Published for SISSA by Springer

Received: September 12, 2018
Accepted: October 22, 2018
Published: October 29, 2018

Relating 't Hooft anomalies of 4d pure Yang-Mills and 2d $\mathbb{CP}^{N-1}$ model

Masahito Yamazaki

Kavli IPMU (WPI), University of Tokyo,
Kashiwa, Chiba 277-8583, Japan

E-mail: masahito.yamazaki@ipmu.jp

Abstract: It has recently been shown that a center-twisted compactification of the four-dimensional pure SU(N) Yang-Mills theory on a three-torus gives rise to the two-dimensional $\mathbb{CP}^{N-1}$-model on a circle with a flavor-twisted boundary condition. We verify the consistency of this statement non-perturbatively at theta angle $\theta = \pi$, in terms of the mixed 't Hooft anomalies for flavor symmetries and the time-reversal symmetry. This provides further support for the approach to the confinement of four-dimensional Yang-Mills theory from the two-dimensional $\mathbb{CP}^{N-1}$-model.

Keywords: Anomalies in Field and String Theories, Confinement

ArXiv ePrint: 1711.04360

https://doi.org/10.1007/JHEP10(2018)172

Contents

1 Introduction

The two-dimensional sigma model with the target space $\mathbb{CP}^{N-1}$ (the $\mathbb{CP}^{N-1}$-model [1–3]) has long been studied as a toy model for the four-dimensional Yang-Mills theory; both theories are asymptotically free and has a dynamically-generated mass gap, for example.

Despite the similarities between the two, it has been long unclear if there could be more direct and quantitative relations between the two. Is the $\mathbb{CP}^{N-1}$-model only a toy model? Or could we use the $\mathbb{CP}^{N-1}$-model to actually solve the four-dimensional Yang-Mills theory, e.g. for an analytical demonstration of confinement of the latter theory?[1]

A positive result in this direction has been recently given in the recent work of the author and K. Yonekura [8]. As we will explain below, it was shown that a center-twisted compactification of four-dimensional pure $\mathrm{SU}(N)$ Yang-Mills theory on a three-torus $T^3 = S^1 \times T^2$, when the size of the two-torus T^2 is small, gives rise to the two-dimensional $\mathbb{CP}^{N-1}$-model[2] on the residual circle S^1, with a flavor-twisted boundary condition studied in [9]. As emphasized in [8] this provides a well-defined weakly-coupled setup with no notorious IR problems (e.g. the Linde problem [10]). This means that we in principle have a hope of analytically-continuing back to the flat space $\mathbb{R}^4$ — we first start in the weakly-coupled region and then sum up the perturbative as well non-perturbative contributions into a well-defined function (e.g. with the help of the Borel-Écalle resummation of the trans-series [11], as applied to an infinite-dimensional setup), and then adiabatically/analytically continue back to the flat $\mathbb{R}^4$ by smoothly changing the size of the torus.[3]

It would be, however, rather non-trivial to carry out this procedure in full generality, and before venturing into the detailed computation one might wish to further check the consistency of the result of [8], preferably in a non-perturbative manner.

[1] The relation between the two theories has been discussed in supersymmetric contexts, see e.g. [4–7]. Our focus here, however, is to study an honest non-supersymmetric theory where there is no protection from supersymmetry.

[2] A care is needed since this $\mathbb{CP}^{N-1}$ does not have the standard Fubini-Study metric.

[3] This strategy of adiabatic continuation from compactification has a rather long history. The literature is too vast to be covered in this short note, see [12–16] for early references in eighties and see [8] for more references. Note that the novelty of the setup of [8] is to consider $T^2 \times S^1$ compactification where there is a hierarchy between the sizes of T^2 and S^1.

JHEP10(2018)172

The goal of this short note is to verify the consistency of this proposal by matching the 't Hooft anomalies [19] of the two theories (in four and two dimensions). While the standard anomaly is lost upon dimensional reduction due to contamination from high-energy states, we can study the recently-found mixed 't Hooft anomaly between time reversal and center symmetry in four dimensions, present for the special value of the theta angle $\theta = \pi$ [17]. We find that this 't Hooft anomaly, upon a twisted compactification on a three-torus T^3, precisely reproduces to the 't Hooft anomaly for a twisted compactification of the two-dimensional $\mathbb{CP}^{N-1}$ model on a circle, as recently derived in [18].

2 From 4d to 2d

Let us first recapitulate the some of the crucial statements from [8]. Consider four-dimensional pure SU(N) Yang-Mills theory on the geometry $\mathbb{R}\times T^3_{ABC} = \mathbb{R}\times S^1_A \times S^1_B \times S^1_C$, with coordinates t, x_A, x_B, x_C. We denote the circumference of the circles by L_A, L_B, L_C, and we choose the periodicity of x_A, x_B, x_C to be 1 ($x_A \sim x_A + 1$ etc.).

In [8] the size of the circles are taken to be

$$L_A, L_B \ll L_C\,. \tag{2.1}$$

In this parameter region the Yang-Mills gauge field along the two-torus $T^2_{AB} = S^1_A \times S^1_B$ is given by the flat connection on the two-torus, which is known to be parametrized by a point of the complex projective space $\mathbb{CP}^{N-1}$ [20–22].[4] After reduction along the two-torus T^2_{AB} we find that the resulting two-dimensional theory on $\mathbb{R}\times S^1_A$ is given by the two-dimensional $\mathbb{CP}^{N-1}$-model, where the theta angle of the four-dimensional theory is identified with that of the two-dimensional theory. In this paper we denote the homogeneous coordinate of $\mathbb{CP}^{N-1}$ as $[z_1,\ldots,z_N]$ (i.e. $[z_1,\ldots,z_N] \sim [cz_1,\ldots,cz_N]$ for $c\in\mathbb{C}^\times$).

Now the crucial ingredient of [8] is to include a $\mathbb{Z}_N$-center symmetry twist in the boundary conditions — we include 't Hooft discrete magnetic flux [23] along the two-torus $T^2_{BC} = S^1_B \times S^1_C$.[5] It was then shown that this reproduces the twisted boundary condition of the two-dimensional theory along the residual circle S^1_C, given by an element of the flavor symmetry of the $\mathbb{CP}^{N-1}$-model;

$$\mathbb{Z}_N^{(B)} : [\ldots, z_k, \ldots] \to \left[\ldots, e^{\frac{2\pi \mathrm{i}}{k}} z_k, \ldots,\right]\,. \tag{2.2}$$

In four-dimensional language, this $\mathbb{Z}_N$ (zero-form) symmetry arises from the four-dimensional $\mathbb{Z}_N$ (one-form) center symmetry by compactification along the circle S^1_B (and hence the notation).[6] Similarly, the four-dimensional $\mathbb{Z}_N$ center symmetry compactified on

[4] The complex projective space $\mathbb{CP}^{N-1}$ arising from the moduli space of flat connections do not have the canonical Fubini-Study metric, and in particular has singularities where new degrees of freedom of W-bosons emerge. The analysis of [8], however, was done at the classical values of the flat connection which are away from these singularities. Such an analysis was enough to demonstrate the existence of fractional instantons and the dynamical recovery of the center symmetry. For this reason we expect that the subtlety coming from the singularities of the $\mathbb{CP}^{N-1}$ does not affect the discrete anomalies discussed in this paper.

[5] Beaware that this is different from the other two-torus $T^2_{AB} = S^1_A \times S^1_B$.

[6] The center symmetry is a one-form symmetry [25], and hence upon a circle compactification generates a zero-form symmetry in addition to a one-form symmetry.

the another cycle S^1_A generates another zero-form global symmetry of the two-dimensional symmetry. This is given by [8]

$$\mathbb{Z}_N^{(A)} : [z_1, \ldots, z_{N-1}, z_N] \to [z_2, \ldots, z_{N-1}, z_1] \,. \tag{2.3}$$

As we will see, this symmetry will play a crucial role in what follows.

3 Twisted compactification of 't Hooft anomaly

Let us now come to the 't Hooft anomalies.

As already mentioned above, the four-dimensional pure $\mathrm{SU}(N)$ Yang-Mills theory has the $\mathbb{Z}_N$ center one-form symmetry; we denote the corresponding two-form discrete $\mathbb{Z}_N$ gauge field as B. In addition the theory has a $\mathbb{Z}_2$ time-reversal symmetry T.

The results of [17] shows that the pure Yang-Mills theory with theta angle $\theta = \pi$ has a mixed 't Hooft anomaly between the center symmetry and the time-reversal symmetry. This means that the partition of the theory $\mathcal{Z}^{4\mathrm{d}}_{\theta=\pi}[(A,B)]$ at theta angle $\theta = \pi$, regarded as a function of the background gauge fields A and B, is not invariant under the time-reversal symmetry:

$$\mathcal{Z}^{4\mathrm{d}}_{\theta=\pi}[\mathsf{T}\cdot(A,B)] = \mathcal{Z}^{4\mathrm{d}}_{\theta=\pi}[(A,B)] \, \exp\left(\frac{\mathrm{i}N}{4\pi}\int B\wedge B\right) \,. \tag{3.1}$$

Let us next compactify the theory onto the geometry $\mathbb{R}\times S^{(A)}_1 \times S^{(B)}_1 \times S^{(C)}_1$. We take the limit $L_A, L_B, L_C \to 0$, while still keep the hierarchy of scales as in (2.1). By decomposing the two-form gauge field B into components, we find the decomposition

$$\begin{aligned} B = \, & B^{(1)}_A dx^A + B^{(1)}_B dx^B + B^{(1)}_C dx^C \\ & + B^{(0)}_{BC} dx^B \wedge dx^C + B^{(0)}_{CA} dx^C \wedge dx^A + B^{(0)}_{AB} dx^A \wedge dx^B \,, \end{aligned} \tag{3.2}$$

where $B^{(1)}_A, B^{(1)}_B, B^{(1)}_C$ and $B^{(0)}_{BC}, B^{(0)}_{CA}, B^{(0)}_{AB}$ are one-forms and zero-forms on the residual $\mathbb{R}$-direction, respectively, and none of them have any non-trivial dependence along the three-torus T^3.

The one-forms $B^{(1)}$ and the zero-forms $B^{(0)}$ play the role of the 'electric field' and 'magnetic field' for the discrete $\mathbb{Z}_N$ center symmetry. The electric objects are Wilson lines (holonomies) around the non-trivial cycles of the three-torus. The magnetic gauge field, on the other hand, represents the 't Hooft magnetic flux along a two-torus [23, 24]. For example, the zero-form field $B^{(0)}_{BC}$ represents the Aharanov-Bohm-type phase penetrating through the two-torus $T_{BC} = S^1_B \times S^1_C$, making the holonomies along S^1_B and S^1_C non-commutative.

Let us substitute the decomposition (3.2) into the expression for the mixed anomaly (3.1). After trivially integrating over the small three-torus directions, we find that the mixed anomaly (3.1) now is expressed as an integral over the residual $\mathbb{R}$-direction:

$$\frac{2\mathrm{i}N}{4\pi}\int_{\mathbb{R}} \left(B^{(1)}_A B^{(0)}_{BC} + B^{(1)}_B B^{(0)}_{CA} + B^{(1)}_C B^{(0)}_{AB} \right) \,. \tag{3.3}$$

The expression appearing here is an analog of the Poynting vector of electromagnetism, but now for the discrete $\mathbb{Z}_N$ center symmetry.

To this point we have not used any information regarding the choice of the boundary conditions. Recall from section 2 we turn on the 't Hooft magnetic flux is turned on along the two-torus $T^2_{BC} = S^1_B \times S^1_C$ directions, and not in other directions involving S^1_A. Moreover, the value of $B^{(0)}_{BC}$ can be derived from the fact that we have one unit of the 't Hooft discrete magnetic flux [8]:

$$U_C U_B = e^{-\frac{2\pi\mathrm{i}}{N}} U_B U_C \,, \tag{3.4}$$

where U_B and U_C denotes the holonomy along the S^1_B and S^1_C.[7] We thus obtain

$$B^{(0)}_{BC} = -\frac{2\pi}{N} \,, \quad B^{(0)}_{CA} = B^{(0)}_{AB} = 0 \,, \tag{3.5}$$

and we arrive at the 't Hooft anomaly

$$\frac{2\mathrm{i}N}{4\pi} \int_{\mathbb{R}} B^{(1)}_A \left(-\frac{2\pi}{N} \right) = -\mathrm{i} \int_{\mathbb{R}} B^{(1)}_A \,. \tag{3.6}$$

In other words, under the time-reversal symmetry the one-dimensional partition function $\mathcal{Z}^{\mathrm{1d}}$ at $\theta = \pi$, as a function of the background gauge field $B^{(1)}_A$, transforms non-trivially as

$$\mathcal{Z}^{\mathrm{1d}}_{\theta=\pi}[\mathsf{T} \cdot B^{(1)}_A] = \mathcal{Z}^{\mathrm{1d}}_{\theta=\pi}[B^{(1)}_A] \exp\left(-\mathrm{i} \int_{\mathbb{R}} B^{(1)}_A \right) . \tag{3.7}$$

It turns out that this is exactly the 't Hooft anomaly derived in [18], which discussed the $\mathbb{Z}_N$-twisted circle compactification of the two-dimensional anomaly discussed in [26]. Indeed, recall that the one-form gauge field $B^{(1)}_A$ (3.2) arises from the reduction of the two-form field along the circle S^1_A, and hence should be associated with the zero-form global symmetry $\mathbb{Z}^{(A)}_N$. As we have seen before, this symmetry acts on the homogeneous coordinates of $\mathbb{CP}^{N-1}$ by (2.3). This is nothing but the $\mathbb{Z}_N$ 'shift symmetry' (denoted by $(\mathbb{Z}_N)_S$) in [18], and hence our result (3.6) coincides with (3.13) of [18]. This completes our discussion of the twisted compactification of the 't Hooft mixed anomalies.

4 Discussion

The result of this note proves that the vacua of the two theories, namely four-dimensional pure SU(N) Yang-Mills theory on a center-twisted three-torus on the one hand, and two-dimensional $\mathbb{CP}^{N-1}$ model on a flavor-twisted circle on the other, are constrained by the same 't Hooft anomaly. In particular we find that neither theory has a trivial vacuum.

It would be interesting to extend the analysis to four-dimensional Yang-Mills theory coupled with matters, say with adjoint or fundamental/anti-fundamental matters.

Our results provides a rather non-trivial non-perturbative consistency check of the proposal of [8], and make it even more plausible the optimistic scenario that the setup

[7] For gauge invariance under the twisted boundary condition a care is needed for the definition of U_C. Note that U_C here is denoted by U'_C in [8].

JHEP10(2018)172

of [8] provides a right direction towards an analytic demonstration of the confinement and the mass gap of the asymptotically-free pure Yang-Mills theory, the holy grail of the subject.

The finding of this note also supports the claim in [8] that four-dimensional pure Yang-Mills theory with theta angle has N metastable vacua,[8] as expected from the presence of the N classical vacua in the $\mathbb{CP}^{N-1}$-model [8, 9, 29]. In addition to theoretical curiosity, this has an interesting implication to the observability of the tensor modes in the recently-proposed axion-type model of inflation [30, 31].

Acknowledgments

This work is inspired by talks and conversations at the KITP conference "Resurgence in Gauge and String Theory" and the KITP program "Resurgent Asymptotics in Physics and Mathematics" (in particular a presentation by Yuya Tanizaki), and the author would like to thank KITP, UCSB for hospitality. He would also like to thank Tatsuhiro Misumi, Norisuke Sakai, Yuya Tanizaki and Mithat Ünsal for valuable discussions. This work is supported by WPI program (MEXT, Japan), by JSPS Program No. R2603, by JSPS KAKENHI Grant No. 15K17634, by JSPS-NRF Joint Research Project, and by NSF under Grant No. PHY-1125915.

References

[1] H. Eichenherr, SU(N) *Invariant Nonlinear σ-models*, *Nucl. Phys.* **B 146** (1978) 215 [*Erratum ibid.* **B 155** (1979) 544] [INSPIRE].

[2] V.L. Golo and A.M. Perelomov, *Solution of the Duality Equations for the Two-Dimensional* SU(N) *Invariant Chiral Model*, *Phys. Lett.* **B 79** (1978) 112 [INSPIRE].

[3] E. Cremmer and J. Scherk, *The Supersymmetric Nonlinear σ-model in Four-Dimensions and Its Coupling to Supergravity*, *Phys. Lett.* **B 74** (1978) 341 [INSPIRE].

[4] S. Cecotti and C. Vafa, *On classification of $N = 2$ supersymmetric theories*, *Commun. Math. Phys.* **158** (1993) 569 [hep-th/9211097] [INSPIRE].

[5] S. Cecotti and C. Vafa, *Exact results for supersymmetric σ-models*, *Phys. Rev. Lett.* **68** (1992) 903 [hep-th/9111016] [INSPIRE].

[6] N. Dorey, *The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*, *JHEP* **11** (1998) 005 [hep-th/9806056] [INSPIRE].

[7] N. Dorey, T.J. Hollowood and D. Tong, *The BPS spectra of gauge theories in two-dimensions and four-dimensions*, *JHEP* **05** (1999) 006 [hep-th/9902134] [INSPIRE].

[8] M. Yamazaki and K. Yonekura, *From 4d Yang-Mills to 2d $\mathbb{CP}^{N-1}$ model: IR problem and confinement at weak coupling*, *JHEP* **07** (2017) 088 [arXiv:1704.05852] [INSPIRE].

[9] G.V. Dunne and M. Ünsal, *Resurgence and Trans-series in Quantum Field Theory: The CP(N-1) Model*, *JHEP* **11** (2012) 170 [arXiv:1210.2423] [INSPIRE].

[8] In the large N limit, this is consistent with the old results of [27, 28] in four dimensions.

[10] A.D. Linde, *Infrared Problem in Thermodynamics of the Yang-Mills Gas*, *Phys. Lett.* **B 96** (1980) 289 [INSPIRE].

[11] J. Écalle, *Les fonctions résurgentes, I–III*, Publications Mathématiques d'Orsay, (1981).

[12] T. Eguchi and H. Kawai, *Reduction of Dynamical Degrees of Freedom in the Large N Gauge Theory*, *Phys. Rev. Lett.* **48** (1982) 1063 [INSPIRE].

[13] A. Gonzalez-Arroyo and M. Okawa, *A Twisted Model for Large N Lattice Gauge Theory*, *Phys. Lett.* **B 120** (1983) 174 [INSPIRE].

[14] A. Gonzalez-Arroyo and M. Okawa, *The Twisted Eguchi-Kawai Model: A Reduced Model for Large N Lattice Gauge Theory*, *Phys. Rev.* **D 27** (1983) 2397 [INSPIRE].

[15] M. Lüscher, *A New Method to Compute the Spectrum of Low Lying States in Massless Asymptotically Free Field Theories*, *Phys. Lett.* **B 118** (1982) 391 [INSPIRE].

[16] M. Lüscher and G. Munster, *Weak Coupling Expansion of the Low Lying Energy Values in the* SU(2) *Gauge Theory on a Torus*, *Nucl. Phys.* **B 232** (1984) 445 [INSPIRE].

[17] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, Time Reversal and Temperature*, *JHEP* **05** (2017) 091 [arXiv:1703.00501] [INSPIRE].

[18] Y. Tanizaki, T. Misumi and N. Sakai, *Circle compactification and 't Hooft anomaly*, *JHEP* **12** (2017) 056 [arXiv:1710.08923] [INSPIRE].

[19] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, *NATO Sci. Ser.* **B 59** (1980) 135.

[20] E. Looijenga, *Root Systems and Elliptic Curves*, *Invent. Math.* **38** (1977) 17.

[21] E. Looijenga, *Invariant Theory For Generalized Root Sytems*, *Invent. Math.* **61** (1980) 1.

[22] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [hep-th/9701162] [INSPIRE].

[23] G. 't Hooft, *A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories*, *Nucl. Phys.* **B 153** (1979) 141 [INSPIRE].

[24] E. Witten, *Supersymmetric index in four-dimensional gauge theories*, *Adv. Theor. Math. Phys.* **5** (2002) 841 [hep-th/0006010] [INSPIRE].

[25] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[26] Z. Komargodski, A. Sharon, R. Thorngren and X. Zhou, *Comments on Abelian Higgs Models and Persistent Order*, arXiv:1705.04786 [INSPIRE].

[27] E. Witten, *Large N Chiral Dynamics*, *Annals Phys.* **128** (1980) 363 [INSPIRE].

[28] E. Witten, *Theta dependence in the large N limit of four-dimensional gauge theories*, *Phys. Rev. Lett.* **81** (1998) 2862 [hep-th/9807109] [INSPIRE].

[29] G.V. Dunne and M. Ünsal, *Continuity and Resurgence: towards a continuum definition of the* $\mathbb{CP}$*(N-1) model*, *Phys. Rev.* **D 87** (2013) 025015 [arXiv:1210.3646] [INSPIRE].

[30] Y. Nomura, T. Watari and M. Yamazaki, *Pure Natural Inflation*, *Phys. Lett.* **B 776** (2018) 227 [arXiv:1706.08522] [INSPIRE].

[31] Y. Nomura and M. Yamazaki, *Tensor Modes in Pure Natural Inflation*, *Phys. Lett.* **B 780** (2018) 106 [arXiv:1711.10490] [INSPIRE].

JHEP10(2018)172