Topologically massive higher spin gauge theories

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ABSTRACT: We elaborate on conformal higher-spin gauge theory in three-dimensional (3D) curved space. For any integer $n > 2$ we introduce a conformal spin-$\frac{n}{2}$ gauge field $h_{(n)} = h_{\alpha_1\dots\alpha_n}$ (with n spinor indices) of dimension $(2 - n/2)$ and argue that it possesses a Weyl primary descendant $C_{(n)}$ of dimension $(1+n/2)$. The latter proves to be divergenceless and gauge invariant in any conformally flat space. Primary fields $C_{(3)}$ and $C_{(4)}$ coincide with the linearised Cottino and Cotton tensors, respectively. Associated with $C_{(n)}$ is a Chern-Simons-type action that is both Weyl and gauge invariant in any conformally flat space. These actions, which for $n = 3$ and $n = 4$ coincide with the linearised actions for conformal gravitino and conformal gravity, respectively, are used to construct gauge-invariant models for massive higher-spin fields in Minkowski and anti-de Sitter space. In the former case, the higher-derivative equations of motion are shown to be equivalent to those first-order equations which describe the irreducible unitary massive spin-$\frac{n}{2}$ representations of the 3D Poincaré group. Finally, we develop $\mathcal{N} = 1$ supersymmetric extensions of the above results.

KEYWORDS: Higher Spin Symmetry, Supergravity Models, Superspaces

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Contents

1 Introduction

A unique feature of three spacetime dimensions (3D) is the existence of topologically massive Yang-Mills and gravity theories. These theories are obtained by augmenting the usual Yang-Mills action or the gravitational action by a gauge-invariant topological mass term. Such a mass term coincides with a Chern-Simons functional in the Yang-Mills case [1–5] and with a Lorentz Chern-Simons term in the case of gravity [4, 5]. The Lorentz Chern-Simons term is required to make the gravitational field possess nontrivial dynamics, for the pure gravity action propagates no local degrees of freedom. The Lorentz Chern-Simons term can be interpreted as the action for 3D conformal gravity [6, 7].[1]

Topologically massive gravity possesses supersymmetric extensions. In particular, $\mathcal{N} = 1$ topologically massive supergravity was constructed in [10, 11]. Its topological mass term is the supersymmetric extension of the gravitational Chern-Simons term, which coincides with the action for $\mathcal{N} = 1$ conformal supergravity [6]. Extended topologically massive supergravity will be briefly discussed in section 7.

Topologically massive $\mathcal{N} = 1$ supergravity, with or without a cosmological term, may be linearised about a maximally supersymmetric solution. The resulting linearised actions for the gravitino and the gravitational field contain higher derivatives. However, the genuine massive states prove to obey first-order differential equations. This paper is devoted to the description of higher-spin extensions of the linearised actions for topologically massive gravity and $\mathcal{N} = 1$ supergravity. In particular, for every (half-)integer spin $n/2$, where $n = 5, 6, \ldots$, we present a gauge-invariant higher-derivative action in Minkowski space that propagates a single massive state of helicity $+n/2$ or $-n/2$ on the mass shell. The action is of the form

$$S_{\text{massive}} = S_{\text{massless}} + S_{\text{CS}} . \tag{1.1}$$

Here S_{massless} denotes the 3D massless spin-$\frac{n}{2}$ gauge action of the Fronsdal-Fang type [12, 13], with no propagating degrees of freedom. The second term in the right-hand side of (1.1) is a conformal spin-$\frac{n}{2}$ gauge action [14, 15] described by a Lagrangian of the schematic form $\mathcal{L}_{\text{CS}} \propto \varphi_{(n)} \partial^{n-1} \varphi_{(n)}$, where $\varphi_{(n)}$ stands for the conformal spin-$\frac{n}{2}$ field. We show that S_{massive} propagates a single massive state described by the equations (2.1). We also present extensions of the actions introduced to anti-de Sitter (AdS) space, as well as their $\mathcal{N} = 1$ supersymmetric generalisations.

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[1]The usual Einstein-Hilbert action for 3D gravity with a cosmological term can also be interpreted as the Chern-Simons action for the anti-de Sitter group [8, 9].

In the case of Minkowski space, our actions (1.1) are in fact contained, at the component level, in the massive supersymmetric higher-spin models proposed in [16, 17]. However, the analysis in [16, 17] was carried out mostly in terms of superfields so that the component actions were not studied. All the massive higher-spin gauge models in AdS, which are presented in this paper, are new.

This paper is organised as follows. In section 2 we review field realisations of the irreducible massive spin-$\frac{n}{2}$ representations ($n = 2, 3 \dots$) of the 3D Poincaré and AdS groups. We also review the structure of on-shell massive higher-spin superfields for both 3D $\mathcal{N} = 1$ Poincaré and AdS supersymmetry. In section 3 we introduce, for any integer $n \geq 2$, a conformal spin-$\frac{n}{2}$ gauge field $\mathfrak{h}_{(n)} = \mathfrak{h}_{\alpha_1 \dots \alpha_n} = \mathfrak{h}_{(\alpha_1 \dots \alpha_n)}$ and argue that it possesses a Weyl primary descendant $\mathfrak{C}_{(n)}$ of dimension $(1 + \frac{n}{2})$ with the following properties: (i) $\mathfrak{C}_{(n)}$ is of the schematic form $\nabla^{n-1} \mathfrak{h}_{(n)}$; (ii) $\mathfrak{C}_{(n)}$ is divergenceless and gauge invariant in an arbitrary conformally flat space. These descendants $\mathfrak{C}_{(n)}$ are constructed in any conformally flat space. Making use of the primary fields $\mathfrak{C}_{(n)}$, we propose Chern-Simons-type actions $S_{\text{CS}}^{(n)} \propto \int \mathrm{d}^3 x \, e \, \mathfrak{h}^{\alpha(n)} \mathfrak{C}_{\alpha(n)}$ which are Weyl and gauge invariant in any conformally flat space, and which are higher-spin extensions of the linearised action for 3D conformal gravity. These conformal higher-spin actions are then used to construct massive higher-spin gauge theories in AdS, described by the actions (4.5a) and (4.5b). In section 4 we study the dynamics of the flat-space counterparts to the gauge theories (4.5a) and (4.5b).

Sections 5 and 6 are devoted to supersymmetric extensions of the results presented in sections 3 and 4. In section 5 we introduce conformal higher-spin gauge superfields $\mathfrak{H}_{\alpha(n)}$ in curved $\mathcal{N} = 1$ superspace. These conformal gauge superfields are argued to possess primary descendants $\mathfrak{W}_{\alpha(n)}$ of dimension $(1 + \frac{n}{2})$ that are locally supersymmetric extensions of the linearised higher-spin super-Cotton tensors [15, 17]. For any conformally flat superspace background, the primary superfields $\mathfrak{W}_{\alpha(n)}$ are explicitly constructed, and are shown to be gauge invariant and conserved. Making use of $\mathfrak{H}_{\alpha(n)}$ and $\mathfrak{W}_{\alpha(n)}$, we construct a higher-spin extension of the action for linearised $\mathcal{N} = 1$ conformal gravity, $\mathbb{S}_{\text{SCS}}^{(n)}[\mathfrak{H}_{(n)}]$, which is given by eq. (5.21). We employ $\mathbb{S}_{\text{SCS}}^{(n)}[\mathfrak{H}_{(n)}]$ to construct massive higher-spin gauge actions in $\mathcal{N} = 1$ AdS superspace, given by eqs. (5.39a) and (5.39b). Section 6 describes the component structure of the supersymmetric higher-spin theories introduced in section 5, with the analysis being restricted to the flat-superspace case. Concluding comments and discussion are given in section 7. The main body of the paper is accompanied by three appendices. Appendix A describes our notation and conventions. Appendix B reviews the Tyutin-Vasiliev action [18]. Appendix C provides two realisations for the higher-spin Cotton tensor in Minkowski space, $C_{\alpha(n)}$, as a descendant of gauge-invariant field strengths corresponding to two different higher-spin massless models.[2]

2 On-shell massive (super)fields

In this section we review the structure of irreducible massive higher-spin (super)fields in Minkowski space and in anti-de Sitter space.

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[2] A similar result in the $\mathcal{N} = 2$ supersymmetric case was given in [16].

2.1 Massive fields

We first recall the definition of on-shell massive fields in Minkowski space. Given a positive integer $n > 1$, a massive field, $\phi_{\alpha_1 \cdots \alpha_n} = \bar{\phi}_{\alpha_1 \ldots \alpha_n} = \phi_{(\alpha_1 \cdots \alpha_n)}$, is a real symmetric rank-n spinor field which obeys the differential conditions [18] (see also [19])

$$\partial^{\beta\gamma}\phi_{\beta\gamma\alpha_1\cdots\alpha_{n-2}} = 0\,, \tag{2.1a}$$

$$\partial^{\beta}{}_{(\alpha_1}\phi_{\alpha_2\ldots\alpha_n)\beta} = m\sigma\phi_{\alpha_1\ldots\alpha_n}\,, \qquad \sigma = \pm 1\,, \tag{2.1b}$$

with m being the mass of the field. In the spinor case, $n = 1$, eq. (2.1a) is absent, and the massive field is defined to obey the Dirac equation (2.1b). It is easy to see that (2.1a) and (2.1b) imply the mass-shell equation

$$\left(\Box - m^2\right)\phi_{\alpha_1\cdots\alpha_n} = 0\,. \tag{2.2}$$

In the spinor case, $n = 1$, eq. (2.2) follows from the Dirac equation (2.1b). The helicity of $\phi_{\alpha(n)}$ is $\lambda = \frac{n}{2}\sigma$, and the spin of $\phi_{\alpha(n)}$ is $n/2$.

It should be remarked that the system of equations (2.1a) and (2.2) is equivalent to the 3D version of the Fierz-Pauli field equations [20]. The general solution to (2.1a) and (2.2) is a superposition of two massive states of helicity $+\frac{n}{2}$ and $-\frac{n}{2}$, respectively. Twenty years ago, Tyutin and Vasiliev [18] constructed Lagrangian formulations for massive higher-spin fields that lead to the equations (2.1a) and (2.1b) on the mass shell. Their actions did not possess gauge invariance. In the present paper, we propose gauge-invariant formulations for massive higher-spin fields in Minkowski space that lead to the equations (2.1a) and (2.1b) on-shell.

In the case of AdS space, massive fields are defined to obey the following equations [21, 22] (see also [23])

$$\nabla^{\beta\gamma}\phi_{\beta\gamma\alpha_1\cdots\alpha_{n-2}} = 0\,, \tag{2.3a}$$

$$\nabla^{\beta}{}_{(\alpha_1}\phi_{\alpha_2\ldots\alpha_n)\beta} = \mu\phi_{\alpha_1\ldots\alpha_n}\,, \tag{2.3b}$$

for some real mass parameter μ. Equation (2.3b) implies that

$$\left(\nabla^a\nabla_a + 2(n+2)\mathcal{S}^2 - \mu^2\right)\phi_{\alpha(n)} = 0\,, \tag{2.4}$$

where the parameter $\mathcal{S}$ is related to the AdS curvature via eq. (3.46). Equation (2.4) can be rewritten in terms of the quadratic Casimir operator of the 3D AdS group SO(2, 2),

$$\mathcal{Q} := \nabla^a\nabla_a - 2\mathcal{S}^2 M^{\gamma\delta}M_{\gamma\delta}, \qquad \left[\mathcal{Q}, \nabla_a\right] = 0\,, \tag{2.5}$$

with $M_{\gamma\delta}$ the Lorentz generators, see appendix A.

Equations (2.3a) and (2.4) constitute the 3D AdS counterpart to the Fierz-Pauli field equations. They describe a reducible representation of the AdS isometry group. Gauge-invariant Lagrangian formulations for massive higher-spin fields in AdS, which lead to the equations (2.3a) and (2.4) on the mass shell, were developed in [24–27], including $\mathcal{N} = 1$

supersymmetric extensions obtained by combining the bosonic and fermionic actions (on-shell supersymmetry). The formulations given in [24–27] are based on Zinoviev's gauge-invariant approach [28] to describe massive higher-spin fields. In the present paper, we propose different gauge-invariant formulations for massive higher-spin fields in AdS that lead to the equations (2.3a) and (2.3b) on-shell.

2.2 Massive superfields

For $n > 0$, a massive superfield $T_{\alpha(n)}$ is defined to be a real symmetric rank-n spinor, $T_{\alpha_1 \cdots \alpha_n} = \bar{T}_{\alpha_1 \dots \alpha_n} = T_{(\alpha_1 \cdots \alpha_n)}$, which obeys the differential conditions [29] (see also [17])

$$D^\beta T_{\beta \alpha_1 \cdots \alpha_{n-1}} = 0 \qquad \Longrightarrow \qquad \partial^{\beta\gamma} T_{\beta\gamma\alpha_1 \dots \alpha_{n-2}} = 0 \, , \tag{2.6a}$$

$$-\frac{\mathrm{i}}{2} D^2 T_{\alpha_1 \dots \alpha_n} = m\sigma T_{\alpha_1 \dots \alpha_n} \, , \qquad\qquad \sigma = \pm 1 \, . \tag{2.6b}$$

Here $D^2 = D^\alpha D_\alpha$, and D_α is the spinor covariant derivative of $\mathcal{N} = 1$ Minkowski superspace. It follows from (2.6a) that

$$-\frac{\mathrm{i}}{2} D^2 T_{\alpha_1 \dots \alpha_n} = \partial^\beta{}_{(\alpha_1} T_{\alpha_2 \dots \alpha_n)\beta} \, , \tag{2.7}$$

and thus $T_{\alpha(n)}$ is an on-shell superfield,

$$\partial^\beta{}_{(\alpha_1} T_{\alpha_2 \dots \alpha_n)\beta} = m\sigma T_{\alpha_1 \dots \alpha_n} \, , \qquad \sigma = \pm 1 \, . \tag{2.8}$$

It follows from (2.6b) that[3]

$$\left(\Box - m^2 \right) T_{\alpha(n)} = 0 \, . \tag{2.9}$$

For the superhelicity of $T_{\alpha(n)}$ we obtain

$$\kappa = \frac{1}{2} \left(n + \frac{1}{2} \right) \sigma \, . \tag{2.10}$$

We define the superspin of $T_{\alpha(n)}$ to be $n/2$. The massive supermultiplet $T_{\alpha(n)}$ contains two ordinary massive fields of the type (2.1), which are

$$\phi_{\alpha_1 \dots \alpha_n} := T_{\alpha_1 \dots \alpha_n}|_{\theta=0} \, , \qquad \phi_{\alpha_1 \dots \alpha_{n+1}} := \mathrm{i}^{n+1} D_{(\alpha_1} T_{\alpha_2 \dots \alpha_{n+1})}|_{\theta=0} \, . \tag{2.11}$$

Their helicity values are $\frac{n}{2}\sigma$ and $\frac{n+1}{2}\sigma$, respectively.

The off-shell gauge-invariant formulations for massive higher-spin $\mathcal{N} = 1$ supermultiplets in Minkowski superspace, which lead to the equations (2.6a) and (2.6b) on the mass shell, were constructed in [17].

In the case of $\mathcal{N} = 1$ AdS supersymmetry, on-shell massive superfields are described by the equations [29]

$$\mathcal{D}^\beta T_{\alpha_1 \cdots \alpha_{n-1}\beta} = 0 \, , \tag{2.12a}$$

$$-\frac{\mathrm{i}}{2} \mathcal{D}^2 T_{\alpha_1 \cdots \alpha_n} = \mu T_{\alpha_1 \cdots \alpha_n} \, , \tag{2.12b}$$

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[3]The equations (2.6a) and (2.9) are the $\mathcal{N} = 1$ supersymmetric extension of the Fierz-Pauli equations.

with μ a real mass parameter and $\mathcal{D}^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha$. Here $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha)$ are the covariant derivatives of the $\mathcal{N} = 1$ AdS superspace, see section 5 for the details. It can be shown that

$$-\frac{1}{4}\mathcal{D}^2\mathcal{D}^2 = \mathcal{D}^a\mathcal{D}_a - 2\mathrm{i}\mathcal{S}\mathcal{D}^2 + 2\mathcal{S}\mathcal{D}^{\alpha\beta}M_{\alpha\beta} - 2\mathcal{S}^2 M^{\alpha\beta}M_{\alpha\beta} . \tag{2.13}$$

This differential operator, which is the square of the operator in the left-hand side of (2.12b), can be expressed via the quadratic Casimir operator[4] of the 3D $\mathcal{N} = 1$ AdS supergroup,

$$\mathbb{Q} = -\frac{1}{4}\mathcal{D}^2\mathcal{D}^2 + \mathrm{i}\mathcal{S}\mathcal{D}^2 , \qquad [\mathbb{Q}, \mathcal{D}_A] = 0 . \tag{2.14}$$

It is worth pointing out that the left-hand side of (2.12b) can be rewritten as

$$-\frac{\mathrm{i}}{2}\mathcal{D}^2 T_{\alpha_1\cdots\alpha_n} = \mathcal{D}_{(\alpha_1}{}^\beta T_{\alpha_2\cdots\alpha_n)\beta} + (n+2)\mathcal{S}T_{\alpha_1\cdots\alpha_n} , \tag{2.15}$$

where we have made use of (2.12a).

In this paper we propose off-shell gauge-invariant formulations for massive higher-spin supermultiplets in $\mathcal{N} = 1$ AdS superspace that lead to the equations (2.12a) and (2.12b) on-shell.

3 Conformal higher-spin fields

The concept of conformal higher-spin field theory was introduced by Fradkin and Tseytlin in four dimensions [32]. (Super)conformal higher-spin field theories in three dimensions were discussed in [14, 33]. In this section, our starting points will be (i) the description of conformal higher-spin gauge fields in Minkowski space given in [14, 15]; and (ii) the approach advocated in [34].

3.1 Conformal gravity

The gravitational field may be described in terms of the torsion-free covariant derivatives

$$\nabla_a = e_a + \omega_a = e_a{}^m\partial_m + \frac{1}{2}\omega_a{}^{bc}M_{bc} , \qquad [\nabla_a, \nabla_b] = \frac{1}{2}R_{ab}{}^{cd}M_{cd} . \tag{3.1}$$

Here $M_{bc} = -M_{cb}$ denotes the Lorentz generators, $e_a{}^m$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$, and $\omega_a{}^{bc}$ the torsion-free Lorentz connection. Finally, $R_{ab}{}^{cd}$ is the Riemann curvature tensor. In three dimensions, $R_{ab}{}^{cd}$ is determined by the Ricci tensor $R_{ab} := \eta^{cd}R_{cadb} = R_{ba}$ and the scalar curvature $R = \eta^{ab}R_{ab}$.

The Weyl tensor is identically zero in three dimensions, which means

$$R_{abcd} = \eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac} - \frac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})R . \tag{3.2}$$

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[4]It is of interest to compare (2.14) with the quadratic Casimir operator of the 4D $\mathcal{N} = 1$ AdS supergroup (given by eq. (29) in [30]), which plays an important role in the quantisation [30] of the massless higher-spin supermultiplets [31] in AdS$_4$.

The role of the Weyl tensor is played by the Cotton tensor $W_{abc} = -W_{bac}$, which is defined in terms of the 3D Schouten tensor $P_{ab} = R_{ab} - \frac{1}{4}\eta_{ab}R$ as follows

$$W_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac} \,. \tag{3.3}$$

Spacetime is conformally flat if and only if the Cotton tensor vanishes [35] (see [36] for a modern proof). The algebraic properties of the Cotton tensor are

$$W_{abc} + W_{bca} + W_{cab} = 0 \,, \qquad W_{ab}{}^b = 0 \,. \tag{3.4}$$

They imply that $W_{ab} := \frac{1}{2}\varepsilon_{acd}W^{cd}{}_b$ is symmetric and traceless,

$$W_{ba} = W_{ab} \,, \qquad W^a{}_a = 0 \,. \tag{3.5}$$

It is also divergenceless,

$$\nabla^a W_{ab} = 0 \,, \tag{3.6}$$

as a consequence of the Bianchi identity $\nabla^b R_{ab} = \frac{1}{2}\nabla_a R$.

The condition of vanishing torsion is invariant under Weyl (local scale) transformations of the form

$$\nabla_a \rightarrow \nabla'_a = e^\sigma \left(\nabla_a + \nabla^b \sigma M_{ba} \right) \,, \tag{3.7}$$

with the parameter $\sigma(x)$ being completely arbitrary. In the infinitesimal case, the Weyl transformation laws of R_{ab} and R are

$$\delta_\sigma R_{ab} = 2\sigma R_{ab} + \nabla_a \nabla_b \sigma + \eta_{ab}\Box\sigma \,, \qquad \delta_\sigma R = 2\sigma R + 4\Box\sigma \,, \tag{3.8}$$

where $\Box = \nabla^c \nabla_c$. The Cotton tensor is a Weyl primary field of weight $+3$,

$$\delta_\sigma W_{ab} = 3\sigma W_{ab} \,. \tag{3.9}$$

In what follows, we often convert every vector index into a pair of spinor ones using the well-known correspondence: a three-vector V_a can equivalently be realised as a symmetric spinor $V_{\alpha\beta} = V_{\beta\alpha}$. The relationship between V_a and $V_{\alpha\beta}$ is as follows:

$$V_{\alpha\beta} := (\gamma^a)_{\alpha\beta}V_a = V_{\beta\alpha} \,, \qquad V_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}V_{\alpha\beta} \,. \tag{3.10}$$

Associated with the traceless part of the Ricci tensor, $R_{ab} - \frac{1}{3}\eta_{ab}R$, and the Cotton tensor, W_{ab}, are the following completely symmetric rank-4 spinors:

$$R_{\alpha\beta\gamma\delta} := (\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}\left(R_{ab} - \frac{1}{3}\eta_{ab}R \right) = R_{(\alpha\beta\gamma\delta)} \,, \tag{3.11}$$

$$W_{\alpha\beta\gamma\delta} := (\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}W_{ab} = W_{(\alpha\beta\gamma\delta)} = \nabla^\rho{}_{(\alpha}R_{\beta\gamma\delta)\rho} \,. \tag{3.12}$$

The Weyl transformation of $R_{\alpha\beta\gamma\delta}$ is

$$\delta_\sigma R_{\alpha\beta\gamma\delta} = 2\sigma R_{\alpha\beta\gamma\delta} + \nabla_{(\alpha\beta}\nabla_{\beta\gamma)}\sigma \,. \tag{3.13}$$

3.2 Conformal gauge fields

A real tensor field $\mathfrak{h}_{\alpha(n)} := \mathfrak{h}_{\alpha_1 \ldots \alpha_n} = \mathfrak{h}_{(\alpha_1 \ldots \alpha_n)}$ is said to be a conformal spin-$\frac{n}{2}$ gauge field if (i) it is Weyl primary of some weight d_n,

$$\delta_\sigma \mathfrak{h}_{\alpha(n)} = d_n \sigma \mathfrak{h}_{\alpha(n)} \; ; \tag{3.14}$$

and (ii) it is defined modulo gauge transformations of the form

$$\delta_\zeta \mathfrak{h}_{\alpha(n)} = \nabla_{(\alpha_1 \alpha_2} \zeta_{\alpha_3 \ldots \alpha_n)} \, , \tag{3.15}$$

with the real gauge parameter $\zeta_{\alpha(n-2)}$ being also Weyl primary. These conditions uniquely fix the Weyl weight of $\mathfrak{h}_{\alpha(n)}$ to be

$$d_n = 2 - \frac{n}{2} \; . \tag{3.16}$$

Starting with $\mathfrak{h}_{\alpha(n)}$ one can construct its descendant, $\mathfrak{C}_{\alpha(n)}$, defined uniquely, modulo a normalisation, by the following the properties:

1. $\mathfrak{C}_{\alpha(n)}$ is of the form $\mathcal{A}\mathfrak{h}_{\alpha(n)}$, where $\mathcal{A}$ is a linear differential operator involving the covariant derivatives, the curvature tensors $R_{\alpha(4)}$ and R and their covariant derivatives.

2. $\mathfrak{C}_{\alpha(n)}$ is Weyl primary of weight $(1 + n/2)$,

$$\delta_\sigma \mathfrak{C}_{\alpha(n)} = \left(1 + \frac{n}{2} \right) \sigma \mathfrak{C}_{\alpha(n)} \; . \tag{3.17}$$

3. The gauge variation of $\mathfrak{C}_{\alpha(n)}$ vanishes if the spacetime is conformally flat,

$$\delta_\zeta \mathfrak{C}_{\alpha(n)} = O\left(W_{(4)} \right) \, , \tag{3.18}$$

where $W_{(4)}$ is the Cotton tensor.

4. $\mathfrak{C}_{\alpha(n)}$ is divergenceless if the spacetime is conformally flat,

$$\nabla^{\beta\gamma} \mathfrak{C}_{\beta\gamma\alpha(n-2)} = O\left(W_{(4)} \right) \; . \tag{3.19}$$

Here and in (3.18), $O\left(W_{(4)} \right)$ stands for contributions containing the Cotton tensor and its covariant derivatives.

We now consider several examples. Given a conformal spin-1 gauge field $\mathfrak{h}_{\alpha\beta} = \mathfrak{h}_{\beta\alpha}$,

$$\delta_\sigma \mathfrak{h}_{\alpha\beta} = \sigma \mathfrak{h}_{\alpha\beta} \, , \tag{3.20}$$

the required Weyl primary descendant is $\mathfrak{C}_{\alpha\beta} = \nabla^\gamma{}_{(\alpha} \mathfrak{h}_{\beta)\gamma}$ and coincides with the gauge-invariant field strength, $\mathfrak{C}_{ab} = \nabla_a \mathfrak{h}_b - \nabla_b \mathfrak{h}_a$, of the one-form $\mathfrak{h}_a$. This implies that $\mathfrak{C}_{\alpha(2)}$ is conserved,

$$\nabla^{\beta\gamma} \mathfrak{C}_{\beta\gamma} = 0 \; . \tag{3.21}$$

Next consider a conformal spin-$\frac{3}{2}$ gauge field $\mathfrak{h}_{\alpha(3)}$ (i.e. conformal gravitino),

$$\delta_\sigma \mathfrak{h}_{\alpha(3)} = \frac{1}{2} \sigma \mathfrak{h}_{\alpha(3)} \; . \tag{3.22}$$

The required Weyl primary descendant is

$$\mathfrak{C}_{\alpha(3)} = \frac{3}{4}\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\mathfrak{h}_{\alpha_3)\beta_1\beta_2} + \frac{1}{4}\Box\mathfrak{h}_{\alpha(3)} + \frac{3}{4}R_{\beta_1\beta_2(\alpha_1\alpha_2}\mathfrak{h}_{\alpha_3)}{}^{\beta_1\beta_2} - \frac{1}{16}R\mathfrak{h}_{\alpha(3)} \ . \qquad (3.23)$$

Its gauge transformation is

$$\delta_\zeta\mathfrak{C}_{\alpha(3)} = -\frac{1}{2}W_{\alpha(3)\beta}\zeta^\beta \ . \qquad (3.24)$$

Computing its divergence gives

$$\nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha} = -\frac{1}{2}W_{\alpha\beta(3)}\mathfrak{h}^{\beta(3)} \ . \qquad (3.25)$$

Our last example is a conformal spin-2 gauge field $\mathfrak{h}_{\alpha(4)}$ (i.e. conformal graviton),

$$\delta_\sigma\mathfrak{h}_{\alpha(4)} = 0 \ . \qquad (3.26)$$

The required Weyl primary descendant of $\mathfrak{h}_{\alpha(4)}$ is

$$\begin{aligned}
\mathfrak{C}_{\alpha(4)} =\ & \frac{1}{2}\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\mathfrak{h}_{\alpha_4)\beta(3)} + \frac{1}{2}\Box\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2\alpha_3\alpha_4)\beta_1} + \left(\nabla_{(\alpha_1}{}^{\beta_1}R_{\alpha_2\alpha_3}{}^{\beta_2\beta_3}\right)\mathfrak{h}_{\alpha_4)\beta(3)} \\
& + \frac{1}{12}\left(\nabla_{(\alpha_1}{}^{\beta_1}R\right)\mathfrak{h}_{\alpha_2\alpha_3\alpha_4)\beta_1} - \frac{1}{12}R\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2\alpha_3\alpha_4)\beta_1} + 2R^{\beta_1\beta_2}{}_{(\alpha_1\alpha_2}\nabla_{\alpha_3}{}^{\beta_3}\mathfrak{h}_{\alpha_4)\beta(3)} \\
& - \frac{3}{4}R^{\beta_1}{}_{\delta(\alpha_1\alpha_2}\nabla^{\delta\beta_2}\mathfrak{h}_{\alpha_3\alpha_4)\beta(2)} \ . \qquad (3.27)
\end{aligned}$$

Its gauge transformation is

$$\begin{aligned}
\delta_\zeta\mathfrak{C}_{\alpha(4)} =\ & \left(\nabla^{\gamma\delta}W_{\gamma(\alpha_1\alpha_2\alpha_3}\right)\zeta_{\alpha_4)\delta} + \frac{1}{2}\left(\nabla_{(\alpha_1\alpha_2}W_{\alpha_3\alpha_4)}{}^{\beta(2)}\right)\zeta_{\beta(2)} - W_{\gamma_1(\alpha_1\alpha_2\alpha_3}\nabla^{\gamma(2)}\zeta_{\alpha_4)\gamma_2} \\
& + \frac{11}{12}W_{\alpha(4)}\nabla^{\beta(2)}\zeta_{\beta(2)} + \frac{1}{2}W^\beta{}_{\gamma(\alpha_1\alpha_2}\nabla_{\alpha_3}{}^\gamma\zeta_{\alpha_4)\beta} \ . \qquad (3.28)
\end{aligned}$$

The divergence of $\mathfrak{C}_{\alpha(4)}$ may be shown to be

$$\begin{aligned}
\nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha(2)} =\ & -\frac{1}{2}\left(\nabla_{\gamma(\alpha_1}W^{\gamma\beta(3)}\right)\mathfrak{h}_{\alpha_2)\beta(3)} + \frac{5}{12}\left(\nabla_{\alpha(2)}W^{\beta(4)}\right)\mathfrak{h}_{\beta(4)} + W_{\alpha(2)}{}^{\beta(2)}\nabla^{\gamma(2)}\mathfrak{h}_{\beta(2)\gamma(2)} \\
& - \frac{3}{2}W_{\gamma_1(\alpha_1}{}^{\beta(2)}\nabla^{\gamma(2)}\mathfrak{h}_{\alpha_2)\gamma_2\beta(2)} - \frac{1}{12}W^{\beta(4)}\nabla_{\alpha(2)}\mathfrak{h}_{\beta(4)} \ . \qquad (3.29)
\end{aligned}$$

Suppose that the spacetime under consideration is conformally flat,

$$W_{\alpha(4)} = 0 \ . \qquad (3.30)$$

Then the tensor $\mathfrak{C}_{\alpha(n)}$ is gauge invariant and conserved,

$$\delta_\zeta\mathfrak{C}_{\alpha(n)} = 0 \,, \qquad (3.31a)$$

$$\nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha(n-2)} = 0 \ . \qquad (3.31b)$$

These properties and the Weyl transformation law (3.17) tell us that the action

$$S_{\text{CS}}^{(n)}[\mathfrak{h}_{\alpha(n)}] = \frac{\mathrm{i}^n}{2^{\lfloor n/2\rfloor+1}}\int \mathrm{d}^3x \, e\,\mathfrak{h}^{\alpha(n)}\mathfrak{C}_{\alpha(n)} \,, \qquad e^{-1} = \det(e_a{}^m) \qquad (3.32)$$

is gauge and Weyl invariant,

$$\delta_\zeta S_{\text{CS}}^{(n)}[\mathfrak{h}_{\alpha(n)}] = 0\,, \qquad \delta_\sigma S_{\text{CS}}^{(n)}[\mathfrak{h}_{\alpha(n)}] = 0\,. \tag{3.33}$$

Here $\lfloor x \rfloor$ denotes the floor function; it coincides with the integer part of a real number $x \geq 0$. The above action is actually Weyl invariant in an arbitrary curved space. Condition (3.30) is required to guarantee the gauge invariance of $S_{\text{CS}}^{(n)}[\mathfrak{h}_{\alpha(n)}]$ for $n > 2$.

It follows from the Weyl transformation law (3.17) that $\nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha(n-2)}$ is a primary field,

$$\delta_\sigma \left(\nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha(n-2)} \right) = \left(2 + \frac{n}{2} \right) \sigma \nabla^{\beta\gamma}\mathfrak{C}_{\beta\gamma\alpha(n-2)}\,. \tag{3.34}$$

This property means that the conservation equation (3.31b) is Weyl invariant.

3.3 Higher-spin Cotton tensor in Minkowski space

The linearised higher-spin Cotton tensor in Minkowski space will be denoted $C_{\alpha(n)}(h)$, while the previous notation $\mathfrak{C}_{\alpha(n)}(\mathfrak{h})$ will be reserved for curved spacetimes. For $n \geq 2$, $C_{\alpha(n)}(h)$ is given by the expression [15]

$$C_{\alpha(n)}(h) := \frac{1}{2^{n-1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} \Box^j \partial_{(\alpha_1}{}^{\beta_1} \ldots \partial_{\alpha_{n-2j-1}}{}^{\beta_{n-2j-1}} h_{\alpha_{n-2j}\ldots\alpha_n)\beta_1\ldots\beta_{n-2j-1}}\,. \tag{3.35}$$

It is a descendant of the conformal field $h_{\alpha(n)}$ defined modulo gauge transformations of the form

$$\delta h_{\alpha(n)} = \partial_{(\alpha_1\alpha_2}\zeta_{\alpha_3\ldots\alpha_n)}\,. \tag{3.36}$$

The field strength is invariant under these gauge transformations,

$$\delta_\zeta C_{\alpha(n)} = 0\,, \tag{3.37}$$

and obeys the Bianchi identity

$$\partial^{\beta\gamma}C_{\beta\gamma\alpha_1\ldots\alpha_{n-2}} = 0\,. \tag{3.38}$$

The higher-spin Chern-Simons action

$$S_{\text{CS}}^{(n)}\left[h_{(n)}\right] = \frac{\mathrm{i}^n}{2^{\lfloor n/2 \rfloor+1}} \int \mathrm{d}^3x\, h^{\alpha(n)}C_{\alpha(n)}(h) \tag{3.39}$$

is conformal and invariant under (3.36).

In the case of even rank, $n = 2s$, with $s = 1, 2, \ldots$, the field strength (3.35) can be shown to coincide with the bosonic higher-spin Cotton tensor given originally by Pope and Townsend [14]. It reduces to the linearised Cotton tensor for $n = 4$, and to the Maxwell field strength for $n = 2$. The fermionic case, $n = 2s+1$, with $s = 2, \ldots$, was not considered in [14]. It was presented for the first time in [15].

It should be pointed out that the conformal spin-3 case, $n = 6$, was studied for the first time in [37]. The spin-3/2 case, $n = 3$, was considered in [38]. The field strength $C_{\alpha(3)}$ is the linearised version of the Cottino vector spinor [10, 39].

The normalisation of $C_{\alpha(n)}(h)$ defined by (3.35) can be explained as follows. The gauge freedom (3.36) allows us to impose a gauge condition

$$\partial^{\beta\gamma} h_{\beta\gamma\alpha(n-2)} = 0 \ . \tag{3.40}$$

Under this gauge condition, the field strength (3.35), with $s = 1, 2, \ldots$, takes the form

$$C_{\alpha(2s)} = \Box^{s-1} \partial^{\beta}{}_{(\alpha_1} h_{\alpha_2 \ldots \alpha_{2s})\beta} = \Box^{s-1} \partial^{\beta}{}_{\alpha_1} h_{\alpha_2 \ldots \alpha_{2s}\beta} \ , \tag{3.41a}$$

$$C_{\alpha(2s+1)} = \Box^{s} h_{\alpha(2s+1)} \ , \tag{3.41b}$$

as a consequence of the identity

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} = 2^{n-1} \ . \tag{3.42}$$

The field strength (3.35) proves to be the general solution to the conservation equation (3.38). This result has recently been proved in [40] in the bosonic case, $n = 2s$, and the proof given is quite nontrivial (see also [41]). An alternative proof, which is valid for arbitrary integer $n > 1$ and is based on supersymmetry considerations, was given in [15].

3.4 Higher-spin Cotton tensor in conformally flat spaces

Now we are in a position to construct $\mathfrak{C}_{\alpha(n)}$ in a curved conformally flat spacetime $\mathcal{M}^3$. Locally, the covariant derivatives ∇_a of $\mathcal{M}^3$ are related to the flat-space ones by

$$\nabla_a = \mathrm{e}^{\sigma} \left(\partial_a + \partial^b \sigma M_{ba} \right) \ , \tag{3.43}$$

for some scale factor σ. The linearised higher-spin Cotton tensor $\mathfrak{C}_{\alpha(n)}$ in $\mathcal{M}^3$ is related to the flat-space one, eq. (3.35), by the rule

$$\mathfrak{C}_{\alpha(n)} = \mathrm{e}^{(1+\frac{n}{2})\sigma} C_{\alpha(n)} \ . \tag{3.44}$$

The higher-spin gauge field $\mathfrak{h}_{\alpha(n)}$ in $\mathcal{M}^3$ and its counterpart $h_{\alpha(n)}$ in Minkowski space are related to each other as

$$\mathfrak{h}_{\alpha(n)} = \mathrm{e}^{(2-n/2)\sigma} h_{\alpha(n)} \ . \tag{3.45}$$

In general, it is a difficult technical problem to express $\mathfrak{C}_{\alpha(n)}$ in terms of the covariant derivatives ∇_a and the gauge potential $\mathfrak{h}_{\alpha(n)}$. As an example, let us consider the case of AdS space, whose geometry is described by covariant derivatives satisfying the algebra

$$\left[\nabla_a, \nabla_b \right] = -4\mathcal{S}^2 M_{ab} \quad \Longleftrightarrow \quad \left[\nabla_{\alpha\beta}, \nabla_{\gamma\delta} \right] = 4\mathcal{S}^2 \Big(\varepsilon_{\gamma(\alpha} M_{\beta)\delta} + \varepsilon_{\delta(\alpha} M_{\beta)\gamma} \Big). \tag{3.46}$$

Here the parameter $\mathcal{S}$ is related to the AdS scalar curvature as $R = -24\mathcal{S}^2$. The Cotton tensor (3.44) for the cases $n = 3, 4, 5$ and 6 proves to be

$$\mathfrak{C}_{\alpha(3)} = \frac{1}{2^2}\left(3\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\mathfrak{h}_{\alpha_3)\beta_1\beta_2} + \mathcal{Q}\mathfrak{h}_{\alpha(3)} - 9\mathcal{S}^2\mathfrak{h}_{\alpha(3)}\right), \tag{3.47}$$

$$\mathfrak{C}_{\alpha(4)} = \frac{1}{2^3}\left(4\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\mathfrak{h}_{\alpha_4)\beta(3)} + 4\mathcal{Q}\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2\alpha_3\alpha_4)\beta_1} - 80\mathcal{S}^2\nabla_{(\alpha_1}{}^{\beta_1}h_{\alpha_2\alpha_3\alpha_4)\beta_1}\right), \tag{3.48}$$

$$\mathfrak{C}_{\alpha(5)} = \frac{1}{2^4}\left(5\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\nabla_{\alpha_4}{}^{\beta_4}\mathfrak{h}_{\alpha_5)\beta(4)} + 10\mathcal{Q}\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\mathfrak{h}_{\alpha_3\alpha_4\alpha_5)\beta(2)} + \mathcal{Q}^2\mathfrak{h}_{\alpha(5)}\right.$$
$$\left. - 330\mathcal{S}^2\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\mathfrak{h}_{\alpha_3\alpha_4\alpha_5)\beta(2)} - 82\mathcal{S}^2\mathcal{Q}\mathfrak{h}_{\alpha(5)} + 1425\mathcal{S}^4\mathfrak{h}_{\alpha(5)}\right), \tag{3.49}$$

$$\mathfrak{C}_{\alpha(6)} = \frac{1}{2^5}\left(6\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\nabla_{\alpha_4}{}^{\beta_4}\nabla_{\alpha_5}{}^{\beta_5}\mathfrak{h}_{\alpha_6)\beta(5)} + 20\mathcal{Q}\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\mathfrak{h}_{\alpha_4\alpha_5\alpha_6)\beta(3)}\right.$$
$$+ 6\mathcal{Q}^2\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2...\alpha_6)\beta_1} - 960\mathcal{S}^2\nabla_{(\alpha_1}{}^{\beta_1}\nabla_{\alpha_2}{}^{\beta_2}\nabla_{\alpha_3}{}^{\beta_3}\mathfrak{h}_{\alpha_4\alpha_5\alpha_6)\beta(3)}$$
$$\left. - 704\mathcal{S}^2\mathcal{Q}\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2...\alpha_6)\beta_1} + 18432\mathcal{S}^4\nabla_{(\alpha_1}{}^{\beta_1}\mathfrak{h}_{\alpha_2...\alpha_6)\beta_1}\right), \tag{3.50}$$

where $\mathcal{Q}$ is the quadratic Casimir of the 3D AdS group, SO(2, 2), given by eq. (2.5). Each of the tensors $\mathfrak{C}_{\alpha(n)}$ given above can be written as $\mathfrak{C}_{\alpha(n)}(\mathfrak{h}_{(n)}) = \mathcal{A}\mathfrak{h}_{\alpha(n)}$, where the linear differential operator $\mathcal{A}$ is symmetric in the sense that

$$\int \mathrm{d}^3x\, e\, \mathfrak{g}^{\alpha(n)}\mathcal{A}\mathfrak{h}_{\alpha(n)} = \int \mathrm{d}^3x\, e\, \mathfrak{h}^{\alpha(n)}\mathcal{A}\mathfrak{g}_{\alpha(n)}\,, \tag{3.51}$$

for arbitrary prepotentials $\mathfrak{g}_{\alpha(n)}$ and $\mathfrak{h}_{\alpha(n)}$. This means that it suffices to prove one of the two properties in (3.31), and then the second property follows.

4 Massive higher-spin actions in maximally symmetric spaces

The conformal higher-spin actions in conformally flat spaces, eq. (3.32), are formulated in terms of the gauge fields $\mathfrak{h}_{\alpha(n)}$. The same gauge field can be used to construct massless Fronsdal-Fang-type actions [12, 13, 42, 43] in maximally symmetric spaces. Such actions however, will involve not only $\mathfrak{h}_{\alpha(n)}$ but also some compensators.

Here we describe these massless higher-spin gauge actions in AdS$_3$ and then use them to construct gauge-invariant models for massive higher-spin fields.

4.1 Massive higher-spin actions in AdS space

There are two types of the higher-spin massless actions, first-order and second-order ones. Given an integer $n \geq 4$, the first-order model is described by real fields $\mathfrak{h}_{\alpha(n)}$, $\mathfrak{y}_{\alpha(n-2)}$ and $\mathfrak{y}_{\alpha(n-4)}$ which are defined modulo gauge transformations of the form

$$\delta_\zeta \mathfrak{h}_{\alpha(n)} = \nabla_{(\alpha_1\alpha_2}\zeta_{\alpha_3...\alpha_n)}\,, \tag{4.1a}$$

$$\delta_\zeta \mathfrak{y}_{\alpha(n-2)} = \frac{1}{n}\nabla^\beta{}_{(\alpha_1}\zeta_{\alpha_2...\alpha_{n-2})\beta} + \mathcal{S}\zeta_{\alpha(n-2)}\,, \tag{4.1b}$$

$$\delta_\zeta \mathfrak{y}_{\alpha(n-4)} = \nabla^{\beta(2)}\zeta_{\alpha(n-4)\beta(2)}\,. \tag{4.1c}$$

The Fang-Fronsdal-type gauge-invariant action, $S_{\rm FF}^{(n)} = S_{\rm FF}^{(n)}[\mathfrak{h}_{(n)}, \mathfrak{y}_{(n-2)}, \mathfrak{y}_{(n-4)}]$, is

$$
\begin{aligned}
S_{\rm FF}^{(n)} = \frac{{\rm i}^n}{2^{\lceil n/2 \rceil}} \int {\rm d}^3 x\, e \Bigg\{ & \mathfrak{h}^{\alpha(n-1)\gamma} \nabla_\gamma{}^\delta \mathfrak{h}_{\delta\alpha(n-1)} + 2(n-2)\mathfrak{y}^{\alpha(n-2)} \nabla^{\beta(2)} \mathfrak{h}_{\alpha(n-2)\beta(2)} \\
& + 4(n-2)\mathfrak{y}^{\alpha(n-3)\gamma} \nabla_\gamma{}^\delta \mathfrak{y}_{\delta\alpha(n-3)} + 2\frac{n(n-3)}{(n-1)}\mathfrak{y}^{\alpha(n-4)} \nabla^{\beta(2)} \mathfrak{y}_{\alpha(n-4)\beta(2)} \\
& - \frac{(n-3)(n-4)}{(n-1)(n-2)}\mathfrak{y}^{\alpha(n-5)\gamma} \nabla_\gamma{}^\delta \mathfrak{y}_{\delta\alpha(n-5)} + (n-2)\mathcal{S}\mathfrak{h}^{\alpha(n)}\mathfrak{h}_{\alpha(n)} \\
& - 4n(n-2)\mathcal{S}\mathfrak{y}^{\alpha(n-2)}\mathfrak{y}_{\alpha(n-2)} - \frac{n(n-3)}{(n-1)}\mathcal{S}\mathfrak{y}^{\alpha(n-4)}\mathfrak{y}_{\alpha(n-4)} \Bigg\} \,.
\end{aligned}
\tag{4.2}
$$

Here $\lceil n/2 \rceil$ stands for the ceiling function, which is equal to s for $n = 2s$ and $s+1$ for $n = 2s+1$, with $s \geq 0$ an integer.

Given an integer $n \geq 4$, the second-order model is described by real fields $\mathfrak{h}_{\alpha(n)}$ and $\mathfrak{y}_{\alpha(n-4)}$ defined modulo gauge transformations of the form

$$
\delta_\zeta \mathfrak{h}_{\alpha(n)} = \nabla_{(\alpha_1\alpha_2}\zeta_{\alpha_3...\alpha_n)} \,,
\tag{4.3a}
$$

$$
\delta_\zeta \mathfrak{y}_{\alpha(n-4)} = \frac{n-2}{n-1}\nabla^{\beta(2)}\zeta_{\alpha(n-4)\beta(2)} \,.
\tag{4.3b}
$$

The Fronsdal-type gauge-invariant action, $S_{\rm F}^{(n)} = S_{\rm F}^{(n)}[\mathfrak{h}_{(n)}, \mathfrak{y}_{(n-4)}]$, is

$$
\begin{aligned}
S_{\rm F}^{(n)} = \frac{{\rm i}^n}{2^{\lfloor n/2 \rfloor+1}} \int {\rm d}^3 x\, e \Bigg\{ & \mathfrak{h}^{\alpha(n)} \Box \mathfrak{h}_{\alpha(n)} - \frac{n}{4}\nabla_{\gamma(2)}\mathfrak{h}^{\gamma(2)\alpha(n-2)} \nabla^{\beta(2)} \mathfrak{h}_{\alpha(n-2)\beta(2)} \\
& - \frac{n-3}{2}\mathfrak{y}^{\alpha(n-4)} \nabla^{\beta(2)} \nabla^{\gamma(2)} \mathfrak{h}_{\alpha(n-4)\beta(2)\gamma(2)} - n(n-6)\mathcal{S}^2\mathfrak{h}^{\alpha(n)}\mathfrak{h}_{\alpha(n)} \\
& - \frac{(n-3)}{n}\Bigg[2\mathfrak{y}^{\alpha(n-4)} \Box \mathfrak{y}_{\alpha(n-4)} - 2(n^2-2n+4)\mathcal{S}^2\mathfrak{y}^{\alpha(n-4)}\mathfrak{y}_{\alpha(n-4)} \\
& + \frac{(n-4)(n-5)}{4(n-2)}\nabla_{\gamma(2)}\mathfrak{y}^{\gamma(2)\alpha(n-6)} \nabla^{\beta(2)} \mathfrak{y}_{\beta(2)\alpha(n-6)} \Bigg] \Bigg\}.
\end{aligned}
\tag{4.4}
$$

Our action (4.2) is a unique gauge-invariant extension to AdS space of the flat-space action given by Tyutin and Vasiliev [18], see appendix B for a review. When n is odd, $n = 2s+1$, (4.2) is the unique gauge-invariant 3D counterpart to the Fang-Fronsdal action in AdS$_4$ [43].[5] When n is even, $n = 2s$, our action (4.4) is the unique gauge-invariant 3D counterpart to the Fronsdal action in AdS$_4$ [42]. The Fronsdal action [42] can also be generalised to d-dimensional AdS backgrounds [51, 52]. Such an action in AdS$_d$ is

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[5]It is worth pointing out that the Fang-Fronsdal action for a massless spin-$(s + \frac{1}{2})$ field [13] is also described in terms of a triplet of fermionic gauge fields, $\Psi_{\alpha(s+1)\dot\alpha(s)}$, $\Psi_{\alpha(s-1)\dot\alpha(s)}$ and $\Psi_{\alpha(s-1)\dot\alpha(s-2)}$ and their conjugates, if one makes use of the two-component spinor notation, see section 6.9 of [44]. More generally, there exist bosonic and fermionic higher-spin triplet models in higher dimensions [45–49]. On-shell supersymmetric formulations for the generalised triplets in diverse dimensions have recently been given in [50].

formulated in terms of a symmetric double-traceless field and it is fixed by the condition of gauge invariance.[6]

Separately, each of the gauge-invariant actions (3.32), (4.2) and (4.4) proves to describe no propagating degrees of freedom. We claim that the following models

$$S_{\text{massive}}^{(2s+1)} = \lambda S_{\text{CS}}^{(2s+1)} \left[\mathfrak{h}_{(2s+1)}\right] + \mu^{2s-1} S_{\text{FF}}^{(2s+1)} \left[\mathfrak{h}_{(2s+1)}, \mathfrak{y}_{(2s-1)}, \mathfrak{y}_{(2s-3)}\right] \tag{4.5a}$$

$$S_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)} \left[\mathfrak{h}_{(2s)}\right] + \mu^{2s-3} S_{\text{F}}^{(2s)} \left[\mathfrak{h}_{(2s)}, \mathfrak{y}_{(2s-4)}\right] \tag{4.5b}$$

describe irreducible massive fields in AdS$_3$. Here the parameter λ is dimensionless, while μ has dimension of mass. Since we do not have a closed form expression for $\mathfrak{C}_{\alpha(n)}$ in AdS$_3$, for arbitrary n, our analysis below will be restricted to the case of Minkowski space, $\mathbb{M}^3$.

4.2 Massive higher-spin actions in Minkowski space: the fermionic case

In this section we study the dynamics of the flat-space counterparts to the gauge theories (4.5a) and (4.5b). In fact, the resulting flat-space actions are contained at the component level in the massive supersymmetric higher-spin models proposed in [16, 17]. However, the analysis in [16, 17] was carried out mostly in terms of superfields so that the component actions were not studied in detail.

We first analyse the flat-space limit of the fermionic model (4.5a). It is described by the action

$$S_{\text{massive}}^{(2s+1)} = \lambda S_{\text{CS}}^{(2s+1)} \left[h_{(2s+1)}\right] + \mu^{2s-1} S_{\text{FF}}^{(2s+1)} \left[h_{(2s+1)}, y_{(2s-1)}, y_{(2s-3)}\right], \tag{4.6}$$

where the massless sector is

$$S_{\text{FF}}^{(2s+1)} = \frac{i}{2}\left(-\frac{1}{2}\right)^s \int d^3x \left\{ h^{\alpha(2s)\gamma}\partial_\gamma{}^\delta h_{\delta\alpha(2s)} + 2(2s-1)y^{\alpha(2s-1)}\partial^{\beta(2)}h_{\alpha(2s-1)\beta(2)} \right.$$

$$+ 4(2s-1)y^{\alpha(2s-2)\gamma}\partial_\gamma{}^\delta y_{\delta\alpha(2s-2)} + \frac{2}{s}(2s+1)(s-1)y^{\alpha(2s-3)}\partial^{\beta(2)}y_{\alpha(2s-3)\beta(2)}$$

$$\left. - \frac{(s-1)(2s-3)}{s(2s-1)}y^{\alpha(2s-4)\gamma}\partial_\gamma{}^\delta y_{\delta\alpha(2s-4)} \right\}. \tag{4.7}$$

The action (4.6) is invariant under the following gauge transformations:

$$\delta_\zeta h_{\alpha(2s+1)} = \partial_{(\alpha_1\alpha_2}\zeta_{\alpha_3...\alpha_{2s+1})}, \tag{4.8a}$$

$$\delta_\zeta y_{\alpha(2s-1)} = \frac{1}{2s+1}\partial^\beta{}_{(\alpha_1}\zeta_{\alpha_2...\alpha_{2s-1})\beta}, \tag{4.8b}$$

$$\delta_\zeta y_{\alpha(2s-3)} = \partial^{\beta(2)}\zeta_{\alpha(2s-3)\beta(2)}. \tag{4.8c}$$

The equations of motion corresponding to the model (4.6) are

$$0 = \mu^{2s-1}\left(\partial^\beta{}_{(\alpha_1}h_{\alpha_2...\alpha_{2s+1})\beta} - (2s-1)\partial_{(\alpha_1\alpha_2}y_{\alpha_3...\alpha_{2s+1})}\right) + \lambda C_{\alpha(2s+1)}, \tag{4.9a}$$

$$0 = \partial^{\beta(2)}h_{\alpha(2s-1)\beta(2)} + 4\partial^\beta{}_{(\alpha_1}y_{\alpha_2...\alpha_{2s-1})\beta} - \frac{(s-1)(2s+1)}{s(2s-1)}\partial_{(\alpha_1\alpha_2}y_{\alpha_3...\alpha_{2s-1})}, \tag{4.9b}$$

$$0 = (2s-1)\partial^{\beta(2)}y_{\alpha(2s-3)\beta(2)} - \frac{2s-3}{2s+1}\partial^\beta{}_{(\alpha_1}y_{\alpha_2...\alpha_{2s-3})\beta}. \tag{4.9c}$$

---

[6]The dynamical equations for massless higher-spin fields in AdS$_d$ were studied by Metsaev [53–56]. For alternative descriptions of massless higher-spin dynamics in AdS$_d$, see [57, 58].

We now demonstrate that the model (4.6) indeed describes an irreducible massive spin-$(s+\frac{1}{2})$ field on the equations of motion. The gauge transformation (4.8c) tells us that $y_{\alpha(2s-3)}$ can be completely gauged away, that is, we are able to impose the gauge condition

$$y_{\alpha(2s-3)} = 0 . \tag{4.10}$$

Then, the residual gauge freedom is described by $\zeta_{\alpha(2s-1)}$ constrained by

$$\partial^{\beta(2)}\zeta_{\alpha(2s-3)\beta(2)} = 0 \quad \Longrightarrow \quad \partial^{\beta}{}_{(\alpha_1}\zeta_{\alpha_2...\alpha_{2s-1})\beta} = \partial^{\beta}{}_{\alpha_1}\zeta_{\alpha_2...\alpha_{2s-1}\beta} . \tag{4.11}$$

In the gauge (4.10), the equation of motion (4.9c) becomes the condition for $y_{\alpha(2s-1)}$ to be divergenceless,

$$\partial^{\beta(2)}y_{\alpha(2s-3)\beta(2)} = 0 \quad \Longrightarrow \quad \partial^{\beta}{}_{(\alpha_1}y_{\alpha_2...\alpha_{2s-1})\beta} = \partial^{\beta}{}_{\alpha_1}y_{\alpha_2...\alpha_{2s-1}\beta} . \tag{4.12}$$

Due to (4.12), the gauge transformation (4.8b) becomes

$$\delta_\zeta y_{\alpha(2s-1)} = \frac{1}{2s+1}\partial^{\beta}{}_{\alpha_1}\zeta_{\alpha_2...\alpha_{2s+1}\beta} . \tag{4.13}$$

Since $y_{\alpha(2s-1)}$ and $\zeta_{\alpha(2s-1)}$ have the same functional type, we are able to completely gauge away the $y_{\alpha(2s-1)}$ field,

$$y_{\alpha(2s-1)} = 0 . \tag{4.14}$$

In accordance with (4.13) and (4.14), the residual gauge freedom is described by the parameter $\zeta_{\alpha(2s-1)}$ constrained by

$$\partial^{\beta}{}_{\alpha_1}\zeta_{\alpha_2...\alpha_{2s-1}\beta} = 0 \quad \Longrightarrow \quad \Box\zeta_{\alpha(2s-1)} = 0 . \tag{4.15}$$

In the gauge (4.14), the equation of motion (4.9b) tells us that $h_{\alpha(2s+1)}$ is divergenceless,

$$\partial^{\beta(2)}h_{\alpha(2s-1)\beta(2)} = 0 \quad \Longrightarrow \quad \partial^{\beta}{}_{(\alpha_1}h_{\alpha_2...\alpha_{2s+1})\beta} = \partial^{\beta}{}_{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta} . \tag{4.16}$$

So far the above analysis has been identical to that given in appendix B of [16] for the massless model (4.7).

Due to (4.16), the Cotton tensor (3.35) reduces to the expression (3.41b). In the gauge (4.14), the equation of motion (4.9a) becomes

$$\mu^{2s-1}\partial^{\beta}{}_{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta} + \lambda\Box^s h_{\alpha(2s+1)} = 0 . \tag{4.17}$$

This equation has two types of solutions, massless and massive ones,

$$\partial^{\beta}{}_{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta} = 0 \quad \Longrightarrow \quad \Box h_{\alpha(2s+1)} = 0 ; \tag{4.18a}$$
$$\mu^{2s-1}h_{\alpha(2s+1)} + \lambda\Box^{s-1}\partial^{\beta}{}_{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta} = 0 . \tag{4.18b}$$

We point out that $\Psi_{\alpha_1...\alpha_{2s+1}} := \partial^{\beta}{}_{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta}$ is completely symmetric and divergenceless, $\Psi_{\alpha_1...\alpha_{2s+1}} = \Psi_{(\alpha_1...\alpha_{2s+1})}$ and $\partial^{\beta\gamma}\Psi_{\beta\gamma\alpha_1...\alpha_{2s-1}} = 0$.

Let us show that the massless solution (4.18a) is a pure gauge degree of freedom. Since both the gauge field $h_{\alpha(2s+1)}$ and the residual gauge parameter $\zeta_{\alpha(2s-1)}$ are on-shell

massless, it is useful to switch to momentum space, by replacing $h_{\alpha(2s+1)}(x) \to h_{\alpha(2s+1)}(p)$ and $\zeta_{\alpha(2s-1)}(x) \to \zeta_{\alpha(2s-1)}(p)$, where the three-momentum p^a is light-like, $p^{\alpha\beta}p_{\alpha\beta} = 0$. For a given three-momentum, we can choose a frame in which the only non-zero component of $p^{\alpha\beta} = (p^{11}, p^{12} = p^{21}, p^{22})$ is $p^{22} = p_{11}$. Then, the conditions $p^\beta{}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta}(p) = 0$ and $p^\beta{}_{\alpha_1}\zeta_{\alpha_2\ldots\alpha_{2s-1}\beta}(p) = 0$ are equivalent to

$$h_{\alpha(2s)2}(p) = 0 \,, \qquad \zeta_{\alpha(2s-2)2}(p) = 0 \,. \tag{4.19}$$

Thus the only non-zero components of $h_{\alpha(2s+1)}(p)$ and $\zeta_{\alpha(2s-1)}(p)$ are $h_{1\ldots1}(p)$ and $\zeta_{1\ldots1}(p)$. The residual gauge freedom, $\delta h_{1\ldots}(p) \propto p_{11}\zeta_{1\ldots1}$, allows us to gauge away the field $h_{\alpha(2s+1)}$ completely.

Thus, it remains to analyse the general solution of the equation (4.18b), which implies

$$\left(\Box^{2s-1} - (m^2)^{2s-1}\right)h_{\alpha(2s+1)} = 0 \,, \qquad m := \left|\frac{\mu}{\lambda^{1/(2s-1)}}\right| \,. \tag{4.20}$$

This equation in momentum space yields

$$\left(1 - \left(\frac{-p^2}{m^2}\right)^{2s-1}\right)h_{\alpha(2s+1)}(p) = 0 \,. \tag{4.21}$$

Since the polynomial equation $z^{2s-1} - 1 = 0$ has only one real root, $z = 1$, the only real solution to (4.21) is $p^2 = -m^2$, from which it follows that $h_{\alpha(2s+1)}$ satisfies the ordinary Klein-Gordon equation,

$$\left(\Box - m^2\right)h_{\alpha(2s+1)} = 0 \,. \tag{4.22}$$

Applying (4.22) to (4.17) reveals that $h_{\alpha(2s+1)}$ satisfies the equation of motion corresponding to a massive spin $(s + \frac{1}{2})$-field with mass m and helicity $\sigma(s + \frac{1}{2})$,

$$\partial^\beta{}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta} = \sigma m h_{\alpha(2s+1)} \,, \qquad \sigma := -\text{sign}(\mu\lambda) \,. \tag{4.23}$$

Finally, for completeness let us recall the proof of the fact that equation (4.23) describes a single propagating degree of freedom. The field $h_{\alpha(2s+1)}$ is on-shell with momentum satisfying $p^2 = -m^2$, we can therefore transform equation (4.23) into momentum space and boost into the rest frame where $p^a = (m, 0, 0) \implies p^1{}_1 = p^2{}_2 = 0,\ p^1{}_2 = -p^2{}_1 = m$,

$$\text{i}h_{\alpha(2s)1}(p) - \sigma h_{\alpha(2s)2}(p) = 0 \,. \tag{4.24}$$

Due to the symmetry of the field $h_{\alpha(2s+1)}$, equation (4.24) states that there is only a single degree of freedom. Taking the independent field component to be $h_{11\ldots1}(p)$ allows us to express all other components in terms of it.

Along with the fermionic model (4.6), which corresponds to $n = 2s + 1$, we could consider a bosonic one described by the action

$$S_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)}\left[h_{(2s)}\right] + \mu^{2s-2}S_{\text{FF}}^{(2s)}\left[h_{(2s)}, y_{(2s-2)}, y_{(2s-4)}\right] \,, \tag{4.25}$$

which corresponds to $n = 2s$. Most of the above analysis would remain valid in this case as well. However, in place of eq. (4.21) we would have

$$\left(1 - \left(\frac{-p^2}{m^2}\right)^{2s-2}\right)h_{\alpha(2s)}(p) = 0 \,. \tag{4.26}$$

This equation has both physical ($p^2 = -m^2$) and tachyonic ($p^2 = m^2$) solutions. Therefore, the model (4.25) is unphysical. This may be interpreted as a consequence of the spin-statistics theorem.

4.3   Massive higher-spin actions in Minkowski space: the bosonic case

Our next goal is to analyse the flat-space limit of the bosonic model (4.5b). It is described by the action

$$S_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)} \left[h_{(2s)} \right] + \mu^{2s-3} S_{\text{F}}^{(2s)} \left[h_{(2s)}, y_{(2s-4)} \right] , \tag{4.27}$$

where the second term is

$$S_{\text{F}}^{(2s)} = \frac{1}{2} \left(\frac{-1}{2} \right)^s \int \mathrm{d}^3 x \left\{ h^{\alpha(2s)} \Box h_{\alpha(2s)} - \frac{s}{2} \partial_{\gamma(2)} h^{\gamma(2)\alpha(2s-2)} \partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} \right.$$

$$- \frac{(2s-3)}{2s} \left[s y^{\alpha(2s-4)} \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} + 2 y^{\alpha(2s-4)} \Box y_{\alpha(2s-4)} \right.$$

$$\left. \left. + \frac{(s-2)(2s-5)}{4(s-1)} \partial_{\gamma(2)} y^{\gamma(2)\alpha(2s-6)} \partial^{\beta(2)} y_{\beta(2)\alpha(2s-6)} \right] \right\} . \tag{4.28}$$

The action (4.27) is invariant under the gauge transformations

$$\delta_\zeta h_{\alpha(2s)} = \partial_{(\alpha_1\alpha_2} \zeta_{\alpha_3\ldots\alpha_{2s})} , \tag{4.29a}$$

$$\delta_\zeta y_{\alpha(2s-4)} = \frac{2s-2}{2s-1} \partial^{\beta(2)} \zeta_{\alpha(2s-4)\beta(2)} . \tag{4.29b}$$

The equations of motion corresponding to (4.27) are

$$0 = \mu^{2s-3} \left(\Box h_{\alpha(2s)} + \frac{1}{2} s \partial^{\beta(2)} \partial_{(\alpha_1\alpha_2} h_{\alpha_3\ldots\alpha_{2s})\beta(2)} + \right.$$

$$\left. - \frac{1}{4}(2s-3) \partial_{(\alpha_1\alpha_2} \partial_{\alpha_3\alpha_4} y_{\alpha_5\ldots\alpha_{2s})} \right) + \lambda C_{\alpha(2s)} , \tag{4.30a}$$

$$0 = \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} + \frac{4}{s} \Box y_{\alpha(2s-4)} +$$

$$- \frac{(s-2)(2s-5)}{2s(s-1)} \partial^{\beta(2)} \partial_{(\alpha_1\alpha_2} y_{\alpha_3\ldots\alpha_{2s-4})\beta(2)} . \tag{4.30b}$$

We will now show that on-shell, the model $S_{\text{massive}}^{(2s)}$ describes a massive spin-s field which propagates a single degree of freedom. As follows from the gauge transformation (4.29b), it is possible to completely gauge away $y_{\alpha(2s-4)}$,

$$y_{\alpha(2s-4)} = 0 . \tag{4.31}$$

Then, the residual gauge freedom is described by a parameter $\zeta_{\alpha(2s-2)}$ constrained by

$$\partial^{\beta(2)} \zeta_{\alpha(2s-4)\beta(2)} = 0 \quad \Longrightarrow \quad \partial^\beta_{(\alpha_1} \zeta_{\alpha_2\ldots\alpha_{2s-2})\beta} = \partial^\beta_{\alpha_1} \zeta_{\alpha_2\ldots\alpha_{2s-2}\beta}. \tag{4.32}$$

In the gauge (4.31), the equation of motion (4.30b) becomes

$$\partial^{\gamma(2)} \partial^{\beta(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} = 0 . \tag{4.33}$$

According to (4.29a), the divergence of $h_{\alpha(2s)}$ transforms as

$$\delta_\zeta \left(\partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} \right) = \partial^{\beta_1\beta_2} \partial_{(\alpha_1\alpha_2} \zeta_{\alpha_3...\alpha_{2s-2}\beta_1\beta_2)} = -\frac{2}{s} \Box \zeta_{\alpha(2s-2)} \tag{4.34}$$

where we have made use of (4.32). Since $\zeta_{\alpha(2s-2)}$ and $\partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)}$ have the same functional type, it is possible to completely gauge away the divergence of $h_{\alpha(2s)}$,

$$\partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} = 0 \quad \Longrightarrow \quad \partial^\beta_{\ (\alpha_1} h_{\alpha_2...\alpha_{2s})\beta} = \partial^\beta_{\ \alpha_1} h_{\alpha_2...\alpha_{2s}\beta} \ . \tag{4.35}$$

Under the gauge conditions imposed, there still remains some residual gauge freedom described by a gauge parameter constrained by (4.32) and $\Box\zeta_{\alpha(2s-2)} = 0$. So far the above analysis has been identical to that given in appendix B of [16] for the massless model (4.28).

As a consequence of (4.35), the Cotton tensor (3.35) reduces to the simple form (3.41a). Making use of the gauge conditions (4.31) and (4.35) in conjunction with eq. (3.41a), the equation of motion (4.30a) becomes

$$\left(\mu^{2s-3} \delta^\beta_{\ \alpha_1} + \lambda \Box^{s-2} \partial^\beta_{\ \alpha_1} \right) \Box h_{\alpha_2...\alpha_{2s}\beta} = 0 \ . \tag{4.36}$$

This equation has two types of solutions, massless and massive ones,

$$\Box h_{\alpha(2s)} = 0 \ ; \tag{4.37a}$$

$$\mu^{2s-3} h_{\alpha(2s)} + \lambda \Box^{s-2} \partial^\beta_{\ \alpha_1} h_{\alpha_2...\alpha_{2s}\beta} = 0 \ . \tag{4.37b}$$

Let us show that the massless solution (4.37a) is a pure gauge degree of freedom. Since both the gauge field $h_{\alpha(2s)}$ and the gauge parameter $\zeta_{\alpha(2s-2)}$ are on-shell massless, it is useful to switch to momentum space by replacing $h_{\alpha(2s)}(x) \to h_{\alpha(2s)}(p)$ and $\zeta_{\alpha(2s-2)}(x) \to \zeta_{\alpha(2s-2)}(p)$, where the three-momentum p^a is light-like, $p^{\alpha\beta} p_{\alpha\beta} = 0$. As in the fermionic case studied in the previous subsection, we can choose a frame in which the only non-zero component of $p^{\alpha\beta} = (p^{11}, p^{12} = p^{21}, p^{22})$ is $p^{22} = p_{11}$. In this frame, the equations (4.32) and (4.35) are equivalent to

$$h_{\alpha(2s-2)22}(p) = 0 \,, \qquad \zeta_{\alpha(2s-4)22}(p) = 0 \ . \tag{4.38}$$

These conditions tell us that the only non-zero components in this frame are $h_{1...1}(p)$, $h_{1...12}(p)$ and $\zeta_{1...1}(p)$, $\zeta_{1...12}(p)$. However, the gauge transformation (4.29a) is equivalent to $\delta h_{1...1}(p) \propto \zeta_{1...1}(p)$ and $\delta h_{1...12}(p) \propto \zeta_{1...12}(p)$, allowing us to completely gauge away the $h_{\alpha(2s)}$ field.

Let us turn to the other equation (4.37b), which implies

$$\left(\Box^{2s-3} - (m^2)^{2s-3} \right) h_{\alpha(2s)} = 0 \,, \qquad m := \left| \frac{\mu}{\lambda^{1/(2s-3)}} \right| \ . \tag{4.39}$$

Here the mass parameter has the same form as in the fermionic case, eq. (4.20). Transforming eq. (4.39) to momentum space gives

$$\left(1 - \left(\frac{-p^2}{m^2} \right)^{2s-3} \right) h_{\alpha(2s)}(p) = 0 \ . \tag{4.40}$$

In complete analogy with the fermionic case considered in the previous subsection, this equation has the unique real solution $p^2 = -m^2$.

It follows that $h_{\alpha(2s)}$ satisfies the Klein-Gordon equation,

$$\left(\Box - m^2\right) h_{\alpha(2s)} = 0 \; . \tag{4.41}$$

As a consequence, the equation of motion (4.37b) leads to

$$\partial^\beta{}_{\alpha_1} h_{\alpha_2 \dots \alpha_{2s} \beta} = \sigma m h_{\alpha(2s)} \, , \qquad \sigma := -\mathrm{sign}(\mu\lambda) \; . \tag{4.42}$$

Therefore $h_{\alpha(2s)}$ is an irreducible on-shell massive field with mass m and helicity $\lambda = \sigma s$. Equation (4.42) implies that $h_{\alpha(2s)}$ describes a single propagating degree of freedom.

5 Conformal higher-spin gauge superfields

Conformal higher-spin gauge superfields in $\mathcal{N} = 1$ Minkowski superspace were introduced in [15, 17], as a by-product of the $\mathcal{N} = 2$ approach of [16]. In this section we start by generalising this concept to the case of $\mathcal{N} = 1$ supergravity, building on the ideas advocated in [34].

5.1 Conformal supergravity

Consider a curved $\mathcal{N} = 1$ superspace, $\mathcal{M}^{3|2}$, parametrised by local real coordinates $z^M = (x^m, \theta^\mu)$, with $m = 0, 1, 2$ and $\mu = 1, 2$, of which x^m are bosonic and θ^μ fermionic. We introduce a basis of one-forms $E^A = (E^a, E^\alpha)$ and its dual basis $E_A = (E_a, E_\alpha)$,

$$E^A = \mathrm{d}z^M E_M{}^A \, , \qquad E_A = E_A{}^M \partial_M \, , \tag{5.1}$$

which will be referred to as the supervielbein and its inverse, respectively. The superspace structure group is $\mathrm{SL}(2, \mathbb{R})$, the double cover of the connected Lorentz group $\mathrm{SO}_0(2, 1)$. The covariant derivatives have the form:

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha) = E_A + \Omega_A \, , \tag{5.2}$$

where

$$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}M_{bc} = -\Omega_A{}^b M_b = \frac{1}{2}\Omega_A{}^{\beta\gamma}M_{\beta\gamma} \tag{5.3}$$

is the Lorentz connection.

The covariant derivatives are characterised by the graded commutation relations

$$[\mathcal{D}_A, \mathcal{D}_B\} = \mathcal{T}_{AB}{}^C \mathcal{D}_C + \frac{1}{2}\mathcal{R}_{AB}{}^{cd}M_{cd} \, , \tag{5.4}$$

where $T_{AB}{}^C$ and $R_{AB}{}^{cd}$ are the torsion and curvature tensors, respectively. To describe supergravity, the covariant derivatives have to obey certain torsion constraints [59] such that the algebra (5.4) takes the form [60]

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2\mathrm{i}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}\mathcal{S}M_{\alpha\beta} \, , \tag{5.5a}$$

$$[\mathcal{D}_a, \mathcal{D}_\beta] = (\gamma_a)_\beta{}^\gamma \left[\mathcal{S}\mathcal{D}_\gamma + \mathrm{i}\mathcal{C}_{\gamma\delta\rho}M^{\delta\rho}\right] - \frac{2}{3}\left[\mathcal{D}_\beta \mathcal{S}\delta_a^c - 2\varepsilon_{ab}{}^c(\gamma^b)_{\beta\gamma}\mathcal{D}^\gamma \mathcal{S}\right]M_c \, , \tag{5.5b}$$

$$[\mathcal{D}_a, \mathcal{D}_b] = \varepsilon_{abc}\Bigg\{\left[\frac{1}{2}(\gamma^c)_{\alpha\beta}\mathcal{C}^{\alpha\beta\gamma} - \frac{2\mathrm{i}}{3}(\gamma^c)^{\beta\gamma}\mathcal{D}_\beta\mathcal{S}\right]\mathcal{D}_\gamma$$

$$+ \left[\frac{1}{2}(\gamma^c)^{\alpha\beta}(\gamma^d)^{\gamma\delta}\mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma\delta)} + \left(\frac{2\mathrm{i}}{3}\mathcal{D}^2\mathcal{S} + 4\mathcal{S}^2\right)\eta^{cd}\right]M_d\Bigg\} \, . \tag{5.5c}$$

Here the scalar $\mathcal{S}$ and the symmetric spinor $\mathcal{C}_{\alpha\beta\gamma} = \mathcal{C}_{(\alpha\beta\gamma)}$ are real. The dimension-2 Bianchi identities imply that

$$\mathcal{D}_\alpha \mathcal{C}_{\beta\gamma\delta} = \mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma\delta)} + \varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma\delta)}\mathcal{S} \quad \Longrightarrow \quad \mathcal{D}^\gamma \mathcal{C}_{\alpha\beta\gamma} = \frac{4}{3}\mathcal{D}_{\alpha\beta}\mathcal{S} \ . \tag{5.6}$$

We use the notation $\mathcal{D}^2 := \mathcal{D}^\alpha \mathcal{D}_\alpha$.

The algebra of covariant derivatives is invariant under the following super-Weyl transformations [61–63]

$$\delta_\sigma \mathcal{D}_\alpha = \frac{1}{2}\sigma \mathcal{D}_\alpha + \mathcal{D}^\beta \sigma M_{\alpha\beta} \ , \tag{5.7a}$$

$$\delta_\sigma \mathcal{D}_a = \sigma \mathcal{D}_a + \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}\mathcal{D}_\gamma \sigma \mathcal{D}_\delta + \varepsilon_{abc}\mathcal{D}^b \sigma M^c \ , \tag{5.7b}$$

with the parameter σ being a real unconstrained superfield, provided the torsion superfields transform as

$$\delta_\sigma \mathcal{S} = \sigma \mathcal{S} - \frac{\mathrm{i}}{4}\mathcal{D}^2 \sigma \ , \qquad \delta_\sigma \mathcal{C}_{\alpha\beta\gamma} = \frac{3}{2}\sigma \mathcal{C}_{\alpha\beta\gamma} - \frac{\mathrm{i}}{2}\mathcal{D}_{(\alpha\beta}\mathcal{D}_{\gamma)}\sigma \ . \tag{5.8}$$

The $\mathcal{N} = 1$ supersymmetric extension of the Cotton tensor (3.3) was constructed in [64]. It is given by the expression

$$\mathcal{W}_{\alpha\beta\gamma} = \left(\frac{\mathrm{i}}{2}\mathcal{D}^2 + 4\mathcal{S}\right)\mathcal{C}_{\alpha\beta\gamma} + \mathrm{i}\mathcal{D}_{(\alpha\beta}\mathcal{D}_{\gamma)}\mathcal{S} \ . \tag{5.9}$$

The super-Weyl transformation of $\mathcal{W}_{\alpha\beta\gamma}$ proves to be

$$\delta_\sigma \mathcal{W}_{\alpha\beta\gamma} = \frac{5}{2}\sigma \mathcal{W}_{\alpha\beta\gamma} \ . \tag{5.10}$$

It can be shown [36] that the curved superspace is conformally flat if and only if $\mathcal{W}_{\alpha\beta\gamma} = 0$.

5.2 Conformal gauge superfields

A real tensor superfield $\mathfrak{H}_{\alpha(n)}$ is said to be a conformal gauge supermultiplet if (i) it is super-Weyl primary of dimension $(1 - n/2)$,

$$\delta_\sigma \mathfrak{H}_{\alpha(n)} = \left(1 - \frac{n}{2}\right)\sigma \mathfrak{H}_{\alpha(n)} \ ; \tag{5.11}$$

and (ii) it is defined modulo gauge transformations of the form

$$\delta_\lambda \mathfrak{H}_{\alpha(n)} = \mathrm{i}^n \mathcal{D}_{(\alpha_1}\lambda_{\alpha_2\ldots\alpha_n)} \ , \tag{5.12}$$

with the gauge parameter $\lambda_{\alpha(n-1)}$ being real but otherwise unconstrained. The super-Weyl weight of $\mathfrak{H}_{\alpha(n)}$, given by $(1 - n/2)$, is uniquely fixed by requiring $\lambda_{\alpha(n-1)}$ and $\delta_\lambda \mathfrak{H}_{\alpha(n)}$ to be super-Weyl primary.

Starting with $\mathfrak{H}_{\alpha(n)}$ one can construct its descendant, $\mathfrak{W}_{\alpha(n)}(\mathfrak{H})$, defined uniquely, modulo a normalisation, by the following the properties:

1. $\mathfrak{W}_{\alpha(n)}$ is of the form $\mathcal{A}\mathfrak{H}_{\alpha(n)}$, where $\mathcal{A}$ is a linear differential operator involving $\mathcal{D}_A$, the torsion tensors $\mathcal{C}_{\alpha\beta\gamma}$ and $\mathcal{S}$ and their covariant derivatives.

2. $\mathfrak{W}_{\alpha(n)}$ is super-Weyl primary of weight $(1 + n/2)$,

$$\delta_\sigma \mathfrak{W}_{\alpha(n)} = \left(1 + \frac{n}{2}\right) \sigma \mathfrak{W}_{\alpha(n)} . \tag{5.13}$$

3. The gauge variation of $\mathfrak{W}_{\alpha(n)}$ vanishes if the superspace is conformally flat,

$$\delta_\lambda \mathfrak{W}_{\alpha(n)} = O\left(\mathcal{W}_{(3)}\right) , \tag{5.14}$$

where $\mathcal{W}_{(3)}$ is the super-Cotton tensor (5.9).

4. $\mathfrak{W}_{\alpha(n)}$ is divergenceless if the superspace is conformally flat,

$$\mathcal{D}^\beta \mathfrak{W}_{\beta\alpha(n-1)} = O\left(\mathcal{W}_{(3)}\right) . \tag{5.15}$$

Here $O\left(\mathcal{W}_{(3)}\right)$ stands for contributions containing the super-Cotton tensor and its covariant derivatives.

As a simple example, we consider a U(1) vector multiplet coupled to supergravity, which corresponds to the $n = 1$ case. This multiplet is described by a real spinor prepotential $\mathfrak{H}_\alpha$ which is super-Weyl primary of weight $1/2$ and is defined modulo gauge transformations $\delta_\lambda \mathfrak{H}_\alpha = \mathrm{i} \mathcal{D}_\alpha \lambda$, where the gauge parameter λ is an unconstrained real superfield. The required super-Weyl primary descendant of weight $3/2$ is given by

$$\mathfrak{W}_\alpha = -\frac{\mathrm{i}}{2} \mathcal{D}^\beta \mathcal{D}_\alpha \mathfrak{H}_\beta - 2\mathcal{S}\mathfrak{H}_\alpha \tag{5.16}$$

and proves to be gauge invariant,

$$\delta_\zeta \mathfrak{W}_\alpha = 0 . \tag{5.17}$$

The field strength obeys the Bianchi identity

$$\mathcal{D}^\alpha \mathfrak{W}_\alpha = 0 . \tag{5.18}$$

For $n > 1$ the right-hand sides of (5.14) and (5.15) are non-vanishing.

Suppose that our background curved superspace $\mathcal{M}^{3|2}$ is conformally flat,

$$\mathcal{W}_{\alpha(3)} = 0 . \tag{5.19}$$

Then the tensor superfield $\mathfrak{W}_{\alpha(n)}$ is gauge invariant and conserved,

$$\delta_\lambda \mathfrak{W}_{\alpha(n)} = 0 , \tag{5.20a}$$

$$\mathcal{D}^\beta \mathfrak{W}_{\beta\alpha(n-1)} = 0 . \tag{5.20b}$$

These properties and the super-Weyl transformation laws (5.11) and (5.13) imply that the action[7]

$$\mathbb{S}_{\mathrm{SCS}}^{(n)}[\mathfrak{H}_{(n)}] = -\frac{\mathrm{i}^n}{2^{\lfloor n/2 \rfloor + 1}} \int \mathrm{d}^{3|2} z \, E \, \mathfrak{H}^{\alpha(n)} \mathfrak{W}_{\alpha(n)}(\mathfrak{H}) , \qquad E^{-1} = \mathrm{Ber}\left(E_A{}^M\right) \tag{5.21}$$

is gauge and super-Weyl invariant,

$$\delta_\lambda \mathbb{S}_{\mathrm{SCS}}^{(n)}\left[\mathfrak{H}_{(n)}\right] = 0 , \qquad \delta_\sigma \mathbb{S}_{\mathrm{SCS}}^{(n)}\left[\mathfrak{H}_{(n)}\right] = 0 . \tag{5.22}$$

We now turn to constructing the linearised higher-spin super-Cotton tensors $\mathfrak{W}_{\alpha(n)}$ on such a conformally flat superspace.

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[7]The super-Weyl transformation of the superspace integration measure is $\delta_\sigma E = -2\sigma E$.

5.3 Higher-spin super-Cotton tensor in Minkowski superspace

In Minkowski superspace, $\mathbb{M}^{3|2}$, the higher-spin super-Cotton tensor [15, 17] is

$$W_{\alpha_1\ldots\alpha_n} = \left(-\frac{\mathrm{i}}{2}\right)^n D^{\beta_1} D_{\alpha_1} \ldots D^{\beta_n} D_{\alpha_n} H_{\beta_1\ldots\beta_n} = W_{(\alpha_1\ldots\alpha_n)}, \tag{5.23}$$

with $D_A = (\partial_a, D_\alpha)$ being the flat-superspace covariant derivatives. This tensor is invariant under the gauge transformation

$$\delta H_{\alpha_1\alpha_2\ldots\alpha_n} = \mathrm{i}^n D_{(\alpha_1} \zeta_{\alpha_2\ldots\alpha_n)}, \tag{5.24}$$

and obeys the conservation identity

$$D^\beta W_{\beta\alpha_1\ldots\alpha_{n-1}} = 0. \tag{5.25}$$

The fact that $W_{\alpha_1\ldots\alpha_n}$ defined by (5.23) is completely symmetric, is a corollary of the identities

$$D^\alpha D_\beta D_\alpha = 0 \quad \Longrightarrow \quad [D_\alpha D_\beta, D_\gamma D_\delta] = 0. \tag{5.26}$$

The normalisation in (5.23) is explained as follows. The gauge freedom (5.24) allows us to impose a gauge condition

$$D^\beta H_{\beta\alpha(n-1)} = 0, \tag{5.27}$$

under which the expression for the super-Cotton tensor simplifies,

$$D^\beta H_{\beta\alpha_1\ldots\alpha_{n-1}} = 0 \quad \Longrightarrow \quad W_{\alpha(n)} = \partial_{\alpha_1}{}^{\beta_1} \ldots \partial_{\alpha_n}{}^{\beta_n} H_{\beta_1\ldots\beta_n}. \tag{5.28a}$$

This result can be fine-tuned to

$$W_{\alpha(2s)} = \Box^s H_{\alpha(2s)}, \tag{5.28b}$$

$$W_{\alpha(2s+1)} = \Box^s \partial^\beta{}_{(\alpha_1} H_{\alpha_2\ldots\alpha_{2s+1})\beta} = \Box^s \partial^\beta{}_{\alpha_1} H_{\alpha_2\ldots\alpha_{2s+1}\beta}, \tag{5.28c}$$

where $s > 0$ is an integer.

For completeness, we also give another representation for the higher-spin super-Cotton tensor derived in [15, 17]:

$$W_{\alpha_1\ldots\alpha_n} := \frac{1}{2^n} \sum_{j=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{2j} \Box^j \partial_{(\alpha_1}{}^{\beta_1} \ldots \partial_{\alpha_{n-2j}}{}^{\beta_{n-2j}} H_{\alpha_{n-2j+1}\ldots\alpha_n)\beta_1\ldots\beta_{n-2j}} \right.$$
$$\left. -\frac{\mathrm{i}}{2} \binom{n}{2j+1} D^2 \Box^j \partial_{(\alpha_1}{}^{\beta_1} \ldots \partial_{\alpha_{n-2j-1}}{}^{\beta_{n-2j-1}} H_{\alpha_{n-2j}\ldots\alpha_n)\beta_1\ldots\beta_{n-2j-1}} \right\}. \tag{5.29}$$

The following higher-spin action [15, 17]

$$\mathbb{S}^{(n)}_{\mathrm{SCS}}\left[H_{(n)}\right] = -\frac{\mathrm{i}^n}{2^{\lfloor n/2 \rfloor + 1}} \int \mathrm{d}^{3|2} z\, H^{\alpha(n)} W_{\alpha(n)}(H) \tag{5.30}$$

is $\mathcal{N} = 1$ superconformal. It is clearly invariant under the gauge transformations (5.24).

5.4 Higher-spin super-Cotton tensor in conformally flat superspaces

Consider a curved conformally flat superspace $\mathcal{M}^{3|2}$. Locally, its covariant derivatives $\mathcal{D}_A$ are related to the flat-space ones by

$$\mathcal{D}_\alpha = e^{\frac{1}{2}\sigma}\left(D_\alpha + \mathcal{D}^\beta \sigma M_{\alpha\beta}\right), \tag{5.31}$$

$$\mathcal{D}_a = e^\sigma \left(\partial_a + \frac{i}{2}(\gamma_a)^{\alpha\beta}\mathcal{D}_\alpha \sigma D_\beta + \partial^b \sigma M_{ba} - \frac{i}{8}(\gamma_a)^{\alpha\beta}\left(\mathcal{D}^\gamma \sigma\right) D_\gamma \sigma M_{\alpha\beta}\right), \tag{5.32}$$

for some scale factor σ. In accordance with (5.13), the higher-spin super-Cotton tensor $\mathfrak{W}_{\alpha(n)}$ in $\mathcal{M}^{3|2}$ is related to the flat-space one, eq. (5.23) or equivalently (5.29), by the rule

$$\mathfrak{W}_{\alpha(n)} = e^{(1+\frac{n}{2})\sigma}W_{\alpha(n)}. \tag{5.33}$$

In general, it is a difficult technical problem to express $\mathfrak{W}_{\alpha(n)}$ in terms of the covariant derivatives $\mathcal{D}_A$ and the gauge prepotential $\mathfrak{H}_{\alpha(n)} = e^{(1-n/2)\sigma}H_{\alpha(n)}$. As an example, we only give expressions for the supersymmetric photino $\mathfrak{W}_\alpha$ and Cottino $\mathfrak{W}_{\alpha(2)}$ tensors in AdS superspace. The geometry of AdS$^{3|2}$ is encoded in the following algebra of covariant derivatives:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\mathcal{D}_{\alpha\beta} - 4i\mathcal{S}M_{\alpha\beta}, \tag{5.34a}$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma] = -2\mathcal{S}\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}, \tag{5.34b}$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma\delta}] = 4\mathcal{S}^2\left(\varepsilon_{\gamma(\alpha}M_{\beta)\delta} + \varepsilon_{\delta(\alpha}M_{\beta)\gamma}\right), \tag{5.34c}$$

with the real parameter $\mathcal{S}$ being the same as in (3.46). The tensors $\mathfrak{W}_\alpha$ and $\mathfrak{W}_{\alpha(2)}$ are expressed in terms of the operator

$$\Delta^\beta{}_\alpha := -\frac{i}{2}\mathcal{D}^\beta \mathcal{D}_\alpha - 2\mathcal{S}\delta^\beta{}_\alpha, \tag{5.35}$$

with the properties

$$\left[\Delta^{\beta_1}{}_{\alpha_1}, \Delta^{\beta_2}{}_{\alpha_2}\right] = \varepsilon_{\alpha_1\alpha_2}\mathcal{S}\left(\mathcal{D}^{\beta_1\beta_2} - 2\mathcal{S}M^{\beta_1\beta_2}\right) - \varepsilon^{\beta_1\beta_2}\mathcal{S}\left(\mathcal{D}_{\alpha_1\alpha_2} - 2\mathcal{S}M_{\alpha_1\alpha_2}\right). \tag{5.36}$$

These properties follow from the identity

$$\mathcal{D}^\beta \mathcal{D}_\alpha \mathcal{D}_\beta = 4i\mathcal{S}\mathcal{D}_\alpha \quad \Longrightarrow \quad \mathcal{D}^\alpha \Delta^\beta{}_\alpha = 0. \tag{5.37}$$

The expressions for $\mathfrak{W}_\alpha$ and $\mathfrak{W}_{\alpha(2)}$ are:

$$\mathfrak{W}_\alpha := \Delta^\beta{}_\alpha \mathfrak{H}_\beta, \tag{5.38a}$$

$$\mathfrak{W}_{\alpha_1\alpha_2} = \Delta^{\beta_1}{}_{(\alpha_1}\Delta^{\beta_2}{}_{\alpha_2)}\mathfrak{H}_{\beta_1\beta_2} - 2\mathcal{S}\Delta^\beta{}_{(\alpha_1}\mathfrak{H}_{\alpha_2)\beta}. \tag{5.38b}$$

5.5 Massive supersymmetric higher-spin theories in AdS superspace

Massive supersymmetric higher-spin actions in AdS involve different massless sectors depending on the value of superspin.

$$\mathbb{S}^{(2s)}_{\text{massive}} = \lambda \mathbb{S}^{(2s)}_{\text{SCS}}\left[\mathfrak{H}_{(2s)}\right] + \mu^{2s-1}\mathbb{S}^{(2s)}_{\text{FO}}\left[\mathfrak{H}_{(2s)}, \mathfrak{Y}_{(2s-2)}\right], \tag{5.39a}$$

$$\mathbb{S}^{(2s+1)}_{\text{massive}} = \lambda \mathbb{S}^{(2s+1)}_{\text{SCS}}\left[\mathfrak{H}_{(2s+1)}\right] + \mu^{2s-1}\mathbb{S}^{(2s+1)}_{\text{SO}}\left[\mathfrak{H}_{(2s+1)}, \mathfrak{X}_{(2s-2)}\right] \tag{5.39b}$$

5.5.1 First-order massless actions

We introduce a gauge theory described by a reducible gauge superfield $\boldsymbol{\mathcal{H}}_{\beta,\alpha_1\ldots\alpha_{n-1}} = \boldsymbol{\mathcal{H}}_{\beta,(\alpha_1\ldots\alpha_{n-1})}$. This superfield is defined modulo gauge transformations of the form

$$\delta\boldsymbol{\mathcal{H}}_{\beta,\alpha_1\ldots\alpha_{n-1}} = \mathrm{i}^n\mathcal{D}_\beta\lambda_{\alpha_1\ldots\alpha_{n-1}} \ . \tag{5.40}$$

A supersymmetric gauge-invariant action of lowest order in derivatives is

$$\mathbb{S}_{\mathrm{FO}}^{(n)} = \frac{\mathrm{i}^{n+1}}{2^{\lceil(n+1)/2\rceil}} \int \mathrm{d}^{3|2}z\, E\, \boldsymbol{\mathcal{H}}^{\beta,\alpha_1\ldots\alpha_{n-1}}\Big(\mathcal{D}^\gamma\mathcal{D}_\beta - 4\mathrm{i}\mathcal{S}\delta^\gamma_\beta\Big)\boldsymbol{\mathcal{H}}_{\gamma,\alpha_1\ldots\alpha_{n-1}} \ . \tag{5.41}$$

The gauge invariance of $\mathbb{S}_{\mathrm{FO}}^{(n)}$ follows from the identity (5.37). Our action (5.41) is a higher-spin AdS extension of the model for the massless gravitino multiplet ($n = 2$) in Minkowski superspace proposed by Siegel [1] (see also [59]).

The gauge superfield $\boldsymbol{\mathcal{H}}_{\beta,\alpha_1\ldots\alpha_{n-1}}$ can be decomposed into irreducible $\mathrm{SL}(2,\mathbb{R})$ superfields

$$\boldsymbol{\mathcal{H}}_{\beta,\alpha_1\ldots\alpha_{n-1}} = \mathfrak{H}_{\beta\alpha_1\ldots\alpha_{n-1}} + \sum_{k=1}^{n-1}\varepsilon_{\beta\alpha_k}\mathfrak{Y}_{\alpha_1\ldots\hat{\alpha}_k\ldots\alpha_{n-1}} \,, \tag{5.42}$$

where $\mathfrak{H}_{\alpha(n)}$ and $\mathfrak{Y}_{\alpha(n-2)}$ are completely symmetric tensor superfields. Then the gauge transformation (5.40) turns into

$$\delta\mathfrak{H}_{\alpha(n)} = \mathrm{i}^n\mathcal{D}_{(\alpha_1}\lambda_{\alpha_2\ldots\alpha_n)} \,, \tag{5.43a}$$

$$\delta\mathfrak{Y}_{\alpha(n-2)} = \frac{\mathrm{i}^n}{n}\mathcal{D}^\beta\lambda_{\beta\alpha_1\ldots\alpha_{n-2}} \ . \tag{5.43b}$$

The supersymmetric gauge-invariant action takes the form

$$\begin{aligned}
\mathbb{S}_{\mathrm{FO}}^{(n)} = \frac{\mathrm{i}^{n+1}}{2^{\lceil(n+1)/2\rceil}} \int \mathrm{d}^{3|2}z\, E\, \Big\{ &\mathfrak{H}^{\beta\alpha(n-1)}\mathcal{D}^\gamma\mathcal{D}_\beta\mathfrak{H}_{\gamma\alpha(n-1)} + 2\mathrm{i}(n-1)\mathfrak{Y}^{\alpha(n-2)}\mathcal{D}^{\beta\gamma}\mathfrak{H}_{\beta\gamma\alpha(n-2)} \\
&+(n-1)\Big(\mathfrak{Y}^{\alpha(n-2)}\mathcal{D}^2\mathfrak{Y}_{\alpha(n-2)} + (-1)^n(n-2)\mathcal{D}_\beta\mathfrak{Y}^{\beta\alpha(n-3)}\mathcal{D}^\gamma\mathfrak{Y}_{\gamma\alpha(n-3)}\Big) \\
&-4\mathcal{S}\mathrm{i}\Big(\mathfrak{H}^{\alpha(n)}\mathfrak{H}_{\alpha(n)} + n(n-1)\mathfrak{Y}^{\alpha(n-2)}\mathfrak{Y}_{\alpha(n-2)}\Big)\Big\} \ .
\end{aligned} \tag{5.44}$$

When n is even, $n = 2s$, this action is the unique gauge-invariant AdS extension of the massless integer superspin action of [17].

5.5.2 Second-order massless actions

The massless half-integer superspin action in AdS is

$$\begin{aligned}
\mathbb{S}_{\mathrm{SO}}^{(2s+1)} = \left(-\frac{1}{2}\right)^s \int \mathrm{d}^{3|2}z\, E\, \Big\{ &-\frac{\mathrm{i}}{2}\mathfrak{H}^{\alpha(2s+1)}\mathbb{Q}\mathfrak{H}_{\alpha(2s+1)} - \frac{\mathrm{i}}{8}\mathcal{D}_\beta\mathfrak{H}^{\beta\alpha(2s)}\mathcal{D}^2\mathcal{D}^\gamma\mathfrak{H}_{\gamma\alpha(2s)} \\
&+\frac{\mathrm{i}}{4}s\mathcal{D}_{\beta\gamma}\mathfrak{H}^{\beta\gamma\alpha(2s-1)}\mathcal{D}^{\rho\lambda}\mathfrak{H}_{\rho\lambda\alpha(2s-1)} - \frac{1}{2}(2s-1)\mathfrak{X}^{\alpha(2s-2)}\mathcal{D}^{\beta\gamma}\mathcal{D}^\delta\mathfrak{H}_{\beta\gamma\delta\alpha(2s-2)} \\
&+\frac{\mathrm{i}}{2}(2s-1)\left[\mathfrak{X}^{\alpha(2s-2)}\mathcal{D}^2\mathfrak{X}_{\alpha(2s-2)} - \frac{s-1}{s}\mathcal{D}_\beta\mathfrak{X}^{\beta\alpha(2s-3)}\mathcal{D}^\gamma\mathfrak{X}_{\gamma\alpha(2s-3)}\right] \\
&+\mathrm{i}s\mathcal{S}\mathfrak{H}^{\beta\alpha(2s)}\mathcal{D}_\beta{}^\gamma\mathfrak{H}_{\gamma\alpha(2s)} + \frac{1}{2}(s+1)\mathcal{S}\mathfrak{H}^{\alpha(2s+1)}\mathcal{D}^2\mathfrak{H}_{\alpha(2s+1)} \\
&+\mathrm{i}s(2s-3)\mathcal{S}^2\mathfrak{H}^{\alpha(2s+1)}\mathfrak{H}_{\alpha(2s+1)} + \frac{(2s-1)\left(s^2-3s-2\right)}{s}\mathcal{S}X^{\alpha(2s-2)}X_{\alpha(2s-2)}\Big\} \,,
\end{aligned} \tag{5.45}$$

where $\mathbb{Q}$ is the quadratic Casimir operator (2.14). One can express $\mathbb{Q}$ in the form

$$\mathbb{Q} = \mathcal{D}^a \mathcal{D}_a - \mathrm{i}\mathcal{S}\mathcal{D}^2 + 2\mathcal{S}\mathcal{D}^{\alpha\beta} M_{\alpha\beta} - 2\mathcal{S}^2 M^{\alpha\beta} M_{\alpha\beta} \, . \tag{5.46}$$

The action (5.45) is invariant under the gauge transformations

$$\delta \mathfrak{H}_{\alpha(2s+1)} = \mathrm{i}\mathcal{D}_{(\alpha_1} \lambda_{\alpha_2 \ldots \alpha_{2s+1})} \, , \tag{5.47a}$$

$$\delta \mathfrak{X}_{\alpha(2s-2)} = \frac{s}{2s+1} \mathcal{D}^{\beta\gamma} \lambda_{\beta\gamma\alpha_1\ldots\alpha_{2s-2}} \, . \tag{5.47b}$$

The action (5.45) is the unique gauge-invariant AdS extension of the massless half-integer superspin action of [17].

5.6 From AdS superspace to AdS space

To conclude this section, we briefly discuss the key aspects of component reduction for supersymmetric field theories formulated in AdS superspace, AdS$^{3|2}$. In general, the action functional of such a theory is given by

$$S = \int \mathrm{d}^{3|2} z \, E \, \mathcal{L} \, , \tag{5.48}$$

where the Lagrangian $\mathcal{L}$ is a scalar superfield. In accordance with the general formalism described in section 6.4 of [44], the isometry transformations of AdS$^{3|2}$ are generated by the Killing vector fields $\xi^A E_A$ which are defined to obey the master equation [65]

$$\left[\xi + \frac{1}{2}\Lambda^{bc} M_{bc}, \mathcal{D}_A\right] = 0 \, , \qquad \xi := \xi^B \mathcal{D}_B = \xi^b \mathcal{D}_b + \xi^\beta \mathcal{D}_\beta \, , \tag{5.49}$$

for some Lorentz superfield parameter $\Lambda^{bc} = -\Lambda^{cb}$. An infinitesimal isometry transformation acts on a tensor superfield T as

$$\delta_\xi T = \left(\xi + \frac{1}{2}\Lambda^{bc} M_{bc}\right) T \, . \tag{5.50}$$

The action (5.48) is invariant under the isometry group of AdS$^{3|2}$.

As shown in [65], the parameters in (5.49) obey the following Killing equations:

$$\mathcal{D}_\alpha \xi_\beta = \frac{1}{2}\Lambda_{\alpha\beta} + \mathcal{S}\xi_{\alpha\beta} = \mathcal{D}_\beta \xi_\alpha \, , \tag{5.51a}$$

$$\mathcal{D}_\beta \xi^{\beta\alpha} + 6\mathrm{i}\xi^\alpha = 0 \, , \qquad\qquad \mathcal{D}_\beta \Lambda^{\beta\alpha} + 12\mathcal{S}\mathrm{i}\xi^\alpha = 0 \, , \tag{5.51b}$$

$$\mathcal{D}_{(\alpha}\xi_{\beta\gamma)} = 0 \, , \qquad\qquad\qquad \mathcal{D}_{(\alpha}\Lambda_{\beta\gamma)} = 0 \, , \tag{5.51c}$$

which imply

$$\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = 0 \, , \tag{5.52a}$$

$$\mathcal{D}^2 \xi_\alpha - 12\mathrm{i}\mathcal{S}\xi_\alpha = 0 \, , \tag{5.52b}$$

$$\mathcal{D}_{\alpha\beta}\xi^\beta + 2\mathcal{S}\xi_\alpha = 0 \, . \tag{5.52c}$$

Equation (5.52a) tells us that ξ_a is a Killing vector, while (5.52c) means that ξ_α is a Killing spinor. The component form of the action (5.48) is computed using the formula (see also [64])

$$S = \frac{1}{4} \int d^3x \, e \left(i\mathcal{D}^2 + 8\mathcal{S} \right) \mathcal{L} \, . \tag{5.53}$$

Here and in what follows, the θ-independent component $T|_{\theta=0}$ of a superfield $T(x,\theta)$ will simply be denoted $T|$. To complete the formalism of component reduction, we only need the following relation

$$\left(\mathcal{D}_a T \right)\big| = \nabla_a T \big| \, , \tag{5.54}$$

where ∇_a is the standard torsion-free covariant derivative of AdS space. Making use of the AdS transformation law $\delta_\xi \mathcal{L} = \xi \mathcal{L}$ in conjunction with the identities (5.51) and (5.52), one may check that the action (5.53) is invariant under arbitrary isometry transformations of the AdS superspace.

6 Supersymmetric higher-spin actions in components

In this section we will describe the component structure of the supersymmetric higher-spin theories introduced in the previous section. Our analysis will be restricted to the flat-superspace case. As in [17], the integration measure[8] for $\mathcal{N} = 1$ Minkowski superspace is defined as follows:

$$\int d^{3|2}z \, L = \frac{i}{4} \int d^3x \, D^2 L \big|_{\theta=0} \, . \tag{6.1}$$

6.1 Superconformal higher-spin action

We start by reducing the superconformal higher-spin action (5.30) to components. The gauge freedom (5.24) can be used to impose a Wess-Zumino gauge

$$H_{\alpha_1 \dots \alpha_n}| = 0 \, , \qquad D^\beta H_{\beta \alpha_1 \dots \alpha_{n-1}}| = 0 \, . \tag{6.2}$$

In this gauge, there remain two independent component fields

$$h_{\alpha_1 \dots \alpha_{n+1}} := i^{n+1} D_{(\alpha_1} H_{\alpha_2 \dots \alpha_{n+1})}\big| \, , \qquad h_{\alpha_1 \dots \alpha_n} := -\frac{i}{4} D^2 H_{\alpha_1 \dots \alpha_n}\big| \, . \tag{6.3}$$

Due to the conservation equation (5.25), the higher-spin super-Cotton tensor (5.29) also has two independent components, which we define as

$$C_{\alpha_1 \dots \alpha_n} := W_{\alpha_1 \dots \alpha_n}\big| \, , \qquad C_{\alpha_1 \dots \alpha_{n+1}} := i^{n+1} D_{(\alpha_1} W_{\alpha_2 \dots \alpha_{n+1})}\big| \, . \tag{6.4}$$

The field strengths $C_{\alpha(n)}$ and $C_{\alpha(n+1)}$ are given in terms of the gauge potentials $h_{\alpha(n)}$ and $h_{\alpha(n+1)}$, respectively, according to eq. (3.35). To prove this statement for $C_{\alpha(n+1)}$, one has to use the identity

$$\binom{n}{2j} + \binom{n}{2j+1} = \binom{n+1}{2j+1} \, . \tag{6.5}$$

---

[8]This definition implies that $\int d^{3|2}z \, V = \int d^3x \, F$, for any scalar superfield $V(x,\theta) = \cdots + i\theta^2 F(x)$.

Reducing the action (5.30) to components gives

$$\mathbb{S}_{\text{SCS}}^{(n)}\left[H_{(n)}\right] = S_{\text{CS}}^{(n)}\left[h_{(n)}\right] + S_{\text{CS}}^{(n+1)}\left[h_{(n+1)}\right] , \tag{6.6}$$

where the conformal higher-spin action $S_{\text{CS}}^{(n)}[h_{(n)}]$ is defined by eq. (3.39).

In the gauge (6.2), the residual gauge freedom is characterised by the conditions

$$D_{(\alpha_1}\lambda_{\alpha_2...\alpha_n)}| = 0 , \qquad D^2\lambda_{\alpha_1...\alpha_{n-1}}| = -2\text{i}\frac{n-1}{n+1}\partial^\beta{}_{(\alpha_1}\lambda_{\alpha_2...\alpha_{n-1})\beta}| . \tag{6.7}$$

At the component level, the remaining independent gauge transformations are generated by $\zeta_{\alpha(n-1)} \propto \lambda_{\alpha(n-1)}|$ and $\zeta_{\alpha(n-2)} \propto \text{i}^n D^\beta \lambda_{\beta\alpha(n-2)}|$.

6.2 Massless first-order model

We now turn to working out the component structure of the first-order model (5.44) in the flat-superspace limit. In Minkowski superspace, the action can be written in the form

$$\mathbb{S}_{\text{FO}}^{(n)} = \frac{\text{i}^{n+1}}{2^{\lceil(n+1)/2\rceil}} \int \text{d}^{3|2}z \left\{ \text{i}H^{\beta\alpha_1...\alpha_{n-1}}\partial_\beta{}^\gamma H_{\gamma\alpha_1...\alpha_{n-1}} + \frac{1}{2}H^{\alpha_1...\alpha_n}D^2 H_{\alpha_1...\alpha_n} \right.$$
$$+2\text{i}(n-1)Y^{\alpha_1...\alpha_{n-2}}\partial^{\beta\gamma} H_{\beta\gamma\alpha_1...\alpha_{n-2}} + (n-1)Y^{\alpha_1...\alpha_{n-2}}D^2 Y_{\alpha_1...\alpha_{n-2}}$$
$$\left.+(-1)^n(n-1)(n-2)D_\beta Y^{\beta\alpha_1...\alpha_{n-3}}D^\gamma Y_{\gamma\alpha_1...\alpha_{n-3}} \right\} . \tag{6.8}$$

It is invariant under the gauge transformations

$$\delta H_{\alpha_1\alpha_2...\alpha_n} = \text{i}^n D_{(\alpha_1}\lambda_{\alpha_2...\alpha_n)} , \tag{6.9a}$$

$$\delta Y_{\alpha_1...\alpha_{n-2}} = \frac{\text{i}^n}{n}D^\beta\lambda_{\beta\alpha_1...\alpha_{n-2}} , \tag{6.9b}$$

with the gauge parameter $\lambda_{\alpha(n-1)}$ being a real unconstrained superfield. When n is even, $n = 2s$, the action (6.8) describes the massless integer superspin model of [17].

The gauge freedom allows us to choose a Wess-Zumino gauge

$$H_{\alpha_1...\alpha_n}| = 0 , \qquad D^\beta H_{\beta\alpha_1...\alpha_{n-1}}| = 0 , \qquad Y_{\alpha_1...\alpha_{n-2}}| = 0 . \tag{6.10}$$

Then, the residual gauge freedom is characterised by the conditions

$$D_{\alpha_1}\lambda_{\alpha_2...\alpha_n}| = 0 , \qquad D^2\lambda_{\alpha_1...\alpha_{n-1}}| = -2\text{i}\frac{n-1}{n+1}\partial^\beta{}_{(\alpha_1}\lambda_{\alpha_2...\alpha_{n-1})\beta}| . \tag{6.11}$$

These conditions imply that there remains only one independent gauge parameter at the component level. We define it as

$$\zeta_{\alpha_1...\alpha_{n-1}}(x) := (-1)^{n+1}\lambda_{\alpha_1...\alpha_{n-1}}| . \tag{6.12}$$

We define the component fields as

$$h_{\alpha_1...\alpha_{n+1}} := \text{i}^{n+1}D_{(\alpha_1}H_{\alpha_2...\alpha_{n+1})}| , \tag{6.13a}$$

$$h_{\alpha_1...\alpha_n} := -\frac{\text{i}}{4}D^2 H_{\alpha_1...\alpha_n}| , \tag{6.13b}$$

$$y_{\alpha_1...\alpha_{n-1}} := \frac{\text{i}^{n+1}}{2n}D_{(\alpha_1}Y_{\alpha_2...\alpha_{n-1})}| , \qquad y_{\alpha_1...\alpha_{n-3}} := \text{i}^{n+1}D^\beta Y_{\beta\alpha_1...\alpha_{n-3}}| , \tag{6.13c}$$

$$Z_{\alpha_1...\alpha_{n-2}} := \frac{\text{i}}{4}D^2 Y_{\alpha_1...\alpha_{n-2}}| . \tag{6.13d}$$

Their gauge transformation laws are

$$\delta h_{\alpha_1 \ldots \alpha_{n+1}} = \partial_{(\alpha_1 \alpha_2} \zeta_{\alpha_3 \ldots \alpha_{n+1})} \,, \tag{6.14a}$$

$$\delta y_{\alpha_1 \ldots \alpha_{n-1}} = \frac{1}{n+1} \partial^\beta{}_{(\alpha_1} \zeta_{\alpha_2 \ldots \alpha_{n-1})\beta} \,, \tag{6.14b}$$

$$\delta y_{\alpha_1 \ldots \alpha_{n-3}} = \partial^{\beta\gamma} \zeta_{\beta\gamma\alpha_1 \ldots \alpha_{n-3}} \,, \tag{6.14c}$$

$$\delta h_{\alpha_1 \ldots \alpha_n} = 0 \,, \tag{6.14d}$$

$$\delta Z_{\alpha_1 \ldots \alpha_{n-2}} = 0 \,. \tag{6.14e}$$

Direct calculations of the component action give

$$
\begin{aligned}
\mathbb{S}_{\mathrm{FO}}^{(n)} = {} & \frac{\mathrm{i}^n}{2^{\lceil n/2 \rceil}} \int \mathrm{d}^3 x \left\{ h^{\alpha_1 \ldots \alpha_n} h_{\alpha_1 \ldots \alpha_n} + Z^{\alpha_1 \ldots \alpha_{n-2}} Z_{\alpha_1 \ldots \alpha_{n-2}} \right\} \\
& + \frac{\mathrm{i}^{n+1}}{2^{\lceil (n+1)/2 \rceil}} \int \mathrm{d}^3 x \left\{ h^{\beta\alpha_1 \ldots \alpha_n} \partial_\beta{}^\gamma h_{\gamma\alpha_1 \ldots \alpha_n} + 2(n-1) y^{\alpha_1 \ldots \alpha_{n-1}} \partial^{\beta\gamma} h_{\beta\gamma\alpha_1 \ldots \alpha_{n-1}} \right. \\
& + 4(n-1) y^{\beta\alpha_1 \ldots \alpha_{n-2}} \partial_\beta{}^\gamma y_{\gamma\alpha_1 \ldots \alpha_{n-2}} + \frac{2(n-2)(n+1)}{n} y^{\alpha_1 \ldots \alpha_{n-3}} \partial^{\beta\gamma} y_{\beta\gamma\alpha_1 \ldots \alpha_{n-3}} \\
& \left. - \frac{(n-2)(n-3)}{n(n-1)} y^{\beta\alpha_1 \ldots \alpha_{n-4}} \partial_\beta{}^\gamma h_{\gamma\alpha_1 \ldots \alpha_{n-4}} \right\} \,. \tag{6.15}
\end{aligned}
$$

The fields $h_{\alpha(n)}$ and $Z_{\alpha(n-2)}$ appear in the action without derivatives. This action can be rewritten in the form

$$
\mathbb{S}_{\mathrm{FO}}^{(n)} = \frac{\mathrm{i}^n}{2^{\lceil n/2 \rceil}} \int \mathrm{d}^3 x \left\{ h^{\alpha(n)} h_{\alpha(n)} + Z^{\alpha(n-2)} Z_{\alpha(n-2)} \right\} + S_{\mathrm{FF}}^{(n+1)} [h_{(n+1)}, y_{(n-1)}, y_{(n-3)}] \,, \tag{6.16}
$$

where $S_{\mathrm{FF}}^{(n+1)}$ is the flat-space version of (4.2), eg. (B.5), with n replaced by $(n+1)$.

6.3 Massive integer superspin action

We are now prepared to read off the component form of a massive integer superspin action that is obtained from (5.39a) in the flat-superspace limit,

$$\mathbb{S}_{\mathrm{massive}}^{(2s)} = \lambda \mathbb{S}_{\mathrm{SCS}}^{(2s)} \left[H_{(2s)} \right] + \mu^{2s-1} \mathbb{S}_{\mathrm{FO}}^{(2s)} \left[H_{(2s)}, Y_{(2s-2)} \right] \,. \tag{6.17}$$

Choosing $n = 2s$ in the component actions (6.6) and (6.16) gives

$$
\begin{aligned}
\mathbb{S}_{\mathrm{massive}}^{(2s)} = {} & \lambda S_{\mathrm{CS}}^{(2s)} \left[h_{(2s)} \right] + \frac{1}{2} \left(-\frac{1}{2} \right)^s \mu^{2s-1} \int \mathrm{d}^3 x \, h^{\alpha(2s)} h_{\alpha(2s)} \\
& + \lambda S_{\mathrm{CS}}^{(2s+1)} \left[h_{(2s+1)} \right] + \mu^{2s-1} S_{\mathrm{FF}}^{(2s+1)} \left[h_{(2s+1)}, y_{(2s-1)}, y_{(2s-3)} \right] \\
& + \frac{1}{2} \left(-\frac{1}{2} \right)^s \mu^{2s-1} \int \mathrm{d}^3 x \, Z^{\alpha(2s-2)} Z_{\alpha(2s-2)} \,. \tag{6.18}
\end{aligned}
$$

It is seen that the $Z_{\alpha(2s-2)}$ field appears only in the third line of (6.18) and without derivatives, and thus $Z_{\alpha(2s-2)}$ is an auxiliary field. Next, the expression in the second line of (6.18) constitutes the massive gauge-invariant spin-$(s + \frac{1}{2})$ action (4.6). The two terms

in the first line of (6.18) involve the $h_{\alpha(2s)}$ field. Unlike $S_{\text{CS}}^{(2s)}[h_{(2s)}]$, the second mass-like term is not gauge invariant. However, the action

$$\widetilde{S}_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)}\left[h_{(2s)}\right] + \frac{1}{2}\left(-\frac{1}{2}\right)^s \mu^{2s-1} \int \mathrm{d}^3x\, h^{\alpha(2s)} h_{\alpha(2s)} \tag{6.19}$$

does describe a massive spin-s field on-shell. Indeed, the equation of motion is

$$\lambda C_{\alpha(2s)} + \mu^{2s-1} h_{\alpha(2s)} = 0 . \tag{6.20}$$

Since $C_{\alpha(2s)}$ is divergenceless, eq. (3.38), the equation of motion implies that $h_{\alpha(2s)}$ is divergenceless, eq. (4.35). As a consequence, $C_{\alpha(2s)}$ takes the simple form given by (3.41a), and the above equation of motion turns into (compare with eq. (4.36))

$$\lambda \Box^{s-1} \partial^{\beta}{}_{\alpha_1} h_{\alpha_2 \ldots \alpha_{2s}\beta} + \mu^{2s-1} h_{\alpha(2s)} = 0 , \tag{6.21}$$

which implies

$$\left(\Box^{2s-1} - \left(m^2\right)^{2s-1}\right) h_{\alpha(2s)} = 0 , \qquad m := \left|\frac{\mu}{\lambda^{1/(2s-1)}}\right| , \tag{6.22}$$

and should be compared with (4.39). Since the polynomial equation $z^{2s-1} - 1 = 0$ has only one real root, $z = 1$, we conclude that (6.22) leads to the Klein-Gordon equation (4.22). As a result, the higher-derivative equation (6.21) reduces to the first-order one, eq. (4.23).

The above component analysis clearly demonstrates that the model (6.17) describes a single massive supermultiplet subject to the equations (2.6a) and (2.6b) with $n = 2s$ on the mass shell. The superfield proof was provided in [17].

6.4 Massless second-order model

Finally we consider the massless half-integer superspin model describe by the action [17]

$$\begin{aligned}
\mathbb{S}_{\text{SO}}^{(2s+1)} = \left(-\frac{1}{2}\right)^s \int \mathrm{d}^{3|2}z \Bigg\{ &-\frac{\mathrm{i}}{2} H^{\alpha(2s+1)} \Box H_{\alpha(2s+1)} - \frac{\mathrm{i}}{8} D_\beta H^{\beta\alpha(2s)} D^2 D^\gamma H_{\gamma\alpha(2s)} \\
&+\frac{\mathrm{i}}{4} s \partial_{\beta\gamma} H^{\beta\gamma\alpha(2s-1)} \partial^{\rho\lambda} H_{\rho\lambda\alpha(2s-1)} - \frac{1}{2}(2s-1) X^{\alpha(2s-2)} \partial^{\beta\gamma} D^\delta H_{\beta\gamma\delta\alpha(2s-2)} \\
&+\frac{\mathrm{i}}{2}(2s-1)\left[X^{\alpha(2s-2)} D^2 X_{\alpha(2s-2)} - \frac{s-1}{s} D_\beta X^{\beta\alpha(2s-3)} D^\gamma X_{\gamma\alpha(2s-3)} \right] \Bigg\} .
\end{aligned} \tag{6.23}$$

It is invariant under the following gauge transformations

$$\delta H_{\alpha(2s+1)} = \mathrm{i} D_{(\alpha_1} \lambda_{\alpha_2 \ldots \alpha_{2s+1})} , \tag{6.24a}$$

$$\delta X_{\alpha(2s-2)} = \frac{s}{2s+1} \partial^{\beta\gamma} \lambda_{\beta\gamma\alpha_1 \ldots \alpha_{2s-2}} . \tag{6.24b}$$

The gauge freedom allows us to choose a Wess-Zumino gauge of the form

$$H_{\alpha(2s+1)}\big| = 0 , \qquad D^\beta H_{\beta\alpha(2s)}\big| = 0 . \tag{6.25}$$

To preserve these conditions, the residual gauge symmetry has to be constrained by

$$D_{(\alpha_1}\lambda_{\alpha_2...\alpha_{2s+1})}\big| = 0\,, \qquad D^2\lambda_{\alpha(2s)}\big| = -\frac{2\mathrm{i}s}{s+1}\partial^\beta_{\ (\alpha_1}\lambda_{\alpha_2...\alpha_{2s})\beta}\big|\,. \tag{6.26}$$

Under the gauge conditions imposed, the independent component fields of $H_{\alpha(2s+1)}$ can be chosen as

$$h_{\alpha(2s+2)} := -D_{(\alpha_1}H_{\alpha_2...\alpha_{2s+2})}\big|\,, \qquad h_{\alpha(2s+1)} := \frac{\mathrm{i}}{4}D^2 H_{\alpha(2s+1)}\big|\,. \tag{6.27}$$

The remaining independent component parameters of $\lambda_{\alpha(2s)}$ can be chosen as

$$\zeta_{\alpha(2s)} := \lambda_{\alpha(2s)}\big|\,, \qquad \xi_{\alpha(2s-1)} := -\mathrm{i}\frac{s}{2s+1}D^\beta\lambda_{\beta\alpha(2s-1)}\big|\,. \tag{6.28}$$

The gauge transformation laws of $h_{\alpha(2s+2)}$ and $h_{\alpha(2s+1)}$ can be shown to be

$$\delta_\zeta h_{\alpha(2s+2)} = \partial_{(\alpha_1\alpha_2}\zeta_{\alpha_3...\alpha_{2s+2})}\,, \tag{6.29a}$$

$$\delta_\xi h_{\alpha(2s+1)} = \partial_{(\alpha_1\alpha_2}\xi_{\alpha_3...\alpha_{2s+1})}\,. \tag{6.29b}$$

We now define the component fields of $X_{\alpha(2s-2)}$ as follows:

$$y_{\alpha(2s-2)} := 2X_{\alpha(2s-2)}\big|\,, \tag{6.30a}$$

$$y_{\alpha(2s-1)} := -\frac{\mathrm{i}}{2}D_{(\alpha_1}X_{\alpha_2...\alpha_{2s-1})}\big|\,, \qquad y_{\alpha(2s-3)} := -\mathrm{i}D^\beta X_{\beta\alpha(2s-3)}\big|\,, \tag{6.30b}$$

$$F_{\alpha(2s-2)} := \frac{\mathrm{i}}{4}X_{\alpha(2s-2)}\big|\,. \tag{6.30c}$$

The gauge transformation laws of $y_{\alpha(2s-2)}$, $y_{\alpha(2s-1)}$ and $y_{\alpha(2s-3)}$ are as follows:

$$\delta_\zeta y_{\alpha(2s-2)} = \frac{2s}{2s+1}\partial^{\beta\gamma}\zeta_{\beta\gamma\alpha(2s-2)}\,, \tag{6.31a}$$

$$\delta_\xi y_{\alpha(2s-1)} = \frac{1}{2s+1}\partial^\beta_{\ (\alpha_1}\xi_{\alpha_2...\alpha_{2s-1})\beta}\,, \tag{6.31b}$$

$$\delta_\xi y_{\alpha(2s-3)} = \partial^{\beta\gamma}\xi_{\beta\gamma\alpha(2s-3)}\,. \tag{6.31c}$$

In principle, we do not need to derive the gauge transformation of $F_{\alpha(2s-2)}$ since this field turns out to be auxiliary.

The bosonic transformation laws (6.29a) and (6.31a) correspond to the massless spin-$(s+1)$ action $S_\mathrm{F}^{(2s+2)}$ defined by eq. (4.28). The fermionic transformation laws (6.29b), (6.31b) and (6.31c) correspond to the massless spin-$(s+\frac{1}{2})$ action $S_\mathrm{FF}^{(2s+1)}$ defined by eq. (4.7).

The component action follows from (6.23) by making use of the reduction rule (6.1). Direct calculations lead to the following bosonic Lagrangian:

$$2(-2)^{s+1}L_\mathrm{bos} = h^{\alpha(2s+2)}\Box h_{\alpha(2s+2)} - \frac{1}{2}(s+1)\partial_{\gamma(2)}h^{\gamma(2)\alpha(2s)}\partial^{\beta(2)}h_{\alpha(2s)\beta(2)}$$
$$-\frac{1}{2}(2s-1)y^{\alpha(2s-2)}\partial^{\beta(2)}\partial^{\gamma(2)}h_{\alpha(2s-2)\beta(2)\gamma(2)} - \frac{(s+1)(2s-1)}{2s}y^{\alpha(2s-4)}\Box y_{\alpha(2s-4)}$$
$$-4s(2s-1)\left[(s+1)F^{\alpha(2s-2)}F_{\alpha(2s-2)} - \frac{s-1}{2s}F^{\alpha(2s-2)}\partial^\beta_{\ (\alpha_1}y_{\alpha_2...\alpha_{2s-2})\beta}\right].$$
$$\tag{6.32}$$

Eliminating the auxiliary field $F_{\alpha(2s-2)}$ leads to

$$
2(-2)^{s+1} L_{\text{bos}} = h^{\alpha(2s+2)} \Box h_{\alpha(2s+2)} - \frac{1}{2}(s+1) \partial_{\gamma(2)} h^{\gamma(2)\alpha(2s)} \partial^{\beta(2)} h_{\alpha(2s)\beta(2)}
$$
$$
- \frac{1}{2}(2s-1) \Bigg[y^{\alpha(2s-2)} \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-2)\beta(2)\gamma(2)} + \frac{2}{s+1} y^{\alpha(2s-2)} \Box y_{\alpha(2s-2)}
$$
$$
+ \frac{(s-1)(2s-3)}{4(s+1)} \partial_{\gamma(2)} y^{\gamma(2)\alpha(2s-4)} \partial^{\beta(2)} y_{\beta(2)\alpha(2s-4)} \Bigg]. \tag{6.33}
$$

This Lagrangian corresponds to the massless spin-$(s+1)$ action $S_{\text{F}}^{(2s+2)}$ obtained from (4.28) by the replacement $s \to s+1$. The fermionic sector of the component action proves to coincide with the massless spin-$(s+\frac{1}{2})$ action, $S_{\text{FF}}^{(2s+1)}[h_{(2s+1)}, y_{(2s-1)}, y_{(2s-3)}]$.

6.5 Massive half-integer superspin action

We now have all of the ingredients at our disposal to read off the component form of the massive half-integer superspin action that is obtained from (5.39b) in the flat-superspace limit,

$$
\mathbb{S}_{\text{massive}}^{(2s+1)} = \lambda \mathbb{S}_{\text{SCS}}^{(2s+1)} \left[H_{(2s+1)} \right] + \mu^{2s-1} \mathbb{S}_{\text{SO}}^{(2s+1)} \left[H_{(2s+1)}, X_{(2s-2)} \right]
$$
$$
\approx \lambda S_{\text{CS}}^{(2s+2)} \left[h_{(2s+2)} \right] + \mu^{2s-1} S_{\text{F}}^{(2s+2)} \left[h_{(2s+2)}, y_{(2s-2)} \right]
$$
$$
+ \lambda S_{\text{CS}}^{(2s+1)} \left[h_{(2s+1)} \right] + \mu^{2s-1} S_{\text{FF}}^{(2s+1)} \left[h_{(2s+1)}, y_{(2s-1)}, y_{(2s-3)} \right]. \tag{6.34}
$$

Here the symbol '$\approx$' indicates that the auxiliary field has been eliminated.

The explicit structure of the component action (6.34) clearly demonstrates that the model

$$
\mathbb{S}_{\text{massive}}^{(2s+1)} = \lambda \mathbb{S}_{\text{SCS}}^{(2s+1)}[H_{(2s+1)}] + \mu^{2s-1} \mathbb{S}_{\text{SO}}^{(2s+1)}[H_{(2s+1)}, X_{(2s-2)}] \tag{6.35}
$$

describes a single massive supermultiplet subject to the equations (2.6a) and (2.6b) with $n = 2s+1$ on the mass shell. The superfield proof was provided in [17].

7 Concluding comments

All massive higher-spin theories in Minkowski space, which have been presented in this paper, were extracted from off-shell supersymmetric field theories. As shown in section 6, all the theories studied in section 4 are contained at the component level in the $\mathcal{N} = 1$ supersymmetric massive higher-spin theories proposed in [17]. The latter models were obtained from the $\mathcal{N} = 2$ supersymmetric massive higher-spin theories of [16] by carrying out the $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction. Furthermore, the off-shell structure of the massless 3D $\mathcal{N} = 2$ supersymmetric higher-spin actions of [16], which constitute one of the two sectors of the $\mathcal{N} = 2$ massive actions, were designed following the pattern of the gauge off-shell formulations for massless 4D $\mathcal{N} = 1$ higher-spin supermultiplets developed in the early 1990s [66, 67].

Our supersymmetric massive higher-spin theories, which are formulated in AdS$^{3|2}$ superspace and are described by the actions (5.39a) and (5.39b), contain two different models

for a massive integer-spin field in AdS at the component level. One of them is the gauge-invariant model (4.5a). The second model is described by the action

$$\widetilde{S}^{(2s)}_{\text{massive}} = \lambda S^{(2s)}_{\text{CS}}\left[\mathfrak{h}_{(2s)}\right] + \frac{1}{2}\left(-\frac{1}{2}\right)^s \mu^{2s-1} \int \mathrm{d}^3x \, e \, \mathfrak{h}^{\alpha(2s)}\mathfrak{h}_{\alpha(2s)}\,, \tag{7.1}$$

which does not possess gauge invariance and which is the AdS uplift of the model (6.19). The action (7.1) leads to the equation of motion

$$\lambda\mathfrak{C}_{\alpha(2s)} + \mu^{2s-1}\mathfrak{h}_{\alpha(2s)} = 0 \quad \implies \quad \nabla^{\beta\gamma}\mathfrak{h}_{\beta\gamma\alpha(2s-2)} = 0\,. \tag{7.2}$$

The action (7.1) can be turned into a gauge-invariant one by making use of the Stückelberg trick. An interesting feature of the model (7.1) is that it is well-defined in an arbitrary conformally flat space.

The models (4.5b) and (7.1) are higher-spin analogues of the two well-known equivalent models for a massive vector field (see [68, 69] and references therein) with Lagrangians

$$\mathcal{L}_{\text{T}} = -\frac{1}{4}F^{ab}F_{ab} + \frac{m}{4}\varepsilon^{abc}V_aF_{bc}\,, \qquad F_{ab} = \partial_aV_b - \partial_bV_a\,, \tag{7.3a}$$

$$\mathcal{L}_{\text{SD}} = \frac{1}{2}f^af_a - \frac{1}{2m}\varepsilon^{abc}f_a\partial_bf_c\,. \tag{7.3b}$$

New duality transformations were introduced in [15] for theories formulated in terms of the linearised higher-spin Cotton tensors $C_{\alpha(n)}$ and their $\mathcal{N}=1$ supersymmetric counterparts $W_{\alpha(n)}$. These duality transformations can readily be generalised to arbitrary conformally flat backgrounds, with $C_{\alpha(n)}$ and $W_{\alpha(n)}$ replaced with $\mathfrak{C}_{\alpha(n)}$ and $\mathfrak{W}_{\alpha(n)}$, respectively.

In the present paper, we have been unable to obtain closed-form expressions for $\mathfrak{C}_{\alpha(n)}$ and $\mathfrak{W}_{\alpha(n)}$ in terms of the covariant derivatives of AdS (super)space for arbitrary n. These are interesting open problems.

The field strengths $\mathfrak{C}_{\alpha(n)}$ and $\mathfrak{W}_{\alpha(n)}$ are the higher-spin extensions of the linearised Cotton and super-Cotton tensors, respectively. The actions (3.32) and (5.21) are the higher-spin extensions of the linearised actions for conformal gravity and supergravity, respectively. An intriguing question is: do nonlinear higher-spin extensions exist? Within the approach initiated in [70, 71], Linander and Nilsson [72] constructed the full nonlinear spin-3 Cotton equation coupled to spin-2. They made use of the frame field description and the Chern-Simons formulation for 3D (super)conformal field theory due to Fradkin and Linetsky [33]. The construction of the nonlinear spin-3 Cotton tensor [72] requires an elimination of certain auxiliary fields, a procedure that becomes extremely difficult for $s > 3$. However, so far this is unexplored territory. There exist nonlinear formulations for the massless spin-3 theory [73, 74], and the generalisation from $s = 3$ to $s > 3$ is shown in [74] to be trivial within the formulation developed. These results indicate that it is possible to construct a nonlinear topologically massive higher-spin field theory. The fundamental results by Prokushkin and Vasiliev [75, 76] should be essential of course. Any attempt to construct a supersymmetric interacting higher-spin theory should inevitably be an extension of the conformal superspace approach [36, 77].

It should be pointed out that the problem of constructing topologically massive higher-spin theories was considered in [78, 79]. However, the nonlinear action proposed possesses only a restricted gauge freedom in the presence of the Lagrange multiplier β that enforces the torsion-free conditions on the spin connections. Alternative approaches are worth pursuing.

So far we have discussed $\mathcal{N} = 1$ topologically massive supergravity and its higher spin extensions. The off-shell formulations for $\mathcal{N}$-extended topologically massive supergravity theories were presented in [80, 81] for $\mathcal{N} = 2$, in [82] for $\mathcal{N} = 3$, and in [82, 83] for the $\mathcal{N} = 4$ case. In all of these theories, the action functional is a sum of two terms, one of which is the action for pure $\mathcal{N}$-extended supergravity (Poincaré or anti-de Sitter) and the other is the action for $\mathcal{N}$-extended conformal supergravity. The off-shell actions for $\mathcal{N}$-extended supergravity theories in three dimensions were given in [59] for $\mathcal{N} = 1$, [60, 84] for $\mathcal{N} = 2$ and [60] for the cases $\mathcal{N} = 3$, 4. The off-shell actions for $\mathcal{N}$-extended conformal supergravity were given in [6] for $\mathcal{N} = 1$, [85] for $\mathcal{N} = 2$, [77] for $\mathcal{N} = 3$, 4, 5, and in [86, 87] for the $\mathcal{N} = 6$ case. Refs. [77, 87] made use of the off-shell formulation for $\mathcal{N}$-extended conformal supergravity proposed in [36]. The on-shell formulation for $\mathcal{N}$-extended conformal supergravity with $\mathcal{N} > 2$ was given in [88]. On-shell approaches to $\mathcal{N}$-extended topologically massive supergravity theories with $4 \leq \mathcal{N} \leq 8$ were presented in [89–93]. It would be interesting to formulate topologically massive higher spin supermultiplets for $\mathcal{N} > 2$.

Note added in proof: the equations (2.3) for massive fields in AdS$_3$ may be realised as equations of motion in the following model

$$S_{\text{massive}}^{(n)}[\mathfrak{h}_{(n)}] = \frac{\mathrm{i}^n}{2^{\lfloor n/2 \rfloor + 1}} \frac{\lambda}{\mu} \int \mathrm{d}^3 x\, e\, \mathfrak{C}^{\alpha(n)}(\mathfrak{h}) \Big\{ \mu \delta^\beta{}_{\alpha_1} + \nabla^\beta{}_{\alpha_1} \Big\} \mathfrak{h}_{\alpha_2 \ldots \alpha_n \beta} \ ,$$

which is invariant under the gauge transformations (3.15) in AdS$_3$. It is $\mathfrak{C}_{\alpha(n)}(\mathfrak{h})$ which plays the role of $\phi_{\alpha(n)}$. The equations (2.12) for massive superfields in AdS$^{3|2}$ may be realised as equations of motion in the following model

$$\mathbb{S}_{\text{massive}}^{(n)}[\mathfrak{H}_{(n)}] = -\frac{\mathrm{i}^n}{2^{\lfloor n/2 \rfloor + 1}} \frac{\lambda}{\mu} \int \mathrm{d}^{3|2} z\, E\, \mathfrak{W}^{\alpha(n)}(\mathfrak{H}) \Big\{ \mu + \frac{\mathrm{i}}{2} \mathcal{D}^2 \Big\} \mathfrak{H}_{\alpha(n)} \ ,$$

which is invariant under the gauge transformations (5.12) in AdS$^{3|2}$. It is $\mathfrak{W}_{\alpha(n)}(\mathfrak{H})$ which plays the role of $T_{\alpha(n)}$. These models, which become (super)conformal in the $\mu \to \infty$ limit, may be viewed as generalisations of the flat-space bosonic constructions of [19, 94].

A Notation and conventions

We follow the notation and conventions adopted in [60]. In particular, the Minkowski metric is $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$. The spinor indices are raised and lowered using the $\mathrm{SL}(2, \mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^{\alpha}_{\beta} \tag{A.1}$$

by the standard rule:

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\psi_{\beta}, \qquad \psi_{\alpha} = \varepsilon_{\alpha\beta}\psi^{\beta}. \tag{A.2}$$

We make use of real gamma-matrices, $\gamma_a := \left((\gamma_a)_\alpha{}^\beta\right)$, which obey the algebra

$$\gamma_a \gamma_b = \eta_{ab}\mathbb{1} + \varepsilon_{abc}\gamma^c, \tag{A.3}$$

where the Levi-Civita tensor is normalised as $\varepsilon^{012} = -\varepsilon_{012} = 1$. The completeness relation for the gamma-matrices reads

$$(\gamma^a)_{\alpha\beta}(\gamma_a)^{\rho\sigma} = -\left(\delta^\rho_\alpha \delta^\sigma_\beta + \delta^\sigma_\alpha \delta^\rho_\beta\right). \tag{A.4}$$

Here the symmetric matrices $(\gamma_a)^{\alpha\beta}$ and $(\gamma_a)_{\alpha\beta}$ are obtained from $\gamma_a = (\gamma_a)_\alpha{}^\beta$ by the rules (A.2). Some useful relations involving γ-matrices are

$$\varepsilon_{abc}(\gamma^b)_{\alpha\beta}(\gamma^c)_{\gamma\delta} = \varepsilon_{\gamma(\alpha}(\gamma_a)_{\beta)\delta} + \varepsilon_{\delta(\alpha}(\gamma_a)_{\beta)\gamma}, \tag{A.5a}$$

$$\mathrm{tr}[\gamma_a\gamma_b\gamma_c\gamma_d] = 2\eta_{ab}\eta_{cd} - 2\eta_{ac}\eta_{db} + 2\eta_{ad}\eta_{bc}. \tag{A.5b}$$

Given a three-vector x_a, it can be equivalently described by a symmetric second-rank spinor $x_{\alpha\beta}$ defined as

$$x_{\alpha\beta} := (\gamma^a)_{\alpha\beta}x_a = x_{\beta\alpha}, \qquad x_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}x_{\alpha\beta}. \tag{A.6}$$

In the 3D case, an antisymmetric tensor $F_{ab} = -F_{ba}$ is Hodge-dual to a three-vector F_a, specifically

$$F_a = \frac{1}{2}\varepsilon_{abc}F^{bc}, \qquad F_{ab} = -\varepsilon_{abc}F^c. \tag{A.7}$$

Then, the symmetric spinor $F_{\alpha\beta} = F_{\beta\alpha}$, which is associated with F_a, can equivalently be defined in terms of F_{ab}:

$$F_{\alpha\beta} := (\gamma^a)_{\alpha\beta}F_a = \frac{1}{2}(\gamma^a)_{\alpha\beta}\varepsilon_{abc}F^{bc}. \tag{A.8}$$

These three algebraic objects, F_a, F_{ab} and $F_{\alpha\beta}$, are in one-to-one correspondence to each other, $F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha\beta}$. The corresponding inner products are related to each other as follows:

$$-F^aG_a = \frac{1}{2}F^{ab}G_{ab} = \frac{1}{2}F^{\alpha\beta}G_{\alpha\beta}. \tag{A.9}$$

The Lorentz generators with two vector indices $(M_{ab} = -M_{ba})$, one vector index (M_a) and two spinor indices $(M_{\alpha\beta} = M_{\beta\alpha})$ are related to each other by the rules: $M_a = \frac{1}{2}\varepsilon_{abc}M^{bc}$ and $M_{\alpha\beta} = (\gamma^a)_{\alpha\beta}M_a$. These generators act on a vector V_c and a spinor Ψ_γ as follows:

$$M_{ab}V_c = 2\eta_{c[a}V_{b]}, \qquad M_{\alpha\beta}\Psi_\gamma = \varepsilon_{\gamma(\alpha}\Psi_{\beta)}. \tag{A.10}$$

B First-order higher-spin model

In this appendix we review the first-order higher-spin model in Minkowski space used by Tyutin and Vasiliev [18] in their formulation for massive higher-spin fields. It is realised in terms of a reducible field $\boldsymbol{h}_{b,\alpha_1...\alpha_{n-2}} = \boldsymbol{h}_{b,(\alpha_1...\alpha_{n-2})}$ which is defined modulo gauge transformations of the form

$$\delta\boldsymbol{h}_{b,\alpha_1...\alpha_{n-2}} = \partial_b\xi_{\alpha_1...\alpha_{n-2}}\,, \qquad \xi_{\alpha_1...\alpha_{n-2}} = \xi_{(\alpha_1...\alpha_{n-2})}\,. \tag{B.1}$$

The structure of this transformation implies that the following action

$$S_{\text{FF}}^{(n)} = -\frac{\mathrm{i}^n}{2^{\lfloor n/2\rfloor}}\int \mathrm{d}^3x\,\varepsilon^{bcd}\boldsymbol{h}_{b,}{}^{\alpha_1...\alpha_{n-2}}\partial_c\boldsymbol{h}_{d,\alpha_1...\alpha_{n-2}} \tag{B.2}$$

is gauge invariant.

The field $\boldsymbol{h}_{\beta\gamma,\alpha_1...\alpha_{n-2}} := (\gamma^b)_{\beta\gamma}\boldsymbol{h}_{b,\alpha_1...\alpha_{n-2}}$ contains three irreducible $\text{SL}(2,\mathbb{R})$ fields that we define as follows:

$$h_{\alpha_1...\alpha_n} := \boldsymbol{h}_{(\alpha_1\alpha_2,\alpha_3...\alpha_n)}\,, \tag{B.3a}$$

$$y_{\alpha_1...\alpha_{n-2}} := \frac{1}{n}\boldsymbol{h}^\beta{}_{(\alpha_1,\alpha_2...\alpha_{n-2})\beta}\,, \tag{B.3b}$$

$$y_{\alpha_1...\alpha_{n-4}} := \boldsymbol{h}^{\beta\gamma,}{}_{\beta\gamma\alpha_1...\alpha_{n-4}}\,. \tag{B.3c}$$

In accordance with (B.1), the gauge transformation laws of these fields are

$$\delta h_{\alpha_1...\alpha_n} = \partial_{(\alpha_1\alpha_2}\xi_{\alpha_3...\alpha_n)}\,, \tag{B.4a}$$

$$\delta y_{\alpha_1...\alpha_{n-2}} = \frac{1}{n}\partial^\beta{}_{(\alpha_1}\xi_{\alpha_2...\alpha_{n-2})\beta}\,, \tag{B.4b}$$

$$\delta y_{\alpha_1...\alpha_{n-4}} = \partial^{\beta\gamma}\xi_{\beta\gamma\alpha_1...\alpha_{n-4}}\,. \tag{B.4c}$$

The action (B.2) turns into

$$S_{\text{FF}}^{(n)} = \frac{\mathrm{i}^n}{2^{\lfloor n/2\rfloor+1}}\int \mathrm{d}^3x \left\{ h^{\beta\alpha_1...\alpha_{n-1}}\partial_\beta{}^\gamma h_{\gamma\alpha_1...\alpha_{n-1}} + 2(n-2)y^{\alpha_1...\alpha_{n-2}}\partial^{\beta\gamma}h_{\beta\gamma\alpha_1...\alpha_{n-2}} \right.$$

$$+ 4(n-2)y^{\beta\alpha_1...\alpha_{n-3}}\partial_\beta{}^\gamma y_{\gamma\alpha_1...\alpha_{n-3}} + 2\frac{n(n-3)}{n-1}y^{\alpha_1...\alpha_{n-4}}\partial^{\beta\gamma}y_{\beta\gamma\alpha_1...\alpha_{n-4}}$$

$$\left. -\frac{(n-3)(n-4)}{(n-1)(n-2)}y^{\beta\alpha_1...\alpha_{n-3}}\partial_\beta{}^\gamma y_{\gamma\alpha_1...\alpha_{n-3}} \right\}\,. \tag{B.5}$$

This is the flat-space limit of the first-order action (4.2). When n is odd, $n = 2s+1$, the functional $S_{\text{FF}}^{(2s+1)}$ coincides with plain 4D → 3D dimensional reduction of the Fang-Fronsdal action [13].

C Higher-spin Cotton tensor as a descendent of gauge-invariant field strengths

The Cotton tensor is defined in terms of the Ricci tensor according to (3.3). The latter determines the equations of motion corresponding to the Einstein-Hilbert action. In this appendix we show that analogous properties hold for the linearised higher-spin Cotton tensor defined by eq. (3.35).

C.1 The first-order case

We begin by demonstrating that the higher-spin Cotton tensor (3.35) is a descendant of gauge-invariant field strengths which determine the equations of motion in the first-order model (B.5). Associated with the dynamical variables $h_{\alpha(n)}, y_{\alpha(n-2)}$ and $y_{\alpha(n-4)}$ are the following gauge-invariant field strengths:

$$F_{\alpha(n)} := \partial_{(\alpha_1}{}^{\beta} h_{\alpha_2...\alpha_n)\beta} - (n-2)\partial_{(\alpha_1\alpha_2} y_{\alpha_3...\alpha_n)}, \tag{C.1a}$$

$$G_{\alpha(n-2)} := \partial^{\beta(2)} h_{\alpha(n-2)\beta(2)} + 4\partial_{(\alpha_1}{}^{\beta} y_{\alpha_2...\alpha_{n-2})\beta} - \frac{n(n-3)}{(n-1)(n-2)}\partial_{(\alpha_1\alpha_2} y_{\alpha_3...\alpha_{n-2})}, \tag{C.1b}$$

$$H_{\alpha(n-4)} := (n-2)\partial^{\beta(2)} y_{\alpha(n-4)\beta(2)} - \frac{n-4}{n}\partial_{(\alpha_1}{}^{\beta} y_{\alpha_2...\alpha_{n-4})\beta}. \tag{C.1c}$$

The equations of motion corresponding to (B.5) are the conditions that these field strengths vanish. Furthermore, the gauge symmetry implies that $F_{\alpha(n)}$, $G_{\alpha(n-2)}$ and $H_{\alpha(n-4)}$ are related to each other via the Noether identity

$$0 = \partial^{\beta(2)} F_{\alpha(n-2)\beta(2)} - \frac{n-2}{n}\partial_{(\alpha_1}{}^{\beta} G_{\alpha_2...\alpha_{n-2})\beta} + \frac{n(n-3)}{(n-1)(n-2)}\partial_{(\alpha_1\alpha_2} H_{\alpha_3...\alpha_{n-2})}. \tag{C.2}$$

We claim that the Cotton tensor $C_{\alpha(n)}(h)$ may be expressed as $C_{\alpha(n)} = (\mathcal{A}_1 F)_{\alpha(n)} + (\mathcal{A}_2 G)_{\alpha(n)} + (\mathcal{A}_3 H)_{\alpha(n)}$, for some linear differential operators $\mathcal{A}_i$ of order $n-2$. A suitable ansatz for such an expression is

$$
\begin{aligned}
C_{\alpha(n)} = &\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_j \Box^j \partial_{(\alpha_1}{}^{\beta_1} \cdots \partial_{\alpha_{n-2j-2}}{}^{\beta_{n-2j-2}} F_{\alpha_{n-2j-1}...\alpha_n)\beta_1...\beta_{n-2j-2}} \\
&+ \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 2} b_k \Box^k \partial_{(\alpha_1}{}^{\beta_1} \cdots \partial_{\alpha_{n-2k-3}}{}^{\beta_{n-2k-3}} \partial_{\alpha_{n-2k-2}\alpha_{n-2k-1}} G_{\alpha_{n-2k}...\alpha_n)\beta_1...\beta_{n-2k-3}} \\
&+ \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 2} c_l \Box^l \partial_{(\alpha_1}{}^{\beta_1} \cdots \partial_{\alpha_{n-2l-4}}{}^{\beta_{n-2l-4}} \partial_{\alpha_{n-2l-3}\alpha_{n-2l-2}} \\
&\quad \times \partial_{\alpha_{n-2l-1}\alpha_{n-2l}} H_{\alpha_{n-2l+1}...\alpha_n)\beta_1...\beta_{n-2l-4}}
\end{aligned}
\tag{C.3}
$$

for some coefficients a_j, b_k and c_l. It may be shown that the values of these coefficients are not unique and that there are $\lfloor \frac{n}{2} \rfloor - 1$ free parameters. For example, when $n = 5$ one may show that the general solution is

$$
\begin{pmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{18}{5}c_0 \\ \frac{1}{2} - \frac{18}{5}c_0 \\ \frac{9}{80} - \frac{36}{25}c_0 \\ \frac{3}{80} - \frac{18}{25}c_0 \\ c_0 \end{pmatrix}.
$$

We may use this freedom to completely eliminate the $\lfloor \frac{n}{2} \rfloor - 1$ coefficients c_l so that only the field strengths $F_{\alpha(n)}$ and $G_{\alpha(n-2)}$ appear in (C.3). This fixes the solution uniquely to

$$a_j = \frac{1}{2^{n-2}} \frac{(n-1)}{(2j+1)} \binom{n-2}{2j} \qquad \text{for } 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 , \qquad \text{(C.4a)}$$

$$b_k = \frac{1}{2^{n-1}} \frac{(n-2)^2}{n(2k+1)} \binom{n-3}{2k} \qquad \text{for } 0 \leq k \leq \left\lceil \frac{n}{2} \right\rceil - 2 , \qquad \text{(C.4b)}$$

$$c_l = 0 \qquad \text{for } 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 . \qquad \text{(C.4c)}$$

The fact that there are $\lfloor \frac{n}{2} \rfloor - 1$ free parameters may be understood as a consequence of the Noether identity (C.2). To see this, observe that, in principle, we may use (C.2) to replace all occurrences of $H_{\alpha(n-4)}$ with $F_{\alpha(n)}$ and $G_{\alpha(n-2)}$ in the ansatz (C.3). There will then be only two sets of independent coefficients, say $\tilde{a}_j$ and $\tilde{b}_k$, whose unique values coincide with those of (C.4a) and (C.4b).

C.2 The second-order case

We now consider the flat-space version of the second-order model (4.4). It is described by the real fields $h_{\alpha(n)}$ and $h_{\alpha(n-4)}$. Associated with these two fields are the following gauge-invariant field strengths:

$$F_{\alpha(n)} = \Box h_{\alpha(n)} + \frac{n}{4} \partial^{\beta(2)} \partial_{(\alpha_1 \alpha_2} h_{\alpha_3 \dots \alpha_n) \beta(2)} - \frac{n-3}{4} \partial_{(\alpha_1 \alpha_2} \partial_{\alpha_3 \alpha_4} y_{\alpha_5 \dots \alpha_n)} , \qquad \text{(C.5a)}$$

$$G_{\alpha(n-4)} = \partial^{\beta(2)} \partial^{\beta(2)} h_{\alpha(n-4)\beta(4)} + \frac{8}{n} \Box y_{\alpha(n-4)} - \frac{(n-4)(n-5)}{n(n-2)} \partial^{\beta(2)} \partial_{(\alpha_1 \alpha_2} y_{\alpha_3 \dots \alpha_{n-4}) \beta(2)} . \qquad \text{(C.5b)}$$

The equations of motion for the model are $F_{\alpha(n)} = 0$ and $G_{\alpha(n-4)} = 0$. The two field strengths are related by the Noether identity

$$\partial^{\beta(2)} F_{\alpha(n-2)\beta(2)} = \frac{(n-3)(n-2)}{4(n-1)} \partial_{(\alpha_1 \alpha_2} G_{\alpha_3 \dots \alpha_{n-2})} . \qquad \text{(C.6)}$$

We claim that the Cotton tensor $C_{\alpha(n)}(h)$ may be written as $C_{\alpha(n)} = (\mathcal{A}_1 F)_{\alpha(n)} + (\mathcal{A}_2 G)_{\alpha(n)}$ where the $\mathcal{A}_i$ are linear differential operators of order $n-3$. A suitable ansatz for such an expression is

$$C_{\alpha(n)} = \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 2} a_j \Box^j \partial_{(\alpha_1}{}^{\beta_1} \cdots \partial_{\alpha_{n-2j-3}}{}^{\beta_{n-2j-3}} F_{\alpha_{n-2j-2} \dots \alpha_n) \beta_1 \dots \beta_{n-2j-3}} \qquad \text{(C.7)}$$

$$+ \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 3} b_k \Box^k \partial_{(\alpha_1}{}^{\beta_1} \cdots \partial_{\alpha_{n-2k-5}}{}^{\beta_{n-2k-5}}$$

$$\times \partial_{\alpha_{n-2k-4}\alpha_{n-2k-3}} \partial_{\alpha_{n-2k-2}\alpha_{n-2k-1}} G_{\alpha_{n-2k} \dots \alpha_n) \beta_1 \dots \beta_{n-2k-5}} ,$$

for some coefficients a_j and b_k. It may be shown that the choice of these coefficients is not unique, and that there are $\lceil \frac{n}{2} \rceil - 2$ free parameters. For example, when $n = 6$ one may show that the general solution is

$$\begin{pmatrix} a_0 \\ a_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} - \frac{10}{3} b_0 \\ \frac{3}{8} + \frac{10}{3} b_0 \\ b_0 \end{pmatrix} \, .$$

We can use this freedom to completely eliminate the $\lceil \frac{n}{2} \rceil - 2$ coefficients b_k so that only the top field strength, $F_{\alpha(n)}$, appears in (C.7). This gives the unique solution

$$a_j = (j + 1) \frac{\binom{n-3}{2j}}{\binom{2j+3}{3}} \frac{n(n-1)}{3 \cdot 2^{n-2}} \qquad \text{for } 0 \le j \le \left\lceil \frac{n}{2} \right\rceil - 2 \,, \qquad \text{(C.8a)}$$

$$b_k = 0 \qquad \text{for } 0 \le k \le \left\lceil \frac{n}{2} \right\rceil - 3 \,. \qquad \text{(C.8b)}$$

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