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Exploring the lambda model of the hybrid superstring

David M. Schmidtt

Instituto de Física Teórica IFT/UNESP,
Rua Dr. Bento Teobaldo Ferraz 271, Bloco II, CEP 01140-070, São Paulo-SP, Brasil

E-mail: david.schmidtt@gmail.com

Abstract: The purpose of this contribution is to initiate the study of integrable deformations for different superstring theory formalisms that manifest the property of (classical) integrability. In this paper we choose the hybrid formalism of the superstring in the background $AdS_2 \times S^2$ and explore in detail the most immediate consequences of its λ-deformation. The resulting action functional corresponds to the λ-model of the matter part of the fairly more sophisticated pure spinor formalism, which is also known to be classical integrable. In particular, the deformation preserves the integrability and the one-loop conformal invariance of its parent theory, hence being a marginal deformation.

Keywords: Integrable Field Theories, Sigma Models, Superstrings and Heterotic Strings

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Contents

1 Introduction

During the last few years, two novel integrable deformations of string and superstring σ-models have received a considerable great deal of attention due, in part, to their potential applications to the AdS/CFT correspondence. The first kind of integrable field theory is known under the name of the η-deformation and leads to a quantum group q-deformation of its parent σ-model S-matrix with a real parameter $q \in \mathbb{R}$. For σ-models on the bosonic cosets F/G and for the Green-Schwarz (GS) superstring σ-model on the background $AdS_5 \times S^5$, their respective deformations were presented in [1–3] as natural generalizations of the Yang-Baxter type deformation of the principal chiral model originally constructed in [4].

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The second kind of integrable field theory is known as the λ-deformation[1] and basically leads to a quantum group q-deformation of its parent σ-model S-matrix [5–7] this time with a root-of-unity parameter $q^N = 1$, for some $N \in \mathbb{Z}$. For the σ-models on the bosonic cosets F/G and for the $AdS_5 \times S^5$ GS superstring σ-model the corresponding deformations were introduced in [8, 9] as generalizations of the deformation of the Non-Abelian T-dual of the principal chiral model initially constructed in [10]. In this paper we will refer to these two kinds of deformed σ-models as η-models and λ-models. These two types of integrable field theories does not seem to be related at first sight or to have something in common as they have very different action functionals and properties but, remarkably, it turns out to be that under certain circumstances they form a sigma model pair (at least classically) under the so-called Poisson-Lie T-duality[2] [11–15]. Recently, the properties of both approaches were combined into the so-called generalized λ-deformations [16], which is the largest family of string theory integrable deformations known to date. This larger theory have been considered in more detail in [17].

Despite of the fact that both deformations preserve the very stringent property of integrability present in their parent σ-models, in the superstring theory context there is a side requirement which is extremely important and that must hold if the geometry associated to the η-model or the λ-model is to be considered as a consistent deformed string theory background, i.e, a physical deformation. Thence of some relevance to quantum superstring theory and to the AdS/CFT correspondence. This extra requirement is that the background fields (metric, dilaton, RR fluxes, B-field etc) of the deformed theory must organize themselves into a solution of an associated set of supergravity equations of motion. Unfortunately, the η-model of the $AdS_5 \times S^5$ GS superstring does not surpasses this test [18] but a milder version of it instead [21]. Fortunately, the λ-models associated to the GS superstrings in the backgrounds $AdS_n \times S^n, n = 2, 3, 5$ do as have been recently shown, respectively, in [19–21]. For previous treatments see [22, 23]. These results are very encouraging and favors the λ-deformations in this respect but also raises a very interesting question: is this result unique to the GS formalism? or does it extends to other approaches to superstring theory as well?. After all, we know that there are also available in the literature the Ramond-Neveu-Schwarz (RNS), the hybrid and the pure spinor (PS) formalism, just to mention the most common formulations, which can be used to suit different needs and purposes.

Due to its inherent simplicity when compared to other approaches, in this paper we choose the hybrid formulation [24] of superstring theoy on $AdS_2 \times S^2$ to initiate the study of this question (although the supercoset $AdS_3 \times S^3$ can be treated along the same lines too [26]). The hybrid formalism is a crossbreeding between the RNS and the GS formalisms combining the advantages of both approaches. For instance, it uses space-time spinors allowing the introduction of RR fields like in the GS approach but in flat space it reduces

[1] Or k-deformation. We will bow, however, to the more common name of λ-deformation (or λ-model) despite of the fact that the true quantum deformation parameter depends on the WZW level $k \in \mathbb{Z}$ and not on the Lagrangian deformation parameter λ.

[2] This classical equivalence might have an interesting physical interpretation if realized on the dual gauge theory side under the light of the AdS/CFT duality.

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to a free theory like in the RNS approach, so quantization is straightforward. A covariant quantization preserving manifest space-time supersymmetry is also possible dispensing the use of the light-cone gauge prevalent to the GS approach. The kappa symmetry proper of the GS superstring is replaced by a world-sheet superconformal invariance related to a BRST symmetry that is used to remove unphysical degrees of freedom etc. Of course, despite of its similitude with the GS formalism (on some respects) it provides a very different approach for treating the superstring. For further details on the properties and applications of this formalism, see for example [24–30] and references therein.

The basic goal of this paper is to explore the most direct consequences of the λ-deformation for this simpler case keeping in mind the $AdS_5 \times S^5$ supercoset for a future work as it requires instead the use of the pure spinor formalism [31, 32], which is fairly more complex and where, as we will see below, the introduction of the deformation is more delicate than in the present situation. One of the goals of working out this kind of problem is to further elucidate and understand the very structure of the λ-deformation itself from an integrable field theory point of view by testing it on different scenarios.

The paper is organized as follows. In section 2, we introduced the λ-model for the hybrid superstring on $AdS_2 \times S^2$ and display its properties, there we disregard the compactification manifold in order to keep the discussion simpler. In section 3, we pursue Dirac's procedure and identify the phase space constraints. In section 4, we prove the classical integrability of the deformed hybrid formalism and show that it has the same integrable structure than the λ-model of the Green-Schwarz formalism. In section 5, we elaborate on the $\lambda \to 0$ limit and show how the deformed theory presents a light-cone splitting on its current algebra revealing a 2d Lorentz invariance suggesting that the theory might be simpler in this limit. In section 6, we provide evidence that the conjectured $N = (2, 2)$ superconformal invariance of the undeformed hybrid superstring might be extended also to its λ-deformed partner. In section 7, we show that the λ-deformation is marginal to one-loop in $1/k$ but exact in the deforming parameter λ, hence preserving the one-loop conformal invariance of the original theory. In section 8 we speculate on a possible deformation of the PS formalism and point out a subtlety of the deformation in the current form. In section 9, we write some concluding remarks and comment on some possible future directions of research. There are two appendices.

2 The λ-model of the hybrid superstring on $AdS_2 \times S^2$

Start by introducing some basic information.[3] Consider the Lie superalgebra $\mathfrak{f} = \mathfrak{psu}(1,1|2)$ and its $\mathbb{Z}_4$ decomposition induced by the automorphism Φ

$$\Phi(\mathfrak{f}^{(m)}) = i^m \mathfrak{f}^{(m)}, \quad \mathfrak{f} = \bigoplus_{i=0}^{3} \mathfrak{f}^{(i)}, \quad [\mathfrak{f}^{(m)}, \mathfrak{f}^{(n)}] \subset \mathfrak{f}^{(m+n) \bmod 4}, \tag{2.1}$$

where $m, n = 0, 1, 2, 3$. From this decomposition we associate the usual twisted loop superagebra

$$\widehat{\mathfrak{f}} = \bigoplus_{n\in\mathbb{Z}} \left(\bigoplus_{i=0}^{3} \mathfrak{f}^{(i)} \otimes z^{4n+i} \right) = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{f}}^{(n)}. \tag{2.2}$$

[3]The 2d notation used in this paper is: $x^{\pm} = t \pm x$, $\partial_{\pm} = \frac{1}{2}(\partial_0 \pm \partial_1)$, $\eta_{\mu\nu} = diag(1, -1)$ and $\epsilon^{01} = 1$. We also have that $a_{\pm} = \frac{1}{2}(a_0 \pm a_1)$.

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The action functional for the λ-models is given by the general expression

$$S_\lambda = S_{F/F}(\mathcal{F}, A_\mu) - \frac{k}{\pi}\int_\Sigma d^2x \left\langle A_+(\Omega - 1)A_- \right\rangle, \quad k \in \mathbb{Z}, \tag{2.3}$$

where $\langle *, * \rangle = Str(*, *)$ is the supertrace in some faithful representation of $\mathfrak{f}$, $\Sigma = \mathbb{R} \times S^1$ is the world-sheet manifold with the topology of a closed string (a cylinder) and

$$\Omega = P^{(0)} + \lambda^{-3}P^{(1)} + \lambda^{-2}P^{(2)} + \lambda^{-1}P^{(3)}, \quad \lambda^{-2} = 1 + \kappa^2/k \tag{2.4}$$

is the omega projector that defines the λ-deformed hybrid superstring. It is worth highlighting the difference with the Ω projector of the λ-deformed GS superstring [9]

$$\Omega = P^{(0)} + \lambda^{-1}P^{(1)} + \lambda^{-2}P^{(2)} + \lambda P^{(3)}. \tag{2.5}$$

The main difference is that while the former introduce a kinetic term for the current components along fermionic coset directions, the latter forbids such a term and this crucial difference has important consequences for the symmetry structure of both theories and also for their quantization. More on this below.

Above, we have that

$$S_{F/F}(\mathcal{F}, A_\mu) = S_{WZW}(\mathcal{F}) - \frac{k}{\pi}\int_\Sigma d^2x \left\langle A_+\partial_-\mathcal{F}\mathcal{F}^{-1} - A_-\mathcal{F}^{-1}\partial_+\mathcal{F} - A_+\mathcal{F}A_-\mathcal{F}^{-1} + A_+A_- \right\rangle, \tag{2.6}$$

where $S_{WZW}(\mathcal{F})$ is the usual WZW model action. Note that the gauge field $A_\pm \in \mathfrak{f}$ takes values on the whole Lie superalgebra. The action (2.3) is universal and each λ-model is characterized simply by the choice of a particular Ω projector. It is important to notice that the action is only gauge invariant with respect to the bosonic gauge group G with Lie algebra $\mathfrak{f}^{(0)} = \mathfrak{u}(1) \times \mathfrak{u}(1)$, hence only the components $A_\pm^{(0)}$ are genuine gauge fields, the other components $A_\pm^{(i)}$, $i = 1, 2, 3$ play the role of auxiliary spectators fields. However, for simplicity we will refer to the whole $A_\pm$ as the gauge field.

In the sigma model limit, which is defined by expanding the group-like Lagrange multiplier near the identity $\mathcal{F} = 1 + \kappa^2\nu/k + \ldots$ with $k \to \infty$ and κ^2 fixed, we find that

$$\Omega = 1 + \frac{\kappa^2}{k}\theta + \ldots, \quad \theta = P^{(2)} + \frac{3}{2}P^{(1)} + \frac{1}{2}P^{(3)}. \tag{2.7}$$

In this limit, the deformed action reduces to the action of the hybrid superstring written in the first order (or non-Abelian Buscher) form

$$S_{\text{hybrid}} = -\frac{\kappa^2}{\pi}\int_\Sigma d^2x \left\langle A_+\theta A_- + \nu F_{+-} \right\rangle + \ldots, \tag{2.8}$$

where the ellipsis denote sub-leading terms of order $1/k$. After using the equations of motion for the Lagrange multiplier field ν and by fixing the gauge $A_\pm = J_\pm = f^{-1}\partial_\pm f$, we recover the usual $AdS_2 \times S^2$ hybrid superstring action functional[4] [24]

$$S_{\text{hybrid}} = -\frac{\kappa^2}{\pi}\int_\Sigma d^2x \left\langle (J_+ - J_+^{(0)})(J_- - J_-^{(0)}) - \frac{1}{2}(J_+^{(1)}J_-^{(3)} - J_+^{(3)}J_-^{(1)}) \right\rangle. \tag{2.9}$$

[4]Notice the presence of a kinetic term along fermionic coset directions which otherwise is absent in the GS formalism.

Alternatively, the gauge field equations of motion are given by

$$A_+ = \left(\Omega^T - D^T\right)^{-1} \mathcal{F}^{-1}\partial_+\mathcal{F}, \quad A_- = -\left(\Omega - D\right)^{-1} \partial_-\mathcal{F}\mathcal{F}^{-1}, \quad D = Ad_{\mathcal{F}}. \tag{2.10}$$

After putting them back into the action (2.3), a deformation of the Non Abelian T-dual of the hybrid action (2.9) is produced $S_{\text{eff}} = S'_{\text{hybrid}} + S_{WZ} + S_{dil}$, with

$$S'_{\text{hybrid}} = -\frac{k}{2\pi}(\lambda^{-4}-1)\int_\Sigma d^2x \big\langle (\hat{J}_+ - \hat{J}_+^{(0)})(\hat{J}_- - \hat{J}_-^{(0)}) + (\hat{J}_+\partial_-\mathcal{F}\mathcal{F}^{-1} - \partial_+\mathcal{F}\mathcal{F}^{-1}\hat{J}_-)\big\rangle \tag{2.11}$$

and where we have introduced the hatted currents

$$\widehat{J}_\pm = (\Omega^T - D^T)^{-1}\mathcal{F}^{-1}\partial_\pm\mathcal{F}. \tag{2.12}$$

A dilaton field is also generated in this process because the action functional is quadratic in the fields A_μ. However, its explicit form is not required for the present level of analysis but its general Ω-dependent form can be found in [9]. The combination of the B-fields coming from the integration and the WZ term are such that the equations of motion are preserved. In [19, 20], an explicit construction of the background fields in the Green-Schwarz formulation for the supercosets $AdS_2 \times S^2$ and $AdS_3 \times S^3$ is presented, there the dilaton receives contributions from the fermionic directions of the auxiliary fields after integration.

By defining the deformed dual currents

$$\begin{aligned} &I_\pm^{(0)} = A_\pm^{(0)}, && I_+^{(1)} = \lambda^{-1/2}A_+^{(1)}, && I_-^{(1)} = \lambda^{-3/2}A_-^{(1)}, \\ &I_\pm^{(2)} = \lambda^{-1}A_\pm^{(2)}, && I_+^{(3)} = \lambda^{-3/2}A_+^{(3)}, && I_-^{(3)} = \lambda^{-1/2}A_-^{(3)}, \end{aligned} \tag{2.13}$$

where the $A_\pm$ are given by (2.10), the equations of motion of the action (2.3), for generic values of λ, becomes exactly those of the hybrid superstring

$$\begin{aligned} \partial_+ I_-^{(0)} - \partial_- I_+^{(0)} + [I_+^{(0)}, I_-^{(0)}] + [I_+^{(1)}, I_-^{(3)}] + [I_+^{(2)}, I_-^{(2)}] + [I_+^{(3)}, I_-^{(1)}] &= 0, \\ D_+^{(0)} I_-^{(3)} + [I_+^{(1)}, I_-^{(2)}] + [I_+^{(2)}, I_-^{(1)}] &= 0, \\ D_-^{(0)} I_+^{(1)} - [I_+^{(2)}, I_-^{(3)}] - [I_+^{(3)}, I_-^{(2)}] &= 0, \\ D_+^{(0)} I_-^{(2)} + [I_+^{(1)}, I_-^{(1)}] &= 0, \\ D_-^{(0)} I_+^{(2)} - [I_+^{(3)}, I_-^{(3)}] &= 0, \\ D_+^{(0)} I_-^{(1)} &= 0, \\ D_-^{(0)} I_+^{(3)} &= 0, \end{aligned} \tag{2.14}$$

where $D_\pm^{(0)}(*) = \partial_\pm(*) + [I_\pm^{(0)}, *]$ is a covariant derivative. The last two equations states that $I_-^{(1)}$ and $I_+^{(3)}$ are covariantly chiral. The whole set of equations (2.14) can be condensed into a Lax pair representation given by

$$\mathscr{L}_+(z) = I_+^{(0)} + zI_+^{(1)} + z^2 I_+^{(2)} + z^3 I_+^{(3)}, \quad \mathscr{L}_-(z) = I_-^{(0)} + z^{-3}I_-^{(1)} + z^{-2}I_-^{(2)} + z^{-1}I_-^{(3)}, \tag{2.15}$$

which is valued in the twisted Lie superalgebra (2.2). Under the action of the $\mathbb{Z}_4$ grading automorphism (2.1), the Lax pair satisfy

$$\Phi(\mathscr{L}_\pm(z)) = \mathscr{L}_\pm(iz). \tag{2.16}$$

There are also two currents defined by

$$\mathscr{J}_+ = -\frac{k}{2\pi}\left(\mathcal{F}^{-1}\partial_+\mathcal{F} + \mathcal{F}^{-1}A_+\mathcal{F} - A_-\right), \quad \mathscr{J}_- = \frac{k}{2\pi}\left(\partial_-\mathcal{F}\mathcal{F}^{-1} - \mathcal{F}A_-\mathcal{F}^{-1} + A_+\right) \tag{2.17}$$

that obey the algebra of two mutually commuting Kac-Moody superalgebras[5]

$$\{\overset{1}{\mathscr{J}}_\pm(x), \overset{2}{\mathscr{J}}_\pm(y)\} = \frac{1}{2}[C_{12}, \overset{1}{\mathscr{J}}_\pm(x) - \overset{2}{\mathscr{J}}_\pm(y)]\delta_{xy} \mp \frac{k}{2\pi}C_{12}\delta'_{xy}, \quad \{\overset{1}{\mathscr{J}}_\pm(x), \overset{2}{\mathscr{J}}_\mp(y)\} = 0 \tag{2.18}$$

no matter what Ω is. They are universal to all λ-models. On-shell, in the sense that the equations (2.10) are used, they reduce to

$$\mathscr{J}_+ = -\frac{k}{2\pi}\left(\Omega^T A_+ - A_-\right), \quad \mathscr{J}_- = -\frac{k}{2\pi}\left(\Omega A_- - A_+\right). \tag{2.19}$$

By defining the special values $z_\pm = \lambda^{\mp 1/2}$ of the spectral parameter, we find that

$$\mathscr{L}_+(z_+) = \Omega^T A_+, \quad \mathscr{L}_-(z_+) = A_-, \quad \mathscr{L}_+(z_-) = A_+, \quad \mathscr{L}_-(z_-) = \Omega A_-. \tag{2.20}$$

Then, the spatial component of the Lax pair, which is defined by $\mathscr{L}_1 = \mathscr{L}_+ - \mathscr{L}_-$, imply the interesting relation between the Lax operator and the Kac-Moody currents

$$\mathscr{L}_1(z_+) = -\frac{2\pi}{k}\mathscr{J}_+, \quad \mathscr{L}_1(z_-) = \frac{2\pi}{k}\mathscr{J}_-. \tag{2.21}$$

From this we see that (set $\mathscr{L}_1 = \mathscr{L}$ to avoid clutter) the Kac-Moody algebra[6] can be written as

$$\{\overset{1}{\mathscr{L}}(z_\pm), \overset{2}{\mathscr{L}}(z_\pm)\} = -[\mathfrak{s}_{12}(z_\pm), \overset{1}{\mathscr{L}}(z_\pm) - \overset{2}{\mathscr{L}}(z_\pm)]\delta_{xy} - 2\mathfrak{s}_{12}(z_\pm)\delta'_{xy}, \quad \mathfrak{s}_{12}(z_\pm) = \pm\frac{\pi}{k}C_{12}. \tag{2.22}$$

One of the goals below is to find operators $\mathfrak{r}/\mathfrak{s}$ such that the Maillet bracket is obeyed

$$\{\overset{1}{\mathscr{L}}(x,z), \overset{2}{\mathscr{L}}(y,w)\} = [\mathfrak{r}_{12}, \overset{1}{\mathscr{L}}(x,z) + \overset{2}{\mathscr{L}}(y,w)]\delta_{xy} - [\mathfrak{s}_{12}, \overset{1}{\mathscr{L}}(x,z) - \overset{2}{\mathscr{L}}(y,w)]\delta_{xy} - 2\mathfrak{s}_{12}\delta'_{xy}, \tag{2.23}$$

and that reduces to (2.22) at the special points $z_\pm$.

After integrating out the gauge fields, the effective λ-model action is invariant under the parity-like transformation defined by

$$\Pi(\mathcal{F}, \Omega, k) = (\mathcal{F}^{-1}, \Omega^{-1}, -k), \tag{2.24}$$

[5] We use $\eta_{AB} = \langle T_A, T_B\rangle$, $C_{12} = \eta^{AB}T_A \otimes T_B$ and denote $\delta_{xy} = \delta(x-y)$, $\delta'_{xy} = \partial_x\delta(x-y)$. See also the appendix A for the tensor index convention.

[6] After imposing the gauge field equations of motion, the Kac-Moody algebra for the currents (2.19) is the same of (2.18). This is a consequence of the protection mechanism [8].

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whose effect on the on-shell Kac-Moody currents is to swap them

$$\Pi \mathscr{J}_{\pm} = \mathscr{J}_{\mp}. \tag{2.25}$$

This last result follows from the identities

$$\Pi A_{+} = \Omega^{T} A_{+}. \quad \Pi A_{-} = \Omega A_{-}. \tag{2.26}$$

The action of Π as given in (2.24) is an important symmetry common to all known λ-models (just change the Ω right above in each case). For instance, it can be exploited to constraint the very form of the λ-beta functions [33, 34].

The action also has a couple of global Poisson-Lie symmetries with conserved charges [35]

$$m(z_{\pm}) = P \exp \left[\pm \frac{2\pi}{k} \int_{S^1} dx \, \mathscr{J}_{\pm}(x) \right] \tag{2.27}$$

associated to the global right/left actions $\delta_R \mathcal{F} = \mathcal{F}\epsilon_R$ and $\delta_L \mathcal{F} = \epsilon_L \mathcal{F}$ of the group F, respectively. These charges are alternatively extracted by evaluating the monodromy matrix

$$m(z) = P \exp \left[- \int_{S^1} dx \mathscr{L}(x, z) \right] \tag{2.28}$$

at the special points $z = z_{\pm}$ and using (2.21). However, these two symmetries are not independent because they obey the relation $\Pi m(z_{\pm}) = m(z_{\mp})$ and play a very important role at the quantum level when putting them in the lattice [35] indicating the presence of a quantum group symmetry.

2.1 From Noether to Poisson-Lie symmetry

One of the main properties of the deformation is that it promotes the global Noether symmetry of the parent σ-model associated to the left action of the group F to a global Poisson-Lie group symmetry in the λ-model[7] [41]. The actions (2.9) and (2.3) have the same Lax pair structure but for the former in (2.15) we replace $I_{\pm}$ by the left-invariant currents $J_{\pm} = f^{-1}\partial_{\pm} f$. Then, both theories have the same associated linear system,[8] namely

$$(\partial_{\pm} + \mathscr{L}_{\pm}(z))\Psi(z) = 0, \tag{2.29}$$

where $\Psi(z)$ is the so-called wave function. This last equation in combination with (2.10) imply that, on-shell, we have the relations[9]

$$f = \Psi(1)^{-1}, \quad \mathcal{F} = \Psi(\lambda^{1/2})\Psi(\lambda^{-1/2})^{-1}. \tag{2.30}$$

Then, the constant right action of the group F on the wave-function $\Psi(z)$ can be lifted to the left action of F on f that leads to the well-known Noether symmetry of the σ-model

[7] For a detailed study of this type of symmetry in the case of the η-models, see [37].

[8] For simplicity we drop the $\overline{x} = (t, x)$ dependence on the quantities, if needed.

[9] This relation is used in [40] to construct the deformed giant magnon solutions of the $AdS_5 \times S^5$ GS superstring in the lambda background.

generated by the charge Q_L, i.e. we have that the variation $\delta_X f = Xf$, $X \in \mathfrak{f}$ can be written in the usual Abelian moment form

$$\delta_X f(\overline{x}) = \left\langle X, \{Q_L, f(\overline{x})\}\right\rangle. \tag{2.31}$$

However, this action is hidden in the dual field $\mathcal{F}$ as can be seen from (2.30) but it can be shown [41] that the infinitesimal right action $\delta\Psi(\lambda^{\pm 1/2}) = \Psi(\lambda^{\pm 1/2})X$, $X \in \mathfrak{f}$ directly on the wave-function can be written instead in the non-Abelian moment form

$$\delta_X \Psi_{(\pm)}(\overline{x}) = \pm\left\langle \overset{2}{X}, \overset{2}{\mathcal{W}}{}^{-1}\{\overset{2}{\mathcal{W}}, \overset{1}{\Psi}_{(\pm)}(\overline{x})\}\right\rangle_2, \tag{2.32}$$

where we have denoted $\Psi(\lambda^{\pm 1/2}) = \Psi_{(\pm)}$. This shows that the global left action of F in the σ-model becomes a Poisson-Lie symmetry in the λ-model generated by the non-Abelian Hamiltonian $\mathcal{W}$, which turns out to be the (right) monodromy matrix. It is important to stress that this kind of symmetry only holds on-shell and that it cannot be lifted off-shell to be a symmetry of the action functional in the usual Noether theorem sense. It is also important to notice that this situation is exactly the same for the hybrid and the Green-Schwarz formalisms and apply as well for the charges (2.27) extracted from the (left) monodromy matrix (2.28). Hence, the λ-deformation naturally introduces a q-deformation on the hybrid formalism.

3 Dirac's procedure: the constraints

In order to construct (2.23) we first need to identify which constraints are first class and which constraints are second class, then we need to use Dirac's procedure. However, we will only focus in identifying them not paying attention to the specific values of the Lagrange multipliers fixed by stability of the constraints and so on.

For the purpose of applying the Dirac procedure, we will need the Kac-Moody current algebra written above in (2.18) plus the basic Poisson brackets

$$\left\{P^A_\pm(x), A^B_\mp(y)\right\} = \frac{1}{2}\eta^{AB}\delta_{xy}, \quad \partial_0\varphi(x) = \int dy \left\{\varphi(x), H(y)\right\}, \tag{3.1}$$

where $P_\mp$ is the momentum field conjugate to the gauge field $A_\pm$.

Step I. *Find the primary constraints and construct the total Hamiltonian.* The primary constraints are given by

$$P_+ \approx 0, \quad P_- \approx 0 \tag{3.2}$$

and the total Hamiltonian density is

$$H_T = H_C - 2\left\langle u_+ P_- + u_- P_+\right\rangle, \tag{3.3}$$

where $u_\pm$ are arbitrary Lagrange multipliers and

$$H_C = -\frac{k}{\pi}\left\langle \left(\frac{\pi}{k}\right)^2 \left(\mathscr{J}_+^2 + \mathscr{J}_-^2\right) - \frac{2\pi}{k}\left(A_+\mathscr{J}_- + A_-\mathscr{J}_+\right) + \frac{1}{2}\left(A_+^2 + A_-^2\right) - A_+\Omega A_-\right\rangle \tag{3.4}$$

is the canonical Hamiltonian.

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Step II. *Find the secondary constraints using H_T, i.e., all relations that are $u_\pm$-independent.* There are only two secondary constraints, they are

$$C_+ = \mathscr{J}_+ + \frac{k}{2\pi}\left(\Omega^T A_+ - A_-\right) \approx 0, \quad C_- = \mathscr{J}_- + \frac{k}{2\pi}\left(\Omega A_- - A_+\right) \approx 0 \tag{3.5}$$

and are completely equivalent to the $A_\pm$ equations of motion written above in (2.10).

The symmetric stress tensor of the action (2.3) is found (after re-installation of the world-sheet metric) by the variation of the action with respect to the 2d metric. It has the following non-zero components

$$T_{\pm\pm} = -\frac{k}{4\pi}\langle(\mathcal{F}^{-1}D_\pm\mathcal{F})^2 + 2A_\pm(\Omega-1)A_\pm\rangle, \tag{3.6}$$

where $D_\pm(*) = \partial_\pm(*) + [A_\pm, *]$ is a covariant derivative. After using the definitions (2.17), we can show that (set $C_0 = C_+ + C_-$, $C_1 = C_+ - C_-$)

$$T_{++} + T_{--} = H_C - \langle A_0 C_0\rangle\,, \tag{3.7}$$

where

$$T_{\pm\pm} = -\frac{k}{\pi}\left\langle \left(\frac{\pi}{k}\right)^2 \mathscr{J}_\pm^2 \pm \frac{\pi}{k}\mathscr{J}_\pm A_1 + \frac{1}{4}A_1^2 + \frac{1}{2}A_\pm(\Omega-1)A_\pm\right\rangle. \tag{3.8}$$

Alternatively, we get the relation

$$H_C = T_{++} + T_{--} + \langle A_0 C_0\rangle\,, \tag{3.9}$$

where

$$\begin{aligned} T_{++} &= -\left\langle \frac{\pi}{k}C_+^2 - C_+(\Omega^T-1)A_+ + \frac{k}{4\pi}A_+(\Omega\Omega^T-1)A_+\right\rangle, \\ T_{--} &= -\left\langle \frac{\pi}{k}C_-^2 - C_-(\Omega-1)A_- + \frac{k}{4\pi}A_-(\Omega^T\Omega-1)A_-\right\rangle, \end{aligned} \tag{3.10}$$

which expresses the canonical Hamiltonian H_C in terms of constraints only. Of course, on the constraint surface we have that

$$T_{\pm\pm} \approx -\frac{k}{4\pi}(\lambda^{-4}-1)\langle A_\pm^{(2)}A_\pm^{(2)} + 2A_\pm^{(1)}A_\pm^{(3)}\rangle, \tag{3.11}$$

where the $A'_\pm s$ are now determined by the conditions $C_\pm \approx 0$ in terms of the other fields (2.10). The Virasoro constraints (more specifically $T_{\pm\pm} \approx 0$) are secondary constraints which appear as the stability conditions to the primary constraints given by the momentum conjugates to the 2d world-sheet metric components.[10] Notice that using the equations of motion (2.14) we can confirm that the stress-tensor components are indeed chiral for any value of λ

$$\partial_\mp T_{\pm\pm} = 0 \rightarrow T_{\pm\pm} = T_{\pm\pm}(x^\pm). \tag{3.12}$$

[10] In the hybrid superstring the Virasoro constraints are not imposed in the same way as they are imposed on the GS approach. Instead, they are implemented through a BRST operator.

By introducing the extended Hamiltonian

$$H_E = H_C - 2\left\langle u_+ P_- + u_- P_+ + \mu_+ C_- + \mu_- C_+ \right\rangle \tag{3.13}$$

and by forcing the stability conditions $\partial_0 C_\pm \approx 0$, we determine all the Lagrange multipliers $u'_\pm s$ but $u_\pm^{(0)}$, which are linked to the bosonic gauge symmetry present in the hybrid theory and generated by the constraint $C_0^{(0)}$ belonging to the grade zero part $\mathfrak{f}^{(0)}$ of the superalgebra $\mathfrak{f}$. Now we have identified the full set of constraints and multipliers and hence the algorithm stops. The last step deals with the information we have found.

Step III. *Separate the first and second class constraints.* There is only one primary first class constraint and it is given by $P_0^{(0)}$. To find the secondary first class ones we must find a way to get rid of the gauge field. The obvious combinations with the less number of gauge field components are

$$\begin{aligned} C &= C_+ + \Omega^T C_- = \mathscr{J}_+ + \Omega^T \mathscr{J}_- + \frac{k}{2\pi}\left(\Omega^T\Omega - 1\right) A_-, \\ \overline{C} &= \Omega C_+ + C_- = \Omega \mathscr{J}_+ + \mathscr{J}_- + \frac{k}{2\pi}\left(\Omega\Omega^T - 1\right) A_+. \end{aligned} \tag{3.14}$$

From this we realize that along the supercoset directions $\mathfrak{f}^{(1)}, \mathfrak{f}^{(2)}$ and $\mathfrak{f}^{(3)}$ all the constraints are second class mimicking the ordinary sigma model on bosonic cosets [8]. This is to be contrasted with the GS formulation in which along the fermionic directions $\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(3)}$ there is a mixture of first class and second class constraints, the first class constraint being associated to the kappa symmetry [41].

Then, we have found the following constraint splitting:

First class constraints:

$$P_0^{(0)}, \quad C_0^{(0)} = \mathscr{J}_+^{(0)} + \mathscr{J}_-^{(0)}. \tag{3.15}$$

Second class constraints:

$$\begin{array}{ccc} P_1^{(0)} & \text{and} & C_-^{(0)}, \\ (P_+^{(1)}, P_+^{(2)}, P_+^{(3)}) & \text{and} & (C^{(1)}, C^{(2)}, C^{(3)}), \\ (P_-^{(1)}, P_-^{(2)}, P_-^{(3)}) & \text{and} & (\overline{C}^{(1)}, \overline{C}^{(2)}, \overline{C}^{(3)}). \end{array} \tag{3.16}$$

By virtue of the protection mechanism [8], we can set all second class constraints strongly to zero and continue using the super Kac-Moody algebra (2.18) at no harm. Then, we are now able to use (in the strong sense) the phase space relations

$$\begin{aligned} A_1^{(0)} &= \frac{2\pi}{k}\mathscr{J}_-^{(0)}, \\ A_\pm^{(1)} &= \alpha(\lambda^{\mp 1}\mathscr{J}_\pm^{(1)} + \lambda^2 \mathscr{J}_\mp^{(1)}), \\ A_\pm^{(2)} &= \alpha(\mathscr{J}_\pm^{(2)} + \lambda^2 \mathscr{J}_\mp^{(2)}), \\ A_\pm^{(3)} &= \alpha(\lambda^{\pm 1}\mathscr{J}_\pm^{(3)} + \lambda^2 \mathscr{J}_\mp^{(3)}), \end{aligned} \tag{3.17}$$

where we have defined

$$\alpha = -\frac{2\pi}{k}\frac{1}{(z_+^4 - z_-^4)}. \tag{3.18}$$

The Poisson algebra generated by the currents (2.13), i.e. the current algebra, is found by means of (3.17) and the Kac-Moody algebra structure of the theory. Their algebra is written at extend in the appendix A.

4 Integrability: the Maillet $\mathfrak{r}/\mathfrak{s}$ bracket

In order to construct the Maillet bracket (2.23), we first impose strongly all the second class constraints (3.17) on the spatial component of the Lax connection $\mathscr{L}_1$ defined by (2.15) and second we extend it outside the constraint surface by adding to it the only first class constraint left behind (3.15). Then, the Hamiltonian or extended Lax operator

$$\mathscr{L}'(z) = \mathscr{L}(z) + \rho(z)C_0^{(0)}, \tag{4.1}$$

is entirely expressed in terms of the components of the Kac-Moody currents $\mathscr{J}_\pm$. The function $\rho(z)$ is completely arbitrary but it must be such that $\mathscr{L}'(z)$ obeys (2.16). However, it can be fixed by requiring that the relations

$$\mathscr{L}'(z_+) = -\frac{2\pi}{k}\mathscr{J}_+, \quad \mathscr{L}'(z_-) = \frac{2\pi}{k}\mathscr{J}_- \tag{4.2}$$

are still valid outside the constraint surface. This last requirement imply that

$$\rho(z) = \alpha(z^4 - z_-^4). \tag{4.3}$$

Hence, we find the extended or Hamiltonian Lax operator

$$\begin{aligned}\mathscr{L}'(z) = \alpha(z^4 - z_-^4)\left\{\mathscr{J}_+^{(0)} + \frac{z_+^3}{z^3}\mathscr{J}_+^{(1)} + \frac{z_+^2}{z^2}\mathscr{J}_+^{(2)} + \frac{z_+}{z}\mathscr{J}_+^{(3)}\right\} \\ + \alpha(z^4 - z_+^4)\left\{\mathscr{J}_-^{(0)} + \frac{z_-^3}{z^3}\mathscr{J}_-^{(1)} + \frac{z_-^2}{z^2}\mathscr{J}_-^{(2)} + \frac{z_-}{z}\mathscr{J}_-^{(3)}\right\}.\end{aligned} \tag{4.4}$$

Of course, by construction it satisfies the property

$$\Phi(\mathscr{L}'(z)) = \mathscr{L}'(iz). \tag{4.5}$$

Notice that if we extend the action of the omega projector (2.4) Ω to the whole complex plane by defining

$$\Omega(z) = P^{(0)} + z^{-3}P^{(1)} + z^{-2}P^{(2)} + z^{-1}P^{(3)}, \tag{4.6}$$

where obviously $\Omega = \Omega(\lambda)$, and use the identities $\Omega(z)\Omega(w) = \Omega(w)\Omega(z) = \Omega(zw)$, we can write quite compactly

$$\mathscr{L}'(z) = f_-(z)\Omega\left(z/z_+\right)\mathscr{J}_+ + f_+(z)\Omega\left(z/z_-\right)\mathscr{J}_-, \quad f_\pm(z) = \alpha(z^4 - z_\pm^4). \tag{4.7}$$

We can profit from this notation and write the Maillet bracket in terms of (z,λ)-dependent projectors acting on the Kac-Moody superalgebras

$$\{\overset{1}{\mathscr{L}'}(x,z),\overset{2}{\mathscr{L}'}(y,w)\} = f_-(z)f_-(w)\overset{1}{\Omega}(z/z_+)\overset{2}{\Omega}(w/z_+)\{\overset{1}{\mathscr{J}}_+(x),\overset{2}{\mathscr{J}}_+(y)\} + f_+(z)f_+(w)\overset{1}{\Omega}(z/z_-)\overset{2}{\Omega}(w/z_-)\{\overset{1}{\mathscr{J}}_-(x),\overset{2}{\mathscr{J}}_-(y)\}, \tag{4.8}$$

which clearly satisfy the condition (2.22) at the special points $z_\pm$. We can also write (2.15) in the compact form

$$\mathscr{L}_+(z) = \Omega^T(z_-/z)\,A_+, \quad \mathscr{L}_-(z) = \Omega(z/z_+)\,A_-. \tag{4.9}$$

To recover the Lax operator $\mathscr{L}_1$ on the constraint surface with the pair $\mathscr{L}_\pm$ given by (2.15), we simply replace in (4.7) the on-shell values of the Kac-Moody currents $\mathscr{J}_\pm$ as given in (2.19).

From the central terms of the Kac-Moody algebras we can immediately isolate the symmetric part of the AKS R-matrix, namely,

$$\mathfrak{s}_{12}(z,w) = \frac{k}{4\pi}\Big[f_-(z)f_-(w)\overset{1}{\Omega}(z/z_+)\overset{2}{\Omega}(w/z_+) - f_+(z)f_+(w)\overset{1}{\Omega}(z/z_-)\overset{2}{\Omega}(w/z_-)\Big]C_{12}. \tag{4.10}$$

There is a special value of the deformation parameter where $\mathfrak{s}_{12}$ simplifies

$$\lim_{\lambda\to 0}\mathfrak{s}_{12}(z,w) = -\frac{\pi}{k}C_{12}^{(00)}. \tag{4.11}$$

This means that in this limit the non-ultralocality of the theory is contained (or tamed or alleviated) within the grade zero part of the superalgebra and that it is not affected by the coset directions. This same result also holds for the GS formalism[11] [41] and the purely bosonic theories [35] and is a version of the Faddeev-Reshetikhin ultralocalization mechanism [36] but now applied to this particular model.

An explicit calculation reveals that

$$\mathfrak{s}_{12}(z,w) = s_0C_{12}^{(00)} + s_1C_{12}^{(13)} + s_2C_{12}^{(22)} + s_3C_{12}^{(31)}, \tag{4.12}$$

where

$$\begin{aligned} s_0(z,w) &= -\frac{\alpha}{2}\left[z^4+w^4-(z_+^4+z_-^4)\right],\\ s_1(z,w) &= \frac{\alpha}{2}\frac{1}{z^3w}\left[1-z^4w^4\right],\\ s_2(z,w) &= \frac{\alpha}{2}\frac{1}{z^2w^2}\left[1-z^4w^4\right],\\ s_3(z,w) &= \frac{\alpha}{2}\frac{1}{zw^3}\left[1-z^4w^4\right]. \end{aligned} \tag{4.13}$$

We can write this compactly as

$$\mathfrak{s}_{12}(z,w) = -\frac{1}{z^4-w^4}\textstyle\sum_{j=0}^{3}\{z^jw^{4-j}C_{12}^{(j,4-j)}\varphi_\lambda^{-1}(w) - z^{4-j}w^jC_{12}^{(4-j,j)}\varphi_\lambda^{-1}(z)\}, \tag{4.14}$$

[11] See also [46] for the first attempt to alleviate the non-ultralocality of the GS superstring.

where

$$\varphi_\lambda^{-1}(z) = -2\alpha \left[\varphi_\sigma^{-1}(z) + \epsilon^2(\lambda)\right], \tag{4.15}$$

is the deformed twisting function and

$$\varphi_\sigma^{-1}(z) = \frac{1}{4}(z^2 - z^{-2})^2, \quad \epsilon^2(\lambda) = -\frac{1}{4}(z_+^2 - z_-^2)^2. \tag{4.16}$$

The first term above is the well-known σ-model twisting function, the second term implements the deformation and is responsible for displacing the poles of $\varphi_\sigma(z)$ along the real axis. Note that the special values $z_\pm$ introduced above are two of the displaced poles of the original sigma model twisting function.

Now, using the symmetric part $\mathfrak{s}_{12}$ as an input in the Maillet bracket, we can solve for the antisymmetric part of the R-matrix

$$\mathfrak{r}_{12}(z,w) = \frac{1}{z^4 - w^4} \textstyle\sum_{j=0}^{3} \{ z^j w^{4-j} C_{12}^{(j,4-j)} \varphi_\lambda^{-1}(w) + z^{4-j} w^j C_{12}^{(4-j,j)} \varphi_\lambda^{-1}(z) \}, \tag{4.17}$$

showing that the deformed hybrid formulation of the superstring is still integrable as it can be put in Maillet's form. The use of the projectors in showing this is quite powerful. All these results fit perfectly within the analysis presented in [41].

4.1 Relation to the λ-model of the GS superstring

To show the equivalence of the deformed hybrid (H) and Green-Schwarz (GS) superstrings at the level of the Maillet brackets, we need to show that for the GS case the extended Lax operator takes exactly the same form as in the hybrid formulation (4.7) with the same projector operator $\Omega!$. This could come as a surprise but we will show this is indeed the case. A similar situation was realized in [45] for the un-deformed traditional sigma models.

For the sake of comparison, we write the projectors associated to both formulations

$$\Omega_H(z) = P^{(0)} + z^{-3} P^{(1)} + z^{-2} P^{(2)} + z^{-1} P^{(3)}, \tag{4.18}$$
$$\Omega_{GS}(z) = P^{(0)} + z^{-1} P^{(1)} + z^{-2} P^{(2)} + z P^{(3)}. \tag{4.19}$$

To show the equivalence we will work in reverse instead. Suppose that (4.4) or (4.7) is given and that we consider the special form for the super Kac-Moody currents

$$\begin{aligned} \mathscr{J}'_\pm = \mp \frac{k}{4\pi} \left[(1 - z_\pm^4)\Pi^{(0)} + 2\mathcal{A}^{(0)}\right] \pm \frac{k}{4\pi} \frac{z_\pm}{2} \left[2(1 - z_\mp^4)\Pi^{(1)} - (3 + z_\mp^4)\mathcal{A}^{(1)}\right] \\ + \frac{k}{4\pi} \left[(z_+^2 - z_-^2)\Pi^{(2)} \mp (z_+^2 + z_-^2)\mathcal{A}^{(2)}\right] \mp \frac{k}{4\pi} \frac{z_\mp}{2} \left[2(1 - z_\pm^4)\Pi^{(3)} + (3 + z_\pm^4)\mathcal{A}^{(2)}\right] \end{aligned} \tag{4.20}$$

in term of a new set of conjugate fields $(\mathcal{A}, \Pi)$. Then, we get that

$$\begin{aligned} \mathscr{L}'_{GS}(z) = \mathcal{A}^{(0)} + \frac{1}{4}(3z + z^{-3})\mathcal{A}^{(1)} + \frac{1}{2}(z^2 + z^{-2})\mathcal{A}^{(2)} + \frac{1}{4}(3z^{-1} + z^3)\mathcal{A}^{(3)} \\ + \frac{1}{2}(1 - z^4)\Pi^{(0)} + \frac{1}{2}(z^{-3} - z)\Pi^{(1)} + \frac{1}{2}(z^{-2} - z^2)\Pi^{(2)} + \frac{1}{2}(z^{-1} - z^3)\Pi^{(3)}, \end{aligned} \tag{4.21}$$

which is nothing but the Hamiltonian GS Lax operator that takes into account the extension by the fermionic constraints proper of the GS formalism [45, 46]. This shows that both formulations has the same extended Lax operator and the same set of $\mathfrak{r}/\mathfrak{s}$ matrices. This is because in the GS case the arbitrary functions multiplying the fermionic constraints are arbitrary and can be fixed by demanding equivalence to the hybrid formalism [45]. This means that as phase spaces the hybrid formulation and the Green-Schwarz formulation of the $AdS_2 \times S^2$ superstring are the same. The only difference being their dynamics and the local symmetries (defined through the Ω-dependence of the constraints in (3.14)) involved. Indeed, notice that the particular combination of projectors

$$(\Omega\Omega^T)_H - 1 = (\lambda^{-4} - 1)(P^{(1)} + P^{(2)} + P^{(3)}), \quad (\Omega\Omega^T)_{GS} - 1 = (\lambda^{-4} - 1)P^{(2)}, \tag{4.22}$$

change dramatically the Dirac analysis of the phase space constraints. In the former case it is conjectured [24] the existence of a quantum $N = (2, 2)$ world-sheet superconformal symmetry that replaces the kappa symmetry[12] (2+2 to be exact) present in the latter case, both gauge symmetries being used to remove un-physical degrees of freedom from the spectrum. However, it is important to realize that not even the classical part of this superconformal symmetry (corresponding to a W-algebra) is manifest in the action fuctional, as can be seen from the Dirac analysis. Below we will show how to construct explicitly the classical generator for this hidden symmetry.

5 Deformed Poisson brackets and the $\lambda \to 0$ limit

The fact that the Poisson brackets of the Lax operator can be put in the Maillet $\mathfrak{r}/\mathfrak{s}$ form, means that we can write in a compact way the Poisson bracket for functions on $\mathcal{L}$ in terms of the usual R bracket associated to the twisted loop Lie superalgebra $\widehat{\mathfrak{f}}$. Namely,

$$\{F, G\}(\mathscr{L}') = (\mathscr{L}', [dF, dG]_R)_{\varphi_\lambda} + \omega(R(dF), dG)_{\varphi_\lambda} + \omega(dF, R(dG))_{\varphi_\lambda}, \tag{5.1}$$

where $R = \pm(\Pi_{\geq 0} - \Pi_{<0})$ is the usual AKS R-matrix,

$$(X, Y)_{\varphi_\lambda} = \int_{S^1} dx \oint_0 \frac{dz}{2\pi i z} \varphi_\lambda(z) \langle X(x, z), Y(x, z) \rangle, \tag{5.2}$$

$$\omega(X, Y)_{\varphi_\lambda} = \int_{S^1} dx \oint_0 \frac{dz}{2\pi i z} \varphi_\lambda(z) \langle X(x, z), \partial_1 Y(x, z) \rangle \tag{5.3}$$

are the twisted inner product and co-cycle, respectively, and[13]

$$\mathscr{L}'_1(z) = I_1^{(0)} + z I_+^{(1)} + z^2 I_+^{(2)} + z^3 I_+^{(3)} - z^{-3} I_-^{(1)} - z^{-2} I_-^{(2)} - z^{-1} I_-^{(3)} + \rho(z) C_0^{(0)} \tag{5.4}$$

is the extended Lax operator written this time in terms of the dual currents. Above, $\Pi_{\geq 0}$ and $\Pi_{<0}$ are projectors along positive/negative powers of z acting on quantities valued in the loop superalgebra $\widehat{\mathfrak{f}}$.

[12] At classical level, after gauge fixing the kappa symmetry there is a global fermionic symmetry leftover, which in the $\lambda \to 0$ can be identified with an exotic global 2d (N, N) extended supersymmetry. The N being the rank of the kappa symmetry that was gauge fixed [47–51].

[13] As a curiosity, note that $\mathscr{L}'_1(z) = R\mathscr{L}'_0(z)$.

The functions on $\mathscr{L}'$ and their associated differentials are defined by the usual relations

$$F(\mathscr{L}') = (F, \mathscr{L}')_{\varphi_\lambda}, \quad \lim_{t\to 0} \frac{d}{dt} F(\mathscr{L}' + tX) = (dF, X)_{\varphi_\lambda}. \tag{5.5}$$

For the current components $I_\pm$, we use $F(\mathscr{L}') = (F, \mathscr{L}')_{\varphi_\lambda}$ with

$$\begin{aligned} F(x,z) = &\varphi_\lambda^{-1}(z)[(1 + z_-^4 z^{-4})\mu^{(0)} + z^{-1}\mu^{(3)} + z^{-2}\mu^{(2)} + z^{-3}\mu^{(1)}] \\ &- \varphi_\lambda^{-1}(z)[z^3\nu^{(3)} + z^2\nu^{(2)} + z\nu^{(1)}] \end{aligned} \tag{5.6}$$

and similarly for the constraint, we use $F(\mathscr{L}') = (F, \mathscr{L}')_{\varphi_\lambda}$ with

$$F(x,z) = \frac{1}{\alpha}\varphi_\lambda^{-1}(z) z^{-4}\eta^{(0)}. \tag{5.7}$$

Above, $\mu, \nu, \eta : S^1 \to \widehat{\mathfrak{f}}$ are test functions that we remove at the end of calculations. By obvious linearity, it follows that the differentials are simply found by setting $F \to dF$. For the positive part, i.e, positive powers of z, we have

$$\begin{aligned} \int_{S^1} dx \langle \eta^{(0)}, C_0^{(0)} \rangle &\to \frac{1}{\alpha}\varphi_\lambda^{-1}(z) z^{-4}\eta^{(0)}, \\ \int_{S^1} dx \langle \mu^{(0)}, I_1^{(0)} \rangle &\to \varphi_\lambda^{-1}(z)(1 + z_-^4 z^{-4})\mu^{(0)}, \\ \int_{S^1} dx \langle \mu^{(3)}, I_+^{(1)} \rangle &\to \varphi_\lambda^{-1}(z) z^{-1}\mu^{(3)}, \\ \int_{S^1} dx \langle \mu^{(2)}, I_+^{(2)} \rangle &\to \varphi_\lambda^{-1}(z) z^{-2}\mu^{(2)}, \\ \int_{S^1} dx \langle \mu^{(1)}, I_+^{(3)} \rangle &\to \varphi_\lambda^{-1}(z) z^{-3}\mu^{(1)}. \end{aligned} \tag{5.8}$$

For the negative part, i.e, negative powers of z, we get

$$\begin{aligned} \int_{S^1} dx \langle \nu^{(3)}, I_-^{(1)} \rangle &\to -\varphi_\lambda^{-1}(z) z^3\nu^{(3)}, \\ \int_{S^1} dx \langle \nu^{(2)}, I_-^{(2)} \rangle &\to -\varphi_\lambda^{-1}(z) z^2\nu^{(2)}, \\ \int_{S^1} dx \langle \nu^{(1)}, I_-^{(3)} \rangle &\to -\varphi_\lambda^{-1}(z) z\nu^{(1)}. \end{aligned} \tag{5.9}$$

Now it is a turn to compute the deformed current algebra for the λ-model of the hybrid superstring. It is written in appendix A below, after using the $\mathfrak{r}/\mathfrak{s}$ approach we find perfect agreement with the more direct and pedestrian computation that follows from the relations that are consequence of (3.17) and the Kac-moody algebra structure of the theory.

In the $\lambda \to 0$ limit a dramatic simplification of the current algebra occurs. The only non-zero brackets being (those involving the constraint remain the same)

$$\begin{aligned} \{\overset{1}{I}{}_1^{(0)}(x), \overset{2}{I}{}_1^{(0)}(y)\} &= -\frac{2\pi}{k}([C_{12}^{(00)}, \overset{2}{I}{}_1^{(0)}(y)]\delta_{xy} - C_{12}^{(00)}\delta'_{xy}), \\ \{\overset{1}{I}{}_1^{(0)}(x), \overset{2}{I}{}_-^{(i)}(y)\} &= -\frac{2\pi}{k}[C_{12}^{(00)}, \overset{2}{I}{}_-^{(i)}(y)]\delta_{xy}, \quad i = 1,2,3 \end{aligned} \tag{5.10}$$

for the brackets involving the grade zero current and

$$
\begin{aligned}
\{\overset{1}{I}{}^{(1)}_{+}(x), \overset{2}{I}{}^{(1)}_{+}(y)\} &= \frac{2\pi}{k}[C^{(13)}_{12}, \overset{2}{I}{}^{(2)}_{+}(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(1)}_{+}(x), \overset{2}{I}{}^{(2)}_{+}(y)\} &= \frac{2\pi}{k}[C^{(13)}_{12}, \overset{2}{I}{}^{(3)}_{+}(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(2)}_{-}(x), \overset{2}{I}{}^{(3)}_{-}(y)\} &= \frac{2\pi}{k}[C^{(22)}_{12}, \overset{2}{I}{}^{(1)}_{-}(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(3)}_{-}(x), \overset{2}{I}{}^{(3)}_{-}(y)\} &= \frac{2\pi}{k}[C^{(31)}_{12}, \overset{2}{I}{}^{(2)}_{-}(y)]\delta_{xy}
\end{aligned}
\tag{5.11}
$$

for the currents along the coset directions. Notice that, very remarkably, the $\pm$ light-cone sectors along the coset directions completely decouple in the sense that the current components $I_{\pm}$ do not mix, manifesting 2d relativistic invariance in this limit. The theory has the same mild non-ultralocality as in the Green-Schwarz case but this time there is no Poisson Casimir and the usual connection to the Pohlmeyer reduction, their associated generalized sine-Gordon models and its mKdV-type integrable hierarchy typical of the GS superstring [41, 49–55] is absent for this case, showing that the λ-deformation is along a different direction in the space of Poisson structures. It would be very interesting to further explore the hybrid formalism in the $\lambda \to 0$ limit, in particular it seems to be it might have simpler OPE's and vertex operators as they depend on the symplectic structure of the theory, which drastically simplifies in this limit. Indeed, the fact that in this limit the $\pm$ sectors decouple (at least classically) suggest that (anti)-chiral objects do not mix either raising the interesting possibility of computing exact OPE's even in a curved background.

6 The $N = (2, 2)$ superconformal algebra

The un-deformed theory (2.9) is conjectured to posses an $N = (2, 2)$ superconformal symmetry at the quantum level [24]. We will restrict here to a purely classical analysis and argue, however, that it is reasonable to expect that this conjecture might be extended to the λ-model as well and this is suggested by the independence of the symmetry algebra structure on the deformation parameter λ. Recall that Poisson brackets only capture the information of the OPE's that corresponds to the classical results (like single contractions), then we will only be able to reproduce the W-algebra structure of the theory. For further details on the conjecture see [24].

Start writing the stress tensor components (3.10) in the form

$$
T_{\pm\pm} = \frac{1}{2\alpha}\left\langle K_{\pm}, K_{\pm}\right\rangle, \quad K_{\pm} = I^{(2)}_{\pm} + I^{(1)}_{\pm} + I^{(3)}_{\pm}. \tag{6.1}
$$

Using the Poisson algebra written down in the appendix A, we find that on the constraint surface (when $C^{(0)}_{0} \approx 0$) and for any λ, we get the usual Virasoro algebra

$$
\{T_{\pm\pm}(x), T_{\pm\pm}(y)\} = \pm(T'_{\pm\pm}(x)\delta_{xy} + 2T_{\pm\pm}(x)\delta'_{xy}), \tag{6.2}
$$

with other brackets being zero.

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To construct the classical chiral generators a more refined analysis of the Lie superalgebra $\mathfrak{f}$ is required, see appendix B for details. Indeed, the current components, of say $I_+^{(3)}$, decompose under the action of the gauge algebra $\mathfrak{f}^{(0)} = \mathfrak{u}(1) \times \mathfrak{u}(1)$ as follows

$$I_+^{(3)} \to \{I_{(++)}^{(3)}, I_{(--)}^{(3)}, I_{(+-)}^{(3)}, I_{(-+)}^{(3)}\}, \tag{6.3}$$

where we have dropped the light-cone index $+$ in favor of the gauge labels. This decomposition imply that the two gauge invariant fermion bilinears defined by

$$G^+ = cI_{(++)}^{(3)} \cdot I_{(--)}^{(3)}, \quad G^- = cI_{(+-)}^{(3)} \cdot I_{(-+)}^{(3)}, \tag{6.4}$$

with c arbitrary, satisfy the chirality condition

$$\partial_- G^\pm = 0 \to G^\pm = G^\pm(x^+) \tag{6.5}$$

by virtue of the last equation of motion in (2.14) as the current $I_+^{(3)}$ is covariantly chiral. Recall that the equations of motion are the same for any value of λ. A similar results is valid for $I_-^{(1)}$.

Now we proceed to compute the classical symmetry algebra for the chiral sector $(+)$. Appendices A and B imply that on the constraint surface (when $C_0^{(0)} \approx 0$) we have the following Poisson brackets

$$\begin{aligned} \{T_{++}(x), G^\pm(y)\} &= G'^\pm(x)\delta_{xy} + 2G^\pm(x)\delta'_{xy}, \\ \{G^+(x), G^-(y)\} &= W(x)\delta_{xy}, \\ \{G^\pm(x), G^\pm(y)\} &= 0, \end{aligned} \tag{6.6}$$

where we have set $c^2\alpha l = 1$ and introduced a new generator

$$W = I_{(++)}^{(3)} \cdot [I_+^{(2)}, I_+^{(3)}]_{(--)}^{(1)} + I_{(--)}^{(3)} \cdot [I_+^{(2)}, I_+^{(3)}]_{(++)}^{(1)}. \tag{6.7}$$

This last generator is a spin-3 current

$$\{T_{++}(x), W(y)\} = 2W'(x)\delta_{xy} + 3W(x)\delta'_{xy}, \tag{6.8}$$

and reveals a classical W-algebra structure.[14] Other Poisson brackets mixing elements of different sectors vanish identically. The important point is that this symmetry algebra is independent of the deformation parameter λ, then it is natural to conjecture that the λ-model has the same $N = (2, 2)$ superconformal symmetry of the original hybrid action (2.9). This is because using either the dual currents $I_\pm$ or the original currents $J_\pm$, the chiral symmetry algebras are quite the same in form and content.

7 Conformal invariance: one-loop beta function

We quickly review the calculation of the one-loop beta function of [24] (see also [39, 59] for the GS formalism) but combined with the simpler constant background current used in [42], in which bosonic and fermionic fluctuations decouple making the calculation simpler. Then, we apply the same strategy to the deformed theory.

[14] This last Poisson bracket matches (up to a sign) the one introduced in [38] to compute the conformal weight Δ of the W-current. Namely, $\{T(x), W(y)\} = -(\Delta - 1)W'(x)\delta_{xy} - \Delta W(x)\delta'_{xy}$.

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7.1 The sigma model

Consider the un-deformed hybrid action[15] given by [24]

$$S_{\text{hybrid}} = -\frac{\kappa^2}{2\pi t_N}\int_\Sigma d^2x \, \langle J_+, \theta J_- \rangle_N \,, \quad \theta = P^{(2)} + (1+s)P^{(1)} + (1-s)P^{(3)}, \tag{7.1}$$

where t_N is the Dynkin index of the defining N-dimensional representation of the superalgebra $\mathfrak{f}$. See appendix A of [42] for further details on the Lie algebraic conventions used through this section.

The fluctuations fields to be used $\eta \in \mathfrak{f}$ are Lie superalgebra valued and are related to the fluctuations of the currents $J_\pm$, through the basic relations

$$\delta J_\pm = \frac{1}{\kappa} D_\pm \eta, \quad f^{-1}\delta f \equiv \frac{1}{\kappa}\eta, \quad \text{and} \quad (\delta D_\pm)\eta = [D_\pm \eta, \eta], \tag{7.2}$$

where $D_\pm = \partial_\pm + [J_\pm, *]$ is a covariant derivative. By fixing the gauge $\eta^{(0)} = 0$, associated to the gauge symmetry of the action, and by using the following specific choice of background field given by

$$f = \exp x^\mu \Theta_\mu \rightarrow J_\pm = \Theta_\pm, \tag{7.3}$$

where $\Theta_\mu \in \mathfrak{f}^{(2)}$ are constant fields satisfying $[\Theta_\mu, \Theta_\nu] = 0$, we find the operators that govern the fluctuations η. Namely,

$$\mathcal{D}_B(x) = (-\partial_+\partial_- + \Theta_+\Theta_-) \quad \text{acting on} \quad \eta^{(2)} \tag{7.4}$$

for the bosonic sector and

$$\mathcal{D}_F(x) = \begin{pmatrix} -\partial_+\partial_- + s\Theta_+\Theta_- & \left(s-\frac{1}{2}\right)\Theta_+\partial_- - \left(s+\frac{1}{2}\right)\Theta_-\partial_+ \\ (s-\frac{1}{2})\Theta_-\partial_+ - (s+\frac{1}{2})\Theta_+\partial_- & -\partial_+\partial_- - s\Theta_+\Theta_- \end{pmatrix} \text{acting on} \begin{pmatrix} \eta^{(1)} \\ \eta^{(3)} \end{pmatrix} \tag{7.5}$$

for the fermionic sector. Notice that this is basically the content of the eq. (4.16) of [24] after some obvious re-arrangements and identifications.

After Wick rotating and gathering all logarithmic divergences, we find the one-loop contribution to the effective Lagrangian in Euclidean signature [24]

$$I^{1-\text{loop}}_{\text{undef}} = -\frac{1}{8\pi}\ln\mu \cdot [Tr^{(0)}_{adj} + Tr^{(2)}_{adj} - (2s^2 + \frac{1}{2})(Tr^{(1)}_{adj} + Tr^{(3)}_{adj})](\Theta\cdot\Theta). \tag{7.6}$$

The theory (7.1) has vanishing one-loop beta function [24] precisely when $s = \pm 1/2$ as a consequence of the vanishing of the dual Coxeter number or, equivalently, the quadratic Casimir operator in the adjoint representation of $\mathfrak{f} = \mathfrak{psu}(1,1|2)$.

7.2 The lambda model

Now we want to discover if the deformation described by (2.3) preserves the one loop conformal invariance of the action (2.9). Recall that our choice above corresponds to $s = 1/2$.

[15] For the choice $s = 1/2$, we recover the hybrid superstring action (2.9).

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After using the gauge field $A_\pm$ equations of motion, we obtain the effective lambda model action[16] (cf. (2.11))

$$S_\lambda = -\frac{k}{4\pi t_N}\int_\Sigma d^2x \left\langle \mathcal{F}^{-1}\partial_+\mathcal{F}[1+2(\Omega-D)^{-1}D]\mathcal{F}^{-1}\partial_-\mathcal{F}\right\rangle_N + S_{WZ} + S_{dil}. \tag{7.7}$$

Using the background field (compared with f above)

$$\mathcal{F} = \exp x^\mu \Lambda_\mu, \tag{7.8}$$

where $\Lambda_\mu \in \mathfrak{f}^{(2)}$ are constant fields satisfying $[\Lambda_\mu, \Lambda_\nu] = 0$, we get the following dual background currents

$$I_\pm^{(2)} \equiv \Theta_\pm = \pm\frac{\lambda}{(1-\lambda^2)}\Lambda_\pm, \quad I_\pm^{(i)} = 0, \quad i = 0,1,3. \tag{7.9}$$

From the equations of motion (2.14), we obtain the operators governing the fluctuations of the bosonic and fermionic sectors. For the bosonic sector we get

$$\mathcal{D}_B(x) = \begin{pmatrix} \partial_- & 0 & 0 & -\Theta_+ \\ 0 & \partial_+ & -\Theta_- & 0 \\ -\Theta_- & \Theta_+ & -\partial_- & \partial_+ \\ 0 & 0 & \partial_- & \partial_+ \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} \widehat{I}_+^{(2)} \\ \widehat{I}_-^{(2)} \\ \widehat{I}_+^{(0)} \\ \widehat{I}_-^{(0)} \end{pmatrix}, \tag{7.10}$$

where the last line right above is the analogue of the gauge fixing condition $\eta^{(0)} = 0$ used in the un-deformed hybrid sigma model. For the fermionic sector, we obtain

$$\mathcal{D}_F(x) = \begin{pmatrix} \partial_- & 0 & \Theta_- & -\Theta_+ \\ 0 & \partial_+ & 0 & 0 \\ 0 & 0 & \partial_- & 0 \\ -\Theta_- & \Theta_+ & 0 & \partial_+ \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} \widehat{I}_+^{(1)} \\ \widehat{I}_-^{(1)} \\ \widehat{I}_+^{(3)} \\ \widehat{I}_-^{(3)} \end{pmatrix}. \tag{7.11}$$

The 1-loop quantum effective Lagrangian in Euclidean signature is then given by

$$\mathcal{L}_E^{\text{eff}} = \mathcal{L}_E^{(0)} + I_{\text{def}}^{1-\text{loop}}, \quad I_{\text{def}}^{1-\text{loop}} = \frac{1}{2}\int_{|p|<\mu}\frac{d^2p}{(2\pi)^2} tr[\log\mathcal{D}_B(p) - \log\mathcal{D}_F(p)], \tag{7.12}$$

where

$$\mathcal{L}_E^{(0)} = \frac{k}{16\pi t_N}\frac{1+\lambda^2}{1-\lambda^2}\langle\Lambda\cdot\Lambda\rangle_N \tag{7.13}$$

and

$$\mathcal{D}_B(p) = \begin{pmatrix} p_- & 0 & 0 & -\Theta_+ \\ 0 & p_+ & -\Theta_- & 0 \\ -\Theta_- & \Theta_+ & -p_- & p_+ \\ 0 & 0 & p_- & p_+ \end{pmatrix}, \quad \mathcal{D}_F(p) = \begin{pmatrix} p_- & 0 & \Theta_- & -\Theta_+ \\ 0 & p_+ & 0 & 0 \\ 0 & 0 & p_- & 0 \\ -\Theta_- & \Theta_+ & 0 & p_+ \end{pmatrix}. \tag{7.14}$$

[16]The dilaton contribution S_{dil} is not necessary at this level of analysis as we are interested only in the quantum scale invariance, not Weyl.

The contributions associated to logarithmic divergences (denoted by the symbol $\doteq$) are

$$\frac{1}{2}\int_{|p|<\mu}\frac{d^2p}{(2\pi)^2}tr[\log \mathcal{D}_B(p)] \doteq -\frac{1}{8\pi}\ln\mu[Tr^{(0)}_{adj}+Tr^{(2)}_{adj}](\Theta\cdot\Theta), \tag{7.15}$$

$$-\frac{1}{2}\int_{|p|<\mu}\frac{d^2p}{(2\pi)^2}tr[\log \mathcal{D}_F(p)] \doteq \frac{1}{8\pi}\ln\mu[Tr^{(1)}_{adj}+Tr^{(3)}_{adj}](\Theta\cdot\Theta). \tag{7.16}$$

Altogether we get, to one-loop in $1/k$ but exact in λ, that

$$I^{1-\text{loop}}_{\text{def}} = -\frac{1}{8\pi}\ln\mu\cdot[Tr^{(0)}_{adj}+Tr^{(2)}_{adj}-(Tr^{(1)}_{adj}+Tr^{(3)}_{adj})](\Theta\cdot\Theta), \tag{7.17}$$

which is proportional to the un-deformed one-loop contribution found above. The proportionality factor being λ-dependent, determined by (7.9) and can be found by writing $\Theta_\pm$ in terms of $\Lambda_\pm$. The λ-deformation besides preserving the underlying integrability of the original hybrid superstring sigma model also preserves its 1-loop conformal invariance, i.e. the coupling λ is marginal to this order. It would be very interesting to follow the lines of [19, 20] and to verify if this λ-model is also Weyl invariant at quantum level by constructing explicitly the background fields and checking if they obey the relevant set of supergravity equations of motion.

8 Digression on the pure spinor λ-model

In this last final section we speculate on the possibility of associating a λ-model to the pure spinor (PS) superstring on the background $AdS_5\times S^5$. One of the main properties of the PS formalism [31, 32] is that on it both the kappa symmetry and the Virasoro constraints characteristics of the GS formalism are replaced by a single BRST symmetry. Fortunately, by demanding BRST invariance of the λ-model of the hybrid superstring plus a term involving the PS ghosts we are able to construct a sensible deformation. Unfortunately, the deformation does not seem to preserve integrability.

8.1 The σ-model of the PS superstring

The pure spinor superstring action on $AdS_5\times S^5$ is given by

$$S=-\frac{\kappa^2}{\pi}\int_\Sigma d^2x\,\langle J_+\theta J_-\rangle-\frac{2\kappa^2}{\pi}\int_\Sigma d^2x\langle w^{(3)}D^{(0)}_-l^{(1)}+\overline{w}^{(1)}D^{(0)}_+\overline{l}^{(3)}+N\overline{N}\rangle, \tag{8.1}$$

where $D^{(0)}_\pm(*)=\partial_\pm(*)+\left[J^{(0)}_\pm,*\right]$ is a covariant derivative, $J_\pm=f^{-1}\partial_\pm f$ is the usual flat current and θ is the same projector used for the hybrid superstring (2.7). This time the Lie superalgebra to be considered is $\mathfrak{psu}(2,2|4)$. Now, $l^{(1)}$ and $\overline{l}^{(3)}$ are ghosts satisfying the pure spinor constraints

$$\left[l^{(1)},l^{(1)}\right]_+=\left[\overline{l}^{(3)},\overline{l}^{(3)}\right]_+=0 \tag{8.2}$$

and $w^{(3)}$ and $\overline{w}^{(1)}$ are their conjugate momenta. It is important to notice that $l^{(1)}$ and $\overline{l}^{(3)}$ are fermionic in character because

$$l^{(1)}=l^\alpha T_\alpha,\quad \overline{l}^{(3)}=l^{\overline{\alpha}}T_{\overline{\alpha}}, \tag{8.3}$$

where $T_\alpha \in \mathfrak{f}^{(1)}$, $T_{\overline{\alpha}} \in \mathfrak{f}^{(3)}$ are fermionic generators of $\mathfrak{f}$, while the components l^α and $l^{\overline{\alpha}}$ are bosonic spinors. Also

$$N = -\left[l^{(1)}, w^{(3)}\right]_+, \quad \overline{N} = -[\overline{l}^{(3)}, \overline{w}^{(1)}]_+, \tag{8.4}$$

are the PS Lorentz currents. They are bosonic and belong to $\mathfrak{f}^{(0)}$.

The action (8.1) is invariant under an on-shell BRST symmetry

$$\delta_B f = f(l^{(1)} + \overline{l}^{(3)}), \quad \delta_B \overline{w}^{(1)} = -J_-^{(1)}, \quad \delta_B w^{(3)} = -J_+^{(3)}, \quad \delta_B l^{(1)} = \delta_B \overline{l}^{(3)} = 0 \tag{8.5}$$

and it is also classical integrable [44, 45] with a Lax pair given by

$$\begin{aligned} \mathcal{L}_+(z) &= J_+^{(0)} + zJ_+^{(1)} + z^2 J_+^{(2)} + z^3 J_+^{(3)} + (z^4 - 1)N, \\ \mathcal{L}_-(z) &= J_-^{(0)} + z^{-3}J_-^{(1)} + z^{-2}J_-^{(2)} + z^{-1}J_-^{(3)} + (z^{-4} - 1)\overline{N}. \end{aligned} \tag{8.6}$$

Now, we proceed to deform this theory.

8.2 The "λ-model" of the PS superstring

In order to construct the lambda model of the pure spinor superstring, we need to find a way to: I) preserve its BRST symmetry and II) preserve its integrability. Our strategy will be to start with I) and later verify if II) is guaranteed by the resulting deformation.

We construct the lambda model for the PS superstring by adding to the λ-deformed hybrid action

$$S_{\text{hybrid}} = S_{F/F}(\mathcal{F}, A_\mu) - \frac{k}{\pi}\int_\Sigma d^2x \left\langle A_+(\Omega - 1)A_- \right\rangle, \tag{8.7}$$

a term proportional to the PS ghosts

$$S_{\text{ghost}} = r\int d^2x \big\langle w^{(3)} D_-^{(0)} l^{(1)} + \overline{w}^{(1)} D_+^{(0)} \overline{l}^{(3)} + sN\overline{N}\big\rangle, \tag{8.8}$$

where r, s are parameters to be determined by BRST symmetry arguments and $D_\pm^{(0)}(*) = \partial_\pm(*) + \left[A_\pm^{(0)}, *\right]$ is a covariant derivative. Namely, we define

$$S_{PS} = S_{\text{hybrid}} + S_{\text{ghost}}. \tag{8.9}$$

In order to find a candidate BRST symmetry we will work on stages. Start by considering the matter part and propose (set $\delta_B = \overline{\delta}$) the following transformations[17]

$$\overline{\delta}\mathcal{F} = -\alpha\mathcal{F} + \mathcal{F}\beta, \quad \overline{\delta}A_+ = D_+\alpha, \quad \overline{\delta}A_- = D_-\beta, \tag{8.10}$$

where α and β are functions of $l^{(1)}$, $\overline{l}^{(3)}$. The variation of the first term in (8.7) is given by

$$\overline{\delta}S_{F/F} = \frac{k}{\pi}\int_\Sigma d^2x \big\langle (\alpha - \beta)\, F_{+-}\big\rangle, \tag{8.11}$$

[17] This is the same method used in [9] to construct the kappa symmetry for the GS case. Notice the resemblance between the kappa and the BRST symmetry in both formulations.

where $F_{+-} \neq 0$ is the curvature of $A_\pm$. The variation of the Ω-dependent part of (8.7) is of the form

$$\begin{aligned}\bar{\delta} S_\Omega = &\frac{k}{\pi}(\lambda-1)\int_\Sigma d^2x \big\langle -cl^{(1)}F^{(3)}_{+-} + b\bar{l}^{(3)}F^{(1)}_{+-}\big\rangle \\ &+\frac{k}{\pi}\lambda(\lambda^{-4}-1)\int_\Sigma d^2x\big\langle cl^{(1)}D^{(0)}_{-}A^{(3)}_{+} + b\bar{l}^{(3)}D^{(0)}_{+}A^{(1)}_{-}\big\rangle,\end{aligned} \tag{8.12}$$

where we have taken

$$\alpha = \lambda c l^{(1)} + b\bar{l}^{(3)}, \quad \beta = cl^{(1)} + \lambda b \bar{l}^{(3)}, \tag{8.13}$$

with b and c arbitrary constants. Using this particular choice we end up with

$$\bar{\delta} S_{\text{hybrid}} = \frac{k}{\pi}\lambda(\lambda^{-4}-1)\int_\Sigma d^2x\big\langle cl^{(1)}D^{(0)}_{-}A^{(3)}_{+} + b\bar{l}^{(3)}D^{(0)}_{+}A^{(1)}_{-}\big\rangle. \tag{8.14}$$

From this expression, we discover that by taking $r = -\frac{k}{\pi}(\lambda^{-4}-1)$ and setting

$$\bar{\delta}\overline{w}^{(1)} = -\lambda b A^{(1)}_{-}, \quad \bar{\delta} w^{(3)} = -\lambda c A^{(3)}_{+}, \quad \bar{\delta} l^{(1)} = \bar{\delta}\bar{l}^{(3)} = 0, \tag{8.15}$$

we obtain for the whole action that

$$\bar{\delta} S_{PS} = \frac{k}{\pi}(\lambda^{-4}-1)\int_\Sigma d^2x\big\langle \overline{N}\bar{\delta}(A^{(0)}_{+} - sN) + N\bar{\delta}(A^{(0)}_{-} - s\overline{N})\big\rangle. \tag{8.16}$$

By setting $s = 1$, we arrive at the desired form

$$\bar{\delta} S_{PS} = \frac{k}{\pi}(\lambda^{-4}-1)\int_\Sigma d^2x\big\langle bA^{(1)}_{+}\big[\bar{l}^{(3)}, \overline{N}\big] + cA^{(3)}_{-}\big[l^{(1)}, N\big]\big\rangle. \tag{8.17}$$

Finally, we notice that the action is BRST invariant $\bar{\delta} S_{PS} = 0$ because, say

$$\big[l^{(1)}, N\big] = \frac{1}{2}\Big[w^{(3)}, \big[l^{(1)}, l^{(1)}\big]_{+}\Big], \tag{8.18}$$

vanishes by virtue of the pure spinor constraints (8.2). This is where the formulation borrows its name.

In the sigma model limit $\lambda \to 1$, the action (8.9) reduces to the first order form

$$S_{PS} = -\frac{\kappa^2}{\pi}\int_\Sigma d^2x\,\langle A_+\theta A_- + \nu F_{+-}\rangle - \frac{2\kappa^2}{\pi}\int_\Sigma d^2x\big\langle w^{(3)}D^{(0)}_{-}l^{(1)} + \overline{w}^{(1)}D^{(0)}_{+}\bar{l}^{(3)} + N\overline{N}\big\rangle, \tag{8.19}$$

which is to be compared with (8.1). However, after taking the limit the action is no longer BRST invariant because in this limit $\alpha = \beta$ and hence $\bar{\delta}\,\langle \nu F_{+-}\rangle = 0$. When compared with (8.11) this term is needed to cancel some contributions of the curvature coming from the variation of S_Ω. Only when we use the ν equations of motion and fix the gauge $A_\pm = f^{-1}\partial_\pm f$, the BRST symmetry is restored, i.e. when we return to the original formulation and to the set of variations (8.5).

Now, we can find the equations of motion by varying the action we have constructed. For the field $\mathcal{F}$, we get

$$\begin{aligned}\delta S_{PS} &= -\frac{k}{\pi}\int_{\Sigma} d^2x \left\langle \mathcal{F}^{-1}\delta\mathcal{F}\left[\partial_+ + \Omega^T A_+ + (\lambda^{-4}-1)N, \partial_- + A_-\right]\right\rangle \\ &= -\frac{k}{\pi}\int_{\Sigma} d^2x \left\langle \delta\mathcal{F}\mathcal{F}^{-1}\left[\partial_+ + A_+, \partial_- + \Omega A_- + (\lambda^{-4}-1)\overline{N}\right]\right\rangle,\end{aligned} \tag{8.20}$$

after using the $A_\pm$ equations of motion

$$\begin{aligned}A_+ &= \left(\Omega^T - D^T\right)^{-1}[\mathcal{F}^{-1}\partial_+\mathcal{F} - (\lambda^{-4}-1)N], \\ A_- &= -\left(\Omega - D\right)^{-1}[\partial_-\mathcal{F}\mathcal{F}^{-1} + (\lambda^{-4}-1)\overline{N}].\end{aligned} \tag{8.21}$$

The ghosts have the same equations as in the un-deformed theory

$$D_+^{(0)}\overline{N} + [\overline{N}, N] = 0, \quad D_-^{(0)}N + [N, \overline{N}] = 0. \tag{8.22}$$

If the deformation is to preserve the integrability, then the two expressions (8.20) for the equations of motion should be equivalent to the evaluation of the curvature

$$[\partial_+ + \mathcal{L}_+(z), \partial_- + \mathcal{L}_-(z)], \tag{8.23}$$

of the Lax pair

$$\begin{aligned}\mathcal{L}_+(z) &= I_+^{(0)} + zI_+^{(1)} + z^2I_+^{(2)} + z^3I_+^{(3)} + (z^4-1)N', \\ \mathcal{L}_-(z) &= I_-^{(0)} + z^{-3}I_-^{(1)} + z^{-2}I_-^{(2)} + z^{-1}I_-^{(3)} + (z^{-4}-1)\overline{N}'\end{aligned} \tag{8.24}$$

at the special values of the spectral paramenter $z = \lambda^{-1/2}$ and $z = \lambda^{1/2}$, respectively.[18] The prime in N' and $\overline{N}'$ is to denote possible re-scalings of the ghosts in terms of the parameter λ similar to the ones required to define the currents $I_\pm$. We conclude that under the present (naive) construction, the pure spinor superstring does not seem to admit a λ-model and more work is to be required. A possibility is to add a new term in order to restore integrability. This new term should, in principle, possess the following properties:

- It must be BRST invariant and gauge invariant, at least under the gauge group generated by the $\mathfrak{f}^{(0)}$ part of the Lie superalgebra $\mathfrak{f} = \mathfrak{psu}(2,2|4)$,
- It must be become a sub-leading correction of the order $O(1/k)$ in the sigma model limit $\lambda \to 1$, where $k \to \infty$ with κ^2 fixed.

By replacing (8.21) back into (8.9) we find that the resulting effective action differs from (2.11) plus the ghosts term action by a non-standard coupling between the currents $\hat{J}_\pm$'s and $N, \overline{N}$. Another hint that perhaps we need to add a new term in order to compensate the extra terms. However, we will leave this problem to be considered more carefully in a companion paper.

[18] This time the currents $I_\pm$ will include contributions from the PS currents.

9 Concluding remarks

In this paper we have studied in detail the λ-model of the hybrid formalism of the superstring in the background[19] $AdS_2 \times S^2$ and showed how it preserves most of the main characteristics of the original σ-model except the one related to the maximal isometry group of the target space, a situation that is common to all λ-models. The presence of Poisson-Lie groups at classical level is a strong signal of a quantum group symmetry F_q, which should appear as the symmetry group of some non-commutative space.[20] From the point of view of string theory the claim is that in the λ-model the isometry group F of the original target space is replaced by F_q with q a phase but without breaking any of the conditions that makes the deformed target space a genuine string background.[21] This is also supported by the recent results of [19–21] showing that the background fields judiciously extracted from the λ-model action functional form a one-parameter family of solutions to the supergravity equations of motion of relevance to each case. In the present situation the target space metric has fermionic directions as well and the explicit construction of it should be much more involved than in the Green-Schwarz formalism. We leave the problem of the explicit construction of the background fields for the near future.

One of the goals of this work was to gain a better understanding of the structure of the λ-deformation itself, in the sense of clarifying its true content from the integrable systems point of view. For further work devoted to this specific question see the papers [40, 41] to which the present results should be added as a complement.

Relying on our findings and on what is known for the GS superstring on $AdS_5 \times S^5$, it is reasonable to expect that the λ-model for the GS superstring on the supercoset $AdS_4 \times \mathbb{C}P^3$ is not only classical integrable but also one-loop conformal invariant. This can be seen from the Lie algebraic properties of semi-symmetric spaces [59] and from the fact that there is no difference in the construction of the Lagrangian in comparison to the case of $AdS_5 \times S^5$. The Lax pair representation is also the same [43] and as a consequence of this the determinant for the fluctuations will be proportional to the quadratic Casimir in the adjoint representation as well.

Finally, an interesting question to be considered is if the Poisson-Lie T-duality that is known between the η and the λ models of the Green-Schwarz superstring has an analogue for the hybrid superstring as well, i.e, if the action (2.9) admits a deformation of the Yang-Baxter type in terms of an R-matrix satisfying the cmYBE as constructed in [2] for the GS formalism. However, it is already known that the η-deformation is not Weyl invariant at the quantum level for the GS case and perhaps an analogue situation could be present in the hybrid superstring as well. A possible way out of this situation in both formulations might be to consider Yang-Baxter deformations in terms of dynamical R-matrices instead of the usual constant ones. Hopefully, they could be general enough as to introduce the necessary freedom required to restore Weyl's symmetry. We will come back to this question elsewhere.

[19] The supercoset $AdS_3 \times S^3$ can be treated along the same lines.

[20] This is certainly an interesting situation to be further explored in the context of the AdS/CFT duality. For an example of this in relation to the η-deformation see [58].

[21] As initially suggested by the vanishing of the beta functions.

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These two appendices gather the most useful algebraic results used for calculations in the body of the paper: the deformed current algebra and the $\mathfrak{psu}(1,1|2)$ Lie superalgebra proper to the $AdS_2 \times S^2$ supercoset.

A Current algebra for the deformed hybrid formulation

The non-zero Poisson brackets for the currents (2.13) or (3.17) can be computed directly from[22] (5.1) by using the identities

$$\begin{aligned}
\{\langle \mu, I_\alpha \rangle, \langle \overline{\mu}, I_\beta \rangle\} =& \big\langle \{\overset{1}{I}_\alpha, \overset{2}{I}_\beta\}, (\mu \otimes \overline{\mu}) \big\rangle_{12}, \\
\langle [\mu, \overline{\mu}], I_\alpha \rangle =& - \big\langle [C_{12}, \overset{2}{I}_\alpha], \mu \otimes \overline{\mu} \big\rangle_{12}, \\
\langle \mu, \overline{\mu} \rangle =& \langle C_{12}, \mu \otimes \overline{\mu} \rangle_{12}\,,
\end{aligned} \tag{A.1}$$

where $\mu, \overline{\mu}$ are the test functions and $\alpha, \beta = \pm$. The upper indices $1, 2$ refer to the copy in a chain of tensor products. For example, $\overset{1}{u} = u \otimes I$, $\overset{2}{u} = I \otimes u$. The lower indices $1, 2$ indicate taking the supertrace on the first or on the second copy of the vector space in the tensor product.

The non-zero current algebra elements are given by

$$\begin{aligned}
\{\overset{1}{I}{}^{(0)}_1(x), \overset{2}{I}{}^{(0)}_1(y)\} &= -\frac{2\pi}{k}([C^{(00)}_{12}, \overset{2}{I}{}^{(0)}_1(y)]\delta_{xy} - C^{(00)}_{12}\delta'_{xy}), \\
\{\overset{1}{I}{}^{(0)}_1(x), \overset{2}{I}{}^{(1)}_\pm(y)\} &= \pm\alpha[C^{(00)}_{12}, \overset{2}{I}{}^{(1)}_\mp(y) - z^4_\mp \overset{2}{I}{}^{(1)}_\pm(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(0)}_1(x), \overset{2}{I}{}^{(2)}_\pm(y)\} &= \pm\alpha[C^{(00)}_{12}, \overset{2}{I}{}^{(2)}_\mp(y) - z^4_\mp \overset{2}{I}{}^{(2)}_\pm(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(0)}_1(x), \overset{2}{I}{}^{(3)}_\pm(y)\} &= \pm\alpha[C^{(00)}_{12}, \overset{2}{I}{}^{(3)}_\mp(y) - z^4_\mp \overset{2}{I}{}^{(3)}_\pm(y)]\delta_{xy},
\end{aligned} \tag{A.2}$$

for $[\mathfrak{f}^{(0)}, \mathfrak{f}^{(i)}]$, $i = 0, 1, 2, 3$.

$$\begin{aligned}
\{\overset{1}{I}{}^{(1)}_+(x), \overset{2}{I}{}^{(1)}_+(y)\} &= \alpha[C^{(13)}_{12}, \overset{2}{I}{}^{(2)}_-(y) - a\overset{2}{I}{}^{(2)}_+(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(1)}_\pm(x), \overset{2}{I}{}^{(1)}_-(y)\} &= -\alpha[C^{(13)}_{12}, \overset{2}{I}{}^{(2)}_\pm(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(1)}_+(x), \overset{2}{I}{}^{(2)}_+(y)\} &= \alpha[C^{(13)}_{12}, \overset{2}{I}{}^{(3)}_-(y) - a\overset{2}{I}{}^{(3)}_+(y)]\delta_{xy}, \\
\{\overset{1}{I}{}^{(1)}_+(x), \overset{2}{I}{}^{(2)}_-(y)\} &= -\alpha[C^{(13)}_{12}, \overset{2}{I}{}^{(3)}_+(y)]\delta_{xy},
\end{aligned} \tag{A.3}$$

[22] The R with the minus sign is the one that reproduce the Poisson brackets computed directly from (3.17).

$$
\begin{aligned}
&\{\overset{1}{I}{}^{(1)}_{-}(x),\overset{2}{I}{}^{(2)}_{\pm}(y)\} = -\alpha[C^{(13)}_{12},\overset{2}{I}{}^{(3)}_{\pm}(y)]\delta_{xy},\\
&\{\overset{1}{I}{}^{(1)}_{\pm}(x),\overset{2}{I}{}^{(3)}_{\pm}(y)\} = \mp\alpha([C^{(13)}_{12},\overset{2}{I}{}^{(0)}_{1}(y)\pm\alpha z_{\pm}^{4}C^{(0)}_{0}(y)]\delta_{xy}-C^{(13)}_{12}\delta'_{xy}),\\
&\{\overset{1}{I}{}^{(1)}_{\pm}(x),\overset{2}{I}{}^{(3)}_{\mp}(y)\} = -\alpha^{2}[C^{(13)}_{12},\overset{2}{C}{}^{(0)}_{0}(y)]\delta_{xy},
\end{aligned}
\tag{A.4}
$$

for $[\mathfrak{f}^{(1)},\mathfrak{f}^{(i)}]$, $i=1,2,3$.

$$
\begin{aligned}
&\{\overset{1}{I}{}^{(2)}_{\pm}(x),\overset{2}{I}{}^{(2)}_{\pm}(y)\} = \mp\alpha([C^{(22)}_{12},\overset{2}{I}{}^{(0)}_{1}(y)\pm\alpha z_{\pm}^{4}C^{(0)}_{0}(y)]\delta_{xy}-C^{(22)}_{12}\delta'_{xy}),\\
&\{\overset{1}{I}{}^{(2)}_{+}(x),\overset{2}{I}{}^{(2)}_{-}(y)\} = -\alpha^{2}[C^{(22)}_{12},\overset{2}{C}{}^{(0)}_{0}(y)]\delta_{xy},\\
&\{\overset{1}{I}{}^{(2)}_{+}(x),\overset{2}{I}{}^{(3)}_{\pm}(y)\} = -\alpha[C^{(22)}_{12},\overset{2}{I}{}^{(1)}_{\pm}(y)]\delta_{xy},\\
&\{\overset{1}{I}{}^{(2)}_{-}(x),\overset{2}{I}{}^{(3)}_{+}(y)\} = -\alpha[C^{(22)}_{12},\overset{2}{I}{}^{(1)}_{-}(y)]\delta_{xy},\\
&\{\overset{1}{I}{}^{(2)}_{-}(x),\overset{2}{I}{}^{(3)}_{-}(y)\} = \alpha[C^{(22)}_{12},\overset{2}{I}{}^{(1)}_{+}(y)-a\overset{2}{I}{}^{(1)}_{-}(y)]\delta_{xy},
\end{aligned}
\tag{A.5}
$$

for $[\mathfrak{f}^{(2)},\mathfrak{f}^{(i)}]$, $i=2,3$ and

$$
\begin{aligned}
&\{\overset{1}{I}{}^{(3)}_{+}(x),\overset{2}{I}{}^{(3)}_{\pm}(y)\} = -\alpha[C^{(31)}_{12},\overset{2}{I}{}^{(2)}_{\pm}(y)]\delta_{xy},\\
&\{\overset{1}{I}{}^{(3)}_{-}(x),\overset{2}{I}{}^{(3)}_{-}(y)\} = \alpha[C^{(31)}_{12},\overset{2}{I}{}^{(2)}_{+}(y)-a\overset{2}{I}{}^{(2)}_{-}(y)]\delta_{xy},
\end{aligned}
\tag{A.6}
$$

for $[\mathfrak{f}^{(3)},\mathfrak{f}^{(3)}]$. We have defined $a\equiv z_{+}^{4}+z_{-}^{4}$.

Notice that $a=-2(2\epsilon^{2}-1)$ for comparison with previous works that make use of $\epsilon^{2}=-\frac{(1-\lambda^{2})^{2}}{4\lambda^{2}}$ as the deformation parameter. In the sigma model limit when $\lambda\to1$, the Poisson brackets above coincide with the current algebra of the matter sector of the pure spinor superstring computed in [57].

Finally, the brackets involving the gauge constraint are the standard ones

$$
\begin{aligned}
&\{\overset{1}{C}{}^{(0)}_{0}(x),\overset{2}{C}{}^{(0)}_{0}(y)\} = -[C^{(00)}_{12},\overset{2}{C}{}^{(0)}_{0}(y)]\delta_{xy},\\
&\{\overset{1}{C}{}^{(0)}_{0}(x),\overset{2}{I}{}^{(0)}_{1}(y)\} = -([C^{(00)}_{12},\overset{2}{I}{}^{(0)}_{1}(y)]\delta_{xy}-C^{(00)}_{12}\delta'_{xy}),\\
&\{\overset{1}{C}{}^{(0)}_{0}(x),\overset{2}{I}{}^{(i)}_{\pm}(y)\} = -[C^{(00)}_{12},\overset{2}{I}{}^{(i)}_{\pm}(y)]\delta_{xy},\quad i=1,2,3.
\end{aligned}
\tag{A.7}
$$

B A basis for the $\mathfrak{psu}(1,1|2)$ Lie superalgebra

For completeness we write the basis presented in [56] and include the fermionic elements used here to construct explicitly the W-element related to the superconformal algebra in section 6.

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The (anti)-commutation relations are (the m's are even while the q's are odd)

$$\begin{aligned}
&[m^\alpha{}_\beta, m^\gamma{}_\delta] = \delta^\gamma{}_\beta m^\alpha{}_\delta - \delta^\alpha{}_\delta m^\gamma{}_\beta, && [\overline{m}^i{}_j, \overline{m}^k{}_n] = \delta^k{}_j \overline{m}^i{}_n - \delta^i{}_n \overline{m}^k{}_j, \\
&[m^\alpha{}_\beta, q^k{}_\gamma] = -\delta^\alpha{}_\gamma q^k{}_\beta + \frac{1}{2}\delta^\alpha{}_\beta q^k{}_\gamma, && [m^\alpha{}_\beta, \overline{q}^\gamma{}_k] = \delta^\gamma{}_\beta \overline{q}^\alpha{}_k - \frac{1}{2}\delta^\alpha{}_\beta \overline{q}^\gamma{}_k, \\
&[\overline{m}^i{}_j, q^k{}_\alpha] = \delta^k{}_j q^i{}_\alpha - \frac{1}{2}\delta^i{}_j q^k{}_\alpha, && [\overline{m}^i{}_j, \overline{q}^\alpha{}_k] = -\delta^i{}_k \overline{q}^\alpha{}_j + \frac{1}{2}\delta^i{}_j \overline{q}^\alpha{}_k, \\
&[q^i{}_\gamma, \overline{q}^\beta{}_j]_+ = l(\delta^i{}_j m^\beta{}_\alpha + \delta^\alpha{}_\beta \overline{m}^i{}_j), && l^2 = -1,
\end{aligned} \tag{B.1}$$

where $\alpha, \beta = 1, 2$ and $i, j = 1, 2$. The bosonic subalgebras $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ are generated by $m^\alpha{}_\beta$ and $\overline{m}^i{}_j$, respectively. There are 8 supercharges $q^k{}_\gamma$, $\overline{q}^\gamma{}_k$.

Under the $\mathbb{Z}_4$ decomposition $\mathfrak{f} = \bigoplus_{i=0}^3 \mathfrak{f}^{(i)}$, the generators split as follows

$$\begin{aligned}
\mathfrak{f}^{(0)} &= span\{m^1{}_1,\ \overline{m}^1{}_1\}, \\
\mathfrak{f}^{(1)} &= span\{q^1{}_1,\ q^2{}_2,\ \overline{q}^1{}_2,\ \overline{q}^2{}_1\}, \\
\mathfrak{f}^{(2)} &= span\{m^1{}_2,\ m^2{}_1,\ \overline{m}^1{}_2,\ \overline{m}^2{}_1\}, \\
\mathfrak{f}^{(3)} &= span\{\overline{q}^1{}_1,\ \overline{q}^2{}_2,\ q^1{}_2,\ q^2{}_1\}.
\end{aligned} \tag{B.2}$$

Consider now the following re-labeling of generators for the fermionic sectors $\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(3)}$, respectively,

$$\begin{aligned}
&T^{(1)}_{(++)} = \overline{q}^1{}_2, && T^{(1)}_{(--)} = \overline{q}^2{}_1, && T^{(1)}_{(+-)} = q^2{}_2, && T^{(1)}_{(-+)} = q^1{}_1, \\
&T^{(3)}_{(++)} = q^1{}_2, && T^{(3)}_{(--)} = q^2{}_1, && T^{(3)}_{(+-)} = \overline{q}^1{}_1, && T^{(3)}_{(-+)} = \overline{q}^2{}_2.
\end{aligned} \tag{B.3}$$

They satisfy the commutation relations with the gauge algebra $\mathfrak{f}^{(0)} = \mathfrak{u}(1) \times \mathfrak{u}(1)$

$$[(\mathfrak{h}_1, \mathfrak{h}_2), T^{(a)}_{(\pm\pm)}] = \frac{1}{2}(\pm 1, \pm 1) T^{(a)}_{(\pm\pm)}, \quad [(\mathfrak{h}_1, \mathfrak{h}_2), T^{(a)}_{(\pm\mp)}] = \frac{1}{2}(\pm 1, \mp 1) T^{(a)}_{(\pm\pm)}, \tag{B.4}$$

where $a = 1, 3$. We have introduced the "vector" $(\mathfrak{h}_1 = m^1{}_1, \mathfrak{h}_2 = \overline{m}^1{}_1)$ in order to exhibit the gauge labels in a compact way.

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