Emergent gravity from relatively local Hamiltonians and a possible resolution of the black hole information puzzle

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ABSTRACT: In this paper, we study a possibility where gravity and time emerge from quantum matter. Within the Hilbert space of matter fields defined on a spatial manifold, we consider a sub-Hilbert space spanned by states which are parameterized by spatial metric. In those states, metric is introduced as a collective variable that controls local structures of entanglement. The underlying matter fields endow the states labeled by metric with an unambiguous inner product. Then we construct a Hamiltonian for the matter fields that is an endomorphism of the sub-Hilbert space, thereby inducing a quantum Hamiltonian of the metric. It is shown that there exists a matter Hamiltonian that induces the general relativity in the semi-classical field theory limit. Although the Hamiltonian is not local in the absolute sense, it has a weaker notion of locality, called *relative locality*: the range of interactions is set by the entanglement present in target states on which the Hamiltonian acts. In general, normalizable states are not invariant under the transformations generated by the Hamiltonian. As a result, a physical state spontaneously breaks the Hamiltonian constraint, and picks a moment of time. The subsequent flow of time can be understood as a Goldstone mode associated with the broken symmetry. The construction allows one to study dynamics of gravity from the perspective of matter fields. The Hawking radiation corresponds to a unitary evolution where entanglement across horizon is gradually transferred from color degrees of freedom to singlet degrees of freedom. The underlying quantum states remain pure as evaporating black holes keep entanglement with early Hawking radiations in the singlet sector which is not captured by the Bekenstein-Hawking entropy.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

There have been many efforts to understand gravity as an emergent phenomenon from various angles [1–20]. The anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [21–23] is a concrete example of emergent gravity, where a gravitational theory in the bulk emerges from a quantum field theory defined at the boundary of the anti-de Sitter space. However, the AdS/CFT correspondence does not directly apply to our universe which does not have a spatial boundary. For our universe, it seems more natural that the bulk spacetime emerges from a theory defined at a temporal boundary in the past or future. The program of the de Sitter space/conformal field theory (dS/CFT) correspondence aims to make this scenario concrete with the guidance from the AdS/CFT correspondence [24–26].

In this paper, we study the possibility in which time and gravity emerge from quantum matter, employing a more microscopic perspective built from the quantum renormalization group (RG) [27, 28]. Quantum RG provides a prescription to construct holographic duals for general quantum field theories based on the intuition that the emergent space direction in the bulk corresponds to a length scale [29–35]. The basic object in quantum RG is wavefunctions defined in the space of couplings. Instead of specifying a quantum field theory in terms of classical values of all couplings allowed by symmetry, a theory is represented as a wavefunction defined in a much smaller subspace of couplings. The subspace is chosen so that all symmetry-allowed operators can be constructed as composites of those operators sourced by the couplings in the subspace. Then, general theories can be represented as coherent linear superpositions of theories defined in the subspace. As a result, the couplings in the subspace are promoted to fluctuating variables. Metric, which sources the energy-momentum tensor, also becomes a dynamical variable whose fluctuations account for composite operators made of the energy-momentum tensor. While the Wilsonian RG flow is a classical flow defined in the full space of couplings, the same exact RG flow can be represented as a quantum evolution of the wavefunction defined in the subspace. The classical flow of the Wilsonian RG is replaced by a sum of all possible RG paths defined in the subspace of couplings. The weight for each RG path is determined by an action which includes a dynamical gravity [36].

In order to realize an emergent time in a manner that a space-like direction emerges in quantum RG, we consider wavefunctions of ordinary quantum matters defined on a space manifold instead of wavefunctions of couplings defined on a spacetime manifold. Metric in Lorentzian quantum field theories determines connectivity of spacetime by setting the strength of derivative terms in local actions. In quantum states of matter fields we consider here, metric with the Euclidean signature is introduced as a collective variable that controls entanglement of matter fields in the space manifold. Namely, local actions for Lorentzian quantum field theories are replaced with short-range entangled states of quantum matters. The metric in quantum states of matter plays the role of a variational parameter that sets the notion of locality (‘short-rangeness’) in how matter fields are entangled in space. More specifically, we consider a set of wavefunctions of matter fields parameterized by Riemannian metric. The space spanned by those states forms a sub-Hilbert space in the full Hilbert space of the matter field.

With wavefunctions of couplings replaced by wavefunctions of matter fields, we consider a unitary evolution of the quantum states. Although an unitary evolution is not same as RG flow, one may still view the former as a coarse graining process in which information accessible to local observers decreases in time through scrambling. In particular, we consider an evolution generated by a Hamiltonian which maps the sub-Hilbert space into the sub-Hilbert space. Since the sub-Hilbert space is parameterized by spatial metric, the Hamiltonian of the matter fields induces a quantum Hamiltonian of the metric. In this way, one can induce quantum theories of metric from matter fields. The main goal of this paper is to address the following questions:

1. Can a matter Hamiltonian induce a quantum theory that becomes Einstein's general relativity at long distances in the classical limit?
2. Is the matter Hamiltonian that gives rise to the general relativity local?
3. What is the nature of time in the emergent gravity?
4. How does a quantum state of matter maintain its purity under an evolution that is dual to a black hole evaporation?

The short answers to these questions are

1. Yes, one can engineer a matter Hamiltonian whose induced dynamics agrees with the general relativity in the semi-classical field theory limit.
2. No, the Hamiltonian is not local in the usual sense. However, it possesses a relative locality in that the range of interactions depends on states on which the Hamiltonian acts.
3. Time arises as a Goldstone mode associated with a spontaneous breaking of the symmetry generated by the Hamiltonian constraint.
4. During black hole evaporations, quantum states stay pure by transferring entanglement from color degrees of freedom to singlet sectors.

The rest of the paper gives long answers to the questions. Here is an outline that may serve as a summary of the paper.

In section 2, we sketch the main idea that is used in the explicit examples constructed in the following sections. This section constitutes a conceptual guide for the rest of the paper.

Section 3 is a warm-up which discusses a toy model from which a minisuperspace quantum cosmology emerges. Although there is no extended space in the toy model, it still contains the essential idea on how time emerges. In section 3.1, we introduce a set of quantum states for N variables. The states in the set are parameterized by two collective variables. Those states labeled by the collective variables span a sub-Hilbert space in the full Hilbert space of the N fundamental variables. Throughout the section, we will focus on the sub-Hilbert space. It becomes the Hilbert space for the induced cosmology in which the two collective variables become the scale factor of a universe and a scalar field,

respectively. The inner product between states in the sub-Hilbert space, which is inherited from the one defined in the full Hilbert space, provides a notion of distance between states with different collective variables. With increasing N , two states with different collective variables become increasingly orthogonal.

In section 3.2, we construct a Hamiltonian for the N variables that is an endomorphism of the sub-Hilbert space, that is, an operator that maps the sub-Hilbert space into the sub-Hilbert space. Through the evolution generated by the Hamiltonian, a state with a definite collective variable evolves into a linear superposition of states with different collective variables in the sub-Hilbert space. The evolution is naturally described as a unitary quantum evolution of wavefunctions defined in the space of the collective variables. Therefore, one can identify a Hamiltonian for the collective variables induced from the matter Hamiltonian. The key result of this subsection is that there exists a Hamiltonian for the N variables which gives rise to a minisuperspace Wheeler-DeWitt Hamiltonian for the collective variables.

In section 3.3, we address the issue of time. In general relativity which includes minisuperspace cosmology, Hamiltonian is a constraint which generates time reparameterization transformations. It is usually assumed that ‘physical states’ are the ones that are invariant under diffeomorphism and are annihilated by the Hamiltonian constraint. This gives rise to the problem of time because ‘physical states’ are stationary, and no change is generated under Hamiltonian evolution. In the present theory of induced quantum cosmology, the problem of time is avoided because there is no normalizable state that satisfies the Hamiltonian constraint in the sub-Hilbert space. This is shown by diagonalizing the matter Hamiltonian numerically. Nonetheless, there exist semi-classical states which are normalizable. They satisfy the Hamiltonian constraint to the leading order in the large N limit, yet break the constraint beyond the leading order. While the semi-classical states do not satisfy the Hamiltonian constraint exactly, they are legitimate states as quantum states of matter. A semi-classical state ‘picks’ a moment of time spontaneously because it is forced to have a finite norm. Non-trivial time evolution of the semi-classical states can be understood as Goldstone modes associated with the weak spontaneous symmetry breaking. In the large N limit, the time evolution of semi-classical states coincides with the classical minisuperspace cosmology.

In the following section, we generalize the discussion on the emergent minisuperspace cosmology to a fully fledged gravity in $(3+1)$ -dimensions. The starting point is an $N \times N$ Hermitian matrix field defined on a three-dimensional spatial manifold. In section 4.1.1, we define a Hilbert space for the induced gravity from the matter field. The full Hilbert space of the matter field is spanned by eigenstates of the matrix field. Within the full Hilbert space, we focus on a sub-Hilbert space spanned by gaussian wavefunctions. Those gaussian wavefunctions, which are singlet under a $SU(N)$ internal symmetry, are parameterized by a Riemannian metric and a scalar field. Namely, we consider a set of $SU(N)$ invariant wavefunctions in which metric and a scalar field enters as collective variables (equivalently, variational parameters) that control how the matter field is entangled in space. General states within the sub-Hilbert space are given by linear superpositions of states with different collective variables.

Section 4.1.2 is devoted to the inner product. The inner product in the full Hilbert space is defined in terms of normal modes of an elliptic differential operator associated with a fiducial Riemannian metric. Although a fiducial metric is introduced to define the inner product in the full Hilbert space, the fiducial metric decouples in the inner product between normalized states in the sub-Hilbert space. Based on this, we show the following properties of the inner product. First, the induced inner product in the sub-Hilbert space is invariant under diffeomorphisms of the collective variables. Second, we show that two states in the sub-Hilbert space are orthogonal unless the two have metrics that give same local proper volume. Third, even for states with same local proper volume, the overlap decays exponentially as the difference in the collective variables increases in the large N limit. This is explicitly shown for states whose metrics are close to the flat Euclidean metric.

In section 4.2, we show that the metric, as a variational parameter, controls the number of degrees of freedom that generate entanglement in space. As quantum states of matter field defined in continuum, the number of degrees of freedom per unit coordinate volume is infinite. Nonetheless, the wavefunctions in the sub-Hilbert space have a short-distance cut-off scale which regularizes the amount of entanglement. It is shown that the von Neumann entanglement entropy of states in the sub-Hilbert space has two contributions. One is the color entanglement entropy which is generated by the matter field in a classical configuration of the collective variables. The color entanglement entropy of a region in space obeys the area law, where the area is measured with the metric associated with the state in the unit of the short-distance cut-off. The amount of color entanglement a region in the manifold can support is not fixed. Rather it is a dynamical quantity that is determined by the metric. With increasing proper volume, the entanglement entropy increases accordingly. In the presence of fluctuations of the collective variables, correlations in the fluctuations give rise to an additional contribution to the von Neumann entanglement entropy, called singlet entanglement entropy. These two contributions can be approximately separated in semi-classical states where fluctuations of the collective variables are small.

In section 4.3, we consider endomorphisms of the sub-Hilbert space. We show that there exist Hermitian operators for the matter field which induce the momentum density operator and a regularized Wheeler-DeWitt Hamiltonian density operator for the collective variables. Those operators for the matter fields generate an evolution of the collective variables once the lapse and the shift are fixed. At long distances and in the large N limit, the evolution coincides with the time evolution of the classical Einstein's gravity in a fixed gauge. The matter Hamiltonian that gives rise to the general relativity has no absolute locality because the Hamiltonian, as a quantum operator, contains operators with arbitrarily long ranges. Nonetheless, it is relatively local in that the range of interactions that survive when applied to a state is limited by the amount of entanglement present in the state. The notion of locality in the Hamiltonian is determined relative to states to which the Hamiltonian is applied.

The fact that the general relativity can emerge from matter fields provides an opportunity to examine the black hole information puzzle from the perspective of the underlying matter field. In section 5, we consider a formation and evaporation of a black hole in the induced theory of gravity. By construction, the time evolution is unitary. The discussion

is centered on how purity of a quantum state can be in principle maintained during an evolution. As a black hole evaporates, the color entanglement entropy across the horizon, which is identified as the Bekenstein-Hawking entropy [37, 38], decreases in time. On the other hand, the singlet entanglement entropy increases because Hawking radiation is emitted in the singlet sector. As a result, entanglement is gradually transferred from the color sector to the singlet sector. This leads to a ‘neutralization’ of entanglement entropy. The full quantum states remain pure as black holes keep the entanglement with early Hawking radiation in the singlet sector which is not captured by the Bekenstein-Hawking entropy. The failure of the Bekenstein-Hawking entropy to account for all available states in a black hole is attributed to a lack of equilibrium. It is argued that localization can arise dynamically because both states and Hamiltonian effectively flow under the time evolution generated by the relatively local Hamiltonian. In section 6, we conclude with a summary and discussions on open questions.

2 The main idea

In this section, we sketch the main idea of the paper that is summarized in figure 1. Our starting point is a Hilbert space of matter fields defined on a spatial manifold with a fixed topology. Within the full Hilbert space ($\mathbb{H}$) of the matter fields, we consider a sub-Hilbert space ($\mathcal{V}$) that is spanned by a set of short-range entangled states. $\mathbb{H}$ is equipped with an inner product, which gives a unique inner product in $\mathcal{V}$. Each basis state of $\mathcal{V}$ is associated with a set of collective variables. Among the collective variables is spatial metric. In this discussion of the conceptual idea, we focus on the metric, ignoring other collective variables. A spatial metric is assigned to each basis state such that the von Neumann entanglement entropy of a basis state for any region in space is proportional to the proper area of the boundary of the region measured with the metric for the basis state (see figure 2). Two states which support different amounts of entanglement in a region are assigned to have different metrics so that they give different proper sizes of the region in proportion to the entanglement. In this sense, the metric can be viewed as a collective variable which controls the amount of entanglement in quantum states of the matter fields. A general state in $\mathcal{V}$ can be expressed as a linear superposition of the basis states, $|\chi\rangle = \int Dg_{\mu\nu} |g_{\mu\nu}\rangle \chi(g_{\mu\nu})$. Here $|g_{\mu\nu}\rangle$ is the basis state associated with $g_{\mu\nu}(x)$, $\chi(g_{\mu\nu})$ is a wavefunction defined in the space of spatial metric, and $Dg_{\mu\nu}$ is a measure that is defined based on the inner product in $\mathcal{V}$. In the limit that the number of matter fields is large, two states with different metrics become orthogonal. This way, the sub-Hilbert space of the matter field is identified as a Hilbert space for spatial metric.

Next we study dynamics of the matter field within the sub-Hilbert space by considering a Hamiltonian $\hat{H}$ that maps $\mathcal{V}$ into $\mathcal{V}$. If an initial state $|\chi\rangle$ is prepared to be in $\mathcal{V}$, $e^{-it\hat{H}} |\chi\rangle$ can be written as a linear superposition of $\{|g_{\mu\nu}(x)\rangle\}$. Because the unitary time evolution is a linear map acting on the sub-Hilbert space, one can identify a differential operator $\mathcal{H}\left(g_{\mu\nu}, \frac{\partial}{\partial g_{\mu\nu}}\right)$ that acts on the wavefunction of metric such that $e^{-i\hat{H}t} |\chi\rangle = \int Dg_{\mu\nu} |g_{\mu\nu}\rangle e^{-i\mathcal{H}\left(g_{\mu\nu}, \frac{\partial}{\partial g_{\mu\nu}}\right)t} \chi(g_{\mu\nu})$. We identify $\mathcal{H}\left(g_{\mu\nu}, \frac{\partial}{\partial g_{\mu\nu}}\right)$ as an induced Hamiltonian

Figure 1. In the full Hilbert space $\mathbb{H}$ of matter fields, a sub-Hilbert space $\mathcal{V}$ is spanned by a set of basis vectors, where each basis vector is associated with a spatial metric. A general state in $\mathcal{V}$ can be written as a linear superposition of the basis vectors, $|\chi\rangle = \int Dg_{\mu\nu} |g_{\mu\nu}\rangle \chi(g_{\mu\nu})$, where $\chi(g_{\mu\nu})$ is a wavefunction defined in the space of spatial metric. An endomorphic Hamiltonian $\hat{H}$ generates a map from $\mathcal{V}$ into $\mathcal{V}$. Therefore, $|\chi'\rangle = e^{-i\hat{H}t}|\chi\rangle$ can be also written as $|\chi'\rangle = \int Dg_{\mu\nu} |g_{\mu\nu}\rangle \chi'(g_{\mu\nu})$. The linear map between $\chi(g_{\mu\nu})$ and $\chi'(g_{\mu\nu})$ can be written as $\chi'(g_{\mu\nu}) = \exp\left[-i t \mathcal{H}\left(g_{\mu\nu}, \frac{\partial}{\partial g_{\mu\nu}}\right)\right] \chi(g_{\mu\nu})$, where $\mathcal{H}\left(g_{\mu\nu}, \frac{\partial}{\partial g_{\mu\nu}}\right)$ is identified as an induced Hamiltonian for the metric.

Figure 2. A metric is associated with each basis state in $\mathcal{V}$ such that the von Neumann entanglement entropy of a region A is proportional to the proper area of ∂A measured with respect to the metric.

of the metric. By requiring that the induced Hamiltonian becomes the Wheeler-DeWitt Hamiltonian in the classical limit, we construct a matter Hamiltonian that induces the general relativity.

The matter Hamiltonian that induces the general relativity turns out to be a non-local Hamiltonian. Yet, it has a weaker notion of locality called relative locality. While the Hamiltonian is non-local as a quantum operator, the range of interaction that survives when applied to a state is determined by the entanglement present in the target state. There is an intuitive way to understand this. The Hamiltonian that induces the general relativity can not be local because one can not have a local gradient term in the Hamiltonian without

introducing a fixed background.¹ Therefore, any background independent theory can not have an absolute notion of locality. On the other hand, the general relativity is reduced to a local effective field theory when fluctuations of the metric are weak. Small fluctuations of metric propagate on top of a saddle point configuration that is dynamically determined, and the notion of locality in the effective field theory is determined by the saddle point metric. In the present construction, the metric is determined by the entanglement present in quantum matter. Therefore, the notion of locality should be set by the amount of entanglement present in states of matter fields.

In the following two sections, we work out examples which elucidate the idea outlined in this section. In section 3, we provide a toy example of quantum mechanical system in zero space dimension. In this model, the spatial metric is reduced to one scale factor, and a minisuperspace quantum cosmology emerges. In section 4, we generalize the construction to a fully fledged gravity in three space dimension. In section 5, we discuss possible implications of the induced gravity for the black hole information puzzle.

3 Emergent minisuperspace cosmology

Based on the general idea outlined in the previous section, in this section we consider a quantum mechanical system of N variables from which a minisuperspace quantum cosmology emerges. We start by defining a sub-Hilbert space of the N variables which becomes the Hilbert space for two collective variables: a scale factor and a scalar. The sub-Hilbert space is spanned by a set of basis vectors labeled by the two collective variables. After examining the kinematic structure of the sub-Hilbert space, we explicitly construct a Hamiltonian of the N variables that induces the Wheeler-DeWitt Hamiltonian of the minisuperspace cosmology for the scale factor and the scalar. Finally, we address the problem of time in quantum cosmology. By numerically diagonalizing the matter Hamiltonian, we show that there is no normalizable state that satisfies the Hamiltonian constraint within the sub-Hilbert space. From this, we conclude that the requirement that a physical state should have a finite norm forces quantum states spontaneously break the Hamiltonian constraint, and the subsequent time evolution arises as a Goldstone mode associated with the broken symmetry.

3.1 Hilbert space

We consider a system of N compact variables whose Hilbert space is spanned by $\{|\phi\rangle \mid 0 \leq \phi_a < 2\pi, a = 1, 2, \dots, N\}$ with the inner product $\langle \phi' | \phi \rangle = \prod_{a=1}^N \delta(\phi'_a - \phi_a)$. Within the full Hilbert space, we consider states which are invariant under permutations of the N flavors. Furthermore, we focus on wavefunctions that depend only on the first harmonics of ϕ_a through $O_c = \sum_{a=1}^N \cos \phi_a$ and $O_s = \sum_{a=1}^N \sin \phi_a$. Wavefunctions that depend on O_c and O_s can be spanned by two-parameter family of ‘plane waves’, $e^{i(k_c O_c + k_s O_s)}$. We denote the sources for O_c and O_s as $k_c = e^{3\alpha} \cosh \frac{3}{\sqrt{2}}\sigma$ and $k_s = e^{3\alpha} \sinh \frac{3}{\sqrt{2}}\sigma$ to label the basis states

¹One alternative possibility for a ultra-local Hamiltonian has been considered in ref. [28].

Figure 3. A tensor representation of the wavefunction $\Psi(\phi; \alpha, \sigma)$. Once the collective variables α, σ are fixed, the tensor defines a wavefunction for ϕ_a .

in terms of two non-compact variables (α, σ) as

$$|\alpha, \sigma\rangle = \int_0^{2\pi} \prod_{a=1}^N d\phi_a |\phi\rangle \Psi(\phi; \alpha, \sigma), \quad (3.1)$$

where the wavefunction is written as

$$\Psi(\phi; \alpha, \sigma) = \frac{1}{(2\pi)^{N/2}} e^{ie^{3\alpha} \left[\cosh \frac{3}{\sqrt{2}} \sigma O_c + \sinh \frac{3}{\sqrt{2}} \sigma O_s \right]}. \quad (3.2)$$

Here the normalization is chosen such that $\langle \alpha, \sigma | \alpha, \sigma \rangle = 1$. $\mathcal{V}$ denotes the sub-Hilbert space spanned by

$$\left\{ |\alpha, \sigma\rangle \mid -\infty < \alpha < \infty, -\infty < \sigma < \infty \right\}. \quad (3.3)$$

The wavefunction $\Psi(\phi; \alpha, \sigma)$ can be viewed as a tensor which depends on ϕ, α, σ as is shown in figure 3. If we considered more general wavefunctions that include higher harmonics, we would have to introduce more collective variables to span the extended Hilbert space. However, we focus on the two-parameter family of basis states in our discussion to keep the form of wavefunction simple. We are mainly interested in constructing a simple example of emergent cosmology to demonstrate the proof of principle discussed in section 2.

The overlap between states in $\mathcal{V}$ is given by

$$\begin{aligned} \langle \alpha', \sigma' | \alpha, \sigma \rangle &= \left[\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(e^{3\alpha} \cosh \frac{3}{\sqrt{2}} \sigma - e^{3\alpha'} \cosh \frac{3}{\sqrt{2}} \sigma') \cos \phi + i(e^{3\alpha} \sinh \frac{3}{\sqrt{2}} \sigma - e^{3\alpha'} \sinh \frac{3}{\sqrt{2}} \sigma') \sin \phi} \right]^N \\ &= [I_0(iR_{\alpha', \sigma'; \alpha, \sigma})]^N, \end{aligned} \quad (3.4)$$

where $I_0(x)$ is the modified Bessel function, and

$$R_{\alpha', \sigma'; \alpha, \sigma} = \sqrt{\left(e^{3\alpha} \cosh \frac{3}{\sqrt{2}} \sigma - e^{3\alpha'} \cosh \frac{3}{\sqrt{2}} \sigma' \right)^2 + \left(e^{3\alpha} \sinh \frac{3}{\sqrt{2}} \sigma - e^{3\alpha'} \sinh \frac{3}{\sqrt{2}} \sigma' \right)^2} \quad (3.5)$$

is a measure of distance between two states. For $N \gg 1$, the Bessel function decays exponentially in R as $[I_0(iR)]^N \approx e^{-N \frac{R^2}{4}}$. Roughly speaking, two states with $R_{\alpha', \sigma'; \alpha, \sigma}$ greater than $N^{-1/2}$ are orthogonal.

In the large N limit, the overlap is proportional to the delta function upto a multiplicative factor,

$$\lim_{N \rightarrow \infty} \langle \alpha', \sigma' | \alpha, \sigma \rangle = \mu(\alpha, \sigma)^{-1} \delta(\alpha' - \alpha) \delta(\sigma' - \sigma), \quad (3.6)$$

where

$$\mu(\alpha, \sigma) = \frac{9Ne^{6\alpha}}{4\pi\sqrt{2}}. \quad (3.7)$$

The overlap defines a natural measure in the space of α, σ ,

$$D\alpha D\sigma \equiv \mu(\alpha, \sigma) d\alpha d\sigma, \quad (3.8)$$

which guarantees that

$$\lim_{N \rightarrow \infty} \int D\alpha D\sigma \langle \alpha', \sigma' | \alpha, \sigma \rangle = 1 \quad (3.9)$$

for any α' and σ' .

3.2 Hamiltonian for induced quantum cosmology

We emphasize that ϕ_a 's, which we call ‘matter fields’, are the only fundamental degrees of freedom. $\{\alpha, \sigma\}$ parameterizes collective modes of the matter fields. If a Hamiltonian for the matter fields generates a dynamical flow within $\mathcal{V}$, the dynamical flow can be understood as an evolution generated by an induced Hamiltonian for the collective variables. In the following, we construct a Hamiltonian for the matter fields which induces a minisuperspace quantum cosmology for the collective variables, where α and σ become the scale factor of a flat universe and a scalar field, respectively.

We first look for a Hamiltonian $\hat{\mathcal{H}}(\alpha, \sigma)$ whose action on $|\alpha, \sigma\rangle$ induces

$$\hat{\mathcal{H}}(\alpha, \sigma) |\alpha, \sigma\rangle = h_{\alpha, \sigma} |\alpha, \sigma\rangle, \quad (3.10)$$

where $h_{\alpha, \sigma}$ is a Wheeler-DeWitt differential operator for the minisuperspace cosmology of a flat three-dimensional universe,

$$h_{\alpha, \sigma} = \frac{1}{2} \left[\kappa^2 e^{-3\alpha} \left(\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \sigma^2} \right) + \frac{e^{3\alpha}}{\kappa^2} V(\sigma) \right] \quad (3.11)$$

with a potential $V(\sigma)$. It is not difficult to construct a Hamiltonian that does the job for a given state $|\alpha, \sigma\rangle$. We try the standard quadratic kinetic term with a potential term,

$$\hat{\mathcal{H}}(\alpha, \sigma) = \frac{1}{2\sqrt{N}} \left[\frac{e^{-3\alpha}}{2} \sum_a \hat{\pi}_a^2 + e^{3\alpha} U(\hat{\phi}, \alpha, \sigma) \right], \quad (3.12)$$

where $\hat{\pi}_a$ is the conjugate momentum for $\hat{\phi}_a$ with the commutation relation $[\hat{\pi}_a, \hat{\phi}_b] = -i\delta_{a,b}$. Requiring that $\hat{\mathcal{H}}(\alpha, \sigma) |\alpha, \sigma\rangle$ is in $\mathcal{V}$ fixes $U(\hat{\phi}, \alpha, \sigma)$ to be

$$\begin{aligned} U(\hat{\phi}, \alpha, \sigma) = & -\frac{1}{2} \sum_a \left(\cos 2\hat{\phi}_a + 1 \right) - \frac{\sinh 3\sqrt{2}\sigma}{2} \sum_{a \neq b} \cos \hat{\phi}_a \sin \hat{\phi}_b \\ & - \frac{\cosh 3\sqrt{2}\sigma + 3}{4} \sum_{a \neq b} \cos \hat{\phi}_a \cos \hat{\phi}_b - \frac{\cosh 3\sqrt{2}\sigma - 3}{4} \sum_{a \neq b} \sin \hat{\phi}_a \sin \hat{\phi}_b. \end{aligned} \quad (3.13)$$

Applying $\hat{\mathcal{H}}(\alpha, \sigma)$ to $|\alpha, \sigma\rangle$, one indeed obtains eq. (3.10) with $V(\sigma) = \frac{1}{18} (\cosh 3\sqrt{2}\sigma - 1)$ and $\kappa^2 = \frac{1}{9\sqrt{N}}$. Eq. (3.10) implies that the action of $\hat{\mathcal{H}}(\alpha, \sigma)$ on $|\alpha, \sigma\rangle$ is equivalent to a differential operator acting on the collective variables. This proves that $\hat{\mathcal{H}}(\alpha, \sigma) |\alpha, \sigma\rangle$ is in

$\mathcal{V}$. The induced differential operator $h_{\alpha,\sigma}$ is $O(\sqrt{N})$ because $\frac{\partial}{\partial\alpha} \sim \frac{\partial}{\partial\sigma} \sim \sqrt{N}$ to the leading order in the large N limit.²

$\hat{\mathcal{H}}(\alpha, \sigma)$, as a quantum operator of the matter fields, depends on α, σ . This means that $\hat{\mathcal{H}}(\alpha, \sigma)$ can not generate the desired dynamics for general states in $\mathcal{V}$, that is, $\hat{\mathcal{H}}(\alpha, \sigma)|\alpha', \sigma'\rangle \neq h_{\alpha', \sigma'}|\alpha', \sigma'\rangle$ if $(\alpha', \sigma') \neq (\alpha, \sigma)$. In order for the Hamiltonian flow to stay within $\mathcal{V}$ for arbitrary initial states in $\mathcal{V}$, one effectively has to choose different Hamiltonians for states with different collective variables. Such a ‘state-dependent’ operator can be realized through a linear map in the large N limit because states with different collective variables are orthogonal in the large N limit as is shown in eq. (3.6). Based on this intuition, we consider the following Hamiltonian,

$$\hat{H} = \frac{1}{2} \int D\alpha D\sigma \left[\hat{\mathcal{H}}(\alpha, \sigma) |\alpha, \sigma\rangle \langle \alpha, \sigma| + |\alpha, \sigma\rangle \langle \alpha, \sigma| \hat{\mathcal{H}}^\dagger(\alpha, \sigma) \right]. \quad (3.14)$$

It is noted that $\hat{P}_{\alpha,\sigma} \equiv |\alpha, \sigma\rangle \langle \alpha, \sigma|$ becomes orthogonal projection operators in the large N limit. $\hat{H}$ is made of the projection operator and $\hat{\mathcal{H}}(\alpha, \sigma)$. In the large N limit, the projection operator first picks a state with a definite $\{\alpha, \sigma\}$ before $\hat{\mathcal{H}}(\alpha, \sigma)$ is applied.³ This way, the operator tailored for each set of collective variables is applied to the state with the corresponding collective variables. Because $\hat{\mathcal{H}}(\alpha, \sigma)$ depends on α, σ , one may regard $\hat{H}$ as a state dependent operator whose action on the Hilbert space depends on states it acts on [39]. However, it is still a linear operator.⁴

Eq. (3.10) allows us to write $\hat{H}$ as

$$\hat{H} = \int D\alpha D\sigma \left(\tilde{H}_{\alpha,\sigma} |\alpha, \sigma\rangle \right) \langle \alpha, \sigma|, \quad (3.15)$$

where $\tilde{H}_{\alpha,\sigma} = \frac{1}{2} (h_{\alpha,\sigma} + h_{\alpha,\sigma}^\dagger)$ and $h_{\alpha,\sigma}^\dagger$ is the Hermitian conjugate of $h_{\alpha,\sigma}$ defined from

$$\int D\alpha D\sigma f^*(\alpha, \sigma) [h_{\alpha,\sigma} g(\alpha, \sigma)] = \int D\alpha D\sigma \left[h_{\alpha,\sigma}^\dagger f(\alpha, \sigma) \right]^* g(\alpha, \sigma). \quad (3.16)$$

It is noted that $h_{\alpha,\sigma}^\dagger$ differs from $h_{\alpha,\sigma}$ only by terms that are at most $O(1)$.

For general states constructed from linear superpositions of $|\alpha, \sigma\rangle$,

$$|\chi\rangle = \int D\alpha D\sigma |\alpha, \sigma\rangle \chi(\alpha, \sigma), \quad (3.17)$$

$\hat{H}$ acts as

$$\hat{H}|\chi\rangle = \int D\alpha D\sigma |\alpha, \sigma\rangle (H_{\alpha,\sigma} \chi(\alpha, \sigma)), \quad (3.18)$$

where

$$H_{\alpha,\sigma} \chi(\alpha, \sigma) \equiv \tilde{H}_{\alpha,\sigma} \int D\alpha' D\sigma' \langle \alpha, \sigma | \alpha', \sigma' \rangle \chi(\alpha', \sigma'). \quad (3.19)$$

²This can be understood from $\int \prod_a d\phi_a \Psi^*(\phi; \alpha, \sigma) \frac{\partial^2}{\partial \alpha^2} \Psi(\phi; \alpha, \sigma) \sim \int \prod_a d\phi_a \Psi^*(\phi; \alpha, \sigma) \frac{\partial^2}{\partial \sigma^2} \Psi(\phi; \alpha, \sigma) \sim N$.

³For a finite N , there is a smearing of $\delta\alpha, \delta\sigma \sim \frac{1}{\sqrt{N}}$.

⁴For example, $\hat{H} = \hat{\tau}_x \frac{1+\hat{\tau}_z}{2} + \hat{\tau}_y \frac{1-\hat{\tau}_z}{2}$ acts as $\hat{\tau}_x$ and $\hat{\tau}_y$ on $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

Figure 4. The filled box represents an operator that acts on a quantum state of the matter field, $|\chi\rangle = \int d\alpha d\sigma |\alpha, \sigma\rangle \chi(\alpha, \sigma)$. If the operator is an endomorphism of $\mathcal{V}$, it can be represented as an operator (represented by the empty box) that acts on the wavefunction $\chi(\alpha, \sigma)$ for the collective variables.

Therefore, the Hamiltonian for the matter fields translates to a linear operator acting on the wavefunction of the collective variables. This is illustrated in figure 4. The induced Hamiltonian for the collective variables consists of two parts. The first is the convolution of the wavefunction with $\langle \alpha, \sigma | \alpha', \sigma' \rangle \approx e^{-\frac{N}{4} R_{\alpha, \sigma; \alpha', \sigma'}^2}$. The convolution smears out sharp features which vary at scales shorter than $\Delta R_{\alpha, \sigma; \alpha', \sigma'} \sim N^{-1/2}$ in the space of α, σ . For slowly varying $\chi(\alpha, \sigma)$ with $\frac{1}{\sqrt{N}} \frac{\partial \ln \chi(\alpha, \sigma)}{\partial \alpha}, \frac{1}{\sqrt{N}} \frac{\partial \ln \chi(\alpha, \sigma)}{\partial \sigma} \ll 1$, $\int D\alpha' D\sigma' \langle \alpha, \sigma | \alpha', \sigma' \rangle \chi(\alpha', \sigma') \propto \chi(\alpha, \sigma)$, and the convolution merely changes the normalization of $\chi(\alpha, \sigma)$. The second is the minisuperspace Wheeler-DeWitt Hamiltonian, $\tilde{H}_{\alpha, \sigma}$. Combined, $H_{\alpha, \sigma}$ can be understood as a regularized Wheeler-DeWitt Hamiltonian.

3.3 Emergent time as a Goldstone mode

Eq. (3.18) implies that $\hat{H}$ induces the Wheeler-DeWitt Hamiltonian of a minisuperspace cosmology. In gravity, time evolution is a part of diffeomorphism, and states that are invariant under diffeomorphism are annihilated by the Hamiltonian. A state that satisfies the Hamiltonian constraint represents a whole history rather than a moment of time. Recovering time from a stationary state is the problem of time in gravity [40, 41]. States that satisfy the Hamiltonian constraint correspond to quantum states with zero energy. For a finite N , however, there is no guarantee that there exists a state with zero energy in $\mathcal{V}$. This is because the configuration space is compact, and the energy level is discrete. In order to check this explicitly, we diagonalize $\hat{H}$ in eq. (3.14). The matrix elements of the Hamiltonian is written as

$$\begin{aligned} \langle \phi' | \hat{H} | \phi \rangle = & \frac{1}{4\sqrt{N}} \int D\alpha D\sigma \left[\left(-\frac{e^{-3\alpha}}{2} \sum_a \frac{\partial^2}{\partial \phi_a^2} + e^{3\alpha} U(\phi', \alpha, \sigma) \right) \Psi(\phi', \alpha, \sigma) \right] \Psi^*(\phi, \alpha, \sigma) \\ & + \frac{1}{4\sqrt{N}} \int D\alpha D\sigma \Psi(\phi', \alpha, \sigma) \left[\left(-\frac{e^{-3\alpha}}{2} \sum_a \frac{\partial^2}{\partial \phi_a^2} + e^{3\alpha} U(\phi, \alpha, \sigma) \right) \Psi^*(\phi, \alpha, \sigma) \right]. \end{aligned} \quad (3.20)$$

Explicit integrations over α, σ result in

$$\begin{aligned} \langle \phi' | \hat{H} | \phi \rangle = & iN \text{sgn}(\Delta_c) \frac{\Theta(\Delta_c^2 - \Delta_s^2)}{2^{N+5} \pi^N (\Delta_c^2 - \Delta_s^2)^{5/2}} \left\{ \frac{4}{9\kappa^2} (\Delta_c^2 + 2\Delta_s^2) \right. \\ & + 9\kappa^2 \left[6\Delta_s \Delta_c (O'_c O'_s + O_c O_s) + \Delta_c^4 - 3\Delta_c^2 (O_c'^2 + O_c^2) - 2\Delta_s^2 (\Delta_c^2 + O_s'^2 + O_s' O_s + O_s^2) \right] \Big\} \\ & + N \frac{\Theta(\Delta_s^2 - \Delta_c^2)}{2^{N+4} \pi^N (\Delta_s^2 - \Delta_c^2)^{5/2}} \left\{ -\frac{2}{9\kappa^2} (\Delta_c^2 + 2\Delta_s^2) + 9\kappa^2 \left[(O_c^2 - O_s^2)^2 + (O_c'^2 - O_s'^2)^2 \right. \right. \\ & \left. \left. + O_c' O_c \Delta_s^2 + (O_s' O_s - O_c' O_c) (O_c'^2 + O_c^2) + O_s'^2 O_c^2 + O_s^2 O_c'^2 - O_s O_s' (O_s'^2 + O_s^2) \right] \right\}, \end{aligned} \quad (3.21)$$

Figure 5. The eigenvalues of $\hat{H}$ whose eigenstates have nonzero projection to $\mathcal{V}$. The x -axis is a label of the eigenvalues sorted in the ascending order, and the y -axis denotes logarithm of the absolute magnitude of eigenvalues. The eigenvalues on the left (right) of the dip are negative (positive). The Hamiltonian has been numerically diagonalized for $N = 3$ by discretizing the compact space of each ϕ_a into L segments, where (a) $L = 12$, (b) $L = 24$, (c) $L = 36$, (d) $L = 48$. With increasing L , the bandwidth of the eigenvalues keeps increasing, which reflects the fact that the spectrum is unbounded both from the above and below in the continuum limit. On the other hand, the eigenvalue that is smallest in magnitude saturates to a nonzero value. This suggests that there is no state in $\mathcal{V}$ with zero energy in the continuum limit.

where $\Delta_c = O'_c - O_c$ and $\Delta_s = O'_s - O_s$ with $O_c = \sum_{a=1}^N \cos \phi_a$, $O_s = \sum_{a=1}^N \sin \phi_a$, $O'_c = \sum_{a=1}^N \cos \phi'_a$, $O'_s = \sum_{a=1}^N \sin \phi'_a$. We note that the matter fields are subject to strong all-to-all interactions.

We numerically diagonalize $\hat{H}$ for $N = 3$. Indeed, all eigenstates which have nonzero projection in $\mathcal{V}$ have non-zero eigenvalues, as is shown in figure 5. This may seem contradictory because $H_{\alpha,\sigma}\chi_0(\alpha,\sigma) = 0$ is a hyperbolic equation, which can be solved once a boundary condition is provided. Solutions to the Wheeler-DeWitt equation formally give zero energy states. The reason why such states do not appear in the spectrum is because they are not normalizable [40, 41].

The fact that there is no normalizable state which satisfies the Hamiltonian constraint means that a physical state in $\mathcal{V}$ inevitably breaks the time translational symmetry by the virtue of having a finite norm. This is analogous to the fact that states that are invariant under spatial translations in the Euclidean space are not normalizable, and physical states

Figure 6. (a) States that satisfy the Hamiltonian constraint are extended in the space of the collective variables, and are not normalizable. (b) Physical states with finite norms spontaneously break the symmetry generated by the Hamiltonian. The Goldstone mode associated with the spontaneously broken symmetry gives rise to a non-trivial time evolution.

(such as wave packets) necessarily break the translational symmetry. However, there exist normalizable semi-classical states which are annihilated by $\hat{H}$ to the leading order in $1/\sqrt{N}$ and break the symmetry only weakly,

$$\chi(\alpha, \sigma) = e^{-\frac{(\alpha-\bar{\alpha})^2 + (\sigma-\bar{\sigma})^2}{2\Delta^2}} e^{\frac{i}{\kappa^2}(\bar{\pi}\alpha + \bar{\pi}_\sigma\sigma)}, \quad (3.22)$$

where $\bar{\alpha}, \bar{\sigma}, \bar{\pi}, \bar{\pi}_\sigma$ are classical coordinates and momenta which satisfy

$$e^{-3\bar{\alpha}}(-\bar{\pi}^2 + \bar{\pi}_\sigma^2) + e^{3\bar{\alpha}}V(\bar{\sigma}) = 0. \quad (3.23)$$

Δ determines the uncertainty of α and σ . With $\kappa^2 \ll \Delta \ll 1$, eq. (3.22) represents a semi-classical state in which both coordinates and momenta are well defined with $\delta\alpha \delta\pi \sim \delta\sigma \delta\pi_\sigma \sim \kappa^2 \sim \frac{1}{\sqrt{N}}$, and the Hamiltonian constraint is satisfied to the leading order in $1/\sqrt{N}$.

Because semi-classical states are not exactly annihilated by $\hat{H}$, they spontaneously break the symmetry generated by the Hamiltonian. The spontaneous symmetry breaking amounts to picking a moment of time in a history. The following evolution of the state generated by $\hat{H}$ creates one-parameter family of states. The evolution can be viewed as a Goldstone mode associated with the spontaneously broken symmetry. This is illustrated in figure 6. We call the parameter along the orbit t . However, t itself is not a physical observable because there is no independent way of measuring t in a closed quantum system. It is merely a parameter that labels a sequence of states generated by the Hamiltonian evolution. What is physical is relation between physical observables, e.g., the value of σ when α takes a certain value.

Now we examine how semi-classical states evolve under $\hat{H}$. Since eq. (3.22) has a fast oscillating phase factor, the convolution integration in eq. (3.19) gives rise to a suppression

in the norm of the wavefunction,

$$\int D\alpha' D\sigma' \langle \alpha, \sigma | \alpha', \sigma' \rangle \chi(\alpha', \sigma') \approx A_\chi \chi(\alpha, \sigma), \quad (3.24)$$

where $A_\chi = e^{-18e^{-6\bar{\alpha}} \left(\frac{\cosh 3\sqrt{2}\bar{\sigma}}{2} \bar{\pi}^2 - \sqrt{2} \sinh 3\sqrt{2}\bar{\sigma} \bar{\pi} \bar{\pi}_\sigma + \cos 3\sqrt{2}\bar{\sigma} \bar{\pi}_\sigma^2 \right)} < 1$, and it is used that $\chi(\alpha, \sigma)$ is sharply peaked at $(\bar{\alpha}, \bar{\sigma})$. As a result, $H_{\alpha, \sigma}$ becomes the Wheeler-DeWitt Hamiltonian upto a multiplicative factor that depends on the wavefunction,

$$H_{\alpha, \sigma} \chi(\alpha, \sigma) \approx A_\chi \tilde{H}_{\alpha, \sigma} \chi(\alpha, \sigma). \quad (3.25)$$

By choosing a lapse that absorbs A_χ , the state after an infinitesimal step of the parameter time can be written as

$$\begin{aligned} |\chi; dt\rangle &= e^{-in^{(1)} A_\chi^{-1} \hat{H} dt} |\chi\rangle \\ &= \int D\alpha^{(0)} D\sigma^{(0)} D\alpha^{(1)} D\sigma^{(1)} D\pi^{(1)} D\pi_\sigma^{(1)} |\alpha^{(1)}, \sigma^{(1)}\rangle e^{\frac{i}{\kappa^2} [\pi^{(1)}(\alpha^{(1)} - \alpha^{(0)}) + \pi_\sigma^{(1)}(\sigma^{(1)} - \sigma^{(0)})]} \\ &\quad e^{-in^{(1)} dt \frac{e^{-3\alpha^{(0)}}}{2\kappa^2} [-\pi^{(1)2} + \pi_\sigma^{(1)2} + e^{6\alpha} V(\sigma^{(0)}) + O(\kappa^2)]} \chi(\alpha^{(0)}, \sigma^{(0)}). \end{aligned} \quad (3.26)$$

Here $n^{(1)}$ determines the speed of the flow along the orbit. $O(\kappa^2)$ represents sub-leading terms that are generated from the measure and the smearing. The measure for the conjugate momenta has been defined as $D\pi D\pi_\sigma \equiv \mu(\alpha, \sigma)^{-1} d\pi d\pi_\sigma$. In the large N limit, eq. (3.26) remains a semi-classical state centered at a different classical configuration. In the next step, we choose the lapse $n^{(2)} A_{\chi(dt)}^{-1}$. Repeating these steps, one obtains a state at parameter time t ,

$$\begin{aligned} |\chi; t\rangle &= e^{-i \int_0^t d\tau n(\tau) \hat{H}} |\chi\rangle \\ &= \int D\alpha(\tau) D\sigma(\tau) D\pi(\tau) D\pi_\sigma(\tau) |\alpha(t), \sigma(t)\rangle e^{iS} \chi(\alpha(0), \sigma(0)), \end{aligned} \quad (3.27)$$

where $n(\tau)$ is a time-dependent speed of time evolution which can be chosen at one's will, and

$$S = \frac{1}{\kappa^2} \int_0^t d\tau \left\{ \pi \partial_\tau \alpha + \pi_\sigma \partial_\tau \sigma - n(\tau) \frac{e^{-3\alpha}}{2} [-\pi^2 + \pi_\sigma^2 + e^{6\alpha} V(\sigma) + O(\kappa^2)] \right\}. \quad (3.28)$$

This is a minisuperspace quantum cosmology for the three-dimensional flat universe with one scalar field. In the large N limit, the classical path dominates the path integration.

There is a sense in which the emergent time in the present theory resembles an internal time generated by relative motions of a subsystem in stationary states [42]. To make the connection, one views $\Psi(\phi; \alpha, \sigma)$ as a wavefunction of an enlarged system that includes not only the matter fields but also the collective variables as independent dynamical degrees of freedom. In this case, eq. (3.10) is understood as the Wheeler-DeWitt equation for the whole system (with a wrong sign in the kinetic term for the matter field). Although the full state is stationary, one defines a time flow in terms of the evolution of the matter fields relative to the collective variables. What is different in the present construction are

two-fold. First, the collective variables are not independent dynamical degrees of freedom. Instead, they describe collective excitations of the matter fields. Accordingly, the quantization of the collective variables follow from that of the matter fields. Second, the inability to find normalizable states in the Hilbert space of the matter fields provides a dynamical mechanism to pick a moment of time in the induced theory of cosmology.

4 Emergent gravity

In this section, we extend the discussion on the emergent minisuperspace cosmology to gravity in $(3 + 1)$ dimensions. The biggest difference from the previous section is that we are now dealing with an infinite dimensional Hilbert space. To be concrete, we consider an $N \times N$ matrix field defined on a three dimensional manifold. Within the full Hilbert space, we define a sub-Hilbert space of the matter field that becomes a Hilbert space for two collective fields: a spatial metric and a scalar field.⁵ The sub-Hilbert space is spanned by a set of basis vectors each of which is labeled by the metric and the scalar field. As variational parameters of wavefunctions of the matter field, the spatial metric sets the notion of locality in how matter fields are entangled in space, while the scalar field determines the range of mutual information in each basis state. After we discuss the covariant regularization of the wavefunctions and the inner product within the sub-Hilbert space, we explain the connection between the collective variables and entanglement in details. Building on the intuitions we learned from the previous two sections, we then construct a matter Hamiltonian that induces the general relativity at long distances in the large N limit.

4.1 Construction of a Hilbert space for metric from matter fields

In this subsection, we define a sub-Hilbert space of a matrix field and an inner product that is invariant under spatial diffeomorphisms.

4.1.1 Hilbert space

We consider an $N \times N$ Hermitian matrix field $\Phi(x)$ defined on a compact three dimensional manifold. The full Hilbert space of the matrix field is spanned by the eigenstates of the field operator, $\hat{\Phi}_{ab}(x)|\Phi\rangle = \Phi_{ab}(x)|\Phi\rangle$. In order to define an inner product in the infinite dimensional Hilbert space, we need to introduce a discrete basis that spans the space of $\Phi_{ab}(x)$. For this, we choose an elliptic differential operator whose eigenvectors form a complete basis,

$$K_{(E,\sigma)} f_n^{(E,\sigma)}(x) = \lambda_n^{(E,\sigma)} f_n^{(E,\sigma)}(x) \quad (4.1)$$

with

$$K_{(E,\sigma)} = \left[-g_E^{\mu\nu} \nabla_\mu^E \nabla_\nu^E + \frac{e^{2\sigma}}{l_c^2} \right]. \quad (4.2)$$

⁵The collective variables are fields in this case.

Here ∇_μ^E is the covariant derivative defined with respect to a Riemannian metric, $g_{E,\mu\nu}(x)$, which is parameterized by a triad,

$$g_{E,\mu\nu}(x) = E_{\mu i}(x)E_\nu^i(x). \quad (4.3)$$

In eq. (4.3), the local Euclidean index i is raised or lowered with $\delta^{ij} = \delta_{ij}$, and repeated indices are summed over $i = 1, 2, 3$. $\sigma(x)$ is a scalar that determines the ‘mass’ in the unit of a fixed length scale, l_c . $\lambda_n^{(E,\sigma)}$ is the n -th eigenvalue and $f_n^{(E,\sigma)}(x)$ is the eigenfunction with the normalization condition, $\int dx |E| f_n^{(E,\sigma)*}(x) f_m^{(E,\sigma)}(x) = \delta_{n,m}$ with $|E| \equiv |\det E_{\mu i}|$. For a choice of $(E_{\mu i}, \sigma)$, the set of eigenvectors $\{f_n^{(E,\sigma)}(x) | n = 1, 2, \dots\}$ forms a complete basis. A general field configuration can be decomposed as $\Phi_{ab}(x) = \sum_n \Phi_{ab,n}^{(E,\sigma)} f_n^{(E,\sigma)}(x)$, where $\Phi_{ab,n}^{(E,\sigma)}$ represents the amplitude of the n -th normal mode in the basis of $\{f_n^{(E,\sigma)}(x)\}$. In order to define an inner product, we choose a fiducial triad and scalar, $(\hat{E}_{\mu i}, \hat{\sigma})$. In terms of the normal mode associated with $K_{(\hat{E}, \hat{\sigma})}$, the inner product is defined to be

$$\langle \Phi' | \Phi \rangle = \prod_{a,b} \prod_n \left[\sqrt{\pi} \delta \left(\Phi_{ab,n}'^{(\hat{E}, \hat{\sigma})} - \Phi_{ab,n}^{(\hat{E}, \hat{\sigma})} \right) \right]. \quad (4.4)$$

Two states with different amplitudes in any of the normal modes are orthogonal. The inner product defines a natural measure for a functional integration of the matter field in terms of the normal modes as

$$D^{(\hat{E}, \hat{\sigma})} \Phi \equiv \prod_{a,b} \prod_n \left[\frac{d\Phi_{ab,n}^{(\hat{E}, \hat{\sigma})}}{\sqrt{\pi}} \right]. \quad (4.5)$$

This guarantees that

$$\int D^{(\hat{E}, \hat{\sigma})} \Phi \langle \Phi' | \Phi \rangle f(\Phi) = f(\Phi') \quad (4.6)$$

for general functional $f(\Phi)$. Obviously, the inner product and the measure depends on the choice of the fiducial triad and scalar, $(\hat{E}_{\mu i}, \hat{\sigma})$. A measure defined in terms of a different triad and scalar field (E, σ) is related to eq. (4.5) through a Jacobian,

$$D^{(\hat{E}, \hat{\sigma})} \Phi = J_{(E,\sigma)}^{(\hat{E}, \hat{\sigma})} D^{(E,\sigma)} \Phi, \quad (4.7)$$

where $J_{(E,\sigma)}^{(\hat{E}, \hat{\sigma})}$ is the determinant of the matrix,

$$a_{mn} = \int dx |\hat{E}| f_m^{(\hat{E}, \hat{\sigma})*}(x) f_n^{(E,\sigma)}(x). \quad (4.8)$$

In general, the Jacobian is not unity. However, in special cases with $|E(x)| = |\hat{E}(x)|$, $J_{(E,\sigma)}^{(\hat{E}, \hat{\sigma})} = 1$ because $(a^{-1})_{nm} = a_{mn}^*$.

Within the full Hilbert space, we focus on singlet states that are invariant under global $SU(N)$ transformations, $\Phi(x) \rightarrow U^\dagger \Phi(x) U$, where U is $SU(N)$ matrix. In particular,

we consider a sub-Hilbert space, $\mathcal{V}$ spanned by a set of basis states that are labeled by $\{E_{\mu i}(x), \sigma(x)\}$,

$$|E, \sigma\rangle = \int D^{(\hat{E}, \hat{\sigma})} \Phi |\Phi\rangle \Psi(\Phi; E, \sigma). \quad (4.9)$$

Here $\Psi(\Phi; E, \sigma)$ is a short-range entangled wavefunction of the matter field in which the metric $(g_{E, \mu\nu})$ and the scalar field (σ) set local structures of entanglement. Wavefunctions for such short-range entangled states can be written as an exponential of a local functional,

$$\Psi(\Phi; E, \sigma) = e^{-\int dx |E(x)| \mathcal{L}[\Phi(x); E_{\mu i}(x), \sigma(x)] - \frac{1}{2} S_0[E, \sigma]} \quad (4.10)$$

in which the triad and the scalar enter as variational parameters. For simplicity, we choose $\mathcal{L}[\Phi(x); E_{\mu i}(x), \sigma(x)]$ to be a gaussian form,⁶

$$\mathcal{L}[\Phi(x); E_{\mu i}(x), \sigma(x)] = \frac{1}{2} \text{tr} \left[\Phi e^{-\Gamma[l_c^2 K_{(E, \sigma)}]} \Phi \right]. \quad (4.11)$$

$\text{tr}[\dots]$ denotes the trace over matrix indices. $\Gamma[l_c^2 \lambda] \equiv \int_{l_c^2}^{\infty} \frac{dt}{t} e^{-\lambda t}$ is the incomplete Gamma function and l_c is the cut-off length scale. $\Gamma[l_c^2 K_{(E, \sigma)}]$ is a regularized derivative operator that creates local entanglement at distance scales larger than l_c . It has the following asymptotic behaviors,

$$\begin{aligned} \Gamma(x) &= -\ln x - \gamma_E + O(x) \quad \text{for } x \ll 1, \\ \Gamma(x) &= \frac{e^{-x}}{x} (1 + O(x^{-1})) \quad \text{for } x \gg 1, \end{aligned} \quad (4.12)$$

where γ_E is the Euler-Mascheroni constant. For modes with eigenvalues $\lambda_n^{(E, \sigma)} \ll l_c^{-2}$, the kernel becomes the usual two-derivative operator, $e^{-\Gamma[l_c^2 K_{(E, \sigma)}]} \sim K_{(E, \sigma)}$. At large wavevectors with $\lambda_n^{(E, \sigma)} \gg l_c^{-2}$, the gradient term is suppressed and one has $e^{-\Gamma[l_c^2 K_{(E, \sigma)}]} \approx 1$. Basically, $e^{-\Gamma[l_c^2 K_{(E, \sigma)}]}$ behaves as a two-derivative term at long distances while it becomes a constant at short distances. Only those modes with wavelengths larger than l_c have non-negligible entanglement in space. A plot of $e^{-\Gamma[l_c^2 K_{(E, \sigma)}]}$ is shown in figure 7. $S_0[E, \sigma]$ is chosen to enforce the normalization condition, $\langle E, \sigma | E, \sigma \rangle = 1$. From

$$\langle E, \sigma | E, \sigma \rangle = \int D^{(E, \sigma)} \Phi J_{(E, \sigma)}^{(\hat{E}, \hat{\sigma})} e^{-\int d^3x |E| \text{tr} \left[\Phi e^{-\Gamma[l_c^2 K_{(E, \sigma)}]} \Phi \right] - S_0[E, \sigma]}, \quad (4.13)$$

we obtain

$$S_0[E, \sigma] = \frac{N^2}{2} \text{Tr} \left(\Gamma[l_c^2 K_{(E, \sigma)}] \right) + \ln J_{(E, \sigma)}^{(\hat{E}, \hat{\sigma})}. \quad (4.14)$$

Here $\text{Tr}(\dots)$ denotes the trace of differential operators. In eq. (4.13), eq. (4.7) is used.

It is noted that the particular choice in eq. (4.11) is not crucial. In order to include more general wavefunctions, one needs to introduce more collective variables which source

⁶The gaussian wavefunction has the $O(N^2)$ internal symmetry.

Figure 7. $e^{-\Gamma[(kl_c)^2 + e^{2\sigma}]}$ plotted as a function for kl_c for (a) $e^\sigma = 10^{-2}$ and (b) $e^\sigma = 10^2$. For $e^\sigma \ll 1$, the kernel disperses quadratically in k at small momenta before the dispersion is lost at large momenta with $k \gg l_c^{-1}$. For $e^\sigma \gg 1$, the momentum dependence is suppressed at all momenta, and the wavefunction becomes a direct product state in space.

Figure 8. A tensor representation of the wavefunction $\Psi(\Phi; E, \sigma)$, where the triad and a scalar field are collective variables of the matter field.

different operators in eq. (4.11). Here we choose the simplest form of wavefunction to have a tractable example.

$\mathcal{L}$ can be understood as local tensors that generate short-range entangled states, $\Psi(\Phi; E, \sigma) \propto \prod_x e^{-dx|E(x)|} \mathcal{L}[\Phi(x); E_{\mu i}(x), \sigma(x)]$. Here $E_{\mu i}$ and σ play the role of variational parameters (see figure 8) [28]. The metric sets the notion of distance in how matter fields are entangled in eq. (4.11). Because the proper cut-off length scale below which the matter field is unentangled is measured with the metric, the metric controls the number of degrees of freedom that participate in entanglement. On the other hand, σ determines the range of mutual information. In the large σ limit, $\Psi(\Phi; E, \sigma)$ becomes a direct product state in real space. The precise connection between entanglement and the collective variables will be established in section 4.2.

We note that $\Psi(\Phi; E, \sigma)$ depends on triad only through $g_{E, \mu\nu}$. Because metric is invariant under local $\text{SO}(3)$ transformations, $E_{\mu i}(x) \rightarrow O_i^j(x) E_{\mu j}(x)$, there is a gauge redundancy in labeling states in $\mathcal{V}$ in terms of triad. Each gauge orbit generated by $\text{SO}(3)$ transformations corresponds to one state in $\mathcal{V}$. Unlike the $\text{SO}(3)$ gauge transformation, a diffeomorphism of the collective variables generates a different state of the matter field in general. In order to see this, we note that $\Psi(\Phi; E, \sigma)$ is invariant upto a multiplicative factor under diffeomorphisms of the collective variables and the matter field. Under a

Figure 9. The curves represent gauge orbits generated by local $SO(3)$ transformations, $E'_{\mu i}(x) = O_i^j(x)E_{\mu j}$, $\sigma'(x) = \sigma(x)$ in the space of $\{E_{\mu i}(x), \sigma(x)\}$, where $O_i^j(x)$ is space dependent $SO(3)$ matrices. Configurations connected by $SO(3)$ transformations represent a same physical state. On the other hand, a diffeomorphism, $\tilde{E}_{\mu i}(x) = E_{\mu i}(x) - \nabla_\mu \xi^\nu E_{\nu i}$, $\tilde{\sigma}(x) = (1 - \xi^\mu \partial_\mu)\sigma(x)$ generally gives a physically distinct state of the matter field.

diffeomorphism generated by an infinitesimal vector field, ξ^μ ,⁷

$$\tilde{E}_{\mu i}(x) = E_{\mu i}(x) - \nabla_\mu \xi^\nu E_{\nu i}, \quad (4.15)$$

$$\tilde{\sigma}(x) = (1 - \xi^\mu \partial_\mu)\sigma(x), \quad (4.16)$$

$$\tilde{\Phi}(x) = (1 - \xi^\mu \partial_\mu)\Phi(x), \quad (4.17)$$

the wavefunction is transformed as

$$\Psi(\tilde{\Phi}; \tilde{E}, \tilde{\sigma}) = \left[J_{(E, \sigma)}^{(\tilde{E}, \tilde{\sigma})} \right]^{\frac{1}{2}} \Psi(\Phi; E, \sigma). \quad (4.18)$$

Therefore $|\tilde{E}, \tilde{\sigma}\rangle$ represents a state in which the matter field is shifted in space, and is in general distinct from $|E, \sigma\rangle$ as a quantum state of the matrix field (see figure 9).

4.1.2 Inner product

The inner product between states in $\mathcal{V}$ is written as

$$\langle E', \sigma' | E, \sigma \rangle = \int D^{(\hat{E}, \hat{\sigma})} \Phi \quad \Psi^*(\Phi; E', \sigma') \Psi(\Phi; E, \sigma). \quad (4.19)$$

While both $D^{(\hat{E}, \hat{\sigma})} \Phi$ and $\Psi^*(\Phi; E', \sigma') \Psi(\Phi; E, \sigma)$ depend on the fiducial metric, $\langle E', \sigma' | E, \sigma \rangle$ does not because the dependence on the fiducial metric in the measure is canceled by the normalization factor in eq. (4.14). This can be seen by rewriting the functional integration

⁷Under the diffeomorphism, the triad is transformed as $\tilde{E}_{\mu i}(x) = (\delta_\mu^\nu - \partial_\mu \xi^\nu) (\delta_i^j - \xi^\alpha \omega_{\alpha i}^j) (1 - \xi^\beta \partial_\beta) E_{\nu j}(x)$, where $\omega_{\alpha i}^j$ is a $SO(3)$ spin connection. The spin connection is determined from the compatibility condition, $\nabla_\mu E_{\nu i} \equiv \partial_\mu E_{\nu i} - \Gamma_{\mu\nu}^\alpha E_{\alpha i} + \omega_{\mu i}^j E_{\nu j} = 0$, where $\Gamma_{\mu\nu}^\alpha$ is the torsionless Christoffel symbol that is compatible with $g_{E, \mu\nu}$. From $\nabla_\mu E_{\nu i} = 0$, one can write $\tilde{E}_{\mu i}(x) = E_{\mu i}(x) - \nabla_\mu \xi^\nu E_{\nu i}$, where the covariant derivative of a space vector is defined as $\nabla_\mu \xi^\nu \equiv \partial_\mu \xi^\nu + \Gamma_{\mu\alpha}^\nu \xi^\alpha$.

in eq. (4.19) in terms of the measure associated with $E_{\mu i}$ or $E'_{\mu i}$. In terms of the measure associated with (E, σ) , eq. (4.19) can be written as

$$\langle E', \sigma' | E, \sigma \rangle = \left[J_{(E, \sigma)}^{(E', \sigma')} \right]^{\frac{1}{2}} e^{-\frac{N^2}{4} \{ \text{Tr}(\Gamma[l_c^2 K_{(E', \sigma')}] + \text{Tr}(\Gamma[l_c^2 K_{(E, \sigma)}]) \}} \times \int D^{(E, \sigma)} \Phi e^{-\frac{1}{2} \int d^3x \text{tr} \Phi \left(|E'| e^{-\Gamma[l_c^2 K_{(E', \sigma')}] + |E| e^{-\Gamma[l_c^2 K_{(E, \sigma)}]}} \right) \Phi}. \quad (4.20)$$

The fiducial metric drops out in eq. (4.20). This has an important consequence: the inner product between states in $\mathcal{V}$ is invariant under spatial diffeomorphisms,

$$\langle E', \sigma' | E, \sigma \rangle = \langle \tilde{E}', \tilde{\sigma}' | \tilde{E}, \tilde{\sigma} \rangle, \quad (4.21)$$

where $\{\tilde{E}_{\mu i}(x), \tilde{\sigma}(x)\}$ and $\{\tilde{E}'_{\mu i}(x), \tilde{\sigma}'(x)\}$ are respectively related to $\{E_{\mu i}(x), \sigma(x)\}$ and $\{E'_{\mu i}(x), \sigma'(x)\}$ through a diffeomorphism in eqs. (4.15)–(4.16). See appendix A for the proof of eq. (4.21).

Once the Gaussian integration is performed in eq. (4.20), the overlap can be written as

$$\langle E, \sigma | E', \sigma' \rangle = e^{-\int dx |E| \delta v_a(x) \mathcal{M}_{ab}(x) \delta v_b(x)} \quad (4.22)$$

to the quadratic order in the difference of the collective variables, $v_a(x) = (h_{E, \mu\nu}(x), \delta\sigma(x))$ with index a running over different collective variables, where $h_{\mu\nu} = g_{E', \mu\nu} - g_{E, \mu\nu}$ and $\delta\sigma = \sigma' - \sigma$. $\mathcal{M}_{ab}(x)$ is a positive kernel which is order of N^2 . In the large N limit, the cubic and higher order terms in δv_a are negligible in eq. (4.22) because $\delta v_a \sim 1/N$.

One can show that eq. (4.20) vanishes identically unless $|E(x)| = |E'(x)|$ at all x . Therefore $\mathcal{M}(x) = \infty$ if $|E(x)| \neq |E'(x)|$. This is because metrics with different local proper volumes support eigenmodes with different normalizations. The mismatch in the normalization of modes with arbitrarily large momenta gives rise to zero overlap if there is any region in space with $|E(x)| \neq |E'(x)|$. The proof is given in appendix B. It automatically follows that two states with metrics which give different global proper volumes are orthogonal.

Two states with $|E(x)| = |E'(x)|$ are not orthogonal in general. Nonetheless, $\langle E', \sigma' | E, \sigma \rangle$ decays exponentially in $h_{\mu\nu}$ and $\delta\sigma$ in the large N limit. This is because each of the N^2 components of the matrix field contributes an overlap which is less than 1 when the collective variables do not match. Since l_c is the only scale, $\mathcal{M}(x) \sim \frac{N^2}{l_c^3} (1 + O(l_c \nabla))$ in eq. (4.22). This form of $\mathcal{M}(x)$ is confirmed through an explicit computation of the overlap between states with metrics close to the Euclidean metric in appendix C. Two states whose collective variables differ by

$$h_\mu^\nu(x) \sim \frac{1}{N}, \quad |\delta\sigma(x)| \sim \frac{e^{-3/2\sigma}}{N} \quad (4.23)$$

or more over a proper volume larger than l_c^3 are nearly orthogonal even when $|E(x)| = |E'(x)|$ (see appendix C). With increasing N , the overlap approaches the delta function upto a normalization factor. In the large N limit, the overlap can be formally written as

$$\lim_{N \rightarrow \infty} \langle E', \sigma' | E, \sigma \rangle = \tilde{\mu}^{-1}(E, \sigma) \prod_x \left[\delta(\sigma'(x) - \sigma(x)) \prod_{(\mu, \nu)} \delta(g_{E', \mu\nu}(x) - g_{E, \mu\nu}(x)) \right], \quad (4.24)$$

Figure 10. A tensor representation of a general state. The thick lines represent the collective variables which are contracted with the wavefunction $\chi(E, \sigma)$.

where $\tilde{\mu}^{-1}(E, \sigma)$ is a measure determined from the determinant of eq. (C.15). The full expression for $\tilde{\mu}(E, \sigma)$ can be in principle computed from eq. (4.20). Here we don't need an explicit form of the measure.

The overlap provides the natural measure for the functional integration over the collective variables. We define the measure from the condition that

$$\int DE_{\mu i} D\sigma \langle E', \sigma' | E, \sigma \rangle = 1 \quad (4.25)$$

for any $E'_{\mu i}$ and σ' . Formally, the measure is written as $DE_{\mu i} D\sigma \equiv \mu(E, \sigma) \prod_x [dE_{\mu i}(x) d\sigma(x)]$ with $\mu(E, \sigma) = \tilde{\mu}(E, \sigma) \prod_x \left[\int dE'_{\mu i}(x) \delta(g_{E', \mu\nu}(x) - g_{E, \mu\nu}(x)) \right]^{-1}$, where the last factor divides out the $SO(3)$ gauge volume. The measure defined by this condition is invariant under diffeomorphism. This can be checked from a series of identities,

$$\begin{aligned} \int DE_{\mu i} D\sigma \langle E', \sigma' | E, \sigma \rangle &= \int DE_{\mu i} D\sigma \langle \tilde{E}', \tilde{\sigma}' | E, \sigma \rangle \\ &= \int D\tilde{E}_{\mu i} D\tilde{\sigma} \langle \tilde{E}', \tilde{\sigma}' | \tilde{E}, \tilde{\sigma} \rangle = \int D\tilde{E}_{\mu i} D\tilde{\sigma} \langle E', \sigma' | E, \sigma \rangle, \end{aligned} \quad (4.26)$$

where $\{\tilde{E}_{\mu i}, \tilde{\sigma}\}$ is related to $\{E_{\mu i}, \sigma\}$ through a diffeomorphism. For the first equality, we use the fact that eq. (4.25) holds for any $E_{\mu i}$ and σ . The second equality is a simple change of variables. For the third equality, we use the fact that the inner product is invariant under diffeomorphism. Eq. (4.26) implies that $DE_{\mu i} D\sigma = D\tilde{E}_{\mu i} D\tilde{\sigma}$.

General states in $\mathcal{V}$ can be expressed as linear superpositions of $|E, \sigma\rangle$,

$$|\chi\rangle = \int DE_{\mu i} D\sigma |E, \sigma\rangle \chi(E, \sigma), \quad (4.27)$$

where $\chi(E, \sigma)$ is invariant under local $SO(3)$ transformations. Its tensor representation is shown in figure 10. It is normalized such that $\int DE'_{\mu i} D\sigma' DE_{\mu i} D\sigma \chi^*(E', \sigma') \langle E', \sigma' | E, \sigma \rangle \chi(E, \sigma) = 1$. In the large N limit, $\langle E', \sigma' | E, \sigma \rangle$ is sharply peaked at $g_{E', \mu\nu} = g_{E, \mu\nu}$, $\sigma' = \sigma$, and the normalization condition reduces to $\int DE_{\mu i} D\sigma |\chi(E, \sigma)|^2 = 1$. Similar to eq. (3.16), we define the Hermitian conjugate of a differential operator acting on the collective variables from

$$\int DE_{\mu i} D\sigma f^*(E, \sigma) H[g(E, \sigma)] = \int DE_{\mu i} D\sigma \left[H^\dagger f(E, \sigma) \right]^* g(E, \sigma). \quad (4.28)$$

4.2 Metric as a collective variable for entanglement

In this subsection, we discuss the physical meaning of the metric and the scalar field as collective variables for the matter field. In particular, we show that the metric controls the

Figure 11. A tensor representation of the density matrix of region A in space.

number of degrees of freedom that are entangled in space, and the scalar field determines the rate at which the mutual information decays in space. Being a wavefunction defined in continuum, the size of the Hilbert space per unit *coordinate volume* is infinite. However, the number of degrees of freedom that contribute to entanglement is controlled by the proper volume measured in the unit of the short-distance cut-off, l_c . The metric sets the notion of distance in the short range entangled states of the matter.

Let us consider a region A in space. For general states in eq. (4.27), the density matrix of the region is given by

$$\rho_A(\Phi'(x_A), \Phi(x_A)) = \int D^{(\hat{E}, \hat{\sigma})} \Phi(x_{\bar{A}}) D E_1 D \sigma_1 D E_2 D \sigma_2 \chi^*(E_1, \sigma_1) \Psi^*(\Phi'(x_A), \Phi(x_{\bar{A}}); E_1, \sigma_1) \Psi(\Phi(x_A), \Phi(x_{\bar{A}}); E_2, \sigma_2) \chi(E_2, \sigma_2), \quad (4.29)$$

where $\bar{A}$ is the complement of A (see figure 11). The replica method allows one to express the von Neumann entanglement entropy as

$$S(A) = - \lim_{n \rightarrow 1} \frac{1}{n-1} (Z_n - 1), \quad (4.30)$$

where $Z_n = \text{Tr}(\rho_A^n)$. The entanglement entropy for general states depends both on $\Psi(\Phi; E, \sigma)$ and $\chi(E, \sigma)$ in a complicated way, where the former represents the wavefunction of the matter field for a fixed collective variable (E, σ) and the latter encodes fluctuations of the collective variables. Here we focus on $\chi(E, \sigma)$ that is peaked at a classical configuration $(\bar{E}, \bar{\sigma})$ with small fluctuations around it. For such semi-classical wavefunctions for the collective variables, the entanglement entropy can be approximately decomposed into two contributions,

$$S(A) \approx S_\Phi(A) + S_{E, \sigma}(A). \quad (4.31)$$

Here $S_\Phi(A)$ is the entanglement entropy of the matter degrees of freedom defined in the classical background collective variables $(\bar{E}, \bar{\sigma})$,

$$S_\Phi(A) = F_A + F_{\bar{A}} - F, \quad (4.32)$$

where

$$F = -\ln \int D^{(\bar{E}, \bar{\sigma})} \Phi e^{-2 \int dx |\bar{E}| \mathcal{L}[\Phi; \bar{E}, \bar{\sigma}]},$$

$$F_{A(\bar{A})} = -\ln \int D^{(\bar{E}, \bar{\sigma})} \Phi e^{-2 \int_{A(\bar{A})} dx |\bar{E}| \mathcal{L}[\Phi; \bar{E}, \bar{\sigma}]} \Big|_{\Phi(x_{\partial A})=0}. \quad (4.33)$$

On the other hand, $S_{E,\sigma}(A)$ is the entanglement generated by correlations between fluctuations of the collective variables,

$$S_{E,\sigma}(A) = -\lim_{n \rightarrow 1} \frac{1}{n-1} (Z_n^{E,\sigma} - 1), \quad (4.34)$$

where

$$Z_n^{E,\sigma} = \int \prod_{j=1}^n DE^j D\sigma^j \left\{ e^{S_0[E^j, \sigma^j]} \tilde{\chi}^* \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^j(x_A), \sigma^j(x_A) \right) \times \right.$$

$$\left. \tilde{\chi} \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^{j-1}(x_A), \sigma^{j-1}(x_A) \right) \Big|_{\substack{\sigma^2, \dots, \sigma^n(x_{\partial A}) = \sigma^1(x_{\partial A}), \\ g_{E^2, \dots, \sigma^n}(x_{\partial A}) = g_{E^1, \sigma^1}(x_{\partial A})}} \right\}. \quad (4.35)$$

with $\tilde{\chi}(E, \sigma) = e^{-\frac{1}{2}S_0[E, \sigma]} \chi(E, \sigma)$. Here $E^0 = E^n$ and $\sigma^0 = \sigma^n$.

The derivation of eqs. (4.31)–(4.35) is given in appendix D. Here we provide an intuitive explanation of the result. When $\chi(E, \sigma) \propto \delta(g_{E, \mu\nu} - g_{\bar{E}, \mu\nu})\delta(\sigma - \bar{\sigma})$, there is no fluctuations in the collective variables. In this case, the entanglement entropy is given by that of $\Psi(\Phi; \bar{E}, \bar{\sigma})$. Because $\Psi(\Phi; \bar{E}, \bar{\sigma})$ is written as an exponential of a local functional, the entanglement entropy is related to the ‘free energy’ difference caused by a Dirichlet boundary condition as is shown in eq. (4.33) [43]. Now, suppose the wavefunction for the collective variables has a small but nonzero width around the semi-classical configuration. As a simple example, let us assume that there are only two configurations of the collective variables, $\chi(E, \sigma) = A\delta(g_{E, \mu\nu} - g_{E^1, \mu\nu})\delta(\sigma - \sigma^1) + B\delta(g_{E, \mu\nu} - g_{E^2, \mu\nu})\delta(\sigma - \sigma^2)$, where (E^1, σ^1) and (E^2, σ^2) are distinct from each other but are close to their average, $(\bar{E}, \bar{\sigma})$. On the one hand, there is an entanglement generated by the matter field whose wavefunction is well approximated by $\Psi(\Phi; \bar{E}, \bar{\sigma})$. This entanglement is given by S_Φ . However, $\Psi(\Phi; \bar{E}, \bar{\sigma})$ does not capture the entire correlation present in the system. There is an additional correlation generated by fluctuations of the collective variables. Since $\Psi(\Phi; E^1, \sigma^1)$ and $\Psi(\Phi; E^2, \sigma^2)$ are almost orthogonal when N is large, these fluctuations of the collective variables give rise to an additional entanglement which is captured by $S_{E,\sigma}$.

In the limits that $e^\sigma \ll 1$ and the linear proper size of A is much larger than l_c , S_Φ is proportional to the area of ∂A and the number of matter fields. When the metric is flat and σ is constant, one can compute S_Φ explicitly. Consider a region, $A = \{(x_1, x_2, x_3) | 0 < x_1 < l, 0 \leq x_2 < l_c, 0 \leq x_3 < l_c\}$ in T^3 with the flat metric $g_{\bar{E}, \mu\nu} = a^2 \delta_{\mu\nu}$ as is shown in figure 12. In the small l_c limit with fixed al_c , the entanglement entropy of region A is given by (see appendix E for derivation)

$$S_\Phi(A) = \frac{\mathcal{A}_{\partial A}}{4\kappa^2}, \quad (4.36)$$

Figure 12. A region with linear size l in the compact space which has T^3 topology.

where $\mathcal{A}_{\partial A}$ is the area of ∂A measured with the metric $g_{\bar{E},\mu\nu}$, and

$$\kappa^2 \equiv \frac{4\pi l_c^2}{N^2}. \quad (4.37)$$

The entanglement entropy is given by the proper area of the boundary measured in the unit of κ^2 . κ is much smaller than the cut-off scale l_c in the large N limit. Although eq. (4.36) has been derived in the flat metric, the same formula is expected to hold for general metrics to the leading order in the limit that the curvature is much smaller than l_c^{-1} . This is because the leading order contribution, which is divergent in the $l_c \rightarrow 0$ limit, comes from short-wavelength modes for which geometry can be regarded locally flat and the WKB approximation is valid.

The entanglement entropy of a fixed region A increases as the proper area of ∂A increases. This can be understood in terms of mode softening with increasing proper volume. The eigenvalue for the mode with momentum $k_\mu = \frac{2\pi}{l_c}(n_1, n_2, n_3)$ with integer n_μ is given by $\lambda_k = \left(\frac{2\pi}{al_c}\right)^2 (n_1^2 + n_2^2 + n_3^2) + \frac{e^{2\sigma}}{l_c^2}$. Because of the cut-off scale, only the modes with wave-numbers, $n_i < a$ contribute significantly to the entanglement entropy as is shown in appendix E. With increasing a , more modes become soft and contribute to entanglement. Therefore, the number of degrees of freedom that generate entanglement in space is a dynamical quantity rather than a fixed number. This has an important consequence. There is no fundamental limit in the amount of information a ‘finite’ region in space can hold because the proper size of the region is a dynamical variable which can be as large as it can be. This may sound unphysical until we think about our universe, which was once of the Planck size yet contained the vast amount of information on the current universe.

$S_\Phi(A)$ is the contribution from the degrees of freedom that carry non-trivial charge under the $SU(N)$ symmetry. The classical metric controls the entanglement encoded in the non-singlet degrees of freedom. For this reason, we call $S_\Phi(A)$ ‘color entanglement entropy’. On the other hand, $S_{E,\sigma}(A)$ is encoded in the wavefunction for the collective variables. It is the entanglement generated by correlations in fluctuations of the singlet collective variables. We call $S_{E,\sigma}(A)$ ‘singlet entanglement entropy’.

In terms of N counting, $S_{E,\sigma}(A)$ is $O(1)$ while $S_\Phi(A)$ is $O(N^2)$ for semi-classical states. However, the singlet entanglement entropy can be larger than the color entanglement entropy in some states. This is because the singlet entanglement entropy can scale with the volume of a region if there exist long-range correlations in fluctuations of the collective variables while color entanglement entropy scales with the area of its boundary. This point will become important in the discussion on black hole evaporation in section 4.

In the von Neumann entanglement entropy, σ doesn't enter to the leading order in the small l_c limit. However, the scalar field plays a more distinct role in determining mutual information. The mutual information between two regions A and B is defined to be $I(A, B) = S(A) + S(B) - S(A \cup B)$. For semi-classical states, the mutual information is again decomposed as $I(A, B) \approx I_\Phi(A, B) + I_{E,\sigma}(A, B)$, where $I_\Phi(A, B)$ ($I_{E,\sigma}(A, B)$) is the contribution from color degrees of freedom (singlet collective variables). In order to examine the relation between the color mutual information and the collective variables, it is useful to write the expression for the color entanglement entropy as

$$S_\Phi(A) = - \lim_{n \rightarrow 1} \frac{1}{n-1} \left(\int D^{(\hat{E}, \bar{\sigma})} \Phi^j O_{\partial A} \prod_j |\Psi(\Phi^j; \bar{E}, \bar{\sigma})|^2 - 1 \right) \quad (4.38)$$

with $O_{\partial A} \propto \prod_{x \in \partial A} \prod_{j=2}^n \int d\lambda_j(x) e^{i \text{Tr}(\lambda_j(x) [\Phi^j(x) - \Phi^1(x)])}$. λ_j is an $N \times N$ Hermitian Lagrangian multiplier which enforces the Dirichlet boundary condition at the boundary. This is only schematic because the measure for $\prod_{x \in \partial A}$ hasn't been specified. However, it is still useful in understanding the connection between the mutual information and the collective variables. Suppose A and B represent infinitesimally small balls centered at x and y , respectively. In the limit that the proper distance between x and y is large, the color mutual information is dominated by the connected correlation function between the fundamental fields inserted at x and y which exhibits the slowest decay in eq. (4.38). A straightforward calculation shows that the color mutual information scales as

$$I_\Phi(A, B) \sim N^2 G[x, y; \bar{E}, \bar{\sigma}]^2, \quad (4.39)$$

where $G[x, y; \bar{E}, \bar{\sigma}]$ is the correlation function of the fundamental field. In the small e^σ limit, $G[x, y; \bar{E}, \bar{\sigma}] \sim \frac{1}{d_{x,y}}$, where $d_{x,y}$ is the proper distance between x and y measured with the metric $g_{\bar{E}, \mu\nu}$. For fixed x and y in the manifold, the proper distance between the points is controlled by the metric, and so does the mutual information. For example, states that support small (large) color mutual information between two points give large (small) proper distance between the points. When e^σ is not negligible, the Green's function decays exponentially at large distances, $G[x, y; \bar{E}, \bar{\sigma}] \sim e^{-\int_x^y \frac{e^\sigma}{l_c} ds}$, where ds is the infinitesimal proper distance along the geodesic that connects x and y . This shows that σ determines the range of entanglement, while the metric sets the notion of locality in how matter fields are entangled in space. In this construction, the connection between entanglement and geometry [44–49] has been encoded as a kinematic building block of the theory.

4.3 Relatively local Hamiltonian

Having understood the kinematic structure of the sub-Hilbert space, now we construct a Hamiltonian of matter field which induces the Wheeler-DeWitt Hamiltonian of the general relativity in the sub-Hilbert space.

Hermitian operators that map $\mathcal{V}$ to $\mathcal{V}$ generate unitary evolutions of the collective variables. One example of such endomorphisms is the momentum density operator for the matter fields,

$$\hat{\mathcal{H}}_\mu(x) = -\frac{1}{2} \left[\left(\nabla_\mu \hat{\Phi}_{ab}(x) \right) \hat{\pi}_{ba}(x) + \hat{\pi}_{ba}(x) \left(\nabla_\mu \hat{\Phi}_{ab}(x) \right) \right], \quad (4.40)$$

where $\hat{\pi}_{ba}(x)$ is the conjugate momentum of $\hat{\Phi}_{ab}(x)$ with the commutator $[\hat{\pi}_{ab}(x), \hat{\Phi}_{cd}(y)] = -i\delta_{ad}\delta_{bc}\delta(x-y)$. Due to eq. (4.18), the action of $\hat{\mathcal{H}}_\mu(x)$ on $|E, \sigma\rangle$ is equivalent to a differential operator that induces a diffeomorphism of the collective variables,

$$\int dx n^\mu(x) \hat{\mathcal{H}}_\mu(x) |E, \sigma\rangle = - \int dx n^\mu(x) \mathcal{H}_\mu^{E, \sigma}(x) |E, \sigma\rangle, \quad (4.41)$$

where

$$\mathcal{H}_\mu^{E, \sigma}(x) = -iE_{\mu i} \nabla_\nu \frac{\delta}{\delta E_{\nu i}} + i(\nabla_\mu \sigma) \frac{\delta}{\delta \sigma(x)}. \quad (4.42)$$

Eq. (4.41) is proven in appendix F. It implies that a shift of the matter field is equivalent to the inverse shift of the collective fields. This follows from the fact that $|E, \sigma\rangle$ is invariant under the simultaneous shift of the matter field and the collective variables. Only relative shifts between the matter field and the collective variables matter. Eq. (4.42) can be viewed as the induced momentum density operator for the collective variables. For general states in eq. (4.27), the operation of eq. (4.40) results in a shift in wavefunctions of the collective variables as is illustrated in figure 13,

$$\begin{aligned} & \int dx n^\mu(x) \hat{\mathcal{H}}_\mu(x) \left[\int DED\sigma |E, \sigma\rangle \chi(E, \sigma) \right] \\ &= \int DED\sigma |E, \sigma\rangle \left[\int dx n^\mu(x) \mathcal{H}_\mu^{E, \sigma}(x) \chi(E, \sigma) \right]. \end{aligned} \quad (4.43)$$

Here it is used that $DED\sigma$ is invariant under diffeomorphism.⁸

Similarly, a Hamiltonian for the matter field whose trajectories stay within $\mathcal{V}$ induces a quantum Hamiltonian for $E_{\mu i}$ and σ . Our goal is to construct a Hamiltonian for the matter field which induces the Einstein's general relativity at long distances in the large N limit. Our strategy is to start with a regularized Wheeler-DeWitt Hamiltonian for the collective variables, and reverse engineer to find the corresponding Hamiltonian for the matter field. We look for a Hamiltonian density whose action on $|E, \sigma\rangle$ leads to

$$\hat{\mathcal{H}}(x; E, \sigma) |E, \sigma\rangle = \tilde{h}^{E, \sigma}(x) |E, \sigma\rangle, \quad (4.44)$$

⁸The derivation goes as follows. From eq. (4.41), we have $e^{-i \int dx n^\mu(x) \hat{\mathcal{H}}_\mu(x)} [\int DED\sigma |E, \sigma\rangle \chi(E, \sigma)] = \int DED\sigma |\tilde{E}, \tilde{\sigma}\rangle \chi(E, \sigma)$, where $\tilde{E}_{\mu i}(x) = E_{\mu i}(x) - \nabla_\mu n^\nu E_{\nu i}$, $\tilde{\sigma}(x) = (1 - n^\mu \nabla_\mu) \sigma(x)$. By shifting $E_{\mu i}(x) \rightarrow \tilde{E}_{\mu i}(x) = E_{\mu i}(x) + \nabla_\mu n^\nu E_{\nu i}$, $\sigma(x) \rightarrow \sigma'(x) = (1 + n^\mu \partial_\mu) \sigma(x)$, and using the fact that the measure is invariant under diffeomorphism, one obtains $e^{-i \int dx n^\mu(x) \hat{\mathcal{H}}_\mu(x)} [\int DED\sigma |E, \sigma\rangle \chi(E, \sigma)] = \int DED\sigma |\tilde{E}, \tilde{\sigma}\rangle \chi(E, \sigma) = \int DED\sigma |E, \sigma\rangle e^{-i \int dx n^\mu(x) \mathcal{H}_\mu^{E, \sigma}(x)} \chi(E, \sigma)$. The linear terms in n^μ give eq. (4.43).

Figure 13. An endomorphism of $\mathcal{V}$ represented by the dark filled box induces a map for the collective variable represented by the empty box.

where $\tilde{h}^{E,\sigma}(x)$ is a regularized Wheeler-DeWitt differential operator [50–52] for the collective variables,

$$\begin{aligned} \tilde{h}^{E,\sigma}(x) = & -\tilde{\kappa}^2 \left(: \frac{G_{ijkl}}{|E|} E_\mu^j E_\nu^l \frac{\delta}{\delta E_{\mu i}(x)} \frac{\delta}{\delta E_{\nu k}(x)} : + : \frac{1}{2|E|F(\sigma)} \frac{\delta}{\delta \sigma(x)} \frac{\delta}{\delta \sigma(x)} : \right) \\ & + \frac{|E|}{\tilde{\kappa}^2} \left(-R + \frac{F(\sigma)}{2} g_E^{\mu\nu}(x) \nabla_\mu \sigma(x) \nabla_\nu \sigma(x) + V(\sigma) + U_3(g_E, \sigma) \right). \end{aligned} \quad (4.45)$$

Here $G_{ijkl} = \frac{1}{4} (\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl})$ is the supermetric for the kinetic term of the triad. $F(\sigma)$ represents a nonlinear term in the kinetic energy of the scalar. R is the curvature scalar for the three-dimensional metric $g_{E,\mu\nu}$. $V(\sigma)$ is a potential for the scalar. $U_3(g_E, \sigma)$ represents terms that involve more than two derivatives for $g_{E,\mu\nu}$ and σ , where the higher-derivative terms are suppressed by $(l_c \nabla)$ compared to the two-derivative terms. $F(\sigma)$, $V(\sigma)$ and $U_3(g_E, \sigma)$ are included for generality, but we do not need to specify their forms for our purpose. It is important to note that the second order functional derivatives in eq. (4.45) needs to be regularized as the derivatives acting on one point in space is ill-defined. Here the derivatives are regularized through a point-splitting scheme based on the heat-Kernel regularization,

$$\begin{aligned} : \frac{G_{ijkl}}{|E|} E_\mu^j E_\nu^l \frac{\delta}{\delta E_{\mu i}(x)} \frac{\delta}{\delta E_{\nu k}(x)} : & \equiv \int dy dz K_{\mu\nu k}(y, z; x, l_c^2) \frac{\delta}{\delta E_{\mu i}(y)} \frac{\delta}{\delta E_{\nu k}(z)}, \\ : \frac{1}{|E|F(\sigma)} \frac{\delta}{\delta \sigma(x)} \frac{\delta}{\delta \sigma(x)} : & \equiv \frac{1}{F(\sigma)} \int dy dz K(y, z; x, l_c^2) \frac{\delta}{\delta \sigma(y)} \frac{\delta}{\delta \sigma(z)}. \end{aligned} \quad (4.46)$$

$K_{\mu\nu k}(y, z; x, t)$ and $K(y, z; x, t)$ spread the two differential operators over the cut-off length scale l_c centered at x . In the heat kernel regularization scheme [53–55], the kernels satisfy the diffusion equation,

$$\frac{\partial}{\partial t} K_{\mu\nu k}(y, z; x, t) = [\nabla_y^2 + \nabla_z^2] K_{\mu\nu k}(y, z; x, t), \quad (4.47)$$

$$\frac{\partial}{\partial t} K(y, z; x, t) = [\nabla_y^2 + \nabla_z^2] K(y, z; x, t) \quad (4.48)$$

with the boundary condition, $K_{\mu\nu k}(y, z; x, 0) = \frac{G_{ijkl}}{|E(x)|} E_\mu^j(x) E_\nu^l(x) \delta(y-x) \delta(z-x)$, $K(y, z; x, 0) = \frac{1}{|E(x)|} \delta(y-x) \delta(z-x)$. In eq. (4.47) and eq. (4.48), ∇_y (∇_z) represents the co-variant derivative acting on coordinate y (z). In the Euclidean space, the regulators become

$$K_{\mu\nu k}(y, z; x, l_c^2) = \frac{1}{(2\sqrt{\pi}l_c)^6 |E|} G_{ijkl} E_\mu^j E_\nu^l e^{-\frac{d_{y,x}^2 + d_{z,x}^2}{4l_c^2}}, \quad K(y, z; x, l_c^2) = \frac{1}{(2\sqrt{\pi}l_c)^6 |E|} e^{-\frac{d_{y,x}^2 + d_{z,x}^2}{4l_c^2}},$$

where $d_{x,y}$ is the proper distance between x and y . In eq. (4.45), $\tilde{\kappa}$ is the Planck scale for the induced gravity, which is a free parameter for now. Below, we show that $\tilde{\kappa}$ should be order of κ in the large N limit if the underlying matter Hamiltonian has a well-defined large N limit.

The Hamiltonian density for the matter field that satisfies eq. (4.44) is given by

$$\begin{aligned} \hat{\mathcal{H}}(x; E, \sigma) = & \frac{1}{\tilde{\kappa}^2} \left[-\frac{\tilde{\kappa}^4}{\kappa^4} : \frac{1}{|E|} \left(G_{ijkl} E_\mu^j E_\nu^l \hat{T}^{\mu i} \hat{T}^{\nu k} + \frac{1}{2F(\sigma)} (\hat{O}_\sigma)^2 \right) : \right. \\ & \left. + |E| \left(-R + \frac{F(\sigma)}{2} g_E^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma + V(\sigma) + U_3(g_E, \sigma) \right) \right] \end{aligned} \quad (4.49)$$

to the leading order in the large N limit, where

$$\begin{aligned} \hat{T}^{\mu i}(x) &\equiv \kappa^2 \frac{\delta}{\delta E_{\mu i}(x)} S[\hat{\Phi}; E, \sigma], \\ \hat{O}_\sigma(x) &\equiv \kappa^2 \frac{\delta}{\delta \sigma(x)} S[\hat{\Phi}; E, \sigma] \end{aligned} \quad (4.50)$$

with $S[\Phi; E, \sigma] = \int dx |E(x)| \mathcal{L}[\Phi(x); E_{\mu i}(x), \sigma(x)] + \frac{1}{2} S_0[E, \sigma]$. $\hat{T}^{\mu i}(x)$ and $\hat{O}_\sigma(x)$ in eq. (4.50) scale as $O(N^0)$ in the large N limit. In eq. (4.49), the double-trace operators [33, 34, 36, 56, 57] are responsible for generating the kinetic terms in $\tilde{h}^{E, \sigma}$. In order for the leading kinetic term and the potential term in eq. (4.49) to scale uniformly in the large N limit, one needs $\frac{\tilde{\kappa}}{\kappa} \sim O(N^0)$. This implies that a matter Hamiltonian which scales as $O(N^2)$ in the large N limit induces a gravity with the Planck scale $\kappa \sim \frac{l_c}{N}$, which also controls the color entanglement entropy through eq. (4.37). From now on, we focus on such Hamiltonians, and set $\tilde{\kappa} = \kappa$.

Eq. (4.44) implies that the evolution of $|E, \sigma\rangle$ generated by $\hat{\mathcal{H}}(x; E, \sigma)$ is reproduced by the differential operator $\tilde{h}^{E, \sigma}(x)$ acting on the collective variables. However, $\hat{\mathcal{H}}(x; E, \sigma)$ is defined with reference to (E, σ) , and it does not act on states with different collective variables in the same way. We encountered the same issue in section 3. In order to induce a background independent Hamiltonian for the collective variables, we use the strategy introduced in the minisuperspace cosmology. Namely, we make a Hamiltonian to effectively depend on the collective variables so that $\hat{\mathcal{H}}(x; E, \sigma)$ associated with a specific collective variable is only applied to the corresponding state, $|E, \sigma\rangle$. This can be implemented by the following Hamiltonian density,

$$\hat{\mathcal{H}}(x) = \frac{1}{2} \int DED\sigma \left[\hat{\mathcal{H}}(x, E, \sigma) \hat{P}_{E, \sigma} + h.c. \right]. \quad (4.51)$$

Here $h.c.$ represents the Hermitian conjugate of the first term. $\hat{\mathcal{H}}(x, E, \sigma)$ is given by eq. (4.49) with $\tilde{\kappa} = \kappa$. $\hat{P}_{E, \sigma}$ is an operator that satisfies

$$\hat{P}_{E, \sigma} \int DE' D\sigma' |E', \sigma'\rangle \chi(E', \sigma') = |E, \sigma\rangle \chi(E, \sigma). \quad (4.52)$$

In eq. (4.51), a general state is first projected to the state with each collective variable (E, σ) , and then the Hamiltonian associated with the collective variables, $\hat{\mathcal{H}}(x; E, \sigma)$ is

Figure 14. When $\hat{\mathcal{H}}(x)$ is applied to a state made of a linear superposition of multiple basis states with different collective variables, each basis state $|E, \sigma\rangle$ is applied by $\hat{\mathcal{H}}(x; E, \sigma)$ which is local with respect to the distance measured with the metric $g_{E, \mu\nu}$.

applied (see figure 14). For any $|E, \sigma\rangle$, $\hat{\mathcal{H}}(x)$ satisfies

$$\hat{\mathcal{H}}(x)|E, \sigma\rangle = \mathcal{H}^{E, \sigma}(x)|E, \sigma\rangle, \quad (4.53)$$

where

$$\mathcal{H}^{E, \sigma}(x) = \frac{1}{2} \left[\tilde{h}^{E, \sigma}(x) + \tilde{h}^{E, \sigma^\dagger}(x) \right] \left\{ 1 + O \left(l_c \kappa^2 \frac{\delta}{\delta E_{\mu i}(x)}, l_c \kappa^2 \frac{\delta}{\delta \sigma(x)} \right) \right\}. \quad (4.54)$$

The construction of $\hat{P}_{E, \sigma}$ and the derivation of eq. (4.54) are in appendix G. In the limit that $l_c \kappa^2 \frac{\delta}{\delta E_{\mu i}(x)}, l_c \kappa^2 \frac{\delta}{\delta \sigma(x)} \ll 1$, the higher derivative terms in eq. (4.54) can be ignored, and $\mathcal{H}(x)$ induces the Wheeler-DeWitt Hamiltonian for the collective variables at long distance scales,

$$\hat{\mathcal{H}}(x) \int DED\sigma |E, \sigma\rangle \chi(E, \sigma) = \int DED\sigma |E, \sigma\rangle \mathcal{H}^{E, \sigma}(x) \chi(E, \sigma). \quad (4.55)$$

Unlike the case with the minisuperspace cosmology discussed in the previous section, it is hard to perform the functional integrations over the collective variables explicitly in eq. (4.51). In the following, we discuss general features of the Hamiltonian, focusing on its locality.

By choosing a space dependent speed of local time evolution, we construct a Hamiltonian,

$$\hat{H}_n(t) \equiv \int dx n(x, t) \hat{\mathcal{H}}(x), \quad (4.56)$$

where $n(x, t)$ in general depends both on space and time. As a Hamiltonian for matter fields, one can ask how local the Hamiltonian is in space. In order to answer this question, we first focus on $\hat{\mathcal{H}}(x; E, \sigma)$ which is a part of $\hat{\mathcal{H}}(x)$. In eq. (4.49), the point-splitting of the kinetic terms and the higher-derivative terms are controlled by l_c . Therefore, $\hat{\mathcal{H}}(x; E, \sigma)$ is local at length scales larger than l_c , if distances are measured with the metric $g_{E, \mu\nu}$. However, there is no absolute sense of locality in $\hat{H}_n(t)$ because the metric that determines proper distances is not fixed. Instead, $\hat{\mathcal{H}}(x)$ is given by the sum of $\hat{\mathcal{H}}(x; E, \sigma) \hat{P}_{E, \sigma}$ over different metrics. To understand this point, let us consider two points, say x_A and x_B in the manifold. Because the metric in $\hat{\mathcal{H}}(x; E, \sigma)$ is determined by the state to which

Figure 15. $\hat{\mathcal{H}}(x_A; E, \sigma)$ has a spread over $\Delta x \sim \frac{l_c}{\sqrt{g_{E,xx}}}$ along x direction in the manifold. In the presence of a large (small) mutual information between x_A and x_B , $g_{E,\mu\nu}$ is small (large) and the spread of the operator inserted at x_A can (can not) reach x_B . Therefore, the strength of coupling between x_A and x_B depends on entanglement of target states which determines the metric.

a target state is first projected by $\hat{P}_{E,\sigma}$, the strength of the coupling between x_A and x_B in the Hamiltonian is determined by the entanglement present in the target state. A state which supports small mutual information between the two points is projected to a state in which the proper distance between the points is large. Accordingly, the operators in eq. (4.49) are spread over a small *coordinate* distance, and the coupling between the points is weak. Conversely, for a state which supports large mutual information between the two points, the metric in $|E, \sigma\rangle$ gives a small proper distance and a large coupling in $\hat{H}_n(t)$. This is illustrated in figure 15. There is no absolute locality because the metric with which locality is defined varies with states. Since the coupling between any two points can be large for long-range entangled states, $\hat{H}_n(t)$ is not a local Hamiltonian as an operator. This is expected because there is no fixed notion of distance in any theory of background independent gravity [16]. Since locality of the Hamiltonian is determined relative to target states, we call $\hat{H}_n(t)$ *relatively local*. We emphasize that this conclusion on relative locality holds generally for Hamiltonians which induce background independent gravity irrespective of specific choice of $\Psi(\Phi; E, \sigma)$.

Ideally, one would hope to fix the regularization scheme and the higher-derivative terms in eq. (4.45) such that $\hat{\mathcal{H}}(x)$ and $\hat{\mathcal{H}}_\mu(x)$ satisfy the hypersurface embedding algebra at the quantum level [58]. The commutators that involve the momentum constraint satisfy the algebra easily. However, it is not clear whether there exists a regularized matter Hamiltonian which obeys the closed algebra at the quantum level. What is guaranteed in this construction is that $\hat{\mathcal{H}}(x)$ and $\hat{\mathcal{H}}_\mu(x)$ satisfy the closed algebra to the leading order in the large N limit within states with slowly varying collective variables in space.

Now we view $\hat{\mathcal{H}}(x)$ and $\hat{\mathcal{H}}_\mu(x)$ as generators of symmetry. States that are invariant under the symmetry, if exist, satisfy

$$\mathcal{H}_\mu^{E,\sigma}(x)\chi(E, \sigma) = 0, \quad (4.57)$$

$$\mathcal{H}^{E,\sigma}(x)\chi(E, \sigma) = 0. \quad (4.58)$$

Eq. (4.57) combined with eq. (4.41) implies that the quantum state of the matter fields is invariant under diffeomorphism. States that are invariant under diffeomorphism are topological because all physical properties are also invariant under diffeomorphism. Such states either have no entanglement at all, or must have infinitely long-range entanglement. Similarly, eq. (4.58) is a condition that a state is invariant under ‘time’ translation. States that

satisfy eqs. (4.57)–(4.58) generally have divergent norms because the amplitude of the wavefunction is conserved under the symmetry transformations that are non-compact [40, 41].

Therefore we consider normalizable states that break the symmetry spontaneously. In particular, we consider normalizable semi-classical states which satisfy the constraints to the leading order in N but break the symmetry only to the sub-leading order,

$$\chi(E, \sigma) = \chi_n e^{-\int dx \left[\sqrt{|\bar{g}|} \frac{\bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} (g_{E, \mu\alpha} - \bar{g}_{\mu\alpha}) e^{-\frac{l_c^2 \bar{\nabla}^2}{2}} (g_{E, \nu\beta} - \bar{g}_{\nu\beta}) + F(\bar{\sigma})(\sigma - \bar{\sigma}) e^{-\frac{l_c^2 \bar{\nabla}^2}{2}} (\sigma - \bar{\sigma}) - \frac{i}{\kappa^2} (\bar{\pi}^{\mu\nu} g_{E, \mu\nu} + \bar{\pi}_\sigma \sigma) \right]} \quad (4.59)$$

Here $\bar{g}_{\mu\nu}(x)$, $\bar{\sigma}(x)$, $\bar{\pi}^{\mu\nu}(x)$ and $\bar{\pi}_\sigma(x)$ are classical collective variables and their conjugate momenta, and χ_n is a normalization constant. $e^{-\frac{l_c^2 \bar{\nabla}^2}{2}}$ suppresses fluctuations of the collective variables with momenta larger than l_c^{-1} , where $\bar{\nabla}_\mu$ is the covariant derivative associated with the metric $\bar{g}_{\mu\nu}$. The wavefunction is manifestly invariant under $\text{SO}(3)$ gauge transformations. Both collective variables and their conjugate momenta are well defined if $\frac{l_c}{\Delta} \gg 1$, $\frac{l_c \bar{\pi}^{\mu\nu}}{\sqrt{\bar{g}}}, \frac{l_c \bar{\pi}_\sigma}{\sqrt{\bar{g}}} \gg \frac{1}{N^2} \left(\frac{l_c}{\Delta}\right)^{3/2}$. In the large N and the long-wavelength limits, eqs. (4.57) and (4.58) become the classical constraint equations [50],

$$\begin{aligned} \frac{1}{\sqrt{|\bar{g}|}} \left(\bar{\pi}^{\mu\nu} \bar{\pi}_{\mu\nu} - \frac{1}{2} (\bar{\pi}_\mu^\mu)^2 + \frac{1}{2F(\bar{\sigma})} \bar{\pi}_\sigma^2 \right) + \sqrt{|\bar{g}|} \left(-\bar{R} + \frac{F(\bar{\sigma})}{2} \bar{g}^{\mu\nu} \nabla_\mu \bar{\sigma} \nabla_\nu \bar{\sigma} + V(\bar{\sigma}) \right) &= 0, \\ 2\nabla_\nu \bar{\pi}^{\mu\nu} - (\nabla^\mu \bar{\sigma}) \bar{\pi}_\sigma &= 0 \end{aligned} \quad (4.60)$$

to the leading order in $1/N, \Delta/l_c, (l_c \nabla), (l_c \pi)$. If the classical collective variables and the conjugate momenta satisfy eq. (4.60), semi-classical states obey the momentum and Hamiltonian constraints approximately.

For such states with weakly broken symmetry, the constraints generate non-trivial evolution by creating Goldstone modes associated with the spontaneously broken symmetry,

$$|\chi(dt)\rangle = e^{-idt \int dx [n^{(1)}(x) \hat{\mathcal{H}}(x) + n^{(1)\mu}(x) \hat{\mathcal{H}}_\mu(x)]} |\chi\rangle, \quad (4.61)$$

where dt is an infinitesimal parameter. $n^{(1)}(x)$ and $n^{(1)\mu}(x)$ are arbitrary functions of x which control the local speed of the unitary transformation and the shift respectively. From eq. (4.43) and eq. (4.55), we obtain

$$\begin{aligned} |\chi(dt)\rangle &= \int DE^{(0)} D\sigma^{(0)} |E^{(0)}, \sigma^{(0)}\rangle e^{-idt \int dx [n^{(1)}(x) \mathcal{H}^{E^{(0)}, \sigma^{(0)}}(x) + n^{(1)\mu}(x) \mathcal{H}_\mu^{E^{(0)}, \sigma^{(0)}}(x)]} \chi(E^{(0)}, \sigma^{(0)}) \\ &= \int DE^{(0)} D\sigma^{(0)} DE^{(1)} D\sigma^{(1)} D\pi^{(1)} D\pi_\sigma^{(1)} |E^{(1)}, \sigma^{(1)}\rangle e^{\frac{i}{\kappa^2} \int dx [\pi^{(1)\mu i} (E_{\mu i}^{(1)} - E_{\mu i}^{(0)}) + \pi_\sigma^{(1)} (\sigma^{(1)} - \sigma^{(0)})]} \times \\ &\quad e^{-\frac{i}{\kappa^2} dt \int dx [n^{(1)}(x) \mathcal{H}[E^{(0)}, \sigma^{(0)}, \pi^{(1)}, \pi_\sigma^{(1)}] + n^{(1)\mu} \mathcal{H}_\mu[E^{(0)}, \sigma^{(0)}, \pi^{(1)}, \pi_\sigma^{(1)}]]} \chi(E^{(0)}, \sigma^{(0)}), \end{aligned} \quad (4.62)$$

where $D\pi D\pi_\sigma \equiv \mu^{-1}(E, \sigma) \prod_x d\pi^{\mu i}(x) d\pi_\sigma(x)$,

$$\begin{aligned} \mathcal{H}[E, \sigma, \pi, \pi_\sigma] = & : \frac{1}{|E|} \left(G_{ijkl} E_\mu^j E_\nu^l \pi^{\mu i} \pi^{\nu k} + \frac{1}{2F(\sigma)} \pi_\sigma^2 \right) : \\ & + |E| \left(-R + \frac{F(\sigma)}{2} g_E^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma + V(\sigma) + U_3(g_E, \sigma) \right) + O(N^{-2}, l_c \pi^3), \\ \mathcal{H}_\mu[E, \sigma, \pi, \pi_\sigma] = & E_{\mu i} \nabla_\nu \pi^{\nu i} - \nabla_\mu \sigma \pi_\sigma. \end{aligned} \quad (4.63)$$

After repeating this step infinitely many times in the $dt \rightarrow 0$ limit, one obtains

$$|\chi(t)\rangle = \int DE(\tau, x) D\sigma(\tau, x) D\pi(\tau, x) D\pi_\sigma(\tau, x) |E(t), \sigma(t)\rangle e^{iS} \chi(E(0), \sigma(0)), \quad (4.64)$$

where

$$S = \frac{1}{\kappa^2} \int_0^t d\tau \int dx \left\{ \pi^{\mu i} \partial_t E_{\mu i} + \pi_\sigma \partial_t \sigma - n \mathcal{H}[E, \sigma, \pi, \pi_\sigma] - n^\mu \mathcal{H}_\mu[E, \sigma, \pi, \pi_\sigma] \right\}. \quad (4.65)$$

The theory describes gravity coupled with a scalar field in a fixed gauge. In the large N limit, the saddle-point configuration which satisfies the classical field equations dominates the path integration.

The theory in eq. (4.65) has three length scales. One is the cut-off length scale, l_c which controls the relative locality of the theory. At length scales larger than l_c , where lengths are measured with a saddle-point configuration of the dynamical metric, the local field theory is valid for the description of fluctuations of the collective variables above the saddle-point configuration. At shorter length scales, non-local effects kick in through the higher-derivative terms in the Hamiltonian. The other length scale is the Planck scale, $\kappa \sim \frac{l_c}{N}$ below which quantum fluctuations of the collective variables become important. Another scale is the curvature of the saddle-point geometry set by the vacuum energy. Here we assume that $V(\sigma)$ is chosen such that the cosmological constant is much smaller than l_c^{-4} , which in general requires a fine tuning of the potential. In the large N limit, the semi-classical field theory approximation is valid for modes whose wavelengths are larger than l_c .

5 Black hole evaporation

5.1 Entanglement neutralization

The discussion in the previous section shows that a relatively local Hamiltonian for the matter field induces a quantum theory for the collective variables, which reduces to Einstein's gravity coupled with a scalar field at long distances in the large N limit. Given an initial state $|\chi\rangle$, the state at parameter time t is given by

$$|\chi(t)\rangle = \mathcal{P}_T e^{-i \int_0^t d\tau \int dx [n(x, \tau) \hat{\mathcal{H}}(x) + n^\mu(x, \tau) \hat{\mathcal{H}}_\mu(x)]} |\chi\rangle, \quad (5.1)$$

where $\mathcal{P}_T$ time-orders the evolution operator. The state at time t can be written in the basis of $|E, \sigma\rangle$ as $|\chi(t)\rangle = \int DE D\sigma |E, \sigma\rangle \chi(t; E, \sigma)$. If the initial state is chosen to be

a semi-classical state in eq. (4.59), $\chi(t; E, \sigma)$ is sharply peaked at a saddle-point path in the large N limit. The saddle-point path, $\{\bar{E}_{\mu i}(x, t), \bar{\sigma}(x, t), \bar{\pi}^{\mu i}(x, t), \bar{\pi}_\sigma(x, t)\}$ solves the classical field equation with the initial condition, $g_{\bar{E}, \mu\nu}(x, 0) = \bar{g}_{\mu\nu}(x)$, $\bar{\sigma}(x, 0) = \bar{\sigma}(x)$, $\bar{E}_i^\nu(x, 0)\bar{\pi}^{\mu i}(x, 0) = \bar{\pi}^{\mu\nu}(x) + \bar{\pi}^{\nu\mu}(x)$, $\bar{\pi}_\sigma(x, 0) = \bar{\pi}_\sigma(x)$.

Suppose the initial condition of the semi-classical state is chosen such that the classical solution describes a gravitational collapse of a spherically symmetric mass shell, which forms a macroscopic black hole with mass M in the asymptotically flat Minkowski space. We assume that the size of the horizon is much larger than the cut-off length scales, $r_H = 2M\kappa^2 \gg l_c$ right after the black hole is formed. Across the horizon, the underlying quantum state supports color entanglement,

$$S_\Phi^i = \frac{\pi r_H^2}{\kappa^2} \sim N^2 \left(\frac{r_H}{l_c} \right)^2. \quad (5.2)$$

We identify eq. (5.2) as the Bekenstein-Hawking entropy [37, 38]. On the other hand, the singlet entanglement entropy is negligible when the black hole is just formed.

In describing the consequent evolution of the black hole, we choose $n^\mu(x, t) = 0$ and $n(x, t) > 0$ at all x, t in eq. (5.1). Far away from the black hole, the lapse is chosen to approach a non-zero constant. Inside the black hole, the lapse is chosen such that time slices do not hit the ‘singularity’, and the theory at each time slice stays within the realm of the semi-classical field theory. As time progresses, the space that connects the interior of the black hole and the asymptotic region is stretched as is illustrated in figure 16.

For a large but finite N , one should include quantum fluctuations of the collective variables in eq. (4.64). Due to quantum fluctuations, the black hole emits Hawking radiation, and its mass decreases in time. Let $\bar{g}'_{\mu\nu}(x, t)$ describe an evaporating black hole geometry that satisfies the classical field equation in the presence of a time dependent black hole mass and the energy-momentum tensor of the Hawking radiation. The rate at which the black hole mass decreases should be self-consistently determined by the Hawking radiation created by small fluctuations around the classical geometry, $g_{\mu\nu}(x, t) = \bar{g}'_{\mu\nu}(x, t) + \delta g_{\mu\nu}(x, t)$, $\sigma(x, t) = \bar{\sigma}'(x, t) + \delta\sigma(x, t)$. For large black holes, the Hawking radiation should be well approximated by the adiabatic approximation because the rate at which the mass decreases is much smaller than r_H^{-1} .

As the black hole evaporates, the horizon shrinks and the color entanglement entropy decreases. On the other hand, the singlet entanglement entropy increases because Hawking radiation is emitted in the form of fluctuations of the collective variables. This is easy to understand in the weakly coupled effective theory for the collective variables. From the perspective of the fundamental matrix field, it is not obvious why Hawking radiation is emitted only in the singlet sector. However, the underlying theory for the matrix field in eq. (4.51) is likely to be a strongly coupled field theory, and it is conceivable that there is only $O(1)$ Hawking radiation [59]. If there was Hawking radiation of $O(N^2)$ color degrees of freedom, the induced theory for the collective variables could not be the semi-classical general relativity which we know is the correct description of eq. (4.51). Therefore the increasing entanglement between the Hawking radiation and the degrees of freedom inside the horizon should be in the singlet sector. In this regard, black hole evaporation can be

Figure 16. A collapsing mass shell (red dashed line) and the horizon (blue solid line) in the in-going Eddington-Finkelstein coordinate system, where the transverse directions (θ, ϕ) are suppressed. The metric outside the collapsing mass shell is given by the Schwarzschild metric, $ds^2 = -\left(1 - \frac{2M}{r'}\right) dv^2 + 2dr' dv + r'^2(d\theta^2 + \sin^2\theta d\phi^2)$ where $t^* = v - r'$. The curves represent time slices which march forward with a constant lapse far from the black hole while avoiding the singularity inside the horizon. $\frac{\partial}{\partial r}$ ($\frac{\partial}{\partial t}$) represents the vector field that is tangential (perpendicular) to each time slice. It is noted that $\frac{\partial}{\partial r}$, which is distinct from $\frac{\partial}{\partial r'}$, is chosen to be space-like everywhere.

viewed as an *entanglement neutralization process* in which entanglement across horizon is transferred from color degrees of freedom to singlet degrees of freedom.

5.2 Late time evolution

The fate of $\chi(t; E, \sigma)$ in the large t limit largely depends on which of the following two possibilities is realized. The first possibility is that eq. (5.1) evolves to a state which ceases to support a well-defined horizon as early as the Page time. The second is that $\chi(t; E, \sigma)$ remains sharply peaked around the time dependent classical metric $\bar{g}'_{\mu\nu}(x, t)$ with a well defined horizon. Resolving this issue is a complicated dynamical question. It may well be that the answer depends on details of initial states. In this section, we consider consequences of the second possibility, assuming that there exists some initial states which continue to support a well-defined horizon throughout the evolution before the size of black hole reaches the cut-off length scale. The reason we focus on the second possibility is because the black hole information puzzle arises in that case [38]. Our goal here is to understand how the puzzle can be in principle resolved in the current framework.

Figure 17. (a) The Penrose diagram of an evaporating black hole. The question mark is intended to represent the unknown fate of the interior of the black hole. (b) The geometry for the quantum state at $t = T$. When the size of the horizon is l_c , the region inside the horizon is stretched to a long funnel with size $L \sim M^{7/2} \kappa^5 l_c^{-1/2}$.

Let us choose the lapse such that at time T the long throat inside the horizon as well as the horizon itself reaches the size of $\sim l_c^2$ in the transverse direction. Up to this point, the local semi-classical description is still valid. The asymptotic time that takes for a large black hole with mass M to evolve to this state scales as $T \sim M^3 \kappa^4$. During this time, the throat inside the horizon stretches to the size, $L \sim \sqrt{g_{rr}} M^3 \kappa^4 \sim M^{7/2} \kappa^5 l_c^{-1/2}$, where $g_{rr} \sim \frac{M \kappa^2}{l_c}$ is used for the metric inside the horizon.

Under the time evolution, the size of the horizon continues to shrink, and so does the color entanglement entropy. At $t = T$, the color entanglement entropy across the horizon becomes $S_\Phi^f \sim \frac{l_c^2}{\kappa^2} \sim N^2$. On the other hand, the Hawking radiation generates a large entanglement across the horizon,

$$S_H \sim S_\Phi^i. \quad (5.3)$$

In the context of the present induced gravity, the ‘information puzzle’ [38, 60–65] can be phrased as the statement that the small color entanglement entropy, which is identified as the Bekenstein-Hawking entropy, can not account for the large entanglement created by the Hawking radiation. However, this is not necessarily paradoxical because the color entanglement entropy captures only a part of the full entanglement. The other part is the singlet entanglement entropy which is supported by correlations between fluctuations of the collective variables.

In this theory, the time evolution is unitary by construction. In order to support the entanglement with the Hawking radiation outside the horizon, at least e^{S_H} states need to be excited inside the horizon. In the large N limit, modes with wavelengths larger than l_c are

described by the weakly coupled field theory, and they have Gaussian fluctuations which are order of δh_ν^μ , $\delta\sigma \sim \frac{1}{N}$.⁹ The volume of the throat inside the horizon is $V \sim M^{7/2} \kappa^5 l_c^{3/2}$ at $t = T$. If all field theory modes with wavelengths larger than l_c are excited, the total number of states that are available inside the horizon is $e^{S_{col}}$ with

$$S_{col} \sim M^{7/2} \kappa^5 l_c^{-3/2} \sim N^2 \left(\frac{r_H}{l_c} \right)^{7/2}. \quad (5.4)$$

For a macroscopic initial black hole with $r_H \gg l_c$, $S_{col} \gg S_H$. The number of field theory modes available inside the horizon is much larger than what the color entanglement entropy (Bekenstein-Hawking entropy) accounts for near the end of evaporation process.¹⁰ The states counted in eq. (5.4) include highly excited states. However, one does not need all of them. In order to account for $S_H \sim N^2 \left(\frac{r_H}{l_c} \right)^2$, it is enough to excite modes with wavelengths larger than $\lambda \sim l_c \left(\frac{r_H}{l_c} \right)^{1/2}$. In the $r_H \gg l_c$ limit, only those excitations whose wavelengths are much larger than l_c are needed to account for the entanglement with the early Hawking radiation.

The large number of singlet states that are not captured by the Bekenstein-Hawking entropy suggests that the quantum state inside the horizon is far from equilibrium. The lack of equilibrium can arise dynamically because of the relative nature of the Hamiltonian. Since the strength of coupling between two given points in the manifold is determined by states, points that were once connected by a strong coupling can dynamically decouple at a later time if the state at later time supports little color entanglement. In this sense, not only states but also the Hamiltonian effectively flow in time under the evolution generated by the relatively local Hamiltonian. The growth of a long geometry inside the horizon with a small contact with the exterior is a form of dynamical localization where the coupling across the horizon becomes weaker in time.

6 Summary and discussion

In this paper, it is shown that a quantum theory of gravity can be induced from quantum matter. Metric is introduced as a collective variable which controls entanglement of matter fields. There exists a Hamiltonian for matter fields whose induced dynamics for metric coincides with the general relativity at long distances in the large N limit. The Hamiltonian that gives rise to the background independent gravity is non-local. However, it has a relative locality in that the range of interactions is controlled by entanglement present in target states. Within the induced theory of gravity, a black hole evaporation can be understood as a unitary process where entanglement of matter is gradually transferred from color degrees of freedom to singlet collective degrees of freedom. We close with some remarks and speculations on open questions.

⁹For example, we can choose $F(\sigma) \sim e^{3\sigma}$, $V(\sigma) \sim l_c^{-2} e^{4\sigma}$ such that $\delta h_\nu^\mu \sim \frac{1}{N}$, $\delta\sigma \sim \frac{e^{-3/2\sigma}}{N}$ for $e^\sigma \ll 1$.

¹⁰It is noted that a renormalization of the Newton's constant by $1/N$ corrections is not enough to incorporate the singlet entanglement entropy within the color entanglement entropy.

Figure 18. One possible outcome of evolving the state in figure 17(b) further in time is a topological phase transition where the region inside the horizon is disconnected from the outer universe.

6.1 Baby universe as a dynamical localization

What happens if we continue to evolve the state in eq. (5.1) beyond time T ? It is hard to answer the question without considering the full theory beyond the local semi-classical approximation. Here we consider possibilities that do not modify the usual rules of quantum mechanics. If the long throat inside the horizon remains attached to the outer space, it gives rise to a long-lived (or stable) remnant (for a review on remnant, see ref. [66] and references there-in). If the region inside the horizon becomes geometrically disconnected from the exterior, a baby universe can form as in figure 18 [67, 68]. From the perspective of the underlying matter fields, a baby universe corresponds to a dynamical localization where the region inside the horizon dynamically decouples from the exterior. Here localization is driven not by disorder but by the relative nature of the Hamiltonian, where the strength of couplings between the interior and the exterior of the horizon dynamically flows to zero at late time.

Since the proper volume inside the horizon can be arbitrarily large, there can be infinitely many different remnants or baby universes. Although this seems unphysical, this is allowed within the present theory because the number of degrees of freedom within a given region of the manifold is not fixed. Any background independent quantum theory of gravity should include such states in the Hilbert space. The presence of infinitely many internal states does not necessarily lead to an infinite production rate if the matrix element between a state with a smooth geometry and a state with a remnant or a baby universe is exponentially suppressed as the volume of the ‘hidden’ space increases. For example, let $E_{\mu i}(x)$ be the flat Euclidean geometry, $g_{E,\mu\nu} = \delta_{\mu,\nu}$, and $E'_{\mu i}(x)$ represent a geometry which coincides with the Euclidean metric for $|x| > R$ but has a long funnel with proper volume $V' \gg R^3$ for $|x| < R$. Let $\hat{O}$ be an operator that has a support within $|x| < R$. The matrix element is given by

$$\langle E', \sigma' | \hat{O} | E, \sigma \rangle \sim \int D^{(\hat{E}, \hat{\sigma})} \Phi \, O \, e^{-\int_{|x| < R} dx \left[|E'| \mathcal{L}(\Phi; E', \sigma') + |E| \mathcal{L}(\Phi; E, \sigma) \right]}. \quad (6.1)$$

If the matrix element is small enough, the net production rate can be suppressed.

Figure 19. A pictorial representation of the overlap between $|0\rangle$ and $|E, \sigma\rangle$. Since $|0\rangle$ is annihilated by the Hamiltonian and momentum constraints, the overlap is invariant under the insertions of the evolution operator, which gives rise to a path integration of the metric and the scalar field in the bulk.

6.2 dS/CFT

By construction, $\hat{\mathcal{H}}_\mu(x)$ and $\hat{\mathcal{H}}(x)$ satisfy the closed algebra [58] to the leading order in the $1/N$ and the derivative expansions. However, the commutator between two Hamiltonian constraints may have an anomaly that involves higher derivative terms and $1/N$ corrections. Whether one can choose a regularization scheme for the matter Hamiltonian such that the algebra is closed exactly is an open question [69–72].

Suppose there exists a state $|0\rangle$ which is annihilated by $\hat{\mathcal{H}}(x)$ and $\hat{\mathcal{H}}_\mu(x)$. The existence of such states doesn't necessarily require that the algebra is closed at the operator level. Although $|0\rangle$ is not normalizable in general, the overlap with a normalizable state, $|E, \sigma\rangle$ is well defined. The overlap, which can be viewed as a wavefunction of universe, is invariant under the insertion of the evolution operator generated by the Hamiltonian and momentum constraints [28],

$$\langle 0|E, \sigma\rangle = \langle 0| e^{-idt \int dx [n(x, \tau) \hat{\mathcal{H}}(x) + n^\mu(x, \tau) \hat{\mathcal{H}}_\mu(x)]} |E, \sigma\rangle. \quad (6.2)$$

Repeated insertions of the evolution operators lead to

$$\langle 0|E, \sigma\rangle = \int DE(\tau, x) D\sigma(\tau, x) D\pi(\tau, x) D\pi_\sigma(\tau, x) \langle 0|E(t), \sigma(t)\rangle e^{iS} \Big|_{E(0, x)=E(x), \sigma(0, x)=\sigma(x)}, \quad (6.3)$$

where the bulk action is given by eq. (4.65). Therefore, the overlap is given by the $(D+1)$ -dimensional path integration with a Dirichlet boundary condition for the collective variables as is represented in figure 19. The bulk path integral can be viewed as the gravitational dual for the generating functional of the non-unitary boundary field theory in the dS/CFT correspondence [24, 25]. If the lapse and the shift are integrated over, the bulk path integration becomes a projection operator which imposes the Hamiltonian and momentum constraints on the boundary state, $|E, \sigma\rangle$.

6.3 Non-gaussian states

The standard lore in the AdS/CFT correspondence is that a bulk theory includes only a small number of fields if the dual boundary theory is in a strong coupling regime and the majority of operators have large scaling dimensions [73]. The number of dynamical fields one has to keep in the bulk is determined by the number of independent operators

from which all other operators can be constructed as composite operators [27]. Although there are in general infinitely many such operators, if most of them acquire large scaling dimensions they correspond to heavy fields in the bulk, which can be integrated out without sacrificing locality in the bulk.

From this perspective, it is rather surprising that a simple bulk theory is obtained from the Gaussian wavefunction defined at a temporal boundary. The reason why we have only dynamical metric and a scalar field in the bulk is because the initial states and the Hamiltonian for the matter field are fine-tuned so that the time evolution generates deformations contained within the sector of the energy-momentum tensor and one scalar operator only. From the point of view of RG flow, we are in the basin of attraction toward a multi-critical point via a fine tuning. In general, there exist ‘relevant’ perturbations which take the flow away from the multi-critical point once perturbations are turned on. In the present work, we simply didn’t consider such perturbations in the initial state. In order to suppress other operators without fine tuning, one probably needs non-Gaussian wavefunctions which describe strongly coupled boundary theories.

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A Diffeomorphism invariance of the inner product

First we note that eq. (4.1) is covariant under diffeomorphism, that is,

$$K_{(\tilde{E}, \tilde{\sigma})} f_n^{(\tilde{E}, \tilde{\sigma})}(x) = \lambda_n^{(E, \sigma)} f_n^{(\tilde{E}, \tilde{\sigma})}(x), \quad (\text{A.1})$$

where $f_n^{(\tilde{E}, \tilde{\sigma})}(\tilde{x}) = f_n^{(E, \sigma)}(x)$ with $\tilde{x} = x + \xi(x)$, and $(\tilde{E}, \tilde{\sigma})$ is related to (E, σ) through the diffeomorphism. This means $\lambda_n^{(\tilde{E}, \tilde{\sigma})} = \lambda_n^{(E, \sigma)}$ and $\text{Tr} \left(\Gamma \left[l_c^2 K_{(\tilde{E}, \tilde{\sigma})} \right] \right) = \text{Tr} \left(\Gamma \left[l_c^2 K_{(E, \sigma)} \right] \right)$. Similarly, $\text{Tr} \left(\Gamma \left[l_c^2 K_{(\tilde{E}', \tilde{\sigma}')} \right] \right) = \text{Tr} \left(\Gamma \left[l_c^2 K_{(E', \sigma')} \right] \right)$. Furthermore, $J_{(\tilde{E}, \tilde{\sigma})}^{(\tilde{E}', \tilde{\sigma}')} = J_{(E, \sigma)}^{(E', \sigma')}$ because the matrix elements in eq. (4.8) are invariant under diffeomorphism,

$$\int d\tilde{x} |\tilde{E}'| f_m^{(\tilde{E}', \tilde{\sigma}')*}(\tilde{x}) f_n^{(\tilde{E}, \tilde{\sigma})}(\tilde{x}) = \int dx |E'| f_m^{(E', \sigma')*}(x) f_n^{(E, \sigma)}(x). \quad (\text{A.2})$$

Finally,

$$\begin{aligned} & \int D^{(\tilde{E}, \tilde{\sigma})} \tilde{\Phi} e^{-\frac{1}{2} \int d^3x \text{tr} \tilde{\Phi} \left(|\tilde{E}'| e^{-\Gamma[l_c^2 K_{(\tilde{E}', \tilde{\sigma}')}] + |\tilde{E}| e^{-\Gamma[l_c^2 K_{(\tilde{E}, \tilde{\sigma})}]} \right)} \tilde{\Phi} \\ &= \int D^{(E, \sigma)} \Phi e^{-\frac{1}{2} \int d^3x \text{tr} \Phi \left(|E'| e^{-\Gamma[l_c^2 K_{(E', \sigma')}] + |E| e^{-\Gamma[l_c^2 K_{(E, \sigma)}]} \right)} \Phi \end{aligned} \quad (\text{A.3})$$

because $D^{(\tilde{E}, \tilde{\sigma})} \tilde{\Phi} = D^{(E, \sigma)} \Phi$ and $\int d^3x \text{tr} \Phi \left(|\tilde{E}'| e^{-\Gamma[l_c^2 K_{(\tilde{E}', \tilde{\sigma}')}] } + |\tilde{E}| e^{-\Gamma[l_c^2 K_{(\tilde{E}, \tilde{\sigma})}] } \right) \Phi$ is invariant under diffeomorphism. This proves that $\langle E', \sigma' | E, \sigma \rangle = \langle \tilde{E}', \tilde{\sigma}' | \tilde{E}, \tilde{\sigma} \rangle$.

B $\langle E', \sigma' | E, \sigma \rangle = 0$ unless $|E'(x)| = |E(x)|$

We prove the title of this appendix. The vanishing overlap between states with different local proper volumes is the result of mismatch in the gaussian wavefunctions for large momentum modes. For modes with momenta much larger than curvature scales, we can use the semi-classical approximation to represent eigenmodes in terms of wavepackets. To keep track of position of wavepackets, we divide the spatial manifold into blocks. Suppose we choose a coordinate system with $0 \leq x_1, x_2, x_3 < l_c$ for a spatial manifold with T^3 topology. The j -th block is denoted as $S_j = \{x | x_j^\mu \leq x^\mu < x_j^\mu + l, \text{ with } \mu = 1, 2, 3\}$, where $x_j^\mu = l j^\mu$ with j^μ being integers. $l \equiv \frac{l_c}{P}$ with an integer P is chosen to be small enough that $g_{E, \mu\nu}(x), \sigma(x), g_{E', \mu\nu}(x), \sigma'(x)$ can be regarded to be constants in each block. A wavepacket is labeled by a block index j and quantized momentum k allowed within the block,

$$f_{jk}^{(E, \sigma)}(x) = \sqrt{\frac{8}{l^3 |E(x_j)|}} \sin(k_1(x^1 - x_j^1)) \sin(k_2(x^2 - x_j^2)) \sin(k_3(x^3 - x_j^3)), \quad \text{if } x \in S_j$$

$$= 0, \quad \text{if } x \notin S.$$
(B.1)

Here momentum is quantized as $k_\mu = \frac{\pi n_\mu}{l}$ with positive integer n_μ . The normalization is chosen such that $\int dx |E| f_{j'k'}^{(E, \sigma)*}(x) f_{jk}^{(E, \sigma)}(x) = \delta_{j', j} \delta_{k', k}$. Field configurations which vanish at the boundaries of the blocks can be decomposed in terms of the wavepackets. Because $f_{jk}^{(E, \sigma)}(x) = \sqrt{\frac{|\hat{E}(x_j)|}{|E(x_j)|}} f_{jk}^{(\hat{E}, \hat{\sigma})}(x)$ and $f_{jk}^{(E', \sigma')}(x) = \sqrt{\frac{|\hat{E}'(x_j)|}{|E'(x_j)|}} f_{jk}^{(\hat{E}', \hat{\sigma}')} (x)$, we have $\Phi_{jk}^{(E, \sigma)} = \sqrt{\frac{|E(x_j)|}{|\hat{E}(x_j)|}} \Phi_{jk}^{(\hat{E}, \hat{\sigma})}$ and $\Phi_{jk}^{(E', \sigma')} = \sqrt{\frac{|E'(x_j)|}{|\hat{E}'(x_j)|}} \Phi_{jk}^{(\hat{E}', \hat{\sigma}')}.$ The wave packets contribute to the overlap in eq. (4.19) as

$$\langle E', \sigma' | E, \sigma \rangle \propto \left[\prod_{jk} \frac{|E(x_j)| |E'(x_j)|}{|\hat{E}(x_j)|^2} \right]^{\frac{1}{4}} e^{-\frac{N^2}{4} \{ \text{Tr}(\Gamma[l_c^2 K_{(E', \sigma')}]) + \text{Tr}(\Gamma[l_c^2 K_{(E, \sigma)}]) \}} \times$$

$$\int \prod_{jk} d\Phi_{jk}^{(\hat{E}, \hat{\sigma})} e^{-\frac{1}{2} \sum_{jk} \left\{ e^{-\Gamma[l_c^2 \lambda_{jk}^{(E', \sigma')}] } \frac{|E'(x_j)|}{|\hat{E}'(x_j)|} + e^{-\Gamma[l_c^2 \lambda_{jk}^{(E, \sigma)}]} \frac{|E(x_j)|}{|\hat{E}(x_j)|} \right\} \left| \Phi_{jk}^{(\hat{E}, \hat{\sigma})} \right|^2},$$
(B.2)

where $\lambda_{jk}^{(E, \sigma)} = g_E^{\mu\nu}(x_j) k_\mu k_\nu + \frac{e^{2\sigma(x_j)}}{l_c^2}$ and $\lambda_{jk}^{(E', \sigma')} = g_{E'}^{\mu\nu}(x_j) k_\mu k_\nu + \frac{e^{2\sigma'(x_j)}}{l_c^2}$. Since $\Gamma[l_c^2 \lambda_{jk}^{(E, \sigma)}], \Gamma[l_c^2 \lambda_{jk}^{(E', \sigma')}] \rightarrow 0$ in the large k limit, the contribution from large momentum modes becomes

$$\langle E', \sigma' | E, \sigma \rangle \propto \left[\prod_j \prod_k' \frac{4|E(x_j)| |E'(x_j)|}{[|E(x_j)| + |E'(x_j)|]^2} \right]^{\frac{1}{4}} e^{-\frac{N^2}{4} \{ \text{Tr}(\Gamma[l_c^2 K_{(E', \sigma')}]) + \text{Tr}(\Gamma[l_c^2 K_{(E, \sigma)}]) \}},$$
(B.3)

where $\prod'_k$ include momenta with $g_E^{\mu\nu}(x_j)k_\mu k_\nu, g_{E'}^{\mu\nu}(x_j)k_\mu k_\nu \gg l_c^{-2}$. It is noted that $\frac{4|E(x_j)||E'(x_j)|}{[|E(x_j)|+|E'(x_j)|]^2} < 1$ unless $|E(x_j)| = |E'(x_j)|$. Since there are infinitely many momentum modes with $g_E^{\mu\nu}(x_j)k_\mu k_\nu, g_{E'}^{\mu\nu}(x_j)k_\mu k_\nu \gg l_c^{-2}$ in each block, $\langle E', \sigma' | E, \sigma \rangle$ vanishes if there is any block in which $|E(x_j)| \neq |E'(x_j)|$. Here we ignored the modes that describe fluctuations of $\Phi(x)$ at the boundaries between blocks. Those modes do not change the conclusion because they form fluctuations of measure zero.

C Overlap

In this appendix, we compute the overlap in eq. (4.20) between states associated with metrics close to the Euclidean metric. Since the overlap is zero for $|E(x)| \neq |E'(x)|$, we consider the case with $|E(x)| = |E'(x)|$. In this case, the exponent in the integrand of eq. (4.20) can be written as

$$\int dx |E| \text{tr} \left[\Phi \frac{e^{-\Gamma[l_c^2 K_{(E', \sigma')}] + e^{-\Gamma[l_c^2 K_{(E, \sigma)}]}}{2} \Phi \right] = \int dx |E| \text{tr} \left[\Phi e^{-\Gamma[l_c^2 K'']} \Phi \right] + O(l_c^2 h_{\mu\nu}^2, l_c^2 \delta\sigma^2), \quad (\text{C.1})$$

where $K'' \equiv \frac{1}{2} [K_{(E', \sigma')} + K_{(E, \sigma)}]$. To the leading order in $h_{\mu\nu} \equiv g_{E', \mu\nu} - g_{E, \mu\nu}$, $\delta\sigma \equiv \sigma' - \sigma$ and l_c , eq. (4.20) becomes

$$\langle E', \sigma' | E, \sigma \rangle \approx e^{-\frac{N^2}{4} \{ \text{Tr}(\Gamma[l_c^2 K_{(E', \sigma')}] + \text{Tr}(\Gamma[l_c^2 K_{(E, \sigma)}]) \}} \int D^{(E, \sigma)} \Phi e^{-\int d^3x |E| \text{tr} \left[\Phi e^{-\Gamma[l_c^2 K'']} \Phi \right]}, \quad (\text{C.2})$$

where we use $J_{(E, \sigma)}^{(E', \sigma')} = 1$ for $|E'(x)| = |E(x)|$. The logarithm of eq. (C.2) can be written as

$$\ln \langle E', \sigma' | E, \sigma \rangle \approx -\frac{N^2}{4} \int_{l_c^2}^{\infty} \frac{dt}{t} \text{Tr} \left(e^{-K_{(E', \sigma')} t} + e^{-K_{(E, \sigma)} t} - 2e^{-K'' t} \right). \quad (\text{C.3})$$

The Kernel with perturbations in the collective variables is written as $K'' = K + \delta K$ and $K_{(E', \sigma')} = K + 2\delta K$, where $K \equiv K_{(E, \sigma)}$. The exponential of the perturbed Kernel can be expressed as

$$\begin{aligned} e^{-(K+\delta K)t} &= e^{-Kt} - \int_0^t d\tau e^{-K(t-\tau)} \delta K e^{-K\tau} \\ &\quad + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-K(t-\tau_1)} \delta K e^{-K(\tau_1-\tau_2)} \delta K e^{-K\tau_2} + O(\delta K^3). \end{aligned} \quad (\text{C.4})$$

The terms linear in δK are all canceled in eq. (C.3) and one obtains

$$\begin{aligned} \ln \langle E', \sigma' | E, \sigma \rangle &\approx -\frac{N^2}{2} \int_{l_c^2}^{\infty} \frac{dt}{t} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \text{Tr} \left(\delta K e^{-K(\tau_1-\tau_2)} \delta K e^{-K(t-\tau_1+\tau_2)} \right) + O(\delta K^3) \\ &= -\frac{N^2}{4} \int_{l_c^2}^{\infty} dt \int_0^t d\tau \text{Tr} \left(\delta K e^{-K\tau} \delta K e^{-K(t-\tau)} \right) + O(\delta K^3), \end{aligned} \quad (\text{C.5})$$

where the cyclic property of the trace has been used. By inserting the complete set of basis of $K_{(E,\sigma)}$ in eq. (C.5), one obtains

$$\begin{aligned} \ln \langle E', \sigma' | E, \sigma \rangle &= -\frac{N^2}{4} \sum_{n,m} \int_{l_c^2}^\infty dt \int_0^t d\tau e^{-(\lambda_m^{(E,\sigma)} - \lambda_n^{(E,\sigma)})\tau - \lambda_n^{(E,\sigma)}t} |\langle n | \delta K | m \rangle|^2 \\ &\quad + O(\delta K^3), \end{aligned} \quad (C.6)$$

where $\langle n | \delta K | m \rangle = \int dx |E| f_n^{(E,\sigma)*}(x) \delta K f_m^{(E,\sigma)}(x)$. The subsequent integrations over τ and t gives

$$\ln \langle E', \sigma' | E, \sigma \rangle \approx -\frac{N^2}{4} \sum_{n,m} \frac{|\langle n | \delta K | m \rangle|^2}{\lambda_m^{(E,\sigma)} - \lambda_n^{(E,\sigma)}} \left(\frac{e^{-l_c^2 \lambda_n^{(E,\sigma)}}}{\lambda_n^{(E,\sigma)}} - \frac{e^{-l_c^2 \lambda_m^{(E,\sigma)}}}{\lambda_m^{(E,\sigma)}} \right). \quad (C.7)$$

Now, we compute eq. (C.7) between a state with the Euclidean metric and a state with small perturbations. Suppose the space manifold has T^3 topology, and $E_{\mu i}$ describes a flat Euclidean metric,

$$g_{E,\mu\nu} = a^2 \delta_{\mu\nu} \quad (C.8)$$

in a coordinate system with $0 \leq x_1, x_2, x_3 < l_c$, where a is a scale factor that determines the proper size of the manifold in the unit of the short-distance cut-off. Let $E_{\mu i}$ be the triad for the flat Euclidean metric, $g_{E,\mu\nu} = a^2 \delta_{\mu\nu}$, and σ is a constant. The eigenvectors of $K_{(E,\sigma)}$ are the plane waves, $f_p(x) = \frac{1}{(al_c)^{3/2}} e^{ipx}$ with discrete momentum, $p_\mu = \frac{2\pi}{l_c} n_\mu$ with integer n_μ and eigenvalue, $\lambda_p = p^2 + \frac{e^{2\sigma}}{l_c^2}$. States with perturbed collective variables are parameterized by the deformed metric and scalar field, $g_{E',\mu\nu}(x) = g_{E,\mu\nu} + h_{\mu\nu}(x)$, $\sigma'(x) = \sigma + \delta\sigma(x)$, where $h_{\mu\nu}(x)$ and $\delta\sigma(x)$ are small perturbations.

In the presence of the perturbation in the collective variables, the kernel is modified to be $K_{(E',\sigma')} = K + 2\delta K$ with

$$\delta K = \frac{1}{2} \left[\frac{1}{|E(x)|} \partial_\mu \left(|E(x)| h^{\mu\nu} \partial_\nu - |E(x)| \frac{h}{2} g_E^{\mu\nu} \partial_\nu \right) + \frac{e^{2\sigma}}{l_c^2} \left(\frac{h}{2} + 2\delta\sigma \right) \right]. \quad (C.9)$$

From eq. (C.7), we write the overlap between the state with the Euclidean metric and a state with the perturbation as

$$\ln \langle E', \sigma' | E, \sigma \rangle = -\frac{N^2 e^{-e^{2\sigma}}}{4} \sum_{p,q} \frac{|\langle q+p | \delta K | q \rangle|^2}{(p+q)^2 - q^2} \left(\frac{e^{-l_c^2 q^2}}{q^2 + l_c^{-2} e^{2\sigma}} - \frac{e^{-l_c^2 (q+p)^2}}{(q+p)^2 + l_c^{-2} e^{2\sigma}} \right) \quad (C.10)$$

to the quadratic order in δK , where

$$\langle q+p | \delta K | q \rangle = \frac{1}{2(al_c)^{3/2}} \left\{ \left[-q^\mu (q^\nu + p^\nu) + \frac{1}{2} (q(q+p) + l_c^{-2} e^{2\sigma}) g_E^{\mu\nu} \right] h_{\mu\nu}(p) + 2e^{2\sigma} l_c^{-2} \delta\sigma(p) \right\}. \quad (C.11)$$

In the large a limit, the summation over the momenta can be done through integration, and we obtain

$$\begin{aligned} \ln \langle E', \sigma' | E, \sigma \rangle \approx & -\frac{N^2}{4^2 l_c^3} e^{-e^2 \sigma} \sum_p \left[I^{\mu\nu\alpha\beta}(p) h_{\mu\nu}(p) h_{\alpha\beta}(-p) \right. \\ & \left. + 2I^{\mu\nu}(p) h_{\mu\nu}(p) \delta\sigma(-p) + I(p) \delta\sigma(p) \delta\sigma(-p) \right] \end{aligned} \quad (\text{C.12})$$

to the leading order in l_c^{-1} , $h_{\mu\nu}$ and $\delta\sigma$. Here $h_{\mu\nu}(p) = \int dx |E| f_p^*(x) h_{\mu\nu}(x)$ and $\delta\sigma(p) = \int dx |E| f_p^*(x) \delta\sigma(x)$ with $f_p(x) = \frac{1}{(al_c)^{3/2}} e^{ipx}$. In the large a limit, $I^{\mu\nu\alpha\beta}(p)$, $I^{\mu\nu}(p)$, $I(p)$ are given by

$$\begin{aligned} I^{\mu\nu\alpha\beta}(p) = & I^{4,1}(p) g_E^{\mu\nu} g_E^{\alpha\beta} + I^{4,2}(p) \left(g_E^{\mu\alpha} g_E^{\nu\beta} + g_E^{\mu\beta} g_E^{\nu\alpha} \right) + I^{4,3}(p) \left(\frac{p^\mu p^\nu}{p^2} g_E^{\alpha\beta} + \frac{p^\alpha p^\beta}{p^2} g_E^{\mu\nu} \right) \\ & + I^{4,4}(p) \left(\frac{p^\nu p^\beta}{p^2} g_E^{\alpha\mu} + \frac{p^\mu p^\beta}{p^2} g_E^{\alpha\nu} + \frac{p^\nu p^\alpha}{p^2} g_E^{\mu\beta} + \frac{p^\mu p^\alpha}{p^2} g_E^{\beta\nu} \right) + I^{4,5}(p) \frac{p^\mu p^\nu p^\alpha p^\beta}{p^4}, \\ I^{\mu\nu}(p) = & I^{2,1}(p) g_E^{\mu\nu} + I^{2,2}(p) \frac{p^\mu p^\nu}{p^2}, \\ I(p) = & I^{0,1}(p), \end{aligned} \quad (\text{C.13})$$

where

$$\begin{aligned} I^{4,1}(p) = & -\frac{\pi^{3/2} (8\sqrt{\pi} e^{3\sigma} C_\sigma - 14e^{2\sigma} + 1)}{24} - \frac{\pi^{3/2} (8\sqrt{\pi} e^\sigma C_\sigma + 2e^{2\sigma} - 1)}{48} (l_c p)^2 + O((l_c p)^4), \\ I^{4,2}(p) = & \frac{\pi^{3/2} (2\sqrt{\pi} e^{3\sigma} C_\sigma - 2e^{2\sigma} + 1)}{6} + \frac{\pi^{3/2} (\sqrt{\pi} e^\sigma C_\sigma - 1)}{12} (l_c p)^2 + O((l_c p)^4), \\ I^{4,3}(p) = & -\frac{\pi^{3/2} - 2\pi^2 e^\sigma C_\sigma}{12} (l_c p)^2 + O((l_c p)^4), \\ I^{4,4}(p) = & \frac{\pi^{3/2} - \pi^2 e^\sigma C_\sigma}{12} (l_c p)^2 + O((l_c p)^4), \\ I^{4,5}(p) = & O((l_c p)^4), \\ I^{2,1}(p) = & \pi^{3/2} e^{2\sigma} - \frac{1}{6} \pi^{3/2} e^\sigma (2\sqrt{\pi} C_\sigma + e^\sigma) (l_c p)^2 + O((l_c p)^4), \\ I^{2,2}(p) = & \frac{1}{3} \pi^2 e^\sigma C_\sigma (l_c p)^2 + O((l_c p)^4), \\ I^{0,1}(p) = & 4\pi^2 e^{3\sigma} C_\sigma - \frac{1}{3} \pi^{3/2} e^\sigma (\sqrt{\pi} C_\sigma + 2e^\sigma) (l_c p)^2 + O((l_c p)^4). \end{aligned} \quad (\text{C.14})$$

Here $C_\sigma \equiv e^{e^2 \sigma} \text{erfc}(e^\sigma)$, where $\text{erfc}(x)$ is the complimentary error function. It has the asymptotic behavior, $\lim_{\sigma \rightarrow -\infty} C_\sigma = 1$, $\lim_{\sigma \rightarrow \infty} C_\sigma = \frac{1}{\sqrt{\pi}} e^{-\sigma}$. For $|E'(x)| = |E(x)|$, $g_E^{\mu\nu} h_{\mu\nu} = 0$ to the leading order in $h_{\mu\nu}$, and only $I^{4,2}$ and $I^{0,1}$ are important in eq. (C.13) in the small momentum limit. The overlap decays exponentially in $h_{\mu\nu}$ and $\delta\sigma$ with the width which is controlled by the eigenvalues of the matrix,

$$\mathcal{M}(p) = \frac{N^2}{4^2 l_c^3} e^{-e^2 \sigma} \begin{pmatrix} I^{\mu\nu\alpha\beta}(p) & I^{\mu\nu}(p) \\ I^{\alpha\beta}(p) & I(p) \end{pmatrix}. \quad (\text{C.15})$$

It is noted that two states with $|h_\mu^\nu(p)|, e^{3/2\sigma}|\delta\sigma(p)| > \frac{l_c^{3/2}}{N}$ for any wavevector $p \ll l_c^{-1}$ are almost orthogonal. In real space, this implies that two states whose collective variables differ by $h_\mu^\nu(x) \sim \frac{1}{N}$, $|\delta\sigma(x)| \sim \frac{e^{-3/2\sigma}}{N}$ over a proper volume l_c^3 are nearly orthogonal. This follows from the fact that if $h_\mu^\nu(x) \sim \frac{1}{N}$ over a proper volume l_c^3 , $h_\mu^\nu(p) \sim \frac{l_c^3}{N\sqrt{(al_c)^{3/2}}}$ for $p^2 < l_c^{-2}$. Therefore, $\ln \langle E', \sigma' | E, \sigma \rangle \sim -\frac{N^2}{l_c^3} \sum_{p^2 < l_c^{-2}} |h_\mu^\nu(p)|^2 \sim -1$. In the large N limit, the overlap becomes proportional to the delta function,

$$\lim_{N \rightarrow \infty} \langle E', \sigma' | E, \sigma \rangle \propto \prod_p \left[\delta(\delta\sigma(p)) \prod_{(\mu, \nu)} \delta(h_{\mu\nu}(p)) \right]. \quad (\text{C.16})$$

D Decomposition of the entanglement entropy for semi-classical states

In this appendix, we derive eqs. (4.31)–(4.35) from eq. (4.30). In manipulating eq. (4.30), it is convenient to rewrite eq. (4.10) as

$$\Psi(\Phi; E, \sigma) = e^{-tr \int dx dy \Phi(x) t_{E, \sigma}(x, y) \Phi(y) - \frac{1}{2} S_0[E, \sigma]}, \quad (\text{D.1})$$

where $t(x, y)$ can be written as

$$t(x, y) = \frac{1}{2} \int dz |E(z)| \left[\frac{e^{2\sigma}}{l_c^2} \delta(z-x) \delta(z-y) + g_E^{\mu\nu}(z) \partial_\mu \delta(z-x) \partial_\nu \delta(z-y) + O(l_c^2 \nabla^4) \right] \quad (\text{D.2})$$

in the small l_c limit with fixed $\frac{e^{2\sigma}}{l_c^2}$. In eq. (D.1), the derivative terms in $\Psi(\Phi; E, \sigma)$ are represented as bi-local couplings. In terms of the bi-local representation of the wavefunction, Z_n in eq. (4.30) is written as

$$\begin{aligned} Z_n = & \int \prod_{j=1}^n \left[D^{(\hat{E}, \hat{\sigma})} \Phi^j D E_1^j D \sigma_1^j D E_2^j D \sigma_2^j \right] \\ & e^{-\sum_j tr \int_{x, y \in \bar{A}} dx dy \Phi^j(x) \left(t_{E_2^j, \sigma_2^j}(x, y) + t_{E_1^j, \sigma_1^j}(x, y) \right) \Phi^j(y)} \times \\ & e^{-\sum_j tr \int_{x, y \in A} dx dy \Phi^j(x) \left(t_{E_2^{j+1}, \sigma_2^{j+1}}(x, y) + t_{E_1^j, \sigma_1^j}(x, y) \right) \Phi^j(y)} \times \\ & e^{-\sum_j tr \int_{x \in \bar{A}, y \in A} dx dy \Phi^j(x) \left(t_{E_1^j, \sigma_1^j}(x, y) + t_{E_1^j, \sigma_1^j}(y, x) \right) \Phi^j(y)} \times \\ & e^{-\sum_j tr \int_{x \in \bar{A}, y \in A} dx dy \Phi^j(x) \left(t_{E_2^j, \sigma_2^j}(x, y) + t_{E_2^j, \sigma_2^j}(y, x) \right) \Phi^{j-1}(y)} \times \\ & \left(\prod_{j=1}^n \tilde{\chi}^*(E_1^j, \sigma_1^j) \tilde{\chi}(E_2^j, \sigma_2^j) \right). \end{aligned} \quad (\text{D.3})$$

Here $\tilde{\chi}(E, \sigma) = e^{-\frac{1}{2} S_0[E, \sigma]} \chi(E, \sigma)$. The variables with replica indices outside the range of $[1, n]$ are identified cyclically with period n : $E_{\mu i}^{n+j} = E_{\mu i}^j$, $\sigma^{n+j} = \sigma^j$. Each term in eq. (D.3) has obvious meaning. The second and third lines represent the bi-local couplings within region $\bar{A}$ and A , respectively. The fourth and fifth lines represent the bi-local couplings between $\bar{A}$ and A in the bra and ket of the wavefunction, respectively. The last line is

the contribution from the collective variables. It is the last three terms that generate non-trivial entanglement.

Now we insert the following expression for the identity inside the integration of eq. (D.3),

$$1 = \prod_{j=1}^n \left\{ \tilde{\mu}^{-1}(E_1^{[j]}, \sigma_1^{[j]}) e^{S_0[E_1^{[j]}, \sigma_1^{[j]}]} \prod_x \left[\delta(\sigma_2^j(x) - \sigma_1^{[j]}(x)) \prod_{(\mu, \nu)} \delta(g_{E_2^j, \mu\nu}(x) - g_{E_1^{[j]}, \mu\nu}(x)) \right] \times \left[\int D(\hat{E}, \hat{\sigma}) \Phi^j e^{-\sum_j \text{tr} \int dxdy \Phi^j(x) t_{E_1^{[j]}, \sigma_1^{[j]}(x, y)} \Phi^j(y) - \sum_j \text{tr} \int dxdy \Phi^j(x) t_{E_2^j, \sigma_2^j(x, y)} \Phi^j(y)} \right]^{-1} \right\}, \quad (\text{D.4})$$

where x, y run over the entire space, and

$$\begin{aligned} \{E^{[j]}(x), \sigma^{[j]}(x)\} &= \{E^j(x), \sigma^j(x)\} && \text{for } x \in \bar{A}, \\ &= \{E^{j-1}(x), \sigma^{j-1}(x)\} && \text{for } x \in A. \end{aligned} \quad (\text{D.5})$$

The functional integration of Φ in the last line is unconstrained, and eq. (4.24) has been used. The integrations over E_2^j, σ_2^j result in

$$\begin{aligned} Z_n &= \int \prod_{j=1}^n [D E^j D \sigma^j] \prod_{j=1}^n \left\{ e^{S_0[E^{[j]}, \sigma^{[j]}]} \tilde{\chi}^* \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^j(x_A), \sigma^j(x_A) \right) \times \right. \\ &\quad \left. \tilde{\chi} \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^{j-1}(x_A), \sigma^{j-1}(x_A) \right) \Big|_{\sigma^2, \dots, n(x) = \sigma^1(x), g_{E^2, \dots, n, \mu\nu}(x) = g_{E^1, \mu\nu}(x), x \in \partial A} \right\} \times \\ &\quad \int \prod_{j=1}^n D(\hat{E}, \hat{\sigma}) \Phi^j \left[e^{-2 \sum_j \text{tr} \int_{x, y \in \bar{A}} dxdy \Phi^j(x) t_{E^j, \sigma^j}(x, y) \Phi^j(y)} \times \right. \\ &\quad e^{-2 \sum_j \text{tr} \int_{x, y \in A} dxdy \Phi^j(x) t_{E^j, \sigma^j}(x, y) \Phi^j(y)} \times \\ &\quad e^{-\sum_j \text{tr} \int_{x \in \bar{A}, y \in A} dxdy \Phi^j(x) \left(t_{E^j, \sigma^j}(x, y) + t_{E^j, \sigma^j}(y, x) \right) \Phi^j(y)} \times \\ &\quad \left. e^{-\sum_j \text{tr} \int_{x \in A, y \in \bar{A}} dxdy \Phi^j(x) \left(t_{E^{[j]}, \sigma^{[j]}(x, y)} + t_{E^{[j]}, \sigma^{[j]}(y, x)} \right) \Phi^{j-1}(y)} \right] \times \\ &\quad \left[\int \prod_{j=1}^n D(\hat{E}, \hat{\sigma}) \Phi^j e^{-2 \sum_j \text{tr} \int_{x, y} dxdy \Phi^j(x) t_{E^{[j]}, \sigma^{[j]}(x, y)} \Phi^j(y)} \right]^{-1}. \end{aligned} \quad (\text{D.6})$$

It is noted that the delta functions in eq. (D.4) twists the boundary condition for the collective variables in eq. (D.6), and force $\sigma^j(x)$ and $g_{E^j, \mu\nu}$ to be independent of j in ∂A .

If $\chi(E, \sigma)$ is sharply peaked at a classical configuration, $\bar{E}(x), \bar{\sigma}(x)$ (and its SO(3) gauge orbits), eq. (D.6) can be approximately factorized into the contribution from the matter fields and the contribution from the collective variables as

$$Z_n \approx Z_n^\Phi(\bar{E}, \bar{\sigma}) Z_n^{E, \sigma}, \quad (\text{D.7})$$

where

$$\begin{aligned}
 Z_n^\Phi(\bar{E}, \bar{\sigma}) = & \int \prod_{j=1}^n D(\bar{E}, \bar{\sigma}) \Phi^j \left[e^{-2 \sum_j \text{tr} \int_{x,y \in \bar{A}} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \times \right. \\
 & e^{-2 \sum_j \text{tr} \int_{x,y \in A} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \times \\
 & e^{-\sum_j \text{tr} \int_{x \in \bar{A}, y \in A} dx dy \Phi^j(x) \left(t_{\bar{E}, \bar{\sigma}}(x,y) + t_{\bar{E}, \bar{\sigma}}(y,x) \right) \Phi^j(y)} \times \\
 & \left. e^{-\sum_j \text{tr} \int_{x \in \bar{A}, y \in A} dx dy \Phi^j(x) \left(t_{\bar{E}, \bar{\sigma}}(x,y) + t_{\bar{E}, \bar{\sigma}}(y,x) \right) \Phi^{j-1}(y)} \right] \times \\
 & \left[\int \prod_{j=1}^n D(\bar{E}, \bar{\sigma}) \Phi^j e^{-2 \sum_j \text{tr} \int_{x,y} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \right]^{-1},
 \end{aligned} \tag{D.8}$$

$$\begin{aligned}
 Z_n^{E, \sigma} = & \int \prod_{j=1}^n D E^j D \sigma^j \left\{ e^{S_0[E^{[j]}, \sigma^{[j]}]} \tilde{\chi}^* \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^j(x_A), \sigma^j(x_A) \right) \times \right. \\
 & \left. \tilde{\chi} \left(E^j(x_{\bar{A}}), \sigma^j(x_{\bar{A}}); E^{j-1}(x_A), \sigma^{j-1}(x_A) \right) \Big|_{\substack{\sigma^2, \dots, n(x_{\partial A}) = \sigma^1(x_{\partial A}), \\ g_{E^2, \dots, n, \mu\nu}(x_{\partial A}) = g_{E^1, \mu\nu}(x_{\partial A})}} \right\}.
 \end{aligned} \tag{D.9}$$

As a result, the entanglement entropy is given by the sum of two contributions,

$$S(A) \approx S_\Phi(A) + S_{E, \sigma}(A), \tag{D.10}$$

where $S_\Phi(A) = -\lim_{n \rightarrow 1} \frac{1}{n-1} (Z_n^\Phi - 1)$ and $S_{E, \sigma}(A) = -\lim_{n \rightarrow 1} \frac{1}{n-1} (Z_n^{E, \sigma} - 1)$.

$S_\Phi(A)$ is the entanglement supported by the matter field defined in the classical geometry set by the collective variables, $\{\bar{E}_{\mu i}, \bar{\sigma}\}$. The third and the fourth lines in eq. (D.8) describe the couplings between A and $\bar{A}$. It penalizes configurations of the matter fields which have discontinuities across the boundary. In the small l_c limit, the cross couplings between A and $\bar{A}$ are localized near the boundary, and it can be replaced with a function that depends on $\Phi^j(x) - \Phi^{j-1}(x)$ at the boundary,

$$\begin{aligned}
 & e^{-\sum_j \text{tr} \int_{x \in \bar{A}, y \in A} dx dy \left[\Phi^j(x) \left(t_{\bar{E}, \bar{\sigma}}(x,y) + t_{\bar{E}, \bar{\sigma}}(y,x) \right) \Phi^j(y) + \Phi^j(x) \left(t_{\bar{E}, \bar{\sigma}}(x,y) + t_{\bar{E}, \bar{\sigma}}(y,x) \right) \Phi^{j-1}(y) \right]} \\
 & \approx \prod_{r=2}^N \prod_{x \in \partial A} F \left(\Phi^r(x) - \Phi^{r-1}(x); \bar{E}(x) \right).
 \end{aligned} \tag{D.11}$$

It is noted that $F(\Phi^r(x) - \Phi^{r-1}(x); \bar{E}(x))$ does not depend on $\bar{\sigma}$ in the small l_c limit because the term associated with σ in eq. (D.2) is ultra-local. The integration measure in the numerator of eq. (D.8) can be decomposed into modes with a Dirichlet boundary condition and modes localized at the boundary as

$$\int \prod_{j=1}^n D(\bar{E}, \bar{\sigma}) \Phi^j = \int \prod_{j=1}^n D(\bar{E}, \bar{\sigma}) \Phi^j \Big|_{\Phi^2, \dots, n(x_{\partial A}) = \Phi^1(x_{\partial A})} \times \prod_{r=2}^n \prod_{x \in \partial A} D(\bar{E}, \bar{\sigma}) \Phi^r(x). \tag{D.12}$$

Here $\prod_{j=1}^n D^{(\bar{E}, \bar{\sigma})} \Phi^j \Big|_{\Phi^{2, \dots, n}(x_{\partial A}) = \Phi^1(x_{\partial A})} \equiv \prod_{j=1}^n \prod_m \prod_{a,b} d\Phi_{ab,m}^{0(\bar{E}, \bar{\sigma})j}$ with $\Phi_{ab,m}^{0(\bar{E}, \bar{\sigma})j}$ representing the m -th normal mode which satisfies the boundary condition, $\Phi^{2, \dots, n}(x_{\partial A}) = \Phi^1(x_{\partial A})$. Eq. (D.12) should be viewed as the defining expression for $\prod_{r=2}^n \prod_{x \in \partial A} D^{(\bar{E}, \bar{\sigma})} \Phi^r(x)$. Since $\prod_r \prod_{x \in \partial A} F(\Phi^r(x) - \Phi^{r-1}(x); \bar{E}(x))$ is sharply peaked at $\Phi^{2, \dots, n}(x_{\partial A}) = \Phi^1(x_{\partial A})$ in the small l_c limit, the integrations in eq. (D.8) can be factorized as

$$Z_n^\Phi(\bar{E}, \bar{\sigma}) \approx C(\bar{E}, \bar{\sigma}) \int \prod_{j=1}^n D^{(\bar{E}, \bar{\sigma})} \Phi^j \left[e^{-2 \sum_j \text{tr} \int_{x,y \in \bar{A}} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \times \right. \\ \left. e^{-2 \sum_j \text{tr} \int_{x,y \in A} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \right]_{\Phi^{2, \dots, n}(x_{\partial A}) = \Phi^1(x_{\partial A})} \times \\ \left[\int \prod_{j=1}^n D^{(\bar{E}, \bar{\sigma})} \Phi^j e^{-2 \sum_j \text{tr} \int_{x,y} dx dy \Phi^j(x) t_{\bar{E}, \bar{\sigma}}(x,y) \Phi^j(y)} \right]^{-1}, \quad (\text{D.13})$$

where

$$C(\bar{E}, \bar{\sigma}) \equiv \int \prod_{x \in \partial A} \left[\prod_{r=2}^n D^{(\bar{E}, \bar{\sigma})} \Phi^r(x) \prod_{r=2}^n F(\Phi^r(x) - \Phi^{r-1}(x); \bar{E}(x)) \right]. \quad (\text{D.14})$$

In order to determine $C(\bar{E}, \bar{\sigma})$, we first note that $Z_n^\Phi(\bar{E}, \bar{\sigma}) = C(\bar{E}, \bar{\sigma})$ in the $\sigma \rightarrow \infty$ limit. This follows from the fact that $\lim_{\sigma \rightarrow \infty} \int \prod_{j=1}^n D^{(\bar{E}, \bar{\sigma})} \Phi^j e^{-2 \int dx |\bar{E}| \mathcal{L}[\Phi^j; \bar{E}, \bar{\sigma}]} \Big|_{\Phi^{2, \dots, n}(x_{\partial A}) = \Phi^1(x_{\partial A})} = \lim_{\sigma \rightarrow \infty} \int \prod_{j=1}^n D^{(\bar{E}, \bar{\sigma})} \Phi^j e^{-2 \int dx |\bar{E}| \mathcal{L}[\Phi^j; \bar{E}, \bar{\sigma}]} = 1$. Because $S_\Phi(A)$ should vanish in the large σ limit, we have $\lim_{n \rightarrow 1} \frac{C(\bar{E}, \infty) - 1}{n-1} = 0$. For a finite $\bar{\sigma}$, we have $C(\bar{E}, \bar{\sigma}) = \left[\tilde{J}_{(\bar{E}, \infty)}^{(\bar{E}, \bar{\sigma})} \right]^{n-1} C(\bar{E}, \infty)$, where $\tilde{J}_{(\bar{E}, \infty)}^{(\bar{E}, \bar{\sigma})}$ is the Jacobian for the change of basis at the boundary. The Jacobian is finite and independent of l_c . For example, for a flat metric with a constant $\bar{\sigma}$, $J_{(\bar{E}, \infty)}^{(\bar{E}, \bar{\sigma})} = 1$ irrespective of the value of $\bar{\sigma}$. As a result, $C(\bar{E}, \bar{\sigma})$ does not give rise to a singular contribution to the entanglement entropy in the small l_c limit. Therefore, $S_\Phi(A)$ can be written as eq. (4.33) to the leading order in l_c [43]. On the other hand, $S_{E, \sigma}(A)$ is the entanglement generated by fluctuations of the collective variables.

E Computation of the entanglement entropy

Here we derive eq. (4.36) for the state with the Euclidean metric in eq. (C.8) and a constant σ . In the three torus, we compute the entanglement entropy for region A which is parameterized by the coordinate, $A = \{(x_1, x_2, x_3) | 0 \leq x_1 < l, 0 \leq x_2 < l_c, 0 \leq x_3 < l_c\}$. In the presence of the Dirichlet boundary condition, $\Phi(0, x^2, x^3) = \Phi(l, x^2, x^3) = 0$, the eigenmodes in region A are modified such that

$$f_k = \sqrt{\frac{2}{a^3 l_c^2 l}} \sin(k_1 x^1) e^{i(k_2 x^2 + k_3 x^3)}, \quad (\text{E.1})$$

where $(k_1, k_2, k_3) = \left(\frac{n_1\pi}{l}, \frac{2n_2\pi}{l_c}, \frac{2n_3\pi}{l_c}\right)$ with eigenvalue $\lambda_k = k^2 + l_c^{-2}e^{2\sigma}$. Here n_1 represents positive integers while n_2, n_3 are general integers. The free energy from region A is given by

$$F_A = -\frac{N^2}{2} \int_{l_c^2}^{\infty} \frac{dt}{t} e^{-e^{2\sigma} l_c^{-2} t} \left[\sum_{n_1=1}^{\infty} e^{-\left(\frac{\pi n_1}{at}\right)^2 t} \right] \left[\sum_{n=-\infty}^{\infty} e^{-\left(\frac{2\pi n}{al_c}\right)^2 t} \right]^2. \quad (\text{E.2})$$

Similarly, the free energy from the complement of A is

$$F_{\bar{A}} = -\frac{N^2}{2} \int_{l_c^2}^{\infty} \frac{dt}{t} e^{-e^{2\sigma} l_c^{-2} t} \left[\sum_{n_1=1}^{\infty} e^{-\left(\frac{\pi n_1}{a(l_c-l)}\right)^2 t} \right] \left[\sum_{n=-\infty}^{\infty} e^{-\left(\frac{2\pi n}{al_c}\right)^2 t} \right]^2. \quad (\text{E.3})$$

Subtracting the free energy without the Dirichlet boundary condition,

$$F = -\frac{N^2}{2} \int_{l_c^2}^{\infty} \frac{dt}{t} e^{-e^{2\sigma} l_c^{-2} t} \left[\sum_{n=-\infty}^{\infty} e^{-\left(\frac{2\pi n}{al_c}\right)^2 t} \right]^3, \quad (\text{E.4})$$

one obtains

$$F_A + F_{\bar{A}} - F = \frac{N^2}{2} \int_{l_c^2}^{\infty} \frac{dt}{t} e^{-e^{2\sigma} l_c^{-2} t} \frac{a^2 l_c^2}{4\pi t} \quad (\text{E.5})$$

in the limit that $a \gg 1$. Integration over t gives eq. (4.36) in the $e^\sigma \rightarrow 0$ limit.

F Proof of eq. (4.41)

We consider the unitary operator that generates diffeomorphism for the matter field,

$$\hat{T}(n) = e^{-i \int dx n^\mu(x) \hat{H}_\mu(x)}, \quad (\text{F.1})$$

where $n^\mu(x)$ is an infinitesimal translation. $\hat{T}(n)$ shifts $\Phi(x)$ by $n^\mu(x)$ as

$$\hat{T}(n)|\Phi\rangle = A(n)|\tilde{\Phi}\rangle, \quad (\text{F.2})$$

where $\tilde{\Phi}(\tilde{x}) = \Phi(x)$ with $\tilde{x}^\mu = x^\mu + n^\mu(x)$. $A(n)$ is a constant which is determined from the normalization condition,

$$\begin{aligned} \delta^{(\hat{E}, \hat{\sigma})}(\Phi' - \Phi) &= \langle \Phi' | \hat{T}(n)^\dagger \hat{T}(n) | \Phi \rangle \\ &= |A(n)|^2 \langle \tilde{\Phi}' | \tilde{\Phi} \rangle = |A(n)|^2 \delta^{(\hat{E}, \hat{\sigma})}(\tilde{\Phi}' - \tilde{\Phi}), \end{aligned} \quad (\text{F.3})$$

where $\delta^{(\hat{E}, \hat{\sigma})}(\Phi' - \Phi) = \prod_{a,b} \prod_n \delta\left(\Phi'_{ab,n} - \Phi_{ab,n}\right)$. Let $\tilde{\sigma}(x) = \hat{\sigma}(x) - n^\mu \nabla_\mu \hat{\sigma}$ and $\tilde{\hat{E}}_{\mu i}(x) = \hat{E}_{\mu i}(x) - \nabla_\mu n^\nu \hat{E}_{\nu i}$. From $\int D^{(\hat{E}, \hat{\sigma})} \Phi \delta^{(\hat{E}, \hat{\sigma})}(\Phi' - \Phi) = \int D^{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})} \Phi \delta^{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}(\Phi' - \Phi)$ and $D^{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})} \Phi = J_{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}^{(\hat{E}, \hat{\sigma})} D^{(\hat{E}, \hat{\sigma})} \Phi$, we have $\delta^{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}(\Phi' - \Phi) = J_{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}^{(\hat{E}, \hat{\sigma})} \delta^{(\hat{E}, \hat{\sigma})}(\Phi' - \Phi)$. On the other hand,

$$\delta^{(\hat{E}, \hat{\sigma})}(\Phi' - \Phi) = \delta^{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}(\tilde{\Phi}' - \tilde{\Phi}) = J_{(\tilde{\hat{E}}, \tilde{\hat{\sigma}})}^{(\hat{E}, \hat{\sigma})} \delta^{(\hat{E}, \hat{\sigma})}(\tilde{\Phi}' - \tilde{\Phi}). \quad (\text{F.4})$$

Eq. (F.3) and eq. (F.4) leads to

$$A(n) = \left[J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} \right]^{1/2}. \quad (\text{F.5})$$

Now we examine how $\hat{T}(n)$ acts on $|E, \sigma\rangle$:

$$\begin{aligned} \hat{T}(n)|E, \sigma\rangle &= \int D^{(\hat{E}, \hat{\sigma})} \Phi \left[J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} \right]^{1/2} |\tilde{\Phi}\rangle \Psi(\Phi; E, \sigma) \\ &= \int D^{(\tilde{E}, \tilde{\sigma})} \tilde{\Phi} \left[J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} J_{(\tilde{E}, \tilde{\sigma})}^{(E, \sigma)} \right]^{1/2} |\tilde{\Phi}\rangle \Psi(\tilde{\Phi}; \tilde{E}, \tilde{\sigma}) \\ &= \int D^{(\hat{E}, \hat{\sigma})} \tilde{\Phi} \left[J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} J_{(\tilde{E}, \tilde{\sigma})}^{(E, \sigma)} \right]^{1/2} |\tilde{\Phi}\rangle \Psi(\tilde{\Phi}; \tilde{E}, \tilde{\sigma}) \\ &= |\tilde{E}, \tilde{\sigma}\rangle. \end{aligned} \quad (\text{F.6})$$

From the first to the second lines, we use $D^{(\hat{E}, \hat{\sigma})} \Phi = D^{(\tilde{E}, \tilde{\sigma})} \tilde{\Phi}$ and eq. (4.18). For the next equality, we use $D^{(\tilde{E}, \tilde{\sigma})} \tilde{\Phi} = J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} D^{(\hat{E}, \hat{\sigma})} \tilde{\Phi}$. In the last equality, we use the fact that eq. (4.8) is invariant under diffeomorphism: $J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})} J_{(\tilde{E}, \tilde{\sigma})}^{(E, \sigma)} = \frac{J_{(\tilde{E}, \tilde{\sigma})}^{(\hat{E}, \hat{\sigma})}}{J_{(E, \sigma)}^{(\tilde{E}, \tilde{\sigma})}} = 1$. The linear terms in n^μ gives eq. (4.41).

G Induced Wheeler-DeWitt Hamiltonian

In this appendix, we prove eq. (4.53). As a first step, we construct the projection operator $\hat{P}_{E, \sigma}$ in eq. (4.51) that satisfies

$$\hat{P}_{E, \sigma} \int DE' D\sigma' |E', \sigma'\rangle \chi(E', \sigma') = |E, \sigma\rangle \chi(E, \sigma). \quad (\text{G.1})$$

We first try $|E, \sigma\rangle \langle E, \sigma|$ as a candidate for the projection operator. However, it does not satisfy eq. (G.1) because the convolution integration in $\int DE' D\sigma' \langle E, \sigma | E', \sigma' \rangle \chi(E', \sigma')$ generates a significant smearing of the wavefunction. In the present case, the smearing modifies $\chi(E, \sigma)$ beyond the wavefunction normalization unlike the case in section 3.2. The difference is caused by the fact that in the Gaussian model κ^{-1} is proportional to the number of matter fields unlike the non-Gaussian model considered for the minisuperspace cosmology. The smearing can be represented as a differential operator,

$$\int DE' D\sigma' \langle E, \sigma | E', \sigma' \rangle \chi(E', \sigma') = \mathcal{S} \left(E, \sigma, \frac{\delta}{\delta E}, \frac{\delta}{\delta \sigma} \right) \chi(E, \sigma), \quad (\text{G.2})$$

where

$$\mathcal{S} \left(E, \sigma, \frac{\delta}{\delta E}, \frac{\delta}{\delta \sigma} \right) = \left[e^{\int dy_1 dy_2 |E(y_1)| |E(y_2)| \mathcal{M}_{ab}^{-1}(y_1, y_2) \frac{\delta}{\delta v_a(y_1)} \frac{\delta}{\delta v_b(y_2)}} \right] \quad (\text{G.3})$$

in the large N limit with $v_a(x) = (g_{E,\mu\nu}(x), \sigma(x))$. $\mathcal{M}^{-1}(y_1, y_2) \sim \frac{1}{|E|^2 N^2} e^{-d_{y_1, y_2}/l_c}$ is given by the inverse of eq. (C.15), which decays exponentially in the proper distance between y_1 and y_2 . Inside [...] of eq. (G.3), it is understood that the functional differentiations are ordered to the right so that they act only on $\chi(E, \sigma)$. However, to the leading order in $1/N^2$, the normal ordering can be ignored because $\mathcal{S} = e^{\int dy_1 dy_2 |E(y_1)| |E(y_2)| \mathcal{M}_{ab}^{-1}(y_1, y_2) \frac{\delta}{\delta v_a(y_1)} \frac{\delta}{\delta v_b(y_2)} + O(N^{-4})}$. The metric differentiation is defined through the chain rule, $\frac{\delta}{\delta g_{E,\mu\nu}(y)} = \frac{\partial E_{\rho j}(y)}{\partial g_{E,\mu\nu}(y)} \frac{\delta}{\delta E_{\rho j}(y)}$, where $\frac{\partial E_{\rho j}(y)}{\partial g_{E,\mu\nu}(y)}$ is evaluated in a fixed gauge for the local $\text{SO}(3)$ symmetry. Since $\chi(E, \sigma)$ depends on the triad only through the metric, $\frac{\delta}{\delta g_{E,\mu\nu}(y)}$ is independent of the gauge choice. In order to undo the smearing introduced in eq. (G.2), we insert the inverse of $\mathcal{S}$ as

$$\hat{P}_{E,\sigma} = |E, \sigma\rangle \mathcal{S}^{-1} \langle E, \sigma|, \quad (\text{G.4})$$

where $\mathcal{S}^{-1}$ is the inverse of $\mathcal{S}$, which can be computed order by order in $1/N^2$. To the leading order, it is given by $\mathcal{S}^{-1} = e^{-\int dy_1 dy_2 |E(y_1)| |E(y_2)| \mathcal{M}_{ab}^{-1}(y_1, y_2) \frac{\delta}{\delta v_a(y_1)} \frac{\delta}{\delta v_b(y_2)} + O(N^{-4})}$. The existence of $\mathcal{S}^{-1}$ relies on the linear independence of $|E, \sigma\rangle$. This follows from the fact that states for a large number (N^2) of matter fields are parameterized by $O(1)$ collective fields. Therefore two states with different collective variables should be linearly independent for a sufficiently large N while they are not necessarily orthogonal. Eq. (G.4) satisfies eq. (4.52) because

$$|E, \sigma\rangle \mathcal{S}^{-1} \langle E, \sigma| \int DE' D\sigma' |E', \sigma'\rangle \chi(E', \sigma') = |E, \sigma\rangle \mathcal{S}^{-1} \mathcal{S} \chi(E, \sigma) = |E, \sigma\rangle \chi(E, \sigma) \quad (\text{G.5})$$

Now it is straightforward to check eq. (4.53). From eq. (4.44) and eq. (4.51), we obtain

$$\begin{aligned} & \hat{\mathcal{H}}(x) \int DE' D\sigma' |E', \sigma'\rangle \chi(E', \sigma') \\ &= \frac{1}{2} \int DED\sigma DE' D\sigma' \left\{ \tilde{h}^{E,\sigma}(x) |E, \sigma\rangle \mathcal{S}^{-1} \langle E, \sigma| + |E, \sigma\rangle \mathcal{S}^{-1} \tilde{h}^{E,\sigma}(x) \langle E, \sigma| \right\} |E', \sigma'\rangle \chi(E', \sigma') \\ &= \frac{1}{2} \int DED\sigma \left\{ \tilde{h}^{E,\sigma}(x) |E, \sigma\rangle \mathcal{S}^{-1} \mathcal{S} + |E, \sigma\rangle \mathcal{S}^{-1} \tilde{h}^{E,\sigma}(x) \mathcal{S} \right\} \chi(E, \sigma) \\ &= \frac{1}{2} \int DED\sigma \left\{ \tilde{h}^{E,\sigma}(x) |E, \sigma\rangle + \mathcal{S} \tilde{h}^{E,\sigma\dagger}(x) \mathcal{S}^{-1} |E, \sigma\rangle \right\} \chi(E, \sigma), \end{aligned} \quad (\text{G.6})$$

where $\mathcal{S}^\dagger = \mathcal{S}$ has been used. The hermiticity of $\mathcal{S}$ follows from the fact that $\int DED\sigma DE' D\sigma' \chi^*(E, \sigma) \langle E, \sigma| E', \sigma'\rangle \chi'(E', \sigma')$ can be either written as $\int DED\sigma \chi^*(E, \sigma) \mathcal{S} \chi'(E, \sigma)$ or $\int DED\sigma [\mathcal{S} \chi(E, \sigma)]^* \chi'(E, \sigma)$.

Since $\mathcal{S}$ does not commute with $\tilde{h}^{E,\sigma\dagger}(x)$, the unsmearing can not be done perfectly. The residual effect of smearing gives rise to higher derivative terms in the collective variable in eq. (G.6). Using the Hausdorff-Campbell formula, one can isolate the residual term as

$$\mathcal{S} \tilde{h}^{E,\sigma\dagger}(x) \mathcal{S}^{-1} = \left[1 + O\left(\frac{l_c^3}{|E| N^2} \frac{\delta}{\delta v_a(x)} \right) \right] \tilde{h}^{E,\sigma\dagger}(x), \quad (\text{G.7})$$

where it is used that operators in $\tilde{h}^{E,\sigma\dagger}(x)$ in eq. (4.45) are centered at x with a spread l_c in space, and $\mathcal{M}^{-1}(x, y)$ decays exponentially beyond the length scale l_c . In terms of the

conjugate momenta,

$$\pi^{\mu i}(x) = i\kappa^2 \frac{\delta}{\delta E_{\mu i}(x)}, \quad \pi_\sigma(x) = i\kappa^2 \frac{\delta}{\delta \sigma(x)}, \quad (\text{G.8})$$

the higher derivative terms in eq. (G.7) are of the order of $O\left(l_c \frac{\pi}{|E|}\right)$ where π denotes either $\pi^{\mu i}$ or π . Therefore, the action of $\hat{\mathcal{H}}(x)$ on $|E, \sigma\rangle$ results in

$$\hat{\mathcal{H}}(x)|E, \sigma\rangle = \frac{1}{2} \left[\tilde{h}^{E, \sigma}(x) + \tilde{h}^{E, \sigma^\dagger}(x) \right] \left\{ 1 + O\left(l_c \kappa^2 \frac{\delta}{\delta E_{\mu i}(x)}, l_c \kappa^2 \frac{\delta}{\delta \sigma(x)}\right) \right\} |E, \sigma\rangle. \quad (\text{G.9})$$

The higher derivative correction from the smearing is small if the conjugate momentum is small compared to the l_c^{-1} .

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