Lifshitz hydrodynamics from Lifshitz black branes with linear momentum

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ABSTRACT: We construct a new class of 4-dimensional $z = 2$ Lifshitz black branes that have a non-zero linear momentum. These are solutions of an Einstein-Proca-dilaton model that can be obtained by Scherk-Schwarz circle reduction of AdS_5 gravity coupled to a free real scalar field. The boundary of a bulk Lifshitz space-time is a Newton-Cartan geometry. We show that the fluid dual to the moving Lifshitz black brane leads to a novel form of Lifshitz hydrodynamics on a Newton-Cartan space-time. Since the linear momentum of the black brane cannot be obtained by a boost transformation the velocity of the fluid or rather, by boundary rotational invariance, its magnitude plays the role of a chemical potential. The conjugate dual variable is mass density. The Lifshitz perfect fluid can be thought of as arising from a Schrödinger perfect fluid with broken particle number symmetry.

Contents

1 Introduction

Hydrodynamics is a powerful tool to describe the long wavelength physics of quantum field theories at finite temperature. Remarkably, holography provides a dual realization of such effective descriptions, typically at strong coupling in the field theory, in terms of the long wavelength dynamics of black holes. The seminal example of this has been the calculation of the viscosity to entropy ratio [1, 2] (see [3] for a review) of strongly coupled plasmas using

black holes via the AdS/CFT correspondence. This deep relation between fluid dynamics and gravity has led to the fluid/gravity correspondence [4, 5] (see [6] for a review) which has sparked numerous novel insights on both sides of the duality.[1]

These developments initially focused on the dual gravitational formulation of relativistic hydrodynamics, since in the standard AdS holography the dual field theories are relativistic. Motivated by applying holography in a wider setting, in particular strongly coupled non-relativistic field theories, more general holographic bulk theories with anisotropic scaling between time and space, characterized by the dynamical exponent z, have been introduced [11–14]. These include include Schrödinger and Lifshitz space-times, and in the latter case[2] there are different bulk realizations (e.g. Einstein-Maxwell-dilaton (EMD) and Einstein-Proca-dilaton (EPD) models) which have distinct physical features. All these holographic models serve to describe different types of non-relativistic field theories, where in the former there is Galilean boost symmetry, while in the latter there is a broken boost symmetry. For possible applications to condensed matter systems, but also to further our understanding of holography in non-AdS setups, it is thus a natural question to generalize the fluid/gravity correspondence to these different classes of non-relativistic field theories. This may also serve as a step towards a more general classification of such field theories.

For theories with Schrödinger symmetries a corresponding version of (conformal) non-relativistic fluid/gravity correspondence was developed in [16–19]. Certain realizations of Lifshitz hydrodynamics and their holographic description have subsequently been considered in [20–22]. Another class of Lifshitz theories and their hydrodynamics was holographically studied in the context of the Einstein-Maxwell-dilaton (EMD) model [23]. In these theories there is an extra bulk U(1) symmetry and since the dilaton runs logarithmically close to the boundary, there is a new scaling exponent on top of the dynamical exponent z.

In this paper we will focus on yet another class of Lifshitz theories, namely those that have Lifshitz symmetry in the bulk and Schrödinger symmetry with broken particle number on the boundary. For a large class of EPD models, it was shown in [24–26] that holography in such bulk theories is dual to non-relativistic field theories of this type, coupled to a background torsional Newton-Cartan geometry that is induced on the boundary. Our aim is therefore to find a gravitational dual realization of the hydrodynamics, or rather the perfect fluid limit, of this class of non-relativistic theories by constructing appropriate Lifshitz black branes.

A classification of the different versions of Lifshitz hydrodynamics will be given in the upcoming work [27] using a field theory perspective. The novel version of Lifshitz hydrodynamics that we find in this paper is a holographic realization of one particular class, which will be also discussed in [27] with field theory examples.[3] We will find in this paper that this version of Lifshitz hydrodynamics requires the construction of a new class

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[1]For asymptotically flat black branes the blackfold approach [7–9] also gives a relation between the long wavelength dynamics of black branes and fluids that live on dynamically embedded surfaces. Applied to D3-branes this has been shown to encapsulate AdS fluid/gravity [10].

[2]See [15] for a review on Lifshitz holography.

[3]See also [28] for a discussion on field theories coupled to torsional Newton-Cartan geometry with broken Schrödinger symmetries.

of four-dimensional $z = 2$ Lifshitz black branes that have a non-zero linear momentum. While, as mentioned above, a large class of general z Lifshitz space-times can be constructed in the EPD model, we restrict for technical reasons to a particular EPD model with $z = 2$ solutions, that can be obtained by Scherk-Schwarz circle reduction of AdS$_5$ gravity coupled to a free real scalar field.

Our new $z = 2$ Lifshitz black brane solutions exhibit the following features:

- The linear momentum of the black brane cannot be obtained by a boost transformation, and hence this class of solutions is physically distinct from unboosted solutions.

- The (squared) magnitude of the boost velocity plays the role of a chemical potential dual to the mass density. Consequently, the mass density occurs asymptotically as an extra parameter on top of the energy, even when the velocity is zero.

- The black brane configurations describe a new class of Lifshitz perfect fluids that are obtained by breaking particle number symmetry in Schrödinger perfect fluids.

In further detail, the thermodynamics for these Lifshitz black branes can be summarized by

$$\mathcal{E} + P = Ts + \frac{1}{2}\rho V^2\,, \tag{1.1}$$

$$\delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho\,, \tag{1.2}$$

where $\mathcal{E}$ is the energy density, P the pressure with equation of state $P = \mathcal{E}$ (2 spatial dimensions with $z = 2$), T temperature, s entropy density, ρ mass density and $V^2 = V^i V^i$ with V^i the fluid velocity.

Outline. The outline of the paper is as follows. In section 2 we introduce the Einstein-Proca-dilaton (EPD) theory which consists of Einstein gravity coupled to a massive vector and a dilaton with arbitrary dilaton-dependent couplings between the massive vector and the dilaton. In section 2.2 we show that this model admits, under some mild restrictions on the dilaton-dependent coupling functions, Lifshitz solutions for any value of the dynamical exponent z. Important for the rest of this work we show in section 2.3 that there is one EPD model that can uplifted to a 5-dimensional AdS gravity theory coupled to a free real scalar field. This specific model will be referred to as the upliftable model and it admits $z = 2$ Lifshitz solutions.

For the higher-dimensional AdS theory it is known how to perform holographic renormalization and by reducing the result to one dimension lower we can obtain the relevant counterterms and near-boundary asymptotic expansions. This reduction is of the Scherk-Schwarz type meaning that the 5-dimensional scalar field is required to come back to itself up to a shift (which is a global symmetry) upon going around the compact 5th dimension. In appendix B we provide the details of the holographic renormalization before and after the dimensional reduction. In section B.5 we give a proof that the reduction is consistent.

The reduction in the bulk is everywhere along a spacelike circle however on the boundary (due to conformal rescaling) the circle is null. Hence from a boundary perspective we are dealing with a null reduction. It is well known that null reductions of Lorentzian geometries,

in this case the boundary of the asymptotically AdS_5 space-time, lead to Newton-Cartan geometries. The details of this null reduction for both the metric and energy-momentum tensor are given in appendix A.

Section 3 is concerned with the construction and properties of Lifshitz black branes with linear momentum. We start with the ansatz in section 3.1 where we also show that the effective action, that reproduces the equations of motion of the EPD model in which our ansatz has been substituted, possesses two scale symmetries. This leads to two Noether charges or first integrals of motion that are constant along the holographic coordinate. The following sections 3.2–3.4 study the properties of the solution near the boundary and near the horizon. In the last two subsections 3.5 and 3.6 we work out the thermodynamic properties of the solution showing that the magnitude of the velocity acts like a chemical potential whose conjugate variable is the mass density. We further derive an Euler-type thermodynamic relation for Lifshitz perfect fluids using the conserved Noether charges and once more using the Killing charges associated with bulk Killing vectors. We end with a discussion of the first law of thermodynamics (summarized in (1.1), (1.2)) for these Lifshitz black branes.

Section 4 can be read independently from sections 2 and 3. It only requires appendix A. It takes the point of view that these Lifshitz perfect fluids can be obtained by dimensional reduction of a relativistic perfect fluid (as discussed in appendix A.3) in the presence of a scalar source that is linearly dependent on the circle coordinate. It presents the Ward identities of a Lifshitz fluid and the expressions for the energy-momentum tensor and mass current at the perfect fluid level. The goal of this section is to derive the first law of our Lifshitz perfect fluids from the requirement that the Ward identities lead to the existence of a conserved entropy current. Finally, we present our conclusions and a number of open questions in section 5.

2 The bulk theory

Lifshitz space-times as a solution of a theory with Einstein gravity coupled to matter fall into 2 classes. These are the Einstein-Proca-dilaton (EPD) theories of [29, 30] and the Einstein-Maxwell dilaton (EMD) theories of [14, 31]. We are interested in Lifshitz black brane solutions of the EPD models and to ultimately use them as a starting point to set up a fluid/gravity correspondence for Lifshitz space-times. Black brane solutions of theories with massive vectors were studied for models without a dilaton in [32–34]. However in such theories the solutions are not analytically known. Nevertheless it is possible to work out the thermodynamics of these solutions by using first integrals of motion (with respect to the holographic radial coordinate) that allows one to relate near-horizon expansions to asymptotic expansions. We will follow a similar approach here.

Regarding Lifshitz black brane solutions of the EMD models, they are known analytically however they have different physical properties due to the presence of an extra bulk U(1) symmetry and the fact that the dilaton is running logarithmically close to the boundary which introduces a new scaling exponent on top of the dynamical exponent z. The fluid/gravity correspondence for these black branes was studied in [23].

In this section we will introduce the EPD model and discuss its Lifshitz solutions. In the last section 2.3 we will show that there is a specific EPD model that can be obtained from dimensional reduction of an action in one dimension higher that admits asymptotically AdS solutions. This so-called upliftable model will be used throughout the rest of this work.

2.1 The EPD model

The general class of 4-dimensional bulk EPD models is described by the following family of actions

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{4} Z(\Phi) F^2 - \frac{1}{2} W(\Phi) B^2 - \frac{x}{2} (\partial \Phi)^2 - V(\Phi) \right), \tag{2.1}$$

where $F = dB$. The equations of motion are

$$G_{MN} = \frac{x}{2} \left(\partial_M \Phi \partial_N \Phi - \frac{1}{2} (\partial \Phi)^2 g_{MN} \right) - \frac{1}{2} V(\Phi) g_{MN}$$
$$+ \frac{1}{2} Z(\Phi) \left(F_{MP} F_N{}^P - \frac{1}{4} F^2 g_{MN} \right) + \frac{1}{2} W(\Phi) \left(B_M B_N - \frac{1}{2} B^2 g_{MN} \right), \tag{2.2}$$

$$\frac{x}{\sqrt{-g}} \partial_M \left(\sqrt{-g} \partial^M \Phi \right) = \frac{1}{4} \frac{dZ}{d\Phi} F^2 + \frac{1}{2} \frac{dW}{d\Phi} B^2 + \frac{dV}{d\Phi}, \tag{2.3}$$

$$\frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g} Z(\Phi) F^{MN} \right) = W(\Phi) B^N. \tag{2.4}$$

The parameter x can always be set equal to one but often it is more convenient to take some other value for it. It will prove convenient to make the following Stückelberg decomposition of the massive vector field

$$B_M = A_M - \partial_M \Xi. \tag{2.5}$$

The scalar Ξ has dimensions of length and all other fields are dimensionless.

2.2 Lifshitz solution

The equations of motion admit the following Lifshitz solutions (with $z > 1$)

$$ds^2 = -\frac{1}{r^{2z}} dt^2 + \frac{1}{r^2} \left(dr^2 + dx^2 + dy^2 \right), \tag{2.6}$$

$$B = A_0 \frac{1}{r^z} dt, \tag{2.7}$$

$$\Phi = \Phi_\star, \tag{2.8}$$

provided that

$$A_0^2 = \frac{2(z-1)}{zZ_0}, \tag{2.9}$$

$$\frac{W_0}{Z_0} = 2z, \tag{2.10}$$

$$V_0 = - \left(z^2 + z + 4 \right), \tag{2.11}$$

$$V_1 = (za + 2b)(z-1). \tag{2.12}$$

where

$$a = \frac{Z_1}{Z_0}, \qquad b = \frac{W_1}{W_0}. \tag{2.13}$$

Above we have used the notation

$$V_0 \equiv V(\Phi_*), \qquad V_1 \equiv \frac{dV}{d\Phi}\Big|_{\Phi=\Phi_*}, \qquad V_2 \equiv \frac{d^2V}{d\Phi^2}\Big|_{\Phi=\Phi_*} \tag{2.14}$$

etc. where $\Phi_\star$ is a constant. Equations (2.10) and (2.11) determine the values of Φ_* and z in terms of the functions appearing in the action. Equation (2.9) fixes A_0, and (2.12) is a condition on the potential in order that Lifshitz is a solution of the family of actions (2.1). We note that there are also solutions of the EPD model with a logarithmically running scalar whose metric is a Lifshitz space-time [29, 30], but since these involve an additional scaling exponent these will not be considered here.

We can without loss of generality always perform a constant shift of Φ and redefine the functions Z, W and V such that for the new Φ the solution has $\Phi_\star = 0$. We will from now on always assume this has been done.

In order to study the boundary fluid properties (or even only thermodynamic properties as we will do here) one needs to understand the near-boundary expansion and the identification in that expansion of all the sources and vevs. We will restrict our attention to a specific model for which this problem has been solved because it can be related to an AdS holographic renormalization problem in one dimension higher. The general features of Lifshitz black brane solutions of other EPD models will have to wait until we have understood fully the problem of performing holographic renormalization for asymptotically Lifshitz solutions of the general class of EPD models (see appendix B.6 for additional comments). The model for which we do have full control of the asymptotic expansion is called the upliftable model and this will be the subject of the next subsection.

2.3 The upliftable model

When we make the choices

$$Z = e^{3\Phi}, \qquad W = 4, \qquad V(\Phi) = 2e^{-3\Phi} - 12e^{-\Phi}, \qquad x = 3. \tag{2.15}$$

the action (2.1) can be uplifted to

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-\mathcal{G}} \left(R + 12 - \frac{1}{2}\partial_\mathcal{M}\psi\partial^\mathcal{M}\psi \right), \tag{2.16}$$

where $\kappa_5^2 = 8\pi G_5$ with G_5 the 5-dimensional Newton's constant and where $\mathcal{M} = (u, M)$. The relation between the 5- and 4-dimensional theories is via a so-called Scherk-Schwarz reduction whereby we demand that the scalar field ψ when going around the compactification circle comes back to itself up to a shift. This is also known as a twisted reduction. This is possible because the higher-dimensional theory has a shift symmetry acting on the scalar. The Scherk-Schwarz reduction leading to (2.1) with the choices (2.15) is obtained

by the following ansatz

$$ds_5^2 = \mathcal{G}_{\mathcal{MN}}dx^{\mathcal{M}}dx^{\mathcal{N}} = \frac{dr^2}{r^2} + \gamma_{AB}dx^A dx^B = e^{-\Phi}g_{MN}dx^M dx^N + e^{2\Phi}\left(du + A_M dx^M\right)^2$$

$$= e^{-\Phi}\left(e^{\Phi}\frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu\right) + e^{2\Phi}\left(du + A_\mu dx^\mu\right)^2, \tag{2.17}$$

$$\psi = 2u + 2\Xi, \tag{2.18}$$

where the four dimensional fields g_{MN}, A_M, Ξ and Φ are independent of the fifth coordinate u which is periodically identified $u \sim u + 2\pi L$. In our normalization the 4-dimensional Newton's constant G_4 obeys $16\pi G_4 = 1$. This means that 5-dimensional Newton's constant G_5 obeys $\frac{2\pi L}{16\pi G_5} = \frac{1}{16\pi G_4} = 1$.

What the Scherk-Schwarz reduction does is that it gauges the shift symmetry of ψ using the Kaluza-Klein vector. In 4 dimensions this results in a covariant derivative acting on Ξ. This covariant derivative can be read as a massive vector field B where B is given by (2.5). The consistency of the reduction will be proven in appendix B.5. We now specialize to the case of the upliftable model (2.15) because for this theory we have full control over the asymptotic solution space.

3   Black branes with linear momentum

The goal of this work is to construct the gravity dual of a Lifshitz perfect fluid. The Lifshitz algebra does not contain a boost generator. We will be interested in cases where the Lifshitz algebra arises from a larger algebra that contains boosts[4] by some explicit symmetry breaking. The bulk Lifshitz space-time has no boost Killing vector and the EPD model has no additional local symmetries that could combine with a space-time transformation to give an additional global symmetry like a Galilean boost.[5] Hence in order to study perfect fluids with a non-zero velocity we cannot simply boost a static Lifshitz black brane and promote the boost velocity to the fluid velocity. If we do that for a static Lifshitz black brane solution of the EPD model we simply describe a static black brane in a moving coordinate system and that is not equivalent to a moving black brane in a static coordinate system because of the absence of a boost symmetry. That means that we need to construct a new class of Lifshitz black branes that has a non-zero velocity or as we shall say non-zero linear momentum. We will construct these solutions near the Lifshitz boundary and near the horizon. We will then construct first integrals of motion to relate near-horizon quantities such as temperature and entropy to near-boundary quantities such as energy and mass density.[6]

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[4]These can only be Galilean or Carrollian boosts as these are compatible with a $z > 1$ scaling. We cannot obtain a Lifshitz algebra by breaking Lorentz boosts because these require $z = 1$.

[5]What we have in mind here is some bulk dual of the mechanism discussed in [28, 35] whereby a boundary space-time transformations combined with a certain U(1) transformation leads to an additional global symmetry. For example Galilean boost symmetries of the Schrödinger equation come about by a combination of a space-time Galilean coordinate transformation and a U(1) phase transformation of the wave function. The latter can be traded for a U(1) transformation of a background gauge field.

[6]If we assume that a Galilean boost symmetry has been broken, the velocity V^i or rather by rotational invariance, its magnitude V^2, will be a chemical potential. On dimensional grounds it follows that the dual

3.1 The ansatz

We assume that the black brane solution admits time and space translation Killing vectors. We can perform a rotation to make sure that the linear momentum is only along the y-direction. The full non-linear solution is thus of the form

$$ds_4^2 = -F_1(r)\frac{dt^2}{r^4} + \frac{1}{F_2(r)}\frac{dr^2}{r^2} + F_3(r)\frac{dx^2}{r^2} + F_4(r)\left(\frac{dy}{r} + N(r)\frac{dt}{r^2}\right)^2, \tag{3.1}$$

$$B = G_1(r)\frac{dt}{r^2} + G_2(r)\left(\frac{dy}{r} + N(r)\frac{dt}{r^2}\right), \tag{3.2}$$

$$\Phi = \Phi(r), \tag{3.3}$$

where we did not fix the r reparametrization invariance. The powers in r are chosen for convenience to match with the Lifshitz scaling of the boundary coordinates t, x and y.

We can go to Eddington-Finkelstein coordinates[7] by defining V and Y coordinates as follows

$$dt = dV + r\,(F_1 F_2)^{-1/2}\,dr, \tag{3.4}$$

$$dy = dY - N\,(F_1 F_2)^{-1/2}\,dr, \tag{3.5}$$

leading to

$$ds_4^2 = -F_1\frac{dV^2}{r^4} - 2\left(\frac{F_1}{F_2}\right)^{1/2}\frac{dV\,dr}{r^3} + F_3\frac{dx^2}{r^2} + F_4\left(\frac{dY}{r} + N\frac{dV}{r^2}\right)^2, \tag{3.6}$$

$$B = G_1\frac{dV}{r^2} + \frac{G_1}{(F_1 F_2)^{1/2}}\frac{dr}{r} + G_2\left(\frac{dY}{r} + N\frac{dV}{r^2}\right). \tag{3.7}$$

Substituting the ansatz (3.1)–(3.3) into the bulk equations of motion (2.2)–(2.4) for (2.15) and integrating the equations to an action leads to the following effective Lagrangian for the equations of motion

$$
\begin{aligned}
L = r^{-5}\,(F_1 F_2 F_3 F_4)^{1/2}&\left[\frac{r^2}{2}\frac{F_1' F_3'}{F_1 F_3} + \frac{r^2}{2}\frac{F_1' F_4'}{F_1 F_4} - r\frac{F_3'}{F_3} - r\frac{F_4'}{F_4} + 2r\frac{F_2'}{F_2} + \frac{r^2}{2}\frac{F_3' F_4'}{F_3 F_4} - 6\right.\\
&+\frac{1}{2}\frac{F_4\,(rN'-N)^2}{F_1} + \frac{1}{2}\frac{Z\,(rG_1'-2G_1)^2}{F_1} + 2\frac{G_1^2}{F_1 F_2} - \frac{3}{2}r^2\Phi'^2 - \frac{V}{F_2} - 2\frac{G_2^2}{F_2 F_4}\\
&\left.+\frac{1}{2}\frac{ZG_2^2(rN'-N)^2}{F_1} - 2\frac{ZG_1 G_2(rN'-N)}{F_1} + r\frac{ZG_2 G_1'(rN'-N)}{F_1} - \frac{1}{2}\frac{Z(rG_2'-G_2)^2}{F_4}\right],
\end{aligned}
\tag{3.8}
$$

where the independent functions are F_1 to F_4, N, G_1, G_2, Φ and their derivatives with respect to r. This effective Lagrangian can also be obtained by substituting the ansatz (3.1)–(3.3) into the bulk action (2.1) with (2.15) and performing a few partial integrations. This

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thermodynamic variable must be a mass density denoted by ρ. We will show that Lifshitz black branes indeed contain such a quantity.

[7]Null geodesics with generalized momenta $\frac{\partial L}{\partial \dot{x}} = 0$ and $\frac{\partial L}{\partial \dot{y}} = 0$ where $L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ correspond to $V = \mathrm{cst}$ and $Y = \mathrm{cst}$.

ansatz is a generalization of a static black brane with zero momentum corresponding to setting $G_2 = N = 0$ and $F_3 = F_4$.

The effective Lagrangian (3.8) has the following two scaling symmetries

$$F_1 \to \lambda^2 F_1\,, \quad F_{3,4} \to \lambda^{-1} F_{3,4}\,, \quad N \to \lambda^{3/2} N\,, \quad G_1 \to \lambda G_1\,, \quad G_2 \to \lambda^{-1/2} G_2\,, \quad (3.9)$$

and

$$F_3 \to \mu^2 F_3\,, \quad F_4 \to \mu^{-2} F_4\,, \quad N \to \mu N\,, \quad G_2 \to \mu^{-1} G_2\,. \quad (3.10)$$

Both of these transformations are symmetries of the ansatz (3.1)–(3.3) provided we transform the coordinates appropriately. For the λ transformation that means that we must rescale the coordinates as

$$t \to \lambda^{-1} t\,, \qquad x \to \lambda^{1/2} x\,, \qquad y \to \lambda^{1/2} y\,, \quad (3.11)$$

while for the μ transformation it means that we must rescale the spatial coordinates as

$$x \to \mu^{-1} x\,, \qquad y \to \mu y\,. \quad (3.12)$$

Using Noether's theorem the associated charges are Q_λ and Q_μ, respectively, and given by

$$Q_\lambda = -2 \frac{\partial L}{\partial F_1'} F_1 + \frac{\partial L}{\partial F_3'} F_3 + \frac{\partial L}{\partial F_4'} F_4 - \frac{3}{2} \frac{\partial L}{\partial N'} N - \frac{\partial L}{\partial G_1'} G_1 + \frac{1}{2} \frac{\partial L}{\partial G_2'} G_2\,, \quad (3.13)$$

$$Q_\mu = -2 \frac{\partial L}{\partial F_3'} F_3 + 2 \frac{\partial L}{\partial F_4'} F_4 - \frac{\partial L}{\partial N'} N + \frac{\partial L}{\partial G_2'} G_2\,. \quad (3.14)$$

Using that L is given by (3.8) these charges can be shown to be equal to

$$Q_\lambda = -\frac{3}{2} r^{-1} Q_N N + r^{-4} (F_1 F_2 F_3 F_4)^{1/2} \left[-\frac{Z G_1}{F_1} \left(r G_1' - 2 G_1 \right) - 2 + r \frac{F_1'}{F_1} - \frac{r}{2} \frac{F_3'}{F_3} - \frac{r}{2} \frac{F_4'}{F_4} \right.$$
$$\left. -\frac{Z G_1 G_2 \left(r N' - N \right)}{F_1} - \frac{1}{2} \frac{Z G_2 \left(r G_2' - G_2 \right)}{F_4} \right]\,, \quad (3.15)$$

and

$$Q_\mu = -r^{-1} Q_N N + r^{-4} (F_1 F_2 F_3 F_4)^{1/2} \left[r \frac{F_3'}{F_3} - r \frac{F_4'}{F_4} - \frac{Z G_2 \left(r G_2' - G_2 \right)}{F_4} \right]\,, \quad (3.16)$$

where we defined the charge Q_N

$$Q_N = \frac{\partial L}{\partial N'} = r^{-3} (F_1 F_2 F_3 F_4)^{1/2} \left[\frac{F_4 \left(r N' - N \right)}{F_1} + \frac{Z G_2^2 \left(r N' - N \right)}{F_1} - 2 \frac{Z G_1 G_2}{F_1} \right.$$
$$\left. + \frac{Z G_2 \left(r G_1' - G_1 \right)}{F_1} \right]\,, \quad (3.17)$$

which results from the fact that L does not depend on N. The Noether charges Q_λ and Q_μ are first integrals of motion and thus independent of the radial coordinate r. This will play an important role later when we derive the thermodynamic properties. We will see that Q_λ relates to the energy and Q_μ to the linear momentum of the black brane.

The ansatz (3.1)–(3.3) has a third global scale symmetry namely

$$t \to \nu^{-1}t\,, \qquad F_1 \to \nu^2 F_1\,, \qquad N \to \nu N\,, \qquad G_1 \to \nu G_1\,. \qquad (3.18)$$

However this transformation does not leave the effective Lagrangian (3.8) invariant because it is not a symmetry of the prefactor. On top of the 3 global symmetries whose parameters are λ, μ and ν the ansatz also has one local symmetry which corresponds to r-reparametrization invariance. This symmetry acts as

$$\delta F_2 = \xi^r F_2' + 2F_2 \left(r^{-1}\xi^r - \partial_r \xi^r \right)\,, \qquad \delta A_I = \xi^r A_I'\,, \qquad (3.19)$$

where A_I is any of the functions appearing in the ansatz that is not F_2 and ξ^r is the local parameter generating the r-reparametrization. This local symmetry can be fixed by choosing a gauge. This local symmetry implies that using the F_2 equation of motion (which is first order and needs to be differentiated with respect to r) and any 6 of the other A_I equations of motion the remaining 7th A_I equation of motion can be derived.

3.2 The asymptotic solution

The 4-dimensional near-boundary expansion follows by dimensional reduction using the reduction ansatz (2.17) and (2.18) as well as the 5-dimensional Fefferman-Graham (FG) expansion (B.4) and (B.5), the details of which are given in appendix B.1.

The ansatz for the black branes with linear momentum are such that all 4-dimensional fields only depend on the radial coordinate r. From the 5-dimensional FG expansion point of view that implies that all sources and vevs must be constants. The only exception to this is of course the fact that ψ is allowed to be linear in the reduction circle coordinate u because we are performing a Scherk-Schwarz reduction. That means that our ansatz forces us to consider a FG expansion in 5D with the following sources and vevs

$$\gamma_{(0)AB} = \text{cst}\,, \qquad \text{with } \gamma_{(0)uu} = 0\,, \qquad (3.20)$$

$$t_{AB} = \text{cst}\,, \qquad (3.21)$$

$$\psi_{(0)} = 2u\,, \qquad \langle O_\psi \rangle = 0\,, \qquad \text{so that } \Xi = 0\,, \qquad (3.22)$$

where t_{AB} obeys the Ward identities (B.28) and (B.29). Setting $\langle O_\psi \rangle = 0$ is a consequence of the Ward identity $\nabla_{(0)A} t^A{}_B = -\langle O_\psi \rangle \partial_B \psi$ for $B = u$ and constant $t^A{}_B$. Since the field Ξ always appears differentiated it makes no difference if we set it equal to zero or equal to some constant. The choice $\gamma_{(0)uu} = 0$ is rather important and is necessary in order that the lower-dimensional theory has a $z = 2$ scaling exponent. This is explained in detail in [25, 36]. It is shown in section 2 of [25] that the reduction in the bulk is everywhere along a spacelike circle (due to $\psi_{(0)} = 2u$) but that this circle is null on the boundary.[8]

It is well known that reductions along null Killing directions turn a Riemannian geometry into a torsional Newton-Cartan (TNC) geometry [25, 37–39]. For details see appendices A.1 and B.4. In particular see the reduction ansatz for the AdS$_5$ boundary

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[8]Here we use a model that is simpler than the one used in [25] but regarding this point the properties are identical.

metric (A.1). In the language of TNC geometry the uu component of the inverse metric is called $\tilde{\Phi}$ which is defined in (A.8) with m_μ the Kaluza-Klein vector associated with the null reduction as given in (A.1). In appendix B.4 it is shown that m_μ combines with χ the source of the bulk scalar Ξ into $M_\mu = m_\mu - \partial_\mu \chi$. Since here we have $\chi = 0$ we can take $M_\mu = m_\mu$. From the inverse metric we know that[9]

$$\gamma^{uu}_{(0)} = 2\tilde{\Phi} \,. \tag{3.23}$$

We will be interested in flat boundaries of the 4-dimensional $z = 2$ Lifshitz space-time. A flat space-time in TNC language means that there exists a coordinate system in which we have [28]

$$\tau_\mu = \delta^t_\mu \,, \qquad M_\mu = 0 \,, \qquad h_{tt} = h_{ti} = 0 \,, \qquad h_{ij} = \delta_{ij} \,. \tag{3.24}$$

This means in particular that $\tilde{\Phi} = 0$. Turning on $\tilde{\Phi}$ corresponds to turning on a Newtonian potential for the boundary theory [28, 40]. We will thus not consider this possibility.

The expansion of the 4-dimensional fields follows from (2.17) and (2.17) which imply that[10]

$$e^{2\Phi} = \gamma_{uu} \,, \tag{3.25}$$

$$A_\mu = \frac{\gamma_{u\mu}}{\gamma_{uu}} \,, \tag{3.26}$$

$$h_{\mu\nu} = (\gamma_{uu})^{1/2} \left(\gamma_{\mu\nu} - \frac{\gamma_{u\mu}\gamma_{u\nu}}{\gamma_{uu}} \right) \,, \tag{3.27}$$

where γ_{AB} is FG expanded using the results of appendix B.1. In order to carry out this reduction we need to know how to reduce the AdS boundary energy-momentum tensor into the language of the energy-momentum tensor of the TNC boundary of the lower-dimensional Lifshitz space-time. The relation between a relativistic energy-momentum tensor t_{MN} and the TNC energy-momentum tensor related via null reduction is explained in appendix A.2 where we derive the following relations

$$t_{uu} = \rho \,, \tag{3.28}$$

$$t_{u\mu} = \tau_\rho T^\rho{}_\mu \,, \tag{3.29}$$

$$t_{\mu\nu} = \hat{h}_{\mu\rho}\hat{h}_{\nu\kappa}h^{\kappa\sigma}T^\rho{}_\sigma - \left(\tau_\nu \hat{h}_{\mu\rho} + \tau_\mu \hat{h}_{\nu\rho} \right) \hat{v}^\sigma T^\rho{}_\sigma + (\hat{v}^\rho\hat{v}^\sigma t_{\rho\sigma}) \tau_\mu\tau_\nu \,, \tag{3.30}$$

where

$$\hat{v}^\rho\hat{v}^\sigma t_{\rho\sigma} = t^{uu} - 4\tilde{\Phi}^2\rho + 4\tilde{\Phi}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma \,. \tag{3.31}$$

Recall that here $\tilde{\Phi} = 0$. The TNC energy-momentum tensor is denoted by $T^\mu{}_\nu$ and the TNC mass density is denoted by ρ. We note that t^{uu} has no lower dimensional interpretation in terms of energy-momentum or mass density. As shown in appendix A.1 it does not appear in any of the Ward identities involving $T^\mu{}_\nu$ and ρ. Hence we will set t^{uu} equal to zero. It

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[9]We warn the reader that the boundary background field $\tilde{\Phi}$ should not be confused with the bulk scalar field Φ.

[10]We warn the reader that we use $h_{\mu\nu}$ both to denote the $\mu\nu$ component of the bulk metric as well as the spatial metric-like quantity (A.2) on the boundary. We hope that this will not cause any confusion.

would appear in 4 dimensions for the first time at order r^2 in that part of the expansion of $h_{\mu\nu}$ that is proportional to $\tau_\mu \tau_\nu$. We refer to [25] for more discussion on the role of t^{uu}.

In order to find out where the momentum flux, the spatial projection of $\tau_\rho T^\rho{}_\mu$, which is one of the quantities of interest, the spatial stress tensor etc. appear upon reduction we need to know what happens with $t_{\mu\nu}$ upon reduction. Clearly in 5 bulk dimensions $t_{\mu\nu}$ appears in $\gamma_{\mu\nu}$ at order r^2. Therefore in order to see it in four bulk dimensions we need to expand A_μ and $h_{\mu\nu}$ up to order r^2. This follows from (3.26) and (3.27) and implies that we need to expand γ_{uu} to order r^6, $\gamma_{u\mu}$ to order r^4 and $\gamma_{\mu\nu}$ to order r^2.

We will now proceed to construct the 5-dimensional solution up to the required order. Since the sources and vevs are constants (with $\psi_{(0)}$ linear in u) we have that $\Box \psi = 0$ implies

$$\sqrt{-\mathcal{G}} r^2 \partial_r \psi = C \,, \tag{3.32}$$

where C is an integration constant. Using

$$\sqrt{-\mathcal{G}} = 1 + O\left(r^6\right) \,, \tag{3.33}$$

it follows that

$$r^{-3} \partial_r \psi = C \left(1 + O\left(r^6\right)\right) \,, \tag{3.34}$$

so that using $\psi_{(4)} = 0$ implies that $C = 0$. Hence $\partial_r \psi = 0$, or in other words $\psi = 2u$ to all orders. With this result the Einstein equation simplifies to

$$G_{\mathcal{M}\mathcal{N}} = 6\mathcal{G}_{\mathcal{M}\mathcal{N}} + 2\delta^u_{\mathcal{M}} \delta^u_{\mathcal{N}} - \mathcal{G}_{\mathcal{M}\mathcal{N}} \gamma^{uu} \,, \tag{3.35}$$

which is equivalent to

$$R_{\mathcal{M}\mathcal{N}} = -4\mathcal{G}_{\mathcal{M}\mathcal{N}} + 2\delta^u_{\mathcal{M}} \delta^u_{\mathcal{N}} \,. \tag{3.36}$$

To find the solution up to order r^6 we make the following ansatz

$$\gamma_{AB} = r^{-2} \left(\gamma_{(0)AB} + r^2 \delta^u_A \delta^u_B - \frac{1}{4} r^4 t_{AB} + r^6 \gamma_{(6)AB} + r^8 \gamma_{(8)AB} + O\left(r^{10}\right) \right) \,. \tag{3.37}$$

The log terms at order $r^2 \log r$ are zero and so it is expected that they are zero to all orders. This is a correct assumption as long as we do not need to put constraints on the sources and vevs coming from the nature of the expansion. The inverse metric reads

$$\gamma^{AB} = r^2 \left(\gamma_{(0)}^{AB} - r^2 \gamma_{(0)}^{Au} \gamma_{(0)}^{Bu} + \frac{1}{4} r^4 t^{AB} + r^6 \sigma_{(6)}^{AB} + r^8 \sigma_{(8)}^{AB} + O\left(r^{10}\right) \right) \,, \tag{3.38}$$

where

$$\sigma_{(6)}^{AB} = -\gamma_{(6)}^{AB} - \frac{1}{4} \gamma_{(0)}^{Au} t^{uB} - \frac{1}{4} \gamma_{(0)}^{Bu} t^{uA} \,, \tag{3.39}$$

$$\sigma_{(8)}^{AB} = -\gamma_{(8)}^{AB} + \gamma_{(0)}^{Au} \gamma_{(6)}^{uB} + \gamma_{(0)}^{Bu} \gamma_{(6)}^{uA} + \frac{1}{16} t^{AC} t_C{}^B + \frac{1}{4} \gamma_{(0)}^{Au} \gamma_{(0)}^{Bu} t^{uu} \,. \tag{3.40}$$

The Christoffel symbols are

$$\Gamma^r_{rr} = -\frac{1}{r} \,, \qquad \Gamma^r_{rA} = 0 \,, \qquad \Gamma^r_{AB} = -\frac{1}{2} r^2 \partial_r \gamma_{AB} \,,$$

$$\Gamma^A_{rr} = 0 \,, \qquad \Gamma^A_{rB} = \frac{1}{2} \gamma^{AC} \partial_r \gamma_{BC} \,, \qquad \Gamma^A_{BC} = 0 \,. \tag{3.41}$$

From this we conclude that

$$R_{rr} = -4r^{-2} + r^4 \left(-12\gamma_{(6)A}^A - 2t^{uu} \right) + r^6 \left(-24\gamma_{(8)A}^A + 18\gamma_{(6)}^{uu} + \frac{1}{2}t^{AB}t_{AB} \right) + O\left(r^8\right).$$
(3.42)

The rr component of (3.36) tells us that $R_{rr} = -4r^{-2}$ so that

$$\gamma_{(6)A}^A = -\frac{1}{6}t^{uu}, \qquad \gamma_{(8)A}^A = \frac{3}{4}\gamma_{(6)}^{uu} + \frac{1}{48}t^{AB}t_{AB}.$$
(3.43)

The rA component of (3.36) brings nothing as both sides are identically zero. Using that

$$R_{AB} = -4r^{-2}\gamma_{(0)AB} - 2\delta_A^u\delta_B^u + r^2t_{AB} + r^4 \left(\frac{1}{4}t^{uu}\gamma_{(0)AB} - 10\gamma_{(6)AB} - \frac{1}{2}\delta_A^u t^u{}_B - \frac{1}{2}\delta_B^u t^u{}_A \right)$$

$$+r^6 \left(-20\gamma_{(8)AB} - \gamma_{(6)}^{uu}\gamma_{(0)AB} - \frac{1}{24}t^{CD}t_{CD}\gamma_{(0)AB} + 6\delta_A^u\gamma_{(6)B}^u + 6\delta_B^u\gamma_{(6)A}^u \right.$$

$$\left. +\frac{1}{2}t_A{}^C t_{CB} + \frac{1}{2}\delta_A^u\delta_B^u t^{uu} \right) + O\left(r^8\right),$$
(3.44)

as well as the equation of motion $R_{AB} = -4\gamma_{AB} + 2\delta_A^u\delta_B^u$, we find that

$$\gamma_{(6)AB} = -\frac{1}{6}\delta_A^u t^u{}_B - \frac{1}{6}\delta_B^u t^u{}_A + \frac{1}{24}t^{uu}\gamma_{(0)AB},$$
(3.45)

$$\gamma_{(8)AB} = -\frac{1}{16}t^{uu}\delta_A^u\delta_B^u - \frac{1}{384}t^{CD}t_{CD}\gamma_{(0)AB} + \frac{1}{32}t_A{}^C t_{BC}.$$
(3.46)

From the reduction (3.25)–(3.27) it follows that

$$\Phi = -\frac{1}{8}r^2\rho + r^4 \left(\frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma - \frac{1}{64}\rho^2 \right) + O\left(r^6\right),$$
(3.47)

$$A_\mu = r^{-2}\tau_\mu + \frac{1}{4}\rho\tau_\mu + r^2 \left(\frac{1}{12}\tau_\rho T^\rho{}_\mu + \frac{1}{16}\rho^2\tau_\mu - \frac{1}{3}\bar{h}_{\mu\rho}T^\rho \right) + O\left(r^4\right),$$
(3.48)

$$h_{\mu\nu} = -r^{-4}\tau_\mu\tau_\nu + r^{-2} \left(\bar{h}_{\mu\nu} - \frac{1}{8}\rho\tau_\mu\tau_\nu \right) - \frac{1}{8}\rho\bar{h}_{\mu\nu} + \frac{1}{4} \left(\tau_\mu\tau_\rho T^\rho{}_\nu + \tau_\nu\tau_\rho T^\rho{}_\mu \right)$$

$$-\left(\frac{3}{128}\rho^2 - \frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma \right) \tau_\mu\tau_\nu + r^2 \left(-\frac{1}{4}\hat{h}_{\mu\rho}\hat{h}_{\nu\kappa}h^{\kappa\sigma}T^\rho{}_\sigma + \frac{1}{12} \left(\tau_\nu\hat{h}_{\mu\rho} + \tau_\mu\hat{h}_{\nu\rho} \right) \hat{v}^\sigma T^\rho{}_\sigma \right.$$

$$+\left(\frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma - \frac{1}{128}\rho^2 \right) \bar{h}_{\mu\nu} - \frac{1}{16}\rho \left(\tau_\mu\tau_\rho T^\rho{}_\nu + \tau_\nu\tau_\rho T^\rho{}_\mu \right)$$

$$\left. +\left(\frac{3}{64}\rho\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma + \frac{1}{64}T^\sigma\tau_\rho T^\rho{}_\sigma - \frac{5}{1024}\rho^3 - \frac{1}{32}t^{uu} \right) \tau_\mu\tau_\nu \right) + O\left(r^4\right).$$
(3.49)

For the interested reader we have included the term t^{uu}. But, as remarked earlier, we will set this independent quantity equal to zero. If we choose the boundary sources to

correspond to a flat TNC boundary as in (3.24) then the expansions become

$$\Phi = -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + O(r^6)\,, \tag{3.50}$$

$$A_t = r^{-2} + \frac{1}{4}\rho + r^2\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + O(r^4)\,, \tag{3.51}$$

$$A_i = -\frac{1}{4}r^2 T^t{}_i + O(r^4)\,, \tag{3.52}$$

$$h_{tt} = -r^{-4} - \frac{1}{8}r^{-2}\rho + \frac{1}{3}T^t{}_t - \frac{3}{128}\rho^2 + O(r^2)\,, \tag{3.53}$$

$$h_{ti} = \frac{1}{4}T^t{}_i + r^2\left(-\frac{1}{12}\delta_{ij}T^j{}_t + \frac{1}{32}\rho T^t{}_i\right) + O(r^4)\,, \tag{3.54}$$

$$h_{ij} = r^{-2}\delta_{ij} - \frac{1}{8}\rho\delta_{ij} + r^2\left(\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2\right)\delta_{ij} - \frac{1}{4}\delta_{ik}T^k{}_j + \frac{1}{8}T^k{}_k\delta_{ij}\right) + O(r^4)\,, \tag{3.55}$$

where in the last expression we used the z-deformed trace Ward identity (equation (A.44) with zero on the right hand side)

$$2T^t{}_t + T^k{}_k = 0\,. \tag{3.56}$$

In this work we are interested in gravitational duals of boundary perfect fluids so without loss of generality we can assume that $T^\mu{}_\nu$ takes the form of a perfect fluid. This form is derived in appendix A.3 by the null reduction of a relativistic perfect fluid. On flat TNC space-time it reads

$$T^t{}_t = -\left(\mathcal{E} + \frac{1}{2}\rho V^2\right)\,, \qquad\qquad T^i{}_t = -\left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^i\,, \tag{3.57}$$

$$T^t{}_i = \rho V_i\,, \qquad\qquad T^j{}_i = \left(P\delta^j_i + \rho V^j V_i\right)\,, \tag{3.58}$$

where $\mathcal{E}$ is the energy density, P the pressure, ρ the mass density and V^i the velocity of the fluid. The z-deformed trace Ward identity tells us that the equation of state is $P = \mathcal{E}$.

It is interesting and insightful to take a closer at look at this V-dependent solution from the 5-dimensional point of view. Using the relations between the lower and higher-dimensional energy-momentum tensors (3.28)–(3.30) we see that the 5-dimensional energy-momentum is given by

$$t_{uu} = \rho\,, \qquad\qquad t_{ut} = -\mathcal{E} - \frac{1}{2}\rho V^2\,, \qquad t_{ui} = \rho V_i\,,$$

$$t_{ti} = -\left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V_i\,, \qquad t_{ij} = P\delta_{ij} + \rho V_i V_j\,, \tag{3.59}$$

with t_{tt} being undetermined. A convenient way of writing this is in terms of $t_{AB}dx^A dx^B$, which can be seen to be equal to

$$t_{AB}dx^A dx^B = \rho\left(du + V_i dx^i - \frac{1}{2}V^2 dt\right)^2 - 2\mathcal{E}dt\left(du + V_i dx^i - \frac{1}{2}V^2 dt\right) \tag{3.60}$$

$$+ P\delta_{ij}\left(dx^i - V^i dt\right)\left(dx^j - V^j dt\right) + \left(t_{tt} - \left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^2\right)dt^2\,.$$

The rest of the solution is fully determined by the following boundary data

$$\gamma_{(0)AB}dx^A dx^B = 2dt du + \delta_{ij}dx^i dx^j \,, \tag{3.61}$$

$$\psi_{(0)} = 2u \,, \qquad \langle O_\psi \rangle = 0 \,. \tag{3.62}$$

If we now perform the following coordinate transformation, which from a lower dimensional point of view is a Galilean boost and a U(1) gauge transformation (acting on the Kaluza-Klein vector m_μ),

$$u = u' - \frac{1}{2}V^2 t' - V_i x'^i \,, \qquad t = t' \,, \qquad x^i = x'^i + V^i t' \,, \tag{3.63}$$

we obtain

$$\begin{aligned} t_{AB}dx^A dx^B = {} & \rho du'^2 - 2\mathcal{E}dt' du' + P\delta_{ij}dx'^i dx'^j \\ & + \left(t_{tt} - \left(\mathcal{E} + P + \frac{1}{2}\rho V^2 \right) V^2 \right) dt'^2 \,, \end{aligned} \tag{3.64}$$

$$\gamma_{(0)AB}dx^A dx^B = 2dt' du' + \delta_{ij}dx'^i dx'^j \,, \tag{3.65}$$

$$\psi_{(0)} = 2u' - V^2 t' - 2V_i x'^i \,, \tag{3.66}$$

$$\langle O_\psi \rangle = 0 \,. \tag{3.67}$$

We thus see that the boundary metric $\gamma_{(0)AB}$ remained invariant and that all the V-dependence now resides in the expression for $\psi_{(0)}$. The $t't'$ component of t_{AB} is not important for the lower dimensional boundary energy-momentum tensor and its Ward identities. It is thus clear that due to the presence of ψ, and the Scherk-Schwarz reduction ansatz $\psi = 2u + \Xi$, solutions with different V^i are not diffeomorphic. We will later see this reflected in the fact that V^2 plays the role of a chemical potential. The ansatz in section 3.1 used rotations to orient the flow in the y-directions. We will see further below that indeed $V^x = 0$.

3.3 The near-horizon solution

The near-horizon expansion is entirely straightforward. Referring to the ansatz (3.6) and (3.7) in EF coordinates we can make the following observations about the behavior of the solution near the horizon.

The horizon is located at the locus where the $r = \mathrm{cst}$ hypersurface becomes null, i.e. at $g^{rr} = 0$. That means that F_2 will have a first order zero at $r = r_h$. Regularity of the metric in EF coordinates, in particular of the component g_{Vr} then tells us that F_1 must also have a first order zero at $r = r_h$. Note that for $N \neq 0$ this is not the locus where ∂_t becomes null. In other words the stationary limit surface $g_{tt} = 0$ comes before the horizon (viewed from outside). Regularity of the massive vector at the horizon forces G_1 to have a first order zero at $r = r_h$. The functions F_4 and N are both regular without any zeros at the horizon, i.e. $F_4(r_h) \neq 0$ and $N(r_h) \neq 0$. The latter quantity can be zero but as we will see in the next subsection that corresponds to a brane without any momentum so we take it to be non-zero. The remaining functions G_2 and Φ are regular at the horizon, but they do not have to be non-vanishing.

A convenient gauge choice to fix the r reparametrization invariance of the ansatz to study the near-horizon horizon solution is to take $F_3 = 1$. In this gauge we will refer to the radial coordinate as R to distinguish it from the radial coordinate r used in the previous subsection.[11] The horizon is now located at $R = R_h$.

The ansatz also has three global scale symmetries (3.11), (3.12) and (3.18) that leave the ansatz invariant. These can be viewed as rescalings of x, y and t. We have used these symmetries to set the asymptotic values of F_1, F_3 and F_4 equal to one. This fixes the asymptotic values of Φ and thus of F_2 (via the asymptotic gauge choice $F_2 = e^{-\Phi}$) as well as of G_1 via the equations of motion. That means that we cannot use these rescaling symmetries a second time to fix parameters in the near-horizon solution. We thus take for the near-horizon solution the following expansion

$$F_1 = f_1 \frac{R - R_h}{R_h} + \dots , \tag{3.68}$$

$$F_2 = h_1 \frac{R - R_h}{R_h} + \dots , \tag{3.69}$$

$$F_3 = 1 , \tag{3.70}$$

$$F_4 = p_0 + p_1 \frac{R - R_h}{R_h} + \dots , \tag{3.71}$$

$$N = n_0 + n_1 \frac{R - R_h}{R_h} + \dots , \tag{3.72}$$

$$G_1 = g_1 \frac{R - R_h}{R_h} + \dots , \tag{3.73}$$

$$G_2 = m_0 + m_1 \frac{R - R_h}{R_h} + \dots , \tag{3.74}$$

$$\Phi = l_0 + l_1 \frac{R - R_h}{R_h} + \dots . \tag{3.75}$$

Most but not all of the coefficients appearing in the near-horizon expansion will be determined by solving the equations of motion of the effective action L in an expansion around $R = R_h$. We studied the solution up to second order in $R - R_h$ and it leaves 8 parameters unfixed. These are f_1, p_0, g_1, m_0, n_0, n_1, l_0 and r_h. The parameter h_1 is fixed by the equations of motion to be[12]

$$h_1 = \frac{2f_1 \left(2e^{-3l_0} - 12e^{-l_0}\right)}{4f_1 - e^{3l_0} \left(g_1 + m_0 n_1\right)^2} , \tag{3.76}$$

where the numerator is $2f_1$ times the potential (2.15) evaluated at $R = R_h$. We expect that most of these parameters will be determined by matching the solution onto the asymptotic region.

There are not many examples known of analytic black brane solutions of the EPD model. However in the context of Schrödinger space-times we can obtain analytic solutions

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[11]We permit ourselves to also use r for the family of gauges parametrized by the ansatz (3.6) and (3.7). We hope that this will not cause any confusion.

[12]To find this result one solves the leading term of the F_1 equation of motion for p_1 and the leading term of the F_3 equation of motion for f_2. The expression then follows from the leading term in the F_4 equation of motion. A similar expression has been observed in [33].

by applying a sequence of duality transformations known as TsT transformations [17] to obtain black brane solutions from known AdS black branes [16, 18, 41]. The resulting Schrödinger black branes have a non-zero charge associated with particle number. Since in Schrödinger holography particle number is realized geometrically this means that these correspond to black branes with a linear momentum along a direction that asymptotically becomes null. If we study these black branes near the horizon in the same coordinates in which the AdS black brane has a flat boundary Minkowski metric written in Cartesian coordinates then we see the exact same near-horizon boundary conditions as we imposed for our Lifshitz black brane.[13]

3.4 Comments on the interpolating solution

We have used different radial gauges in the near-horizon region ($F_3 = 1$) and in the asymptotic region ($F_2 = e^{-\Phi}$). The two coordinates are related via the coordinate transformation

$$h_{xx} = R^{-2} \,, \tag{3.77}$$

where h_{xx} is given in (3.55). In order to write both the near-horizon and the near-boundary expansion in the same gauge it is convenient to rewrite the expansions (3.50)–(3.55) in terms of the radial coordinate R. This can be done as follows. The expansions (3.50)–(3.55) in terms of the ansatz functions correspond to

$$F_1 = 1 + \frac{1}{8}r^2\rho + r^4\left(-\frac{1}{3}T^t{}_t + \frac{3}{128}\rho^2\right) + O(r^6) \,, \tag{3.78}$$

$$F_2 = e^{-\Phi} = 1 + \frac{1}{8}r^2\rho + r^4\left(\frac{1}{6}T^t{}_t + \frac{3}{128}\rho^2\right) + O(r^6) \,, \tag{3.79}$$

$$F_3 = 1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 + \frac{1}{8}\rho V^2\right) + O(r^6) \,, \tag{3.80}$$

$$F_4 = 1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 - \frac{1}{8}\rho V^2\right) + O(r^6) \,, \tag{3.81}$$

$$N = \frac{1}{4}r^3\rho V + O(r^5) \,, \tag{3.82}$$

$$G_1 = 1 + \frac{1}{4}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + O(r^6) \,, \tag{3.83}$$

$$G_2 = -\frac{1}{4}r^3\rho V + O(r^5) \,, \tag{3.84}$$

$$\Phi = -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + O(r^6) \,, \tag{3.85}$$

where we remind that $V = V^y$ and $V^x = 0$. The change of gauge (3.77) implies that we define R asymptotically as

$$R^{-2} = r^{-2}\left(1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 + \frac{1}{8}\rho V^2\right) + O(r^6)\right). \tag{3.86}$$

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[13]More explicitly if we use equation (62) of [41] setting $\xi = V$ the TsT transformation (113)–(115) provides us with a $z = 2$ Schrödinger black brane solution of some EPD model. If we then perform the coordinate transformation $t = T - X$ and $2\xi = 2V = T + X$ we find that the near-horizon geometry has exactly the same properties as the Lifshitz black brane solution studied here.

We can invert this order by order to obtain $r = r(R)$ up to any desired power of R. Inverting (3.86) up to order R^6 we find

$$r = R \left(1 - \frac{1}{16} R^2 \rho + \frac{1}{8} R^4 \left(\frac{1}{3} T^t{}_t + \frac{1}{64} \rho^2 + \frac{1}{2} \rho V^2 \right) + O(R^6) \right) . \qquad (3.87)$$

This can be used to express (3.1)–(3.3) with the above expansions for the various functions as an asymptotic solution that is written in terms of the same radial coordinate R as the near-horizon solution. If we carry out these steps we obtain the following expressions for the ansatz functions in the new gauge

$$F_1 = 1 + \frac{3}{8} R^2 \rho + \frac{1}{2} R^4 \left(-T^t{}_t + \frac{9}{64} \rho^2 - \frac{1}{2} \rho V^2 \right) + O(R^6) , \qquad (3.88)$$

$$F_2 = 1 + \frac{3}{8} R^2 \rho + R^4 \left(-\frac{1}{6} T^t{}_t + \frac{11}{128} \rho^2 - \frac{1}{2} \rho V^2 \right) + O(R^6) , \qquad (3.89)$$

$$F_3 = 1 , \qquad (3.90)$$

$$F_4 = 1 - \frac{1}{4} R^4 \rho V^2 + O(R^6) , \qquad (3.91)$$

$$N = \frac{1}{4} R^3 \rho V + O(R^5) , \qquad (3.92)$$

$$G_1 = 1 + \frac{3}{8} R^2 \rho + \frac{1}{2} R^4 \left(\frac{9}{64} \rho^2 - \frac{1}{4} \rho V^2 \right) + O(R^6) , \qquad (3.93)$$

$$G_2 = -\frac{1}{4} R^3 \rho V + O(R^5) , \qquad (3.94)$$

$$\Phi = -\frac{1}{8} R^2 \rho - \frac{1}{6} R^4 T^t{}_t + O(R^6) . \qquad (3.95)$$

In order to find an interpolating solution we thus need to solve the equations of motion of (3.8) in the $F_3 = 1$ gauge such that near the horizon the solution looks like (3.68)–(3.75) while near the boundary it looks like (3.88)–(3.95). It would be interesting to study the interpolating solution numerically. For the purposes of this work we do not need this explicit solution, but we will need to assume that it exists.

We also see from the asymptotic solution that even for $V = 0$ we still can turn on the ρ deformation. Hence static Lifshitz black branes can have a non-zero mass density. Furthermore, even though the full non-linear solution breaks rotational symmetries the near-boundary solution has an asymptotic Killing vector for rotations. Hence rotations are spontaneously broken.

3.5 Thermodynamics

The most general Killing vector that (3.1) admits is of the form

$$K^M = (\partial_t)^M + A_1 (\partial_x)^M + A_2 (\partial_y)^M , \qquad (3.96)$$

where A_1 and A_2 are constants. The norm is given by

$$||K||^2 = -\frac{F_1}{r^4} + \frac{F_4}{r^2} \left(\frac{N}{r} + A_2 \right)^2 + A_1^2 \frac{F_3}{r^2} . \qquad (3.97)$$

In order to find the generator of the horizon we demand that $||K||^2$ vanishes at $R = R_h$ which will be the case if and only if

$$A_1 = 0\,, \qquad A_2 = -\frac{N(R_h)}{R_h}\,. \tag{3.98}$$

Hence the horizon generator which we will denote by X^M is given by

$$X^M = (\partial_t)^M - \frac{N(R_h)}{R_h}\,(\partial_y)^M\,. \tag{3.99}$$

We thus see that there is a chemical potential $-N(R_h)/R_h$ associated with the motion in the y-direction.

The metric and vector field expanded near the horizon read

$$ds_4^2 = -\tilde\rho^2 d\tilde t^2 + d\tilde\rho^2 + \frac{1}{R_h^2}dx^2 + \frac{p_0}{R_h^2}\left(dy + \frac{N(R_h)}{R_h}dt\right)^2\,, \tag{3.100}$$

$$B = \frac{1}{2}g_1\left(h_1 f_1^{-1}\right)^{1/2}\tilde\rho^2 d\tilde t + \frac{m_0}{R_h}\left(dy + \frac{N(R_h)}{R_h}dt\right)\,, \tag{3.101}$$

where we defined

$$\tilde\rho = 2\left(\frac{R - R_h}{h_1 R_h}\right)^{1/2}\,, \tag{3.102}$$

$$\tilde t = \frac{1}{2}\left(f_1 h_1\right)^{1/2} R_h^{-2} t\,. \tag{3.103}$$

We next ask which linear functions $f(t,x,y)$ solve the equation $X^M \partial_M f = 0$. These are x and $y + \frac{N(R_h)}{R_h}t$. The metric induced on the common intersection of the hyperplanes $x = \text{cst}$ and $y + \frac{N(R_h)}{R_h}t = \text{cst}$, after Wick rotating the time coordinate $t = -it_E$, is called the bolt and is given by

$$ds^2|_{\text{bolt}} = \frac{F_1}{R^4}dt_E^2 + \frac{dR^2}{F_2 R^2}\,. \tag{3.104}$$

We expand this metric around $R = R_h$ with a periodic t_E demanding the absence of conical singularities. Because we are on the hyperplane $y + \frac{N(R_h)}{R_h}t = \text{cst}$ this forces us to also Wick rotate $y = -iy_E$ and make it periodic as well in agreement with the interpretation of $-N(R_h)/R_h$ as a chemical potential. The inverse temperature is the periodicity of t_E. The temperature and entropy density are given by

$$T = \frac{1}{4\pi R_h^2}\left(f_1 h_1\right)^{1/2}\,, \tag{3.105}$$

$$s = 4\pi\frac{(p_0)^{1/2}}{R_h^2}\,, \tag{3.106}$$

where we used units in which $16\pi G_N = 1$.

In the Wick rotated geometry t_E and y_E are periodic. The thermal cycle parametrized by t_E is contractible while the cycle parametrized by $y_E + \frac{N(R_h)}{R_h}t_E$ is non-contractible.

Hence we can compute $\oint_{R=R_h} B$ where we integrate along the cycle parametrized by $y_E + \frac{N(R_h)}{R_h} t_E$. The result is

$$\oint_{R=R_h} B = \frac{4\pi m_0 n_0}{(f_1 h_1)^{1/2}} \,. \tag{3.107}$$

In general using our ansatz the massive vector field can be written as

$$B^M = -R^2 \frac{G_1}{F_1} \left((\partial_t)^M - \frac{N}{R} (\partial_y)^M \right) + R \frac{G_2}{F_4} (\partial_y)^M \,. \tag{3.108}$$

It thus follows that for $m_0 = 0$ the massive vector field B^M is proportional to the horizon generator X^M at $R = R_h$.

It can be shown by using the near-horizon solution that the charges Q_λ and Q_μ (3.15) and (3.16) are such that

$$Q_\lambda - \frac{3}{2} Q_\mu = Ts \,, \tag{3.109}$$

with T, s given in (3.105), (3.106). Using the asymptotic form of the solution (3.50)–(3.55) with (3.57) and (3.58) to compute the left hand side of (3.109) we conclude that

$$\mathcal{E} + P = Ts + \frac{1}{2} \rho V^2 \,. \tag{3.110}$$

The equations of state follows from (3.56)

$$P = \mathcal{E} \,. \tag{3.111}$$

We have thus been able to derive the thermodynamic relations without knowing the full solution analytically using the Noether charges Q_λ and Q_μ. This is similar to what has been done in [33, 34]. We will see further below that we can also derive the first law of thermodynamics without having full analytic control of the solution. All that we need to know is the near-horizon expansion, the near-boundary expansion and the existence of an interpolating solution. We assume the latter to be the case. It would be interesting to provide numerical evidence for the interpolating solution.

3.6 Charges

The goal of this subsection is to find an alternative derivation of (3.110) which can be thought of as an integral form in terms of the renormalized on-shell action and certain horizon charges. The second goal is to find additional relations between near-boundary and near-horizon quantities. In particular we will show that the velocity $V^y = V$ is equal to the chemical potential $-N(R_h)/R_h$.

In order to define the black brane charges we use the boundary diffeomorphism Ward identity which on a flat TNC geometry reads (B.92). Given a boundary Killing vector K^μ in the sense that

$$\mathcal{L}_K \tau_\mu = 0 \,, \qquad \mathcal{L}_K \bar{h}_{\mu\nu} = 0 \,, \qquad \mathcal{L}_K \tilde{\Phi} = 0 \,, \tag{3.112}$$

it can be shown (see [28, 35]) that we find the conserved current

$$\partial_\mu \left(K^\nu T^\mu{}_\nu \right) = 0 \,. \tag{3.113}$$

The conserved charge associated with the boundary Killing vector K^μ is thus

$$Q_K = -\int_{t=\text{cst}} dxdy K^\nu T^t{}_\nu \,. \qquad (3.114)$$

For our case the integrand is independent of x and y and so it is better to consider the charge per unit boundary volume. We will often write $\int_{t=\text{cst}} dxdy$ as a formal integral that we never really perform. We can always divide the charges by it. We will assume that K^μ is the μ component of a bulk Killing vector K^M.

Using the definitions of the vevs in (B.48), (B.49) and (B.61), (B.62),[14] as well as the boundary energy-momentum tensor in (B.77) and (B.90) we find that

$$T^t{}_\nu = -\lim_{r\to 0} r^{-2} \left(T_{\mu\nu} E_0^\nu + \mathcal{T}^\rho E_\rho^0 B_\mu \right) \,. \qquad (3.115)$$

Using (B.41) and (B.43) we find that for purely radial solutions (no dependence on boundary coordinates)

$$T_{\mu\nu} E_0^\nu + \mathcal{T}^\rho E_\rho^0 B_\mu = \frac{1}{\sqrt{-h}} \mathcal{L}_{\text{bdry}}^{\text{os}} E_\mu^0 + 2 K_{\mu\nu} E_0^\nu + e^{3\Phi} n^M E_0^\nu F_{M\nu} B_\mu \,, \qquad (3.116)$$

where $\mathcal{L}_{\text{bdry}}^{\text{os}}$ is the on-shell value of the counterterm Lagrangian (B.33) including the Gibbons-Hawking boundary term, i.e.

$$\mathcal{L}_{\text{bdry}}^{\text{os}} = \sqrt{-h}\left(2K - 5e^{-\Phi/2} + e^{\Phi/2} B_\rho B^\rho \right) \,. \qquad (3.117)$$

The extrinsic curvature K is given by $K = h^{\mu\nu} K_{\mu\nu}$ where $K_{\mu\nu}$ is the $\mu\nu$ component of $K_{MN} = -\frac{1}{2}\mathcal{L}_n h_{MN} = \nabla_M n_N - n_M n^K \nabla_K n_N$ with the unit normal vector n_M given by $n_M = -(g^{rr})^{-1/2}\delta_M^r$. Since the Killing vector K^M is a boundary Killing vector we have $K^M n_M = 0$. Further we employ a radial gauge choice such that $E_0^M n_M = 0$. Using these results we can write

$$Q_K = \int_{t=\text{cst}} dxdy \lim_{r\to 0} r^{-2} \left(\frac{1}{\sqrt{-h}} \mathcal{L}_{\text{bdry}}^{\text{os}} K^M E_M^0 + n^M E^{N0} Z_{NM} \right) \,, \qquad (3.118)$$

where $Z_{NM} = -Z_{MN}$,[15] is given by

$$Z_{NM} = 2\nabla_N K_M + e^{3\Phi} F_{NM} K^P B_P \,. \qquad (3.119)$$

The integrand is over a $t = \text{cst}$ hypersurface. Its timelike unit normal is given by

$$u_M = U\delta_M^t \,, \qquad U = r^{-2}\left(1 + \frac{1}{16}r^2\rho + r^4\left(-\frac{1}{6}T^t{}_t + \frac{5}{2}\frac{1}{256}\rho^2 \right) + O(r^6) \right) \,, \qquad (3.120)$$

where we used the boundary expansions of section 3.2. It can be shown using these same expansions that

$$U^t = E^{t0} + O(r^8) \,, \qquad U^i = O(r^4) \,, \qquad (3.121)$$

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[14] The quantity $\alpha_{(0)}$ defined in (B.67) equals unity because for our solutions $\Phi = O(r^2)$ so that $\phi = 0$ as follows from (B.57).

[15] The antisymmetry follows from the fact that K_M is also assumed to be a bulk Killing vector.

where $E^{t0} = U^{-1} + O(r^8)$. We also have that $E^{i0} = O(r^4)$. Using these results together with the near-boundary expansion of Z_{NM} it can be proven that we can replace E^{N0} by u^N everywhere in the integrand of Q_K, i.e. we can write

$$Q_K = \int_{t=\text{cst}} dxdy \lim_{r \to 0} r^{-2} \left(\frac{1}{\sqrt{-h}} \mathcal{L}^{\text{os}}_{\text{bdry}} K^M u_M + n^M u^N Z_{NM} \right) . \tag{3.122}$$

Let us define the projector $P^N_M = \delta^N_M + u_M u^N$ which projects onto the $t = \text{cst}$ hypersurface whose metric we will denote by H_{IJ}, i.e. using the ADM decomposition we obtain

$$ds^2 = -U^2 dt^2 + H_{IJ} \left(dx^I + u^I dt \right) \left(dx^J + u^J dt \right) . \tag{3.123}$$

Let us furthermore define $Z^M = u^N Z_{NM}$. We can derive the following identity

$$P^N_M \nabla_N Z^M = \frac{1}{\sqrt{-H}} \partial_I \left(\sqrt{H} Z^I \right) . \tag{3.124}$$

Hence it follows that

$$\int_{t=\text{cst}} d^3x \sqrt{H} P^N_M \nabla_N Z^M = \int_{t=\text{cst}} dxdy \int_\epsilon^{R_h} dR \partial_R \left(\sqrt{H} Z^R \right) , \tag{3.125}$$

where we used the radial R coordinate of section 3.4, i.e. the $F_3 = 1$ gauge, with a cut-off boundary at $R = \epsilon$ and the horizon at $R = R_h$. The integration measure in terms of the ansatz functions can be written as

$$\sqrt{H} = r^{-3} F_2^{-1/2} F_4^{1/2} . \tag{3.126}$$

It follows that

$$\int_{t=\text{cst}} d^3x \sqrt{H} P^N_M \nabla_N Z^M = \int_{t=\text{cst}} dxdy R^{-2} n_M Z^M|_{R=\epsilon}$$
$$+ \int_{t=\text{cst}} dxdy R^{-3} F_2^{-1/2} F_4^{1/2} Z^R|_{R=R_h} . \tag{3.127}$$

We conclude that the charge Q_K can be written as

$$Q_K = \int_{t=\text{cst}} dxdy R^{-2} \frac{1}{\sqrt{-h}} \mathcal{L}^{\text{os}}_{\text{bdry}} K^M U_M|_{R=\epsilon} + \int_{t=\text{cst}} dxdy \int_\epsilon^{R_h} dR \sqrt{H} P^N_M \nabla_N Z^M$$
$$+ \int_{t=\text{cst}} dxdy R^{-2} F_4^{1/2} X^N Y^P Z_{NP}|_{R=R_h} , \tag{3.128}$$

where we send ϵ to zero. In the horizon integral X^P is the horizon generator (3.99) and Y^N is given by (in the EF coordinates of (3.6) with $F_3 = 1$)

$$Y^N = R^3 \left(\frac{F_2}{F_1} \right)^{1/2} \delta^N_R . \tag{3.129}$$

The vector Y is a null vector that satisfies $X \cdot Y = -1$ at the horizon.

Using the equations of motion (2.2)–(2.4) with (2.15) as well as the fact that the Killing vector K is a symmetry of the matter fields which means that

$$\mathcal{L}_K B_M = 0, \qquad \mathcal{L}_K \Phi = 0, \tag{3.130}$$

it can be shown that

$$P_M^N \nabla_N Z^M = \frac{1}{\sqrt{-g}} \mathcal{L}_{\text{bulk}}^{\text{os}} K^M u_M. \tag{3.131}$$

The charge can now be written as

$$Q_K = K^t T S_E^{\text{os}} + \int_{t=\text{cst}} d^2x \sqrt{\sigma} X^N Y^P Z_{NP}|_{R=R_h}, \tag{3.132}$$

where we used that $K^M u_M = K^t U$ with $\sqrt{-g} = U\sqrt{H}$ and where we defined $\sqrt{\sigma} = F_4^{1/2} R^{-2}$ which is the determinant of the metric on the $t = \text{cst}$ and $R = \text{cst}$ submanifold. In this expression for the charge S_E^{os} is the Euclidean on-shell action, i.e.

$$T S_E^{\text{os}} = \lim_{\epsilon \to 0} \left[\int_\epsilon^{R_h} dR \int_{t=\text{cst}} d^2x \mathcal{L}_{\text{bulk}}^{\text{os}} + \int_{t=\text{cst}} d^2x \mathcal{L}_{\text{bdry}}^{\text{os}}|_{R=\epsilon} \right]. \tag{3.133}$$

Equation (3.132) is the result we were looking for. It expresses the asymptotic charge associated with the Killing vector K^M in terms of a horizon integral and the Euclidean on-shell action.

This result can be used to compute the charges associated with the Killing vectors ∂_t and ∂_y twice, once near the boundary using (3.114) and once at the horizon using (3.132). Near the boundary we find

$$Q_{\partial_t} = -\int_{t=\text{cst}} dx dy T^t{}_t = \int_{t=\text{cst}} dx dy \left(\mathcal{E} + \frac{1}{2} \rho V^2 \right), \tag{3.134}$$

$$Q_{\partial_y} = -\int_{t=\text{cst}} dx dy T^t{}_y = -\int_{t=\text{cst}} dx dy \rho V. \tag{3.135}$$

Using (3.132) we can derive the following relation

$$Q_{\partial_t} - T S_E^{\text{os}} = T \int_{t=\text{cst}} dx dy s - \frac{N(R_h)}{R_h} Q_{\partial_y}, \tag{3.136}$$

where we used (3.105) and (3.106).

The momentum Q_{∂_y} can be written in terms the Noether charge Q_μ defined in (3.16) via

$$\frac{N(R_h)}{R_h} Q_{\partial_y} = \int_{t=\text{cst}} dx dy Q_\mu. \tag{3.137}$$

This can be proven by computing the left and right hand side at the horizon where for the left hand side we use the integral form given in (3.132). The Noether charge Q_μ can also be computed near the boundary where it gives $Q_\mu = \rho V^2$. Hence with (3.135) we conclude that

$$\frac{N(R_h)}{R_h} = -V^y = -V, \tag{3.138}$$

i.e. the chemical potential is the velocity of the fluid. From this and (3.136) it follows that the Euclidean on-shell action relates to the pressure as follows

$$TS_E^{\text{os}} = -\int_{t=\text{cst}} dxdy P \,. \tag{3.139}$$

We believe that similar arguments allow one to derive the first law of thermodynamics for these Lifshitz holographic fluids. For example using arguments similar to those of [42] that do not require an explicit knowledge of the interpolating solution. However there are quite compelling arguments that fix the first law in a more straightforward manner so we will refrain from using a more general approach. One of these arguments uses the Ward identities of the dual holographic fluid and the existence of an entropy current. This will be discussed in the next section. The other argument results from the assumption that the pressure only depends on temperature and chemical potential. Given, say, a numerical solution this could be tested by evaluating (3.139). For us this is a rather minor assumption because it is essentially assuming that a solution with a horizon generated by X^M exists. If we assume that $P = P(T, V^2)$ we can vary it and use (3.110) to derive

$$\left(\frac{\partial P}{\partial T}\right)_{V^2} = s \,, \qquad \left(\frac{\partial P}{\partial V^2}\right)_T = \frac{1}{2}\rho \,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho \,, \tag{3.140}$$

where the latter relation is the first law for our holographic Lifshitz perfect fluid. More will be said about this in the next section.

4 Lifshitz perfect fluids

This section is independent from holography and derives the Lifshitz perfect fluid from dimensional reduction. In appendix A.3 we have discussed the null reduction of a relativistic perfect fluid. This gives rise to a Galilean perfect fluid. If furthermore the relativistic fluid is scale invariant, i.e. conformal, the lower-dimensional Galilean perfect fluid has a $z = 2$ Schrödinger invariance. The $z = 2$ Schrödinger algebra contains the $z = 2$ Lifshitz algebra as a subalgebra. Hence a Lifshitz invariant system can be obtained by starting with a Schrödinger invariant system and breaking the generators that are part of the Schrödinger algebra but not of the Lifshitz algebra. One of these symmetries is particle number N. By breaking N explicitly the $z = 2$ Schrödinger algebra reduces to the $z = 2$ Lifshitz algebra.[16] This is precisely what our holographic model for Lifshitz invariant field theories does.

We have shown that the 4-dimensional bulk theory follows from Scherk-Schwarz reduction of a 5-dimensional AdS-gravity model coupled to a scalar field. This scalar field leads to an additional source in the dual field theory and, as derived in section B.2, the corresponding diffeomorphism Ward identity, reads

$$\nabla_A t^A{}_B = -\langle O_\psi \rangle \partial_B \psi \,. \tag{4.1}$$

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[16]In the Schrödinger algebra the commutator between Galilean boosts G_i and momenta P_i reads $[P_i, G_j] = \delta_{ij}N$ so by breaking N keeping P_i intact we break G_i. Further special conformal symmetries K in the Schrödinger algebra satisfy the commutation relation $[K, P_i] = -G_i$ so that breaking G_i leads to broken K symmetries. Hence by breaking N we loose the G_i and K generators as well and we are left with the Lifshitz algebra.

Here we will be interested in flat space only so the left hand side is simply $\nabla_A t^A{}_B = \partial_A t^A{}_B$. The Scherk-Schwarz reduction tells us that

$$\psi = 2u - 2\chi\,, \tag{4.2}$$

$$\langle O_\psi \rangle = -\frac{1}{2}\langle O_\chi \rangle\,, \tag{4.3}$$

where χ and $\langle O_\chi \rangle$ are independent of u. If we now set the 4-dimensional scalar source $\chi = 0$ we obtain the 4-dimensional Ward identities (see also (B.92) and (B.93))

$$\partial_\mu T^\mu{}_\nu = 0\,, \tag{4.4}$$

$$\partial_\mu T^\mu = \langle O_\chi \rangle\,, \tag{4.5}$$

where we used (A.40) and (A.41). We thus see that the mass current T^μ is not conserved due to the presence of $\langle O_\chi \rangle$. The $z = 2$ scale Ward identity follows from (A.44) with $t^A{}_A = 0$ which for the case of a flat TNC space-time (3.24) reads

$$2T^t{}_t + T^i{}_i = 0\,. \tag{4.6}$$

The null reduction also implies the identities (A.32) and (A.33) which on a flat TNC space-time read

$$T^t{}_i = T^i\,, \qquad T^i{}_j = T^j{}_i\,. \tag{4.7}$$

The null reduction in the presence of the scalar source ψ as written in (4.1)–(4.3) gives rise to a system that breaks Galilean boost symmetries and particle number. This is due to the fact the ψ in (4.2) breaks these symmetries. What we are left with is a $z = 2$ Lifshitz invariant system in one dimension lower.

We will now apply the Lifshitz Ward identities (4.4)–(4.7) to the case of a $d = z = 2$ perfect fluid where $\mathcal{E} = P$ and V^i are functions of t, x^i. It has been shown that the form of $T^\mu{}_\nu$ and T^μ for the null reduction of a relativistic perfect fluid take the form (A.57) and (A.58). We now consider the fluid equations as follows from the Ward identities and demand that there exists a conserved entropy current. The latter requirement will tell us what the thermodynamic relations for a Lifshitz perfect fluid are.

On flat TNC space-time the form of the fluid energy-momentum tensor and mass current for a perfect fluid are given by (3.57) and (3.58). The fluid equations are thus given by the Ward identities which read[17]

$$0 = \partial_t \left(\mathcal{E} + \frac{1}{2}\rho V^2 \right) + \partial_i \left(\left(\mathcal{E} + P + \frac{1}{2}\rho V^2 \right) V^i \right)\,, \tag{4.8}$$

$$0 = \partial_t \left(\rho V_i \right) + \partial_i \left(P \delta^j_i + \rho V^j V_i \right)\,, \tag{4.9}$$

$$\langle O_\chi \rangle = \partial_t \rho + \partial_i \left(\rho V_i \right)\,. \tag{4.10}$$

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[17]If in the holographic setup we would make the fluid variables functions of the boundary coordinates we would have to correct the energy-momentum tensor by derivatives of the fluid variables. The Einstein equations will then lead to Ward identities for this corrected boundary energy-momentum tensor. At leading order in derivatives it will however reduce to the Ward identities for a perfect fluid.

These equations can be used to rewrite the equation for energy conservation (4.8) as

$$\partial_t \mathcal{E} + V^i \partial_i \mathcal{E} + \left(\mathcal{E} + P - \frac{1}{2}\rho V^2 \right) \partial_i V^i - \frac{1}{2} V^2 \left(\partial_t \rho + V^i \partial_i \rho \right) = 0 \,. \qquad (4.11)$$

This gives rise to an equation for conservation of entropy,

$$\partial_t s + \partial_i \left(s V^i \right) = 0 \,, \qquad (4.12)$$

provided we take

$$\mathcal{E} + P = Ts + \frac{1}{2}\rho V^2 \,, \qquad (4.13)$$

$$\delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho \,. \qquad (4.14)$$

These two equations together with the equation of state $P = \mathcal{E}$ (which follows from (4.6)) describe the thermodynamic properties of a Lifshitz invariant system obtained by breaking particle number symmetries. What we see here is a realization of a Lifshitz perfect fluid where the velocity or rather, due to rotational symmetries, V^2, plays the role of a chemical potential.[18] The thermodynamically conjugate variable is the mass density ρ. From the first law (4.14) it follows that

$$\delta P = s\delta T + \frac{1}{2}\rho\delta V^2 \,, \qquad (4.15)$$

so that pressure is a function of T and the chemical potential V^2.

We see here that the way in which we realize Lifshitz hydrodynamics is quite different from what has been discussed in [20]. The approach in [20] is to start with a $z = 1$ relativistic perfect fluid and to break Lorentz symmetries by adding higher derivative interactions that break the symmetry of the energy-momentum tensor. One can then take a non-relativistic limit to obtain systems with $z \neq 1$ that break Galilean boost symmetries. This leads to a model where Galilean boosts are broken at higher orders in a derivative expansion. On the other hand here we realize Lifshitz symmetries by breaking particle number and hence Galilean boosts already at the perfect fluid level. In [27] we will present more examples of Lifshitz hydrodynamics from a field theory perspective.

As a final comment we note that in order to solve the $d + 2$ equations (4.8)–(4.10) we need to know what $\langle O_\chi \rangle$ is in terms of the fluid variables ρ, V^i and $\mathcal{E}$. Explicit examples will be given in [27].

5 Discussion and outlook

We have shown that there is a new class of Lifshitz perfect fluids in which Galilean boosts are broken at the perfect fluid level. The holographic dual description is realized by a moving black brane solution of the EPD model. The motion of the black brane is not obtained by applying a boost transformation to a static black brane but follows from

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[18]A similar extension of the first law of thermodynamics involving a fluid with boost momentum was seen in [43] in the proposed effective theory for the dynamics of helicoidal black p-branes using the blackfold construction [7, 8].

constructing a new class of solutions corresponding to Lifshitz black branes with linear momentum. From the dual field theory point of view the boundary fluid can be obtained by a twisted null reduction of relativistic fluid in the background of a free scalar source that depends linearly on the null circle. From the lower-dimensional point of view this corresponds to a Schrödinger fluid with broken particle number symmetry.

In this work we restricted our attention to a specific EPD model for which we obtained the counterterms and near-boundary expansion by dimensional reduction from AdS holography coupled to a free real scalar. In order to consider similar solutions of other EPD models we need to be able to write down the counterterms and near-boundary expansions for general EPD models. Despite a lot of effort the situation is presently still not fully understood. There are different proposals [44, 45] and [26, 28] (see [46, 47] for earlier work) that share certain similarities but that also have some differences. A comparison between [44, 45] and [26, 28] is made in appendix B.6. We believe that more work needs to be done before we can state what the near-boundary expansion and counterterms are for a given EPD model in the general class that admits Lifshitz solutions. This general analysis includes asymptotically Lifshitz solutions with hyperscaling violation exponent θ and the charge hyperscaling violation exponent introduced in [29, 30].

A special subset of the EPD models are those for which $W = 0$ so that the bulk vector field becomes a Maxwell gauge potential with a U(1) gauge symmetry. It has been shown in [23] that the corresponding global U(1) symmetry in the boundary theory leads to mass conservation. For the EMD model we know the black brane solutions that are dual to perfect fluids analytically [14, 31]. For the solutions of the EPD models with $W \neq 0$ we only know the solution near the boundary and near the horizon but we do not know that interpolating solution. Hence we have to resort to arguments based on the existence of conserved Noether charges that are a consequence of various ansatz symmetries that allows one to relate near-boundary and near-horizon properties of the solution as was done in [33, 34]. Here we followed a similar approach and we added to this various integral forms of the asymptotic charges related to the existence of Killing vectors. It would be interesting to see how far one can push this kind of analysis beyond the perfect fluid level. In other words it is worth exploring if it is possible to construct bulk solutions in which the fluid variables such as the temperature and velocity become slowly varying functions of the boundary coordinates in such a way that we can extract all the relevant boundary properties from the near-horizon and near-boundary features. Further, it would be nice to have numerical confirmation about the interpolating solution we have assumed to exist.

We also remark that it would be interesting to study the role of charge in the boundary Lifshitz hydrodynamics by adding additional U(1) vector gauge fields to the bulk description like in [48]. For this the recent results in [49] on non-relativistic electrodynamics coupled to TNC could be relevant.

Finally, another interesting direction to pursue is to use Hořava-Lifshitz gravity theories as bulk theories in holography [50, 51] and examine the connection with Lifshitz hydrodynamics [52, 53]. It would be worthwhile to pursue this further in the light of the results of this paper. In particular in connection to dynamical NC geometry [54] and finite temperature states in the 3-dimensional Chern-Simons Schrödinger gravity that was recently constructed [55].

Acknowledgments

A Torsional Newton-Cartan geometry and non-relativistic field theory

Here we summarize results obtained in [24–26, 28, 35, 40, 54, 56, 57] regarding the most general formulation of torsional Newton-Cartan (TNC) geometry. We will focus only on those aspects that are needed for the purposes of this work. Since here we encounter TNC geometry through null reduction of the AdS$_5$ boundary metric we will study its properties in this context.

A.1 Null reduction of metric

Consider the null reduction ansatz for the metric

$$ds^2 = \gamma_{AB}dx^A dx^B = 2\tau_\mu dx^\mu \left(du - m_\nu dx^\nu\right) + h_{\mu\nu}dx^\mu dx^\nu \,, \tag{A.1}$$

where $A = (u, \mu)$ and with

$$h_{\mu\nu} = \delta_{ab}e_\mu^a e_b^\nu \,, \tag{A.2}$$

in which $a = 1, \ldots, d$. The metric $h_{\mu\nu}$ has vanishing determinant. The reduction ansatz is the most general metric for which $\gamma_{uu} = 0$. It is assumed that ∂_u is a null Killing vector of γ_{AB}. The fields τ_μ and e_μ^a are the vielbeins of the $d + 1$ dimensional TNC geometry. The metric (A.1) preserves the following local tangent space transformations

$$\delta\tau_\mu = 0 \,, \tag{A.3}$$

$$\delta e_\mu^a = \tau_\mu \lambda^a + \lambda^a{}_b e_\mu^b \,, \tag{A.4}$$

$$\delta m_\mu = \partial_\mu \sigma + \lambda_a e_\mu^a \,. \tag{A.5}$$

The local σ transformation requires $\delta u = \sigma$. The transformations with local parameter λ^a correspond to tangent space Galilean boosts (G) and transformations with local parameter $\lambda^a{}_b$ correspond to tangent space rotations (J). The metric components $\gamma_{\mu u} = \tau_\mu$ and $\gamma_{\mu\nu} = \bar{h}_{\mu\nu}$ where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu m_\nu - \partial_\nu \tau_\mu \,, \tag{A.6}$$

are invariant under these local transformations.

The inverse metric is

$$\gamma^{uu} = 2\tilde{\Phi}\,, \qquad \gamma^{u\mu} = -\hat{v}^\mu\,, \qquad \gamma^{\mu\nu} = h^{\mu\nu}\,, \tag{A.7}$$

where

$$\tilde{\Phi} = -v^\rho m_\rho + \frac{1}{2}h^{\rho\sigma}m_\rho m_\sigma\,, \tag{A.8}$$

$$\hat{v}^\mu = v^\mu - h^{\mu\nu}m_\nu\,, \tag{A.9}$$

$$h^{\mu\nu} = \delta^{ab}e_a^\mu e_b^\nu\,. \tag{A.10}$$

The inverse vielbeins v^μ and e_a^μ are defined through

$$v^\mu e_\mu^a = 0\,, \tag{A.11}$$

$$v^\mu \tau_\mu = -1\,, \tag{A.12}$$

$$e_a^\mu \tau_\mu = 0\,, \tag{A.13}$$

$$e_a^\mu e_\mu^b = \delta_a^b\,. \tag{A.14}$$

It is sometimes useful to work with the Galilean boost invariant vielbeins τ_μ, $\hat{e}_\mu^a$ and their inverse $\hat{v}^\mu$, e_a^μ where $\hat{e}_\mu^a$ is defined by

$$\hat{e}_\mu^a = e_\mu^a - m_\nu e^{\nu a}\tau_\mu\,. \tag{A.15}$$

These satisfy the orthogonality relations

$$\hat{v}^\mu \hat{e}_\mu^a = 0\,, \tag{A.16}$$

$$\hat{v}^\mu \tau_\mu = -1\,, \tag{A.17}$$

$$e_a^\mu \tau_\mu = 0\,, \tag{A.18}$$

$$e_a^\mu \hat{e}_\mu^b = \delta_a^b\,. \tag{A.19}$$

Finally we will often use the spatial metric $\hat{h}_{\mu\nu}$ defined by

$$\hat{h}_{\mu\nu} = \delta_{ab}\hat{e}_\mu^a \hat{e}_\nu^b = \bar{h}_{\mu\nu} + 2\tilde{\Phi}\tau_\mu\tau_\nu\,. \tag{A.20}$$

The inverse vielbeins transform as

$$\delta v^\mu = \lambda^a e_a^\mu\,, \tag{A.21}$$

$$\delta e_a^\mu = \lambda_a{}^b e_b^\mu\,. \tag{A.22}$$

A torsionful affine connection $\Gamma_{\mu\nu}^\rho$ that is invariant under the local tangent space symmetries (G, J) and that satisfies metric compatibility, in the sense of

$$\nabla_\mu \tau_\nu = 0\,, \tag{A.23}$$

$$\nabla_\mu h^{\nu\rho} = 0\,, \tag{A.24}$$

is given by

$$\bar{\Gamma}_{\mu\nu}^\rho = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right)\,. \tag{A.25}$$

The bar on Γ is supposed to emphasize that the connection is not unique and we have chosen a particular realization. We will later encounter another affine connection that is metric compatible. We note that in [58] it is shown that at the linearized level the connection (A.25) (which is linear in m_μ) appears when applying the Noether procedure to gauging the space-time symmetries in theories with Galilean symmetries.

A.2 Null reduction of energy-momentum tensor

In [25, 26, 28, 35, 57] (see also [58]) the coupling prescriptions of non-relativistic field theories to torsional Newton-Cartan (TNC) backgrounds have been worked out both directly in field theory and from Lifshitz holography. The results of course agree (see e.g. [28]). Here we briefly review these results and derive them from null reduction as done in [25].

The TNC energy-momentum tensor (EMT) is defined as the response to varying the TNC fields via

$$\delta_{\text{bg}}S = \int d^{d+1}xe \left[-\tau_\nu T^\nu{}_\mu \delta\hat{v}^\mu - \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho \delta h^{\rho\sigma} \right. $$
$$\left. + \frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\delta h^{\rho\sigma} + \tau_\mu T^\mu \delta\tilde{\Phi}\right], \tag{A.26}$$

where e is the determinant of the 3 by 3 matrix (τ_μ, e^a_μ) which is both boost and rotation invariant. We can alternatively define an energy-momentum tensor by varying the unhatted TNC fields via

$$\delta_{\text{bg}}S = \int d^{d+1}xe \left[-\mathcal{T}_\mu \delta v^\mu + \frac{1}{2}\mathcal{T}_{\mu\nu}\delta h^{\mu\nu} + T^\mu \delta m_\mu\right]. \tag{A.27}$$

The two are related via

$$h^{\nu\rho}\mathcal{T}_{\rho\mu} - v^\nu \mathcal{T}_\mu = T^\nu{}_\mu + T^\nu m_\mu. \tag{A.28}$$

According to the null reduction of [25] the energy momentum tensor $T^\mu{}_\nu$ and mass current T^μ are related to the higher-dimensional energy-momentum tensor t^{AB} via[19]

$$t^{\mu u} = 2\tilde{\Phi}T^\mu - \hat{v}^\sigma T^\mu{}_\sigma, \tag{A.29}$$

$$t^{\mu\nu} = -\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho. \tag{A.30}$$

The latter relation implies due to the symmetry of $t^{\mu\nu}$

$$-\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho + \hat{v}^\nu T^\mu - h^{\nu\rho}T^\mu{}_\rho = 0, \tag{A.31}$$

from which we read off the boost and rotation Ward identities

$$0 = -\hat{h}_{\mu\nu}T^\mu + \tau_\mu h^{\rho\sigma}\hat{h}_{\nu\sigma}T^\mu{}_\rho, \tag{A.32}$$

$$0 = \hat{h}_{\mu\rho}\hat{h}_{\nu\lambda}h^{\lambda\sigma}T^\rho{}_\sigma - (\mu \leftrightarrow \nu). \tag{A.33}$$

The definitions (A.29) and (A.30) imply

$$-\frac{1}{2}t^{AB}\delta\gamma_{AB} = -\tau_\nu T^\nu{}_\mu \delta\hat{v}^\mu - \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho \delta h^{\rho\sigma} $$
$$+ \frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\delta h^{\rho\sigma} + \tau_\mu T^\mu \delta\tilde{\Phi}, \tag{A.34}$$

---

[19]Since t^{AB} is the response to varying γ_{AB} there is no need for t^{uu} since $\gamma_{uu} = 0$.

in agreement with the definition of $T^\mu{}_\nu$ and T^μ as the response to varying the TNC invariants $\hat{v}^\mu$, $h^{\mu\nu}$ and $\tilde{\Phi}$ as given in (A.26). The relation between the higher and lower dimensional energy-momentum tensors holds for any reduction given that γ_{AB} admits a null Killing vector. No additional assumptions such as hypersurface orthogonality of ∂_u are needed.

The higher-dimensional energy-momentum tensor corresponds to the boundary theory of a bulk AdS_5 space-time and is thus traceless. Further by boundary diffeomorphism invariance it satisfies a Ward identity for local diffeomorphism invariance. Upon reduction these give rise to Ward identities for local scale and diffeomorphism invariance. To this end it is useful to consider[20] $t^A{}_B$, i.e.

$$t^u{}_u = 2\tilde{\Phi}\tau_\mu T^\mu - \hat{v}^\nu \tau_\mu T^\mu{}_\nu \,, \tag{A.38}$$

$$t^u{}_\nu = 2\tilde{\Phi}\tau_\nu T^\mu{}_\nu - \hat{v}^\sigma \hat{h}_{\nu\rho} T^\rho{}_\sigma + \tau_\nu \hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma} \,, \tag{A.39}$$

$$t^\mu{}_u = T^\mu \,, \tag{A.40}$$

$$t^\mu{}_\nu = T^\mu{}_\nu \,, \tag{A.41}$$

where $\hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma}$, which contains t^{uu}, is unspecified in terms of lower dimensional quantities as it will drop out of the Ward identities. We can derive the following identities

$$\nabla_A t^A{}_u = \partial_\mu \left(e T^\mu\right) \,, \tag{A.42}$$

$$\nabla_A t^A{}_\mu = e^{-1}\partial_\nu \left(e T^\nu{}_\mu\right) + T^\rho{}_\nu \left(\hat{v}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu \partial_\mu \tilde{\Phi} \,, \tag{A.43}$$

$$t^A{}_A = -2\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2\tilde{\Phi}\tau_\mu T^\mu \,. \tag{A.44}$$

If we are dealing with a relativistic and scale invariant theory, i.e. $\nabla_A t^A{}_B = t^A{}_A = 0$ we find the diffeomorphism, $\mathrm{U}(1)$ and the $z = 2$ version of the local dilatation Ward identities as given in [25, 26, 28, 35].

The diffeomorphism Ward identity (A.43) can also be written in a TNC covariant form using the connection (A.25) as done in [26]. Instead of the connection (A.25) we can also take the Riemann-Cartan connection of [59], that we will denote by $\check{\Gamma}^\rho_{\mu\nu}$, given by

$$\check{\Gamma}^\lambda_{\mu\rho} = -\hat{v}^\lambda \partial_\mu \tau_\rho + \frac{1}{2}h^{\nu\lambda}\left(\partial_\mu \hat{h}_{\rho\nu} + \partial_\rho \hat{h}_{\mu\nu} - \partial_\nu \hat{h}_{\mu\rho}\right) - h^{\nu\lambda}\tau_\rho K_{\mu\nu} \,, \tag{A.45}$$

where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\hat{v}}\hat{h}_{\mu\nu}$ is the extrinsic curvature. This connection obeys

$$\check{\nabla}_\mu \tau_\nu = 0 \,, \qquad \check{\nabla}_\mu \hat{h}_{\nu\rho} = 0 \,, \qquad \check{\nabla}_\mu \hat{v}^\nu = 0 \,, \qquad \check{\nabla}_\mu h^{\nu\rho} = 0 \,, \tag{A.46}$$

and the relation (A.43) becomes

$$\nabla_A t^A{}_\nu = \check{\nabla}_\mu T^\mu{}_\nu + 2\check{\Gamma}^\rho_{[\mu\rho]}T^\mu{}_\nu - 2\check{\Gamma}^\mu_{[\nu\rho]}T^\rho{}_\mu + \tau_\mu T^\mu \partial_\nu \tilde{\Phi} \,. \tag{A.47}$$

This is the most compact and TNC covariant way of writing the diffeomorphism Ward identity.

---

[20]Sometimes it is useful to express t_{AB} in terms of lower-dimensional quantities via

$$t_{\mu\nu} = \hat{h}_{\rho\nu}T^\rho{}_\mu + \hat{h}_{\rho\mu}T^\rho{}_\nu - \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma}h^{\sigma\lambda}T^\rho{}_\lambda + \tau_\mu \tau_\nu \hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma} \,, \tag{A.35}$$

$$t_{\mu u} = \tau_\rho T^\rho{}_\mu \,, \tag{A.36}$$

$$t_{uu} = \tau_\rho T^\rho \,. \tag{A.37}$$

In [59] it was shown that TTNC geometry (but not the more general TNC geometry) can be obtained by projecting the higher-dimensional metric compatibility conditions involving the Levi-Civita connection onto the surface orthogonal to ∂_u (null reduction) in the sense that the TNC metric compatibility conditions follow from the projection of the higher-dimensional metric compatibility conditions only when ∂_u is hypersurface orthogonal. However, the condition that the TNC metric compatibility conditions follows by projection is somewhat artificial. Here we see that the diffeomorphism Ward identity takes the required form for any field theory on a TNC geometry and not just TTNC geometry. At no point in the analysis did we assume anything about τ_μ.

A.3 Null reduction of a perfect relativistic fluid

Since we are interested in non-relativistic versions of the fluid/gravity correspondence we study here the null reduction of a relativistic fluid.[21]

A relativistic perfect fluid is given by a conserved energy-momentum tensor t_{AB} that is of the form

$$t_{AB} = (E + P) U_A U_B + P \gamma_{AB} \,, \tag{A.48}$$

where U_A satisfies $U_A U^A = -1$. Consider the following parametrization of U_A,

$$U_u^2 = \frac{\rho}{E + P} \,, \tag{A.49}$$

$$h^{\mu\nu} U_\nu = U_u \left(\hat{v}^\mu - u^\mu \right) \,, \tag{A.50}$$

$$\hat{v}^\mu U_\mu = \frac{1}{2} U_u \left(\hat{h}_{\mu\nu} u^\mu u^\nu + 2\tilde{\Phi} + U_u^{-2} \right) \,, \tag{A.51}$$

where u^μ satisfies $\tau_\mu u^\mu = -1$. It follows that

$$U_\mu = -\frac{1}{2} U_u \tau_\mu \left(\hat{h}_{\rho\sigma} u^\rho u^\sigma + 2\tilde{\Phi} + U_u^{-2} \right) - U_u \hat{h}_{\mu\nu} u^\nu \tag{A.52}$$

$$= -U_u \left[\frac{1}{2} \tau_\mu \left(h_{\rho\sigma} u^\rho u^\sigma + U_u^{-2} \right) + h_{\mu\nu} u^\nu + m_\mu \right] \,, \tag{A.53}$$

which (except for the U_u^{-2} term) takes the form of the velocity of a point particle. The components of U^μ are given by

$$U^u = -\frac{1}{2} U_u \left(\hat{h}_{\mu\nu} u^\mu u^\nu - 2\tilde{\Phi} + U_u^{-2} \right) \,, \tag{A.54}$$

$$U^\mu = -U_u u^\mu \,. \tag{A.55}$$

Further redefine the energy density E as

$$E = 2\mathcal{E} + P \,. \tag{A.56}$$

Using the above results it follows that $T^\mu{}_\nu$ and T^μ are given by

$$T^\mu{}_\nu = \left(\mathcal{E} + P + \rho\tilde{\Phi} + \frac{1}{2} \rho \hat{h}_{\lambda\kappa} u^\lambda u^\kappa \right) u^\mu \tau_\nu + P \delta^\mu_\nu + \rho u^\mu \hat{h}_{\nu\rho} u^\rho$$

$$= \left(\mathcal{E} + P + \frac{1}{2} \rho h_{\lambda\kappa} u^\lambda u^\kappa \right) u^\mu \tau_\nu + P \delta^\mu_\nu + \rho u^\mu h_{\nu\rho} u^\rho + \rho u^\mu m_\nu \,, \tag{A.57}$$

$$T^\mu = -\rho u^\mu \,. \tag{A.58}$$

---

[21]This has also been done in [60] but our approach differs in that we do not need to introduce what is called a null fluid in [60].

This can be shown to agree with the notion of a Galilean perfect fluid as given in [61]. On flat NC space-time as defined in (3.24) the velocity u^μ can be parameterized as $u^t = -1$ and $u^i = -V^i$, so that the resulting energy-momentum tensor becomes (3.57), (3.58).

The null reduction ansatz has a local U(1) symmetry which is the diffeomorphism $\delta u = -\xi^u$ and $\delta m_\mu = -\partial_\mu \xi^u$. If we act with this diffeomorphism on U^A and U_A via

$$\delta U_A = \xi^B \partial_B U_A + U_B \partial_A \xi^B , \qquad \delta U^A = \xi^B \partial_B U^A - U^B \partial_B \xi^A , \qquad (A.59)$$

with $\xi^A = \delta_u^A \xi^u$ we see that U_u and U^μ are U(1) invariant. It follows from (A.55) that the fluid velocity u^μ is particle number invariant.

The $z = 2$ trace Ward identity reads (for $d = 2$ spatial dimensions)

$$t^A{}_A = -2\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2\tilde{\Phi}\tau_\mu T^\mu = -2\mathcal{E} + 2P = 0 . \qquad (A.60)$$

The null reduction only leads to theories with $z = 2$ scaling relations.

B Holographic renormalization of the upliftable model

As discussed in section 2.3 the EPD model with

$$Z = e^{3\Phi} , \qquad W = 4 , \qquad V(\Phi) = 2e^{-3\Phi} - 12e^{-\Phi} , \qquad x = 3 . \qquad (B.1)$$

can be obtained from a Scherk-Schwarz reduction of the 5-dimensional action

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-\mathcal{G}} \left(R + 12 - \frac{1}{2}\partial_\mathcal{M}\psi\partial^\mathcal{M}\psi \right) , \qquad (B.2)$$

where $\kappa_5^2 = 8\pi G_5$ with G_5 the 5-dimensional Newton's constant and where $\mathcal{M} = (u, M)$. The consistency of this reduction will be shown in section B.5.

In this appendix we will first perform the holographic renormalization in 5 dimensions for those asymptotically locally AdS space-times that have a boundary metric obeying the null reduction ansatz of section A.1. We then subsequently reduce the result to obtain the counterterms and near-boundary expansions in 4 dimensions for asymptotically locally $z = 2$ Lifshitz space-times.

B.1 Fefferman-Graham expansions and counterterms

By using the results of [25, 36, 62, 63][22] we can obtain the solution to the equations of motion of (B.2) (that are given further below in (B.18) and (B.19)) expressed as an asymptotic series in radial gauge, i.e. as a Fefferman-Graham (FG) expansion [64]. The result reads[23]

$$\mathcal{G}_{\mathcal{MN}}dx^\mathcal{M}dx^\mathcal{N} = \frac{dr^2}{r^2} + \gamma_{AB}dx^A dx^B , \qquad (B.3)$$

$$\gamma_{AB} = \frac{1}{r^2}\left[\gamma_{(0)AB} + r^2\gamma_{(2)AB} + r^4\log r\,\gamma_{(4,1)AB} + r^4\gamma_{(4)AB} + O(r^6\log r)\right], \quad (B.4)$$

$$\psi = \psi_{(0)} + r^2\psi_{(2)} + r^4\log r\,\psi_{(4,1)} + r^4\psi_{(4)} + O(r^6\log r) , \qquad (B.5)$$

---

[22]We set $\hat{\chi} = 0$ and redefine $\hat{\phi} = \psi$ in [25].

[23]We will denote here and further below by $a_{(n,m)}$ the coefficient at order $r^n(\log r)^m$ of the field $r^\Delta a$ where $r^{-\Delta}$ is the leading term in the expansion of a with the exception of the $a_{(n,0)}$ term which we will simply denote as $a_{(n)}$.

where the coefficients are given by

$$\gamma_{(2)AB} = -\frac{1}{2}\left(R_{(0)AB} - \frac{1}{2}\partial_A\psi_{(0)}\partial_B\psi_{(0)}\right) + \frac{1}{12}\gamma_{(0)AB}\left(R_{(0)} - \frac{1}{2}(\partial\psi_{(0)})^2\right), \quad \text{(B.6)}$$

$$\psi_{(2)} = \frac{1}{4}\Box_{(0)}\psi_{(0)}, \quad \text{(B.7)}$$

at second order and by

$$\gamma_{(4,1)AB} = \frac{1}{4}\nabla_{(0)}^C\left(\nabla_{(0)A}\gamma_{(2)BC} + \nabla_{(0)B}\gamma_{(2)AC} - \nabla_{(0)C}\gamma_{(2)AB}\right) - \frac{1}{4}\nabla_{(0)A}\nabla_{(0)B}\gamma_{(2)C}^C$$
$$+\gamma_{(2)AC}\gamma_{(2)B}^C - \frac{1}{2}\partial_{(A}\psi_{(0)}\nabla_{(0)B)}\psi_{(2)} - \gamma_{(0)AB}\left(\frac{1}{4}\gamma_{(2)}^{CD}\gamma_{(2)CD} + \frac{1}{2}\psi_{(2)}^2\right), \quad \text{(B.8)}$$

$$\psi_{(4,1)} = -\frac{1}{4}\left[\Box_{(0)}\psi_{(2)} + 2\psi_{(2)}\gamma_{(2)A}^A + \frac{1}{2}\partial^A\psi_{(0)}\nabla_{(0)A}\gamma_{(2)B}^B - \gamma_{(2)}^{AB}\nabla_{(0)A}\partial_B\psi_{(0)}\right.$$
$$\left.-\partial^A\psi_{(0)}\nabla_{(0)}^B\gamma_{(2)AB}\right], \quad \text{(B.9)}$$

at order $r^4\log r$. We note that the quantity $\gamma_{(4,1)AB}$ is traceless. Indices of the expansion coefficients are raised and lowered with the AdS boundary metric $\gamma_{(0)AB}$. At order r^4 we have that $\gamma_{(4)AB}$ is constrained by

$$\gamma_{(4)A}^A = \frac{1}{4}\gamma_{(2)AB}\gamma_{(2)}^{AB} - \frac{1}{2}\psi_{(2)}^2, \quad \text{(B.10)}$$

$$\nabla_{(0)}^B\gamma_{(4)AB} = \psi_{(4)}\partial_A\psi_{(0)} - \frac{1}{2}\psi_{(2)}\nabla_{(0)A}\psi_{(2)} - \frac{1}{4}\gamma_{(2)}^{BC}\nabla_{(0)A}\gamma_{(2)BC}$$
$$-\frac{1}{4}\gamma_{(2)AC}\nabla_{(0)}^C\gamma_{(2)B}^B + \frac{1}{2}\gamma_{(2)}^{BC}\nabla_{(0)B}\gamma_{(2)AC} + \frac{1}{2}\gamma_{(2)A}^C\nabla_{(0)}^B\gamma_{(2)BC}. \quad \text{(B.11)}$$

Following [62] we write the coefficient $\gamma_{(4)AB}$ as

$$\gamma_{(4)AB} = X_{AB} - \frac{1}{4}t_{AB}, \quad \text{(B.12)}$$

where t_{AB} is the boundary energy-momentum tensor defined in (B.24). The trace and divergence of t_{AB} will be given below together with the explicit form of X_{AB}. In the expansion for the scalar we have that $\psi_{(4)}$ is a fully arbitrary function of the boundary coordinates.

The complete action with Gibbons-Hawking and local counterterms (using minimal subtraction) is given by

$$S_{\text{ren}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}}d^5x\sqrt{-\mathcal{G}}\left(R + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi\right) + \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}K + S_{\text{ct}}, \quad \text{(B.13)}$$

where γ denotes the determinant of the metric γ_{AB} on the cut off boundary $\partial\mathcal{M}$, the extrinsic curvature K is given by

$$K = \gamma^{AB}K_{AB}, \qquad K_{AB} = -\frac{1}{2}\mathcal{L}_n\gamma_{AB}, \qquad n^{\mathcal{M}} = -r\delta_r^{\mathcal{M}}, \quad \text{(B.14)}$$

and where

$$S_{\text{ct}} = \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}\left(-\frac{1}{4}\left(R_{(\gamma)} + 12 - \frac{1}{2}\partial_A\psi\partial^A\psi\right) - \frac{1}{2}\mathcal{A}\log r\right), \quad \text{(B.15)}$$

with

$$\mathcal{A} = -\frac{1}{4}\left(Q^{AB}Q_{AB} - \frac{1}{3}Q^2 + \frac{1}{2}\left(\Box_{(\gamma)}\psi\right)^2\right),\tag{B.16}$$

$$Q_{AB} = R_{(\gamma)AB} - \frac{1}{2}\partial_A\psi\partial_B\psi\,.$$

B.2 One-point functions

To compute one-point functions, we write the total variation of $S_{\mathrm{ren}} = S_{\mathrm{bulk}} + S_{\mathrm{GH}} + S_{\mathrm{ct}}$ as

$$\delta S_{\mathrm{ren}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}}d^5x\sqrt{-\mathcal{G}}\left(\mathcal{E}_{\mathcal{MN}}\delta\mathcal{G}^{\mathcal{MN}} + \mathcal{E}_\psi\delta\psi\right)$$
$$+\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}\left(\frac{1}{2}T_{AB}\delta\gamma^{AB} + T_\psi\delta\psi\right),\tag{B.17}$$

where $\mathcal{E}_{\mathcal{MN}}$ and $\mathcal{E}_\psi$ are the equations of motion

$$\mathcal{E}_{\mathcal{MN}} = G_{\mathcal{MN}} - 6\mathcal{G}_{\mathcal{MN}} - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial_{\mathcal{N}}\psi + \frac{1}{4}\mathcal{G}_{\mathcal{MN}}(\partial\psi)^2\,,\tag{B.18}$$

$$\mathcal{E}_\psi = \Box\psi\,,\tag{B.19}$$

and where

$$T_{AB} = -2(K-3)\gamma_{AB} + 2K_{AB} - Q_{AB} + \frac{1}{2}h_{AB}Q + \log r\,T_{AB}^{(A)}\,,\tag{B.20}$$

$$T_\psi = -n^M\partial_M\psi - \frac{1}{2}\Box_{(\gamma)}\psi + \log r\,T_\psi^{(A)}\,.\tag{B.21}$$

Here we defined

$$T_{AB}^{(A)} = -\frac{2\kappa_5^2}{\sqrt{-\gamma}}\frac{\delta A}{\delta\gamma^{AB}}\,,\qquad T_\psi^{(A)} = -\frac{\kappa_5^2}{\sqrt{-\gamma}}\frac{\delta A}{\delta\psi}\,,\tag{B.22}$$

with

$$A = \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}\mathcal{A}\,.\tag{B.23}$$

From the expansions it follows that $\sqrt{-\gamma} = r^{-4}\sqrt{-\gamma_{(0)}} + O(r^{-2})$, $\delta\gamma^{AB} = r^2\delta\gamma_{(0)}^{AB} + O(r^4)$, $\delta\psi = \delta\psi_{(0)} + O(r^2)$, which is used to obtain the following one-point functions (we take the cut-off boundary at $r = \epsilon$)

$$t_{AB} = \frac{4\kappa_5^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S_{\mathrm{ren}}^{\mathrm{on\text{-}shell}}}{\delta\gamma_{(0)}^{AB}} = \lim_{\epsilon\to 0}\epsilon^{-2}T_{AB} = -4\gamma_{(4)AB} + 4X_{AB}\,,\tag{B.24}$$

$$\langle\mathcal{O}_\psi\rangle = \frac{2\kappa_5^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S_{\mathrm{ren}}^{\mathrm{on\text{-}shell}}}{\delta\psi_{(0)}} = \lim_{\epsilon\to 0}\epsilon^{-4}T_\psi = 4\psi_{(4)} + \psi_{(2)}\gamma_{(2)A}^A + 3\psi_{(4,1)}\,,\tag{B.25}$$

where

$$X_{AB} = \frac{1}{2}\gamma_{(2)AC}\gamma_{(2)B}^C - \frac{1}{4}\gamma_{(2)C}^C\gamma_{(2)AB} + \frac{1}{8}\gamma_{(0)AB}\mathcal{A}_{(0)} - \frac{3}{4}\gamma_{(4,1)AB}\,,\tag{B.26}$$

with

$$\mathcal{A}_{(0)} = \lim_{\epsilon\to 0}\epsilon^{-4}\mathcal{A} = \left(\gamma_{(2)A}^A\right)^2 - \gamma_{(2)}^{AB}\gamma_{(2)AB} - 2\psi_{(2)}^2\,.\tag{B.27}$$

Using equations (B.10) and (B.11) we can compute the trace and divergence of the boundary energy-momentum tensor and the result is

$$t^A{}_A = \mathcal{A}_{(0)}\,,\tag{B.28}$$

$$\nabla_{(0)A}t^A{}_B = -\langle\mathcal{O}_\psi\rangle\partial_B\psi_{(0)}\,.\tag{B.29}$$

B.3 Dimensional reduction of the action

The Scherk-Schwarz reduction leading to (2.1) with the choices (2.15) is obtained by the following reduction ansatz

$$ds_5^2 = \mathcal{G}_{\mathcal{M}\mathcal{N}}dx^{\mathcal{M}}dx^{\mathcal{N}} = \frac{dr^2}{r^2} + \gamma_{AB}dx^A dx^B = e^{-\Phi}g_{MN}dx^M dx^N + e^{2\Phi}\left(du + A_M dx^M\right)^2$$

$$= e^{-\Phi}\left(e^{\Phi}\frac{dr^2}{r^2} + h_{\mu\nu}dx^{\mu}dx^{\nu}\right) + e^{2\Phi}\left(du + A_{\mu}dx^{\mu}\right)^2, \tag{B.30}$$

$$\psi = 2u + 2\Xi, \tag{B.31}$$

where all the functions are independent of the fifth coordinate u which is periodically identified, so $u \sim u + 2\pi L$. The only exception is the term linear in ψ which means that upon going around the reduction circle ψ comes back to itself up to a constant shift. This is allowed because shifting ψ is a global symmetry of the higher-dimensional theory. The consistency of the reduction is proven in section B.5.

After reduction the four dimensional action is

$$S = \int d^4x\sqrt{-g}\left(R - \frac{3}{2}\partial_M\Phi\partial^M\Phi - \frac{1}{4}e^{3\Phi}F_{MN}F^{MN} - 2B_M B^M - V\right)$$

$$+2\int d^3x\sqrt{-h}K + S_{\text{ct}}, \tag{B.32}$$

$$S_{\text{ct}} = 2\int_{\partial\mathcal{M}}d^3x\sqrt{-h}\left[-\frac{1}{4}e^{\Phi/2}\left(R_{(h)} - \frac{3}{2}\partial_{\mu}\Phi\partial^{\mu}\Phi - \frac{1}{4}e^{3\Phi}F_{\mu\nu}F^{\mu\nu} - 2B_{\mu}B^{\mu} + 10e^{-\Phi}\right)\right]$$

$$-\log r\int_{\partial\mathcal{M}}d^3x\sqrt{-h}e^{-\Phi/2}\mathcal{A}, \tag{B.33}$$

where

$$B_M = A_M - \partial_M\Xi, \tag{B.34}$$

$$F_{MN} = \partial_M B_N - \partial_N B_M, \tag{B.35}$$

$$V = 2e^{-3\Phi} - 12e^{-\Phi}, \tag{B.36}$$

and where we used that $\frac{2\pi L}{2\kappa_5^2} = 1$.

The total variation can be written as

$$\delta S_{\text{ren}} = \int_{\mathcal{M}}d^4x\sqrt{-g}\left(\mathcal{E}_{MN}\delta g^{MN} + \mathcal{E}^N\delta B_N + \mathcal{E}_{\Phi}\delta\Phi\right)$$

$$+\int_{\partial\mathcal{M}}d^3x\sqrt{-h}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^{\nu}\delta B_{\nu} + T_{\Phi}\delta\Phi\right), \tag{B.37}$$

with

$$\mathcal{E}_{MN} = G_{MN} + \frac{1}{8}e^{3\Phi}g_{MN}F_{PQ}F^{PQ} - \frac{1}{2}e^{3\Phi}F_{MP}F_N{}^P + g_{MN}B_P B^P - 2B_M B_N$$

$$+\frac{3}{4}g_{MN}\partial_P\Phi\partial^P\Phi - \frac{3}{2}\partial_M\Phi\partial_N\Phi + \frac{1}{2}g_{MN}V, \tag{B.38}$$

$$\mathcal{E}_{\Phi} = 3\Box\Phi - \frac{3}{4}e^{3\Phi}F_{MN}F^{MN} + 6e^{-3\Phi} - 12e^{-\Phi}, \tag{B.39}$$

$$\mathcal{E}^N = \nabla_M\left(e^{3\Phi}F^{MN}\right) - 4B^N, \tag{B.40}$$

and

$$T_{\mu\nu} = -2K h_{\mu\nu} + 2K_{\mu\nu} - e^{\Phi/2} G_{(h)\mu\nu} + 5e^{-\Phi/2} h_{\mu\nu}$$
$$+ \frac{1}{2} e^{7\Phi/2} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{8} e^{7\Phi/2} h_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - e^{\Phi/2} h_{\mu\nu} B_\rho B^\rho + 2e^{\Phi/2} B_\mu B_\nu$$
$$+ \frac{1}{2} e^{\Phi/2} \left(\nabla_\mu^{(h)} \partial_\nu \Phi - h_{\mu\nu} \Box_{(h)} \Phi \right) + \frac{7}{4} e^{\Phi/2} \partial_\mu \Phi \partial_\nu \Phi - e^{\Phi/2} h_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi \,, \quad \text{(B.41)}$$

$$T_\Phi = -3n^M \partial_M \Phi - \frac{1}{4} e^{\Phi/2} R_{(h)} - \frac{3}{8} e^{\Phi/2} \partial_\mu \Phi \partial^\mu \Phi - \frac{3}{2} e^{\Phi/2} \Box_{(h)} \Phi$$
$$+ \frac{7}{16} e^{7\Phi/2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^{\Phi/2} B_\mu B^\mu + \frac{5}{2} e^{-\Phi/2} \,, \quad \text{(B.42)}$$

$$\mathcal{T}^\nu = -e^{3\Phi} n_M F^{M\nu} - \frac{1}{2} \nabla_\mu^{(h)} \left(e^{7\Phi/2} F^{\mu\nu} \right) + 2e^{\Phi/2} B^\nu \,, \quad \text{(B.43)}$$

where the extrinsic curvature is

$$K = h^{\mu\nu} K_{\mu\nu} \,, \qquad K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n h_{\mu\nu} \,, \qquad n^M = -r e^{-\Phi/2} \delta_r^M \,. \quad \text{(B.44)}$$

These expressions are correct up to $\log r$ terms since we did not vary those counterterms.

B.4 Sources and vevs

We write the 4-dimensional metric in (B.30) as

$$ds^2 = e^\Phi \frac{dr^2}{r^2} + h_{\mu\nu} dx^\mu dx^\nu = e^\Phi \frac{dr^2}{r^2} - E^0 E^0 + \delta_{ab} E^a E^b \,. \quad \text{(B.45)}$$

In order to compute the vevs we use the identity [25, 26]

$$\frac{1}{2} T_{\mu\nu} \delta h^{\mu\nu} + \mathcal{T}^\nu \delta B_\nu + T_\Phi \delta\Phi = \mathcal{S}_\mu^0 \delta E_0^\mu + \mathcal{S}_\mu^a \delta E_a^\mu + \mathcal{T}_\varphi \delta\varphi + \mathcal{T}^a \delta A_a + \mathcal{T}_\Xi \delta\Xi + \mathcal{T}_\Phi \delta\Phi \,, \quad \text{(B.46)}$$

which holds up to a total derivative, where we used that $B_\nu = A_\nu - \partial_\nu \Xi$, $A_a = E_a^\mu A_\mu$ and where φ is defined by [26]

$$\varphi = E_0^\nu A_\nu - \alpha(\Phi) \,, \quad \text{(B.47)}$$

with $\alpha = e^{-3\Phi/2}$ for the particular model studied here [25] and where

$$\mathcal{S}_\mu^0 = -\left(T_{\mu\nu} E_0^\nu + \mathcal{T}^\rho E_\rho^0 A_\mu \right) \,, \quad \text{(B.48)}$$

$$\mathcal{S}_\mu^a = \left(T_{\mu\nu} E^{\nu a} - \mathcal{T}^\rho E_\rho^a A_\mu \right) \,, \quad \text{(B.49)}$$

$$\mathcal{T}_\varphi = \mathcal{T}^\nu E_\nu^0 \,, \quad \text{(B.50)}$$

$$\mathcal{T}_\Phi = T_\Phi + \mathcal{T}^\nu E_\nu^0 \frac{d\alpha}{d\Phi} \,, \quad \text{(B.51)}$$

$$\mathcal{T}^a = \mathcal{T}^\nu E_\nu^a \,, \quad \text{(B.52)}$$

$$\mathcal{T}_\Xi = e^{-1} \partial_\mu \left(e \mathcal{T}^\mu \right) \,. \quad \text{(B.53)}$$

The 4-dimensional sources are defined as the leading terms in the expansions of the bulk fields appearing on the right hand side of (B.46). We find the sources $v^\mu, e_a^\mu, m_\mu, \phi, \chi$

defined via

$$E_0^\mu \simeq -r^2 \alpha_{(0)}^{-1/3} v^\mu \,, \tag{B.54}$$

$$E_a^\mu \simeq r \alpha_{(0)}^{1/3} e_a^\mu \,, \tag{B.55}$$

$$A_\mu - \alpha(\Phi) E_\mu^0 \simeq -m_\mu \,, \tag{B.56}$$

$$\Phi \simeq \phi \,, \tag{B.57}$$

$$\Xi \simeq -\chi \,, \tag{B.58}$$

$$\varphi \simeq r^2 \alpha_{(0)}^{-1/3} v^\mu m_\mu \,, \tag{B.59}$$

$$A_a \simeq -r \alpha_{(0)}^{1/3} e_a^\mu m_\mu \,. \tag{B.60}$$

Likewise the vevs are defined as the leading terms in the expansions of the objects that are the responses to the variations written in (B.46), i.e. we define the vevs $S_\mu^0, S_\mu^a, T^0, T^a, \langle O_\phi \rangle, \langle O_\chi \rangle$

$$\mathcal{S}_\mu^0 \simeq r^2 \alpha_{(0)}^{2/3} S_\mu^0 \,, \tag{B.61}$$

$$\mathcal{S}_\mu^a \simeq r^3 S_\mu^a \,, \tag{B.62}$$

$$\mathcal{T}_\varphi \simeq -r^2 \alpha_{(0)}^{2/3} T^0 \,, \tag{B.63}$$

$$\mathcal{T}^a \simeq -r^3 T^a \,, \tag{B.64}$$

$$\mathcal{T}_\Phi \simeq r^4 \alpha_{(0)}^{1/3} \langle O_\phi \rangle \,, \tag{B.65}$$

$$\mathcal{T}_\Xi \simeq -r^4 \alpha_{(0)}^{1/3} \langle O_\chi \rangle \,, \tag{B.66}$$

where

$$\alpha_{(0)} = e^{-3\phi/2} \,. \tag{B.67}$$

Using (B.45), (B.30) as well as the definitions of the 4-dimensional sources (B.54)–(B.56) we can derive the following relation between the 5-dimensional boundary metric $\gamma_{(0)AB}$ and the 4-dimensional sources τ_μ, m_μ and e_μ^a,

$$ds^2 = \gamma_{(0)AB} dx^A dx^B = 2\tau_\mu dx^\mu (du - m_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu \,, \tag{B.68}$$

where $h_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b$ which is the form of a null reduction ansatz for a reduction along u as discussed in section A.1. The fact that the boundary metric of the 5-dimensional asymptotically locally AdS space-time must have a null circle means that the source ϕ which appears in the expansion of Φ is not independent of the other sources. This can be seen by noting that $\gamma_{uu} = e^{2\Phi}$, so that the 5-dimensional FG expansion via (B.4) and (B.6) tells us that

$$e^{2\phi} = \gamma_{(2)uu} = -\frac{1}{2} R_{(0)uu} + 1 = -\frac{1}{4} (\epsilon^{\mu\nu\rho} \tau_\mu \partial_\nu \tau_\rho)^2 + 1 \,, \tag{B.69}$$

where the epsilon tensor is given by $\epsilon^{\mu\nu\rho} = e^{-1} \varepsilon^{\mu\nu\rho}$ where e is the determinant of the TNC vielbein matrix (τ_μ, e_μ^a) and $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita symbol. For more details we refer to [25]. The consequence of this is that the variation of the on-shell with respect to ϕ gives zero since nothing depends on ϕ.

We now relate the 5-dimensional vevs to the 4-dimensional vevs. For all solutions obeying the reduction ansatz the variation of the on-shell action can be written in both a 5-dimensional and a 4-dimensional notation. From a 5-dimensional perspective we have

$$
\delta S_{\text{ren}}^{\text{on-shell}} = \lim_{\epsilon \to 0} \frac{1}{2\kappa_5^2} \int_{r=\epsilon} d^4 x \sqrt{-\gamma} \left(\frac{1}{2} T_{AB} \delta \gamma^{AB} + T_\psi \delta\psi \right)
$$

$$
= \int_{\partial \mathcal{M}} d^3 x e \left(\frac{1}{2} t_{AB} \delta\gamma_{(0)}^{AB} + \langle O_\psi \rangle \delta\psi_{(0)} \right), \tag{B.70}
$$

where we used the fact that $\sqrt{-\gamma_{(0)}} = e = \det\left(\tau_\mu, e_\mu^a\right)$ as follows from (B.68) and the fact that nothing depends on u so that we can perform the u integral. At the same time from a 4-dimensional perspective we also have, using (B.37), (B.46),

$$
\delta S_{\text{ren}}^{\text{on-shell}} = \lim_{\epsilon \to 0} \int_{r=\epsilon} d^3 x \sqrt{-h} \left(\frac{1}{2} T_{\mu\nu} \delta h^{\mu\nu} + \mathcal{T}^\nu \delta B_\nu + T_\Phi \delta\Phi \right) \tag{B.71}
$$

$$
= \int_{\partial \mathcal{M}} d^3 x e \left(-S_\mu^0 \delta v^\mu + S_\mu^a \delta e_a^\mu + T^0 \delta m_0 + T^a \delta m_a + \langle O_\chi \rangle \delta\chi + \langle \tilde{O}_\phi \rangle \delta\phi \right),
$$

with $m_0 = -v^\mu m_\mu$, $m_a = e_a^\mu m_\mu$ and where we used

$$
\psi_{(0)} = 2u - 2\chi, \qquad \langle O_\psi \rangle = -\tfrac{1}{2} \langle O_\chi \rangle, \tag{B.72}
$$

so that $\delta\psi_{(0)} = -2\delta\chi$ and where furthermore $\tilde{O}_\phi$ is given by

$$
\tilde{O}_\phi = O_\phi - \frac{1}{2} \left[v^\mu \left(S_\mu^0 + T^0 m_\mu \right) + e_a^\mu \left(S_\mu^a + T^a m_\mu \right) \right] = 0, \tag{B.73}
$$

which must vanish because of the comment below (B.69). The extra terms added to O_ϕ come from the variation of ϕ due to the $\alpha_{(0)}(\phi)$ factors in (B.54)–(B.56). Equating (B.71) with (B.70) we obtain

$$
\frac{1}{2} t_{AB} \delta\gamma_{(0)}^{AB} + \langle O_\psi \rangle \delta\psi_{(0)} = -S_\mu^0 \delta v^\mu + S_\mu^a \delta e_a^\mu + T^0 \delta m_0 + T^a \delta m_a + \langle O_\chi \rangle \delta\chi + \langle \tilde{O}_\phi \rangle \delta\phi,
$$
$$\tag{B.74}$$

up to total derivatives. The right hand side can be rewritten as follows

$$
-S_\mu^0 \delta v^\mu + S_\mu^a \delta e_a^\mu + T^0 \delta m_0 + T^a \delta m_a + \langle O_\chi \rangle \delta\chi + \langle \tilde{O}_\phi \rangle \delta\phi =
$$

$$
-\tau_\nu T_\chi{}^\nu{}_\mu \delta\hat{v}_\chi^\mu - \left(\tau_{(\mu} \hat{h}_{\nu)\rho}^\chi \hat{v}_\chi^\sigma T_\chi{}^\rho{}_\sigma \right) \delta h^{\mu\nu} + \frac{1}{2} \left(\hat{h}_{\mu\rho}^\chi \hat{h}_{\nu\lambda}^\chi h^{\lambda\sigma} T_\chi{}^\rho{}_\sigma \right) \delta h^{\mu\nu} + \tau_\mu T^\mu \delta\tilde{\Phi}_\chi
$$

$$
+ \left(\langle O_\chi \rangle - \frac{1}{e} \partial_\mu \left(e T^\mu \right) \right) \delta\chi + \left(\hat{e}_{\chi\mu}^a T^\mu - \tau_\nu e^{\mu a} T_\chi{}^\nu{}_\mu \right) \delta M_a - \frac{1}{2} \hat{e}_\chi^{[a} e^{b]\mu} T_\chi{}^\nu{}_\mu \left(\hat{e}_{\chi\rho a} \delta e_b^\rho - \hat{e}_{\chi\rho b} \delta e_a^\rho \right),
$$
$$\tag{B.75}$$

where $\hat{v}_\chi^\mu$, $\hat{e}_{\chi\mu}^a$ and $\tilde{\Phi}_\chi$ are given by (A.8), (A.9) and (A.15) but with m_μ replaced by M_μ which is

$$
M_\mu = m_\mu - \partial_\mu \chi. \tag{B.76}
$$

This does not affect their orthonormality properties. Further we defined $M_a = e_a^\mu M_\mu$ and

$$
T_\chi{}^\mu{}_\nu = -\left(S_\nu^0 + T^0 \partial_\nu \chi \right) v^\mu + \left(S_\nu^a + T^a \partial_\nu \chi \right) e_a^\mu, \tag{B.77}
$$

$$
T^\mu = -T^0 v^\mu + T^a e_a^\mu. \tag{B.78}
$$

The definitions of the 4-dimensional sources (B.68) and (B.72) imply that they transform under the local symmetries as TNC fields

$$\delta e_\mu^a = \tau_\mu \lambda^a + \lambda^a{}_b e_\mu^b \,, \tag{B.79}$$

$$\delta m_\mu = \partial_\mu \sigma + \lambda_a e_\mu^a \,, \tag{B.80}$$

$$\delta \chi = \sigma \,, \tag{B.81}$$

for the same reasons as discussed in section A.1. From this we conclude that

$$\delta e_a^\mu = \lambda_a{}^b e_b^\mu \,, \tag{B.82}$$

$$\delta M_a = \lambda_a + \lambda_a{}^b M_b \,, \tag{B.83}$$

so that we must have the off-shell Ward identities

$$\hat{e}_{\chi\mu}^a T^\mu = \tau_\nu e^{\mu a} T_\chi{}^\nu{}_\mu \,, \tag{B.84}$$

$$0 = \hat{e}_{\chi\nu}^{[a} e^{b]\mu} T_\chi{}^\nu{}_\mu \,, \tag{B.85}$$

$$\langle O_\chi \rangle = \frac{1}{e} \partial_\mu \left(e T^\mu \right) \,. \tag{B.86}$$

Hence we obtain the following relation between the 5- and 4-dimensional vevs

$$\frac{1}{2} t_{AB} \delta \gamma_{(0)}^{AB} + \langle O_\psi \rangle \delta \psi_{(0)} = -\tau_\nu T_\chi{}^\nu{}_\mu \delta \hat{v}_\chi^\mu - \left(\tau_{(\mu} \hat{h}_{\nu)\rho}^\chi \hat{v}_\chi^\sigma T_\chi{}^\rho{}_\sigma \right) \delta h^{\mu\nu}$$
$$+ \frac{1}{2} \left(\hat{h}_{\mu\rho}^\chi \hat{h}_{\nu\lambda}^\chi h^{\lambda\sigma} T_\chi{}^\rho{}_\sigma \right) \delta h^{\mu\nu} + \tau_\mu T^\mu \delta \tilde{\Phi}_\chi \,. \tag{B.87}$$

Using the same reasoning as in section A.2 we conclude from this that the relation between the 5- and 4-dimensional vevs can be summarized as

$$t^{\mu u} = 2 \tilde{\Phi} T^\mu - \hat{v}^\nu \left(T_\chi{}^\mu{}_\nu - T^\mu \partial_\nu \chi \right) \,, \tag{B.88}$$

$$t^{\mu\nu} = -\hat{v}^\mu T^\nu + h^{\mu\rho} \left(T_\chi{}^\nu{}_\rho - T^\nu \partial_\rho \chi \right) \,. \tag{B.89}$$

Note that $T_\chi{}^\mu{}_\nu - T^\mu \partial_\nu \chi$ is independent of χ because we absorbed $T^\mu \partial_\nu \chi$ into the definition of $T_\chi{}^\mu{}_\nu$ (see also (B.77)). Put another way we can use equations (A.35)–(A.41) with

$$T^\mu{}_\nu = T_\chi{}^\mu{}_\nu - T^\mu \partial_\nu \chi \,. \tag{B.90}$$

The Ward identities are then obtained by the dimensional reduction of (B.28) and (B.29) using (B.72) and equations (A.35)–(A.41) with $T^\mu{}_\nu = T_\chi{}^\mu{}_\nu - T^\mu \partial_\nu \chi$. On a flat boundary with $\tau_\mu = \delta_\mu^t$, $h_{\mu\nu} = \delta_{ij} \delta_\mu^i \delta_\nu^j$, $m_\mu = 0$ and $\chi = 0$ this becomes

$$2 T^t{}_t + T^i{}_i = 0 \,, \tag{B.91}$$

$$\partial_\mu T^\mu{}_\nu = 0 \,, \tag{B.92}$$

$$\partial_\mu T^\mu = \langle O_\chi \rangle \,. \tag{B.93}$$

B.5 Consistency of the reduction

In this subsection we will show that the Scherk-Schwarz reduction (B.30) and (B.31) is consistent. We performed the reduction at the level of the action in section B.3. It remains to show that also the equations of motion of the 5-dimensional action reduce correctly. The 5-dimensional equations of motion (B.18) and (B.19) can be written as

$$R_{\mathcal{MN}}^{(5)} = -4\mathcal{G}_{\mathcal{MN}} + \frac{1}{2}\partial_{\mathcal{M}}\psi\partial_{\mathcal{N}}\psi\,, \tag{B.94}$$

$$0 = \partial_{\mathcal{M}}\left(\sqrt{-\mathcal{G}}\mathcal{G}^{\mathcal{MN}}\partial_{\mathcal{N}}\psi\right)\,, \tag{B.95}$$

where the superscript on the Ricci tensor is used to distinguish its MN component from the 4-dimensional Ricci tensor $R_{MN}^{(4)}$.

The Kaluza-Klein ansatz for the metric (B.30) tells us that

$$\mathcal{G}_{MN} = e^{-\Phi}g_{MN} + e^{2\Phi}A_M A_N\,, \quad \mathcal{G}_{Mu} = e^{2\Phi}A_M\,, \quad \mathcal{G}_{uu} = e^{2\Phi}\,, \tag{B.96}$$

$$\mathcal{G}^{MN} = e^{\Phi}g^{MN}\,, \qquad\qquad g^{Mu} = -e^{\Phi}A^M\,, \quad \mathcal{G}^{uu} = e^{-2\Phi} + e^{\Phi}A^M A_M\,. \tag{B.97}$$

Further we have $\sqrt{-\mathcal{G}} = e^{-\Phi}\sqrt{-g}$. The reduction of the 5-dimensional Ricci tensor follows from standard results on circle reductions of gravity (see for example [65]). The components of the 5-dimensional Ricci tensor can be written as follows

$$R_{uu}^{(5)} = -e^{3\Phi}\Box\Phi + \frac{1}{4}e^{6\Phi}F^2\,, \tag{B.98}$$

$$R_{uM}^{(5)} = R_{uu}^{(5)}A_M + \frac{1}{2}\nabla^N\left(e^{3\Phi}F_{MN}\right)\,, \tag{B.99}$$

$$R_{MN}^{(5)} = A_M R_{uN}^{(5)} + A_N R_{uM}^{(5)} - A_M A_N R_{uu}^{(5)} + R_{MN}^{(4)} - \frac{3}{2}\partial_M\Phi\partial_N\Phi$$
$$+ \frac{1}{2}g_{MN}\Box\Phi - \frac{1}{2}e^{3\Phi}F_{MP}F_N{}^P\,. \tag{B.100}$$

Using the Scherk-Schwarz reduction ansatz for ψ given in (B.31) we also have

$$R_{uu}^{(5)} = -4e^{2\Phi} + 2\,, \tag{B.101}$$

$$R_{uM}^{(5)} = -4e^{2\Phi}A_M + 2\partial_M\Xi\,, \tag{B.102}$$

$$R_{MN}^{(5)} = -4e^{-\Phi}g_{MN} - 4e^{2\Phi}A_M A_N + 2\partial_M\Xi\partial_N\Xi\,. \tag{B.103}$$

It is now straightforward to verify that combining (B.98) and (B.101) leads to the equation of motion for Φ given in (B.39). Continuing with (B.99) and (B.102) we obtain the equation of motion for $B_M = A_M - \partial_M\Xi$ given in (B.39). Finally the equations (B.100) and (B.100) lead to the trace-reversed versions of the Einstein equation given in (B.38). We also have the 5-dimensional equation of motion for ψ. This can be seen to reduce to $\partial_M\left(\sqrt{-g}B^M\right) = 0$ which is a consequence of (B.39). We have hereby shown that the reduction (B.30) and (B.31) is consistent.

B.6 Comparison to other approaches

The works [44, 45] and [26, 28] both study asymptotically locally Lifshitz solutions of the EPD model. The setup of [44, 45] includes in principle what we refer to as the upliftable

model but does not study it explicitly. Below we will make a first attempt at a comparison between the two approaches. We will do this for the general class of EPD models for as much as possible. Some statements will however be more specific for the case of the upliftable model.

In the notation of [44, 45] the solution to the equations of motion of the EPD model near a Lifshitz boundary is written as

$$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j \,, \tag{B.104}$$

$$A = A_i dx^i \,, \qquad B = A - d\omega \,. \tag{B.105}$$

The U(1) gauge transformations have been partially fixed by setting $A_r = 0$. In [26, 28] we make the same gauge choice only for $z = 2$. In our notation we would replace the i index by a μ index and replace r by $\log r$. Further ω here is denoted by Ξ,[24] and γ_{ij} is called $h_{\mu\nu}$ here.

In [44, 45] a radial gauge ($F_2 = 1$) is employed for the metric while in [26, 28] we allow for a general function in the rr component of the metric. For example for the upliftable model it is more natural to work with a gauge in which[25] $F_2 = e^{-\Phi}$ so that the 5-dimensional uplifted asymptotically locally AdS metric is written in radial gauge. This difference is more than just a matter of choice because we have shown in [25] that one cannot transform to the $F_2 = 1$ gauge unless the leading term in the expansion of Φ, that we call ϕ, vanishes.[26] This is not always the case and when we impose this extra condition it leads via (B.69) to the condition that τ_μ is hypersurface orthogonal. Hence if we make the assumption that the asymptotically locally Lifshitz boundary conditions of [25] are compatible with radial gauge we need to put the source $\phi = 0$.

In [44, 45] the metric γ_{ij} is written in the ADM decomposition as

$$\gamma_{ij}dx^i dx^j = -n^2 dt^2 + \sigma_{ab}\left(dx^a + n^a dt\right)\left(dx^b + n^b dt\right) \,, \tag{B.106}$$

where a labels the number of spatial dimensions which here is $d = 2$. In order to make contact with the way we set up the definition of the sources we write the ADM decomposition in terms of vielbeins as follows

$$\gamma_{ij}dx^i dx^j = -E^0 E^0 + \delta_{\underline{ab}}E^{\underline{a}}E^{\underline{b}}, \tag{B.107}$$

where underlined indices $\underline{a}$ refer to flat tangent space indices that take as many values as there are spatial coordinates. We keep here with the notation of [44, 45]. Since these works do not use tangent space indices we introduced these underlined indices only in this section for the sake of comparison. We can take without loss of generality

$$E^0 = ndt \,, \qquad E^{\underline{a}} = e^a_{\underline{a}}(dx^a + n^a dt) \,. \tag{B.108}$$

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[24]The source $\omega_{(0)}$ of the Stückelberg field ω is what we call χ.

[25]Similar gauge choices for F_2 are also important for some other EPD models that do not admit an uplift (see the $z = 2$ and $\Delta = 0$ cases discussed in [26]).

[26]For general EPD models we set $\Phi \simeq r^\Delta \phi$ [26, 28] where the value of Δ depends on the details of the EPD action. To the best of our knowledge this Δ parameter does not appear explicitly in [44, 45]. However there is a comment below their equation (5.25) stating that the asymptotic form of the dilaton depends on the potential which is essentially allowing for a Δ in the fall-off of the dilaton.

This allows us to establish the following dictionary between the sources in [44, 45] and those defined in [26, 28]

$$\tau_\mu = \left(n_{(0)}\,,0\right)\,,\qquad v^\mu = n_{(0)}^{-1}\left(-1\,,n_{(0)}^a\right)\,,\tag{B.109}$$

$$h_{\mu\nu}dx^\mu dx^\nu = g_{(0)ab}\left(dx^a + n_{(0)}^a dt\right)\left(dx^b + n_{(0)}^b dt\right)\,.\tag{B.110}$$

Hence in [44, 45] the source τ_μ is always taken to be hypersurface orthogonal.

In the work [26, 28] we also introduce the Newton-Cartan vector m_μ as a source or rather the U(1) invariant combination $M_\mu = m_\mu - \partial_\mu \chi$. By fixing local tangent space transformations (with parameter λ_a) we can fix all but one component of M_μ. The remaining component is related to $\tilde{\Phi}_\chi$ which is defined by (A.8) with m_μ replaced by M_μ. The scalar $\tilde{\Phi}$ has scaling weight $2(z-1)$. Similarly there is a scalar source in the work of [44, 45] that is denoted by ψ following [46]. The main difference between the approach of [44, 45] and [26, 28] lies in the fact that the dilatation weight of ψ denoted by Δ_- does not in general agree with the dilatation weight of $\tilde{\Phi}$. The number of sources (when comparing both approaches in radial gauge and taking τ_μ to be hypersurface orthogonal) thus agrees but for one of them the scaling dimensions differ.

To see where $\tilde{\Phi}$ appears in our near-boundary expansion we consider purely radial solutions, like we studied in section 3.2. Recall that in section 3.2 we set $\tilde{\Phi} = 0$ by hand. If we do not do this then we obtain, using the results of appendices A.1 and B.1,

$$\gamma_{(2)AB} = \delta_A^u \delta_B^u - \frac{1}{3}\tilde{\Phi}\gamma_{(0)AB}\,,\tag{B.111}$$

$$\gamma_{(4,1)AB} = \frac{4}{3}\tilde{\Phi}\delta_A^u \delta_B^u - \frac{2}{3}\tilde{\Phi}^2\gamma_{(0)AB}\,,\tag{B.112}$$

$$\gamma_{(4)AB} = -\frac{1}{2}\tilde{\Phi}\delta_A^u \delta_B^u + \frac{5}{18}\tilde{\Phi}^2\gamma_{(0)AB} - \frac{1}{4}t_{AB}\,,\tag{B.113}$$

$$\psi_{(2)} = \psi_{(4,1)} = \psi_{(4)} = 0\,.\tag{B.114}$$

Components such as $\gamma_{(4,1)AB}$ correspond to logarithmic terms in the expansion. This implies that for example the expansions of the matter fields become

$$\Phi = \frac{2}{3}\tilde{\Phi}r^2 \log r - \frac{1}{8}\left(\rho + 2\tilde{\Phi}\right)r^2 + \dots\,,\tag{B.115}$$

$$A_\mu = r^{-2}\tau_\mu - \frac{4}{3}\log r \tilde{\Phi}\tau_\mu + \frac{1}{6}\tilde{\Phi}\tau_\mu + \frac{1}{4}\rho\tau_\mu + \dots\,,\tag{B.116}$$

where the dots denote subleading terms.

It would be nice to make a direct comparison between the case studied here where V is a sum of two exponential potentials as given in (2.15). However the case where V is the sum of two exponentials is not explicitly studied in [44, 45] so this would have to be worked out first.[27]

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[27]There exists another model used in [25] that contains two dilatons that can be obtained by a similar Scherk-Schwarz reduction as used in the present work admitting a $z = 2$ Lifshitz solution. This model is related in [36] (see around equation (6.40)) to an EPD model with a single exponential potential, which is a case explicitly worked out in that paper. This is done by setting a linear combination of these two scalars equal to a constant, which, however, is not a consistent truncation of the model discussed in [36]. The relation (6.40) of [36] only holds asymptotically at leading order in a near-boundary expansion.

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