The moduli space of vacua of $\mathcal{N} = 2$ class $\mathcal{S}$ theories

Dan Xie and Kazuya Yonekura

*School of Natural Sciences, Institute for Advanced Study, 1 Einstein Drive,
Princeton, NJ 08540, U.S.A.*

E-mail: dxie@ias.edu, yonekura@ias.edu

ABSTRACT: We develop a systematic method to describe the moduli space of vacua of four dimensional $\mathcal{N} = 2$ class $\mathcal{S}$ theories including Coulomb branch, Higgs branch and mixed branches. In particular, we determine the Higgs and mixed branch roots, and the dimensions of the Coulomb and Higgs components of mixed branches. They are derived by using generalized Hitchin's equations obtained from twisted compactification of 5d maximal Super-Yang-Mills, with local degrees of freedom at punctures given by (nilpotent) orbits. The crucial thing is the holomorphic factorization of the Seiberg-Witten curve and reduction of singularity at punctures. We illustrate our method by many examples including $\mathcal{N} = 2$ SQCD, T_N theory and Argyres-Douglas theories.

Contents

1 Introduction

The structure of moduli space of vacua plays a crucial role in studying the dynamics of supersymmetric field theory. Sometimes one can find exact solutions by analyzing various properties of the moduli space, such as the singularity structure, asymptotical behavior, etc; and the typical example is the Seiberg-Witten solutions of $\mathcal{N} = 2$ theory [1, 2].

The full moduli space of four dimensional $\mathcal{N} = 2$ theory has the general structure

$$\bigcup_{\alpha} \mathcal{C}_{\alpha} \times \mathcal{H}_{\alpha}, \tag{1.1}$$

where α labels the components of branches, and $\mathcal{C}_{\alpha}$ are parametrized by Coulomb moduli fields and $\mathcal{H}_{\alpha}$ are parametrized by Higgs moduli fields. The $\mathcal{C}_{\alpha}$ are special Kahler manifolds and the $\mathcal{H}_{\alpha}$ are hyperkahler manifolds, and their metrics do not mix due to $\mathcal{N} = 2$ supersymmetry [3, 4]. There is usually a pure Coulomb branch, which we simply denote as $\mathcal{C}$; sometimes, there is also a pure Higgs branch $\mathcal{H}$ which touches with the Coulomb branch at a single point (such as SCFT points). Mixed branches are emanating from special loci of the Coulomb branch as shown in figure 1, and the corresponding submanifolds on the Coulomb branch are called the Higgs or mixed branch roots [4].

The Coulomb branch of $\mathcal{N} = 2$ theory has been studied extensively, and the Seiberg-Witten solution can often be found easily by using the brane construction [5–7], or using the relation to integrable systems [8, 9]. However, the pure Higgs branches are much less studied, see e.g., [4, 10–15] for some examples. Furthermore, the full structure of moduli spaces including mixed branches was studied only in very few examples such as $\mathcal{N} = 2$ SQCD [4, 10] using the non-renormalization theorem and exact solution on the Coulomb branch. (See also the discussion using type IIA brane construction [5, 16–18].) But the method there is based on solving the equations of motion of Lagrangians explicitly, and it is not easily generalized to more complicated quivers or non-Lagrangian theories. The aim of this paper is to give a systematic method to study the full structure of moduli space which is applicable to a large class of theories including non-Lagrangian theories.

We are going to study the full moduli space of $\mathcal{N} = 2$ theories obtained by twisted compactification of 6d $\mathcal{N} = (2, 0)$ A_{N-1} theory on a Riemann surface, called class $\mathcal{S}$ theories [19–23].[1] Instead of using field theory method, we are going to use the geometric method developed in [26–28] (see also [29]) to determine various branches.

Let's first recall some of the basic features of M5 brane construction which leads to this class of theories. There are five scalars φ_I ($I = 1, 2, 3, 4, 5$) in the adjoint representation of $A_{N-1} = \mathrm{SU}(N)$ which describe the transverse fluctuation of M5 branes. Four dimensional theory is derived by twisted compactification of the 6d theory on a Riemann surface C, and one get a Higgs field Φ in the canonical (or cotangent) bundle of C which describes the Coulomb branch deformations. There are other real scalars $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ which are

---

[1]By class $\mathcal{S}$ theories, we mean four dimensional theories which can be derived from compactifying 6d theory on a Riemann surface with all kinds of defects, such as regular defects, irregular defects, etc. This class of theories include many well studied theories such as SQCD, Argyres-Douglas theory [24], Maldacena-Nunez theory [25], etc.
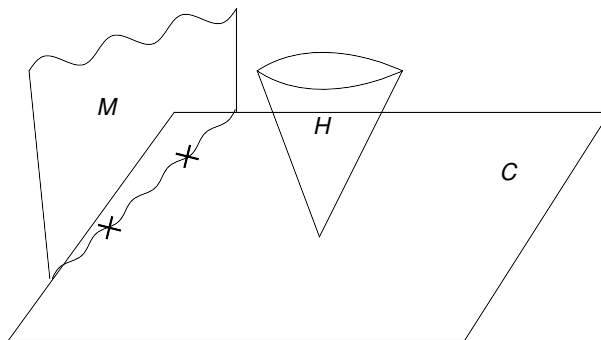
Figure 1. The vacuum structure of $\mathcal{N} = 2$ theory: C: Coulomb branch; H: Higgs branch; M: Mixed branch.

in the trivial bundle of C and describe the Higgs branch deformations. Generically, the Coulomb branch and Higgs branch are determined as follows:

- The Coulomb branch is described by turning on only the deformations in Φ, and the Seiberg-Witten curve is then given by the spectral curve of Φ:

$$0 = \det(xI_N - \Phi) = x^N + \sum_{i=2}^{N} \phi_i(z)x^{N-i}, \qquad (1.2)$$

here ϕ_i are degree i holomorphic differential on Riemann surface.

- The pure Higgs branch is described by turning on only $\vec{\varphi}$ deformations (we can get another scalar from $B_{\mu\nu}$ field on M5 brane [16].).

- Mixed branches are described by turning on both $\vec{\varphi}$ and Φ.

Let's discuss the mixed branch in more detail. The deformation in $\vec{\varphi}$ is simple: they can be diagonalized with constant eigenvalues:

$$\vec{\varphi} = \text{diag}(\vec{a}^{(1)}I_{n_1}, \cdots, \vec{a}^{(k)}I_{n_k}), \qquad (1.3)$$

where n_s are integers such that $\sum_{s=1}^{k} n_s = N$, I_{n_s} are $n_s \times n_s$ the unit matrices, and $\vec{a}^{(s)}$ are constants. Now the crucial thing is that brane dynamics requires the commuting condition $[\Phi, \vec{\varphi}] = 0$, which impies that the Seiberg-Witten curve has to be factorized as:

$$\det(xI_N - \Phi) = x^N + \sum_{i=2}^{N} \phi_i x^{N-i} = 0 \rightarrow \prod_{s=1}^{k} \left(x^{n_s} + \sum_{i=1}^{n_s} \phi_{s,i}(z)x^{n_s-i} \right) = 0, \qquad (1.4)$$

here $\phi_{s,i}(z)$ are various holomorphic differentials, and n_s are the same as above.

There are more variations due to punctures on the Riemann surface C. Regular punctures are classified by Young diagrams with total number of boxes N, and one can define an ordering between two Young diagrams. In the mixed branch, a Young diagram Y^D of SU(N) theory is split into a collection of Young diagrams Y_s^D of SU(n_s) theory, see figure 2. (The superscript D will be explained later in this paper.) One can form a new
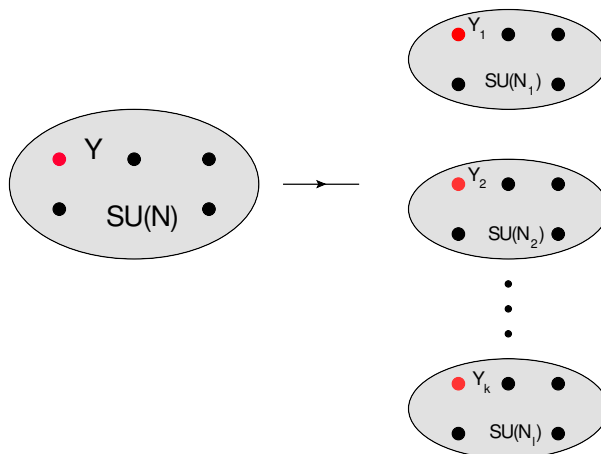
Figure 2. The Higgs and mixed branch roots can be described as: the $SU(N)$ Hitchin system is decomposed into $SU(n_1) \times SU(n_2) \times \ldots \times SU(n_k) \times U(1)^{k-1}$ Hitchin systems, and the regular punctures of $SU(N)$ are decomposed into sums of regular punctures of lower rank Hitchin systems.

Young diagram Y'^D with N boxes by assembling Y_s^D together, and the constraint on Y'^D is that

$$Y'^D \leq Y^D, \tag{1.5}$$

where we used the ordering between two Young diagrams. There is a mixed branch corresponding to each collection Y_s^D.

Unlike regular singularity, the constraint from irregular singularities is more rigid, as the boundary condition of Φ is fixed at the puncture and sometimes the curve can not be factorized in an arbitrary way. The factorization pattern of Seiberg-Witten curve of equation (1.4) has to be consistent with the boundary condition on the irregular puncture.

This paper is organized as follows. In section 2, we give a general algorithm of how to find mixed branch roots, and how to count the dimensions of the Coulomb and Higgs factors. In section 3 we derive those rules following the strategy outlined above. In section 4 and appendix A, we reproduce the results of $\mathcal{N} = 2$ SQCD by our method. In section 5, theories defined using regular singularities are discussed. In section 6, we study Argyres-Douglas theories. Finally in section 7, we give a short discussion.

2 General rules for finding mixed branches

In general, the moduli space of $\mathcal{N} = 2$ supersymmetric field theory has the general form

$$\bigcup_\alpha \mathcal{C}_\alpha \times \mathcal{H}_\alpha, \tag{2.1}$$

where α runs over all the components of the moduli space, and $\mathcal{C}_\alpha$ and $\mathcal{H}_\alpha$ are Coulomb factor and Higgs factor, respectively. In other words, $\mathcal{C}_\alpha$ is parametrized by vevs of vector multiplets, and $\mathcal{H}_\alpha$ is parametrized by vevs of hypermultiplets. For Lagrangian theory, the Higgs branch can be found using classical equations of motion due to non-renormalization

theorem [4]. Coulomb branch is much more difficult to determine because one needs to take into account various quantum corrections, and the result can be elegantly summarized using Seiberg-Witten curve.

In this section we would like to state the rules of how to find the roots of these mixed branches $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$ and give the formulas for the dimensions of the Coulomb factor $\mathcal{C}_\alpha$ and the Higgs factor $\mathcal{H}_\alpha$. Derivations of the rules are given in the next section 3. One of the branches is the pure Coulomb branch $\mathcal{C} \times \{0\}$, which we denote as just $\mathcal{C}$. Then, the root of the branch $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$ is the intersection of it with $\mathcal{C}$.

2.1 Pure Coulomb branch of class $\mathcal{S}$ theories

The Seiberg-Witten curve describing the Coulomb branch $\mathcal{C}$ is given by the general form [19, 20, 23],

$$0 = x^N + \sum_{i=2}^{N} \phi_i(z) x^{N-i}, \tag{2.2}$$

where z is the coordinate of the base Riemann surface C, and x is the coordinate of the fiber of the canonical (or cotangent) bundle $K = T^*C$. The $\phi_i(z)$ is a section of the line bundle $K^{\otimes i}$, i.e., it is an i-th differential on C. Neglecting punctures, its contribution to the dimension of Coulomb moduli can be found using the Riemann-Roch theorem,

$$\dim(\phi_i) = \dim H^0(C, K^{\otimes i}) = (2i - 1)(g - 1), \tag{2.3}$$

where we used the vanishing theorem $H^0(C, K^{1-i}) = 0$ for $i > 1$.

On the Riemann surface C, we can also add various regular and irregular punctures. Let us recall some properties of them. We consider local properties near one puncture, and we take the coordinate so that the puncture is at $z = 0$ for regular singularity.

Regular singularities. Regular (or tame) singularities [19] are classified by a partition of N, $[m_1, \cdots, m_\ell]$, which satisfies $N = m_1 + \cdots + m_\ell$. Without loss of generality we take $m_1 \geq m_2 \cdots \geq m_\ell$. This partition can be represented by a Young diagram Y such that the a-th column has m_a boxes (i.e., its hight is m_a). In our convention, a full puncture is represented by $[1, 1, \ldots, 1]$, and a simple puncture is represented by $[N - 1, 1]$.

The order of pole p_i at the regular singularity of the differential $\phi_i(z)$ can be easily found from the Young diagram[2]

$$\phi_i(z) = \mathcal{O}(z^{-p_i}), \qquad p_i = i - s_i, \tag{2.4}$$

where s_i is the height of the ith box of the Young diagram. The counting is from left to right and from bottom to top. See figure 3 for examples of Young diagrams.

The local contribution of a regular puncture to the Coulomb branch dimension is

$$d(Y^D) = \sum_{i=2}^{N} p_i = \frac{1}{2}\left(N^2 - \sum_{a=1}^{\ell} m_a^2\right), \tag{2.5}$$

---

[2]Unless otherwise stated, we take mass parameters of regular singularities to be zero.
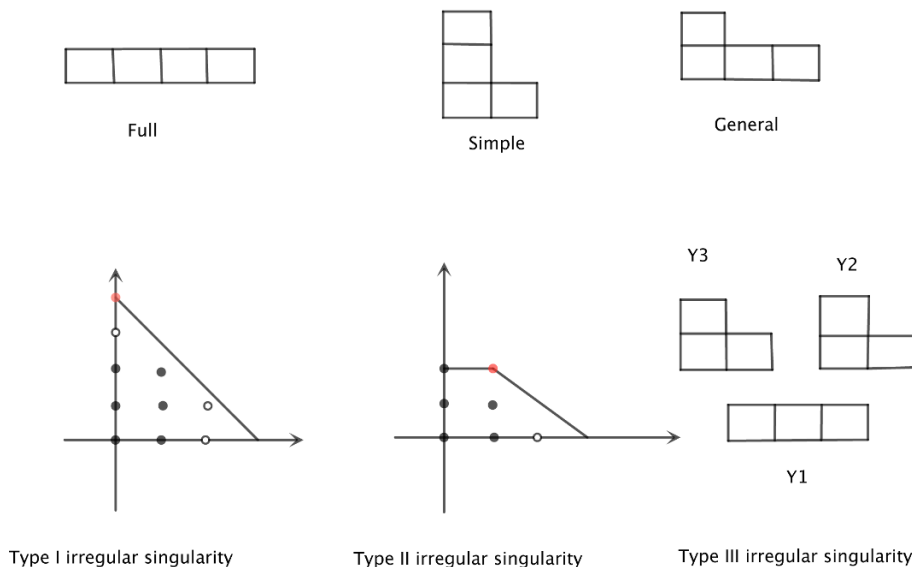
Figure 3. Upper: regular puncture are represented by Young diagrams. Bottom: two types of irregular singularities are represented by Newton polygon, and a third type is represented by a collection of Young diagrams.

where Y^D is the Young diagram obtained by transposing Y, i.e., the rows of Y^D are columns of Y. The reason for using Y^D in (2.5) will become clearer in the next section.

For later use, let us define partial ordering of two Young diagrams. We define

$$Y^D \geq Y'^D \iff p_i \geq p_i^{'} \iff \sum_{b=1}^{a} \tilde{m}_a \geq \sum_{b=1}^{a} \tilde{m}_a'. \tag{2.6}$$

where $Y^D = [\tilde{m}_a]$. The second equivalence can be checked by using (2.4). Notice that this is only a partial ordering: it is possible that neither $Y^D \geq Y'^D$ nor $Y^D \leq Y'^D$.

Irregular singularities. We can also have irregular (or wild) singularities, which are needed for asymptotically free theories and Argyres-Douglas theories. The irregular singularities which are relevant for SQCD will be discussed in detail in appendix A. The irregular singularities which are relevant for Argyres-Douglas theories can also be summarized by a Newton polygon, see figure 3, and the local contribution to the Coulomb moduli can also be found from Newton polygon, for details, see [23]. The leading order eigenvalues of x can be read from the slop of the Newtwon polygon:

$$\text{Type I}: \quad x \sim \frac{1}{z^{2+\frac{k}{N}}}(1, \omega_N, \ldots, \omega_N^{N-1})$$

$$\text{Type II}: \quad x \sim \frac{1}{z^{2+\frac{k}{N-1}}}(0, 1, \omega_{N-1}, \ldots, \omega_{N-1}^{N-2})$$

$$\text{Type III}: x \sim \frac{A_n}{z^n} + \frac{A_{n-1}}{z^{n-1}} + \ldots + \frac{A_1}{z} \tag{2.7}$$

here k is the height of the marked point in the Newton polygon of figure 3, $\omega_\ell = \exp(2\pi i/\ell)$, and the eigenvalue degeneracy of A_i is determined by the Young diagrams Y_i.

The major feature of irregular singularity is that the leading order behavior is fixed as the UV data, and we can not change it by tuning the Coulomb branch moduli. For regular singularities, one can actually tune the Coulomb branch moduli to change the singularity.

2.2 Mixed branches

Now we can state our general rules for finding the roots of mixed branches $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$.

Step 1. The first step is to choose a partition of N: $X = [n_1, n_2, \cdots, n_k]$. Then we take the curve (2.2) to be of factorized form

$$x^N + \sum_{i=2}^{N} \phi_i(z) x^{N-i} = 0 \rightarrow \prod_{s=1}^{k} \left(x^{n_s} + \sum_{i=1}^{n_s} \phi_{s,i}(z) x^{n_s-i} \right) = 0, \tag{2.8}$$

where $\phi_{s,i}$ are holomorphic differentials on the Riemann surface, and we also have $\sum_s \phi_{s,1} = 0$ due to the traceless condition.

The existence of type I and type II irregular singularities would constrain the possible factorization X:

- Type I irregular singularity: define d as the maximal common divisor of (k, N). Then the maximal partition X is

$$X_{\text{max}} = [\frac{N}{d}, \frac{N}{d}, \dots, \frac{N}{d}], \tag{2.9}$$

and other possible factorization is derived by combining the columns of X.

- Type II irregular singularity: define d as the maximal common divisor of $(k, N-1)$, then the maximal partition X is

$$X_{\text{max}} = [\frac{N-1}{d}, \frac{N-1}{d}, \dots, \frac{N-1}{d}, 1] \tag{2.10}$$

and other partitions are found by combining columns expect the last column with height one.

When there are more than one irregular singularity, X is constrained to be compatible with all irregular singularities.

Step 2. The second step is to take into account local punctures. After the above global factorization, the SU(N) theory is split into a sum of SU(n_s) (and U(1)) theories. For each regular puncture, a dual Young diagram Y^D of the SU(N) theory is split into a sum of diagrams Y_s^D for SU(n_s) theories. The important point is that these Y_s^D are not uniquely determined from Y^D. A choice of the collection Y_s^D corresponds to the choice of a mixed branch we consider.

One can form a Young diagram of N boxes Y'^D by combining Y_s^D. The columns of Y'^D are taken to be the columns of Y_s^D. Then the only constraint on Y_s^D is

$$Y'^D \le Y^D. \tag{2.11}$$

Step 3. We consider every possible set of X and Y_s^D, and find the curve consistent with the above conditions. If (2.8) cannot be further factorized in a generic curve, this is the root of a branch $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$, where the label α is specified as $\alpha = \{X, Y_s^D\}$. If (2.8) is factorized in a generic curve, this is just a boundary of some other mixed branch and we discard this case.

Now let us give the formulas for the dimensions of the branches.

Coulomb factor. Once the factorization and local data are given, the dimension of the Coulomb factor $\mathcal{C}_\alpha$ can be easily found from the factorized Seiberg-Witten curve: each factor is the spectral curve of a lower rank SU(n_s) or U(1) Hitchin system. Each U(1) Hitchin system contributes dimension g to the Coulomb branch dimension.

Here we give an explicit formula for the Coulomb branch factor if there are only regular punctures. First, let us define local contribution of a puncture p as

$$\dim_{\mathbb{C}} \mathcal{C}_\alpha(p) \equiv \sum_{s=1}^k d(Y_s^D), \tag{2.12}$$

where $d(Y_s^D)$ is defined as in (2.5). Then, the total dimension is given by

$$\dim_{\mathbb{C}} \mathcal{C}_\alpha = (k-1)g + \sum_{s=1}^k (n_s^2 - 1)(g-1) + \sum_p \dim_{\mathbb{C}} \mathcal{C}_\alpha(p), \tag{2.13}$$

where k is the number of elements in the partition $X = [n_1, \cdots, n_k]$. The local contribution of the irregular singularity can be found using the method discussed in [23].

Higgs factor. The dimension of the Higgs factor $\mathcal{H}_\alpha$ has several contributions. There is global contribution due to the factorization of the Seiberg-Witten curve. The quaternionic dimension of this part is simply $k - 1$.

There is also contribution from local punctures, and an explicit form of the regular puncture contribution can be easily written down. Defining Y' as the dual Young diagram of Y'^D, the local Higgs branch dimension is given by

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha(p) = d(Y') - d(Y). \tag{2.14}$$

This local contribution is always non-negative due to the constraint on Y' given by (2.11), which is actually equivalent to $Y' \geq Y$. There is no local contribution to Higgs branch from type I and type II irregular singularity. The total dimension of the Higgs branch is given by (assuming there is only regular puncture):

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha = k - 1 + \sum_p \dim_{\mathbb{H}} \mathcal{H}_\alpha(p). \tag{2.15}$$

3 Vacuum structure of class $\mathcal{S}$ theories

In this section, we derive the rules for finding mixed branches stated in the previous section. The reader who is only interested in applications of the rules can skip this section. The essential ingredients are generalized Hitchin's equations and 5d maximal Super-Yang-Mills (SYM) twisted on the Riemann surface C [26–28], $T^\rho[\mathrm{SU}(N)]$ theories of Gaiotto and Witten [30] placed at punctures [22], and irregular singularity classified in [23].

3.1 5d maximal SYM and generalized Hitchin's equations

The class $\mathcal{S}$ theories are defined as the low energy limit of 6d $\mathcal{N} = (2,0)$ theory compactified on a Riemann surface C with punctures. If we further compactify it on a circle S^1 so that the 6d theory is placed on $\mathbb{R}^3 \times S^1 \times C$, we get an S^1 compactification of four dimensional theory (in the small area limit of C). The same configuration can be looked at in a different way. We first compactify the theory on S^1 and get 5d maximal SYM, then we study the 5d theory on $\mathbb{R}^3 \times C$. Actually, the 5d maximal SYM might be enough to compute (BPS or protected) quantities of the 6d $\mathcal{N} = (2,0)$ theory on S^1 (see e.g., [31, 32]). Then, denoting the are of the Riemann surface as $\mathcal{A}$, we claim (see [28] for more discussions and the scope of validity of the claim),[3]

Vacuum moduli space of 4d class $\mathcal{S}$ theory on $\mathbb{R}^3 \times S^1$

$=$ Vacuum moduli space of 6d class $\mathcal{N} = (2,0)$ theory on $\mathbb{R}^3 \times S^1 \times C|_{\mathcal{A} \to 0}$

$=$ Vacuum moduli space of 5d maximal SYM on $\mathbb{R}^3 \times C|_{\mathcal{A} \to 0}$. (3.1)

Furthermore, in this relation, the Coulomb branch and Higgs branch are exchanged,

Coulomb (Higgs) branch of 4d class $\mathcal{S}$ theory on $\mathbb{R}^3 \times S^1$

$=$ Higgs (Coulomb) branch of 5d maximal SYM on $\mathbb{R}^3 \times C$. (3.2)

The Coulomb branch of 4d theory on S^1 is a Hyperkahler manifold. The Coulomb branch has a distinguished complex structure which we denote as I. This complex structure does not depend on the radius of S^1 [33], so we can learn the moduli space of 4d theory using complex structure I of Coulomb branch of 4d theory on S^1.[4] Moreover, in complex structure I there is a fibration structure which can be identified with the Seiberg-Witten fibration over the Coulomb moduli space.

Now we are going to describe the structure of the moduli space of vacua using the 5d theory. The 5d maximal SYM contains gauge fields A_M ($M = 0, 1, 2, 3, 4$) and five real adjoint scalars φ_I ($I = 1, 2, 3, 4, 5$). It must also be twisted on the Riemann surface so that half of the supersymmetry is preserved by the compactification. Then, taking the coordinate of $\mathbb{R}^3 \times C$ as (x^μ, z) where $\mu = 0, 1, 2$ and z is a complex coordinate on C, the fields of the 5d SYM are given by

$$(\vec{\varphi}, A_\mu), \quad (\Phi, A_{\bar{z}}). \tag{3.3}$$

Here, $\Phi (= \varphi_4 + i\varphi_5)$ is a complex adjoint field, which takes values in $K \otimes \mathrm{ad}(E)$, where $K = T^*C$ is the canonical bundle and $\mathrm{ad}(E)$ is the vector bundle in the adjoint representation of the gauge group. The fact that Φ is a section of the canonical bundle K comes from the twisting of $\mathrm{SO}(2)_R \subset \mathrm{SO}(5)_R$ R-symmetry to preserve eight supercharges. The $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ are real adjoint fields in $\mathrm{ad}(E)$, A_μ is the gauge field on $\mathbb{R}^3$ and $A_{\bar{z}}$ is the gauge field on C. The $\vec{\varphi}$ is a triplet of the $\mathrm{SU}(2)_R = \mathrm{SO}(3)_R \subset \mathrm{SO}(5)_R$ R-symmetry. The

---

[3]See also [12] for finite area effects.
[4]This is true not only for the Coulomb branch but also for the entire vacuum moduli space, up to the caveat discussed in section 4 of [28].

combinations $(\vec{\varphi}, A_\mu)$ and $(\Phi, A_{\bar{z}})$ are taken so that each of them form a single multiplet of the supersymmetry. An important point is that $(\Phi, A_{\bar{z}})$ corresponds to the Coulomb moduli fields of the 4d class $\mathcal{S}$ theory, and $(\vec{\varphi}, A_\mu)$ corresponds to the Higgs moduli fields of the 4d theory.

The classical equations for supersymmetric vacua of the 5d SYM are given by generalized Hitchin's equations [26] (see also [29]),[5] which in general can describe theories with four supercharges. In the case of eight supercharges (i.e., 4d $\mathcal{N} = 2$), they are simplified as [28]

$$F_{z\bar{z}} + [\Phi, \Phi^\dagger] = 0 \tag{3.4}$$

$$D_{\bar{z}}\Phi = 0 \tag{3.5}$$

$$D_z\vec{\varphi} = D_{\bar{z}}\vec{\varphi} = 0 \tag{3.6}$$

$$[\vec{\varphi}, \Phi] = 0 \tag{3.7}$$

$$[\varphi_i, \varphi_j] = 0 \quad (i, j = 1, 2, 3) \tag{3.8}$$

where $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$.

The Coulomb branch of the 4d theory (or equivalently the Higgs branch of 5d theory) is described by the first two equations which are precisely the Hitchin's equations [34, 35]. This branch can be determined using the classical Hitchin's equations due to nonrenormalization theorem.

The classical configuration of the Higgs branch of the 4d theory (or Coulomb branch of 5d theory) can be described by the expectation values of the scalar fields φ_i, whose solutions are just constant matrices, and there are various quantum corrections to the metric of the moduli space which are difficult to calculate. However, for our purpose of studying the mixed branch structure and counting dimensions of moduli space, we can still use the classical equations due to the nonrenormalization theorem discussed in [28]. Due to the commuting condition between Φ and φ_i, we get the breaking pattern on the Hitchin's equations, and therefore determine the mixed branch roots.

3.2 Factorization of the Seiberg-Witten curve

Let us solve these equations in detail. From (3.8), the $\vec{\varphi}$ are simultaneously diagonalized by SU(N) gauge transformations. Let us assume that this is done. Then we get

$$\vec{\varphi} = \begin{pmatrix} \vec{a}^{(1)} I_{n_1} & 0 & \dots & 0 \\ 0 & \vec{a}^{(2)} I_{n_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \vec{a}^{(k)} I_{n_k} \end{pmatrix} \tag{3.9}$$

Here $N = n_1 + n_2 + \cdots + n_k$ defines a partition of N, $X = [n_s] = [n_1, \cdots, n_k]$. The I_{n_s} are $n_s \times n_s$ unit matrices, and $\vec{a}^{(s)} = (a_1^{(s)}, a_2^{(s)}, a_3^{(s)})$ are vectors with $\vec{a}^{(s)} \neq \vec{a}^{(t)}$ for $s \neq t$ and

---

[5]In the generalized Hitchin system, the fields $\Phi_1 = (\varphi_1 + i\varphi_2)/\sqrt{2}$, $\Phi_2 = \Phi$ and $A_{\bar{z}}$, are used. We must supplement them with φ_3 and A_μ to describe the complete moduli space of vacua.

$\sum_{s=1}^{k} n_s \vec{a}^{(s)} = 0$. These $\vec{a}^{(s)}$ are constants because of (3.6). The SU(N) gauge symmetry is broken to a subgroup $H = \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_k) \times \mathrm{U}(1)^{k-1}$. Furthermore, (3.6) and (3.7) gives a constraint

$$\Phi = \begin{pmatrix} \Phi^{(1)} & 0 & \dots & 0 \\ 0 & \Phi^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \Phi^{(k)} \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} A_{\bar{z}}^{(1)} & 0 & \dots & 0 \\ 0 & A_{\bar{z}}^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_{\bar{z}}^{(k)} \end{pmatrix} \tag{3.10}$$

where $\Phi^{(s)}$ and $A_{\bar{z}}^{(s)}$ are $n_s \times n_s$ matrices. Then (3.4) and (3.5) are equivalent to

$$F_{z\bar{z}}^{(s)} + [\Phi^{(s)}, (\Phi^{(s)})^{\dagger}] = 0, \quad D_{\bar{z}} \Phi^{(s)} \equiv \partial_{\bar{z}} \Phi^{(s)} + [A_{\bar{z}}^{(s)}, \Phi^{(s)}] = 0, \tag{3.11}$$

for each $s = 1, \cdots, k$. Therefore, each pair $(A_{\bar{z}}^{(s)}, \Phi^{(s)})$ satisfies Hitchin's equations. If the vevs of $(A_{\bar{z}}^{(s)}, \Phi^{(s)})$ completely break the SU(n_s) gauge symmetries, the unbroken gauge group is U(1)$^{k-1}$. Then, taking the zero mode of A_μ ($\mu = 0, 1, 2$) on C and dualizing these gauge fields to scalar fields on $\mathbb{R}^3$, we get dual scalars $b^{(s)}$ ($s = 1, \cdots, k$) with $\sum_s n_s b^{(s)} = 0$.

For the moment, we neglect punctures. Then, the general structure is as follows. Mixed branches are labelled by the partition of N, $X = [n_s]$. The Higgs factor $\mathcal{H}_\alpha^{\mathrm{bulk}}$ consists of the moduli fields $(\vec{a}^{(s)}, b^{(s)})$ which contribute to the quaternionic dimension of $\mathcal{H}_\alpha$ as $k - 1$ (or real dimension $4(k-1)$). Here we added the word "bulk" to $\mathcal{H}_\alpha^{\mathrm{bulk}}$ because we are neglecting punctures. The Coulomb factor $\mathcal{C}_\alpha$ consists of the solutions of Hitchin's equations with the gauge group $H = \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_k) \times \mathrm{U}(1)^{k-1}$.

If generic solutions of Hitchin's equations for $(A_{\bar{z}}^{(s)}, \Phi^{(s)})$ do not break SU(n_s) completely, the above branch is just a boundary of a more larger branch. For example, consider a simple case where the only solution to the SU(n_s) Hitchin's equations is trivial, $(A_{\bar{z}}^{(s)}, \Phi^{(s)}) = 0$. This can happen e.g., when the genus is $g = 0$ and there are no punctures. Then, we can smoothly go to another branch as

$$\vec{a}^{(s)} I_{n_s} \to \mathrm{diag}(\vec{a}^{(s,1)}, \cdots, \vec{a}^{(s,n_s)}) \tag{3.12}$$

while satisfying (3.4)–(3.8). This means that we can go to the branch in which $n_s \to 1 + \cdots + 1$. More generally, if $(A_{\bar{z}}^{(s)}, \Phi^{(s)})$ breaks SU(n_s) to a nontrivial subgroup $H_s' \subset \mathrm{SU}(n_s)$, we can smoothly turn on a vev of $\vec{\varphi}$ in the Cartan subalgebra of the unbroken group H_s'. The integer n_s is further partitioned to $n_s = n_{s,1} + n_{s,2} + \cdots$. So we do not consider the original partition $[n_s]$ as a separate mixed branch.

The spectral curve of the original Hitchin system is given as

$$0 = \det(xI_N - \Phi) = x^N + \sum_{i=2}^{N} \phi_i(z) x^{N-i}. \tag{3.13}$$

Under the decomposition (3.10), the curve is factorized as:

$$\det(xI_N - \Phi) = x^N + \sum_{i=2}^{N} \phi_i x^{N-i} = 0 \to \prod_{s=1}^{k} \left(x^{n_s} + \sum_{i=1}^{n_s} \phi_{s,i}(z) x^{n_s-i} \right) = 0. \tag{3.14}$$

Constraint on factorization from Irregular singularity. If there are irregular singularities, the factorization pattern of the SW curve is constrained because the irregular singularity can not factorize in an arbitrary way. The reason is that the leading order term is fixed and define the UV theory, so we can not tune the leading order behavior of the SW curve. Near the irregular singularity, the curve has the following form (we put the singularity at $z = \infty$):

$$
\begin{aligned}
\text{Type I}: \qquad & (x^{N/d} + z^{k/d})^d = 0, \\
\text{Type II}: \quad & x(x^{(N-1)/d} + z^{k/d})^d = 0, \\
\text{Type III}: \qquad & (x + z^{n-2})^N = 0,
\end{aligned}
\tag{3.15}
$$

where in the Type I (or type II) case, d is the maximal common divisor of k and N (or $N-1$), and we have neglected all the coefficients. From these, one can easily see the rules stated in section 2.

3.3 Local moduli from regular singularities

Regular punctures are realized [22, 28] by coupling the 3d $\mathcal{N} = 4$ $T^\rho[\mathrm{SU}(N)]$ theories [30] to the 5d SYM. These theories are extended in the $\mathbb{R}^3$ direction, and they are localized at punctures on the Riemann surface C.

3.3.1 $T^\rho[\mathrm{SU}(N)]$ theories

We first review some properties of $T^\rho[\mathrm{SU}(N)]$ necessary for our purposes; see [22, 30] for more details.

Moduli space of $T[\mathrm{SU}(N)]$ theory. First, let us recall some of the important properties of the $T[\mathrm{SU}(N)]$ theory, i.e., $\rho = 0$. It has Coulomb branch and Higgs branch, and there are $\mathrm{SU}(N)_\mathcal{C} \times \mathrm{SU}(N)_\mathcal{H}$ global symmetries where $\mathrm{SU}(N)_\mathcal{C}$ acts on the Coulomb branch and $\mathrm{SU}(N)_\mathcal{H}$ acts on the Higgs branch. Since it has 3d $\mathcal{N} = 4$ supersymmetry, both of the branches are hyperkahler, and hence there are hyperkahler moment maps $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ and $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ of the flavor symmetries $\mathrm{SU}(N)_\mathcal{C}$ and $\mathrm{SU}(N)_\mathcal{H}$, respectively. They take values in the Lie algebra of $\mathrm{SU}(N)_\mathcal{C} \times \mathrm{SU}(N)_\mathcal{H}$. It is convenient to define holomorphic moment maps as $\nu_h = \nu_1 + i\nu_2$ and $\mu_h = \mu_1 + i\nu_2$. Then, they are nilpotent matrices, i.e., $\nu_h^N = \mu_h^N = 0$, or equivalently their eigenvalues are zero, when there is no mass deformation. (Mass deformation will be important later.) Vacuum moduli space is parametrized by μ_h and ν_h.

Mixed branches of the $T[\mathrm{SU}(N)]$ theory are given as follows. Let us take a partition of N, $Y' = [m'_a] = [m'_1, m'_2 \cdots, m'_\ell]$. Corresponding to each partition, we can define an embedding of $\mathrm{SU}(2)$ into $\mathrm{SU}(N)$,

$$
\rho' : \mathrm{SU}(2) \to \mathrm{SU}(N), \tag{3.16}
$$

such that the fundamental (N-dimensional) representation of $\mathrm{SU}(N)$ is decomposed into irreducible representations of $\mathrm{SU}(2)$ as

$$
\mathbf{N} \to \mathbf{m'_1} + \mathbf{m'_2} + \cdots + \mathbf{m'_\ell}, \tag{3.17}
$$

where $\mathbf{m}_a$ is the m_a dimensional (spin $(m_a - 1)/2$) representation of SU(2). Similarly, we can consider the dual Young diagram of Y' which we call $Y'^D = [\tilde{m}'_a]$, and we can define another embedding $\rho'_D : \mathrm{SU}(2) \to \mathrm{SU}(N)$.

Define $e' = \rho'(\sigma^+)$ and $e'_D = \rho'_D(\sigma^+)$, i.e., the image of the raising operator σ^+ of SU(2). Then, they are nilpotent matrices (because acting raising operators too many times gives zero). Mixed branches are classified by Y', and the branch labeled by Y' is given as

$$(\nu_h, \mu_h) \in O_{e'_D} \times O_{e'} \tag{3.18}$$

where, for a given matrix A, O_A is the orbit of A by the action of the complexified group $\mathrm{SU}(N)_{\mathbb{C}} = \mathrm{SL}(N)$, $O_A = \{gAg^{-1} : g \in \mathrm{SU}(N)_{\mathbb{C}}\}$. This means that the mixed branch structure $\bigcup_\alpha \mathcal{C}_\alpha \times \mathcal{H}_\alpha$ of the $T[\mathrm{SU}(N)]$ theory is given as $\alpha = Y'$, $\mathcal{C}_\alpha = O_{e'_D}$ and $\mathcal{H}_\alpha = O_{e'}$.

Let us compute the dimension of the orbit $O_{e'}$. It is computed as the number of linearly independent generators of SU(N) which do not commute with $e' = \rho'(\sigma^+)$. The adjoint representation of SU(N) is decomposed under SU(2) as

$$\mathbf{N^2 - 1} \to \left[\bigoplus_{a=1}^{\ell} \bigoplus_{j=1}^{m'_a - 1} (\mathbf{2j + 1}) \right] \bigoplus [(\ell - 1) \cdot \mathbf{1}] \bigoplus \left[2 \bigoplus_{a<b} \bigoplus_{k=1}^{m'_b} (\mathbf{m'_a + m'_b - 2k + 1}) \right]$$
$$\equiv \bigoplus_c (\mathbf{2j_c + 1}), \tag{3.19}$$

where c in the last equation runs over irreducible representations which appear in the decomposition. Corresponding to this decomposition, the generators of SU(N) are given as $T'_{c,m}$, $-j_c \le m \le j_c$ such that $[\rho(\sigma^3/2), T'_{c,m}] = m T'_{c,m}$, $[\rho(\sigma^+), T'_{c,m}] \propto T'_{c,m+1}$ etc. Generators corresponding to the highest spin state in each irreducible representation commutes with $\rho(\sigma^+)$, i.e., $[\rho(\sigma^+), T'_{c,j_c}] = 0$. So the complex dimension of $O_{e'}$ is given as

$$\dim_{\mathbb{C}} O_{e'} = (N^2 - 1) - \left(\sum_{a=1}^{\ell} (m'_a - 1) + (\ell - 1) + 2 \sum_{b=1}^{\ell} (b-1) m'_b \right)$$
$$= N^2 + N - 2 \sum_a a m'_a = \left(N^2 - \sum_a \tilde{m}'^2_a \right) \equiv 2 d(Y'), \tag{3.20}$$

where we used the relation between $Y' = [m'_a]$ and $Y'^D = [\tilde{m}'_a]$. Similarly, the complex dimension of $O_{e'_D}$ is given by $\dim_{\mathbb{C}} O_{e'_D} = 2 d(Y'^D)$.

Moduli space of $T^\rho[\mathrm{SU}(N)]$ theory. An easy way to obtain the $T^\rho[\mathrm{SU}(N)]$ is as follows. Take an embedding $\rho : \mathrm{SU}(2) \to \mathrm{SU}(N)$ corresponding to a partiton of N, $Y = [m_a]$. In the $T[\mathrm{SU}(N)]$ theory, we give a vev $\langle \mu_h \rangle = \rho(\sigma^+) \equiv e$ to the moment map μ_h. Then, in the low energy limit, the $T[\mathrm{SU}(N)]$ theory flows to the $T^\rho[\mathrm{SU}(N)]$ theory with some Goldstone multiplets associated to the spontaneous breaking of the flavor $\mathrm{SU}(N)_{\mathcal{H}}$ symmetry by the vev $\langle \mu_h \rangle = e$. By Goldstone multiplets, we mean Goldstone bosons and their superpartners.

The Goldstone multiplets are given by the orbit of $e = \rho(\sigma^+)$. Eliminating them, we get

$$\mu_h = \rho(\sigma^+) + \sum_{c,m} T_{c,m} \mu_{c,m}$$

$$\rightarrow \rho(\sigma^+) + \sum_c T_{c,-j_c} \mu_{c,-j_c} \tag{3.21}$$

where $T_{c,m}$ is defined according the decomposition as in (3.19) using $Y = [m_a]$. In the above, we used the fact that $\mu_{c,m}$ for $m > -j_c$ are Goldstone multiplets and we have eliminated them. Therefore, the Higgs branch of the $T^\rho[\mathrm{SU}(N)]$ theory is parametrized by the fields $\mu_{c,-j_c}$. A matrix of the form (3.21) is an element of the so called Slodowy slice $\mathcal{S}_\rho$. The Higgs branch of the $T^\rho[\mathrm{SU}(N)]$ theory is parametrized by this μ_h in (3.21).

A matrix of the form (3.21) can take values in the orbit $O_{e'}$ for some $e' = \rho'(\sigma)$ if and only if e can be reached as as a limit of elements of $O_{e'}$. In other words, e must be contained in the closure of $O_{e'}$, denoted as $\bar{O}_{e'}$. So we must have $O_e \subset \bar{O}_{e'}$. It is known that $O_e \subset \bar{O}_{e'}$ if and only if the corresponding partitions $Y = [m_a]$ and $Y = [m_a']$ satisfy $Y \leq Y'$, where the partial ordering between partitions of N is defined in (2.6).

Now we can describe the mixed branch structure $\bigcup_\alpha \mathcal{C}_\alpha \times \mathcal{H}_\alpha$ of the $T^\rho[\mathrm{SU}(N)]$ theory. The branches are labeled by a partition $Y' = [m_a']$ of N, which satisfies $Y' \geq Y$. The Coulomb and Higgs components are given as

$$\mathcal{C}_\alpha(T^\rho[\mathrm{SU}(N)]) = O_{e'_D} \tag{3.22}$$

$$\mathcal{H}_\alpha(T^\rho[\mathrm{SU}(N)]) = O_{e'} \cap S_\rho, \tag{3.23}$$

where e' and e'_D are the nilpotent elements corresponding to $Y' = [m_a']$ and the dual $Y'^D = [\tilde{m}_a']$ as before, and S_ρ is the Slodowy slice (3.21). The quaternionic dimensions (or half the complex dimension) of them are given as

$$\dim_{\mathbb{H}} \mathcal{C}_\alpha(T^\rho[\mathrm{SU}(N)]) = \frac{1}{2} \dim_{\mathbb{C}} O_{e'_D} = d(Y'^D). \tag{3.24}$$

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha(T^\rho[\mathrm{SU}(N)]) = \frac{1}{2} (\dim_{\mathbb{C}} O_{e'} - \dim_{\mathbb{C}} O_e) = d(Y') - d(Y). \tag{3.25}$$

The subtraction of $\dim_{\mathbb{C}} O_e$ in $\dim_{\mathbb{H}} \mathcal{H}_\alpha$ comes from the fact that we have eliminated the Goldstone multiplets in (3.21).

For example, the pure Coulomb branch $\mathcal{C} \times \{0\}$ is given by the orbit O_{e_D} corresponding to the partition $Y^D = [\tilde{m}_a]$ dual to $Y = [m_a]$. This will reproduce the usual rule of regular singularities given in (2.4).

Notice that the constraint $Y' \geq Y$ is equivalent to $Y'^D \leq Y^D$.[6] So in all the branches, we must have the constraint

$$\nu_h \in \bar{O}_{e_D}. \tag{3.26}$$

---

[6]If two partitions $[\tilde{m}_a]$ and $[\tilde{m}_a']$ satisfy $[\tilde{m}_a'] \leq [\tilde{m}_a]$, then $[m_a'] \geq [m_a]$. This claim is proved as follows. Examining the Young diagram, one can see that $\sum_{b=1}^a m_b' = N - \sum_b \max\{\tilde{m}_b' - a, 0\}$. Then by taking c such that $\tilde{m}_c' = a$ and $\tilde{m}_{c+1}' < a$, we get $\sum_b \max\{\tilde{m}_b' - a, 0\} = \sum_{b=1}^c (\tilde{m}_b' - a) \leq \sum_{b=1}^c (\tilde{m}_b - a) \leq \sum_b \max\{\tilde{m}_b - a, 0\}$, where we have used the definition of the partial ordering (2.6). Thus we get $\sum_{b=1}^a m_a' \geq \sum_{b=1}^a m_a$ for $a = 1, 2, \cdots$, proving $[m_a'] \geq [m_a]$.

– 14 –

3.3.2 $T^\rho[\mathrm{SU}(N)]$ coupled to 5d SYM

Now let us couple the $T^\rho[\mathrm{SU}(N)]$ theory to the 5d SYM at a point $p \in C$ which will be a puncture. We gauge the Coulomb branch $\mathrm{SU}(N)_\mathcal{C}$ symmetry of $T^\rho[\mathrm{SU}(N)]$ by the gauge group of the 5d SYM. In particular, this coupling includes a superpotential coupling between the holomorphic moment map ν_h and the complex adjoint field $\varphi_h = \varphi_1 + i\varphi_2$,

$$W \supset \mathrm{tr}(\varphi_h(p)\nu_h), \tag{3.27}$$

where $\varphi_h(p)$ is the value of φ_h evaluated at the point p.

There are two important effects of this coupling. First, by giving the vev (3.9), this coupling gives a mass term of ν_h. Then the Higgs branch moment map μ_h is no longer a nilpotent matrix, but it is such that its characteristic polynomial is given by [28]

$$\det(xI_N - \mu_h) = \det(xI_N - \varphi_h), \tag{3.28}$$

where x is an arbitrary variable. (When $\varphi_h = 0$, this equation means that all the eigenvalues of μ_h are zero, and hence μ_h is nilpotent.)

The other effect of (3.27) is that this term gives a source of Φ. In the bulk superpotential of the 5d SYM, there is also a coupling of the form $W \supset \int d^2z \, \mathrm{tr}(\varphi_h D_{\bar{z}}\Phi)$ and the equation of motion of φ_h gives $D_{\bar{z}}\Phi \sim \delta^2(z)\nu_h$, where we have assumed that the point p is located at $z = 0$. Then we get a pole of Φ as[7]

$$\Phi \sim \frac{\nu_h}{z} + \mathcal{O}(1) \quad (z \to 0). \tag{3.29}$$

Suppose that the ν_h is in the orbit $\nu \in O_{e_D}$ corresponding to the partition $[\tilde{m}_a]$. Then, combining the pole term ν_h/z and also the order one term $\mathcal{O}(1)$ in the above equation, one can check that the singularity of the spectral curve is precisely given by (2.4). The constraint (3.26) ensures that the singularities cannot be stronger than (2.4).

Now we can study the local contribution $\mathcal{C}_\alpha(p) \times \mathcal{H}_\alpha(p)$ from the puncture p to the mixed branch $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$ of the 4d field theory.

Coulomb factor. The commutation relation $[\Phi, \vec{\varphi}] = 0$ requires that the residue ν_h of the pole of Φ at the puncture as given in (3.29) must be of the form

$$\nu_h = \begin{pmatrix} \nu_h^{(1)} & 0 & \ldots & 0 \\ 0 & \nu_h^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \nu_h^{(k)} \end{pmatrix}. \tag{3.30}$$

Furthermore, each of the block $\nu_h^{(s)}$ must be nilpotent. Thus there is a partition of n_s, $Y_s^D = [\tilde{m}_{s,a}]$, the corresponding embedding $\rho_D^{(s)} : \mathrm{SU}(2) \to \mathrm{SU}(n_s)$ and the nilpotent

---

[7]Strictly speaking, (3.29) is not an actual solution of Hitchin's equations if $[\nu_h, \nu_h^\dagger] \neq 0$. We must use the usual argument that imposing (3.4) and dividing by the gauge group is equivalent to dividing by complexified gauge group. Then, using complexified gauge transformation to set $A_{\bar{z}} = 0$, we get (3.29).

element $e(s)_D = \rho_D^{(s)}(\sigma^+)$ such that

$$\nu_h^{(s)} \in O_{e(s)_D} \quad (s = 1, 2, \cdots, k). \tag{3.31}$$

Then, the singularity of the spectral curve is specified by Y_s^D.

Let us combine the partitions Y_s^D to define $Y'^D = [\tilde{m}'_a]$ whose columns are the columns of Y_s^D. Because of the constraint (3.26), this partition must satisfy $Y'^D \le Y^D$.

The Coulomb factor $\mathcal{C}_\alpha(p)$ is parametrized by $\nu_h^{(s)}$. However, the following point must be taken into account. As explained in section 3.1, we compactified the 4d theory on S^1. Then, in the low energy 3d theory, a U(1) massless gauge multiplet is dualized to two real scalars (or equivalently one complex scalar). Therefore, the dimension of Coulomb moduli space $\mathcal{C}_\alpha^{(3d)}$ is doubled from the 4d Coulomb moduli space $\mathcal{C}_\alpha^{(4d)}$. Having this distinction between the 3d and 4d theory in mind, we conclude that

$$\mathcal{C}_\alpha^{(3d)}(p) = \bigoplus_{s=1}^{k} O_{e(s)_D} \tag{3.32}$$

and

$$\dim_{\mathbb{C}} \mathcal{C}_\alpha^{(4d)}(p) = \dim_{\mathbb{H}} \mathcal{C}_\alpha^{(3d)}(p) = \sum_{s=1}^{k} d(Y_s^D). \tag{3.33}$$

This is what we have written in (2.12).

Higgs factor. The μ_h must satisfy (3.28). Furthemore, the ν_h satisfies (3.30) and (3.31). From these facts, it is natural to expect the following (see section 2.3 and 2.4 of [22]). For each s, let $Y_s = [m_{s,a}]$ be the partition of n_s dual to Y_s^D, $\rho^{(s)} : \mathrm{SU}(2) \to \mathrm{SU}(n_s)$ be the corresponding embedding, and $e(s) = \rho^{(s)}(\sigma^+)$. Let $e'' = \bigoplus_s e(s)$ be the block diagonal $N \times N$ matrix. Then we define the matrix

$$\varphi_h + e'' = \begin{pmatrix} a_h^{(1)} I_{n_1} + e(1) & 0 & \dots & 0 \\ 0 & a_h^{(2)} I_{n_2} + e(2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_h^{(k)} I_{n_k} + e(k) \end{pmatrix}, \tag{3.34}$$

where $a_h^{(s)} = a_1^{(s)} + i a_2^{(s)}$. Suppose that $a_h^{(s)}$ are generic so that $a_h^{(s)} \ne a_h^{(t)}$ for $s \ne t$. In this case, we propose that μ_h is in the orbit of $\varphi_h + e''$,

$$\mu_h \in \mathcal{H}_\alpha(p) \equiv O_{\varphi_h + e''} \cap S_\rho. \tag{3.35}$$

The complex dimension of the orbit $O_{\varphi_h + e''}$ is given by

$$\begin{aligned} \dim_{\mathbb{C}} O_{\varphi_h + e''} &= \dim_{\mathbb{C}} O_{\varphi_h} + \sum_{s=1}^{k} \dim_{\mathbb{C}} O_{e(s)} \\ &= \left(N^2 - \sum_{s=1}^{k} n_s^2 \right) + \sum_{s=1}^{k} \left(n_s^2 - \sum_a \tilde{m}_{s,a}^2 \right) \\ &= 2d(Y') \end{aligned} \tag{3.36}$$

where $Y' = [m'_a]$ is the partition of N dual to $Y'^D = [\tilde{m}'_a]$.

We can perform a consistency check by taking the limit $\varphi_h \to 0$. In this case, the $T^\rho[SU(N)]$ theory at the puncture has no (holomorphic) mass term, and hence we have

$$\mu_h \in O_{e'} \cap S_\rho, \tag{3.37}$$

where $e' = \rho'(\sigma^+)$ is a nilpotent element corresponding to the partition $Y' = [m'_a]$. Therefore, we should get

$$\lim_{\varphi_h \to 0} O_{\varphi_h + e''} = O_{e'}. \tag{3.38}$$

Actually, the dimension of $O_{\varphi_h + e''}$ computed above is the same as the dimension of $O_{e'}$. This is required because, even if $\varphi_h = \varphi_1 + i\varphi_2 \to 0$, this limit should be smooth as long as φ_3 is generic.[8]

We conclude that the quaterionic dimension of $\mathcal{H}_\alpha(p)$ is given as

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha(p) = d(Y') - d(Y). \tag{3.39}$$

This is the formula written in (2.14).

4 $\mathcal{N} = 2$ SUSY QCD

In this section, we reproduce the results of the moduli space of vacua in SQCD obtained by Seiberg and Witten for SU(2) [2] and by Argyres, Plesser and Seiberg for SU(N) [4].

4.1 SU(2)

We study the SU(2) SQCD with N_f flavors. Although these models are simple, they illustrate our method without technical complication.

We consider massless cases. In addition to the regular singularities discussed in the previous sections, we also need some irregular singularities. The curve is $x^2 + \phi_2(z) = 0$, and the singularities we use are given as

$$A : \phi_2 = \mathcal{O}(z^{-1}), \tag{4.1}$$

$$B : \phi_2 = \frac{\Lambda^2}{z^3} + \mathcal{O}(z^{-2}), \tag{4.2}$$

$$C : \phi_2 = \frac{\Lambda^2}{z^4} + \mathcal{O}(z^{-2}), \tag{4.3}$$

where Λ is a parameter which roughly corresponds to the dynamical scales of field theory. The first singularity A is the regular singularity corresponding to the partition $Y = [m_a] = [1,1]$ or equivalently its dual partition is $Y^D = [\tilde{m}_a] = [2]$. The second and third singularities B and C are irregular singularities which are obtained [20] from the M-theory uplift of type IIA brane configurations [5]. More details on those types of irregular singularities are discussed in appendix A, but here we simply claim that (4.2) and (4.3) have no local contribution to the Higgs branch.

Let us summarize the important points.

---

[8]The φ_3 does not appear in the complex structure of $T^\rho[SU(N)]$, but it appears in the metric. If we use SU(2)$_R$ symmetry, the case $\varphi_h = 0$ and φ_3=generic can be transformed into φ_h=generic.

- If the curve is factorized as $x^2 + \phi_2 = (x + f(z))(x - f(z))$, there is a contribution to the quaternionic dimension of the Higgs branch as $+1$ coming from the term $k - 1$ in (2.15), i.e., the curve is factorized according to the partition $X = [n_s] = [1, 1]$.

- If there is a singularity of type B, then it is impossible to factorize the curve as $x^2 + \phi_2 = (x + f(z))(x - f(z))$ since we cannot have $\Lambda^2/z^3 + \cdots = (f(z))^2$ for some single valued function $f(z)$. On the other hand, in the case of the singularity of type C, there is no such local obstacle.

- For a singularity of type A, we can have local contribution to the Higgs branch if and only if the $\mathcal{O}(z^{-1})$ term vanishes. In this case we have $Y'^D = [\tilde{m}'_a] = [1, 1]$, and the local contribution to the quaternionic dimension, calculated by the formula (2.14), is $\frac{1}{2}(2^2 - 1^2 - 1^2) = 1$.

Zero flavor $N_f = 0$. The theory with no flavor is realized by putting two irregular singularities of type B, (4.2), on a Riemann sphere parametrized by $z \in \mathbb{C} \cup \{\infty\}$. Putting the singularities at $z = 0$ and $z = \infty$, the curve is

$$0 = x^2 - \left(\frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \right), \tag{4.4}$$

where u is the Coulomb modulus. (Note that near $z \to \infty$, we must use the coordinates $z' = 1/z$ and $x' = -z^2 x$.) We cannot factorize the curve, and there are no local contributions to the Higgs branch from the punctures. Thus there is no Higgs branch, reproducing the result of field theory.

One flavor $N_f = 1$. The theory with one massless flavor is realized by putting one irregular singularity of type B, (4.2), and one irregular singularity of type C, (4.3), on a Riemann sphere. Putting singularities at $z = 0$ and $z = \infty$, the curve is

$$0 = x^2 - \left(\frac{\Lambda^2}{z^4} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \right). \tag{4.5}$$

In this case, there is no Higgs branch as in the case of $N_f = 0$.

Two flavors $N_f = 2$. The theory with two massless flavors can be realized in two different ways. The first way is to put two type C singularities. The second way is to put one type B irregular singularity and two type A singularities, (4.1).

First, let us study the first realization. We use two type C singularities. Putting singularities at $z = 0$ and $z = \infty$, the curve is

$$0 = x^2 - \left(\frac{\Lambda^2}{z^4} + \frac{u}{z^2} + \Lambda^2 \right). \tag{4.6}$$

The curve can be factorized if and only if $u = \pm 2\Lambda^2$. In this case, we have

$$0 = \left(x + \left(\frac{\Lambda}{z^2} \pm \Lambda \right) \right) \left(x - \left(\frac{\Lambda}{z^2} \pm \Lambda \right) \right). \tag{4.7}$$

Therefore, the Higgs branch is emanating from the Coulomb branch points $u = \pm 2\Lambda^2$ and its quaternionic dimension is 1. This is exactly as was found by Seiberg and Witten [2].

Next, let us study the second realization. We take regular punctures of type A at $z = 0, 1$ and a puncture of type B at $z = \infty$. The curve is

$$0 = x^2 - \left(\frac{\Lambda^2}{z} + \frac{u'}{z(z-1)} \right). \tag{4.8}$$

We cannot factorize the curve in this case because of the existence of the singularity of type B. However, there are contributions to the Higgs branch dimension from regular punctures. Notice that

$$\left(\frac{\Lambda^2}{z} + \frac{u'}{z(z-1)} \right) \rightarrow \begin{cases} \dfrac{\Lambda^2 - u'}{z} & z \rightarrow 0 \\ \dfrac{u'}{z-1} & z \rightarrow 1 \end{cases} \tag{4.9}$$

Then, we can have local contribution to the Higgs branch from the puncture at $z = 0$ when $u' = \Lambda^2$, and from the puncture at $z = 1$ when $u' = 0$. At each of these points, we have the Higgs branch with the quaternionic dimension 1. Therefore, we find the same structure as the case of the above first realization if we identify the moduli parameters as $u = 2u' - \Lambda^2$.

It is interesting that the way the Higgs branch appears is quite different in the two realizations. In the first realization, the Higgs branch comes from the "bulk" contribution, while in the second realization the Higgs branch comes from local contributions from punctures. However, the final results are the same in both cases.

Three flavors $N_f = 3$. The theory with three massless flavors is realized by putting two regular singularities of type A at $z = 0, 1$ and one irregular singularity of type C at $z = \infty$. The curve is

$$0 = x^2 - \left(\Lambda^2 + \frac{u}{z(z-1)} \right). \tag{4.10}$$

Note that the local contributions to the Higgs branch from the punctures at $z = 0, 1$ are both turned on when $u = 0$. Furthermore, only in this case the curve is factorized,

$$0 = \left(x + \Lambda^2 \right) \left(x - \Lambda^2 \right). \tag{4.11}$$

Therefore, the Higgs branch is emanating from the single point $u = 0$ and the quaternionic dimension is 3. This is exactly as in [2].

Four flavors $N_f = 4$. The theory with four massless flavors is realized by putting four regular singularities of type A at $z = 0, q, 1, \infty$. The curve is

$$0 = x^2 - \left(\frac{u}{z(z-q)(z-1)} \right). \tag{4.12}$$

All the local contributions to the Higgs branch and the factorization of the curve is possible if and only if $u = 0$. Therefore, the Higgs branch is emanating from $u = 0$ and has the quaternionic dimension 5. This point is the superconformal point of the theory.

Mass deformation and AD point. Let us briefly describe mass deformation. Generic deformation would typically eliminate the Higgs branch. Consider the first realization of SU(2) theory with two flavors. The curve turning on generic mass deformation is

$$x^2 = \frac{\Lambda^2}{z^4} + \frac{m_1 \Lambda}{z^3} + \frac{u}{z^2} + \frac{m_2 \Lambda}{z} + \Lambda^2. \tag{4.13}$$

The above curve can be factorized if and only if

$$m_1 = m_2 = m, \qquad \frac{u}{\Lambda^2} = \frac{m^2}{4\Lambda^2} + 2,$$

$$m_1 = -m_2 = m, \quad \frac{u}{\Lambda^2} = \frac{m^2}{4\Lambda^2} - 2. \tag{4.14}$$

So for non-zero $m = m_1 = \pm m_2$, there is only one Higgs branch of dimension 1, which is consistent with the fact that the flavor symmetry is broken to U(2) by the mass term. The above curve can be further factorized for the following special value of m:

$$x^2 = \frac{\Lambda^2}{z^4}(z \pm 1)^4, \quad m_1 = m_2 = m = \pm 4\Lambda,$$

$$x^2 = \frac{\Lambda^2}{z^4}(z \pm i)^4, \quad m_1 = -m_2 = m = \pm 4i\Lambda. \tag{4.15}$$

There should be new massless particles here, but our method tells us that the Higgs branch dimension does not change at this point. This suggests that the new massless particles might be nonlocal relative to the massless particles which are already present in (4.14). This point is actually the Argyres-Douglas point as discussed in [10].

4.2 SU(N)

In this section we study the SU(N) SQCD. First, let us summarize the result of Argyres, Plesser and Seiberg [4] obtained by field theory methods.

- There is a baryonic Higgs branch for $N_f \geq N$, and the quaternionic dimension is $N(N_f - N) + 1$. There is no Coulomb direction in this branch.

- There are non-baryonic mixed branches with a label $0 \leq r \leq \min([\frac{N_f}{2}], N - 2)$. The complex dimension of the Coulomb factor $\mathcal{C}_r$ is $N - r - 1$, and the quaternionic dimension of the Higgs factor $\mathcal{H}_r$ is $r(N_f - r)$. The pure Coulomb branch is the case $r = 0$.

Here we only treat the conformal theory $N_f = 2N$ which only requires regular singularities. Other cases are discussed in appendix A.

4.2.1 $N_f = 2N$

This theory is described by a Riemann sphere with two simple punctures ($Y = [m_a] = [N - 1, 1]$ or $Y^D = [\tilde{m}_a] = [2, 1^{N-2}]$) and two full punctures ($[Y] = [m_a] = [1^N]$ or $Y^D = [\tilde{m}_a] = [N]$). We put two full punctures at $z = 0, \infty$, and two simple punctures at $z = 1, q$. The Seiberg-Witten curve is

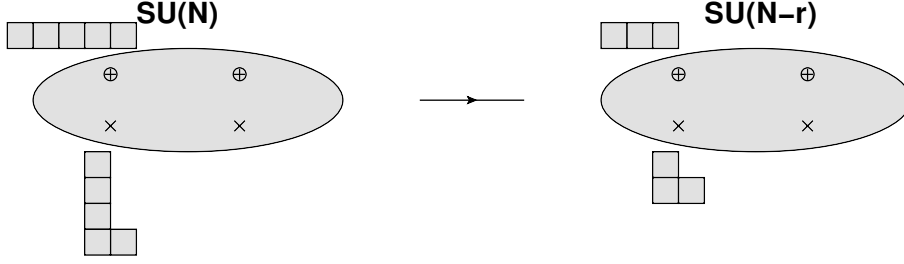$$0 = x^N + \sum_{i=2}^{N} \frac{u_i}{z^{i-1}(z-1)(z-q)} x^{N-i}. \tag{4.16}$$

Figure 4. The factorization patter of rth mix branch of the SU(N) theory with $2N$ flavors.

Non-baryonic branch. Suppose that some of the u_i is nonzero. Then we can define r ($0 \leq r \leq N - 2$) such that $u_{N-r} \neq 0$ and $u_{N-r+1} = \cdots = u_N = 0$. Then the curve is factorized as

$$0 = x^r \left(x^{N-r} + \sum_{i=2}^{N-r} \frac{u_i}{z^{i-1}(z-1)(z-q)} x^{N-i} \right).$$
(4.17)

This means that we have factorization corresponding to the partition $X = [n_s] = [N-r, 1^r]$.

In the second step, we should choose the local puncture type on the SU($N - r$) factor. However, it is easy to see that to have nonzero u_{N-r}, the punctures have to be chosen as in the figure 4: the full (simple) puncture of SU(N) becomes the full (simple) puncture of SU($N - r$). We have the following data on the split of punctures:

$$\begin{aligned}
\text{Full puncture}: \quad & Y_f^D = [N], & Y_f'^D = [N-r, 1^r] \\
\text{Simple puncture}: \quad & Y_s^D = [2, 1^{N-2}], & Y_s'^D = [2, 1^{N-2}]
\end{aligned}$$
(4.18)

Using the formula (2.14), local contribution to the Higgs component from a full or simple puncture is given by

$$\dim_{\mathbb{H}} \mathcal{H}_r(\text{full}) = d(Y_f') - d(Y_f) = \frac{1}{2} \left(N^2 - (N-r)^2 - r \cdot 1^2 \right)$$
(4.19)

$$\dim_{\mathbb{H}} \mathcal{H}_r(\text{simple}) = d(Y_s') - d(Y_s) = 0.$$
(4.20)

Then, the formula (2.15) gives the total Higgs branch dimension as

$$\begin{aligned}
\dim_{\mathbb{H}} \mathcal{H}_r &= r + 2 \dim_{\mathbb{H}} \mathcal{H}_r(\text{full}) + 2 \dim_{\mathbb{H}} \mathcal{H}_r(\text{simple}) \\
&= r(2N - r),
\end{aligned}$$
(4.21)

where we have used $k = r + 1$. This is exactly what was found in [4] with $N_f = 2N$.

Baryonic branch. In the above we have assumed $u_{N-r} \neq 0$ for some r. The remaining possibility is to have all the u_i to be zero. The curve is just $0 = x^N$, which corresponds to the partition $X = [n_s] = [1^N]$. At all the punctures we automatically have $Y'^D = [\tilde{m}_a'] = [1^N]$.

Local contribution from a full or simple puncture is given by

$$\dim_{\mathbb{H}} \mathcal{H}(\text{full}) = \frac{1}{2} \left(N^2 - N \cdot 1^2 \right)$$
(4.22)

$$\dim_{\mathbb{H}} \mathcal{H}(\text{simple}) = \frac{1}{2} \left(N^2 - N \cdot 1^2 \right) - \frac{1}{2} \left(N^2 - 2^2 - (N-2) \cdot 1^2 \right).$$
(4.23)

Then, the formula (2.15) gives

$$\dim_{\mathbb{H}} \mathcal{H} = (N - 1) + 2 \dim_{\mathbb{H}} \mathcal{H}_r(\text{full}) + 2 \dim_{\mathbb{H}} \mathcal{H}_r(\text{simple})$$
$$= N^2 + 1. \tag{4.24}$$

This is exactly what was found in [4] with $N_f = 2N$.

5 Theories defined using regular singularities

Our method can be also applied to theories whose Lagrangian description is not yet know. One of such theories is the so called T_N theory. Pure Higgs branch $\mathcal{H}$ (i.e., branch where there are no Coulomb moduli fields) are studied in [11, 14]. But it is not so easy to study the mixed branch structures directly by field theory. Here we use our method to study these structures. The case of the partition $X = [n_s] = [1^N]$ was studied in [28].

5.1 T_N theory

The T_N theory is realized by using a Riemann sphere with three full punctures. Each full puncture has $SU(N)_{\mathcal{H}}$ Higgs branch symmetry, so the T_N theory has $SU(N)_A \times SU(N)_B \times SU(N)_C$ flavor symmetry.

In this case, what are discussed in sections 2 and 3 suggest the following interpretations. First, the factorization of the curve (2.8) is realized by giving vevs to Higgs branch operators of the T_N theory so that the low energy theory is given by smaller T_N theories [14]

$$T_N \to T_{n_1} + T_{n_2} + \cdots + T_{n_k} + (\text{free fields}). \tag{5.1}$$

Here the free fields come from some $k - 1$ moduli fields responsible for the above breaking which roughly represent the relative "positions" of T_{n_s} on moduli space,[9] and also the Goldstone multiplets associated to the spontaneous breaking $[SU(N)]^3 \to [S(U(n_1) \times \cdots \times U(n_k))]^3$. There are $3 \times \frac{1}{2}(N^2 - \sum_{s=1}^{k} n_s^2)$ Goldstone multiplets (in quaternionic dimension).

Next, we consider the partition $n_s \to Y_s^D = [\tilde{m}_{s,a}]$ at three punctures, which we call A, B and C respectively. Thus, for each s, we have $Y_{s,A}^D, Y_{s,B}^D$ and $Y_{s,C}^D$, and their dual partitions $Y_{s,A}, Y_{s,B}$ and $Y_{s,C}$. Then, we can Higgs the punctures of the T_{n_s} theory by giving nilpotent vevs to the moment maps to get more general three-punctured theories specified by these partitions [22, 36, 37]

$$T_{n_s} \to T_{n_s}(Y_{s,A}, Y_{s,B}, Y_{s,C}) + (\text{Goldstone multiplets}), \tag{5.2}$$

where the Goldstone multiplets are the ones associated to the spontaneous breaking of $[SU(n_s)]^3$ by nilpotent vevs. As discussed in section 3, each puncture contributes $d(Y_s) = \frac{1}{2}(n_s^2 - \sum_a \tilde{m}_{s,a}^2)$ to the quaternionic dimension of Goldstone multiplets. Combining all the

---

[9]In terms of the M5 brane realization of the T_N theory, this means that the N M5 branes are separated as $N \to [n_1, n_2, \cdots, n_s]$.

low energy free fields in the above process gives the dimension of the Higgs factor $\mathcal{H}_\alpha$ given in (2.15). Then we go to the Coulomb branch of the remaining theory

$$\sum_{s=1}^{k} T_{n_s}\left(Y_{s,A}, Y_{s,B}, Y_{s,C}\right). \tag{5.3}$$

This process gives the mixed branches of the T_N theory.

For this process to define a mixed branch of the T_N theory, $T_{n_s}\left(Y_{s,A}, Y_{s,B}, Y_{s,C}\right)$ must be such that its generic curve cannot be factorized. If some of n_s is 1, we interpret the $T_{n=1}$ theory to be empty.

The above picture of the vacua structure suggests that we can find all the lower three punctured sphere theory on the Coulomb branch moduli space of the T_N theory.

5.2 Maldacena-Nunez theory

$\mathcal{N} = 2$ Maldacena-Nunez theories [25, 38] are defined by a genus g Riemann surface with no punctures at all. Therefore, we only need to consider a partition $X = [n_s]$ of the curve $0 = x^N + \sum_i \phi_i(z) x^{N-i}$.

If $g = 0$, the only possible curve is just $0 = x^N$, since there are no nontrivial holomorphic sections of ϕ_i. Hence the only branch allowed by our rule is the case $X = [1^N]$. This is the Higgs branch with quaternionic dimension $N - 1$. There is no Coulomb branch.

If $g = 1$, then the most general curve is given by $0 = \prod_{i=1}^{N}(x - c_i)$ where c_i are constants with $\sum_{i=1}^{N} c_i = 0$. Therefore, also in this case, the only possible partition allowed is $X = [1^N]$. The quaterionic dimension of the Higgs branch is $N - 1$. In fact, the Coulomb branch and Higgs branch are combined into a manifold with complex dimension $3(N - 1)$, which agrees with the field theory result as the theory is just $\mathcal{N} = 4$ SYM.

If $g > 1$, we can have the following factorization:

$$0 = \prod_{s=1}^{k}\left(x^{n_s} + \sum_{i=1}^{n_s} \phi_{s,i}(z) x^{n_s-i}\right), \tag{5.4}$$

This means all the partitions $[n_a]$ are possible, and one can easily find the dimensions of the Higgs and Coulomb factor using our general formula.

Let's remark an interesting point for the branch labeled by the partition $X = [1^N]$. In this branch, we have the maximal Higgs branch deformations. Unlike the sphere case, we still have Coulomb branch deformations which are described by the curve

$$(x - h_1)(x - h_2)\dots(x - h_N) = 0, \tag{5.5}$$

and h_i are sections of the canonical bundle. This means that there is no pure Higgs branch for this class of theories. See [28] for more detailed discussions on this case.

6 Argyres-Douglas theory

Here we use our method to determine the Higgs branch of Argyres-Douglas theory [23, 24, 39]. We will recover and extend the results obtained in [13]. The theories we consider are discussed in [23, 40, 41].

6.1 (A_1, A_N) theory

This theory is described by a sphere with one irregular singularity which we put at $z = \infty$.

N=2n-1. The Seiberg-Witten curve is

$$x^2 = z^{2n} + c_2 z^{2n-2} + \cdots + c_n z^n + c_{n+1} z^{n-1} + u_{n+2} z^{n-2} + \ldots + u_{2n} \qquad (6.1)$$

where c_i $(i = 2, \cdots, n)$ are coupling parameters of the theory, c_{n+1} is a mass parameter, and u_i $(i = n+2, \cdots, 2n)$ are Coulomb branch moduli.

To find the Higgs branch, we need to consider the factorization of the Seiberg-Witten curve. In fact, there is a factorization happening at a single point on the moduli space:

$$x^2 - (z^{2n} + c_2 z^{2n-2} + \cdots + c_{n+1} z^{n-1} + u_{n+2} z^{n-2} + \ldots + u_{2n})$$
$$= \left(x + z^n + \sum_{i=2}^{n} a_i z^{n-i} \right) \left(x - z^n - \sum_{i=2}^{n} a_i z^{n-i} \right). \qquad (6.2)$$

There are $2n - 1$ equations for $2n - 1$ parameters c_{n+1}, u_i $(i = n+2, \cdots, 2n)$ and a_i $(i = 2, \cdots, n)$ when c_i $(i = 2, \cdots, n)$ are fixed. By tuning c_{n+1} appropriately, factorization occurs for certain values of u_i determined by c_i uniquely. For example, for the SCFT case $(c_i = 0)$, we find a one dimensional Higgs branch coming out from $u_i = 0$.

N=2n. The curve for this theory is

$$x^2 = z^{2n+1} + c_2 z^{2n-1} + \cdots + c_{n+1} z^n + u_{n+2} z^{n-1} + \cdots + u_{2n+1} \qquad (6.3)$$

It is impossible to factorize the curve because of the existence of the leading term z^{2n+1}. Therefore there is no Higgs branch.

6.2 (A_1, D_N) theory

This theory is described by one irregular singularity at $z = \infty$ and one regular singularity at $z = 0$.

N=2n+2. The curve is

$$x^2 = z^{2n} + c_1 z^{2n-1} + \cdots + c_{n+1} z^{n-1} + u_{n+2} z^{n-2} + \cdots + \frac{u_{2n+1}}{z} + \frac{m^2}{z^2}, \qquad (6.4)$$

where c_i $(i = 1, \cdots, n)$ are some parameters, c_{n+1} and m are mass parameters and u_i $(i = n+2, \cdots, 2n+1)$ are Coulomb moduli.

We only consider the case in which $m = 0$ and also c_{n+1} is tuned appropriately so that the curve can be factorized. Then, if $u_{2n+1} = 0$, we get a local contribution to the Higgs factor $\mathcal{H}_\alpha$ from the puncture at $z = 0$. Therefore, we have a mixed branch at $u_{2n+1} = 0$. The Coulomb factor $\mathcal{C}_\alpha$ is the same as the (A_1, A_{2n-1}) theory discussed above, and the Higgs factor $\mathcal{H}_\alpha$ has quaternionic dimension 1.

At a single point on the Coulomb moduli space, we can factorized the curve as

$$x^2 - z^{2n} + c_1 z^{2n-1} + \cdots + c_{n+1} z^{n-1} + u_{n+2} z^{n-2} + \cdots + \frac{u_{2n+1}}{z}$$

$$= \left(x + z^n + \sum_{i=2}^{n} a_i z^{n-i} \right) \left(x - z^n - \sum_{i=2}^{n} a_i z^{n-i} \right). \tag{6.5}$$

At this point, we must also set $u_{2n+1} = 0$. There is a Higgs branch coming out of this point. The quaternionic dimension is 2 by counting the contributions from the bulk and the puncture. In particular, the Higgs branch at the SCFT point $(a_i = 0)$ has dimension 2, which agrees with the result in [13].

N=2n+3. The curve is

$$x^2 = z^{2n+1} + c_1 z^{2n} + \cdots + c_{n+1} z^n + u_{n+2} z^{n-1} + \cdots + \frac{u_{2n+2}}{z} + \frac{m^2}{z^2}, \tag{6.6}$$

Again we consider the case $m = 0$. Then, there is a mixed branch defined by $u_{2n+2} = 0$. The contribution to the Higgs factor $\mathcal{H}_\alpha$ comes from the puncture at $z = 0$, and the Coulomb factor $\mathcal{C}_\alpha$ is the same as the (A_1, A_{2n}) theory.

6.3 (A_1, E_N) theory

The Seiberg-Witten curve at the E_N type SCFT point is

$$
\begin{aligned}
E_6: & \quad x^3 + z^4 = 0, \\
E_7: & \quad x^3 + xz^3 = 0, \rightarrow x(x^2 + z^3) = 0, \\
E_8: & \quad x^3 + z^5 = 0.
\end{aligned}
\tag{6.7}
$$

It is easy to see that only the curve for E_7 theory is factorizable, and there is only one type of factorization with $X = [2, 1]$, so the dimension of Higgs branch is one.

7 Conclusion

We have developed a general method to find the vacua structure of class $\mathcal{S}$ theory using generalized Hitchin equation. We derived our results using the geometric method from six dimensional $(2, 0)$ theory. For those 4d theories with lagrangian description, it is interesting to check our result using field theory calculations. The holomorphic factorization of Seiberg-Witten curve and reduction of singularity at punctures play a crucial role in our method, and a field theory interpretation of this fact would be welcome.

Let's summarize the possible moduli space structure of $\mathcal{N} = 2$ theories:

- Typically, there is a Coulomb branch, a Higgs branch and mixed branches. Such examples include the theory defined by a sphere with regular punctures.

- Sometimes there is only a Coulomb branch. The typical example is the pure SU(2) SYM and (A_1, A_{N-1}) AD theory with even N.

- There is only a Higgs branch for the free field theory such as bifundamentals.

- There is no pure Higgs branch, namely only a Coulomb and mixed branches exist. This situation happens e.g. for the theory defined on a higher genus Riemann surface.

We focused on the theories obtained from A_{N-1} type $\mathcal{N} = (2,0)$ theory in this paper, but our method is also applied to other theories using D [42, 43] and E [44] type $(2,0)$ theory. It would be interesting to extend our study to those other types of theories.

The singularity structure on the Coulomb branch of $\mathcal{N} = 2$ theory is crucial for the discovery of Seiberg-Witten solution [1, 2]. New massless particles appear at those singularities. The loci where Higgs or mixed branches appear must have massless particles, but the appearance of new massless particles does not necessary mean that there is a mixed branch. The method described in this paper detects mixed branches rather than massless particles. It would be very interesting to develop a systematic way to find massless particles and their types (e.g., mutually local or nonlocal, abelian or nonabelian, etc.). The study of effective field theory on the Higgs branch roots is particular interesting, as those singularities might survive under $\mathcal{N} = 1$ deformation [4]. We leave the study of those effective theories and $\mathcal{N} = 1$ deformation to the future.

Acknowledgments

A   More on SQCD

In this appendix we study the SU(N) SQCD with general numbers of flavors $N_f < 2N$.

A.1   Irregular singularities

We will need some irregular singularities, so we review them.

The rules.   From a brane realization of SU(N) SQCD with $K < N$ flavors [5], one can see that there exists an irregular singularity of the form [20]

$$\phi_i(z) = \frac{\Lambda^{N-K}}{z^{1+N-K}} \delta_{i,N-K} + \mathcal{O}(z^{-i}), \tag{A.1}$$

where Λ is a parameter which is roughly a dynamical scale of the corresponding field theory. This singularity has the flavor symmetry U(K).

A slight generalization of this singularity is the following. We take a partition of K, $Y = [m_a]$ where $K = m_1 + \cdots + m_\ell$. Then we have a puncture labeled by (K, Y). Consider the dual partition $Y^D = [\tilde{m}_a]$. The singularity is now given by

$$\phi_i(z) = \frac{\Lambda^{N-K}}{z^{1+N-K}} \delta_{i,N-K} + \mathcal{O}(z^{-p_i}), \tag{A.2}$$

where

$$p_i = \begin{cases} i, & 2 \le i \le N - K, \\ i + 1 - a, & (N - K) + \sum_{b=1}^{a-1} \tilde{m}_b < i \le (N - K) + \sum_{b=1}^a \tilde{m}_b \end{cases} \tag{A.3}$$

If $K = N - 1$, we need to shift x in the curve (2.2) so that the x^{N-1} term is cancelled.

The rules discussed in section 2.2 is slightly changed as follows. Take a partition $X = [n_s]$ used in the factorization of the curve (2.8). One of the numbers in $[n_s]$, say n_r, must satisfy $n_r \ge N - K$. Then, we take a partition of $n_r - N + K$, denoted as $Y_r^D = [\tilde{m}_{r,a}]$ where $\sum_a \tilde{m}_{s,a} = n_r - N + K$, and assume

$$\phi_{r,i} = \frac{\Lambda^{N-K}}{z^{1+N-K}} \delta_{i,N-K} + \mathcal{O}(z^{-p_{r,i}})$$

$$p_{r,i} = \begin{cases} 0 & i = 1 \\ i & 2 \le i \le N - K, \\ i + 1 - a & (N - K) + \sum_{b=1}^{a-1} \tilde{m}_{r,b} < i \le (N - K) + \sum_{b=1}^a \tilde{m}_{r,b} \end{cases} \tag{A.4}$$

The other blocks, i.e., n_s with $s \ne r$, has a partition of n_s denoted as $Y_s^D = [\tilde{m}_{s,a}]$, and the singularities are given similar to (2.4) as in the regular puncture case. We combine the partitions $Y_s^D = [\tilde{m}_{r,a}]$ and $Y_s^D = [\tilde{m}_{s,a}]$ ($s \ne r$) as discussed in section 2 to get a partition of K denoted as $Y'^D = [\tilde{m}'_a]$ where $\sum_a \tilde{m}'_a = K$. Then we require that $Y'^D \le Y^D$.

The local contribution to the quaternionic dimension of $\mathcal{H}_\alpha$ is given by the formula

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha(p) = d(Y') - d(Y), \tag{A.5}$$

where Y' is the partition of K dual to Y'^D, and $d(Y)$ is defined as

$$d(Y) = \frac{1}{2} \left(K^2 - \sum_{a=1}^\ell \tilde{m}_a^2 \right). \tag{A.6}$$

We stress that the relevant partitions are partitions of K instead of N in this irregular case.

Derivation. The above rules are derived as follows. If we use complex gauge transformations $\Phi \to g\Phi g^{-1}$, $g \in \mathrm{SU}(N)_{\mathbb{C}}$, to make the Higgs field of the Hitchin system Φ block diagonal, the irregular singularity is given as (assuming that the singularity is at $z = 0$),

$$\Phi \to \frac{1}{z} \begin{pmatrix} \nu_h & 0 \\ 0 & 0 \end{pmatrix} + \Lambda \begin{pmatrix} 0 & 0 \\ 0 & W(z) \end{pmatrix} \tag{A.7}$$

where ν_h is a $K \times K$ nilpotent matrix in the orbit $\nu_h \in \bar{O}_{Y^D}$, and $W(z)$ is an $(N - K) \times (N - K)$ matrix whose eigenvalues are given as

$$W(z) \to \frac{1}{z^{1+1/(N-K)}} \mathrm{diag}(1, \omega_{N-K}, \cdots, \omega_{N-K}^{N-K-1}) + \cdots \tag{A.8}$$

where $\omega_{N-K} = \exp(2\pi i/(N-K))$. The first $K \times K$ block behaves just as regular singularity. The commutation relation $[\Phi, \vec{\varphi}] = 0$ requires that the block which commutes with $W(z)$

must be proportional to an identity matrix. Along the lines of the discussion around (A.4), this block of $\vec{\varphi}$ is an $n_r \times n_r$ unit matrix with $n_r \geq N - K$.

In [28], it was proposed that the holomorphic moment map μ_h of the U(K) flavor symmetry is related to φ_h as

$$\det(xI_N - \varphi_h) = \det\left(xI_N - \begin{pmatrix} \mu_h & 0 \\ 0 & 0 \end{pmatrix} + \frac{\operatorname{tr}\mu_h}{N}I_N\right). \tag{A.9}$$

Also from this, we can see that ϕ_h must be proportional to the unit matrix in the block which contains $W(z)$.

Now the matrices ν_h and μ_h are just as in the regular case. The rules stated above can be found as in the regular singularity.

A.2 Mixed branches of SQCD

Now we study mixed branches of SQCD using the irregular singularities discussed above.

A.2.1 $N_f < N$

Before going to the general cases, let us consider the case $N_f < N$ in a simple realization as a warm-up. We proceed as if $N_f < N - 1$, but the case $N_f = N - 1$ can be obtained just by shifting x to cancel the x^{N-1} term in the curve.

On a Riemann sphere, we put the singularity (A.1) with $K = N_f$ at $z = 0$, and with $K = 0$ at $z = \infty$. The curve is

$$0 = x^N + \sum_{i=2}^{N} \frac{u_i}{z^i}x^{N-i} + \frac{\Lambda^{N-N_f}}{z^{1+N-N_f}}x^{N_f} + \frac{\Lambda^N}{z^{N-1}}. \tag{A.10}$$

Because of the existence of the singularity at $z = \infty$ (i.e., the last term Λ^N/z^{N-1} in the above equation), it is impossible to factorize the curve, so we have $X = [n_s] = [N]$. All the contributions to the Higgs factor comes from the puncture at $z = 0$.

Let us apply (A.4) to the puncture at $z = 0$ with $K = N_f$. Because of the existence of the last term Λ^N/z^{N-1} in (A.10) which has a pole of order $N - 1$ at $z = 0$, one can check that the partition must be of the form $Y'^D = [\tilde{m}'_a] = [N_f - r, r]$, where $0 \leq r \leq [N_f/2]$. Then the curve is given as

$$0 = x^N + \sum_{i=2}^{N-r} \frac{u_i}{z^i}x^{N-i} + \frac{\Lambda^{N-N_f}}{z^{1+N-N_f}}x^{N_f} + \frac{\Lambda^N}{z^{N-1}}. \tag{A.11}$$

The dimension of the Coulomb branch $\mathcal{C}_r$ is $N - r - 1$ spanned by $u_2, \cdots, u_{N-r}$. The quaternionic dimension of the Higgs component $\mathcal{H}_r$ is computed by using (A.5) as

$$\dim_{\mathbb{H}}\mathcal{H}_r = \frac{1}{2}\left(N_f^2 - (N_f - r)^2 - r^2\right) = r(N_f - r). \tag{A.12}$$

This is exactly as was found by Argyres, Plesser and Seiberg [4].

A.2.2 $N_f = N_1 + N_2$

Now let us consider more general cases. We take $N_f = N_1 + N_2$, and assume that $N_{1,2} < N$.

Using the singularity (A.7) for $K = N_1, N_2$ at $z = 0, \infty$ respectively, the curve is

$$0 = x^N + \sum_{i=2}^{N} \frac{u_i}{z^i} x^{N-i} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} x^{N_1} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} x^{N_2}. \tag{A.13}$$

When factorizing the curve as in (2.8), there are two possibilities.

1. One factor contains both the higher singular terms at $z = 0, \infty$, i.e., the factorization is of the form

$$0 = \left(x^{n_1} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} x^{N_1-N+n_1} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} x^{N_2-N+n_1} + \cdots \right)(\cdots). \tag{A.14}$$

2. Higher singular terms at $z = 0$ and $z = \infty$ are contained in separate blocks,

$$0 = \left(x^{n_1} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} x^{N_1-N+n_1} + \cdots \right) \left(x^{n_2} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} x^{N_2-N+n_2} + \cdots \right)(\cdots). \tag{A.15}$$

   This is possible only if $N_f = N_1 + N_2 \geq N$.

We call the first branch as mesonic branch and the second branch as baryonic branch.

Mesonic branch. Assume that the curve is factorized as in (A.14). Then the factors which do not contain the higher pole terms have only regular singularities at $z = 0$ and ∞. Then, one can see that the only possibility is the factorization of the form

$$0 = x^{N-n_1} \left(x^{n_1} + \sum_{i=2}^{n_1} \frac{u_i}{z^i} x^{n_1-i} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} x^{N_1-N+n_1} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} x^{N_2-N+n_1} \right). \tag{A.16}$$

Then the partition specifying the factorization is given by $X = [n_s] = [n_1, 1^{N-n_1}]$.

First, let us consider the case $n_1 > \max\{N - N_1, N - N_2\}$. In this case, we can assume $u_{n_1} \neq 0$, since otherwise the curve would be further factorized according to the partition $[n_s] = [n_1 - 1, 1^{N-n_1+1}]$ if $u_{n_1} = 0$. Then, the condition $u_{n_1} \neq 0$ requires that the partitions $Y'^D = [\tilde{m}'_a]$ of N_1 and N_2 at $z = 0$ and $z = \infty$ must be $[N_1 - N + n_1, 1^{N-n_1}]$ and $[N_2 - N + n_1, 1^{N-n_1}]$, respectively. Thus we get

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_\alpha =& (N - n_1) + \frac{1}{2} \left(N_1^2 - (N_1 - N + n_1)^2 - (N - n_1) \cdot 1^2 \right) \\ &+ \frac{1}{2} \left(N_2^2 - (N_2 - N + n_1)^2 - (N - n_1) \cdot 1^2 \right) \\ =& r(N_f - r), \end{aligned} \tag{A.17}$$

where we have defined $r = N - n_1$ and $N_f = N_1 + N_2$. From the constraint $\max\{N - N_1, N - N_2\} < n_1 \leq N$, we get $0 \leq r < \min\{N_1, N_2\}$

Next, let us consider the case $n_1 = \max\{N - N_1, N - N_2\}$. Without loss of generality, we assume $N_1 \geq N_2$ and $n_1 = N - N_2$. The curve is

$$0 = x^{N_2} \left(x^{N-N_2} + \sum_{i=2}^{N-N_2} \frac{u_i}{z^i} x^{N-N_2-i} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} x^{N_1-N_2} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} \right). \qquad \text{(A.18)}$$

Here we do not assume that u_{N-N_2} is nonzero.

Now, the situation is somewhat similar to section A.2.1. The existence of the last term $\Lambda^{N-N_2}/z^{-1+N-N_2}$ makes it impossible to factorize the curve further. Furthermore, because of this last term, the partition $Y'^D = [\tilde{m}'_a]$ of N_1 at $z = 0$ must be of the form $[N_1 - N_2 - r', r', 1^{N_2}]$, where $0 \leq r' \leq [(N_1 - N_2)/2]$. The partition $Y'^D = [\tilde{m}'_a]$ of N_2 at $z = \infty$ is given by $[1^{N_2}]$. The nonzero Coulomb moduli are given by $u_2, \cdots, u_{N-N_2-r'}$, so the dimension of the Coulomb factor $\mathcal{C}_\alpha$ is $N - N_2 - r' - 1$. The dimension of the Higgs component $\mathcal{H}_\alpha$ is

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha = N_2 + \frac{1}{2} \left(N_1^2 - (N_1 - N_2 - r')^2 - r'^2 - N_2 \right) + \frac{1}{2} \left(N_2^2 - N_2 \right)$$
$$= r(N_f - r), \qquad \text{(A.19)}$$

where we have defined $r = r' + N_2$. It satisfies $\min\{N_1, N_2\} = N_2 \leq r \leq [N_f/2]$.

Summarizing what we have found above, mesonic mixed Higgs-Coulomb branches are labelled by r, $0 \leq r \leq [N_f/2]$. The Higgs branch dimension is $r(N_f - r)$, while the Coulomb branch dimension is $N - r - 1$. This is exactly as was found in [4].

Baryonic branch. Next let us consider the case (A.15). In this case, one can check that the only possible curve allowed by the singularities is given by

$$0 = x^{N_f-N} \left(x^{N-N_1} + \frac{\Lambda^{N-N_1}}{z^{1+N-N_1}} \right) \left(x^{N-N_2} + \frac{\Lambda^{N-N_2}}{z^{-1+N-N_2}} \right). \qquad \text{(A.20)}$$

So all the Coulomb moduli are fixed. This factorization corresponds to the partition $X = [n_s] = [N - N_1, N - N_2, 1^{N_f-N}]$.

Expanding this curve, we find that $u_{2N-N_f} = \Lambda^{2N-N_f} \neq 0$. Then, the partitions $Y'^D = [\tilde{m}'_a]$ of N_1 and N_2 at $z = 0$ and $z = \infty$ are given by $[N - N_2, 1^{N_f-N}]$ and $[N - N_1, 1^{N_f-N}]$, respectively. Therefore we get

$$\dim_{\mathbb{H}} \mathcal{H}_\alpha = (N_f - N + 1) + \frac{1}{2} \left(N_1^2 - (N - N_2)^2 - (N_f - N) \right)$$
$$+ \frac{1}{2} \left(N_2^2 - (N - N_1)^2 - (N_f - N) \right)$$
$$= N(N_f - N) + 1. \qquad \text{(A.21)}$$

This reproduce the result of [4].

It is remarkable that the final results do not depend on N_1 and N_2 separately, but depend only on the combination $N_f = N_1 + N_2$, although each bulk and local contributions to $\dim_{\mathbb{H}} \mathcal{H}_\alpha$ are different for different pairs (N_1, N_2). This means that different brane constructions lead to the same low energy 4d theory as expected.

There are also constructions of SQCD with $N_f \geq N$ by using two regular punctures (one of them is full and the other is simple), and one irregular puncture. We leave that case for the reader.

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485] [`hep-th/9407087`] [INSPIRE].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [`hep-th/9408099`] [INSPIRE].

[3] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ Supergravity - Matter Systems*, *Nucl. Phys.* **B 255** (1985) 569 [INSPIRE].

[4] P.C. Argyres, M.R. Plesser and N. Seiberg, *The Moduli space of vacua of $N = 2$ SUSY QCD and duality in $N = 1$ SUSY QCD*, *Nucl. Phys.* **B 471** (1996) 159 [`hep-th/9603042`] [INSPIRE].

[5] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [`hep-th/9703166`] [INSPIRE].

[6] K. Landsteiner, E. Lopez and D.A. Lowe, *N=2 supersymmetric gauge theories, branes and orientifolds*, *Nucl. Phys.* **B 507** (1997) 197 [`hep-th/9705199`] [INSPIRE].

[7] A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, *M theory and Seiberg-Witten curves: Orthogonal and symplectic groups*, *Nucl. Phys.* **B 504** (1997) 175 [`hep-th/9705232`] [INSPIRE].

[8] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [`hep-th/9510101`] [INSPIRE].

[9] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys.* **B 459** (1996) 97 [`hep-th/9509161`] [INSPIRE].

[10] P.C. Argyres, M.R. Plesser and A.D. Shapere, *N=2 moduli spaces and $N = 1$ dualities for $SO(n(c))$ and $USp(2n(c))$ superQCD*, *Nucl. Phys.* **B 483** (1997) 172 [`hep-th/9608129`] [INSPIRE].

[11] D. Gaiotto, A. Neitzke and Y. Tachikawa, *Argyres-Seiberg duality and the Higgs branch*, *Commun. Math. Phys.* **294** (2010) 389 [`arXiv:0810.4541`] [INSPIRE].

[12] D. Gaiotto, G.W. Moore and Y. Tachikawa, *On 6d N=(2,0) theory compactified on a Riemann surface with finite area*, *Prog. Theor. Exp. Phys.* **2013** (2013) 013B03 [`arXiv:1110.2657`] [INSPIRE].

[13] P.C. Argyres, K. Maruyoshi and Y. Tachikawa, *Quantum Higgs branches of isolated $N = 2$ superconformal field theories*, *JHEP* **10** (2012) 054 [`arXiv:1206.4700`] [INSPIRE].

[14] K. Maruyoshi, Y. Tachikawa, W. Yan and K. Yonekura, *N=1 dynamics with T_N theory*, *JHEP* **10** (2013) 010 [`arXiv:1305.5250`] [INSPIRE].

[15] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli et al., *Infinite Chiral Symmetry in Four Dimensions*, [`arXiv:1312.5344`] [INSPIRE].

[16] K. Hori, H. Ooguri and Y. Oz, *Strong coupling dynamics of four-dimensional $N = 1$ gauge theories from M-theory five-brane*, *Adv. Theor. Math. Phys.* **1** (1998) 1 [`hep-th/9706082`] [inSPIRE].

[17] T. Nakatsu, K. Ohta, T. Yokono and Y. Yoshida, *Higgs branch of $N = 2$ SQCD and M-theory branes*, *Nucl. Phys.* **B 519** (1998) 159 [`hep-th/9707258`] [inSPIRE].

[18] T. Nakatsu, K. Ohta and T. Yokono, *On the baryonic branch root of $N = 2$ MQCD*, *Phys. Rev.* **D 58** (1998) 26003 [`hep-th/9712029`] [inSPIRE].

[19] D. Gaiotto, *N=2 dualities*, *JHEP* **08** (2012) 034 [`arXiv:0904.2715`] [inSPIRE].

[20] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin Systems and the WKB Approximation*, `arXiv:0907.3987` [inSPIRE].

[21] O. Chacaltana and J. Distler, *Tinkertoys for Gaiotto Duality*, *JHEP* **11** (2010) 099 [`arXiv:1008.5203`] [inSPIRE].

[22] O. Chacaltana, J. Distler and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d N=(2,0) theories*, *Int. J. Mod. Phys.* **A 28** (2013) 1340006 [`arXiv:1203.2930`] [inSPIRE].

[23] D. Xie, *General Argyres-Douglas Theory*, *JHEP* **01** (2013) 100 [`arXiv:1204.2270`] [inSPIRE].

[24] P.C. Argyres and M.R. Douglas, *New phenomena in* SU(3) *supersymmetric gauge theory*, *Nucl. Phys.* **B 448** (1995) 93 [`hep-th/9505062`] [inSPIRE].

[25] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [`hep-th/0007018`] [inSPIRE].

[26] D. Xie, *M5 brane and four dimensional $N = 1$ theories I*, *JHEP* **04** (2014) 154 [`arXiv:1307.5877`] [inSPIRE].

[27] D. Xie and K. Yonekura, *Generalized Hitchin system, Spectral curve and $\mathcal{N} = 1$ dynamics*, *JHEP* **01** (2014) 001 [`arXiv:1310.0467`] [inSPIRE].

[28] K. Yonekura, *Supersymmetric gauge theory, (2,0) theory and twisted 5d super-Yang-Mills*, *JHEP* **01** (2014) 142 [`arXiv:1310.7943`] [inSPIRE].

[29] G. Bonelli, S. Giacomelli, K. Maruyoshi and A. Tanzini, *N=1 Geometries via M-theory*, *JHEP* **10** (2013) 227 [`arXiv:1307.7703`] [inSPIRE].

[30] D. Gaiotto and E. Witten, *S-duality of Boundary Conditions In $N = 4$ Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [`arXiv:0807.3720`] [inSPIRE].

[31] M.R. Douglas, *On $D = 5$ super Yang-Mills theory and (2,0) theory*, *JHEP* **02** (2011) 011 [`arXiv:1012.2880`] [inSPIRE].

[32] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *M5-Branes, D4-branes and Quantum 5D super-Yang-Mills*, *JHEP* **01** (2011) 083 [`arXiv:1012.2882`] [inSPIRE].

[33] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three-dimensions*, `hep-th/9607163` [inSPIRE].

[34] N.J. Hitchin, *The Selfduality equations on a Riemann surface*, *Proc. Lond. Math. Soc.* **55** (1987) 59.

[35] N.J. Hitchin, *Stable bundles and integrable systems*, *Duke Math. J.* **54** (1987) 91 [inSPIRE].

[36] F. Benini, S. Benvenuti and Y. Tachikawa, *Webs of five-branes and $N = 2$ superconformal field theories*, *JHEP* **09** (2009) 052 [`arXiv:0906.0359`] [inSPIRE].

[37] Y. Tachikawa, *N=2 supersymmetric dynamics for pedestrians*, arXiv:1312.2684 [INSPIRE].

[38] D. Gaiotto and J. Maldacena, *The Gravity duals of N = 2 superconformal field theories*, *JHEP* **10** (2012) 189 [arXiv:0904.4466] [INSPIRE].

[39] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New N = 2 superconformal field theories in four-dimensions*, *Nucl. Phys.* **B 461** (1996) 71 [hep-th/9511154] [INSPIRE].

[40] G. Bonelli, K. Maruyoshi and A. Tanzini, *Wild Quiver Gauge Theories*, *JHEP* **02** (2012) 031 [arXiv:1112.1691] [INSPIRE].

[41] S. Cecotti, A. Neitzke and C. Vafa, *R-Twisting and 4d/2d Correspondences*, arXiv:1006.3435 [INSPIRE].

[42] Y. Tachikawa, *Six-dimensional D(N) theory and four-dimensional SO-USp quivers*, *JHEP* **07** (2009) 067 [arXiv:0905.4074] [INSPIRE].

[43] O. Chacaltana and J. Distler, *Tinkertoys for the D_N series*, *JHEP* **02** (2013) 110 [arXiv:1106.5410] [INSPIRE].

[44] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the E_6 Theory*, arXiv:1403.4604 [INSPIRE].