
J_{42}	0	0	0	-19.021429	-320.23817
J_{43}	0	0	0	0.89070327	9.183764
J_{44}	0	0	0	0	21.040337
J_{45}	0	0	0	0	-1.4008206

Table 3. Numerical results for the first five terms of the ε-expansion of the master integrals J_{36}-J_{45} at the kinematic point $s = -\frac{12769}{840}m^2$, $t = \frac{10}{11}m^2$.

10   Conclusions

In this article we gave a detailed account on the analytic calculation of the master integrals for the planar double box integral relevant to top-pair production with a closed top loop. The planar double box integral depends on two scales and involves several elliptic sub-sectors. We showed that the associated elliptic curves can be extracted from the maximal cuts. We demonstrated that the system involves three inequivalent elliptic curves. To the best of our knowledge, this is the first time an integral involving more than one elliptic curve has been calculated. Elliptic polylogarithms are by definition iterated integrals on a single elliptic curve. Since the system discussed in this article involves three elliptic curves, we do not expect that our results are naturally expressible in terms of elliptic polylogarithms. We showed that the system of differential equations can be transformed to a form linear in ε, where the ε^0-term is strictly lower-triangular. This system of differential equations is easily solved to any desired order in ε. We expressed our results in terms of iterated integrals and discussed the occurring integration kernels. We believe that the techniques used in this paper are applicable to a wider class of Feynman integrals.

Acknowledgments

A   Master topologies

In this appendix we show diagrams of all master topologies. In total there 27 master topologies. A master topology may contain several master integrals. The total number of master integrals is 44. The number of master integrals within a master topology can be inferred from table 1. The diagrams for the master topologies are shown in figures 3–6.
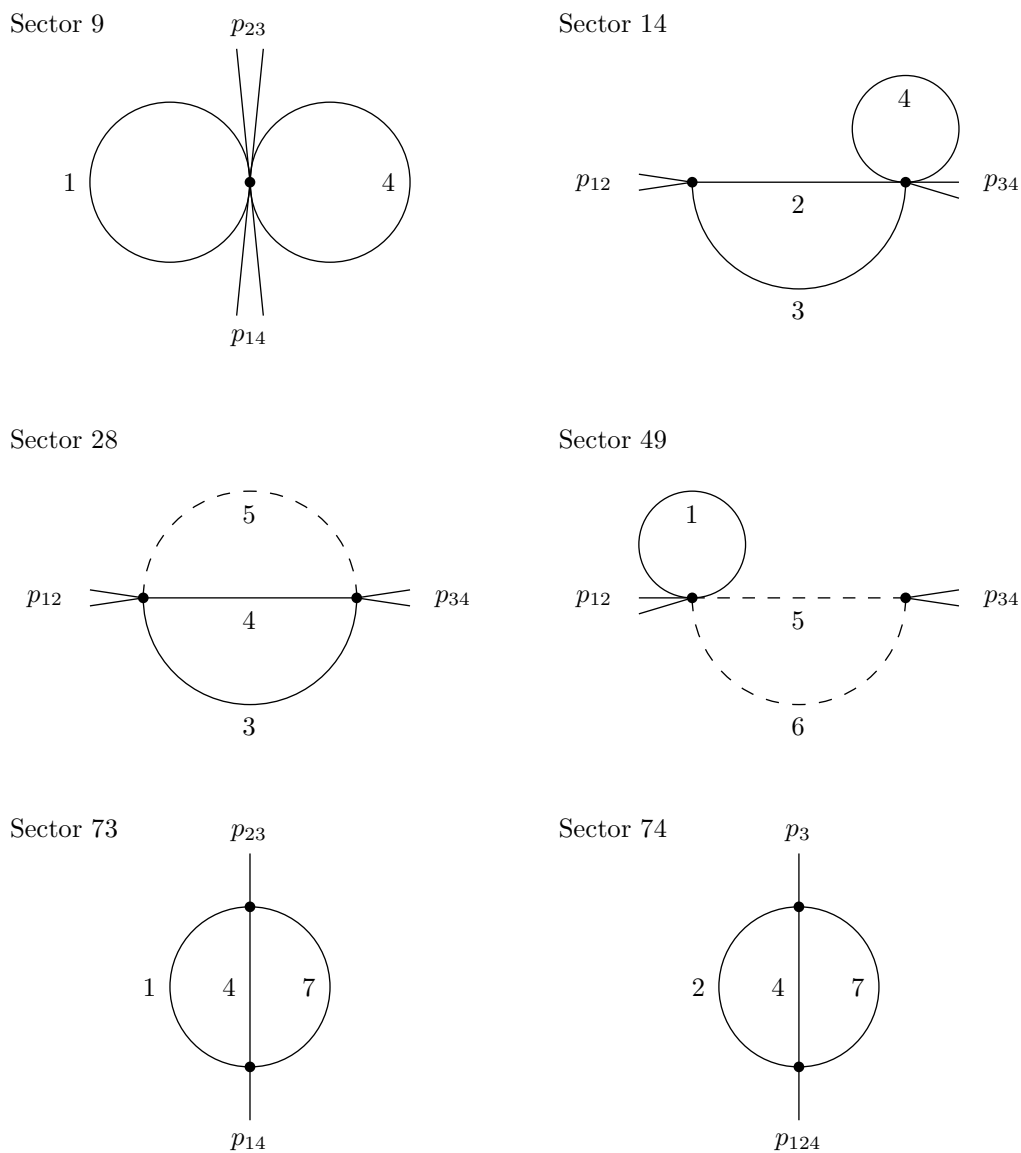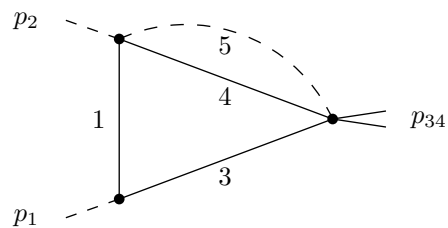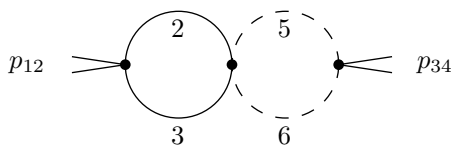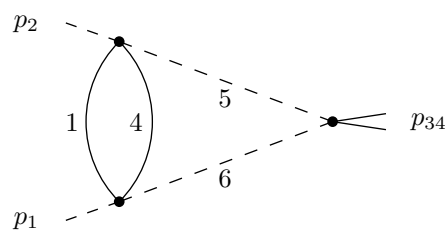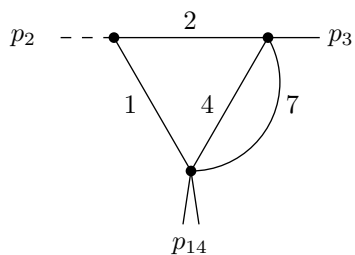
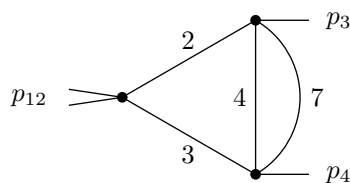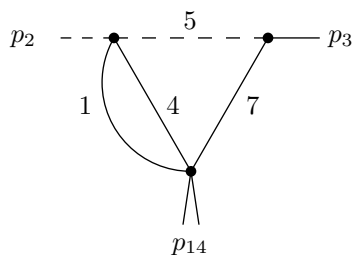

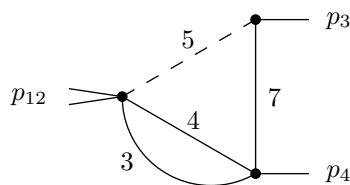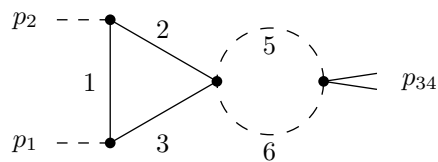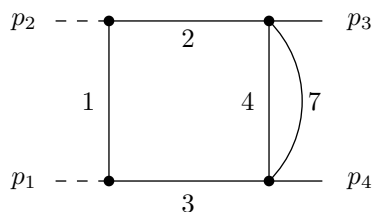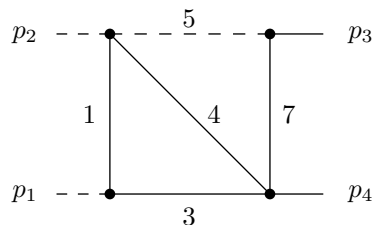**Figure 3**. Master topologies (part 1).

Sector 15
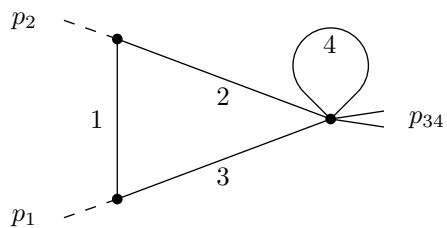
Sector 29

Sector 54

Sector 57

Sector 75

Sector 78

Sector 89

Sector 92

Figure 4. Master topologies (part 2).
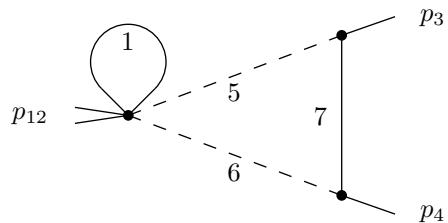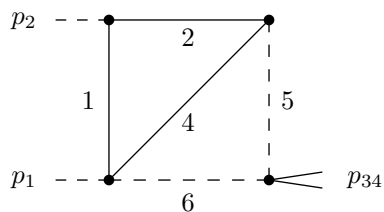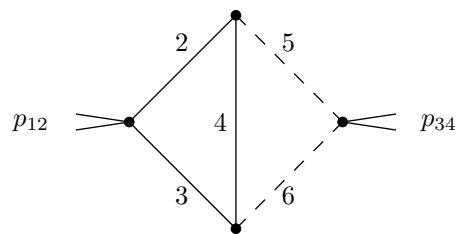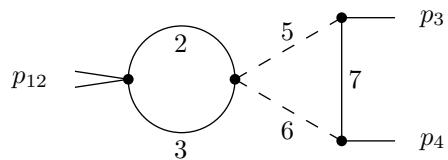
Figure 5. Master topologies (part 3).

Sector 63

Sector 119
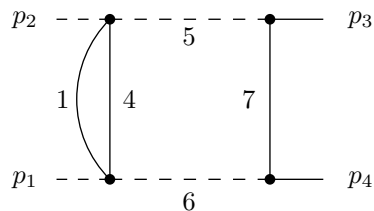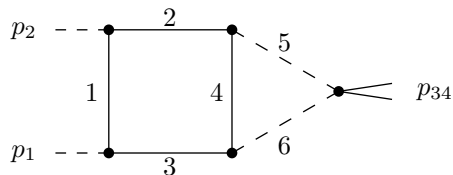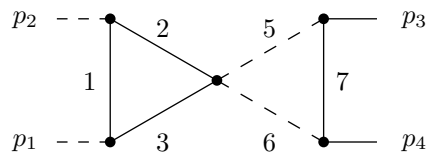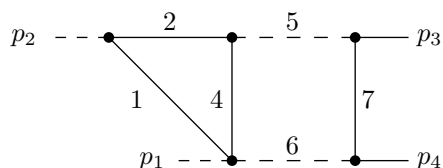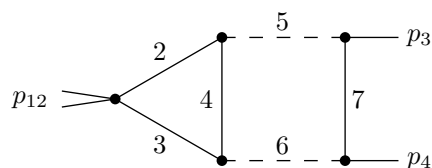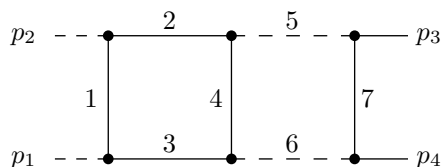
Sector 123

Sector 126

Sector 127

Figure 6. Master topologies (part 4).

B   The extra relation

In this appendix we give the relation which can be used to eliminate in sector 93 one of the five integrals $I_{1011101}, I_{2011101}, I_{1021101}, I_{1012101}, I_{1011201}$. The relation reads

$$48 \left(D-3\right)\left(D-4\right)^2\left(D-5\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)^2\left(m^2-s-t\right)m^6 I_{1011101}$$

$$+32\left(D-3\right)\left(D-4\right)\left(D-5\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)^2\left(m^2-s-t\right)m^8 I_{2011101}$$

$$+32\left(D-3\right)\left(D-4\right)\left(D-5\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)^2\left(m^2-s-t\right)m^8 I_{1021101}$$

$$+16\left(D-3\right)\left(D-4\right)\left(D-5\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)^2\left(2m^4-2m^2t-3m^2s+st\right)m^6$$

$$I_{1012101}$$

$$+64\left(D-3\right)\left(D-4\right)\left(D-5\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)s^2m^8 I_{1011201}$$

$$+32\left(D-3\right)\left(D-4\right)\left(2D-9\right)\left(4m^2-s\right)^2\left(m^2-t\right)\left[-4s-6m^2+\left(s+2m^2\right)D\right]m^6 s I_{0021101}$$

$$+16\left(D-3\right)^2\left(D-4\right)\left(2D-9\right)\left(3D-10\right)\left(4m^2-s\right)^2\left(m^2-t\right)sm^6 I_{0011101}$$

$$-6\left(D-3\right)\left(D-4\right)^3\left(3D-10\right)\left(4m^2-s\right)^2\left(m^2-t\right)^3 sm^2 I_{1001101}$$

$$
\begin{aligned}
&+4\,(D-2)\,(D-3)\,(D-4)\,(2\,D-9)\,\left(4\,m^2-s\right)^2\left(m^2-t\right)^2\left[15\,t-15\,m^2+8\,s\right.\\
&\left.+\left(3\,m^2-3\,t-2\,s\right)D\right]m^4 I_{1101001}\\
&-8\,(D-2)\,(D-3)\,(D-4)\,(D-5)\,(2\,D-9)\,\left(4\,m^2-s\right)^2\left(m^2-t\right)^2 s m^4 I_{1011100}\\
&+2\,(D-3)\,(D-4)\,(2\,D-9)\,(3\,D-8)\,\left(4\,m^2-s\right)^2 m^4\left(m^2-t\right)\left[15\,t-15\,m^2+8\,s\right.\\
&\left.+\left(-3\,t-2\,s+3\,m^2\right)D\right]I_{0101001}\\
&-4\,(D-3)\,(D-4)\,\left(4\,m^2-s\right)^2\left[-432\,m^6 s-1890\,m^6 t-270\,m^2 t^3+810\,m^8+1350\,m^4 t^2\right.\\
&+48\,st^3+192\,m^4 st-288\,m^2 st^2+\left(798\,m^6 t-24\,st^3-92\,m^4 st+114\,m^2 t^3-570\,m^4 t^2\right.\\
&\left.-342\,m^8+156\,m^2 st^2+216\,m^6 s\right)D+\left(3\,st^3-12\,m^2 t^3-84\,m^6 t-27\,m^6 s+60\,m^4 t^2\right.\\
&\left.\left.-21\,m^2 st^2+36\,m^8+13\,m^4 st\right)D^2\right]m^2 I_{2001001}\\
&-2\,(D-3)\,(D-4)\,(3\,D-8)\,\left(4\,m^2-s\right)^2\left[48\,st^2-540\,m^4 t+270\,m^6-144\,m^4 s-24\,m^2 st\right.\\
&+270\,m^2 t^2+\left(228\,m^4 t+16\,m^2 st-114\,m^2 t^2+72\,m^4 s-114\,m^6-24\,st^2\right)D+\left(3\,st^2\right.\\
&\left.\left.+12\,m^2 t^2-9\,m^4 s+12\,m^6-2\,m^2 st-24\,m^4 t\right)D^2\right]m^2 I_{1001001}\\
&-16\,(D-3)\,(2\,D-9)\,\left(m^2-t\right)\left[-140\,m^2 s^2 t+520\,m^4 st+1120\,m^8+28\,m^2 s^3-856\,m^6 s\right.\\
&-1120\,m^6 t+20\,ts^3+428\,m^4 s^2+\left(-19\,m^2 s^3-224\,m^4 st+432\,m^6 s+58\,m^2 s^2 t-238\,m^4 s^2\right.\\
&+544\,m^6 t-544\,m^8-9\,ts^3\big)D+\left(24\,m^4 st-56\,m^6 s-6\,m^2 s^2 t+3\,m^2 s^3-64\,m^6 t+64\,m^8\right.\\
&\left.\left.+ts^3+34\,m^4 s^2\right)D^2\right]m^4 I_{0021100}\\
&-8\,(D-3)\,(2\,D-9)\,(3\,D-8)\,\left(m^2-t\right)\left[128\,m^2 s^2+560\,m^6+220\,m^2 st-560\,m^4 t-20\,s^2 t\right.\\
&-388\,m^4 s+\left(-75\,m^2 s^2-272\,m^6+9\,s^2 t+208\,m^4 s-104\,m^2 st+272\,m^4 t\right)D+\left(12\,m^2 st\right.\\
&\left.\left.-s^2 t+11\,m^2 s^2-28\,m^4 s-32\,m^4 t+32\,m^6\right)D^2\right]m^4 I_{0011100}\\
&-(D-2)^2\left[-768\,s^3 t^2+57600\,m^8 t-28800\,m^6 t^2+20304\,m^8 s-5592\,m^4 s^2 t+288\,m^4 s^3\right.\\
&+4272\,m^4 st^2-16896\,m^6 st-28800\,m^{10}+3804\,m^2 s^2 t^2-2052\,m^6 s^2+960\,tm^2 s^3\\
&+\left(20080\,m^{10}+20080\,m^6 t^2-14776\,m^8 s+4970\,m^4 s^2 t-224\,m^4 s^3+1367\,m^6 s^2\right.\\
&\left.-1688\,m^4 st^2-40160\,m^8 t-3329\,m^2 s^2 t^2+624\,s^3 t^2-776\,tm^2 s^3+10448\,m^6 st\right)D\\
&+\left(3576\,m^8 s+9248\,m^8 t-2064\,m^6 st+955\,m^2 s^2 t^2-301\,m^6 s^2+206\,tm^2 s^3+58\,m^4 s^3\right.\\
&\left.+24\,m^4 st^2-4624\,m^6 t^2-1422\,m^4 s^2 t-4624\,m^{10}-168\,s^3 t^2\right)D^2+\left(-288\,m^8 s+22\,m^6 s^2\right.\\
&+15\,s^3 t^2+352\,m^6 t^2-704\,m^8 t-5\,m^4 s^3+32\,m^4 st^2-18\,tm^2 s^3-90\,m^2 s^2 t^2+132\,m^4 s^2 t\\
&\left.\left.+128\,m^6 st+352\,m^{10}\right)D^3\right]I_{1001000}=0.
\end{aligned}
$$
(B.1)

C The modular forms relevant to the elliptic curve $E^{(a)}$

In this appendix we write for simplicity ψ_1 for $\psi_1^{(a)}$ and τ_6 for $\tau_6^{(a)}$. We further write

$$
r_n = \exp\left(\frac{2\pi i}{n}\right)
$$
(C.1)

for the n-th root of unity.

Let us first introduce a set of modular forms for the congruence subgroup $\Gamma_1(6)$, which was already encountered in the calculation of the sunrise / kite system [21, 26]. We consider

the set

$$\left\{ \frac{\psi_1}{\pi}, f_1, f_2, f_3, g_{2,0}, g_{2,1}, g_{2,9} \right\}. \tag{C.2}$$

ψ_1/π is a modular form of modular weight 1, f_i is of modular weight i and the modular weight of $g_{2,j}$ is 2. In this article we use as boundary point $(x, y) = (0, 1)$ (or equivalently $(s, t) = (\infty, m^2)$). Therefore we choose our periods such that $y = 1$ (or $t = m^2$) for the curve $E^{(a)}$ corresponds to $\tau_6 = i\infty$ and $q_6 = 0$. This corresponds to a q_6-expansion around the cusp $t = m^2$. The Hauptmodul is given by

$$y - 1 = -8 \frac{\eta(\tau_6)^3 \eta(6\tau_6)^9}{\eta(2\tau_6)^3 \eta(3\tau_6)^9}. \tag{C.3}$$

For the modular forms from the set (C.2) we give a representation in the form of an eta-quotient (where such a representation is known), a representation in the form of a polynomial in the two generators e_1 and e_2 of the Eisenstein subspace $\mathcal{E}_1(\Gamma_1(6))$, a representation in terms of generalised Eisenstein series and a representation in terms of $\overline{\mathrm{E}}_{n;m}$-functions. We have

$$\begin{aligned}
\frac{\psi_1}{\pi} &= \frac{1}{2} \frac{\eta(2\tau_6)\,\eta(3\tau_6)^6}{\eta(\tau_6)^2\,\eta(6\tau_6)^3} \\
&= e_1 + 2e_2 \\
&= E_1(\tau_6; \chi_0, \chi_1) + 2E_1(2\tau_6; \chi_0, \chi_1) \\
&= \frac{1}{2} + \frac{1}{\sqrt{3}} \overline{\mathrm{E}}_{0;0}(r_6; 1; q_6), \\
f_1 &= \frac{1}{2}(y + 3) \frac{\psi_1}{\pi} \\
&= 6e_2 \\
&= 6E_1(2\tau_6; \chi_0, \chi_1) \\
&= 1 - \sqrt{3} \overline{\mathrm{E}}_{0;0}(r_3; 1; q_6) + \sqrt{3} \overline{\mathrm{E}}_{0;0}(r_6; 1; q_6), \\
f_2 &= -\frac{1}{4}(3y^2 - 10y - 9) \frac{\psi_1^2}{\pi^2} \\
&= -12e_1^2 + 36e_1 e_2 + 12e_2^2 \\
&= 4B_{2,2}(\tau_6) - 10B_{2,3}(\tau_6) + 8B_{2,6}(\tau_6) \\
&= 1 - 6\overline{\mathrm{E}}_{0;-1}(r_2; 1; q_6) + 2\overline{\mathrm{E}}_{0;-1}(r_3; 1; q_6) - 8\overline{\mathrm{E}}_{0;-1}(r_6; 1; q_6), \\
f_3 &= -\frac{3}{2} y(y - 1)(y - 9) \frac{\psi_1^3}{\pi^3} \\
&= -12 \frac{\eta(\tau_6)^4 \eta(2\tau_6) \eta(6\tau_6)^5}{\eta(3\tau_6)^4} \\
&= 216\left(e_1^3 - 2e_1^2 e_2 - e_1 e_2^2 + 2e_2^3\right) \\
&= -12E_3(\tau_6; \chi_0, \chi_1) + 12E_3(2\tau_6; \chi_0, \chi_1) \\
&= -\frac{9}{2}\sqrt{3} \overline{\mathrm{E}}_{-2;0}(r_3; 1; q_6) + \frac{1}{2}\sqrt{3} \overline{\mathrm{E}}_{-2;0}(r_6; 1; q_6),
\end{aligned}$$

$$
\begin{aligned}
g_{2,0} &= \frac{6\psi_1^2}{2\pi i W_y}\frac{1}{y} = -\frac{1}{2}\left(y-1\right)\left(y-9\right)\frac{\psi_1^2}{\pi^2}\\
&= -8\frac{\eta\left(2\tau_6\right)^4\eta\left(6\tau_6\right)^4}{\eta\left(\tau_6\right)^2\eta\left(3\tau_6\right)^2}\\
&= -24\left(e_1^2-e_2^2\right)\\
&= -4B_{2,2}(\tau_6)-8B_{2,3}(\tau_6)+4B_{2,6}(\tau_6)\\
&= 4\overline{\mathrm{E}}_{0;-1}\left(r_3;1;q_6\right)-4\overline{\mathrm{E}}_{0;-1}\left(r_6;1;q_6\right),\\
g_{2,1} &= \frac{6\psi_1^2}{2\pi i W_y}\frac{1}{y-1} = -\frac{1}{2}y\left(y-9\right)\frac{\psi_1^2}{\pi^2}\\
&= \frac{\eta\left(\tau_6\right)^3\eta\left(2\tau_6\right)^3}{\eta\left(3\tau_6\right)\eta\left(6\tau_6\right)}\\
&= -18e_1^2+18e_1e_2+36e_2^2\\
&= -3B_{2,2}(\tau_6)-9B_{2,3}(\tau_6)+9B_{2,6}(\tau_6)\\
&= 1-3\overline{\mathrm{E}}_{0;-1}\left(r_2;1;q_6\right)-9\overline{\mathrm{E}}_{0;-1}\left(r_6;1;q_6\right),\\
g_{2,9} &= \frac{6\psi_1^2}{2\pi i W_y}\frac{1}{y-9} = -\frac{1}{2}y\left(y-1\right)\frac{\psi_1^2}{\pi^2}\\
&= \frac{\eta\left(\tau_6\right)^7\eta\left(6\tau_6\right)^7}{\eta\left(2\tau_6\right)^5\eta\left(3\tau_6\right)^5}\\
&= -6e_1^2+18e_1e_2-12e_2^2\\
&= 5B_{2,2}(\tau_6)-5B_{2,3}(\tau_6)+B_{2,6}(\tau_6)\\
&= -3\overline{\mathrm{E}}_{0;-1}\left(r_2;1;q_6\right)+4\overline{\mathrm{E}}_{0;-1}\left(r_3;1;q_6\right)-\overline{\mathrm{E}}_{0;-1}\left(r_6;1;q_6\right),
\end{aligned}
\tag{C.4}
$$

Here we used the notation $B_{2,N}(\tau)=E_2(\tau)-NE_2(N\tau)$ and

$$
e_1=E_1\left(\tau_6;\chi_0,\chi_1\right),\quad e_2=E_1\left(2\tau_6;\chi_0,\chi_1\right),
\tag{C.5}
$$

where χ_0 and χ_1 denote primitive Dirichlet characters with conductors 1 and 3, respectively. They are given in terms of the Kronecker symbol by

$$
\chi_0=\left(\frac{1}{n}\right),\quad \chi_1=\left(\frac{-3}{n}\right).
\tag{C.6}
$$

The definition of the generalised Eisenstein series is as in [21, 127]. The two Eisenstein series $\{e_1,e_2\}$ give a basis for the modular forms of modular weight 1 for the Eisenstein subspace $\mathcal{E}_1(\Gamma_1(6))$.

The $\overline{\mathrm{E}}_{n;m}$-functions are defined by [17, 24]

$$
\overline{\mathrm{E}}_{n;m}\left(x;y;q\right)=\begin{cases}\frac{1}{i}\left[\mathrm{ELi}_{n;m}\left(x;y;q\right)-\mathrm{ELi}_{n;m}\left(x^{-1};y^{-1};q\right)\right], & n+m\text{ even},\\[2mm]\mathrm{ELi}_{n;m}\left(x;y;q\right)+\mathrm{ELi}_{n;m}\left(x^{-1};y^{-1};q\right), & n+m\text{ odd}.\end{cases}
\tag{C.7}
$$

and

$$
\mathrm{ELi}_{n;m}\left(x;y;q\right)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^j}{j^n}\frac{y^k}{k^m}q^{jk}.
\tag{C.8}
$$

Products of modular forms are again modular forms. In particular

$$f_4 = f_1^4 \tag{C.9}$$

and

$$g_{3,1} = g_{2,1} \frac{\psi_1}{\pi},$$

$$g_{4,j} = g_{2,j} \left(\frac{\psi_1}{\pi} \right)^2, \qquad j \in \{0, 1, 9\},$$

$$h_{3,0} = \frac{1}{3} f_3,$$

$$h_{4,0} = \frac{1}{3} f_3 \frac{\psi_1}{\pi},$$

$$h_{4,1} = \frac{2}{3} f_1 f_3 - \frac{4}{3} f_3 \frac{\psi_1}{\pi}. \tag{C.10}$$

This shows that all elements of the set in eq. (5.74) are modular forms.

D   Boundary constants

In this appendix we list the boundary constants at the point $(\tilde{x}, \tilde{y}) = (0, 0)$ (or equivalently $(s, t) = (\infty, m^2)$ or $(x, y) = (0, 1)$). The boundary constants are given by

$$J_1|_{(\tilde{x}, \tilde{y})=(0,0)} = 1 + \zeta_2 \varepsilon^2 - \frac{2}{3} \zeta_3 \varepsilon^3 + \frac{7}{4} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_2|_{(\tilde{x}, \tilde{y})=(0,0)} = \zeta_2 \varepsilon^2 + 2 \zeta_3 \varepsilon^3 + \frac{19}{4} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_3|_{(\tilde{x}, \tilde{y})=(0,0)} = -\zeta_2 \varepsilon^2 - 11 \zeta_3 \varepsilon^3 - \frac{29}{2} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_4|_{(\tilde{x}, \tilde{y})=(0,0)} = -6 \zeta_3 \varepsilon^3 - \frac{13}{2} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_5|_{(\tilde{x}, \tilde{y})=(0,0)} = 1 - \frac{8}{3} \zeta_3 \varepsilon^3 - 3 \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_6|_{(\tilde{x}, \tilde{y})=(0,0)} = 3 \zeta_2 \varepsilon^2 + \left[\frac{21}{2} \zeta_3 - 18 \zeta_2 \ln(2) \right] \varepsilon^3 + \left[72 \operatorname{Li}_4 \left(\frac{1}{2} \right) - 39 \zeta_4 + 36 \zeta_2 \ln^2(2) \right.$$
$$\left. + 3 \ln^4(2) \right] \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_7|_{(\tilde{x}, \tilde{y})=(0,0)} = 3 \zeta_2 \varepsilon^2 + \left[\frac{21}{2} \zeta_3 - 18 \zeta_2 \ln(2) \right] \varepsilon^3 + \left[72 \operatorname{Li}_4 \left(\frac{1}{2} \right) - 39 \zeta_4 + 36 \zeta_2 \ln^2(2) \right.$$
$$\left. + 3 \ln^4(2) \right] \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_8|_{(\tilde{x}, \tilde{y})=(0,0)} = 6 \zeta_2 \varepsilon^2 + [21 \zeta_3 - 36 \zeta_2 \ln 2] \varepsilon^3 + \left[144 \operatorname{Li}_4 \left(\frac{1}{2} \right) - 78 \zeta_4 + 72 \zeta_2 \ln^2(2) \right.$$
$$\left. + 6 \ln^4(2) \right] \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_9|_{(\tilde{x}, \tilde{y})=(0,0)} = -3 \zeta_3 \varepsilon^3 - \frac{5}{4} \zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{10}|_{(\tilde{x},\tilde{y})=(0,0)} = -4\zeta_3\varepsilon^3 + 4\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{11}|_{(\tilde{x},\tilde{y})=(0,0)} = \zeta_2\varepsilon^2 + 2\zeta_3\varepsilon^3 + \frac{9}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{12}|_{(\tilde{x},\tilde{y})=(0,0)} = \zeta_2\varepsilon^2 - 4\zeta_3\varepsilon^3 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{13}|_{(\tilde{x},\tilde{y})=(0,0)} = -2\zeta_3\varepsilon^3 + \frac{5}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{14}|_{(\tilde{x},\tilde{y})=(0,0)} = \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{15}|_{(\tilde{x},\tilde{y})=(0,0)} = \left[\frac{11}{2}\zeta_3 - 6\zeta_2\ln(2)\right]\varepsilon^3 + \left[24\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{23}{2}\zeta_4 + 12\zeta_2\ln^2(2) + \ln^4(2)\right]\varepsilon^4$$
$$+\,\mathcal{O}\left(\varepsilon^5\right),$$

$$J_{16}|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{1}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{17}|_{(\tilde{x},\tilde{y})=(0,0)} = \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{18}|_{(\tilde{x},\tilde{y})=(0,0)} = -\zeta_2\varepsilon^2 + \left[\frac{9}{4}\zeta_3 + 3\zeta_2\ln(2)\right]\varepsilon^3 + \left[-12\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{75}{8}\zeta_4 - 6\zeta_2\ln^2(2)\right.$$
$$\left. -\frac{1}{2}\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{19}|_{(\tilde{x},\tilde{y})=(0,0)} = -\frac{1}{2}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{20}|_{(\tilde{x},\tilde{y})=(0,0)} = -4\zeta_2\varepsilon^2 - 5\zeta_3\varepsilon^3 - \frac{99}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{21}|_{(\tilde{x},\tilde{y})=(0,0)} = -3\zeta_3\varepsilon^3 - \frac{5}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{22}|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{7}{4}\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{23}|_{(\tilde{x},\tilde{y})=(0,0)} = 6\zeta_3\varepsilon^3 + 9\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{24}|_{(\tilde{x},\tilde{y})=(0,0)} = \left[-\frac{7}{4}\zeta_3 - 3\zeta_2\ln(2)\right]\varepsilon^3 + \left[\frac{15}{4}\zeta_4 + 18\zeta_2\ln^2(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{25}|_{(\tilde{x},\tilde{y})=(0,0)} = \left[-\frac{23}{6}\zeta_3 + 2\zeta_2\ln(2)\right]\varepsilon^3 + \left[-8\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3\zeta_4 - 4\zeta_2\ln^2(2) - \frac{1}{3}\ln^4(2)\right]\varepsilon^4$$
$$+\,\mathcal{O}\left(\varepsilon^5\right),$$

$$J_{26}|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{7}{4}\zeta_2\varepsilon^2 + \left[\frac{35}{8}\zeta_3 - \frac{27}{2}\zeta_2\ln(2)\right]\varepsilon^3 + \left[42\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 19\zeta_4 + 39\zeta_2\ln^2(2)\right.$$
$$\left. +\frac{7}{4}\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{27}|_{(\tilde{x},\tilde{y})=(0,0)} = \left[-\frac{7}{2}\zeta_3 - 6\zeta_2\ln(2)\right]\varepsilon^3 + \left[\frac{105}{4}\zeta_4 + 36\zeta_2\ln^2(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{28}|_{(\tilde{x},\tilde{y})=(0,0)} = \left[-\frac{1}{3}\zeta_3 - 4\zeta_2\ln(2)\right]\varepsilon^3 + \left[16\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 12\zeta_4 + 8\zeta_2\ln^2(2) + \frac{2}{3}\ln^4(2)\right]\varepsilon^4$$
$$+\,\mathcal{O}\left(\varepsilon^5\right),$$

$$J_{29}|_{(\tilde{x},\tilde{y})=(0,0)} = \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{30}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{7}{2}\,\zeta_2\varepsilon^2 + \left[\frac{35}{4}\,\zeta_3 - 27\,\zeta_2\ln(2)\right]\varepsilon^3 + \left[84\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{77}{4}\,\zeta_4 + 78\,\zeta_2\ln^2(2)\right.$$
$$\left. + \frac{7}{2}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{32}\big|_{(\tilde{x},\tilde{y})=(0,0)} = -4\,\zeta_2\varepsilon^2 - 5\,\zeta_3\varepsilon^3 - \frac{139}{4}\,\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{33}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{105}{4}\,\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{34}\big|_{(\tilde{x},\tilde{y})=(0,0)} = 4\,\zeta_2\varepsilon^2 + \left[\frac{11}{2}\,\zeta_3 - 6\,\zeta_2\ln(2)\right]\varepsilon^3 + \left[24\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{2}\,\zeta_4 + 12\,\zeta_2\ln^2(2)\right.$$
$$\left. + \ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{35}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{3}{4}\,\zeta_2\varepsilon^2 + \left[\frac{21}{8}\,\zeta_3 - \frac{9}{2}\,\zeta_2\ln(2)\right]\varepsilon^3 + \left[18\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{33}{2}\,\zeta_4 + 9\,\zeta_2\ln^2(2)\right.$$
$$\left. + \frac{3}{4}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{36}\big|_{(\tilde{x},\tilde{y})=(0,0)} = -\frac{39}{2}\,\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{37}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{38}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \frac{17}{2}\,\varepsilon^4\zeta_4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{39}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{40}\big|_{(\tilde{x},\tilde{y})=(0,0)} = -3\,\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{41}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \left[8\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{79}{4}\,\zeta_4 - 8\,\zeta_2\ln^2(2) + \frac{1}{3}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{42}\big|_{(\tilde{x},\tilde{y})=(0,0)} = -24\,\zeta_3\varepsilon^3 - 102\,\zeta_4\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{43}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \left[\frac{49}{12}\,\zeta_3 + 7\,\zeta_2\ln(2)\right]\varepsilon^3 + \left[8\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{109}{4}\,\zeta_4 - 50\,\zeta_2\ln^2(2)\right.$$
$$\left. + \frac{1}{3}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{44}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \left[\frac{16}{3}\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{31}{12}\,\zeta_4 - \frac{16}{3}\,\zeta_2\ln^2(2) + \frac{2}{9}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right),$$

$$J_{45}\big|_{(\tilde{x},\tilde{y})=(0,0)} = \left[-\frac{56}{3}\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{11}{12}\,\zeta_4 + \frac{56}{3}\,\zeta_2\ln^2(2) - \frac{7}{9}\,\ln^4(2)\right]\varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right). \tag{D.1}$$

E Supplementary material

Attached to this article is an electronic file in ASCII format with `Maple` syntax, defining the quantities

$$\texttt{I\_basis}, \quad \texttt{U}, \quad \texttt{Uinv}, \quad \texttt{A}, \quad \texttt{integration\_kernels}, \quad \texttt{J}.$$

`I_basis` is a vector defining the basis $\vec{I}$ as listed in the fourth column of table 1, `U` gives the transformation matrix to the basis $\vec{J}$, i.e.

$$\vec{J} = U\vec{I}, \tag{E.1}$$

`Uinv` denotes the matrix inverse of U. The quantity `A` defines the matrix

$$A = A^{(0)} + \varepsilon A^{(1)} \tag{E.2}$$

appearing in the differential equation

$$d\vec{J} = A\vec{J}. \tag{E.3}$$

The entries of A are linear combinations of integration kernels (with symbolic names), whose explicit expressions are defined in `integration_kernels`. Finally, `J` contains the results for the master integrals up to order ε^4 in terms of iterated integrals, including the boundary constants for the boundary point $(x, y) = (0, 1)$.

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