

Supersymmetric SYK model: bi-local collective superfield/supermatrix formulation

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ABSTRACT: We discuss the bi-local collective theory for the $\mathcal{N} = 1, 2$ supersymmetric Sachdev-Ye-Kitaev (SUSY SYK) models. We construct a bi-local superspace, and formulate the bi-local collective superfield theory of the one-dimensional SUSY vector model. The bi-local collective theory provides systematic analysis of the SUSY SYK models. We find that this bi-local collective theory naturally leads to supermatrix formulation in the bi-local superspace. This supermatrix formulation drastically simplifies the analysis of the SUSY SYK models. We also study $\mathcal{N} = 1$ bi-local superconformal generators in the supermatrix formulation, and find the eigenvectors of the superconformal Casimir. We diagonalize the quadratic action in large N expansion.

KEYWORDS: $1/N$ Expansion, Field Theories in Lower Dimensions, Superspaces, Conformal Field Theory

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1 Introduction

The Sachdev-Ye-Kitaev (SYK) model was proposed in [1] and recently has been studied vigorously not only in the context of AdS/CFT [2–5] but also in the context of non-Fermi liquids [6–8]. The SYK model is a quantum mechanical model of N fermions with

disordered interaction. In large N diagrammatics, the dominance of “melonic” diagrams make the model solvable at strong coupling limit [2–5, 9–12]. Also, this model features emergent reparametrization symmetry in the strict strong coupling limit after disorder average [3, 4, 9–12]. This reparametrization symmetry is broken spontaneously and explicitly at strong but finite coupling limit, which leads to Schwarzian effective action for Pseudo-Nambu-Goldstone modes [3, 4, 9–12]. Due to this mode, the SYK model is maximally chaotic, and the Lyapunov exponent of out-of-time-ordered correlator saturates chaos bound [4, 11]. The same feature has been found in unitary quantum mechanical model of fermi tensors without disorder [13–21]. In tensor models, the “melonic” diagrams also dominate in large N , which leads to maximal chaos like the SYK model [13–15, 19]. This maximal chaos [22–24] indicates that both quantum mechanical models could be dual to gravity theory near horizon limit of extremal black hole, and the dual models have been proposed to be dilaton gravity [25, 26], Liouville theories [27] and 3D gravity [28]. Because of these attractive features, the generalizations of the SYK and the tensor models have been studied in various context (e.g., random matrix behavior [29–34], flavor [35, 36], lattice generalization in higher dimensions [37–45], Schwarzian effective action [46–49] and supersymmetry [50, 51], massive field instead of random coupling [52, 53], higher point function [54] and $1/N$ corrections [55, 56].)

Most generalizations of the SYK model share the same feature: bi-local in time space. This bi-local structure is naturally appears in SYK model because the SYK model is essentially a large N vector model. One of the systematic analysis of such large N models was introduced as collective field theory in [57], which captures invariant physical degrees of freedom and provides the effective action thereof. The collective field theory has successfully analyzed the large N models in the context of AdS/CFT [58–67]. Especially, a bi-local collective field theory for three-dimensional $U(N)/O(N)$ vector model gave rich understanding of higher spin AdS_4/CFT_3 correspondence [68–78]. However, in collective field theory, the bi-local structure is not restricted to space-time. In general, one can construct bi-local space of other abstract space in addition to spacetime. For example, in the bi-local thermofield CFT [77], the bi-local field is given by $\Psi(x_1, a; x_2, b)$ where x_1 and x_2 corresponds to spacetime as usual. $a, b (= 1, 2)$ represents labels of two copies of system in thermofield CFT, which corresponds to CFT lives on the left and right boundary of eternal black hole. Furthermore, we have also constructed bi-local field $\Psi(\tau_1, a; \tau_2, b)$ from the time (τ_1, τ_2) and replica space $(a, b = 1, 2, \dots, n)$ in the SYK model [10].

In this paper, we will develop the bi-local collective superfield theory¹ for one-dimensional vector model by constructing bi-local superspace, especially will focus on supersymmetric SYK model introduced by [50]. This bi-local collective superfield theory enable us to analyze the effective action of SUSY SYK model in large N systematically. Furthermore, in the bi-local collective theory, the matrix structure in the bi-local space naturally appears so that the bi-local collective theory can be seen as a matrix theory in the bi-local space. Hence, one can analyze the SUSY SYK model in the supermatrix formulation. This supermatrix formulation drastically simplifies analysis. We find that $\mathcal{N} = 1$

¹Note that the collective theory for large N supermatrix model was already studied in [79–81].

superconformal generators becomes simple matrices in the supermatrix formulation. We also study the large N classical solution and the large N expansion of the collective action of the $\mathcal{N} = 1$ SUSY SYK model. In particular, the quadratic action in large N expansion can be easily diagonalized in the supermatrix formulations. Furthermore, the interaction term in the SUSY SYK model can be understood as the inner product in the supermatrix formulation. Furthermore, this also help diagonalize the rest of the quadratic action. We also emphasize that our formulation is not restricted to the SUSY SYK model. We develop a general framework to analyze large N SUSY vector models as supermatrix theory in the bi-local superspace. Hence, this can be applied the generalization of the SUSY SYK models as well as other SUSY vector models.

The outline of the paper is as follows. In section 2, we develop the bi-local collective superfield theory for one-dimensional $\mathcal{N} = 1$ SUSY vector models, and we systematically study the collective superfield theory for $\mathcal{N} = 1$ SUSY SYK model. $\mathcal{N} = 1$ bi-local superconformal generators and eigenfunctions of superconformal Casimir is analyzed in section 3. In section 4, using these eigenfunctions, we diagonalize the quadratic action of the collective action for $\mathcal{N} = 1$ SUSY SYK model in large N . In section 5, we also develop the bi-local collective superfield theory for $\mathcal{N} = 2$ SUSY vector models and discuss its application to SYK model. In section 6, we give our conclusion and future work.

Note added: while this draft was under preparation, a related article [82, 83] appeared in arXiv.

2 $\mathcal{N} = 1$ supersymmetric SYK model

2.1 Bi-local superspace, superfield and supermatrix

Let us start with doubling the superspace (τ, θ) to construct bi-local superspace:

$$(\tau, \theta) \longrightarrow (\tau_1, \theta_1; \tau_2, \theta_2) \tag{2.1}$$

In this super bi-local space, superfields A can be expanded as

$$A(\tau_1, \theta_1; \tau_2, \theta_2) \equiv A_0(\tau_1, \tau_2) + \theta_1 A_1(\tau_1, \tau_2) - A_2(\tau_1, \tau_2)\theta_2 - \theta_1 A_3(\tau_1, \tau_2)\theta_2 \tag{2.2}$$

where the lowest component A_0 could be either Grassmannian even or odd. This choice of the signs and the ordering of Grassmann variables will lead to a natural definition of a supermatrix and its multiplication. Furthermore, it is useful to call the superfield A to be Grassmannian odd (or, even) if the component A_1 and A_2 are Grassmannian odd (or, even, respectively). i.e.,

$$A^\mp(\tau_1, \theta_1; \tau_2, \theta_2) = A_0^\pm(\tau_1, \tau_2) + \theta_1 A_1^\mp(\tau_1, \tau_2) - A_2^\mp(\tau_1, \tau_2)\theta_2 - \theta_1 A_3^\pm(\tau_1, \tau_2)\theta_2 \tag{2.3}$$

Note that the lowest component of Grassmannian odd superfield is a Grassmannian even and vice versa. We will see later that this unusual definition is related to the fact that the star product (matrix multiplication) in the bi-local superspace is a Grassmannian odd operation.

Now, we define a star product (matrix multiplication) \otimes in the bi-local superspace of two superfields A and B by

$$(A \otimes B)(\tau_1, \theta_1; \tau_2, \theta_2) \equiv \int A(\tau_1, \theta_1; \tau_3, \theta_3) d\tau_3 d\theta_3 B(\tau_3, \theta_3; \tau_2, \theta_2) \quad (2.4)$$

where the star product \star of the components fields is the usual matrix multiplication of the bi-local space (τ_1, τ_2) . i.e., $(A_i \star B_j)(\tau_1, \tau_2) \equiv \int d\tau_3 A_i(\tau_1, \tau_3) B_j(\tau_3, \tau_2)$. Note that we place the (Grassmannian odd) measure between the two superfields to obtain a consistent star product \otimes for all superfields. For example, the star product of two Grassmannian odd superfields is

$$\begin{aligned} (A^- \otimes B^-)(\tau_1, \theta_1; \tau_2, \theta_2) &\equiv \int A^-(\tau_1, \theta_1; \tau_3, \theta_3) d\tau_3 d\theta_3 B^-(\tau_3, \theta_3; \tau_2, \theta_2) \\ &= (A_0^+ \star B_1^- + A_2^- \star B_0^+) + \theta_1 (A_1^- \star B_1^- + A_3^+ \star B_0^+) \\ &\quad - (A_0^+ \star B_3^+ + A_2^- \star B_2^-) \theta_2 - \theta_1 (A_1^- \star B_3^+ + A_3^+ \star B_2^-) \theta_2 \end{aligned} \quad (2.5)$$

This star product in bi-local superspace simplifies in the supermatrix formulation. We represent the superfields A as a supermatrix as follow. i.e.,

$$A^\mp \equiv \begin{pmatrix} A_1^\mp & A_3^\pm \\ A_0^\pm & A_2^\mp \end{pmatrix} \quad (2.6)$$

In this definition of supermatrix, Grassmannian odd (even) superfield corresponds to Grassmannian odd (even) supermatrix. e.g.,

$$A = \underbrace{A_0}_{\text{Grassmannian Odd (Even)}} + \theta_1 \underbrace{A_1}_{\text{Grassmannian Even (Odd)}} + \dots \iff \underbrace{\begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix}}_{\text{Grassmannian Even (Odd)}} \quad (2.7)$$

Then, the star product \otimes in the bi-local superspace becomes a simple matrix product:

$$(A \otimes B)(\tau_1, \theta_1; \tau_2, \theta_2) = \begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} \otimes \begin{pmatrix} B_1 & B_3 \\ B_0 & B_2 \end{pmatrix} \quad (2.8)$$

where the multiplication between component fields is the star product \star in the bi-local space (τ_1, τ_2) . One can easily see that the identity supermatrix gives the expected delta function in the bi-local superspace. i.e.,

$$\mathbb{I}(\tau_1, \theta_1; \tau_2, \theta_2) \equiv \begin{pmatrix} \delta(\tau_1 - \tau_2) & 0 \\ 0 & \delta(\tau_1 - \tau_2) \end{pmatrix} = (\theta_1 - \theta_2) \delta(\tau_1 - \tau_2) \quad (2.9)$$

Furthermore, the natural definition of the trace in the bi-local superspace is consistent with the supertrace of a supermatrix. i.e.,

$$\int d\tau_1 d\theta_1 \delta(\tau_{12}) [A_0(\tau_1, \tau_2) + \theta_1 A_1(\tau_1, \tau_2) - A_2(\tau_1, \tau_2) \theta_1] = \text{tr } A_1 - (-1)^{|A|} \text{tr } A_2 = \text{str } A \quad (2.10)$$

where $(-1)^{|A|}$ is 1 if the supermatrix A is Grassmannian even and $(-1)^{|A|}$ is -1 if A is Grassmannian odd. Also, it is useful to define the superdeterminant (Berezinian) of the supermatrix. For our formulation, since the supermatrix is not restricted to be Grassmannian even, the supermatrix is defined by

$$\begin{aligned} \text{Ber}(A) &= \text{Ber} \begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} & (2.11) \\ &\equiv \begin{cases} \text{Ber}(A) = \det(A_1 - A_3 A_2^{-1} A_0) \det(A_2)^{-1} & (A: \text{Grassmannian even}) \\ \text{Ber}(\mathcal{J}A) = \det(A_0 - A_2 A_3^{-1} A_1) \det(-A_3)^{-1} & (A: \text{Grassmannian odd}) \end{cases} \end{aligned}$$

where the constant supermatrix \mathcal{J} is defined by

$$\mathcal{J} \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (2.12)$$

2.2 Calculus of bi-local collective superfield and supermatrix

Before formulating the bi-local collective superfield theory, we clarify our conventions for the calculus of superfields. First of all, we define the functional derivatives of the same superfield by

$$\frac{\delta f(\tau, \theta)}{\delta f(\tau', \theta')} \equiv (\theta' - \theta) \delta(\tau' - \tau) \quad (2.13)$$

We also define a change of variables and a chain rule for a superfield:

$$\delta f(\tau, \theta) \equiv \int \delta g(\tau', \theta') d\tau' d\theta' \frac{\delta f(\tau, \theta)}{\delta g(\tau', \theta')} \quad (2.14)$$

$$\frac{\delta}{\delta f(\tau, \theta)} \equiv \int \frac{\delta g(\tau', \theta')}{\delta f(\tau, \theta)} d\tau' d\theta' \frac{\delta}{\delta g(\tau', \theta')} \quad (2.15)$$

Note that we chose this unusual position of the Grassmannian odd measure to allow for uniform formulation independent of whether f and g are Grassmannian odd or even. This can easily be generalized to the bi-local collective superfields which could be Grassmannian odd or even. For example, one can check that this definition is consistent with the change of variables and the chain rule:

$$\delta f_\alpha(\tau, \theta) = \sum_\beta \int \delta f_\beta(\tau', \theta') d\tau' d\theta' \frac{\delta f_\alpha(\tau, \theta)}{\delta f_\beta(\tau', \theta')} = \int \delta f_\alpha(\tau', \theta') d\tau' d\theta' (\theta' - \theta) \delta(\tau' - \tau) \quad (2.16)$$

$$\frac{\delta}{\delta f_\alpha(\tau, \theta)} = \sum_\beta \int \frac{\delta f_\beta(\tau', \theta')}{\delta f_\alpha(\tau, \theta)} d\tau' d\theta' \frac{\delta}{\delta f_\beta(\tau', \theta')} = \int (\theta - \theta') \delta(\tau - \tau') d\tau' d\theta' \frac{\delta}{\delta f_\alpha(\tau', \theta')} \quad (2.17)$$

where α runs over some complete basis.

Furthermore, let us consider a change of variables and a chain rule for the bi-local superfield. In general, it is natural to define

$$\frac{\delta F(\tau_1, \theta_1; \tau_2, \theta_2)}{\delta F(\tau_3, \theta_3; \tau_4, \theta_4)} \equiv (\theta_3 - \theta_1)(\theta_4 - \theta_2) \delta(\tau_3 - \tau_1) \delta(\tau_4 - \tau_2) \quad (2.18)$$

Note that the r.h.s. could be different depending on the symmetry of a superfield or supermatrix. Also, we find that the following convention for the change of variables and the chain rule of the bi-local superfield is consistent.

$$\delta F(\tau_1, \theta_1; \tau_2, \theta_2) = \int \delta G(\tau_3, \theta_3; \tau_4, \theta_4) d\tau_4 d\theta_4 d\tau_3 d\theta_3 \frac{\delta F(\tau_1, \theta_1; \tau_2, \theta_2)}{\delta G(\tau_3, \theta_3; \tau_4, \theta_4)} \quad (2.19)$$

$$\frac{\delta}{\delta F(\tau_1, \theta_1; \tau_2, \theta_2)} = \int \frac{\delta G(\tau_3, \theta_3; \tau_4, \theta_4)}{\delta F(\tau_1, \theta_1; \tau_2, \theta_2)} d\tau_4 d\theta_4 d\tau_3 d\theta_3 \frac{\delta}{\delta G(\tau_3, \theta_3; \tau_4, \theta_4)} \quad (2.20)$$

For example, in this notation, we have

$$\delta((F \otimes G)(\tau_1, \theta_1; \tau_2, \theta_2)) = (\delta F \otimes G)(\tau_1, \theta_1; \tau_2, \theta_2) + (F \otimes \delta G)(\tau_1, \theta_1; \tau_2, \theta_2) \quad (2.21)$$

2.3 Bi-local collective superfield theory: Jacobian

For the collective action for the SUSY vector model (e.g., supersymmetric SYK models), we first study the Jacobian which appears in the transformation from the fundamental superfield to the bi-local collective superfield. Let us consider a superfield in $\mathcal{N} = 1$ SUSY SYK model:

$$\psi^i(\tau, \theta) \equiv \chi^i(\tau) + \theta b^i(\tau) \quad (i = 1, 2, \dots, N) \quad (2.22)$$

where χ^i is a Majorana fermion, and b^i is a boson. This superfield transforms in the fundamental representation of $O(N)$:

$$\psi^i(\tau, \theta) \longrightarrow O^{ij} \psi^j(\tau, \theta) \quad (2.23)$$

It is natural to define a bi-local collective superfield which is invariant under $O(N)$ by

$$\Psi(\tau_1, \theta_1; \tau_2, \theta_2) \equiv \frac{1}{N} \psi^i(\tau_1, \theta_1) \psi^i(\tau_2, \theta_2) \quad (2.24)$$

It is important to note that the bi-local superfield is anti-symmetric in the bi-local superspace. i.e.,

$$\Psi(\tau_1, \theta_1; \tau_2, \theta_2) = -\Psi(\tau_2, \theta_2; \tau_1, \theta_1) \quad (2.25)$$

When changing variables in the path integral from the fundamental superfield to bi-local collective superfield, we will get a non-trivial Jacobian. To obtain the Jacobian, it is useful to consider the following identity for an arbitrary functional $F[\Psi]$.

$$\sum_i \int \mathcal{D}\psi \frac{\delta}{\delta \psi^i(\tau_1, \theta_1)} [\psi^i(\tau_2, \theta_2) F[\Psi] e^{-S}] = 0 \quad (2.26)$$

Using the chain rule of the bi-local superfield in (2.20), we have

$$\begin{aligned} \psi^i(\tau_2, \theta_2) \frac{\delta}{\delta \psi^i(\tau_1, \theta_1)} &= \int \psi^i(\tau_2, \theta_2) \frac{\delta \Psi(\tau_3, \theta_3; \tau_4, \theta_4)}{\delta \psi^i(\tau_1, \theta_1)} d\tau_4 d\theta_4 d\tau_3 d\theta_3 \frac{\delta}{\delta \Psi(\tau_3, \theta_3; \tau_4, \theta_4)} \\ &= 2 \int \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} \end{aligned} \quad (2.27)$$

Hence, recalling our convention (2.13), (2.26) can be written as

$$\begin{aligned}
 N(\theta_1 - \theta_2)\delta(\tau_1 - \tau_2)\langle F \rangle + 2 \int \left\langle \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta F[\Psi]}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} \right\rangle \\
 - 2 \int \left\langle \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta S[\Psi]}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} F[\Psi] \right\rangle = 0
 \end{aligned} \tag{2.28}$$

where we used the fact that the superfield $\frac{\delta}{\delta \psi^i(\tau, \theta)}$ is Grassmannian even.

On the other hand, one can also utilize a similar identity in the bi-local collective representation:

$$\int \mathcal{D}\psi \int d\tau_3 d\theta_3 \frac{\delta}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} [\Psi(\tau_2, \theta_2; \tau_3, \theta_3) \mathfrak{J} F[\Psi] e^{-S}] = 0 \tag{2.29}$$

where $\mathfrak{J} = \mathfrak{J}[\Psi]$ is the Jacobian for the bi-local collective representation. Then, we have

$$\begin{aligned}
 \frac{1}{2}(\theta_1 - \theta_2)\delta(\tau_1 - \tau_2)\langle F[\Psi] \rangle + \int \left\langle \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta \log \mathfrak{J}}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} F[\Psi] \right\rangle \\
 + \int \left\langle \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta F}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} \right\rangle \\
 - \int \left\langle \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta S}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} F[\Psi] \right\rangle = 0
 \end{aligned} \tag{2.30}$$

Note that we used

$$\begin{aligned}
 \frac{\delta \Psi(\tau_1, \theta_1; \tau_2, \theta_2)}{\delta \Psi(\tau_3, \theta_3; \tau_4, \theta_4)} &\equiv \frac{1}{2}(\theta_3 - \theta_1)(\theta_4 - \theta_2)\delta(\tau_3 - \tau_1)\delta(\tau_4 - \tau_2) \\
 &\quad - \frac{1}{2}(\theta_3 - \theta_2)(\theta_4 - \theta_1)\delta(\tau_3 - \tau_2)\delta(\tau_4 - \tau_1)
 \end{aligned} \tag{2.31}$$

which is imposed by anti-symmetry of the bi-local superfield Ψ in (2.25). As usual in supersymmetry, we do not have divergence proportional to $\delta(\tau - \tau)$ unlike what appears in the bosonic bi-local collective field theory [63, 65, 68, 74]. In our formulation, this naturally comes from the fact that the analogous $(\theta - \theta)\delta(\tau - \tau)$ for superspace, vanishes. From (2.28) and (2.30) for an arbitrary functional of $F[\Psi]$, we obtain a functional differential equation for the Jacobian \mathfrak{J} :

$$\frac{N-1}{2}(\theta_1 - \theta_2)\delta(\tau_1 - \tau_2) = \int \Psi(\tau_2, \theta_2; \tau_3, \theta_3) d\tau_3 d\theta_3 \frac{\delta \log \mathfrak{J}}{\delta \Psi(\tau_1, \theta_1; \tau_3, \theta_3)} \tag{2.32}$$

This differential equation can easily be solved using the supermatrix formulation in section 2.1. In the supermatrix formulation, it is trivial to conclude that

$$\log \mathfrak{J} = -\frac{N-1}{2} \text{str} \log \Psi(\tau_1, \theta_1; \tau_2, \theta_2) \tag{2.33}$$

We emphasize that anti-symmetry of the bi-local superfield² leads to a term $\frac{1}{2}(\theta_1 - \theta_2)\delta(\tau_1 - \tau_2)$ in (2.30), which shifts large N to $N - 1$. This shift of large N in the Jacobian was

²We thank to Robert de Mello Koch for pointing out this.

already observed in non-supersymmetric bi-local collective field theory [74], and it was shown to play an important role in matching one-loop free energies of higher spin theories and vector models [74, 84–87]. Though this shift is not crucial for the discussion in this paper, it is essential to obtain the exact result. For example, one can consider a free one-dimensional $\mathcal{N} = 1$ SUSY vector model for which one knows the exact answer.³ We confirm that the shift $N - 1$ gives the correct one-point function of bi-local superfield (or, invariant two-point function of fundamental superfields) (See appendix A).

2.4 Bi-local collective superfield theory for $\mathcal{N} = 1$ SUSY SYK model

In [50], the action of the supersymmetric SYK model is given by

$$\mathcal{L} = \sum_i \left[\frac{1}{2} \chi^i \partial \chi^i - \frac{1}{2} b^i b^i + i \sum_{1 \leq j < k \leq N} C_{ijk} b^i \chi^j \chi^k \right] \quad (2.34)$$

where C_{ijk} is a random coupling constant, and is totally anti-symmetric in its indices. After the disorder average of the random coupling constant C_{ijk} over a Gaussian distribution,⁴ one has an effective action [50]:

$$S_{\text{eff}} = \int d\tau \left(\frac{1}{2} \chi^i \partial \chi^i - \frac{1}{2} b^i b^i \right) - \frac{J}{2N^2} \int d\tau_1 d\tau_2 [b^i(\tau_1) b^i(\tau_2)] [\chi^j(\tau_1) \chi^j(\tau_2)]^2 - \frac{J}{N^2} \int [b^i(\tau_1) \chi^i(\tau_2)] [\chi^j(\tau_1) b^j(\tau_2)] [\chi^k(\tau_1) \chi^k(\tau_2)] . \quad (2.35)$$

Note that the disorder average leads to an emergent $O(N)$ symmetry. As before, we define the (fundamental) superfield by

$$\psi^i(\tau, \theta) \equiv \chi^i(\tau) + \theta b^i(\tau) \quad (2.36)$$

we will express the effective action in terms of the bi-local collective superfield given by

$$\begin{aligned} \Psi(\tau_1, \theta_1; \tau_2, \theta_2) &\equiv \frac{1}{N} \sum_{i=1}^N \psi^i(\tau_1, \theta_1) \psi^i(\tau_2, \theta_2) \\ &= \frac{1}{N} \sum_{i=1}^N [\chi^i(\tau_1) \chi^i(\tau_2) + \theta_1 b^i(\tau_1) \chi^i(\tau_2) + \chi^i(\tau_1) b^i(\tau_2) \theta_2 + \theta_1 b^i(\tau_1) b^i(\tau_2) \theta_2] \end{aligned} \quad (2.37)$$

In terms of supermatrix notation, the bi-local superfield can be represented as

$$\Psi(\tau_1, \theta_1; \tau_2, \theta_2) = \frac{1}{N} \sum_{i=1}^N \begin{pmatrix} b^i(\tau_1) \chi^i(\tau_2) & -b^i(\tau_1) b^i(\tau_2) \\ \chi^i(\tau_1) \chi^i(\tau_2) & -\chi^i(\tau_1) b^i(\tau_2) \end{pmatrix} \quad (2.38)$$

Recall that the bi-local superfield is anti-symmetric in the bi-local superspace (see (2.25)). As a supermatrix, the bi-local supermatrix has the following symmetry. i.e.,

$$\mathcal{J} \Psi^{\text{st}} \mathcal{J} = \Psi \quad (2.39)$$

³We also thank to Robert de Mello Koch for raising this issue and confirming the result.

⁴Rigorously, we perform annealed average instead of a quenched average. For a proper quenched average, one has to use the replica trick, which was also done for non-supersymmetric bi-local collective field theory in [10].

where A^{st} is the supertranspose of a supermatrix A defined by

$$A^{\text{st}} \equiv \begin{pmatrix} A_1^t & (-1)^{|A|} A_0^t \\ -(-1)^{|A|} A_3^t & A_2^t \end{pmatrix} \quad (2.40)$$

and the matrix \mathcal{J} is given in (2.12).

For the collective action, it is useful to define a superderivative matrix:

$$\begin{aligned} \mathfrak{D}(\tau_1, \theta_1; \tau_2, \theta_2) &\equiv \mathbb{D}_{\theta_1}(\theta_1 - \theta_2)\delta(\tau_1 - \tau_2) = \delta(\tau_1 - \tau_2) - \theta_1 \partial_{\tau_1} \delta(\tau_1 - \tau_2) \theta_2 \\ &= \begin{pmatrix} 0 & \partial_1 \delta(\tau_1 - \tau_2) \\ \delta(\tau_1 - \tau_2) & 0 \end{pmatrix} \end{aligned} \quad (2.41)$$

where the superderivative \mathbb{D}_{θ_1} is defined by

$$\mathbb{D}_{\theta_1} \equiv \partial_{\theta_1} + \theta_1 \partial_{\tau_1} \quad (2.42)$$

Note that the superderivative matrix \mathfrak{D} is Grassmannian odd supermatrix. Using the supermatrix formulation, one can easily check that

$$(\mathfrak{D} \otimes A)(\tau_1, \theta_1; \tau_2, \theta_2) = \begin{pmatrix} \partial_{\tau_1} A_0(\tau_1, \tau_2) & \partial_{\tau_1} A_2(\tau_1, \tau_2) \\ A_1 & A_3 \end{pmatrix} \quad (2.43)$$

and, therefore, the supertrace of the supermatrix leads to the kinetic term:

$$\text{str}(\mathbb{D} \otimes \Psi) = \int d\tau_1 \left[\partial_{\tau_1} \chi^i(\tau_1) \chi^i(\tau_2) \Big|_{\tau_2 \rightarrow \tau_1} + b^i(\tau_1) b^i(\tau_1) \right] \quad (2.44)$$

As an aside, the superderivative matrix has a similar property as the ordinary superderivative. i.e.,

$$(\mathfrak{D} \otimes \mathfrak{D})(\tau_1, \theta_1; \tau_2, \theta_2) = \partial_{\tau_1} \begin{pmatrix} \delta(\tau_1 - \tau_2) & 0 \\ 0 & \delta(\tau_1 - \tau_2) \end{pmatrix} = \partial_{\tau_1} \mathbb{I}(\tau_1, \theta_1; \tau_2, \theta_2) \quad (2.45)$$

where $\mathbb{I}(\tau_1, \theta_1; \tau_2, \theta_2)$ is the identity supermatrix. Hence, one can immediately obtain the bi-local collective action for the SUSY SYK model.

$$S_{\text{col}} = -\frac{N}{2} \text{str}[\mathfrak{D} \otimes \Psi] + \frac{N}{2} \text{str} \log \Psi - \frac{JN}{6} \int d\tau_1 d\theta_1 d\tau_2 d\theta_2 [\Psi(\tau_1, \theta_1; \tau_2, \theta_2)]^3 \quad (2.46)$$

Also, one can rewrite the collective action completely in terms of supermatrix notation.

$$S_{\text{col}} = \frac{N}{2} \text{str} \left[-\mathfrak{D} \otimes \Psi + \log \Psi - \frac{J}{3} \Psi \otimes [\Psi]^2 \right] \quad (2.47)$$

where we define $[\Psi]^2(\tau_1, \theta_1; \tau_2, \theta_2) \equiv [\Psi(\tau_1, \theta_1; \tau_2, \theta_2)]^2$. Note that it is also straightforward to generalize this into general q case, which we present in section D. Note that in this paper we drop the shift in N found in (2.33) for simplicity because it does not have an effect on our discussions. But, one should take this into account for the sub-leading calculations in $1/N$.

2.5 Large N classical solution

At large N , the variation with respect to the bi-local superfield gives the large N classical solution. Note that in the supermatrix notation, the variation of the collective action (2.47) can easily be performed.⁵ Hence, one can immediately obtain the large N saddle-point equation of the collective action:

$$-\mathfrak{D} + \Psi^{-1} - J\Psi^2 = 0 \quad (2.49)$$

or equivalently, by multiplying supermatrix Ψ , we have

$$-\mathfrak{D} \otimes \Psi + \mathbb{I} - J[\Psi^2] \otimes \Psi = 0 \quad (2.50)$$

The most general ansatz for a scaling solution is given [50] by

$$\begin{aligned} \Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) &= \frac{c_1 \text{sgn}(\tau_{12} - \theta_1 \theta_2)}{|\tau_{12} - \theta_1 \theta_2|^{2\Delta_1}} + \theta_{12} \frac{c_2 + c_3 \text{sgn}(\tau_{12})}{|\tau_{12} - \theta_1 \theta_2|^{2\Delta_2}} \\ &= \frac{c_1}{|\tau_{12}|^{2\Delta_1}} \left[\text{sgn}(\tau_{12}) + 2\Delta_1 \frac{\theta_1 \theta_2}{|\tau_{12}|} \right] + \theta_{12} \frac{c_2 + c_3 \text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_2}} \\ &= \begin{pmatrix} \frac{c_2 + c_3 \text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_2}} & -\frac{2\Delta_1 c_1}{|\tau_{12}|^{2\Delta_1+1}} \\ \frac{c_1 \text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_1}} & -\frac{c_2 + c_3 \text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_2}} \end{pmatrix} \\ &\equiv \begin{pmatrix} c_2 f_{2\Delta_2}^s(\tau_{12}) + c_3 f_{2\Delta_2}^a(\tau_{12}) & -2\Delta_1 c_1 f_{2\Delta_1+1}^s(\tau_{12}) \\ c_1 f_{2\Delta_1}^a(\tau_{12}) & -c_2 f_{2\Delta_2}^s(\tau_{12}) - c_3 f_{2\Delta_2}^a(\tau_{12}) \end{pmatrix} \end{aligned} \quad (2.51)$$

where we define $\theta_{12} = \theta_1 - \theta_2$ and

$$f_\mu^s(\tau) \equiv \frac{1}{|\tau|^\mu}, \quad f_\mu^a(\tau) \equiv \frac{\text{sgn}(\tau)}{|\tau|^\mu} \quad (2.52)$$

Note that c_1 is Grassmannian even while c_2 and c_3 are Grassmannian odd. Moreover, $[\Psi]^2$ can also be expressed as a supermatrix:

$$\begin{aligned} &[\Psi_{\text{cl}}]^2(\tau_1, \theta_1; \tau_2, \theta_2) \\ &= \frac{c_1^2}{|\tau_{12}|^{4\Delta_1}} \left[1 + \theta_1 \theta_2 \frac{4\Delta_1 \text{sgn}(\tau_{12})}{|\tau_{12}|} \right] + \theta_{12} \frac{2c_1(c_2 \text{sgn}(\tau_{12}) + c_3)}{|\tau_{12}|^{2\Delta_1+2\Delta_2}} \\ &= \begin{pmatrix} 2c_1 [c_2 f_{2\Delta_1+2\Delta_2}^a(\tau_{12}) + c_3 f_{2\Delta_1+2\Delta_2}^s(\tau_{12})] & -4\Delta_1 c_1^2 f_{4\Delta_1+1}^a(\tau_{12}) \\ c_1^2 f_{4\Delta_1}^s(\tau_{12}) & -2c_1 [c_2 f_{2\Delta_1+2\Delta_2}^a(\tau_{12}) + c_3 f_{2\Delta_1+2\Delta_2}^s(\tau_{12})] \end{pmatrix} \end{aligned} \quad (2.53)$$

Using the integrals

$$\int d\tau \frac{1}{|\tau|^\lambda} e^{i w \tau} = 2w^{\lambda-1} \int_0^\infty dx x^{-\lambda} \cos x = 2w^{\lambda-1} \Gamma(1-\lambda) \sin \frac{\pi\lambda}{2} \quad (2.54)$$

$$\int d\tau \frac{\text{sgn}(\tau)}{|\tau|^\lambda} e^{i w \tau} = 2iw^{\lambda-1} \int_0^\infty dx x^{-\lambda} \sin x = 2iw^{\lambda-1} \Gamma(1-\lambda) \cos \frac{\pi\lambda}{2} \quad (2.55)$$

⁵It is sometimes simpler to vary the collective action in terms of superfield notation. For instance, the variation of the third term in (2.46) can be expressed as

$$\frac{JN}{2} \int d\tau_1 d\theta_1 d\tau_2 d\theta_2 \delta\Psi(\tau_1, \theta_1; \tau_2, \theta_2) [\Psi(\tau_1, \theta_1; \tau_2, \theta_2)]^2 = \frac{JN}{2} \text{str}(\delta\Psi \otimes [\Psi]^2) \quad (2.48)$$

we can Fourier transform $f_\lambda^s(\tau)$ and $f_\lambda^a(\tau)$ into $\tilde{f}_\lambda^s w^{\lambda-1}$ and $\tilde{f}_\lambda^a(w)w^{\lambda-1}$, respectively. In addition, one can write the star product of f 's in terms of $\tilde{f}_\lambda^s(w)$ and $\tilde{f}_\lambda^a(w)$ as follows

$$(f_\lambda^{p_1} \star f_\nu^{p_2})(\tau_1, \tau_2) = \frac{1}{2\pi} \int dw e^{-iw\tau_{12}} \tilde{f}_\lambda^{p_1} \tilde{f}_\nu^{p_2} w^{\lambda+\nu-2} \quad (p_1, p_2 = s, a) \quad (2.56)$$

where

$$\tilde{f}_\lambda^s \equiv 2\Gamma(1-\lambda) \sin \frac{\pi\lambda}{2}, \quad \tilde{f}_\lambda^a \equiv 2i\Gamma(1-\lambda) \cos \frac{\pi\lambda}{2} \quad (2.57)$$

Thus, the third term in (2.50) can be written as

$$\begin{aligned} & [\Psi]^2 \otimes \Psi(\tau_1, \theta_1; \tau_2, \theta_2) \\ &= \frac{1}{2\pi i} \int dw e^{-iw\tau_{12}} \left(\begin{array}{cc} 2c_1 \left[c_2 \tilde{f}_{2\Delta_1+2\Delta_2}^a + c_3 \tilde{f}_{2\Delta_1+2\Delta_2}^s \right] & -4\Delta_1 c_1^2 \tilde{f}_{4\Delta_1+1}^a \\ c_1^2 \tilde{f}_{4\Delta_1}^s & -2c_1 \left[c_2 \tilde{f}_{2\Delta_1+2\Delta_2}^a + c_3 \tilde{f}_{2\Delta_1+2\Delta_2}^s \right] \end{array} \right) \\ & \times \left(\begin{array}{cc} c_2 \tilde{f}_{2\Delta_2}^s + c_3 \tilde{f}_{2\Delta_2}^a & -2\Delta_1 c_1 \tilde{f}_{2\Delta_1+1}^s \\ c_1 \tilde{f}_{2\Delta_1}^a & -c_2 \tilde{f}_{2\Delta_2}^s - c_3 \tilde{f}_{2\Delta_2}^a \end{array} \right) \end{aligned} \quad (2.58)$$

where the matrix multiplication in the integrand is ordinary matrix multiplication. Recalling the action of the bi-local superderivative, the first term of (2.50) becomes

$$\begin{aligned} \mathfrak{D} \otimes \Psi_{\text{cl}} &= \begin{pmatrix} \partial_{\tau_1} \Psi_{\text{cl},0}(\tau_1, \tau_2) & \partial_{\tau_1} \Psi_{\text{cl},2}(\tau_1, \tau_2) \\ \Psi_{\text{cl},1}(\tau_1, \tau_2) & \Psi_{\text{cl},3}(\tau_1, \tau_2) \end{pmatrix} \\ &= \frac{1}{2\pi i} \int dw e^{-iw\tau_{12}} \begin{pmatrix} -iw c_1 \tilde{f}_{2\Delta_1}^a & iw [c_2 \tilde{f}_{2\Delta_2}^s + c_3 \tilde{f}_{2\Delta_2}^a] \\ c_2 \tilde{f}_{2\Delta_2}^s + c_3 \tilde{f}_{2\Delta_2}^a & -2\Delta_1 c_1 \tilde{f}_{2\Delta_1+1}^s \end{pmatrix} \end{aligned} \quad (2.59)$$

while the second term of (2.50) is trivially given by

$$\mathbb{I} = \frac{1}{2\pi i} \int dw e^{-iw\tau_{12}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.60)$$

Now, we will consider the strong coupling limit:

$$\frac{w}{J} \ll 1 \quad (2.61)$$

Note that the constants c_1, c_2 and c_3 should be scaled with J as follows

$$c_1 \sim J^{-2\Delta_1}, \quad c_2, c_3 \sim J^{-2\Delta_2+\frac{1}{2}} \quad (2.62)$$

Requiring positive conformal dimensions, matching the power-laws of the diagonal elements of the classical equation (2.50) gives

$$\Delta_1 = \frac{1}{6} \quad \text{or} \quad 2\Delta_1 + 4\Delta_2 = 2 \quad (2.63)$$

Let us consider the first case. i.e.,

$$\Delta_1 = \frac{1}{6} \quad (2.64)$$

We match the leading terms of the diagonal elements in the classical equation (2.50). In this case, the off-diagonal elements from $[\Psi]^2 \otimes \Psi$ diverge in the strong coupling limit for $\Delta_2 < \frac{2}{3}$. This divergence cannot be eliminated by tuning the coefficients. Moreover, for $\Delta_2 > \frac{2}{3}$, these terms vanish in the strong coupling limit. However, since we want reparametrization symmetry in the strict strong coupling limit, we had better not treat $[\Psi]^2 \otimes \Psi$ as a perturbation. Hence, we find that the only solution is given by

$$1 - 2\sqrt{3}\pi c_1^3 J = 0 \quad , \quad c_2 = c_3 = 0 \quad (2.65)$$

Note that we do not have to find Δ_2 because $c_2 = c_3 = 0$. Also, note that the kinetic term $\mathfrak{D} \otimes \Psi$ is a perturbation in the strong coupling limit as in the non-supersymmetric SYK model.

Next, we analyze the second case. I.e.,

$$2\Delta_1 + 4\Delta_2 = 2 \quad (2.66)$$

For this case, the off-diagonal elements contain divergent terms of order $\mathcal{O}(w^{-\Delta_1})$ in the strong coupling limit. To remove this divergence, we choose

$$c_3 = ic_2 \cot \frac{\pi\Delta_1}{2} \quad (2.67)$$

But, in this case, one cannot solve the diagonal and off-diagonal classical solution simultaneously.

To summarize, the classical solution is found to be

$$\Psi_{\text{cl}} = c \frac{\text{sgn}(\tau_{12} - \theta_1\theta_2)}{|\tau_{12} - \theta_1\theta_2|^{1/3}} = c \begin{pmatrix} 0 & -\frac{1}{3|\tau_{12}|^{4/3}} \\ \frac{\text{sgn}(\tau_{12})}{|\tau_{12}|^{1/3}} & 0 \end{pmatrix} \quad (2.68)$$

where

$$1 - 2\sqrt{3}\pi c^3 J = 0 \quad (2.69)$$

This classical solution was already found in [50], and corresponds to a vacuum with definite fermion number.

2.6 Large N expansion and quadratic action

Now, we expand the collective action (2.47) for the bi-local superfield:

$$\Psi(\tau_1, \theta_1; \tau_2, \theta_2) \equiv \Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) + \sqrt{\frac{2}{N}} \Phi(\tau_1, \theta_1; \tau_2, \theta_2) \quad (2.70)$$

where $\Phi(\tau_1, \theta_1; \tau_2, \theta_2)$ is a bi-local fluctuation around the classical solution Ψ_{cl} given by

$$\begin{aligned} \Phi(\tau_1, \theta_1; \tau_2, \theta_2) &= \varphi(\tau_1, \tau_2) + \theta_1 \eta_1(\tau_1, \tau_2) - \eta_2(\tau_1, \tau_2) \theta_2 - \theta_1 \sigma(\tau_1, \tau_2) \theta_2 \\ &= \begin{pmatrix} \eta_1(\tau_1, \tau_2) & \sigma(\tau_1, \tau_2) \\ \varphi(\tau_1, \tau_2) & \eta_2(\tau_1, \tau_2) \end{pmatrix} \end{aligned} \quad (2.71)$$

Note that the anti-symmetry of the bi-local field in (2.25) leads to

$$\varphi(\tau_1, \tau_2) = -\varphi(\tau_2, \tau_1) \quad (2.72)$$

$$\sigma(\tau_1, \tau_2) = \sigma(\tau_2, \tau_1) \quad (2.73)$$

$$\eta_1(\tau_1, \tau_2) = \eta_2(\tau_2, \tau_1) \quad (2.74)$$

or, equivalently, we have

$$\mathcal{J}\Phi^{\text{st}}\mathcal{J} = \Phi \quad (2.75)$$

From the supermatrix notation, one can easily obtain the quadratic action:

$$S_{\text{col}}^{(2)} = -\frac{1}{2}\text{str}(\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi) - J \int d\tau_1 d\theta_1 d\tau_2 d\theta_2 \Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) [\Phi(\tau_1, \theta_1; \tau_2, \theta_2)]^2 \quad (2.76)$$

From the classical equation, the inverse supermatrix is given by

$$\Psi_{\text{cl}}^{-1}(\tau_3, \theta_3; \tau_2, \theta_2) = -J[\Psi_{\text{cl}}(\tau_3, \theta_3; \tau_2, \theta_2)]^2 = Jc^2 \begin{pmatrix} 0 & -\frac{2}{3}f_{5/3}^a(\tau_{12}) \\ f_{2/3}^s(\tau_{12}) & 0 \end{pmatrix} \quad (2.77)$$

Hence, one can write the kinetic term as

$$\begin{aligned} & \frac{1}{2}\text{str}(\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi) \\ &= -\frac{J^2 c^4}{2}\text{tr} \left(\frac{4}{9}f_{5/3}^a \star \varphi \star f_{5/3}^a \star \varphi - f_{2/3}^s \star \sigma \star f_{2/3}^s \star \sigma + \frac{4}{3}f_{2/3}^s \star \eta_1 \star f_{5/3}^a \star \eta_2 \right) \end{aligned} \quad (2.78)$$

where the cross terms are cancelled because of the supertrace. Also, the classical solution can be written as

$$\begin{aligned} \Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) &= c \begin{bmatrix} \frac{\text{sgn}(\tau_{12})}{\tau_{12}^{1/3}} + \frac{\theta_1 \theta_2}{3\tau_{12}^{4/3}} \\ f_{1/3}^a(\tau_{12}) - \theta_1 \left(-\frac{1}{3}f_{4/3}^s(\tau_{12}) \right) \theta_2 \end{bmatrix} \\ &= c \begin{pmatrix} 0 & -\frac{1}{3}f_{4/3}^s(\tau_{12}) \\ f_{1/3}^a(\tau_{12}) & 0 \end{pmatrix} \end{aligned} \quad (2.79)$$

The square of bi-local fluctuation can be also written using the supermatrix notation:

$$[\Phi(\tau_1, \theta_1; \tau_2, \theta_2)]^2 = \begin{pmatrix} 2[\varphi\eta_1](\tau_1, \tau_2) & 2[\varphi\sigma](\tau_1, \tau_2) + 2[\eta_1\eta_2](\tau_1, \tau_2) \\ [\varphi^2](\tau_1, \tau_2) & 2[\varphi\eta_2](\tau_1, \tau_2) \end{pmatrix} \quad (2.80)$$

which leads to

$$\begin{aligned} & J \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \bar{\Psi}(\tau_1, \theta_1; \tau_2, \theta_2) [\Phi]^2(\tau_1, \theta_1; \tau_2, \theta_2) \\ &= Jc \text{tr} \left(-\frac{1}{3}f_{4/3}^s \star [\varphi^2] - 2f_{1/3}^a \star [\varphi\sigma] - 2f_{1/3}^a \star [\eta_1\eta_2] \right) \end{aligned} \quad (2.81)$$

In conclusion, the quadratic action can be manipulated as follows.

$$\begin{aligned} S^{(2)} &= \frac{Jc}{4\sqrt{3}\pi} \text{tr} \left(-\frac{4}{9}f_{5/3}^a \star \varphi \star f_{5/3}^a \star \varphi + f_{2/3}^s \star \sigma \star f_{2/3}^s \star \sigma - \frac{4}{3}f_{2/3}^s \star \eta_1 \star f_{5/3}^a \star \eta_2 \right. \\ &\quad \left. + \frac{4\sqrt{3}\pi}{3}f_{4/3}^s \star [\varphi^2] + 8\sqrt{3}\pi f_{1/3}^a \star [\varphi\sigma] + 8\sqrt{3}\pi f_{1/3}^a \star [\eta_1\eta_2] \right) \end{aligned} \quad (2.82)$$

In the section 4, we will diagonalize this quadratic action. Though we express the quadratic action in terms of component fields for pedagogical purposes, we will not use this expression (2.82) in terms of component fields for the diagonalization of the quadratic action. Instead, we find that the collective action of $\mathcal{N} = 1$ SUSY SYK model can completely be written in term of the supermatrix notation:

$$S_{\text{col}}^{(2)} = -\frac{1}{2} \text{str} \left(\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi + 2J\Phi \otimes [\Psi_{\text{cl}}\Phi] \right) \quad (2.83)$$

We will see that It is much easier to diagonalize the quadratic action.

3 $\mathcal{N} = 1$ bi-local superconformal algebra

3.1 Bi-local $\mathcal{N} = 1$ superconformal generators

In non-supersymmetric SYK models, it is useful to find eigenfunctions of the Casimir of the $SL(2)$ algebra in order to diagonalize the quadratic action because the Casimir commutes with the kernel of the quadratic action. Similarly, in the SUSY SYK model, it is important to consider generators of the $\mathcal{N} = 1$ superconformal algebra given by

$$\mathcal{P}_a = \partial_{\tau_a} \quad (3.1)$$

$$\mathcal{K}_a = \tau_a^2 \partial_{\tau_a} + \frac{1}{3} \tau_a + \tau_a \theta_a \partial_{\theta_a} \quad (3.2)$$

$$\mathcal{D}_a = \tau_a \partial_{\tau_a} + \frac{1}{2} \theta_a \partial_{\theta_a} + \frac{1}{6} \quad (3.3)$$

$$\mathcal{Q}_a = \partial_{\theta_a} - \theta_a \partial_{\tau_a} \quad (3.4)$$

$$\mathcal{S}_a = \tau_a \partial_{\theta_a} - \tau_a \theta_a \partial_{\tau_a} - \frac{1}{3} \theta_a \quad (3.5)$$

where $a = 1, 2$. Note that the $\frac{1}{3}$ factors appear because the fermion has conformal dimension $\frac{1}{6}$. We define bi-local superconformal generator as follows.

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad (\mathcal{L} \in \{ \mathcal{P}, \mathcal{K}, \mathcal{D}, \mathcal{Q}, \mathcal{S} \}) \quad (3.6)$$

which satisfy

$$\begin{aligned} [\mathcal{P}, \mathcal{K}] &= 2\mathcal{D}, & \{ \mathcal{Q}, \mathcal{Q} \} &= -2\mathcal{P}, & [\mathcal{D}, \mathcal{Q}] &= -\frac{1}{2}\mathcal{Q}, & [\mathcal{P}, \mathcal{Q}] &= 0 \\ [\mathcal{D}, \mathcal{P}] &= -\mathcal{P}, & \{ \mathcal{Q}, \mathcal{S} \} &= -2\mathcal{D}, & [\mathcal{D}, \mathcal{S}] &= \frac{1}{2}\mathcal{S}, & [\mathcal{K}, \mathcal{S}] &= 0 \\ [\mathcal{D}, \mathcal{K}] &= \mathcal{K}, & \{ \mathcal{S}, \mathcal{S} \} &= -2\mathcal{K}, & [\mathcal{K}, \mathcal{Q}] &= -\mathcal{S}, & [\mathcal{P}, \mathcal{S}] &= \mathcal{Q} \end{aligned} \quad (3.7)$$

The Casimir is given by

$$\mathcal{C} = \mathcal{D}^2 - \frac{1}{2}(\mathcal{P}\mathcal{K} + \mathcal{K}\mathcal{P}) + \frac{1}{4}(\mathcal{S}\mathcal{Q} - \mathcal{Q}\mathcal{S}) = \mathcal{D}^2 - \frac{1}{2}\mathcal{D} - \mathcal{K}\mathcal{P} + \frac{1}{2}\mathcal{S}\mathcal{Q} \quad (3.8)$$

Now, we will translate the generators as differential operators acting on superfields into supermatrices notation. Let us consider a superfield

$$A^{\mp}(\tau_1, \tau_2) = A_0^{\pm} + \theta_1 A_1^{\mp} - A_2^{\mp} \theta_2 - \theta_1 A_3^{\pm} \theta_2 = \begin{pmatrix} A_1^{\mp} & A_3^{\pm} \\ A_0^{\pm} & A_2^{\mp} \end{pmatrix}. \quad (3.9)$$

where we omit the bi-local time coordinates for a while. For example, one can consider the action of \mathcal{K}_1 and \mathcal{K}_2 in (3.2) on the superfield A^\mp :

$$\begin{aligned} \mathcal{K}_1 A^\mp &= \left(\tau_1^2 \partial_{\tau_1} + \frac{1}{3} \tau_1 \right) A_0^\pm + \theta_1 \left(\tau_1^2 \partial_{\tau_1} + \frac{4}{3} \tau_1 \right) A_1^\mp \\ &\quad - \left(\tau_1^2 \partial_{\tau_1} + \frac{1}{3} \tau_1 \right) A_2^\mp \theta_2 - \theta_1 \left(\tau_1^2 \partial_{\tau_1} + \frac{4}{3} \tau_1 \right) A_3^\pm \theta_2 \end{aligned} \quad (3.10)$$

$$\begin{aligned} \mathcal{K}_2 A^\mp &= \left(\tau_2^2 \partial_{\tau_2} + \frac{1}{3} \tau_2 \right) A_0^\pm + \theta_1 \left(\tau_2^2 \partial_{\tau_2} + \frac{1}{3} \tau_2 \right) A_1^\mp \\ &\quad - \left(\tau_2^2 \partial_{\tau_2} + \frac{4}{3} \tau_2 \right) A_2^\mp \theta_2 - \theta_1 \left(\tau_2^2 \partial_{\tau_2} + \frac{4}{3} \tau_2 \right) A_3^\pm \theta_2 \end{aligned} \quad (3.11)$$

From the view point of super matrix, this can be written as

$$\mathcal{K}_1 A^\mp = \mathbb{K} \otimes A^\mp \quad , \quad \mathcal{K}_2 A^\mp = A^\mp \otimes \mathbb{K}^\sharp \quad (3.12)$$

where A^\sharp is the composite operation of the parity transpose and supertranspose of a supermatrix A . Namely, the parity transpose of a supermatrix A is defined by

$$A = \begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} \implies A^\pi = \begin{pmatrix} A_2 & A_0 \\ A_3 & A_1 \end{pmatrix} \quad (3.13)$$

We define A^\sharp by

$$A^\sharp = (A^\pi)^{\text{st}} = \begin{pmatrix} A_2^t & (-1)^{|A|} A_3^t \\ -(-1)^{|A|} A_0^t & A_1^t \end{pmatrix} \quad (3.14)$$

Recall that $|A|$ denotes the parity of the supermatrix A . Repeating the same calculation for the other generators, we find that

$$\mathcal{L}_1 A = \mathbb{L} \otimes A \quad , \quad \mathcal{L}_2 A = (-1)^{|\mathcal{L}| \cdot (|A|+1)} A \otimes \mathbb{L}^\sharp \quad (\mathcal{L}_a \in \{\mathcal{P}_a, \dots, \mathcal{S}_a\} \quad , \quad \mathbb{L} \in \{\mathbb{P}, \mathbb{K}, \mathbb{D}, \mathbb{Q}, \mathbb{S}\}) \quad (3.15)$$

where the supermatrices $\{\mathbb{P}, \mathbb{K}, \mathbb{D}, \mathbb{Q}, \mathbb{S}\}$ are defined by

$$\mathbb{P} \equiv \begin{pmatrix} \partial_{\tau_1} \delta(\tau_1 - \tau_2) & 0 \\ 0 & \partial_{\tau_1} \delta(\tau_1 - \tau_2) \end{pmatrix} \quad (3.16)$$

$$\mathbb{K} \equiv \begin{pmatrix} (\tau_1^2 \partial_{\tau_1} + \frac{4}{3} \tau_1) \delta(\tau_1 - \tau_2) & 0 \\ 0 & (\tau_1^2 \partial_{\tau_1} + \frac{1}{3} \tau_1) \delta(\tau_1 - \tau_2) \end{pmatrix} \quad (3.17)$$

$$\mathbb{D} \equiv \begin{pmatrix} (\tau_1 \partial_{\tau_1} + \frac{2}{3}) \delta(\tau_1 - \tau_2) & 0 \\ 0 & (\tau_1 \partial_{\tau_1} + \frac{1}{6}) \delta(\tau_1 - \tau_2) \end{pmatrix} \quad (3.18)$$

$$\mathbb{Q} \equiv \begin{pmatrix} 0 & -\partial_{\tau_1} \delta(\tau_1 - \tau_2) \\ \delta(\tau_1 - \tau_2) & 0 \end{pmatrix} \quad (3.19)$$

$$\mathbb{S} \equiv \begin{pmatrix} 0 & (-\tau_1 \partial_{\tau_1} - \frac{1}{3}) \delta(\tau_1 - \tau_2) \\ \tau_1 \delta(\tau_1 - \tau_2) & 0 \end{pmatrix} \quad (3.20)$$

Note that $|\mathcal{L}|$ is the usual parity of the generator while $|A|$ is the parity as a supermatrix.⁶ Hence, the action of the bi-local superconformal generator on the superfield can be represented as follows

$$\mathcal{L}A = \mathbb{L} \otimes A + (-1)^{|\mathcal{L}| \cdot (|A|+1)} A \otimes \mathbb{L}^\sharp \quad (\mathcal{L} \in \{\mathcal{P}, \mathcal{K}, \mathcal{D}, \mathcal{Q}, \mathcal{S}\}, \mathbb{L} \in \{\mathbb{P}, \mathbb{K}, \mathbb{D}, \mathbb{Q}, \mathbb{S}\}) \quad (3.21)$$

Note that the supermatrix generators are

$$|\mathbb{P}| = |\mathbb{K}| = |\mathbb{D}| = 0 \quad , \quad |\mathbb{Q}| = |\mathbb{S}| = 1 \quad (3.22)$$

Especially, \mathbb{P} and \mathbb{Q} satisfy

$$\mathbb{P}^\sharp = -\mathbb{P} \quad , \quad \mathbb{Q}^\sharp = \mathbb{Q} \quad , \quad \mathbb{Q} \otimes \mathbb{Q} = \mathbb{P} \quad , \quad (3.23)$$

and therefore, the action of \mathcal{P} and \mathcal{Q} are simply given by

$$\mathcal{P}A = \mathbb{P} \otimes A - A \otimes \mathbb{P} \quad , \quad \mathcal{Q}A = \mathbb{Q} \otimes A + (-1)^{|A|+1} A \otimes \mathbb{Q} \quad (3.24)$$

3.2 Eigenfunctions of superconformal Casimir

In non-supersymmetric SYK model, it is natural to use new coordinates given by

$$t = \frac{1}{2}(\tau_1 + \tau_2) \quad , \quad z = \frac{1}{2}(\tau_1 - \tau_2) \quad (3.25)$$

In fact, this is the simplest example of the bi-local map found in [68, 75, 76, 78] for the duality between higher spin theory in AdS₄ and free vector model CFT₃. This bi-local map can be obtained by comparing the bi-local conformal generators for $O(N)/U(N)$ vector fields and conformal generators for higher spin fields. But, the bi-local space of (non-supersymmetric) SYK model is so simple that we need not do such calculations.⁷ For the rest of Grassmannian odd coordinates, we do not transform, but we will relabel the coordinates by

$$\begin{aligned} \theta_1 &= \zeta_0 \quad , \quad \theta_2 = \zeta_1 \\ \partial_{\zeta_0} &= \partial_{\theta_1} \quad , \quad \partial_{\zeta_1} = \partial_{\theta_2} \end{aligned} \quad (3.26)$$

Under this bi-local map, the superconformal generators can be expressed by

$$\mathcal{P} = \partial_t \quad (3.27)$$

$$\begin{aligned} \mathcal{K} &= (t^2 + z^2)\partial_t + 2tz\partial_z + t(\zeta_0\partial_{\zeta_0} + \zeta_1\partial_{\zeta_1}) + z(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) + \frac{2}{3}t \\ &= (-t^2 + z^2)\partial_t + 2tD + z(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) \end{aligned} \quad (3.28)$$

$$\mathcal{D} = t\partial_t + z\partial_z + \frac{1}{2}\zeta_0\partial_{\zeta_0} + \frac{1}{2}\zeta_1\partial_{\zeta_1} + \frac{1}{3} \quad (3.29)$$

$$\mathcal{Q} = -\frac{1}{2}\zeta_0(\partial_t + \partial_z) + \frac{1}{2}\zeta_1(-\partial_t + \partial_z) + \partial_{\zeta_0} + \partial_{\zeta_1} \quad (3.30)$$

$$\begin{aligned} \mathcal{S} &= (t+z)\partial_{\zeta_0} - (-t+z)\partial_{\zeta_1} - \frac{1}{2}\zeta_0(t+z)(\partial_t + \partial_z) - \frac{1}{2}\zeta_1(-t+z)(-\partial_t + \partial_z) \\ &\quad - \frac{1}{3}(\zeta_0 + \zeta_1) \end{aligned} \quad (3.31)$$

⁶Recall that parity of A as a supermatrix is opposite to the ‘‘usual parity’’ of A as a superfield.

⁷On the other hand, bi-local map of superspace might be non-trivial because there could be a mixing between τ_1, τ_2 and θ_1, θ_2 . For $\mathcal{N} = 1$ SUSY SYK model, such a mixing does not seem to be natural.

and the corresponding Casimir operator is found to be

$$\begin{aligned} \mathcal{C} = & -\frac{1}{18} + \frac{2}{3}z\partial_z + z^2(-\partial_t^2 + \partial_z^2) - z\partial_t(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) + \left(z\partial_z + \frac{1}{6}\right)(\zeta_0\partial_{\zeta_0} + \zeta_1\partial_{\zeta_1}) \\ & + \frac{1}{2}\zeta_0\zeta_1\partial_{\zeta_1}\partial_{\zeta_0} - z\partial_{\zeta_1}\partial_{\zeta_0} - \frac{1}{6}\partial_z\zeta_0\zeta_1 - \frac{1}{4z}(-z^2\partial_t^2 + z^2\partial_z^2)\zeta_0\zeta_1 \\ & - \left(\frac{1}{2}z\partial_z + \frac{1}{6}\right)(\zeta_0\partial_{\zeta_1} + \zeta_1\partial_{\zeta_0}) - \frac{1}{2}z\partial_t(\zeta_0\partial_{\zeta_1} - \zeta_1\partial_{\zeta_0}) \end{aligned} \quad (3.32)$$

Now, we will find (super-)eigenfunctions for the Casimir:

$$\mathcal{C}A(t, z, \zeta_0, \zeta_1) = \Lambda A(t, z, \zeta_0, \zeta_1) \quad (3.33)$$

where the (super-)eigenfunction is given by

$$A(t, z, \zeta_0, \zeta_1) = A_0(t, z) + \zeta_0 A_1(t, z) - A_2(t, z)\zeta_1 - \zeta_0 A_3(t, z)\zeta_1 \quad (3.34)$$

First, we will focus on bosonic⁸ eigenfunction, that is, A_0 is Grassmannian even. Then, acting with the Casimir on the eigenfunction, we have

$$\begin{aligned} \mathcal{C}A^- = & \left[-\frac{1}{18}A_0 + \frac{2}{3}z\partial_z A_0 + z^2(-\partial_t^2 + \partial_z^2)A_0 + zA_3\right] \\ & + \zeta_0 \left[\frac{1}{9}A_1 + \frac{5}{3}z\partial_z A_1 + z^2(-\partial_t^2 + \partial_z^2)A_1 - z\partial_t A_1 - \frac{1}{2}z\partial_z A_2 - \frac{1}{2}z\partial_t A_2 - \frac{1}{6}A_2\right] \\ & - \left[\frac{1}{9}A_2 + \frac{5}{3}z\partial_z A_2 + z^2(-\partial_t^2 + \partial_z^2)A_2 + z\partial_t A_2 - \frac{1}{2}z\partial_z A_1 + \frac{1}{2}z\partial_t A_1 - \frac{1}{6}A_1\right] \zeta_1 \\ & - \zeta_0 \left[\frac{7}{9}A_3 + \frac{8}{3}z\partial_z A_3 + z^2(-\partial_t^2 + \partial_z^2)A_3 + \frac{1}{6}\partial_z A_0 + \frac{1}{4z}(-z^2\partial_t^2 + z^2\partial_z^2)A_0\right] \zeta_1 \end{aligned} \quad (3.35)$$

Note that A_0 (and, A_1) and A_3 (A_2 , respectively) are mixed. For A_0 and A_3 , we will use the following ansatz which is similar to non-supersymmetric SYK model [9, 10]:

$$A_0 = e^{-i\omega t} z^{\frac{1}{6}} J_\nu(\omega z) \quad (3.36)$$

$$A_3 = a_3 e^{-i\omega t} z^{-\frac{5}{6}} J_\nu(\omega z) \quad (3.37)$$

We find that there are two solutions given by

$$a_3 = \frac{1}{2} \left(\frac{1}{6} \pm \nu \right) \quad (3.38)$$

and the corresponding eigenvalues are

$$\mathcal{C}A^- = \nu \left(\nu \pm \frac{1}{2} \right) A^- \quad (3.39)$$

Since \mathcal{Q} commutes with the Casimir, $\mathcal{Q}A^-$ is also an eigenfunction if A^- is an eigenfunction. However, since that the parity of $\mathcal{Q}A^-$ is opposite to A , $\mathcal{Q}A^-$ is a fermionic eigenfunction.

⁸Recall that bosonic bi-local superfield corresponds to Grassmannian odd supermatrix A^- .

Furthermore, A_0 and A_3 components of the bosonic eigenvectors can determine the A_1 and A_2 components of the fermionic eigenfunction because of parity. This is also easily seen by the action of \mathcal{Q} on the (bosonic) eigenfunction:

$$\begin{aligned} \mathcal{Q}A^- &= A_1 + A_2 + \zeta_0 \left(-\frac{1}{2}\partial_t A_0 - \frac{1}{2}\partial_z A_0 + A_3 \right) - \left(\frac{1}{2}\partial_t A_0 - \frac{1}{2}\partial_z A_0 + A_3 \right) \zeta_1 \\ &\quad - \zeta_0 \left(-\frac{1}{2}\partial_t A_2 - \frac{1}{2}\partial_z A_2 + \frac{1}{2}\partial_t A_1 - \frac{1}{2}\partial_z A_1 \right) \zeta_1 . \end{aligned} \quad (3.40)$$

In the same way, one can also find the A_0 and A_3 components of the fermionic eigenfunctions. i.e., The action of the Casimir on the fermionic eigenfunction is

$$\begin{aligned} \mathcal{C}A^+ &= \left[-\frac{1}{18}A_0 + \frac{2}{3}z\partial_z A_0 + z^2(-\partial_t^2 + \partial_z^2)A_0 - zA_3 \right] \\ &\quad + \zeta_0 \left[\frac{1}{9}A_1 + \frac{5}{3}z\partial_z A_1 + z^2(-\partial_t^2 + \partial_z^2)A_1 - z\partial_t A_1 + \frac{1}{2}z\partial_z A_2 + \frac{1}{2}z\partial_t A_2 + \frac{1}{6}A_2 \right] \\ &\quad - \left[\frac{1}{9}A_2 + \frac{5}{3}z\partial_z A_2 + z^2(-\partial_t^2 + \partial_z^2)A_2 + z\partial_t A_2 - \frac{1}{2}z\partial_z A_1 - \frac{1}{2}z\partial_t A_1 + \frac{1}{6}A_1 \right] \zeta_1 \\ &\quad - \zeta_0 \left[\frac{7}{9}A_3 + \frac{8}{3}z\partial_z A_3 + z^2(-\partial_t^2 + \partial_z^2)A_3 - \frac{1}{6}\partial_z A_0 - \frac{1}{4z}(-z^2\partial_t^2 + z^2\partial_z^2)A_0 \right] \zeta_1 . \end{aligned} \quad (3.41)$$

Using an ansatz

$$A_0 = e^{-iwt} z^{\frac{1}{6}} J_\nu(wz) \quad (3.42)$$

$$A_3 = a_3 e^{-iwt} z^{-\frac{5}{6}} J_\nu(wz), \quad (3.43)$$

we find that

$$a_3 = -\frac{1}{2} \left(\frac{1}{6} \pm \nu \right) \quad (3.44)$$

$$\mathcal{C}A^+ = \nu \left(\nu \pm \frac{1}{2} \right) A^+ \quad (3.45)$$

Now, $\mathcal{Q}A^+$ gives A_1 and A_2 components of the bosonic eigenfunctions. e.g.,

$$\begin{aligned} \mathcal{Q}A^+ &= A_1 - A_2 + \zeta_0 \left(-\frac{1}{2}\partial_t A_0 - \frac{1}{2}\partial_z A_0 - A_3 \right) - \left(-\frac{1}{2}\partial_t A_0 + \frac{1}{2}\partial_z A_0 + A_3 \right) \zeta_1 \\ &\quad - \zeta_0 \left(-\frac{1}{2}\partial_t A_2 - \frac{1}{2}\partial_z A_2 - \frac{1}{2}\partial_t A_1 + \frac{1}{2}\partial_z A_1 \right) \zeta_1 \end{aligned} \quad (3.46)$$

We will also utilize the fermionic eigenfunctions of the Casimir in diagonalizing the quadratic action involved with fermi components in section 4.2. We summarize all eigenfunctions in appendix B.

4 Diagonalization of the quadratic action

In this section, we will diagonalize the quadratic action in (2.83). For this, one can directly diagonalize the kernel as in [9] by using eigenfunctions for the Casimir found in the previous section because the classical solution (anti-)commutes with superconformal generators. i.e.,

$$[\mathcal{L}, \Psi_{\text{cl}}] = \mathbb{L} \otimes \Psi_{\text{cl}} + \Psi_{\text{cl}} \otimes \mathbb{L}^\sharp \quad (\mathcal{L} \in \{\mathcal{P}, \mathcal{K}, \mathcal{D}, \mathcal{Q}, \mathcal{S}\}, \mathbb{L} \in \{\mathbb{P}, \mathbb{K}, \mathbb{D}, \mathbb{Q}, \mathbb{S}\}) \quad (4.1)$$

We give this direct diagonalization in appendix C because they involve tedious integrations. Instead, we present the diagonalization in a pedagogical way based on an observation from the result of the direct evaluation.

The basic idea is to diagonalize separately two terms in the quadratic action

$$S_{\text{col}}^{(2)} = -\frac{1}{2} \text{str} (\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi + 2J\Phi \otimes [\Psi_{\text{cl}}\Phi]) . \quad (4.2)$$

Indeed, we will see that the second term

$$\text{str} [\Phi \otimes [\Psi_{\text{cl}}\Phi]] \quad (4.3)$$

is nothing but the inner product of two eigenfunctions. In addition, in order to diagonalize the first term

$$\text{str} [\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi] , \quad (4.4)$$

we will use a similar calculation as in [10]. That is, for each eigenfunction $u_{\nu w}$, we will find a function $\tilde{u}_{\nu w}$ such that

$$\Psi_{\text{cl}} \otimes \tilde{u}_{\nu w} \otimes \Psi_{\text{cl}} = g(\nu) u_{\nu w} \quad (4.5)$$

where w is a frequency related to the eigenvalue of \mathcal{P} , and ν is a representation of the superconformal algebra. In addition, $g(\nu)$ is a function of ν , which will determine the spectrum of the SUSY SYK model.

4.1 Eigenfunctions of the quadratic action: bosonic components

Eigenfunctions: we begin with eigenfunction $u_{\nu w}^1$ of the superconformal Casimir in (B.1). This can be written as

$$e^{-i\omega t} z^{\frac{1}{6}} J_\nu(|wz|) \begin{pmatrix} 0 & -\frac{\nu - \frac{1}{6}}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \quad (4.6)$$

Here, we demand that the eigenfunction $u_{\nu w}^1$ obeys the symmetry of the supermatrix of the $\mathcal{N} = 1$ SYK model in (2.39). i.e.,

$$\mathcal{J} u_{\nu w}^{\text{st}} \mathcal{J} = u_{\nu w} \quad (4.7)$$

In general, we also have a second solution involved with $J_{-\nu}$ because the superconformal Casimir related to this eigenfunction is reduced to Bessel's differential equation. For the given ν and w , we have such an eigenfunction in the same representation in (B.2) given by

$$e^{-i\omega t} z^{\frac{1}{6}} J_{-\nu}(|wz|) \begin{pmatrix} 0 & -\frac{\nu - \frac{1}{6}}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \quad (4.8)$$

where we also demand the symmetry of the eigenfunction in (4.7). Hence, one has to find a relative coefficient of the eigenfunctions (4.6) and (4.8) to diagonalize the kernel of the quadratic action. This coefficient is usually determined by boundary condition. In particular, it is useful to think of the IR boundary condition (i.e., $z \rightarrow \infty$). From the asymptotic behavior of the Bessel function, we have

$$J_\nu(z) + \xi J_{-\nu}(z) \approx \sqrt{\frac{2}{\pi z}} \left(\cos \frac{\pi}{2}(\nu + 1/2) + \xi \sin \frac{\pi}{2}(\nu + 1/2) \right) \cos z + \sqrt{\frac{2}{\pi z}} \left(\sin \frac{\pi}{2}(\nu + 1/2) + \xi \cos \frac{\pi}{2}(\nu + 1/2) \right) \sin z \quad (4.9)$$

where ξ is a relative coefficient. In the non-supersymmetric SYK model, after direct diagonalization of the kernel, it turns out that the eigenfunction behaves like $z^{-\frac{1}{2}} \cos z$ in large z . In this section, we demand the generalized boundary condition thereof by brute force, but we also confirmed in appendix C that this eigenfunctions indeed diagonalizes the quadratic action. In addition to the asymptotic behavior $z^{-\frac{1}{2}} \cos z$, it would also possible to demand $z^{-\frac{1}{2}} \sin z$ in large z . Hence, demanding those two boundary conditions, we generalize the function $Z_\nu(z)$ introduced in [9]:

$$Z_\nu^\mp(z) \equiv J_\nu(z) + \xi_{\pm\nu} J_{-\nu}(z) \quad (4.10)$$

where ξ_ν is defined by

$$\xi_\nu \equiv \frac{\tan \frac{\pi\nu}{2} + 1}{\tan \frac{\pi\nu}{2} - 1} \quad (4.11)$$

Note that at large z , they behave as

$$Z_\nu^-(z) \sim \frac{\cos z}{\sqrt{z}} \quad , \quad Z_\nu^+(z) \sim \frac{\sin z}{\sqrt{z}} \quad (4.12)$$

Now, we will consider UV boundary condition ($z \rightarrow 0$). In [9], the Bessel's differential equation from the Casimir operator was interpreted as a Schrodinger-like equation to claim that a real ν corresponds to a discrete bound state, and pure imaginary ν 's are consist of continuum spectrum. Likewise, one can also expect that there are bound states for real ν . Furthermore, we can also demand that the such eigenfunctions do not diverge as z goes to zero. This gives a discrete series of possible ν 's for each Z_ν^\mp . i.e.,

$$Z_\nu^-(z) \quad : \quad \nu = 2n + \frac{3}{2} \quad (n = 0, 1, 2, \dots) \quad (4.13)$$

$$Z_\nu^+(z) \quad : \quad \nu = 2n + \frac{1}{2} \quad (n = 0, 1, 2, \dots) \quad (4.14)$$

Now, since there are two independent linear combination of (4.6) and (4.8), we have to determine which UV/IR boundary condition is possible for them. For this, we utilize the zero mode of the kernel involved with reparametrization. In non-supersymmetric SYK model, the zero mode can be evaluated [12] by

$$u_0(\tau_1, \tau_2) \equiv \left. \frac{\delta \Psi_{\text{cl},f}(\tau_1, \tau_2)}{\delta f(\tau)} \right|_{f(\tau)=\tau} \quad (4.15)$$

where Ψ_{cl} is the large N classical solution of non-supersymmetric SYK model, and $\Psi_{\text{cl},f}$ is transformed classical solution by reparametrization $f(\tau)$. i.e.,

$$\Psi_{\text{cl},f} = |f'(\tau_1)f'(\tau_2)|^{\frac{1}{q}} \Psi_{\text{cl}}(f(\tau_1), f(\tau_2)) \quad (4.16)$$

In the SUSY SYK model, one can quickly obtain the zero mode from the classical solution in (2.68) by using the reparametrization instead of super-reparametrization. We found

$$u_0 \sim \begin{pmatrix} 0 & -\frac{4}{3|\tau_{12}|} \\ \text{sgn}(\tau_{12}) & 0 \end{pmatrix} \quad (4.17)$$

It was already known that this zero mode corresponds to the eigenfunction $Z_{\frac{3}{2}}^-(z)$ [50]. On the other hand, we have two types of eigenfunctions (B.1) or (B.5). For $\nu = \frac{1}{2}$ or $\nu = \frac{3}{2}$, we found that only (B.1) with $\nu = \frac{3}{2}$ can become the zero mode in (4.17). Hence, we can deduce that (B.1) satisfy the boundary condition of Z_{ν}^- , and therefore, we can write the eigenfunction as

$$u_{\nu w}^1(t, z) = e^{-iwt} |Jz|^{\frac{1}{6}} Z_{\nu}^- (|wz|) \begin{pmatrix} 0 & -\frac{\nu-\frac{1}{6}}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \quad (4.18)$$

or equivalently,

$$u_{\nu w}^1(\tau_1, \tau_2) = \frac{e^{-\frac{iw}{2}(\tau_1+\tau_2)}}{\sqrt{8\pi}} \left| J \frac{\tau_1 - \tau_2}{2} \right|^{\frac{1}{6}} Z_{\nu}^- \left(\left| \frac{w}{2}(\tau_1 - \tau_2) \right| \right) \begin{pmatrix} 0 & -\frac{\nu-\frac{1}{6}}{2|\frac{1}{2}(\tau_1-\tau_2)|} \\ \text{sgn}(\tau_1 - \tau_2) & 0 \end{pmatrix} \quad (4.19)$$

where the representation ν can be either a pure imaginary continuum value or a discrete real value for UV boundary condition as in [9]. i.e.,

$$\nu = \frac{3}{2} + 2n \quad (n = 0, 1, 2, \dots) \quad (4.20)$$

$$\nu = ir \quad (r \geq 0) \quad (4.21)$$

For the other UV/IR boundary condition, we have the eigenfunction (B.6) corresponding to Z_{ν}^+ :

$$u_{\nu w}^2(t, z) = \frac{e^{-iwt}}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} Z_{\nu}^+ (|wz|) \begin{pmatrix} 0 & \frac{\nu+\frac{1}{6}}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \quad (4.22)$$

or equivalently,

$$u_{\nu w}^2(\tau_1, \tau_2) = \frac{e^{-\frac{iw}{2}(\tau_1+\tau_2)}}{\sqrt{8\pi}} \left| J \frac{\tau_1 - \tau_2}{2} \right|^{\frac{1}{6}} Z_{\nu}^+ \left(\left| \frac{w}{2}(\tau_1 - \tau_2) \right| \right) \begin{pmatrix} 0 & \frac{\nu+\frac{1}{6}}{2|\frac{1}{2}(\tau_1-\tau_2)|} \\ \text{sgn}(\tau_1 - \tau_2) & 0 \end{pmatrix} \quad (4.23)$$

where we also demanded the symmetry of eigenfunctions in (4.7), and the representation ν 's are

$$\nu = \frac{1}{2} + 2n \quad (n = 0, 1, 2, \dots) \quad (4.24)$$

$$\nu = ir \quad (r \in \mathbb{R}) \quad (4.25)$$

Diagonalization of the second term: it is useful to find orthogonality of the functions Z_ν^\mp 's because the second term in the quadratic action in (2.83) is, in fact, reduced to an inner product of Z_ν^\mp 's. i.e.,

$$\text{str} (u_{\nu,w} \otimes [\Psi_{\text{cl}} u_{\nu',w'}]) \sim \delta(w+w') \int_0^\infty \frac{dz}{z} Z_\nu^\alpha(z) Z_{\nu'}^{\alpha'}(z) \quad (4.26)$$

where $\alpha, \alpha' = \mp$. First, it is easy to see that Z_ν^- is orthogonal to $Z_{\nu'}^+$ because they have different eigenvalues for Casimir. By a similar analysis to [9], we found that

$$\int_0^\infty \frac{dz}{|z|} \overline{Z_\nu^\alpha(|wz|)} Z_{\nu'}^{\alpha'}(|w'z|) = \delta_{\alpha\alpha'} N_\nu \delta(\nu - \nu') \quad (4.27)$$

where

$$N_\nu = \begin{cases} \frac{1}{2\nu} & (\nu = \frac{3}{2} + 2n \text{ for } Z^-, \text{ or } \nu = \frac{1}{2} + 2n \text{ for } Z^+ \quad (n = 0, 1, 2, \dots)) \\ \frac{2 \sin \pi \nu}{\nu} & (\text{for } \nu = ir \quad (r \in \mathbb{R})) \end{cases} \quad (4.28)$$

For real ν , Z_ν^\mp is a real function so that we can immediately see that (4.26) is diagonalized. On the other hand, for pure imaginary value $\nu = ir$, the complex conjugate of the function Z_ν^\mp can be written as

$$\overline{Z_\nu^\mp} = J_{-\nu}(z) + \xi_{\mp\nu} J_\nu(z) = \xi_{\mp\nu} [J_\nu(z) + \xi_{\pm\nu} J_{-\nu}(z)] = \xi_{\mp\nu} Z_\nu^\mp(z) \quad (4.29)$$

where we used a useful identity for ξ_ν :

$$\xi_{-\nu} \xi_\nu = 1 \quad (4.30)$$

Hence, we have

$$\int_0^\infty \frac{dz}{|z|} Z_{ir}^\mp(|wz|) Z_{ir'}^\mp(|w'z|) = \tilde{N}_{ir}^\mp \delta(r - r') \quad (\tilde{N}_{ir}^\mp \equiv \xi_{\pm ir} N_{ir}) \quad (4.31)$$

and, (4.26) is also diagonalized. We emphasize that (4.26) leads to an induced inner product for the supermatrix formulation. i.e.,

$$\langle u_{\nu,w}, u_{\nu',w'} \rangle \equiv \text{str} (u_{\nu,w} \otimes [\Psi_{\text{cl}} u_{\nu',w'}]) \quad (4.32)$$

Diagonalization of the first term: next, let us consider the first term in (2.83). To diagonalize it, for each $u_{\nu w}$, we will find a function $\tilde{u}_{\nu w}$ such that

$$\Psi_{\text{cl}} \otimes \tilde{u}_{\nu w} \otimes \Psi_{\text{cl}} = g(\nu) u_{\nu w} \quad (4.33)$$

where $g(\nu)$ is a function of ν . In appendix C, one can directly find \tilde{u} for each $u_{\nu w}^1$ and $u_{\nu w}^2$. But, in this section, we present a new method to find \tilde{u} .

Suppose that there exist $\tilde{u}_{\nu w}$ to satisfy (4.33). Then, the first term in (2.83) becomes

$$\text{str} (u_{\nu'w'} \otimes \Psi_{\text{cl}}^{-1} \otimes u_{\nu w} \otimes \Psi_{\text{cl}}^{-1}) = \frac{1}{g(\nu)} \text{str} (u_{\nu'w'} \otimes \tilde{u}_{\nu w}) \quad (4.34)$$

one may find a function $\tilde{u}_{\nu w}$ such that

$$\tilde{u}_{\nu w}(\tau_1, \tau_2) = [\Psi_{\text{cl}} \tilde{u}_{\nu w}](\tau_1, \tau_2) \quad (4.35)$$

where the product on the r.h.s. is the usual product of superfields. Then, (4.34) becomes

$$\text{str} (u_{\nu' w'} \otimes \Psi_{\text{cl}} \otimes u_{\nu w} \otimes \Psi_{\text{cl}}) = \langle u_{\nu' w'}, \tilde{u}_{\nu w} \rangle \quad (4.36)$$

where $\langle \cdot, \cdot \rangle$ is the induced inner product of supermatrix defined in (4.32). Hence, if the first term is diagonalized by $u_{\nu w}$, we should have

$$\tilde{u}_{\nu w}(\tau_1, \tau_2) \sim [\Psi_{\text{cl}} u_{\nu w}](\tau_1, \tau_2) \quad (4.37)$$

Of course, this is confirmed by direct calculation for $q = 3$ case as well as general q case where Ψ_{cl} on the r.h.s. of (4.37) and (4.32) is replaced by Ψ_{cl}^{q-2} . The remaining calculation is to fix the coefficient and the function $g(\nu)$ where one cannot avoid evaluating integrations. We found that

$$\tilde{u}_{\nu w}^1(\tau_1, \tau_2) = -\frac{2A}{\sqrt{8\pi}} J \frac{e^{-\frac{iw}{2}(\tau_1 + \tau_2)}}{\left|\frac{J}{2}(\tau_1 - \tau_2)\right|^{\frac{1}{6}}} Z_{\nu}^{-} \left(\left| \frac{w}{2}(\tau_1 - \tau_2) \right| \right) \begin{pmatrix} 0 & -\frac{\nu + \frac{1}{6}}{2|\frac{1}{2}(\tau_1 - \tau_2)|} \text{sgn}(\tau_1 - \tau_2) \\ 1 & 0 \end{pmatrix} \quad (4.38)$$

$$\tilde{u}_{\nu w}^2(\tau_1, \tau_2) = \frac{2A}{\sqrt{8\pi}} J \frac{e^{-\frac{iw}{2}(\tau_1 + \tau_2)}}{\left|\frac{J}{2}(\tau_1 - \tau_2)\right|^{\frac{1}{6}}} Z_{\nu}^{+} \left(\left| \frac{w}{2}(\tau_1 - \tau_2) \right| \right) \begin{pmatrix} 0 & \frac{\nu - \frac{1}{6}}{2|\frac{1}{2}(\tau_1 - \tau_2)|} \text{sgn}(\tau_1 - \tau_2) \\ 1 & 0 \end{pmatrix} \quad (4.39)$$

where $A = \left(\frac{1}{4\sqrt{3}\pi}\right)^{\frac{1}{3}}$ and

$$g_1(\nu) = -2^{-\frac{1}{3}} \frac{\Gamma\left(\frac{5}{3}\right) \Gamma\left(\frac{5}{12} - \frac{\nu}{2}\right) \Gamma\left(\frac{5}{12} + \frac{\nu}{2}\right)}{\Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{13}{12} - \frac{\nu}{2}\right) \Gamma\left(\frac{1}{12} + \frac{\nu}{2}\right)} \quad (4.40)$$

$$g_2(\nu) = -2^{-\frac{1}{3}} \frac{\Gamma\left(\frac{5}{3}\right) \Gamma\left(\frac{5}{12} - \frac{\nu}{2}\right) \Gamma\left(\frac{5}{12} + \frac{\nu}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{3} - \frac{h}{2}\right) \Gamma\left(\frac{13}{12} + \frac{h}{2}\right)} \quad (4.41)$$

which agrees with [50]. Note that $\tilde{u}_{\nu w}$'s in (4.38) and (4.39) have different symmetry from $u_{\nu w}$. i.e.,

$$\mathcal{J} \otimes \tilde{u}_{\nu w}^{\text{st}} \otimes \mathcal{J} = -\tilde{u}_{\nu w} \quad (4.42)$$

This can be easily seen from the definition of $\tilde{u}_{\nu w}$ in (4.33):

$$\mathcal{J} (\Psi_{\text{cl}} \otimes \tilde{u}_{\nu w} \otimes \Psi_{\text{cl}})^{\text{st}} \otimes \mathcal{J} = -\Psi_{\text{cl}} \otimes \mathcal{J} \otimes \tilde{u}_{\nu w}^{\text{st}} \otimes \mathcal{J} \otimes \Psi_{\text{cl}} = \Psi_{\text{cl}} \otimes \tilde{u}_{\nu w} \otimes \Psi_{\text{cl}} \quad (4.43)$$

Now, we expand the fluctuation Φ in (2.83) in terms of $u_{\nu w}^1$ and $u_{\nu w}^2$:

$$\Phi = \sum_w \left[\sum_{\substack{\nu=2n+\frac{3}{2} \\ n=0,1,\dots}} \mathcal{A}_{\nu w}^1 u_{\nu w}^1 + \sum_{\substack{\nu=2n+\frac{1}{2} \\ n=0,1,\dots}} \mathcal{A}_{\nu w}^2 u_{\nu w}^2 + \sum_{\substack{\nu=ir \\ r \geq 0}} (\mathcal{A}_{\nu w}^1 u_{\nu w}^1 + \mathcal{A}_{\nu w}^2 u_{\nu w}^2) \right] \quad (4.44)$$

Note that the reality condition of the component fields leads to

$$\overline{\Psi} = -\Psi \quad (4.45)$$

which imposes the following constraint.

$$\overline{\mathcal{A}_{\nu w}^1} = -\mathcal{A}_{\nu-w}^1 \quad \text{for } \nu = 2n + \frac{3}{2} \quad (n = 0, 1, 2, \dots) \quad (4.46)$$

$$\overline{\mathcal{A}_{\nu w}^2} = -\mathcal{A}_{\nu-w}^2 \quad \text{for } \nu = 2n + \frac{1}{2} \quad (n = 0, 1, 2, \dots) \quad (4.47)$$

$$\overline{\mathcal{A}_{\nu w}^1} = -\xi_\nu \mathcal{A}_{\nu-w}^1 \quad \text{for } \nu = ir \quad (r \geq 0) \quad (4.48)$$

$$\overline{\mathcal{A}_{\nu w}^2} = -\xi_{-\nu} \mathcal{A}_{\nu-w}^2 \quad \text{for } \nu = ir \quad (r \geq 0) \quad (4.49)$$

Then, we found that the quadratic action in (2.83) can be written as

$$\begin{aligned} S_{\text{col}}^{(2)} = & \frac{2J}{2^{\frac{2}{3}} 3^{\frac{1}{6}} \pi^{\frac{1}{3}}} \sum_{w \geq 0} \sum_{\nu=2n+\frac{3}{2} \text{ or } \nu=ir} \nu N_\nu \frac{1-g_1(\nu)}{g_1(\nu)} |\mathcal{A}_{\nu w}^1|^2 \\ & + \frac{2J}{2^{\frac{2}{3}} 3^{\frac{1}{6}} \pi^{\frac{1}{3}}} \sum_{w \geq 0} \sum_{\nu=2n+\frac{1}{2} \text{ or } \nu=ir} \nu N_\nu \frac{1-g_2(\nu)}{g_2(\nu)} |\mathcal{A}_{\nu w}^2|^2 \end{aligned} \quad (4.50)$$

where we absorbed the factor $\xi_{\pm\nu}$ in the normalization $\tilde{N}_\nu^\pm = \xi_{\pm\nu} N_\nu$ into the reality condition. This leads to two-point function of bi-local collective superfields (or, invariant four-point function of fundamental superfield). The summation over $\nu = ir$ can be understood as a contour integral along the imaginary axis. Repeating the same procedure in [10, 11], one can expect that the contour integral will pick up simple poles from $1 - g_1(\nu)$ and $1 - g_2(\nu)$ and the residues from other simple poles will cancel with the contribution from discrete series of ν . Hence, the half of the spectrum of the $\mathcal{N} = 1$ SUSY SYK model is given by two equations

$$g_1(\nu) = 1 \quad , \quad g_2(\nu) = 1 \quad (4.51)$$

which was shown in [50].

One can also diagonalize the quadratic action with the following fermionic eigenfunctions:

$$u_{\nu w}^3(t, z) = e^{-iwt} |Jz|^{\frac{1}{6}} Z_\nu^- (|wz|) \begin{pmatrix} 0 & \mathcal{B}_{\nu w} \frac{\nu - \frac{1}{6}}{2|z|} \\ \mathcal{B}_{\nu w} \text{sgn}(z) & 0 \end{pmatrix} \quad (4.52)$$

$$u_{\nu w}^4(t, z) = \frac{e^{-iwt}}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} Z_\nu^+ (|wz|) \begin{pmatrix} 0 & -\mathcal{B}_{\nu w} \frac{\nu + \frac{1}{6}}{2|z|} \\ \mathcal{B}_{\nu w} \text{sgn}(z) & 0 \end{pmatrix} \quad (4.53)$$

where $\mathcal{B}_{\nu w}$ is a Grassmannian odd constant. Comparing to $u_{\nu w}^1$ and $u_{\nu w}^2$ in (4.18) and (4.22), one can see that the only difference is the sign of $\theta_1 \theta_2$ components. Moreover, because $\mathcal{B}_{\nu w}$ is Grassmannian odd, one can end up with the same calculations as those in bosonic Grassmannian eigenfunctions except for an overall minus sign.

4.2 Eigenfunctions of the quadratic action: fermionic components

After obtaining the bosonic eigenfunctions and the corresponding eigenvalues for the kernel, the diagonalization by fermionic components of bosonic eigenfunction is straightforward because of supersymmetry. In this section, we work out this diagonalization in detail. Also, we double-checked a part of the diagonalization by direct calculation in appendix C.

We claim that $\mathcal{Q}u_{\nu w}^a$ ($a = 3, 4$) diagonalize the quadratic action with the same eigenvalue as $u_{\nu w}^a$. First, note that the classical solution Ψ_{cl} is annihilated by the bi-local supercharge \mathcal{Q} which we have discussed in (3.1)

$$\mathcal{Q}\Psi_{\text{cl}} = \mathcal{Q}\otimes\Psi_{\text{cl}} + \Psi_{\text{cl}}\otimes\mathcal{Q} = 0 \quad (4.54)$$

where \mathbb{Q} is defined in (3.19).

Now, we will find an analogous identity to (4.33). We will act with $\mathcal{Q}\mathbb{B}\otimes$ on the both sides of (4.33) where \mathbb{B} is a constant Grassmannian odd supermatrix defined by

$$\mathbb{B} = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{pmatrix} \quad \mathcal{B} : \text{Grassmannian odd constant.} \quad (4.55)$$

Note that the supermatrix \mathbb{B} commutes with \mathcal{Q}, \mathcal{P} and Ψ_{cl} . Using (3.24) and (4.54), it becomes

$$\begin{aligned} g\mathcal{Q}(\mathbb{B}\otimes u) &= \mathcal{Q}(\Psi_{\text{cl}}\otimes\mathbb{B}\otimes\tilde{u}\otimes\Psi_{\text{cl}}) = \mathcal{Q}\otimes\Psi_{\text{cl}}\otimes\mathbb{B}\otimes\tilde{u}\otimes\Psi_{\text{cl}} - \Psi_{\text{cl}}\otimes\mathbb{B}\otimes\tilde{u}\otimes\Psi_{\text{cl}}\otimes\mathcal{Q} \\ &= -(\Psi_{\text{cl}}\otimes\mathcal{Q}\otimes\mathbb{B}\otimes\tilde{u}\otimes\Psi_{\text{cl}} - \Psi_{\text{cl}}\otimes\mathbb{B}\otimes\tilde{u}\otimes\mathcal{Q}\otimes\Psi_{\text{cl}}) = -\Psi_{\text{cl}}\otimes[\mathcal{Q}(\mathbb{B}\otimes\tilde{u})]\otimes\Psi_{\text{cl}} \end{aligned} \quad (4.56)$$

where we omit ν and w . Hence, for the given $u_{\nu w}$, $\mathcal{Q}(\mathbb{B}\otimes u_{\nu w})$ and $\mathcal{Q}(\mathbb{B}\otimes\tilde{u}_{\nu w})$ satisfy (4.33) with the same $g(\nu)$, but with an additional minus sign. i.e.,

$$g(\nu)\mathcal{Q}(\mathbb{B}\otimes u_{\nu w}) = -\Psi_{\text{cl}}\otimes\mathcal{Q}(\mathbb{B}\otimes\tilde{u}_{\nu w})\otimes\Psi_{\text{cl}} \quad (4.57)$$

This simplify the first term in (2.83), and we need to evaluate $\text{str}[\mathcal{Q}(\mathbb{B}\otimes u)\otimes\mathcal{Q}(\mathbb{B}\otimes\tilde{u})]$. Using (3.23) and (3.24), we have

$$\begin{aligned} \text{str}[\mathcal{Q}(\mathbb{B}\otimes u)\otimes\mathcal{Q}(\mathbb{B}\otimes\tilde{u})] &= \text{str}[(\mathcal{Q}\otimes\mathbb{B}\otimes u - \mathbb{B}\otimes u\otimes\mathcal{Q})\otimes(\mathcal{Q}\otimes\mathbb{B}\otimes\tilde{u} - \mathbb{B}\otimes\tilde{u}\otimes\mathcal{Q})] \\ &= -\text{str}[-\mathcal{Q}\otimes\mathcal{Q}\otimes(\mathbb{B}\otimes u)\otimes(\mathbb{B}\otimes\tilde{u}) + (\mathbb{B}\otimes u)\otimes\mathcal{Q}\otimes\mathcal{Q}\otimes(\mathbb{B}\otimes\tilde{u})] \\ &\quad + \text{str}[\mathcal{Q}\otimes(\mathbb{B}\otimes u)\otimes\mathcal{Q}\otimes(\mathbb{B}\otimes\tilde{u}) - \mathcal{Q}\otimes(\mathbb{B}\otimes u)\otimes\mathcal{Q}\otimes(\mathbb{B}\otimes\tilde{u})] \\ &= -\text{str}[\mathcal{P}\otimes(\mathbb{B}\otimes u)\otimes(\mathbb{B}\otimes\tilde{u}) - (\mathbb{B}\otimes u)\otimes\mathcal{P}\otimes(\mathbb{B}\otimes\tilde{u})] \\ &= -\text{str}[(\mathcal{P}(\mathbb{B}\otimes u))\otimes(\mathbb{B}\otimes\tilde{u})] \end{aligned} \quad (4.58)$$

where we used the following property of the supertrace in the second line

$$\text{str}(XY) = (-1)^{|X||Y|}\text{str}(YX) \quad (4.59)$$

Therefore, the first term in the quadratic action can be written as

$$\begin{aligned} &-\frac{1}{2}\text{str}[\mathcal{Q}(\mathbb{B}_{\nu'w'}\otimes u_{\nu'w'})\otimes\Psi_{\text{cl}}^{-1}\otimes\mathcal{Q}(\mathbb{B}_{\nu w}\otimes u_{\nu w})\otimes\Psi_{\text{cl}}^{-1}] \\ &= -\frac{1}{2g(\nu)}\text{str}[(\mathcal{P}(\mathbb{B}_{\nu'w'}\otimes u_{\nu'w'}))\otimes\mathbb{B}_{\nu w}\otimes\tilde{u}_{\nu w}] \end{aligned} \quad (4.60)$$

and, this corresponds to diagonalization of Grassmannian odd eigenfunctions in the previous section.

In a similar way, one can also show the $\mathcal{Q}u$ will diagonalize the second term of (2.83). For this, we need to move the differential operator \mathcal{Q} by using integration by parts in the superspace integration. But, in the supermatrix formulation, this is nothing but property of supertrace. e.g.,

$$\begin{aligned} \text{str}[(\mathcal{Q}X)\otimes Y] &= \text{str}[\mathcal{Q}\otimes X\otimes Y] + (-1)^{|X|+1}\text{str}[X\otimes\mathcal{Q}\otimes Y] \\ &= (-1)^{|X|+1}\text{str}[X\otimes(\mathcal{Q}Y)] \end{aligned} \tag{4.61}$$

Thus, the inner product of two $\mathcal{Q}(\mathbb{B}\otimes u)$ is given by

$$\begin{aligned} \langle \mathcal{Q}(\mathbb{B}\otimes u), \mathcal{Q}(\mathbb{B}\otimes u) \rangle &= -\text{str}[\mathbb{B}\otimes u\otimes\mathcal{Q}(\Psi_{\text{cl}}\mathcal{Q}(\mathbb{B}\otimes u))] \\ &= -\text{str}[\mathbb{B}\otimes u\otimes(\Psi_{\text{cl}}\mathcal{Q}^2(\mathbb{B}\otimes u))] = \text{str}[\mathbb{B}\otimes u\otimes(\Psi_{\text{cl}}\mathcal{P}(\mathbb{B}\otimes u))] \\ &= \int d\tau_1 d\theta_1 d\tau_2 d\theta_2 \Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) \\ &\quad \times [\mathbb{B}\otimes u](\tau_1, \theta_1; \tau_2, \theta_2)[\mathcal{P}(\mathbb{B}\otimes u)](\tau_1, \theta_1; \tau_2, \theta_2) \end{aligned} \tag{4.62}$$

In the same way as before, we expand the fluctuation Φ in terms of $\mathcal{Q}(\mathbb{B}_{\nu w}^1 \star u_{\nu w}^3)$ and $\mathcal{Q}(\mathbb{B}_{\nu w}^1 \star u_{\nu w}^4)$, and the diagonalization is exactly the same as those of $u_{\nu w}^3$ and $u_{\nu w}^4$ which we shortly discussed before.

5 $\mathcal{N} = 2$ supersymmetric SYK model

In this section, we will generalize $\mathcal{N} = 1$ bi-local collective superfield theory to $\mathcal{N} = 2$ case.

5.1 Bi-local chiral/anti-chiral superspace, superfield and supermatrix

We begin with the bi-local superspace for $\mathcal{N} = 2$ SUSY vector models. At first glance, it seems that we have a larger Grassmannian space because there are two Grassmannian coordinates θ and $\bar{\theta}$. However, since we will focus on the chiral or anti-chiral superfields, the construction is almost the same as for $\mathcal{N} = 1$ case. First, let us focus on superfield A which is chiral with respect to the first superspace and anti-chiral in the second superspace:

$$\bar{D}_1 A(\tau_1, \theta_1, \bar{\theta}_1; \tau_2, \bar{\theta}_2, \theta_2) = D_2 A(\tau_1, \theta_1, \bar{\theta}_1; \tau_2, \bar{\theta}_2, \theta_2) = 0 \tag{5.1}$$

where the superderivatives are given by

$$D \equiv \partial_\theta + \bar{\theta}\partial_\tau, \quad \bar{D} \equiv \partial_{\bar{\theta}} + \theta\partial_\tau \tag{5.2}$$

Hence, the superfield A depends only on $(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)$ where

$$\sigma \equiv \tau + \theta\bar{\theta}, \quad \bar{\sigma} \equiv \tau - \theta\bar{\theta} \tag{5.3}$$

and, one can expand the superfield A as follows.

$$A(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) = A_0(\sigma_1, \bar{\sigma}_2) + \theta_1 A_1(\tau_1, \bar{\sigma}_2) - A_2(\sigma_1, \tau_2)\bar{\theta}_2 - \theta_1 A_3(\tau_1, \tau_2)\bar{\theta}_2 \tag{5.4}$$

$$= A_0(\sigma_1, \bar{\sigma}_2) + \theta_1 A_1(\sigma_1, \bar{\sigma}_2) - A_2(\sigma_1, \bar{\sigma}_2)\bar{\theta}_2 - \theta_1 A_3(\sigma_1, \bar{\sigma}_2)\bar{\theta}_2 \tag{5.5}$$

This bi-local superfield naturally appears in the $U(N)$ vector models because chiral superfields and anti-chiral superfields transform in the fundamental and anti-fundamental representations of $U(N)$, respectively so that they form a $U(N)$ invariant bi-local field. Hence, it is natural to construct the following bi-local superspace for such bi-local $U(N)$ superfields.

$$(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) \tag{5.6}$$

Now, we will define a star product in this bi-local superspace. However, it is difficult to construct the consistent star product of two chiral/anti-chiral bi-locals because the first and the second superspace have opposite chirality. Hence, we also introduce conjugate anti-chiral/chiral bi-local super field:

$$\bar{B}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2) = \bar{B}_0(\bar{\sigma}_1, \sigma_2) + \bar{\theta}_1 \bar{B}_1(\bar{\sigma}_1, \sigma_2) - \bar{B}_2(\bar{\sigma}_1, \sigma_2)\theta_2 - \bar{\theta}_1 \bar{B}_3(\bar{\sigma}_1, \sigma_2)\theta_2 \tag{5.7}$$

We found that a consistent star product between $A(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)$ and $\bar{B}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)$ is given by

$$A \bar{\otimes} \bar{B} \equiv \int A(\sigma_1, \theta_1; \bar{\sigma}_3, \bar{\theta}_3) d\tau_3 d\bar{\theta}_3 \bar{B}(\bar{\sigma}_3, \bar{\theta}_3; \sigma_2, \theta_2) \tag{5.8}$$

which was already recognized in [50] to analyze the Schwinger-Dyson equation. Similarly, we also define

$$\bar{B} \otimes A \equiv \int \bar{B}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_3, \theta_3) d\tau_3 d\theta_3 A(\sigma_3, \theta_3; \bar{\sigma}_2, \bar{\theta}_2) \tag{5.9}$$

Note that $A \bar{\otimes} \bar{B}$ is a chiral/chiral superfield while $\bar{B} \otimes A$ is an anti-chiral/anti-chiral superfield. As in $\mathcal{N} = 1$ case, the punchline is that the supermatrix formulation drastically simplifies this complicated star product in the bi-local superspace into matrix multiplication. First, we represent the bi-local superfields A and \bar{B} as the following supermatrix.

$$\begin{aligned} A(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) &= A_0(\sigma_1, \bar{\sigma}_2) + \theta_1 A_1(\sigma_1, \bar{\sigma}_2) - A_2(\sigma_1, \bar{\sigma}_2)\bar{\theta}_2 - \theta_1 A_3(\sigma_1, \bar{\sigma}_2)\bar{\theta}_2 \\ &\equiv \begin{pmatrix} A_1(\sigma_1, \bar{\sigma}_2) & A_3(\sigma_1, \bar{\sigma}_2) \\ A_0(\sigma_1, \bar{\sigma}_2) & A_2(\sigma_1, \bar{\sigma}_2) \end{pmatrix} \end{aligned} \tag{5.10}$$

$$\begin{aligned} \bar{B}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2) &= \bar{B}_0(\bar{\sigma}_1, \sigma_2) + \bar{\theta}_1 \bar{B}_1(\bar{\sigma}_1, \sigma_2) - \bar{B}_2(\bar{\sigma}_1, \sigma_2)\theta_2 - \bar{\theta}_1 \bar{B}_3(\bar{\sigma}_1, \sigma_2)\theta_2 \\ &\equiv \begin{pmatrix} \bar{B}_1(\bar{\sigma}_1, \sigma_2) & \bar{B}_3(\bar{\sigma}_1, \sigma_2) \\ \bar{B}_0(\bar{\sigma}_1, \sigma_2) & \bar{B}_2(\bar{\sigma}_1, \sigma_2) \end{pmatrix} \end{aligned} \tag{5.11}$$

Then, one can show that the star product of superfields becomes the following matrix product:

$$A \bar{\otimes} \bar{B} = \begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} \bar{\otimes} \begin{pmatrix} \bar{B}_1 & \bar{B}_3 \\ \bar{B}_0 & \bar{B}_2 \end{pmatrix}, \quad \bar{B} \star A = \begin{pmatrix} \bar{B}_1 & \bar{B}_3 \\ \bar{B}_0 & \bar{B}_2 \end{pmatrix} \otimes \begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} \tag{5.12}$$

These matrix products \otimes and $\bar{\otimes}$ are a combination of the usual matrix product and star product \star in bi-local time space (τ_1, τ_2) like the $\mathcal{N} = 1$ case:

$$\begin{pmatrix} A_1 & A_3 \\ A_0 & A_2 \end{pmatrix} \bar{\otimes} \begin{pmatrix} \bar{B}_1 & \bar{B}_3 \\ \bar{B}_0 & \bar{B}_2 \end{pmatrix} = \begin{pmatrix} (A_1 \star \bar{B}_1 + A_3 \star \bar{B}_0) & (A_1 \star \bar{B}_3 + A_3 \star \bar{B}_2) \\ (A_0 \star \bar{B}_1 + A_2 \star \bar{B}_0) & (A_0 \star \bar{B}_3 + A_2 \star \bar{B}_2) \end{pmatrix} \tag{5.13}$$

However, in the star product \star between components, we replace σ or $\bar{\sigma}$ in the intermediate integration variables with τ . i.e.,

$$(A_1 \star \bar{B}_1)(\sigma_1, \sigma_2) \equiv \int d\tau_3 A_1(\sigma_1, \tau_3) d\tau_3 \bar{B}_1(\tau_3, \sigma_2) \quad (5.14)$$

It is natural to consider chiral/chiral (or, anti-chiral/anti-chiral) supermatrices, too. They also follow the same multiplication rule in the supermatrix formulation. In general, the star product of supermatrices A and B is possible when the chirality of the second index of A is the same as the chirality of the first index of B :

$$A_{u,v} \otimes^v B_{v,w} = C_{u,w} \quad (u, v, w \in \{\text{chiral}, \text{anti-chiral}\}) \quad (5.15)$$

Before discussing the $\mathcal{N} = 2$ collective superfield theory, let us present useful formulae for the calculus of the bi-local superfield in $\mathcal{N} = 2$ which generalize the formulae of section 2.2. First, the functional derivative of the same fundamental superfield is given by

$$\frac{\delta f(\sigma, \theta)}{\delta f(\sigma', \theta')} = (\theta' - \theta) \delta(\sigma' - \sigma) \quad , \quad \frac{\delta \bar{f}(\bar{\sigma}, \bar{\theta})}{\delta \bar{f}(\bar{\sigma}', \bar{\theta}')} = (\bar{\theta}' - \bar{\theta}) \delta(\bar{\sigma}' - \bar{\sigma}) \quad (5.16)$$

We define the change of variables and chain rule for the fundamental superfield as follows.

$$\delta f_\alpha(\sigma, \theta) = \sum_\beta \int \delta f_\beta(\sigma', \theta') d\sigma' d\theta' \frac{\delta f_\alpha(\sigma, \theta)}{\delta f_\beta(\sigma', \theta')} \quad (5.17)$$

$$\frac{\delta}{\delta f_\alpha(\sigma, \theta)} = \sum_\beta \int \frac{\delta f_\beta(\sigma', \theta')}{\delta f_\alpha(\sigma, \theta)} d\sigma' d\theta' \frac{\delta}{\delta f_\beta(\sigma', \theta')} \quad (5.18)$$

$$\delta \bar{f}_\alpha(\bar{\sigma}, \bar{\theta}) = \sum_\beta \int \delta \bar{f}_\beta(\bar{\sigma}', \bar{\theta}') d\bar{\sigma}' d\bar{\theta}' \frac{\delta \bar{f}_\alpha(\bar{\sigma}, \bar{\theta})}{\delta \bar{f}_\beta(\bar{\sigma}', \bar{\theta}')} \quad (5.19)$$

$$\frac{\delta}{\delta \bar{f}_\alpha(\bar{\sigma}, \bar{\theta})} = \sum_\beta \int \frac{\delta \bar{f}_\beta(\bar{\sigma}', \bar{\theta}')}{\delta \bar{f}_\alpha(\bar{\sigma}, \bar{\theta})} d\bar{\sigma}' d\bar{\theta}' \frac{\delta}{\delta \bar{f}_\beta(\bar{\sigma}', \bar{\theta}')} \quad (5.20)$$

where α, β label some basis, and the summation runs over a complete basis. For bi-local superfields, we have the analogous formulae:

$$\frac{\delta F(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)}{\delta F(\sigma_3, \theta_3; \bar{\sigma}_4, \bar{\theta}_4)} \equiv (\theta_3 - \theta_1)(\bar{\theta}_4 - \bar{\theta}_2) \delta(\sigma_3 - \sigma_1) \delta(\bar{\sigma}_4 - \bar{\sigma}_2) \quad (5.21)$$

$$\frac{\delta \bar{F}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)}{\delta \bar{F}(\bar{\sigma}_3, \bar{\theta}_3; \sigma_4, \theta_4)} \equiv (\bar{\theta}_3 - \bar{\theta}_1)(\theta_4 - \theta_2) \delta(\bar{\sigma}_3 - \bar{\sigma}_1) \delta(\sigma_4 - \sigma_2) \quad (5.22)$$

$$\delta F_\alpha(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) = \sum_\beta \int \delta F_\beta(\sigma_3, \theta_3; \bar{\sigma}_4, \bar{\theta}_4) d\bar{\sigma}_4 d\bar{\theta}_4 d\sigma_3 d\theta_3 \frac{\delta F_\alpha(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)}{\delta F_\beta(\sigma_3, \theta_3; \bar{\sigma}_4, \bar{\theta}_4)} \quad (5.23)$$

$$\frac{\delta}{\delta F_\alpha(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)} = \sum_\beta \int \frac{\delta F_\beta(\sigma_3, \theta_3; \bar{\sigma}_4, \bar{\theta}_4)}{\delta F_\alpha(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)} d\bar{\sigma}_4 d\bar{\theta}_4 d\sigma_3 d\theta_3 \frac{\delta}{\delta F_\beta(\sigma_3, \theta_3; \bar{\sigma}_4, \bar{\theta}_4)} \quad (5.24)$$

$$\delta \bar{F}_\alpha(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2) = \sum_\beta \int \delta \bar{F}_\beta(\bar{\sigma}_3, \bar{\theta}_3; \sigma_4, \theta_4) d\sigma_4 d\theta_4 d\bar{\sigma}_3 d\bar{\theta}_3 \frac{\delta \bar{F}_\alpha(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)}{\delta \bar{F}_\beta(\bar{\sigma}_3, \bar{\theta}_3; \sigma_4, \theta_4)} \quad (5.25)$$

$$\frac{\delta}{\delta \bar{F}_\alpha(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)} = \sum_\beta \int \frac{\delta \bar{F}_\beta(\bar{\sigma}_3, \bar{\theta}_3; \sigma_4, \theta_4)}{\delta \bar{F}_\alpha(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)} d\sigma_4 d\theta_4 d\bar{\sigma}_3 d\bar{\theta}_3 \frac{\delta}{\delta \bar{F}_\beta(\bar{\sigma}_3, \bar{\theta}_3; \sigma_4, \theta_4)} \quad (5.26)$$

5.2 $\mathcal{N} = 2$ bi-local collective superfield theory

Consider Grassmannian odd chiral and anti-chiral superfields

$$\bar{D}\psi^i = 0 \quad , \quad D\bar{\psi}_i = 0 \quad (i = 1, 2, \dots, N) \quad (5.27)$$

In terms of component fields, we have

$$\psi^i(\sigma, \theta) \equiv \chi^i(\sigma) + \theta b^i(\tau) \quad (i = 1, 2, \dots, N) \quad (5.28)$$

$$\bar{\psi}_i(\bar{\sigma}, \bar{\theta}) \equiv \bar{\chi}_i(\bar{\sigma}) + \bar{\theta} \bar{b}_i(\tau) \quad (i = 1, 2, \dots, N) \quad (5.29)$$

where $\chi, \bar{\chi}$ are complex fermions while b, \bar{b} are complex bosons. They transform in the fundamental and anti-fundamental representation of $U(N)$, respectively:

$$\psi^i(\sigma, \theta) , \bar{\psi}_i(\bar{\sigma}, \bar{\theta}) \quad \longrightarrow \quad U^i_j \psi^j(\sigma, \theta) , \bar{U}_i^j \bar{\psi}_j(\bar{\sigma}, \bar{\theta}) \quad (5.30)$$

We define bi-local superfields and their conjugate:

$$\begin{aligned} \Psi(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) &\equiv \frac{1}{N} \psi^i(\sigma_1, \theta_1) \bar{\psi}_i(\bar{\sigma}_2, \bar{\theta}_2) \\ \bar{\Psi}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2) &\equiv \frac{1}{N} \bar{\psi}^i(\bar{\sigma}_1, \bar{\theta}_1) \psi_i(\sigma_2, \theta_2) \end{aligned} \quad (5.31)$$

Note that Ψ and $\bar{\Psi}$ are related by complex conjugation:

$$\overline{[\Psi(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)]} = -\bar{\Psi}(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) \quad (5.32)$$

where this is not the complex conjugation of supermatrix but that of a superfield. As a supermatrix, it can be written as

$$\Psi(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) = \frac{1}{N} \begin{pmatrix} b^i(\sigma_1, \theta_1) \bar{\chi}_i(\bar{\sigma}_2, \bar{\theta}_2) & -b^i(\sigma_1, \theta_1) \bar{b}_i(\bar{\sigma}_2, \bar{\theta}_2) \\ \chi^i(\sigma_1, \theta_1) \bar{\chi}_i(\bar{\sigma}_2, \bar{\theta}_2) & -\chi^i(\sigma_1, \theta_1) \bar{b}_i(\bar{\sigma}_2, \bar{\theta}_2) \end{pmatrix} \quad (5.33)$$

The complex conjugate relation of the bi-local superfields in (5.31) can be translated into the following relation in the supermatrix formulation.

$$\mathcal{J} \Psi^{\text{st}} \mathcal{J} = \bar{\Psi} \quad (5.34)$$

Hence, Ψ and $\bar{\Psi}$ are not independent degrees of freedom, like a hermitian matrix. For the bi-local collective action, we need to evaluate a Jacobian coming from the non-trivial transformation of path integral measure. As in section 2.3, we will use the following identities for arbitrary functional $F[\Psi]$ of Ψ .

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \frac{\delta}{\delta \psi^i(\sigma_1, \theta_1)} [\psi^i(\sigma_2, \theta_2) F[\Psi] e^{-S}] = 0 \quad (5.35)$$

$$\int \mathcal{D}\psi \int d\bar{\sigma}_3 d\bar{\theta}_3 \frac{\delta}{\delta \Psi(\sigma_1, \theta_1; \bar{\sigma}_3, \bar{\theta}_3)} [\Psi(\sigma_2, \theta_2; \bar{\sigma}_3, \bar{\theta}_3) \mathfrak{J} F[\Psi] e^{-S}] = 0 \quad (5.36)$$

and, is similar for $\bar{\Psi}$. In the same procedure as before, we can obtain functional differential equations for the Jacobian:

$$N(\theta_1 - \theta_2)\delta(\sigma_1 - \sigma_2) = \int \Psi(\sigma_2, \theta_2; \bar{\sigma}_3, \bar{\theta}_3) d\bar{\sigma}_3 d\bar{\theta}_3 \frac{\delta \log \mathfrak{J}}{\delta \Psi(\sigma_1, \theta_1; \bar{\sigma}_3, \bar{\theta}_3)} \quad (5.37)$$

$$N(\bar{\theta}_1 - \bar{\theta}_2)\delta(\bar{\sigma}_1 - \bar{\sigma}_2) = \int \bar{\Psi}(\bar{\sigma}_2, \bar{\theta}_2; \sigma_3, \theta_3) d\sigma_3 d\theta_3 \frac{\delta \log \mathfrak{J}}{\delta \bar{\Psi}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_3, \theta_3)} \quad (5.38)$$

As usual, this can be solved by

$$\log \mathfrak{J} = -\frac{N}{2} \text{str} \log \Psi \bar{\otimes} \bar{\Psi} \quad (5.39)$$

Note that the Jacobian \mathfrak{J} should be a function of $\Psi \bar{\otimes} \bar{\Psi}$ or $\bar{\Psi} \otimes \Psi$ because this is the only allowed combination, and they are related to

$$\log \mathfrak{J} = -\frac{N}{2} \text{str} \log \Psi \bar{\otimes} \bar{\Psi} = -\frac{N}{2} \text{str} \log(-\bar{\Psi} \otimes \Psi) \quad (5.40)$$

Moreover, when analyzing the collective action later, one might be tempted to treat Ψ and $\bar{\Psi}$ as if they are independent variables. This seems to give the correct result, with certain prescriptions, as usual. However, rigorously speaking, they are not independent, and one should take this into account. For example, a functional derivative with respect to Ψ will act on $\bar{\Psi}$ in the Jacobian. For this, it is helpful to use

$$\mathcal{J}^{\text{st}}(\Psi^{\text{st}})^{\text{st}} \mathcal{J}^{\text{st}} = -\mathcal{J} \Psi \mathcal{J} \quad (5.41)$$

in addition to the fact that supertrace is invariant under the supertranspose. Also, we do not have a shift in N because the bi-local collective superfield does not have symmetry analogous to (2.25). This was already seen in higher dimensional $U(N)$ vector models [63, 65, 74], and has been shown to be consistent for matching one-loop free energy of higher spin AdS/ $U(N)$ vector model [74, 84–87].

Now, to express the kinetic term, we will find the supermatrix representation of the superderivative.

$$D_1 A(\sigma_1, \theta_1, \bar{\sigma}_2, \bar{\theta}_2) = \begin{pmatrix} 2\partial_{\tau_1} A_0(\tau_1; \bar{\sigma}_2) & 2\partial_{\tau_1} A_2(\tau_1, \tau_2) \\ A_1(\bar{\sigma}_1; \bar{\sigma}_2) & A_3(\bar{\sigma}_1; \tau_2) \end{pmatrix} \equiv \mathfrak{D} \otimes A \quad (5.42)$$

Note the chiral superderivative is (Grassmannian odd) anti-chiral/chiral supermatrix:

$$\mathfrak{D}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2) \equiv \begin{pmatrix} 0 & 2\partial_{\tau_1} \delta(\bar{\sigma}_1 - \sigma_2) \\ \delta(\bar{\sigma}_1 - \sigma_2) & 0 \end{pmatrix} \quad (5.43)$$

Hence, the chiral superderivative can be multiplied to $\bar{\Psi}$ from the left by star product $\bar{\otimes}$. In the same way, one can also define the anti-chiral superderivative as follows.

$$\bar{\mathfrak{D}}(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2) \equiv \begin{pmatrix} 0 & 2\partial_{\tau_1} \delta(\sigma_1 - \bar{\sigma}_2) \\ \delta(\sigma_1 - \bar{\sigma}_2) & 0 \end{pmatrix} \quad (5.44)$$

which satisfy

$$(\bar{\mathfrak{D}}\bar{\otimes}\mathfrak{D})(\sigma_1, \theta_1; \sigma_2, \theta_2) = 2\partial_{\tau_1} \begin{pmatrix} \delta(\sigma_1 - \sigma_2) & 0 \\ 0 & \delta(\sigma_1 - \sigma_2) \end{pmatrix} = 2\partial_{\tau_1}\mathbb{I}(\sigma_1, \theta_1; \sigma_2, \theta_2) \quad (5.45)$$

$$(\mathfrak{D}\bar{\otimes}\bar{\mathfrak{D}})(\bar{\sigma}_1, \bar{\theta}_1; \bar{\sigma}_2, \bar{\theta}_2) = 2\partial_{\tau_1} \begin{pmatrix} \delta(\bar{\sigma}_1 - \bar{\sigma}_2) & 0 \\ 0 & \delta(\bar{\sigma}_1 - \bar{\sigma}_2) \end{pmatrix} = 2\partial_{\tau_1}\bar{\mathbb{I}}(\bar{\sigma}_1, \bar{\theta}_1; \bar{\sigma}_2, \bar{\theta}_2) \quad (5.46)$$

Then, in the supermatrix notation, the kinetic term can easily be written with the superderivative matrix as follows.

$$\text{str}(\mathfrak{D}\otimes\Psi) = \text{str}(\bar{\mathfrak{D}}\bar{\otimes}\bar{\Psi}) = \int d\tau_1 \left[2\partial_{\tau_1}\psi^i(\tau_1)\bar{\psi}_i(\tau_2)|_{\tau_2\rightarrow\tau_1} + b^i(\tau_1)\bar{b}_i(\tau_1) \right] \quad (5.47)$$

Therefore, like $\mathcal{N} = 1$ case, the bi-local collective action for $\mathcal{N} = 2$ SYK model is given by

$$S_{\text{col}} = -N\text{str}(\mathfrak{D}\otimes\Psi) + \frac{N}{2}\text{str}\log(\Psi\bar{\otimes}\bar{\Psi}) - \frac{JN}{3}\int d\tau_1 d\theta_1 d\tau_2 d\bar{\theta}_2 [\Psi(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)]^3 \quad (5.48)$$

$$= N\text{str} \left[-\mathfrak{D}\otimes\Psi + \frac{1}{2}\log(\Psi\bar{\otimes}\bar{\Psi}) - \frac{J}{3}\bar{\Psi}\bar{\otimes}[\Psi]^2 \right] \quad (5.49)$$

The rest of calculation is parallel to $\mathcal{N} = 1$ case except that the large N classical solution need not to be anti-symmetric, which admits a one-parameter family of solutions depending on “spectral asymmetry” \mathcal{E} [5, 88]. Also, since the collective action as a supermatrix in (5.49) contains both Ψ and $\bar{\Psi}$ which are not independent, one need additional care. Practically, it is useful to go back and forth between the supermatrix notation (5.49) and the superfield notation (5.48). For example, the superfield notation is useful in varying the interaction term because one can easily change Ψ into $\bar{\Psi}$. i.e.,

$$\int d\tau_1 d\theta_1 d\tau_2 d\bar{\theta}_2 [\Psi(\sigma_1, \theta_1; \bar{\sigma}_2, \bar{\theta}_2)]^3 = \int d\tau_1 d\bar{\theta}_1 d\tau_2 d\theta_2 [\bar{\Psi}(\bar{\sigma}_1, \bar{\theta}_1; \sigma_2, \theta_2)]^3 \quad (5.50)$$

This is a trivial identity from the point of view of the superfield notation, which leads to an identity that can also be proven in the supermatrix notation:

$$\text{str}[\bar{\Psi}\bar{\otimes}[\Psi]^2] = \text{str}[\Psi\bar{\otimes}[\bar{\Psi}]^2] \quad (5.51)$$

Varying the collective action with respect to Ψ and multiplying Ψ from the right, one can obtain the Schwinger-Dyson equation for the $\mathcal{N} = 2$ SYK model [50]:

$$-\mathfrak{D}\otimes\Psi + \mathbb{I} - [\Psi]^2\bar{\otimes}\Psi = 0 \quad (5.52)$$

One can also study $\mathcal{N} = 2$ bi-local superconformal generators and its representation for the supermatrix formulation. Moreover, after finding the eigenfunctions for the Casimir operators, one can diagonalize the quadratic action to find all spectrum as in $\mathcal{N} = 1$ SUSY SYK model. We leave them to future work.

6 Conclusion

In this work, we formulated the bi-local collective superfield theory for one-dimensional $\mathcal{N} = 1, 2$ SUSY vector models. We showed that this bi-local collective theory can be reformulated as supermatrix theory in the bi-local superspace. This drastically simplify the analysis of the $\mathcal{N} = 1$ SUSY SYK model. We also studied the bi-local superconformal generators and its representation in the supermatrix formulation. Using them, we diagonalize the quadratic action of the $\mathcal{N} = 1$ SUSY SYK model. We also developed the bi-local collective superfield theory for $\mathcal{N} = 2$ SYK model, and also connected it to supermatrix formulation. The rich structures of the supermatrix formulation could provide deeper understanding on the SUSY SYK models.

In section 2.3, we easily obtain the shift in large N by -1 which would be advantage of supersymmetry. Otherwise, one needs careful analysis of the differential equation for Jacobian. We showed that this shift in N is not only important in matching free energy in the higher spin AdS/CFT but also in getting correct result in large N expansion (See appendix A). Though we did not evaluate various observables by utilizing supersymmetry in this work, the simplicity of supermatrix formulation and the supersymmetry will enable us to calculate various observables exactly. We leave that to future work.

As mentioned in the introduction, this bi-local construction is not restricted to space-time or superspace. The bi-local collective (super)field theory would shed light on the generalization of the SYK models like higher dimensional generalization by lattice. It is highly interesting to construct $\mathcal{N} = 4$ bi-local superspace and its supermatrix formulation. Also, one might be able to generalize the bi-local superspace into higher-dimensional vector models in the context of higher spin AdS/CFT.

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A $\frac{1}{N}$ corrections in one-dimensional free SUSY vector model

In this appendix, we show that the shift of N by -1 indeed gives the correct one-point function of the bi-local collective superfield (or, invariant two-point function of the fundamental superfield) for a free theory. Consider a one-dimensional free vector model:

$$S_{\text{free}} = \int d\tau \left[\frac{1}{2} \chi^i \partial \chi^i - \frac{1}{2} b^i b^i \right] \tag{A.1}$$

Because it is a free theory, we expect the exact one-point function of the bi-local field will be

$$\langle \Psi(\tau_1, \theta_1; \tau_2, \theta_2) \rangle = \left\langle \frac{1}{N} \psi^i(\tau_1, \theta_1) \psi^i(\tau_2, \theta_2) \right\rangle = \frac{1}{2} (\text{sgn}(\tau_{12}) - \theta_1 2\delta(\tau_{12})\theta_2) \quad (\text{A.2})$$

The corresponding bi-local collective action for the free theory is given by

$$S_{\text{col}} = \text{str} \left[-\frac{N}{2} \mathfrak{D} \otimes \Psi + \frac{N-1}{2} \log \Psi \right] \quad (\text{A.3})$$

One can easily check that the large N classical solution is the same as exact answer.

$$\Psi_{\text{cl}}(\tau_1, \theta_1; \tau_2, \theta_2) = \frac{1}{2} (\text{sgn}(\tau_{12}) - \theta_1 2\delta(\tau_{12})\theta_2) = \begin{pmatrix} 0 & \delta(\tau_{12}) \\ \frac{1}{2} \text{sgn}(\tau_{12}) & 0 \end{pmatrix} \quad (\text{A.4})$$

However, when we expand the bi-local superfield around the classical solution in large N

$$\Psi = \Psi_{\text{cl}} + \frac{1}{\sqrt{N}} \Phi \quad (\text{A.5})$$

the collective action (A.3) generates vertices which comes from

$$\frac{N-1}{2} \text{str} \log \Psi \quad (\text{A.6})$$

and, there should be no $\frac{1}{N}$ correction to (A.4) from those vertices. At large N , the collective action can be expanded as

$$\begin{aligned} S_{\text{col}} &= -\frac{\sqrt{N}}{2} \text{str} (\mathfrak{D} \otimes \Phi) + \frac{N-1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{mN^{\frac{m}{2}}} \text{str} [(\Psi_{\text{cl}} \otimes \Phi)^{\otimes m}] \\ &= \frac{\sqrt{N}}{2} \text{str} [\Psi_{\text{cl}}^{-1} \otimes \Phi - \mathfrak{D} \otimes \Phi] - \frac{1}{4} \text{str} (\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi) \\ &\quad + \frac{1}{2\sqrt{N}} \text{str} \left[-\Psi_{\text{cl}}^{-1} \otimes \Phi + \frac{1}{3} (\Psi_{\text{cl}}^{-1} \otimes \Phi)^{\otimes 3} \right] + \mathcal{O}(N^{-1}) \end{aligned} \quad (\text{A.7})$$

First, one can easily calculate the inverse of the classical solution from (A.4), and it turns out to be equal to the matrix superderivative in (2.41).

$$\Psi_{\text{cl}}^{-1} = \mathfrak{D} \quad (\text{A.8})$$

In fact, this is the large N Schwinger-Dyson equation for the free collective superfield theory. Then, from the quadratic action of order $\mathcal{O}(N^0)$, one can read off the two-point function of the bi-local fluctuation. Furthermore, one can easily show that

$$\langle (\Psi_{\text{cl}}^{-1} \otimes \Phi \otimes \Psi_{\text{cl}}^{-1} \otimes \Phi)(\tau_1, \tau_2) \rangle = \begin{pmatrix} \delta(\tau_{12}) & 0 \\ 0 & \delta(\tau_{12}) \end{pmatrix} \quad (\text{A.9})$$

Now, the leading correction to the one-point function of the bi-local collective superfield is given by

$$\frac{1}{2N} \left\langle \Phi(\tau_1, \theta_1; \tau_2, \theta_2) \text{str} \left[-\Psi_{\text{cl}}^{-1} \otimes \Phi + \frac{1}{3} (\Psi_{\text{cl}}^{-1} \otimes \Phi)^{\otimes 3} \right] \right\rangle \quad (\text{A.10})$$

Using a property of the supertrace and (A.9), one can easily see that this correction vanishes. If it were not for the shift in N by (-1) , this correction would not vanish, and therefore would not give the exact one-point function which one can expect in free theory. Though this shift does not have any influence in the main text of this paper, it would be important in evaluating $\frac{1}{N}$ corrections to correlation functions or the free energy.

B Casimir eigenfunctions

In this appendix, we present the (bosonic and fermionic) eigenfunctions of the superconformal Casimir operators discussed in section 3.2.

B.1 Bosonic eigenfunctions

- *Eigenvalue of Casimir: $\nu(\nu - \frac{1}{2})$*

$$\Gamma_{\nu w}^1 = e^{-iwt} z^{\frac{1}{6}} J_\nu(wz) \begin{pmatrix} 0 & -\frac{\nu - \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.1})$$

$$\Gamma_{\nu w}^2 = e^{-iwt} z^{\frac{1}{6}} J_{-\nu}(wz) \begin{pmatrix} 0 & -\frac{\nu - \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \Gamma_{\nu w}^2 = e^{-iwt} z^{\frac{1}{6}} Y_\nu(wz) \begin{pmatrix} 0 & -\frac{\nu - \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.2})$$

$$\Gamma_{\nu w}^3 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [J_\nu(wz) \mathbb{1} + i J_{\nu-1}(wz) \sigma_3] \quad (\text{B.3})$$

$$\Gamma_{\nu w}^4 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [Y_\nu(wz) \mathbb{1} + i Y_{\nu-1}(wz) \sigma_3]$$

$$\text{or} \quad \Gamma_{\nu w}^4 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [J_{-\nu}(wz) \mathbb{1} - i J_{-\nu+1}(wz) \sigma_3] \quad (\text{B.4})$$

- *Eigenvalue of Casimir: $\nu(\nu + \frac{1}{2})$*

$$\Gamma_{\nu w}^5 = e^{-iwt} z^{\frac{1}{6}} J_\nu(wz) \begin{pmatrix} 0 & \frac{\nu + \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.5})$$

$$\Gamma_{\nu w}^6 = e^{-iwt} z^{\frac{1}{6}} J_{-\nu}(wz) \begin{pmatrix} 0 & \frac{\nu + \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \Gamma_{\nu w}^6 = e^{-iwt} z^{\frac{1}{6}} Y_\nu(wz) \begin{pmatrix} 0 & \frac{\nu + \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.6})$$

$$\Gamma_{\nu w}^7 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [J_\nu(wz) \mathbb{1} - i J_{\nu+1}(wz) \sigma_3] \quad (\text{B.7})$$

$$\Gamma_{\nu w}^8 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [Y_\nu(wz) \mathbb{1} - i Y_{\nu+1}(wz) \sigma_3]$$

$$\text{or} \quad \Gamma_{\nu w}^8 = \frac{i}{2} w e^{-iwt} z^{\frac{1}{6}} [J_{-\nu}(wz) \mathbb{1} + i J_{-\nu-1}(wz) \sigma_3] \quad (\text{B.8})$$

- *Action of supercharge:*

$$\mathcal{Q}\Gamma_{\nu w}^1 = (iw) \frac{i}{2} e^{-iwt} z^{\frac{1}{6}} [J_{\nu-1}(wz) \mathbb{1} - i J_\nu(wz) \sigma_3] \quad (\text{B.9})$$

$$\mathcal{Q}\Gamma_{\nu w}^3 = (iw) \frac{i}{2} e^{-iwt} z^{\frac{1}{6}} J_{\nu+1}(wz) \begin{pmatrix} 0 & \frac{\nu - \frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.10})$$

$$\mathcal{Q}\Gamma_{\nu w}^5 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}[-J_{\nu+1}(wz)\mathbb{1} - iJ_{\nu}(wz)\sigma_3] \quad (\text{B.11})$$

$$\mathcal{Q}\Gamma_{\nu w}^7 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}J_{\nu}(wz)\begin{pmatrix} 0 & -\frac{\nu+\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.12})$$

B.2 Fermionic eigenfunctions

- *Eigenvalue of Casimir: $\nu(\nu - \frac{1}{2})$*

$$\Omega_{\nu w}^1 = e^{-iwt}z^{\frac{1}{6}}J_{\nu}(wz)\begin{pmatrix} 0 & \frac{\nu-\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.13})$$

$$\Omega_{\nu w}^2 = e^{-iwt}z^{\frac{1}{6}}J_{-\nu}(wz)\begin{pmatrix} 0 & \frac{\nu-\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \Omega_{\nu w}^2 = e^{-iwt}z^{\frac{1}{6}}Y_{\nu}(wz)\begin{pmatrix} 0 & \frac{\nu-\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.14})$$

$$\Omega_{\nu w}^3 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[J_{\nu}(wz)\mathbb{1} + iJ_{\nu-1}(wz)\sigma_3] \quad (\text{B.15})$$

$$\Omega_{\nu w}^4 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[Y_{\nu}(wz)\mathbb{1} + iY_{\nu-1}(wz)\sigma_3]$$

$$\text{or} \quad \Omega_{\nu w}^4 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[J_{-\nu}(wz)\mathbb{1} - iJ_{-\nu+1}(wz)\sigma_3] \quad (\text{B.16})$$

- *Eigenvalue of Casimir: $\nu(\nu + \frac{1}{2})$*

$$\Omega_{\nu w}^5 = e^{-iwt}z^{\frac{1}{6}}J_{\nu}(wz)\begin{pmatrix} 0 & -\frac{\nu+\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.17})$$

$$\Omega_{\nu w}^6 = e^{-iwt}z^{\frac{1}{6}}J_{-\nu}(wz)\begin{pmatrix} 0 & -\frac{\nu+\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \Omega_{\nu w}^6 = e^{-iwt}z^{\frac{1}{6}}Y_{\nu}(wz)\begin{pmatrix} 0 & -\frac{\nu+\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.18})$$

$$\Omega_{\nu w}^7 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[J_{\nu}(wz)\mathbb{1} - iJ_{\nu+1}(wz)\sigma_3] \quad (\text{B.19})$$

$$\Omega_{\nu w}^8 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[Y_{\nu}(wz)\mathbb{1} - iY_{\nu+1}(wz)\sigma_3]$$

$$\text{or} \quad \Omega_{\nu w}^8 = \frac{i}{2}we^{-iwt}z^{\frac{1}{6}}[J_{-\nu}(wz)\mathbb{1} + iJ_{-\nu-1}(wz)\sigma_3] \quad (\text{B.20})$$

- *Action of supercharge:*

$$\mathcal{Q}\Omega_{\nu w}^1 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}[-iJ_{\nu}(wz)\mathbb{1} + J_{\nu-1}(wz)\sigma_3] \quad (\text{B.21})$$

$$\mathcal{Q}\Omega_{\nu w}^3 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}J_{\nu}(wz)\begin{pmatrix} 0 & \frac{\nu-\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.22})$$

$$\mathcal{Q}\Omega_{\nu w}^5 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}[-iJ_{\nu}(wz)\mathbb{1} - J_{\nu+1}(wz)\sigma_3] \quad (\text{B.23})$$

$$\mathcal{Q}\Omega_{\nu w}^7 = (iw)\frac{i}{2}e^{-iwt}z^{\frac{1}{6}}J_{\nu}(wz)\begin{pmatrix} 0 & -\frac{\nu+\frac{1}{6}}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{B.24})$$

C Direct diagonalization

In this appendix, we will diagonalize the quadratic action following [9, 10]. In 4.1, we already showed that the second term in the quadratic action (2.83) corresponds to the inner product of two eigenfunctions. Hence, we will focus on the first term of the quadratic action. For each $u_{\nu w}^a$ ($a = 1, 2$), we will find $\tilde{u}_{\nu w}^a$ such that

$$\Psi_{\text{cl}} \otimes \tilde{u}_{\nu w} \otimes \Psi_{\text{cl}} \otimes = g(\nu) u_{\nu w} \quad (\text{C.1})$$

where we will use the known functions $g(\nu)$'s in [50]. (See (4.40) and (4.41).) Because of the symmetry of $\tilde{u}_{\nu w}$ in (4.42), we have the following ansatz.

$$\tilde{u}_{\nu w}(\tau_1, \tau_2) \sim \begin{pmatrix} 0 & \mu f_{\frac{7}{6}}^a(\tau_{12}) \\ f_{\frac{1}{6}}^s(\tau_{12}) & 0 \end{pmatrix} \quad (\text{C.2})$$

One component of the l.h.s. in (C.1) is

$$\begin{aligned} & \int \frac{d\tau_3 d\tau_4}{|\frac{1}{2}(\tau_3 - \tau_4)|^{\frac{7}{6}}} \frac{\text{sgn}(\tau_{13}) \text{sgn}(\tau_{42}) Z_\nu(|\frac{w}{2}(\tau_3 - \tau_4)|) \text{sgn}(\tau_3 - \tau_4)}{|\tau_1 - \tau_3|^{\frac{1}{3}} |\tau_4 - \tau_2|^{\frac{1}{3}}} \\ &= -2e^{-iwt_0} \int dt dz \frac{|z - z_0|^{\frac{1}{3}} Z_\nu(|wz|) \text{sgn}(z)}{|z|^{\frac{7}{6}}} \frac{e^{-i w|z - z_0|t} \text{sgn}(t + 1) \text{sgn}(t - 1)}{|t^2 - 1|^{\frac{1}{3}}} \\ &= -4e^{-iwt_0} \int dz \frac{|z - z_0|^{\frac{1}{3}} Z_\nu(|wz|) \text{sgn}(z)}{|z|^{\frac{7}{6}}} \\ & \quad \times \left[\int_1^\infty dt \frac{\cos w|z - z_0|t}{|t^2 - 1|^{\frac{1}{3}}} - \int_0^1 dt \frac{\cos w|z - z_0|t}{|1 - t^2|^{\frac{1}{3}}} \right] \\ &= 2\sqrt{\pi} \left(\frac{2}{w}\right)^{\frac{1}{6}} \Gamma\left(\frac{2}{3}\right) e^{-iwt_0} \int dz \frac{|z - z_0|^{\frac{1}{6}} Z_\nu(|wz|) \text{sgn}(z)}{|z|^{\frac{7}{6}}} \\ & \quad \times \left[J_{\frac{1}{6}}(|w(z - z_0)|) + Y_{-\frac{1}{6}}(|w(z - z_0)|) \right] \end{aligned} \quad (\text{C.3})$$

up to a trivial factor. Here, we defined

$$t \equiv \frac{1}{2}(\tau_3 + \tau_4), \quad z \equiv \frac{1}{2}(\tau_3 - \tau_4) \quad (\text{C.4})$$

$$t_0 \equiv \frac{1}{2}(\tau_1 + \tau_2), \quad z_0 \equiv \frac{1}{2}(\tau_1 - \tau_2) \quad (\text{C.5})$$

In the last line, we used eq. (3.771) in [89]:

$$\int_0^1 dx \frac{\cos ax}{(x^2 - 1)^b} = \frac{\sqrt{\pi}}{2} \left(\frac{a}{2}\right)^{b - \frac{1}{2}} \Gamma(1 - b) J_{\frac{1}{2} - b}(a) \quad (a > 0, \Re b < 1) \quad (\text{C.6})$$

$$\int_1^\infty dx \frac{\cos ax}{(1 - x^2)^b} = -\frac{\sqrt{\pi}}{2} \left(\frac{a}{2}\right)^{b - \frac{1}{2}} \Gamma(1 - b) Y_{b - \frac{1}{2}}(a) \quad (a > 0, \Re b > 0) \quad (\text{C.7})$$

In the same way, we found that the other component becomes

$$\begin{aligned}
 & \int \frac{d\tau_3 d\tau_4}{|\frac{1}{2}(\tau_3 - \tau_4)|^{\frac{1}{6}}} \frac{Z_\nu(|\frac{w}{2}(\tau_3 - \tau_4)|)}{|\tau_1 - \tau_3|^{\frac{4}{3}} |\tau_4 - \tau_2|^{\frac{4}{3}}} \\
 &= 2e^{-iwt_0} \int \frac{dtdz}{|z|^{\frac{1}{6}} |t^2 - (z - z_0)^2|^{\frac{4}{3}}} \frac{e^{-iwt} Z_\nu(|wz|)}{|z|^{\frac{1}{6}} |t^2 - (z - z_0)^2|^{\frac{4}{3}}} \\
 &= \frac{2}{9} A J^{\frac{5}{6}} c^2 e^{-iwt_0} \int dtdz \frac{Z_\nu(|wz|)}{|z|^{\frac{1}{6}} |z - z_0|^{\frac{5}{3}} |t^2 - 1|^{\frac{4}{3}}} \frac{e^{-iw|z - z_0|t}}{|t^2 - 1|^{\frac{4}{3}}} \\
 &= 4e^{-iwt_0} \int dz \frac{Z_\nu(|wz|)}{|z|^{\frac{1}{6}} |z - z_0|^{\frac{5}{3}}} \\
 &\quad \times \left[\int_1^\infty dt \frac{\cos w|z - z_0|t}{|t^2 - 1|^{\frac{4}{3}}} + \int_0^1 dt \frac{\cos w|z - z_0|t}{|1 - t^2|^{\frac{4}{3}}} \right] \\
 &= 2\sqrt{\pi} \left(\frac{w}{2}\right)^{\frac{5}{6}} \Gamma\left(-\frac{1}{3}\right) e^{-iwt_0} \int dz \frac{Z_\nu(|wz|)}{|z|^{\frac{1}{6}} |z - z_0|^{\frac{5}{6}}} \\
 &\quad \times \left[J_{-\frac{5}{6}}(|w(z - z_0)|) - Y_{\frac{5}{6}}(|w(z - z_0)|) \right] \tag{C.8}
 \end{aligned}$$

up to trivial factors.

Now, we will use Fourier transformation of each Bessel function with appropriate factors. That is, in the l.h.s. of (C.1), we will consider the Fourier transformations of the following six functions.

$$|z - z_0|^{\frac{1}{6}} J_{\frac{1}{6}}(|w(z - z_0)|), \quad |z - z_0|^{\frac{1}{6}} Y_{-\frac{1}{6}}(|w(z - z_0)|), \quad |z|^{-\frac{7}{6}} Z_\nu(|wz|) \tag{C.9}$$

$$|z - z_0|^{-\frac{5}{6}} J_{-\frac{5}{6}}(|w(z - z_0)|), \quad |z - z_0|^{-\frac{5}{6}} Y_{\frac{5}{6}}(|w(z - z_0)|), \quad |z|^{-\frac{1}{6}} Z_\nu(|wz|) \tag{C.10}$$

while on the r.h.s. we need the Fourier transformation of the following function.

$$|z_0|^{\frac{1}{6}} Z_\nu(|wz_0|) \tag{C.11}$$

The Fourier transformation of these functions can be performed by using the following integrals (e.g., see eq. (6.699) in [89]).

$$\begin{aligned}
 I : \int dx x^\nu e^{ipx} J_\nu(|x|) &= 2 \int dx x^\nu \cos px J_\nu(x) \\
 &= \frac{2^{1+\nu} \Gamma(\frac{1}{2} + \nu)}{\sqrt{\pi} |p^2 - 1|^{\nu + \frac{1}{2}}} [\theta(1 - |p|) - \sin \pi \nu \theta(|p| - 1)] \tag{C.12}
 \end{aligned}$$

$$\begin{aligned}
 II : \int dx x^\nu e^{ipx} J_{-\nu}(|x|) &= 2 \int dx x^\nu \cos px J_{-\nu}(x) \\
 &= \frac{2^{1+\nu} \sqrt{\pi}}{\Gamma(\frac{1}{2} - \nu)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + \nu, \frac{1}{2}; p^2\right) \theta(1 - |p|) \tag{C.13}
 \end{aligned}$$

$$\begin{aligned}
 III : \int dx x^\nu e^{ipx} Y_\nu(|x|) &= 2 \int dx x^\nu \cos px \frac{\cos \pi \nu J_\nu(x) - J_{-\nu}(x)}{\sin \pi \nu} \\
 &= \frac{2^\nu \Gamma(\frac{1}{2} + \nu)}{\sqrt{\pi} |p^2 - 1|^{\nu + \frac{1}{2}}} [\theta(1 - |p|) - \sin \pi \nu \theta(|p| - 1)] \\
 &\quad - \frac{2^{1-\nu} \sqrt{\pi} |p|^{2\nu+1}}{\sin \pi \nu \Gamma(\frac{1}{2} + \nu) |p^2 - 1|^{\nu + \frac{1}{2}}} \theta(|p| - 1)
 \end{aligned} \tag{C.14}$$

$$\begin{aligned}
 IV : \int dx |x|^\mu e^{ipx} J_\nu(|x|) &= 2 \int_0^\infty dx x^\mu \cos px J_\nu(x) \\
 &= \frac{2^{1-\nu} \Gamma(1 + \mu + \nu) \cos[\frac{\pi}{2}(1 + \mu + \nu)]}{\Gamma(\nu + 1) |p|^{1+\mu+\nu}} \times \\
 &\quad \times F\left(\frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, \nu + 1; \frac{1}{p^2}\right) \theta(|p| - 1) \\
 &\quad + \frac{2^{1+\mu} \Gamma(\frac{1+\mu+\nu}{2})}{\Gamma(\frac{\nu-\mu+1}{2})} F\left(\frac{1 + \mu + \nu}{2}, \frac{1 + \mu - \nu}{2}, \frac{1}{2}; p^2\right) \theta(1 - |p|)
 \end{aligned} \tag{C.15}$$

$$\begin{aligned}
 V : \int dx |x|^\mu e^{ipx} J_{-\nu}(|x|) &= 2 \int_0^\infty dx x^\mu \cos px J_{-\nu}(x) \\
 &= \frac{2^{1+\nu} \Gamma(1 + \mu - \nu) \cos[\frac{\pi}{2}(1 + \mu - \nu)]}{\Gamma(-\nu + 1) |p|^{1+\mu-\nu}} \times \\
 &\quad \times F\left(\frac{1 + \mu - \nu}{2}, \frac{2 + \mu - \nu}{2}, -\nu + 1; \frac{1}{p^2}\right) \theta(|p| - 1) \\
 &\quad + \frac{2^{1+\mu} \Gamma(\frac{1+\mu-\nu}{2})}{\Gamma(\frac{-\nu-\mu+1}{2})} F\left(\frac{1 + \mu - \nu}{2}, \frac{1 + \mu + \nu}{2}, \frac{1}{2}; p^2\right) \theta(1 - |p|)
 \end{aligned} \tag{C.16}$$

$$\begin{aligned}
 VI : \int dx |x|^\mu \text{sgn}(z) e^{ipx} J_\nu(|x|) &= 2i \int_0^\infty dx x^\mu \sin px J_\nu(x) \\
 &= i 2^{1-\nu} \frac{\Gamma(\nu + \mu + 1) \sin[\frac{\pi}{2}(1 + \mu + \nu)]}{\Gamma(\nu + 1) |p|^{\nu+\mu+1}} \times \\
 &\quad \times F\left(\frac{2 + \mu + \nu}{2}, \frac{1 + \mu + \nu}{2}, \nu + 1; \frac{1}{p^2}\right) \theta(|p| - 1) \\
 &\quad + i 2^{2+\mu} \text{sgn}(p) |p| \frac{\Gamma(\frac{2+\mu+\nu}{2})}{\Gamma(\frac{\nu-\mu}{2})} \times \\
 &\quad \times F\left(\frac{2 + \mu + \nu}{2}, \frac{2 + \mu - \nu}{2}, \frac{3}{2}; p^2\right) \theta(1 - |p|)
 \end{aligned} \tag{C.17}$$

Substituting these Fourier modes into (C.3) and (C.8), one can perform the integration with respect to z . The $e^{-i\omega t_0}$ factor can be easily obtained. By comparing the rest of the components on the both sides of (C.1), we found that

$$\mu = -\frac{1}{2} \left(\nu + \frac{1}{6} \right) \quad \text{for } u_{\nu w}^1 \tag{C.18}$$

$$\mu = \frac{1}{2} \left(\nu - \frac{1}{6} \right) \quad \text{for } u_{\nu w}^2 \tag{C.19}$$

and, thus we also confirmed our claim in (4.37). Using there $u_{\nu w}$'s, we obtain the eigenvalues of the kernel by evaluating the inner product. We find that

$$-\frac{J\nu}{2^{\frac{2}{3}}3^{\frac{1}{6}}\pi^{\frac{1}{3}}}\tilde{N}_{\nu}^{-}\left(\frac{1}{g_1(\nu)}-1\right) \quad \text{for } u_{\nu w}^1 \quad (\text{C.20})$$

$$\frac{J\nu}{2^{\frac{2}{3}}3^{\frac{1}{6}}\pi^{\frac{1}{3}}}\tilde{N}_{\nu}^{+}\left(\frac{1}{g_2(\nu)}-1\right) \quad \text{for } u_{\nu w}^2 \quad (\text{C.21})$$

Now, we will confirm a part of diagonalization of the quadratic action (i.e., the second term in (2.83)) by $\mathcal{Q}\mathbb{B}\otimes u_{\nu w}^a$ ($a=3,4$). Explicitly, we obtain

$$\mathcal{Q}\mathbb{B}\otimes u_{\nu w}^3 = \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [Z_{\nu-1}^{+}(|wz|)\text{sgn}(z)\sigma_{\mathbf{3}} - iZ_{\nu}^{-}(|wz|)\mathbb{B}\otimes] & (w > 0) \\ e^{i|w|t} [Z_{\nu-1}^{+}(|wz|)\text{sgn}(z)\mathbb{B}\otimes\sigma_{\mathbf{3}} + iZ_{\nu}^{-}(|wz|)\mathbb{B}\otimes] & (w < 0) \end{cases} \quad (\text{C.22})$$

where $\sigma_{\mathbf{3}}$ is a Pauli-like supermatrix (i.e., $\sigma_{\mathbf{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) and

$$\nu = \frac{3}{2} + 2n \quad (n = 0, 1, 2, \dots) \quad (\text{C.23})$$

$$\nu = ir \quad (r \in \mathbb{R}) \quad (\text{C.24})$$

In component, we have

$$v_{\nu w}^{11} = \mathcal{B} \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [Z_{\nu-1}^{+}(|wz|)\text{sgn}(z) - iZ_{\nu}^{-}(|wz|)] & (w > 0) \\ e^{i|w|t} [Z_{\nu-1}^{+}(|wz|)\text{sgn}(z) + iZ_{\nu}^{-}(|wz|)] & (w < 0) \end{cases} \quad (\text{C.25})$$

$$v_{\nu w}^{12} = \mathcal{B} \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [-Z_{\nu-1}^{+}(|wz|)\text{sgn}(z) - iZ_{\nu}^{-}(|wz|)] & (w > 0) \\ e^{i|w|t} [-Z_{\nu-1}^{+}(|wz|)\text{sgn}(z) + iZ_{\nu}^{-}(|wz|)] & (w < 0) \end{cases} \quad (\text{C.26})$$

We will evaluate

$$2J^{\frac{4}{3}}c \int d\tau_1 d\tau_2 f_{1/3}^a \eta_1(\tau_1, \tau_2) \eta_2(\tau_1, \tau_2) \quad (\text{C.27})$$

where we expand the η 's in terms of v^{11} and v^{12} . i.e.,

$$\eta_1 = \sum_{w \geq 0} \sum_{\substack{\nu=ir \\ r \geq 0}} \mathcal{B}_{\nu w}^1 v_{\nu w}^{11} + \sum_{w \geq 0} \sum_{\substack{\nu=2n+\frac{3}{2} \\ n=0,1,\dots}} \mathcal{B}_{\nu w}^1 v_{\nu w}^{11} + \text{c.c.} \quad (\text{C.28})$$

$$\eta_2 = \sum_{w \geq 0} \sum_{\substack{\nu=ir \\ r \geq 0}} \mathcal{B}_{\nu w}^1 v_{\nu w}^{12} + \sum_{w \geq 0} \sum_{\substack{\nu=2n+\frac{3}{2} \\ n=0,1,\dots}} \mathcal{B}_{\nu w}^1 v_{\nu w}^{12} + \text{c.c.} \quad (\text{C.29})$$

In order to evaluate these integrals, we need an identity

$$Z_{\nu-1}^{+} = J_{\nu-1} + \xi_{-\nu+1} J_{-\nu+1} = \partial_z Z_{\nu}^{-} + \frac{\nu}{z} Z_{\nu}^{-} \quad (\text{C.30})$$

where we used

$$\xi_{-\nu+1} = -\xi_{\nu} \quad (\text{C.31})$$

The identity (C.30) enables us to evaluate the following integral.

$$\begin{aligned} \int_0^\infty dz (Z_{\nu-1}^+(z)Z_\mu^-(z) + Z_\nu^-(z)Z_{\mu-1}^+(z)) &= Z_\nu^- Z_\mu^- \Big|_0^\infty + (\nu + \mu) \int_0^\infty \frac{dz}{z} Z_\nu^-(z)Z_\mu^-(z) \\ &= 2\nu \tilde{N}_\nu^- \delta(\nu - \mu) \end{aligned} \tag{C.32}$$

Then, we find that (C.27) is

$$\begin{aligned} 2J^{\frac{4}{3}}c \int d\tau_1 d\tau_2 f_{1/3}^a \eta_1(\tau_1, \tau_2) \eta_2(\tau_1, \tau_2) \\ = \frac{2J}{2^{\frac{2}{3}} 3^{\frac{1}{6}} \pi^{\frac{1}{6}}} \left[\sum_{\substack{\nu=ir \\ r \geq 0}} + \sum_{\substack{\nu=2n+\frac{3}{2} \\ n=0,1,\dots}} \right] \sum_{w \geq 0} B_{\nu w}^1 B_{\nu, -w}^1 (-iw) \nu \tilde{N}_\nu^- \end{aligned} \tag{C.33}$$

For the other modes, one can repeat the same evaluation. $Q_{\nu w}^4$ is given by

$$Q_{\nu w}^4 = \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [Z_{\nu+1}^- (|wz|) \text{sgn}(z) \boldsymbol{\sigma}_3 + iZ_\nu^+ (|wz|) \mathbb{1}] & (w > 0) \\ e^{i|w|t} [Z_{\nu+1}^- (|wz|) \text{sgn}(z) \boldsymbol{\sigma}_3 - iZ_\nu^+ (|wz|) \mathbb{1}] & (w < 0) \end{cases} \tag{C.34}$$

where

$$\nu = \frac{1}{2} + 2n \quad (n = 0, 1, 2, \dots) \tag{C.35}$$

$$\nu = ir \quad (r \in \mathbb{R}) \tag{C.36}$$

In components, we have

$$v_{\nu w}^{21} = \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [Z_{\nu+1}^+ (|wz|) \text{sgn}(z) + iZ_\nu^- (|wz|)] & (w > 0) \\ e^{i|w|t} [Z_{\nu+1}^+ (|wz|) \text{sgn}(z) - iZ_\nu^- (|wz|)] & (w < 0) \end{cases} \tag{C.37}$$

$$v_{\nu w}^{22} = \frac{|w|}{2} \frac{1}{\sqrt{8\pi}} |Jz|^{\frac{1}{6}} \begin{cases} e^{-i|w|t} [-Z_{\nu+1}^+ (|wz|) \text{sgn}(z) + iZ_\nu^- (|wz|)] & (w > 0) \\ e^{i|w|t} [-Z_{\nu+1}^+ (|wz|) \text{sgn}(z) - iZ_\nu^- (|wz|)] & (w < 0) \end{cases} \tag{C.38}$$

D $\mathcal{N} = 1$ SUSY SYK model: general q

In this appendix, we discuss the eigenvectors of the $\mathcal{N} = 1$ SUSY SYK model for the general q case. Since the idea is the same as the $q = 3$ case, we present only important results. For the general q case, since the fundamental superfield has dimension $\frac{1}{2q}$, the appropriate $\mathcal{N} = 1$ superconformal generators are given by

$$\mathcal{P}_a = \partial_{\tau_a} \tag{D.1}$$

$$\mathcal{K}_a = \tau_a^2 \partial_{\tau_a} + 2\Delta_a \tau_a + \tau_a \theta_a \partial_{\theta_a} \tag{D.2}$$

$$\mathcal{D}_a = \tau_a \partial_{\tau_a} + \frac{1}{2} \theta_a \partial_{\theta_a} + \Delta_a \tag{D.3}$$

$$\mathcal{Q}_a = \partial_{\theta_a} - \theta_a \partial_{\tau_a} \quad (\text{D.4})$$

$$\mathcal{S}_a = \tau_a \partial_{\theta_a} - \tau_a \theta_a \partial_{\tau_a} - 2\Delta_a \theta_a \quad (\text{D.5})$$

where $a = 1, 2$ and $\Delta_a \equiv \frac{1}{2q}$ ($a = 1, 2$). The bi-local superconformal generators are defined by

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad (L \in \{P, K, D, \mathcal{Q}, S\}) \quad (\text{D.6})$$

and the associated Casimir is

$$\mathcal{C} = \mathcal{D}^2 - \frac{1}{2}(\mathcal{P}\mathcal{K} + \mathcal{K}\mathcal{P}) + \frac{1}{4}(\mathcal{S}\mathcal{Q} - \mathcal{Q}\mathcal{S}) = \mathcal{D}^2 - \frac{1}{2}\mathcal{D} - \mathcal{K}\mathcal{P} + \frac{1}{2}\mathcal{S}\mathcal{Q} \quad (\text{D.7})$$

Via the bi-local map in (3.25) and (3.26), the superconformal generators are represented as

$$\mathcal{P} = \partial_t \quad (\text{D.8})$$

$$\begin{aligned} \mathcal{K} &= (t^2 + z^2)\partial_t + 2tz\partial_z + t(\zeta_0\partial_{\zeta_0} + \zeta_1\partial_{\zeta_1}) + z(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) + \frac{2}{q}t \\ &= (-t^2 + z^2)\partial_t + 2tD + z(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) \end{aligned} \quad (\text{D.9})$$

$$\mathcal{D} = t\partial_t + z\partial_z + \frac{1}{2}\zeta_0\partial_{\zeta_0} + \frac{1}{2}\zeta_1\partial_{\zeta_1} + \frac{1}{q} \quad (\text{D.10})$$

$$\mathcal{Q} = -\frac{1}{2}\zeta_0(\partial_t + \partial_z) + \frac{1}{2}\zeta_1(-\partial_t + \partial_z) + \partial_{\zeta_0} + \partial_{\zeta_1} \quad (\text{D.11})$$

$$\begin{aligned} \mathcal{S} &= (t+z)\partial_{\zeta_0} - (-t+z)\partial_{\zeta_1} - \frac{1}{2}\zeta_0(t+z)(\partial_t + \partial_z) - \frac{1}{2}\zeta_1(-t+z)(-\partial_t + \partial_z) \\ &\quad - \frac{1}{q}(\zeta_0 + \zeta_1) \end{aligned} \quad (\text{D.12})$$

and, the Casimir can be written as

$$\begin{aligned} \mathcal{C} &= \frac{1}{q^2} - \frac{1}{2q} + \frac{2}{q}z\partial_z + z^2(-\partial_t^2 + \partial_z^2) - z\partial_t(\zeta_0\partial_{\zeta_0} - \zeta_1\partial_{\zeta_1}) + \left(z\partial_z + \frac{1}{2q}\right)(\zeta_0\partial_{\zeta_0} + \zeta_1\partial_{\zeta_1}) \\ &\quad + \frac{1}{2}\zeta_0\zeta_1\partial_{\zeta_1}\partial_{\zeta_0} - z\partial_{\zeta_1}\partial_{\zeta_0} - \frac{1}{2q}\partial_z\zeta_0\zeta_1 - \frac{1}{4z}(-z^2\partial_t^2 + z^2\partial_z^2)\zeta_0\zeta_1 \\ &\quad - \left(\frac{1}{2}z\partial_z + \frac{1}{2q}\right)(\zeta_0\partial_{\zeta_1} + \zeta_1\partial_{\zeta_0}) - \frac{1}{2}z\partial_t(\zeta_0\partial_{\zeta_1} - \zeta_1\partial_{\zeta_0}) \end{aligned} \quad (\text{D.13})$$

In the same way as in section 3.2, we obtain the following eigenfunctions of the Casimir:

- *Eigenvalue of Casimir:* $\nu(\nu - \frac{1}{2})$

$$\Upsilon_{\nu w}^1 = e^{-iwt} z^{\frac{1}{2} - \frac{1}{q}} J_\nu(wz) \begin{pmatrix} 0 & -\frac{\nu - (\frac{1}{2} - \frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{D.14})$$

$$\Upsilon_{\nu w}^2 = e^{-iwt} z^{\frac{1}{2} - \frac{1}{q}} J_{-\nu}(wz) \begin{pmatrix} 0 & -\frac{\nu - (\frac{1}{2} - \frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix}$$

$$\text{or } \Upsilon_{\nu w}^2 = e^{-iwt} z^{\frac{1}{2} - \frac{1}{q}} Y_\nu(wz) \begin{pmatrix} 0 & -\frac{\nu - (\frac{1}{2} - \frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix} \quad (\text{D.15})$$

- *Eigenvalue of Casimir: $\nu(\nu + \frac{1}{2})$*

$$\Upsilon_{\nu w}^3 = e^{-iwt} z^{\frac{1}{2}-\frac{1}{q}} J_\nu(wz) \begin{pmatrix} 0 & \frac{\nu+(\frac{1}{2}-\frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix} \tag{D.16}$$

$$\Upsilon_{\nu w}^4 = e^{-iwt} z^{\frac{1}{2}-\frac{1}{q}} J_{-\nu}(wz) \begin{pmatrix} 0 & \frac{\nu+(\frac{1}{2}-\frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix}$$

$$\text{or } \Gamma_{\nu w}^2 = e^{-iwt} z^{\frac{1}{2}-\frac{1}{q}} Y_\nu(wz) \begin{pmatrix} 0 & \frac{\nu+(\frac{1}{2}-\frac{1}{q})}{2z} \\ 1 & 0 \end{pmatrix} \tag{D.17}$$

The other eigenfunctions are also similar to those in appendix B.

For general q case, one can the collective action for the $\mathcal{N} = 1$ SUSY SYK model is given by

$$S_{\text{col}} = \frac{N}{2} \text{str} \left[-\mathfrak{D} \otimes \Psi + \log \Psi - \frac{J}{q} \Psi \otimes [\Psi]^{q-1} \right] \tag{D.18}$$

Note that the additional factor comes from the i 's in the action with disorder interaction which makes the Lagrangian real. The large N saddle point equation is given by

$$\mathbb{I} - J[\Psi]^{q-1} \otimes \Psi = 0 \tag{D.19}$$

where we take the strong coupling limit. Using (2.56) and 2.56, one can easily evaluate the classical solution [50]

$$\Psi_{\text{cl}} = c \begin{pmatrix} 0 & -\frac{1}{q} f_{1/q+1}^s(\tau_{12}) \\ f_{1/q}^a(\tau_{12}) & 0 \end{pmatrix} \quad c = \left[\frac{\tan \frac{\pi}{2q}}{2\pi J} \right]^{\frac{1}{q}} \tag{D.20}$$

and the eigenfunction of the quadratic action are found to be

$$u_{\nu w}^1(t, z) = \frac{1}{\sqrt{8\pi}} e^{-iwt} |Jz|^{\frac{1}{2}-\frac{1}{q}} Z_\nu^-(|wz|) \begin{pmatrix} 0 & -\frac{\nu-(\frac{1}{2}-\frac{1}{q})}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \tag{D.21}$$

$$u_{\nu w}^2(t, z) = \frac{1}{\sqrt{8\pi}} e^{-iwt} |Jz|^{\frac{1}{2}-\frac{1}{q}} Z_\nu^-(|wz|) \begin{pmatrix} 0 & \frac{\nu+(\frac{1}{2}-\frac{1}{q})}{2|z|} \\ \text{sgn}(z) & 0 \end{pmatrix} \tag{D.22}$$

We also confirm that

$$\tilde{u}_{\nu w}^1(\tau_1, \tau_2) = -\frac{A_q}{\sqrt{8\pi}} J \frac{e^{-\frac{iw}{2}(\tau_1+\tau_2)}}{\left| \frac{J}{2}(\tau_1 - \tau_2) \right|^{\frac{1}{2}-\frac{1}{q}}} Z_\nu^- \left(\left| \frac{w}{2}(\tau_1 - \tau_2) \right| \right) \begin{pmatrix} 0 & -\frac{\nu+(\frac{1}{2}-\frac{1}{q})}{2\left| \frac{J}{2}(\tau_1 - \tau_2) \right|} \text{sgn}(\tau_1 - \tau_2) \\ 1 & 0 \end{pmatrix}$$

$$\sim [(\Psi_{\text{cl}})^{q-2} u_{\nu w}] \tag{D.23}$$

for some constant A_q .

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References

- [1] S. Sachdev and J. Ye, *Gapless spin fluid ground state in a random, quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339 [[cond-mat/9212030](#)] [[INSPIRE](#)].
- [2] S. Sachdev, *Holographic metals and the fractionalized Fermi liquid*, *Phys. Rev. Lett.* **105** (2010) 151602 [[arXiv:1006.3794](#)] [[INSPIRE](#)].
- [3] A. Kitaev, *Hidden correlations in the Hawking radiation and thermal noise*, KITP Seminar, February 12,, University of California, U.S.A. (2015), <http://online.kitp.ucsb.edu/online/joint98/kitaev/>,
- [4] A. Kitaev, *A simple model of quantum holography*, talk given at the *KITP Program: entanglement in strongly-correlated quantum matter*, April 6–July 2, University of California, U.S.A. (2015), [part 1](#) and [part 2](#) available online.
- [5] S. Sachdev, *Bekenstein-Hawking entropy and strange metals*, *Phys. Rev. X* **5** (2015) 041025 [[arXiv:1506.05111](#)] [[INSPIRE](#)].
- [6] O. Parcollet and A. Georges, *Non-Fermi-liquid regime of a doped mOtt insulator*, *Phys. Rev. B* **59** (1999) 5341.
- [7] A. Georges, O. Parcollet and S. Sachdev, *Mean field theory of a quantum Heisenberg spin glass*, *Phys. Rev. Lett.* **85** (2000) 840.
- [8] A. Georges, O. Parcollet and S. Sachdev, *Quantum fluctuations of a nearly critical heisenberg spin glass*, *Phys. Rev. B* **63** (2001) 134406.
- [9] J. Polchinski and V. Rosenhaus, *The spectrum in the Sachdev-Ye-Kitaev model*, *JHEP* **04** (2016) 001 [[arXiv:1601.06768](#)] [[INSPIRE](#)].
- [10] A. Jevicki, K. Suzuki and J. Yoon, *Bi-local holography in the SYK model*, *JHEP* **07** (2016) 007 [[arXiv:1603.06246](#)] [[INSPIRE](#)].
- [11] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 106002 [[arXiv:1604.07818](#)] [[INSPIRE](#)].
- [12] A. Jevicki and K. Suzuki, *Bi-local holography in the SYK model: perturbations*, *JHEP* **11** (2016) 046 [[arXiv:1608.07567](#)] [[INSPIRE](#)].
- [13] E. Witten, *An SYK-like model without disorder*, [arXiv:1610.09758](#) [[INSPIRE](#)].
- [14] R. Gurau, *The complete $1/N$ expansion of a SYK-like tensor model*, *Nucl. Phys. B* **916** (2017) 386 [[arXiv:1611.04032](#)] [[INSPIRE](#)].
- [15] I.R. Klebanov and G. Tarnopolsky, *Uncolored random tensors, melon diagrams and the Sachdev-Ye-Kitaev models*, *Phys. Rev. D* **95** (2017) 046004 [[arXiv:1611.08915](#)] [[INSPIRE](#)].
- [16] F. Ferrari, *The large D limit of planar diagrams*, [arXiv:1701.01171](#) [[INSPIRE](#)].
- [17] H. Itoyama, A. Mironov and A. Morozov, *Rainbow tensor model with enhanced symmetry and extreme melonic dominance*, *Phys. Lett. B* **771** (2017) 180 [[arXiv:1703.04983](#)] [[INSPIRE](#)].
- [18] H. Itoyama, A. Mironov and A. Morozov, *Ward identities and combinatorics of rainbow tensor models*, *JHEP* **06** (2017) 115 [[arXiv:1704.08648](#)] [[INSPIRE](#)].
- [19] P. Narayan and J. Yoon, *SYK-like tensor models on the lattice*, *JHEP* **08** (2017) 083 [[arXiv:1705.01554](#)] [[INSPIRE](#)].
- [20] A. Mironov and A. Morozov, *Correlators in tensor models from character calculus*, *Phys. Lett. B* **774** (2017) 210 [[arXiv:1706.03667](#)] [[INSPIRE](#)].

- [21] I.R. Klebanov and G. Tarnopolsky, *On large- N limit of symmetric traceless tensor models*, *JHEP* **10** (2017) 037 [[arXiv:1706.00839](#)] [[INSPIRE](#)].
- [22] S.H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *JHEP* **03** (2014) 067 [[arXiv:1306.0622](#)] [[INSPIRE](#)].
- [23] D.A. Roberts, D. Stanford and L. Susskind, *Localized shocks*, *JHEP* **03** (2015) 051 [[arXiv:1409.8180](#)] [[INSPIRE](#)].
- [24] J. Maldacena, S.H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* **08** (2016) 106 [[arXiv:1503.01409](#)] [[INSPIRE](#)].
- [25] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [[arXiv:1606.01857](#)] [[INSPIRE](#)].
- [26] K. Jensen, *Chaos in AdS_2 holography*, *Phys. Rev. Lett.* **117** (2016) 111601 [[arXiv:1605.06098](#)] [[INSPIRE](#)].
- [27] G. Mandal, P. Nayak and S.R. Wadia, *Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models*, [arXiv:1702.04266](#) [[INSPIRE](#)].
- [28] S.R. Das, A. Jevicki and K. Suzuki, *Three dimensional view of the SYK/AdS duality*, *JHEP* **09** (2017) 017 [[arXiv:1704.07208](#)] [[INSPIRE](#)].
- [29] J.S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S.H. Shenker et al., *Black Holes and Random Matrices*, *JHEP* **05** (2017) 118 [[arXiv:1611.04650](#)] [[INSPIRE](#)].
- [30] A.M. García-García and J.J.M. Verbaarschot, *Spectral and thermodynamic properties of the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 126010 [[arXiv:1610.03816](#)] [[INSPIRE](#)].
- [31] C. Krishnan, S. Sanyal and P.N. Bala Subramanian, *Quantum chaos and holographic tensor models*, *JHEP* **03** (2017) 056 [[arXiv:1612.06330](#)] [[INSPIRE](#)].
- [32] T. Li, J. Liu, Y. Xin and Y. Zhou, *Supersymmetric SYK model and random matrix theory*, *JHEP* **06** (2017) 111 [[arXiv:1702.01738](#)] [[INSPIRE](#)].
- [33] C. Krishnan, K.V.P. Kumar and S. Sanyal, *Random matrices and holographic tensor models*, *JHEP* **06** (2017) 036 [[arXiv:1703.08155](#)] [[INSPIRE](#)].
- [34] T. Kanazawa and T. Wettig, *Complete random matrix classification of SYK models with $\mathcal{N} = 0, 1$ and 2 supersymmetry*, *JHEP* **09** (2017) 050 [[arXiv:1706.03044](#)] [[INSPIRE](#)].
- [35] D.J. Gross and V. Rosenhaus, *A generalization of Sachdev-Ye-Kitaev*, *JHEP* **02** (2017) 093 [[arXiv:1610.01569](#)] [[INSPIRE](#)].
- [36] R. Gurau, *Quenched equals annealed at leading order in the colored SYK model*, [arXiv:1702.04228](#) [[INSPIRE](#)].
- [37] Y. Gu, X.-L. Qi and D. Stanford, *Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models*, *JHEP* **05** (2017) 125 [[arXiv:1609.07832](#)] [[INSPIRE](#)].
- [38] M. Berkooz, P. Narayan, M. Rozali and J. Simón, *Higher dimensional generalizations of the SYK model*, *JHEP* **01** (2017) 138 [[arXiv:1610.02422](#)] [[INSPIRE](#)].
- [39] S. Banerjee and E. Altman, *Solvable model for a dynamical quantum phase transition from fast to slow scrambling*, *Phys. Rev. B* **95** (2017) 134302 [[arXiv:1610.04619](#)] [[INSPIRE](#)].
- [40] G. Turiaci and H. Verlinde, *Towards a 2d QFT analog of the SYK model*, [arXiv:1701.00528](#) [[INSPIRE](#)].

- [41] M. Berkooz, P. Narayan, M. Rozali and J. Simón, *Comments on the random Thirring model*, *JHEP* **09** (2017) 057 [[arXiv:1702.05105](#)] [[INSPIRE](#)].
- [42] S.-K. Jian and H. Yao, *Solvable SYK models in higher dimensions: a new type of many-body localization transition*, [arXiv:1703.02051](#) [[INSPIRE](#)].
- [43] Y. Gu, A. Lucas and X.-L. Qi, *Energy diffusion and the butterfly effect in inhomogeneous Sachdev-Ye-Kitaev chains*, *SciPost Phys.* **2** (2017) 018 [[arXiv:1702.08462](#)] [[INSPIRE](#)].
- [44] S. Chaudhuri, V.I. Giraldo-Rivera, A. Joseph, R. Loganayagam and J. Yoon, *Abelian tensor models on the lattice*, [arXiv:1705.01930](#) [[INSPIRE](#)].
- [45] Y. Chen, H. Zhai and P. Zhang, *Tunable quantum chaos in the Sachdev-Ye-Kitaev model coupled to a thermal bath*, *JHEP* **07** (2017) 150 [[arXiv:1705.09818](#)] [[INSPIRE](#)].
- [46] S. Förste and I. Golla, *Nearly AdS₂ SUGRA and the super-Schwarzian*, *Phys. Lett.* **B 771** (2017) 157 [[arXiv:1703.10969](#)] [[INSPIRE](#)].
- [47] D. Stanford and E. Witten, *Fermionic localization of the Schwarzian theory*, *JHEP* **10** (2017) 008 [[arXiv:1703.04612](#)] [[INSPIRE](#)].
- [48] V.V. Belokurov and E.T. Shavgulidze, *The exact solution of the Schwarzian theory*, [arXiv:1705.02405](#) [[INSPIRE](#)].
- [49] T.G. Mertens, G.J. Turiaci and H.L. Verlinde, *Solving the Schwarzian via the conformal bootstrap*, *JHEP* **08** (2017) 136 [[arXiv:1705.08408](#)] [[INSPIRE](#)].
- [50] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, *Supersymmetric Sachdev-Ye-Kitaev models*, *Phys. Rev.* **D 95** (2017) 026009 [[arXiv:1610.08917](#)] [[INSPIRE](#)].
- [51] C. Peng, M. Spradlin and A. Volovich, *A supersymmetric SYK-like tensor model*, *JHEP* **05** (2017) 062 [[arXiv:1612.03851](#)] [[INSPIRE](#)].
- [52] T. Nishinaka and S. Terashima, *A note on Sachdev-Ye-Kitaev like model without random coupling*, [arXiv:1611.10290](#) [[INSPIRE](#)].
- [53] C. Peng, *Vector models and generalized SYK models*, *JHEP* **05** (2017) 129 [[arXiv:1704.04223](#)] [[INSPIRE](#)].
- [54] D.J. Gross and V. Rosenhaus, *The bulk dual of SYK: cubic couplings*, *JHEP* **05** (2017) 092 [[arXiv:1702.08016](#)] [[INSPIRE](#)].
- [55] V. Bonzom, L. Lionni and A. Tanasa, *Diagrammatics of a colored SYK model and of an SYK-like tensor model, leading and next-to-leading orders*, *J. Math. Phys.* **58** (2017) 052301 [[arXiv:1702.06944](#)] [[INSPIRE](#)].
- [56] S. Dartois, H. Erbin and S. Mondal, *Conformality of $1/N$ corrections in SYK-like models*, [arXiv:1706.00412](#) [[INSPIRE](#)].
- [57] A. Jevicki and B. Sakita, *The quantum collective field method and its application to the planar limit*, *Nucl. Phys.* **B 165** (1980) 511 [[INSPIRE](#)].
- [58] A. Jevicki and B. Sakita, *Loop space representation and the large- N behavior of the one plaquette Kogut-Susskind hamiltonian*, *Phys. Rev.* **D 22** (1980) 467 [[INSPIRE](#)].
- [59] A. Jevicki and B. Sakita, *Collective field approach to the large- N limit: Euclidean field theories*, *Nucl. Phys.* **B 185** (1981) 89 [[INSPIRE](#)].
- [60] A. Jevicki, *Nonperturbative collective field theory*, *Nucl. Phys.* **B 376** (1992) 75 [[INSPIRE](#)].

- [61] A. Jevicki, *Development in 2 – D string theory*, talk given at the *Workshop on string theory, gauge theory and quantum gravity*, April 28–29, Trieste, Italy (1993), [hep-th/9309115](#) [[INSPIRE](#)].
- [62] J. Avan and A. Jevicki, *Collective field theory of the matrix vector models*, *Nucl. Phys. B* **469** (1996) 287 [[hep-th/9512147](#)] [[INSPIRE](#)].
- [63] R. de Mello Koch and J.P. Rodrigues, *Systematic $1/N$ corrections for bosonic and fermionic vector models without auxiliary fields*, *Phys. Rev. D* **54** (1996) 7794 [[hep-th/9605079](#)] [[INSPIRE](#)].
- [64] R. de Mello Koch, A. Jevicki and J.P. Rodrigues, *Collective string field theory of matrix models in the BMN limit*, *Int. J. Mod. Phys. A* **19** (2004) 1747 [[hep-th/0209155](#)] [[INSPIRE](#)].
- [65] S.R. Das and A. Jevicki, *Large- N collective fields and holography*, *Phys. Rev. D* **68** (2003) 044011 [[hep-th/0304093](#)] [[INSPIRE](#)].
- [66] A. Jevicki and J. Yoon, *Field theory of primaries in W_N minimal models*, *JHEP* **11** (2013) 060 [[arXiv:1302.3851](#)] [[INSPIRE](#)].
- [67] A. Jevicki and J. Yoon, *S_N orbifolds and string interactions*, *J. Phys. A* **49** (2016) 205401 [[arXiv:1511.07878](#)] [[INSPIRE](#)].
- [68] R. de Mello Koch, A. Jevicki, K. Jin and J.P. Rodrigues, *AdS_4/CFT_3 construction from collective fields*, *Phys. Rev. D* **83** (2011) 025006 [[arXiv:1008.0633](#)] [[INSPIRE](#)].
- [69] A. Jevicki, K. Jin and Q. Ye, *Collective dipole model of AdS/CFT and higher spin gravity*, *J. Phys. A* **44** (2011) 465402 [[arXiv:1106.3983](#)] [[INSPIRE](#)].
- [70] A. Jevicki, K. Jin and Q. Ye, *Bi-local Model of AdS/CFT and Higher Spin Gravity*, in the proceedings of the 11th *Workshop on Non-Perturbative Quantum Chromodynamics*, June 6–10, Paris, France (2011), [arXiv:1112.2656](#) [[INSPIRE](#)].
- [71] R. de Mello Koch, A. Jevicki, K. Jin, J.P. Rodrigues and Q. Ye, *$S = 1$ in $O(N)/HS$ duality*, *Class. Quant. Grav.* **30** (2013) 104005 [[arXiv:1205.4117](#)] [[INSPIRE](#)].
- [72] A. Jevicki, K. Jin and Q. Ye, *Perturbative and non-perturbative aspects in vector model/higher spin duality*, *J. Phys. A* **46** (2013) 214005 [[arXiv:1212.5215](#)] [[INSPIRE](#)].
- [73] D. Das, S.R. Das, A. Jevicki and Q. Ye, *Bi-local construction of $Sp(2N)/dS$ higher spin correspondence*, *JHEP* **01** (2013) 107 [[arXiv:1205.5776](#)] [[INSPIRE](#)].
- [74] A. Jevicki, K. Jin and J. Yoon, *$1/N$ and loop corrections in higher spin AdS_4/CFT_3 duality*, *Phys. Rev. D* **89** (2014) 085039 [[arXiv:1401.3318](#)] [[INSPIRE](#)].
- [75] R. de Mello Koch, A. Jevicki, J.P. Rodrigues and J. Yoon, *Canonical formulation of $O(N)$ vector/higher spin correspondence*, *J. Phys. A* **48** (2015) 105403 [[arXiv:1408.4800](#)] [[INSPIRE](#)].
- [76] R. de Mello Koch, A. Jevicki, J.P. Rodrigues and J. Yoon, *Holography as a gauge phenomenon in higher spin duality*, *JHEP* **01** (2015) 055 [[arXiv:1408.1255](#)] [[INSPIRE](#)].
- [77] A. Jevicki and J. Yoon, *Bulk from bi-locals in thermo field CFT*, *JHEP* **02** (2016) 090 [[arXiv:1503.08484](#)] [[INSPIRE](#)].
- [78] A. Jevicki and K. Suzuki, *Thermofield duality for higher spin Rindler gravity*, *JHEP* **02** (2016) 094 [[arXiv:1508.07956](#)] [[INSPIRE](#)].
- [79] A. Jevicki and J.P. Rodrigues, *Supersymmetric collective field theory*, *Phys. Lett. B* **268** (1991) 53 [[INSPIRE](#)].

- [80] J.P. Rodrigues and A.J. van Tonder, *Marinari-Parisi and supersymmetric collective field theory*, *Int. J. Mod. Phys. A* **8** (1993) 2517 [[hep-th/9204061](#)] [[INSPIRE](#)].
- [81] R. de Mello Koch and J.P. Rodrigues, *The collective field theory of a singular supersymmetric matrix model*, *Phys. Rev. D* **51** (1995) 5847 [[hep-th/9410012](#)] [[INSPIRE](#)].
- [82] J. Murugan, D. Stanford and E. Witten, *More on supersymmetric and 2d analogs of the SYK model*, *JHEP* **08** (2017) 146 [[arXiv:1706.05362](#)] [[INSPIRE](#)].
- [83] C. Peng, M. Spradlin and A. Volovich, *Correlators in the $\mathcal{N} = 2$ supersymmetric SYK model*, [arXiv:1706.06078](#) [[INSPIRE](#)].
- [84] S. Giombi and I.R. Klebanov, *One loop tests of higher spin AdS/CFT*, *JHEP* **12** (2013) 068 [[arXiv:1308.2337](#)] [[INSPIRE](#)].
- [85] S. Giombi, I.R. Klebanov and B.R. Safdi, *Higher spin AdS_{d+1}/CFT_d at one loop*, *Phys. Rev. D* **89** (2014) 084004 [[arXiv:1401.0825](#)] [[INSPIRE](#)].
- [86] S. Giombi, I.R. Klebanov and A.A. Tseytlin, *Partition functions and Casimir energies in higher spin AdS_{d+1}/CFT_d*, *Phys. Rev. D* **90** (2014) 024048 [[arXiv:1402.5396](#)] [[INSPIRE](#)].
- [87] S. Giombi, I.R. Klebanov and Z.M. Tan, *The ABC of higher-spin AdS/CFT*, [arXiv:1608.07611](#) [[INSPIRE](#)].
- [88] R.A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen and S. Sachdev, *Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography*, *Phys. Rev. B* **95** (2017) 155131 [[arXiv:1612.00849](#)] [[INSPIRE](#)].
- [89] I. Gradshteyn and I. Ryzhik, *Table of integrals, series and products*, 5th edition, Associated Press, New York U.S.A. (1996).