

Boundaries and defects of $\mathcal{N} = 4$ SYM with 4 supercharges. Part II: Brane constructions and $3d$ $\mathcal{N} = 2$ field theories

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ABSTRACT: We study the vacuum moduli spaces of $3d$ $\mathcal{N} = 2$ supersymmetric quantum field theories by applying the formalism developed in our previous paper [1]. The $3d$ theories can be realized by branes in type IIB string theory, which in a decoupling limit reduce to $4d$ $\mathcal{N} = 4$ super-Yang-Mills theory on an interval with BPS defects inserted. The moduli space of a given $3d$ theory is obtained by solving a generalization of Nahm's equations with appropriate boundary/junction conditions, along with help from the S-duality of type IIB string theory. Our classical computations reproduce many known results about the quantum-corrected moduli spaces of $3d$ theories, e.g. $U(N_c)$ theories with N_f flavors with mass and FI parameters turned on. In particular, our methods give first-principles derivations of several results in the literature, including the s -rule, quantum splitting of classical Coulomb branches, the lifting of the Coulomb branch by non-Abelian instantons, quantum merging of Coulomb and Higgs branches, and phase transitions from re-ordering 5-branes.

KEYWORDS: Supersymmetry and Duality, Brane Dynamics in Gauge Theories

ARXIV EPRINT: [1406.5501](https://arxiv.org/abs/1406.5501)

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1 Introduction

String theory enjoys many deep connections with quantum field theory. For example, the solitonic objects of string theory such as D-branes support gauge fields living on the brane worldvolume. One particularly clear realization of the connection between string theory and field theory comes about through the Hanany-Witten brane construction of field theories [2].

This construction turns out to be an enormously powerful technique for studying supersymmetric field theories; see [3] for a comprehensive review. Many properties of the field theory can be understood pictorially in terms of a brane diagram. For example, the vacuum moduli space of the field theory is often simply the space of allowed geometrical arrangements of the branes; by counting the degrees of freedom of the brane motions, one counts the dimension of the moduli space.

Unfortunately, the naive prescription of counting geometric brane motions is not complete; to correctly determine the dimension of moduli space, some additional constraints must be imposed. The most well-known constraint is the Hanany-Witten s -rule [2]. However, in systems with reduced supersymmetry there are additional rules, which seem ad hoc from the point of view of the brane diagrams [3, 4]; these extra constraints are necessary to incorporate the effect of instanton-generated superpotentials [5]. Moreover, the

brane diagram is not enough for understanding the detailed structure of the vacuum moduli space, such as the merging of the Coulomb and the Higgs branches for non-Abelian theories. Clearly, it would be desirable to have a framework where these rules and results can be derived systematically, while retaining the connection to the brane diagram.

The point of view we advocate is that one can regard the brane diagram as a set of rules for constructing a particular system of localized defects coupled to bulk degrees of freedom. In [1], we performed a study of interface conditions for $\mathcal{N} = 4$ super-Yang-Mills theory, and explicitly constructed UV Lagrangians for such defect systems. These defect systems realize $3d \mathcal{N} = 2$ field theories in the IR, and can be constructed from type IIB brane configurations with D3-branes (which support the $4d \mathcal{N} = 4$ theory) suspended between 5-brane defects.

Our goal in the present paper is to study the vacuum moduli spaces of $3d \mathcal{N} = 2$ theories in terms of these defect systems. The moduli spaces in question can be identified with the solution spaces of a generalization of Nahm's monopole equations (a dimensional reduction of the Donaldson-Uhlenbeck-Yau equations) with a certain set of boundary conditions which were described explicitly in [1].

A crucial ingredient in our analysis is the S-duality of type IIB string theory (and of the $4d \mathcal{N} = 4$ theory on the D3-branes). The classical moduli space computation is potentially subject to quantum effects, but in many situations of interest, we can find an S-duality frame where at least some part of the moduli space is free from quantum corrections. As we will see, this happens when the gauge symmetry is completely broken. In this S-duality frame, our *classical* computation then gives the *quantum-corrected* moduli space. For $\mathcal{N} = 4$ theories in $3d$, this is nothing but the usual statement of mirror symmetry [6]. A similar situation holds for $\mathcal{N} = 2$ theories, although with many subtle differences from the $\mathcal{N} = 4$ case.

To place our results in context, we might recall the analogous situation for type IIA brane constructions and their $4d$ field theories. In that case, the strong coupling limit of the $4d$ field theory can be studied by lifting the IIA configuration to M-theory [7]. The brane configuration becomes a single M5-brane wrapped on a complex curve which turns out to be the Seiberg-Witten curve of the $4d$ theory. For type IIB brane constructions, we do not have the lift to M-theory as a tool. Instead, the strong coupling limit of the IIB construction is best understood by performing an S-duality. A detailed description of the defect system, combined with S-duality, will be sufficient to understand many features of the moduli spaces of the $3d$ field theories (previously studied in [8–10]).

The rest of this paper is organized as follows. We summarize our methods in section 2. We realize $3d \mathcal{N} = 2$ theories from brane configurations, and the vacuum moduli space of the $3d$ theory can be identified with the moduli space of the bulk BPS equations with appropriate boundary conditions corresponding to the arrangement of 5-branes. We then apply our formalism to a variety of $3d$ field theories with $U(N_c)$ gauge groups. First, we work out some examples with $\mathcal{N} = 4$ $3d$ supersymmetry in detail in section 3; these simple examples are sufficient to demonstrate most of the technical issues associated with our methods. We then proceed to apply our techniques to $\mathcal{N} = 2$ Abelian field theories in section 4, and to non-Abelian $U(N_c)$ gauge theories in sections 6 and 7. We will also

study in section 8 a few more examples with product gauge groups which exhibit quantum merging of Higgs and Coulomb branches.

2 Moduli space from generalized Nahm equations

In this section we summarize our general method for studying the vacuum moduli spaces of $3d \mathcal{N} = 2$ theories. Our analysis will rely on two crucial ingredients, S-duality and holomorphy. The latter will be expressed mathematically in terms of a complex gauge quotient.

We begin with the standard brane configurations for the $3d \mathcal{N} = 2$ theory (section 2.1), and point out the subtleties of $3d \mathcal{N} = 2$ mirror symmetry, as well as the limitations of the cartoonish brane descriptions (section 2.2). We will then write down the generalized BPS equations following our previous paper [1] (section 2.3), and comment on technical (but important) issues of gauge symmetry breaking and stability in the rest of this section.

2.1 $3d \mathcal{N} = 2$ theory from type IIB brane constructions

In type IIB string theory, we can engineer $3d \mathcal{N} = 2$ field theories by suspending D3-branes between NS5 and D5 branes. The NS5 and D5 branes may be oriented in directions consistent with the supercharges preserved by the $3d$ field theory. We consider two allowed types of NS5 brane which we call NS5 and NS5', and two types of D5 brane, which we call D5 and D5' [4, 8].¹ Our convention for orienting the branes is as follows.

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	○						
D5	○	○	○		○	○	○			
D5'	○	○	○				○	○	○	
NS5	○	○	○					○	○	○
NS5'	○	○	○		○	○				○

(2.1)

In general, D3-branes suspended between NS5-type branes contribute $3d$ vector multiplets while D5-branes intersecting D3-branes contribute quarks. The detailed rules for relating the brane construction to field theory may be found in the review [3].

An example of the kind of system we will be analyzing is illustrated in figure 1. The figure shows a brane construction which realizes $3d$ U(1) gauge theory with 3 flavors, in the limit where the lengths of the D3-brane segments (extended in x_3) are taken to zero.

2.2 Quantum corrections, mirror symmetry and S-duality

In many cases, the brane construction gives a recipe for writing down a supersymmetric field theory, for which the Lagrangian is largely constrained by symmetry considerations.

¹We can consider D5-branes/NS5-branes oriented at more general angles (rotated by the same angle in 47 and 58-planes) while preserving 4 supercharges (see e.g. appendix A3 of [1]). In practice this will correspond to adding a finite mass to the adjoint chiral multiplets. While our formalism includes such D5-branes, we will not discuss them in this paper since we are mostly interested in the IR behavior of $3d$ theories where the precise coefficients of the relevant deformations are not important.

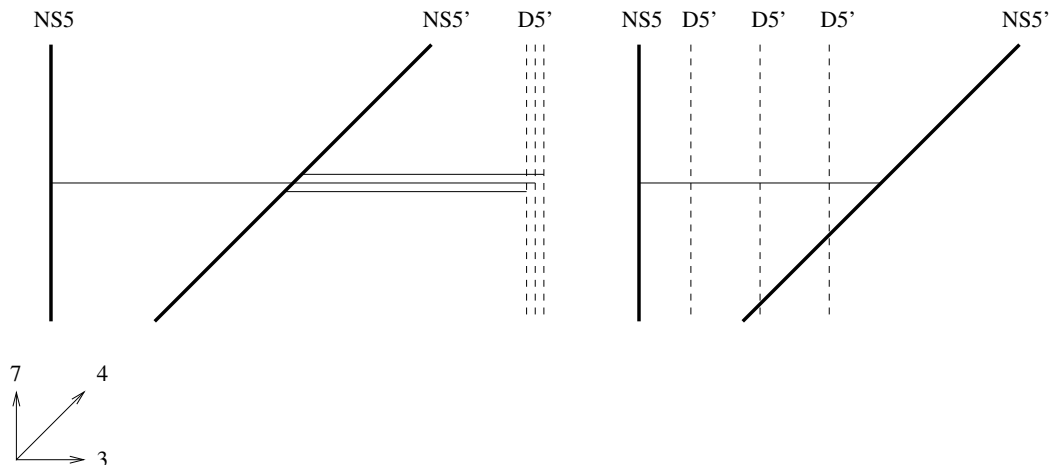


Figure 1. The brane construction of $\mathcal{N} = 2$ $U(1)$ $N_f = 3$ theory in two different brane orderings, related by Hanany-Witten transitions. Some properties of the moduli space can be understood by allowing the D3-brane to break into segments when crossing a 5-brane, and then counting the possible motions of the D3 segments.

Given a Lagrangian, one can determine the classical moduli space of vacua by solving the F- and D-flatness conditions. However, this moduli space can be subject to corrections at the quantum level, and it is the heart of this paper to understand these quantum effects.

2.2.1 $3d$ $\mathcal{N} = 4$ theories

Let us first recall the case with eight supercharges (i.e. $3d$ $\mathcal{N} = 4$ theory), for which the structure of moduli space, though intricate, is highly constrained. The $\mathcal{N} = 4$ moduli spaces fall into distinct parts, a Higgs branch, where the gauge symmetry is broken, and a Coulomb branch, which preserves an unbroken $U(1)^N$ gauge symmetry, as well as mixed branches which are a direct product of a Higgs branch and a Coulomb branch with some number of unbroken $U(1)$'s. The constraints on the $\mathcal{N} = 4$ moduli space partly stem from the existence of an $SO(3)_H \times SO(3)_V$ R-symmetry which acts separately on the hypermultiplets and the vectormultiplets. The moduli space metric is hyperKähler, and can essentially be determined from the asymptotic structure and singularity structure of the moduli space. Moreover, (in part) as a consequence of this global symmetry, the mixed branches are always a direct product of a Coulomb part and a Higgs part (for a proof, see the argument in the $4d$ context in [11]).

The classical field theory computation of the $\mathcal{N} = 4$ moduli space is not reliable on all the branches of vacua. On the Higgs branch, because the vectors are massive, all the remaining fields are simply free and the classical geometry of the Higgs branch is quantum mechanically exact. The Coulomb branch and mixed branches, on the other hand, have unbroken gauge symmetry and are therefore subject to both perturbative corrections and nonperturbative instanton corrections [5].

It is often said that $\mathcal{N} = 4$ mirror symmetry (which is understood as S-duality of a IIB brane construction) exchanges the Higgs and Coulomb branches of two mirror pair

theories A and B. What is really meant by this statement is that the whole moduli space is exchanged by mirror symmetry, but the extent of gauge symmetry and its breaking is different for the branches identified as mirror pairs in theory A and B. Because of this, different branches are computable classically on the two sides of the mirror. If the mirror pair is known, then the Higgs branch of theory A (which is classically exact) determines the Coulomb branch of theory B (which is subject to quantum corrections), and vice versa.

From the brane construction, it is easy to understand this aspect of the duality; D5-brane interfaces which break some amount of gauge symmetry are exchanged with NS5-brane interfaces which do not. The Coulomb branch is associated with the movement of D3 segments stretched between a pair of NS5 branes, and the Higgs branch is associated with the movement of a D3-brane segment stretched between a pair of D5 branes. A D3 segment stretched between an NS5 and the D5, on the other hand, is completely fixed in its position and do not give rise to moduli. The $SO(3) \times SO(3)$ global symmetry acts geometrically by rotating the 456 coordinates and the 789 coordinates separately. In this picture, mirror symmetry is a consequence of S-duality, which exchanges D5 and NS5 branes. Clearly, in the brane construction, it exchanges Coulomb and Higgs branch moduli.

In the case of $\mathcal{N} = 4$ supersymmetry, one can also identify the Coulomb branch as part of the moduli space that is lifted by turning on Fayet-Iliopoulos terms, and the Higgs branch as the part that is lifted by turning on masses to the matter fields. This is a consequence of the fact that these deformations are charged under the $SO(3) \times SO(3)$ global symmetry. From the brane point of view, the FI terms displace the NS5 branes relative to each other, so that D3-branes stretched between the NS5 branes do not preserve supersymmetry, lifting the Coulomb branch. Similarly, the mass terms move the D5-branes relative to each other, lifting the Higgs branch.

2.2.2 3d $\mathcal{N} = 2$ theories

Much of the structure of $\mathcal{N} = 4$ theories extends to $\mathcal{N} = 2$ theories but with various caveats. Some of these subtleties were discussed in [8–10] but we choose to take our own perspective on some of these concepts which we will explain below. A portion of the moduli space can be considered a “Higgs branch” if charged matter fields have expectation values there, and all the gauge symmetry is broken spontaneously. Similarly there can be a “Coulomb branch” if the scalars in the vector multiplet are nonvanishing, and some Abelian subgroup of the gauge group is left unbroken. There can also be mixed branches with both Coulomb and Higgs components.

However, unlike the $\mathcal{N} = 4$ case, where separate $SO(3)_H$ and $SO(3)_V$ global symmetries act on the vector multiplets and hypermultiplets, in $\mathcal{N} = 2$ there is in general only $SO(2)$ R-symmetry, and the structure of the moduli space is much less constrained by symmetry. In this paper, for simplicity we only consider the case where the global symmetry is $SO(2) \times SO(2)$. Even in this case, however, the $SO(2) \times SO(2)$ global symmetry does not sharply distinguish between Coulomb branch and Higgs branch moduli, and the moduli spaces can exhibit considerable complexity.

Because S-duality still has an action on 1/4 BPS brane configurations, there is a notion of mirror symmetry mapping an electric theory A to a magnetic theory B with

$\mathcal{N} = 2$ supersymmetry. Some aspects of this mirror symmetry were also discussed in [8–10]. There are, however, several important differences in the way mirror symmetry acts in the $\mathcal{N} = 2$ theories as compared with the $\mathcal{N} = 4$ prototype. One important distinction is that unlike in the $\mathcal{N} = 4$ theories, the Coulomb branch of model A and the Higgs branch of model B do not map on to one another one-to-one (and similarly for the Higgs branch of model A and the Coulomb branch of model B).

Both the Higgs and the Coulomb branches can receive quantum corrections. Some components of the Coulomb branch can be lifted by superpotentials generated by instanton effects. It is also possible for some of the branches of moduli space to merge quantum mechanically. [8–10]. However, the complex structure of the Higgs branch will be free of instanton effects.² For the purposes of this paper, we will limit our discussion to the complex structure of moduli space, to avoid details of the geometry which depend on the moduli space metric. Some interesting recent work on the moduli space metric for closely related intersecting brane systems appeared in [12] with implications for $3d$ field theory as studied in [13].

Some of these issues can be made apparent by looking at mirror symmetry from the point of view of type IIB brane configurations [4, 8]. A wide class of $\mathcal{N} = 2$ theories in 2+1 dimensions can be engineered by suspending D3 brane intervals between NS5, NS5', D5, and D5' branes. In this brane construction, mirror symmetry follows from the S-duality of the type IIB string theory. The $SO(2)_{45} \times SO(2)_{78}$ global symmetry can be understood as rotation in the 45 plane and the 78 plane for the branes oriented according to (2.1).

It is natural to identify the components of the moduli space associated with moving the D3 branes stretched between an NS5 and an NS5' (or an NS5) brane as being associated with the Coulomb branch. Similarly, D3 segments stretched between a D5 and a D5' (or a D5) brane should be interpreted as being part of the Higgs branch since D3 broken on D5's breaks the gauge symmetry associated with the D3 brane. These two branches naturally map onto each other under S-duality of type IIB string theory.

In the $\mathcal{N} = 2$ construction, however, there are additional moduli associated with the D3 segments stretched between an NS5' and a D5, or between an NS5 and a D5'. These branches, which were not present in the $\mathcal{N} = 4$ construction, behave somewhat differently than the two branches we described above. The $U(1)$ gauge symmetry living on such a D3-brane segment is broken by the D5/D5' boundary conditions, so these moduli are naturally Higgs branch moduli. However, S-duality maps the NS5'-D5 configuration to D5'-NS5, so the corresponding mirror symmetry maps Higgs branch moduli to Higgs branch moduli and not to the Coulomb branch.

It is also instructive to think about how the various branches of moduli space are affected by mass deformations. In $\mathcal{N} = 4$ theories, the Coulomb branch is lifted by turning on FI terms but is not lifted by masses. The Higgs branch, on the other hand, is lifted by the mass terms but is unlifted by the FI term. In the case of $\mathcal{N} = 2$ supersymmetry,

²For instantons to correct the superpotential, they need to have two fermionic zero-modes. 1/2 BPS instantons in $\mathcal{N} = 2$ theories have two such zero-modes, which are the goldstini from spontaneously breaking half the supersymmetry. If the gauge symmetry is fully broken, however, the 1/2 BPS instantons will not exist, and the superpotential will not be corrected.

brane	phase	gauge symmetry	$SO(2)_{45}$	$SO(2)_{78}$	FI	real mass
NS5-NS5'	Coulomb	partly unbroken	neutral	neutral	lifted	unlifted
D5-D5'	Higgs	broken	neutral	neutral	unlifted	lifted
NS5-NS5	Coulomb	partly unbroken	neutral	charged	lifted	unlifted
D5-D5	Higgs	broken	charged	neutral	unlifted	lifted
NS5'-NS5'	Coulomb	partly unbroken	neutral	charged	lifted	unlifted
D5'-D5'	Higgs	broken	charged	neutral	unlifted	lifted
NS5-D5'	Higgs	broken	neutral	charged	unlifted	unlifted
NS5'-D5	Higgs	broken	charged	neutral	unlifted	unlifted

Table 1. Classification of phases of moduli-space components associated with the motion of D3 branes in the Hanany-Witten-like brane construction.

on the other hand, the Coulomb branch is still lifted by the FI term, but not all of the Higgs branch is necessarily lifted by the real mass terms (in this paper we consider only parity-preserving real mass terms). The reason is that the moduli associated to the D3 branes stretched between an NS5 brane and a D5' are not lifted either by the FI or the mass term.

It should be clear from these considerations that one should not think of Higgs and Coulomb branches as being mapped onto one another under mirror symmetry for theories with $\mathcal{N} = 2$ supersymmetry. These aspects of the action of $\mathcal{N} = 2$ mirror symmetry are summarized in table 1.

As mentioned previously, further complications arise because in some cases quantum effects can completely blur the distinction between Higgs branches and Coulomb branches; this was called “quantum merging” of the Higgs and Coulomb branches³ in [10]. We will encounter a number of examples where this occurs; in our analysis it is intrinsically tied to the non-Abelian nature of the $4d \mathcal{N} = 4$ defect system, and is hard to understand from the point of view of the brane cartoon. We will overcome this limitation with the help of a more refined analysis of the BPS equations of the defect system.

2.3 Bulk BPS equations

The brane constructions give rise to defect theories with translation invariance broken in one direction which we will label as y throughout this paper. Because the D3-brane has the $3+1$ -dimensional $\mathcal{N} = 4$ theory living on its worldvolume, this defect theory consists of the $4d \mathcal{N} = 4$ theory living on a sequence of intervals with the defects realized as supersymmetric boundary conditions for each interval. In the cases we will consider, the boundary conditions will correspond to various combinations of NS5 and D5-branes, possibly oriented in different directions compatible with preserving 4 supercharges.

In the traditional brane drawing analysis, one studies the moduli space by allowing the D3-branes to break on the 5-brane defects; in this picture, the allowed motions of the D3-brane segments are the moduli of the theory. In this paper we attempt to analyze

³Strictly speaking, it is the Coulomb branch and a mixed Higgs-Coulomb branch which merge.

the same systems from the point of view of the $4d$ defect theory. For non-Abelian gauge theories, the defect analysis is crucially different from the brane cartoon.

Let us summarize the bulk BPS equations of the defect system [1]. The bulk $\mathcal{N} = 4$ theory consists of a gauge field A_μ , $\mu = 0, 1, 2, 3$, and six adjoint scalars X_i , $i = 4, \dots, 9$, where the numbering of the indices reflects the relationship between the 3+1-dimensional $\mathcal{N} = 4$ theory and the 9+1-dimensional $\mathcal{N} = 1$ SYM theory. In addition to the bosonic fields, the theory contains fermions. In this paper the fermions play no role except that we demand that some of their variations under supersymmetry vanish. As in [1] we choose to take the fields A_μ, X_i to be anti-Hermitian, to be consistent with the mathematical literature on Nahm equations.

Away from any boundaries, the scalars of the 3+1-dimensional field theory must satisfy certain equations to preserve four supersymmetries, with the assumptions of 2+1-dimensional Lorentz invariance and the existence of a $U(1) \times U(1)$ global symmetry. These equations are conveniently written in terms of three complex equations

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y} = 0, \tag{2.2}$$

$$\frac{\mathcal{D}\mathcal{Y}}{\mathcal{D}y} = 0, \tag{2.3}$$

$$[\mathcal{X}, \mathcal{Y}] = 0, \tag{2.4}$$

and one real equation,

$$\frac{d}{dy}(\mathcal{A} - \bar{\mathcal{A}}) - [\mathcal{A}, \bar{\mathcal{A}}] + [\mathcal{X}, \bar{\mathcal{X}}] + [\mathcal{Y}, \bar{\mathcal{Y}}] = 0, \tag{2.5}$$

where

$$\mathcal{X} \equiv X^4 + iX^5, \tag{2.6}$$

$$\mathcal{Y} \equiv X^7 + iX^8, \tag{2.7}$$

$$\mathcal{A} \equiv A_3 + iX^6. \tag{2.8}$$

When $\mathcal{Y} = 0$, these equations reduce to Nahm's equations [14], and our analysis here generalizes the relation between D-branes and Nahm's equations discovered in [15].

When applied to $\mathcal{N} = 4$ SYM on an interval with defects, we must subject the bulk equation to boundary and junction conditions. The conditions relevant for our purposes were studied in our earlier paper [1] and are briefly summarized in appendix B.

2.4 Gauge symmetry breaking and X_9

In addition to the generalized Nahm equations which involve the scalars $X_{4,5,6,7,8}$, we have one more adjoint scalar X_9 , which combines with the field strength of the $3d$ gauge field A_μ (the “dual scalar” φ) into a $3d$ linear multiplet. The BPS equation for the X_9 reads

$$D_3 X_9 = 0, \tag{2.9}$$

$$[X_6, X_9] = 0, \tag{2.10}$$

$$[\mathcal{X}, X_9] = 0, \tag{2.11}$$

$$[\mathcal{Y}, X_9] = 0. \tag{2.12}$$

Although it is not clearly evident in the $4d \mathcal{N} = 4$ analysis, to be consistent with $3d$ supersymmetry, when the field X_9 is a modulus it must be accompanied by the dual photon φ , and so each freely varying component of X_9 always gives rise to two real moduli (or one complex modulus).

In the analysis we will do later, we will keep track of the dimension of the moduli space due to the variation of X_9 and φ , but we will not try to determine more detailed properties of the moduli space, such as the complex structure, when X_9 and φ are involved. The reason is that when X_9 is active there is always some amount of unbroken gauge symmetry, and in general there will be quantum corrections. Hence the complex structure as computed classically will be wrong anyway. In some cases, one can overcome this problem by combining the classical analysis with the S-duality of $4d \mathcal{N} = 4$ theory. In other cases, we will also find that despite the existence of quantum corrections, our method still computes the dimension of moduli space correctly. In particular, this will be the case for mixed branches in non-Abelian theories.

Later in the paper, when counting moduli, we will introduce the variable \mathcal{Z} to keep track of the field X_9 and the dual scalar φ — one can think of it as forming the complex combination (monopole operator) $\mathcal{Z} \sim e^{X_9+i\varphi}$. One should, however, always keep in mind the existence of the quantum corrections.

2.5 $G_{\mathbb{C}}$ quotient and stability

The analysis of the moduli space of equations (2.2)–(2.5) can be drastically simplified by the method of a complex gauge quotient. Recall that both the bulk equations and the boundary conditions can be split into naturally complex and naturally real equations. The complex equations are invariant not just under the gauge symmetry G but rather have a larger gauge symmetry $G_{\mathbb{C}}$, the complexified version of G . The real equations are however only invariant under G . Specifically, we may take $\mathcal{X} \rightarrow g^{-1}\mathcal{X}g$, $\mathcal{Y} \rightarrow g^{-1}\mathcal{Y}g$, $\mathcal{A} \rightarrow g^{-1}\mathcal{A}g + g^{-1}dg$, where g is valued in the complexified gauge group $G_{\mathbb{C}}$. On the other hand, the real equation (2.5) is only invariant under the real gauge symmetry G and transforms nontrivially under $G_{\mathbb{C}}$.

There is a beautiful mathematical result that it is possible to simply ignore the real equations completely (modulo the subtleties to be mentioned in the next paragraph), and instead solve only the complex equations, but with a quotient by the complexified gauge group $G_{\mathbb{C}}$. The point is that we can find a true solution of the full system of equations in the closure of the $G_{\mathbb{C}}$ orbit of a point satisfying only the complex equations.

However, there is an important caveat which is that given a point p in the solution space of the complex equations, it is not necessarily guaranteed that there exists a point in the $G_{\mathbb{C}}$ orbit of p which actually satisfies the real equation; the points for which the appropriate gauge transformation does not exist are said to be “unstable”. The notion of stability was introduced by Mumford [16, 17], and essentially it amounts to classifying singular points of moduli spaces. There are many results in geometric invariant theory which give various criteria for determining the stability properties of a point in moduli space. For algebraic varieties, the definition of an unstable point is that the closure of

its $G_{\mathbb{C}}$ orbit includes the origin.⁴ The stable points have closed $G_{\mathbb{C}}$ orbits and a finite stabilizer (that is, the elements of $G_{\mathbb{C}}$ which leave the point fixed form a finite group). The semistable points satisfy the property that the gauge orbits have a closure which is nonempty but does not include the origin.

One way of thinking about the stability criterion is that we should identify points as gauge equivalent if they are related by the closure of the gauge orbit (not just finite gauge transformations). In some examples, what may appear to be an entire branch of moduli space turns out to be gauge equivalent to a single point, and therefore should not be counted as a separate branch.

2.5.1 Example: the conifold

We can illustrate the stability issue with an example familiar to physicists, the conifold. The complex structure of the conifold may be expressed by the algebraic equation

$$z_1 z_2 - z_3 z_4 = 0. \tag{2.13}$$

A second alternative description is given by “solving” the algebraic equation with the variables A_i, B_j :

$$z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1, \tag{2.14}$$

provided that one mods out by the complex gauge symmetry $U(1)_{\mathbb{C}} = GL(1, \mathbb{C}) = \mathbb{C}^*$

$$A_i \rightarrow \lambda A_i, \quad B_j \rightarrow \lambda^{-1} B_j. \tag{2.15}$$

A third description is given by taking the four complex variables A_i, B_j and mod out only by a (real) gauge symmetry $G = U(1)$

$$A_i \rightarrow e^{i\theta} A_i, \quad B_j \rightarrow e^{-i\theta} B_j, \tag{2.16}$$

and also impose a D-term equation,

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0, \tag{2.17}$$

where signs in the D-term equation are determined by the $U(1)$ charges of the A, B fields.

The three constructions are equivalent for semistable points but not for unstable points. In the second formulation, there is a family of solutions with $A_i = 0$, $B_j =$ arbitrary, and it might seem that this corresponds to a $1d$ branch of moduli space with the complex structure of $\mathbb{C}P^1$ (after modding out by $G_{\mathbb{C}}$). But this branch sits at $z_i = 0$, the conifold singularity, which should be a single point in the first description. The key fact is that this branch consists of unstable points, because we can take $B_j \rightarrow \lambda^{-1} B_j$ for $\lambda \rightarrow \infty$, so the closure of the gauge orbit contains the origin $A_i = B_j = 0$. From the third description,

⁴The origin for our problem is locus where $X_a = c_a \mathbb{I}$ (c_a : constant, $a = 1, \dots, 6$). The shift by such constant identity components clearly preserves the existence/non-existence of the solution to the real equation. In string theory language, this shift represents the center-of-mass modes of the D3-branes, which decouples from the relative positions of the D3-branes.

we see that $A_i = 0$ and $B_j \neq 0$ violates the D-term equation, unless we deform it by an appropriate Fayet-Iliopoulos term.

On the other hand, the semistable points have at least one nonzero A and one nonzero B — we see that in this situation a $G_{\mathbb{C}}$ gauge can always be found to satisfy the D-term equation of the third description.

Note that the issue of stability does not arise, for the most part, in the usual construction of monopoles through Nahm's equations. It turns out that in the case of purely D5-like boundary conditions, the corresponding solutions do not have unstable points, and so one can use the complex gauge quotient freely (see [18, 19]).

However, because we are interested in more general boundary conditions, in particular those which can include NS5-branes, the issue of stability will reappear in the analysis we do later. Namely, we are allowed to follow the procedure of choosing a convenient gauge in $G_{\mathbb{C}}$ and to solve only the complex Nahm equation. This gauge transformation will not preserve the real Nahm equation or the X_6 boundary condition in general, but this is all right, provided we restrict attention to the semistable points (which is equivalent to the statement that the $G_{\mathbb{C}}$ gauge orbit includes a solution of the full Nahm system).

Under some circumstances, the unstable points of moduli space are also important. In particular, when we deform the real Nahm equation by an FI term, it is not always possible to gauge-transform an arbitrary solution of the complex equations to a solution of the full system, and it becomes necessary to study the real equations explicitly. In the conifold example, the corresponding statement is that the singularity admits a small resolution.

3 $3d \mathcal{N} = 4$ gauge theories

We begin our analysis of defect systems of $\mathcal{N} = 4$ super Yang-Mills by studying configurations with 8 supercharges [20, 21]. This will serve as a simple demonstration of our methods before we proceed to the 1/4 BPS case. Several important features arise already in the 1/2 BPS analysis; among these are subtleties in gauge fixing as well as the role played by stability of the $G_{\mathbb{C}}$ quotient.

Of course, the systems with only D3 and D5 branes have already been studied extensively; they are the standard Nahm system for monopoles in $U(N)$ gauge theory, where N is the number of D5-branes, and the number of D3-brane segments is the total monopole charge.

3.1 $\mathcal{N} = 4, N_c = 1, N_f = 0$

We begin with a very simple example, the $\mathcal{N} = 4$ with $U(1)$ gauge symmetry and no hypermultiplet matter. From the point of view of brane configurations, one constructs this theory by suspending a single D3-brane between two NS5-branes. Because the NS5-branes are extended in the 789 directions, the D3-brane has three real moduli corresponding to the scalar fields $X_{7,8,9}$, and one compact real modulus from the dual scalar associated with the 2+1-dimensional gauge field.

The configuration of one D3-brane suspended between two D5-branes, shown in figure 2, is the S-dual of pure $\mathcal{N} = 4$ $U(1)$ gauge theory. Its moduli space is simply $\mathbb{R}^3 \times S^1$,

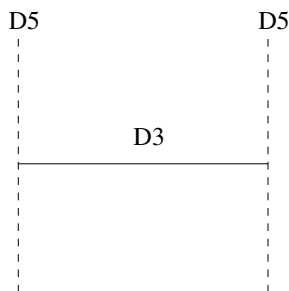


Figure 2. The S-dual of pure U(1) $\mathcal{N} = 4$ gauge theory.

with three noncompact dimensions coming from the motions of the D3-brane in $X_{4,5,6}$ and one compact modulus from the Wilson line $\int A_3 dy$.

What might be a little unclear is how to understand this moduli space from the point of view of $G_{\mathbb{C}}$, as naively one might think that one can gauge away X_6 and A_3 . The point is that this gauge transformation is possible when the branes on one side of the configuration are NS5-like, but not when they are D5-like on both sides. We are constrained to allow only gauge transformations satisfying $g = 1$ at a D5-type boundary condition. But the gauge transformation which sets $A_3 = 0$, for example, is of the form

$$g(y) = \exp\left(\int_0^y dy' A_3(y')\right), \tag{3.1}$$

which will only satisfy $g = 1$ at $y = 0$, where we hereafter (unless explicitly stated) take the position of the leftmost D5 to be at $y = 0$.

Nevertheless, we can still use the complex gauge transformation $G_{\mathbb{C}}$ to choose the gauge where A_3 is equal to its average value over the interval,

$$g(y) = \exp\left(\int_0^y dy' (\mathcal{A}(y') - \langle \mathcal{A} \rangle)\right), \tag{3.2}$$

which does satisfy $g = 1$ at both boundaries. The same considerations apply for the complexified gauge transformations. For this reason we cannot gauge away \mathcal{A} , and instead find that the average value of \mathcal{A} on the interval is a modulus. Note that because the gauge transformation is periodic under shifts of $\langle \mathcal{A} \rangle$ by $2\pi i$, the part of moduli space corresponding to VEVs of \mathcal{A} in this case has the topology of a cylinder. It combines with the VEV of \mathcal{X} to give a two-complex-dimensional moduli space.

3.2 $\mathcal{N} = 4, N_c = 1, N_f = 2$

Let us next add fundamental hypermultiplets, and consider the Abelian theory with 2 flavors. This theory, also called $T[\text{SU}(2)]$, is a well-known example for three-dimensional mirror symmetry. A brane realization of this theory is shown in figure 3, which maps to itself under S-duality, modulo the HW transition. The defect theory is defined on intervals which we label with Roman numerals. In this example, we have three such regions, labeled I, II, III, such that the gauge group is U(1) in regions I and III and U(2) in region II.

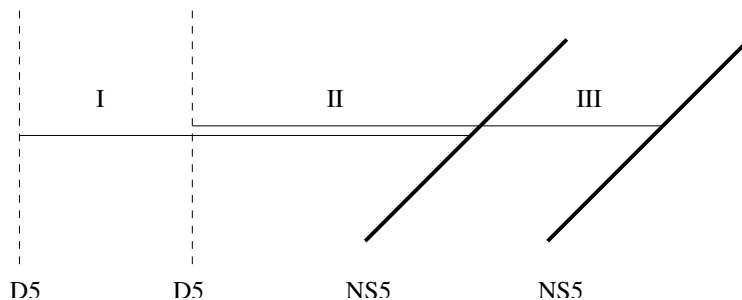


Figure 3. U(1) $\mathcal{N} = 4$ theory with two flavors.

In our framework, this example is simple because we can use complex gauge transformations to set $\mathcal{A} = 0$ everywhere. Then the complex part of the generalized Nahm equations simply imply that \mathcal{X} and \mathcal{Y} are piecewise constant commuting matrices.

Crossing from region I into region II, we find that

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.3}$$

$$\mathcal{Y}_{\text{II}} = 0. \tag{3.4}$$

At the II–III interface we have $\mathcal{X}_{\text{II}} = AB$ and $\mathcal{X}_{\text{III}} = BA = 0$. This fixes two eigenvalues of \mathcal{X}_{II} to vanish. If the bifundamentals A, B break the gauge symmetry, we have to set $\mathcal{Y}_{\text{III}} = \mathcal{Z}_{\text{III}} = 0$. So we have a two-dimensional branch of moduli space. In particular, the condition that both eigenvalues of \mathcal{X}_{II} vanish is equivalent to the conditions $d = -a$, and $a^2 + bc = 0$. This Higgs branch moduli space is the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, and because the gauge symmetry is completely broken, we expect it to be quantum mechanically exact.

There is another class of solutions where $A = B = 0$ and there is an unbroken U(1) gauge symmetry. On this branch we have $a = b = c = d = 0$ but \mathcal{Y}_{III} and \mathcal{Z}_{III} are free to vary. So we have a two-dimensional Coulomb branch with U(1) gauge symmetry, but we do not trust this branch in detail because it is subject to quantum corrections.

Because this theory is self-dual under S-duality, a natural conjecture is that the two $2d$ branches are exchanged by the duality; therefore, the quantum corrected Coulomb branch is also the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. This analysis can be generalized to any number of flavors, as described in appendix A.

3.3 Nahm pole

In our previous example (and many other examples below), the fact that we can take $\mathcal{A} = 0$ by a complex gauge transformation simplifies the analysis dramatically. The remaining equations are algebraic and thus the moduli space computation reduces to a problem of linear algebra.

As we move on to more complicated examples, however, it becomes necessary to properly take into account the singularities of the complex gauge field. When Nahm poles are included, it is not possible to set $\mathcal{A} = 0$ by a non-singular gauge transformation, but it is often still possible to choose a simple form which makes the problem essentially algebraic.

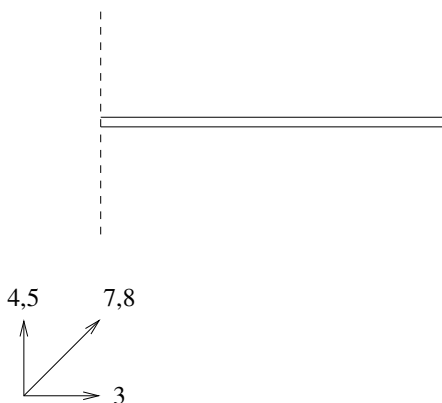


Figure 4. Two D3-branes ending on a D5-brane.

To explain this, we consider a simple noncompact example where two semi-infinite D3-branes end on a D5-brane, as shown pictorially in figure 4. There is a U(2) gauge theory with $\mathcal{N} = 4$ supersymmetry living on the D3-branes. The boundary conditions corresponding to the D5-brane are that there is a Nahm pole singularity at the location of the D5-brane on the left end.

We can choose the gauge where the complex gauge field has only singular contribution

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2y} & 0 \\ 0 & -\frac{1}{2y} \end{pmatrix}. \tag{3.5}$$

Then solving the Nahm equations we find

$$\mathcal{X} = \begin{pmatrix} a & \frac{1}{y} \\ by & a \end{pmatrix}. \tag{3.6}$$

Note that this solution captures the leading behavior of the fields for $y \rightarrow 0$. The subleading terms which satisfy Nahm’s equations can be removed by a $G_{\mathbb{C}}$ gauge transformation with $g(0) = 1$.⁵ In this analysis, we are allowing $g(\infty)$ to be arbitrary.

For the Nahm pole, the matrix-valued fields $X_{4,5,6}$ do not commute with each other. Thus they cannot be simultaneously diagonalized, and it is not possible to simply interpret their eigenvalues as the positions of D3-branes. This is the essential feature of the defect analysis which is hard to capture in the brane drawing.

3.4 $\mathcal{N} = 4, N_c = 2, N_f = 0$

The standard SU(2) Nahm 2-monopole construction also has an interesting problem of gauge fixing. This is equivalent to the brane construction of pure U(2) gauge theory with $\mathcal{N} = 4$ supersymmetry, as shown in figure 5. At both D5 boundaries, we are only allowed to do gauge transformations where $g|_{\partial M} = 1$. This means that we are not allowed to choose the gauge (3.5). This is a situation where the complex gauge formalism is less useful.

⁵In particular, one can check that $\mathcal{X} = \begin{pmatrix} a & \frac{1}{y} \\ by & c \end{pmatrix}$ also solves Nahm’s equations, but there is enough residual gauge symmetry to set $c = a$.

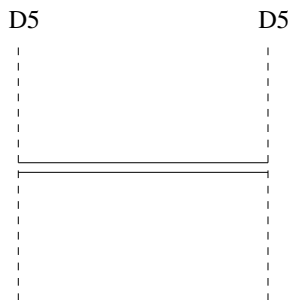


Figure 5. S-dual of pure $U(2)$ $\mathcal{N} = 4$ gauge theory.

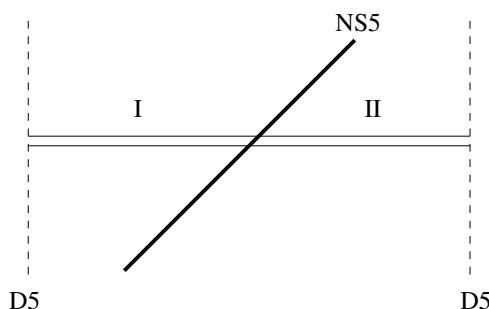


Figure 6. S-dual of $U(2)$ $\mathcal{N} = 4$ gauge theory with one flavor.

Instead, we have to solve the system including the subleading terms, without the ability to choose a simple gauge. The $SU(2)$ solution of Nahm’s equations can be written in terms of special functions (the so-called Euler top functions); see [22] for a pedagogical review.

The moduli space is 4-complex-dimensional. When the monopoles are well-separated, it is possible to understand the dimension intuitively. There is an \mathbb{R}^3 corresponding to the 3 center-of-mass coordinates for the 2 monopoles, one (positive) real coordinate corresponding to the monopole separation, 3 Euler angles rotating the 2-monopole configuration (the monopole solutions are not axisymmetric), and an S^1 from the $U(1)$ gauge framing of the monopoles.

3.5 $\mathcal{N} = 4$, $N_c = 2$, $N_f = 1$

Another interesting system corresponds to adding a flavor to the $U(2)$ $\mathcal{N} = 4$ gauge theory by adding an NS5-brane in the interval (in the S-dual representation). This is represented by the brane configuration shown in figure 6. This theory should also have a four complex-dimensional Coulomb branch and no Higgs branch.

In this example a crucial simplification occurs compared to the the previous case, because we are allowed to use the $G_{\mathbb{C}}$ quotient. In the real formulation of this problem, the fields $X_{4,5,6}$ are again described by the complicated Euler top functions, but if we are only interested in the complex structure, we *can* choose a $G_{\mathbb{C}}$ gauge where the fields take a simpler form.

The key difference between this example and the previous one is that there is an NS5-brane, so we are allowed to do gauge transformations which are discontinuous at the NS5. This means that it is easy to satisfy $g = 1$ at both D5-boundaries while choosing a convenient gauge in the bulk. We place the NS5 at $y = 0$, and we place the D5 and D5' at $y = \pm 1$. Then we can use a complex gauge transformation to set

$$\mathcal{A}_I = \begin{pmatrix} \frac{1}{2(y+1)} & 0 \\ 0 & -\frac{1}{2(y+1)} \end{pmatrix}, \tag{3.7}$$

$$\mathcal{A}_{II} = \begin{pmatrix} \frac{1}{2(1-y)} & 0 \\ 0 & -\frac{1}{2(1-y)} \end{pmatrix}, \tag{3.8}$$

and we can solve Nahm's equations with

$$\mathcal{X}_I = \begin{pmatrix} a & \frac{1}{y+1} \\ b(y+1) & a \end{pmatrix}, \tag{3.9}$$

$$\mathcal{X}_{II} = \begin{pmatrix} a' & \frac{1}{1-y} \\ b'(1-y) & a' \end{pmatrix}. \tag{3.10}$$

At the NS5, we have to satisfy $\mathcal{X}_I = AB$ and $\mathcal{X}_{II} = BA$ where A, B are 2×2 matrices. This forces $a' = a$ and $b' = b$, so in this gauge \mathcal{X} is actually continuous across the boundary, with

$$\mathcal{X}_{y=0} = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}. \tag{3.11}$$

Note that one solution is given by $B = \mathbb{I}$ and $A = \mathcal{X}_{y=0}$; however, this is not the only form of A and B which solves the constraint. There is a two-complex-dimensional family of solutions, given by rigid $GL(2)$ rotations, one of which is generated by \mathbb{I} and the other is generated by $\mathcal{X}_{y=0}$. These transform A, B nontrivially but do not transform \mathcal{X} .

The total moduli space is four complex dimensional. The point we wish to emphasize is that in the case with an NS5 brane we were able to dramatically simplify the problem by using $G_{\mathbb{C}}$ and in fact it was unnecessary to consider the Euler top functions.

3.6 Constraints from the s -rule

Continuing to more general situations with NS5-branes, we might consider the case where 2 D3-branes end on a D5 on one side and on an NS5 on the other side, as shown in figure 7. Of course, this is the situation which is well-known to be excluded by the Hanany-Witten s -rule.

The absence of a solution is easy to see using $G_{\mathbb{C}}$. Because there is an NS5-brane on the right, we are free to choose the form (3.5) for the gauge field. Then the complex scalar which solves Nahm's equations takes the form (3.6). However, we also have to impose the boundary conditions at the NS5-brane at a finite distance in y from the Nahm pole at the D5-brane. The NS5-brane boundary conditions imply that $\mathcal{X} = 0$ at the NS5. There is no solution of Nahm's equations with a pole on the D5 which satisfies this condition at the NS5 on the right.

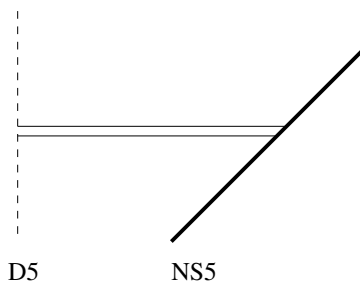


Figure 7. *s*-rule violating configuration.

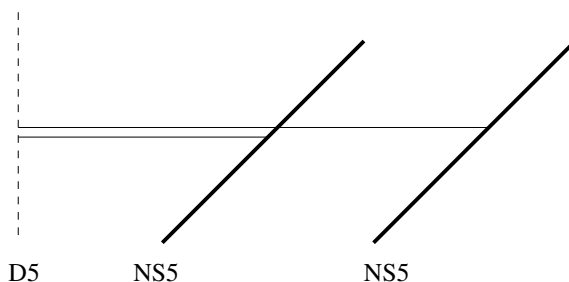


Figure 8. Adding a second NS5-brane satisfies the *s*-rule.

One might wonder if this argument can be circumvented by imposing a standard Dirichlet boundary condition on the D5 with 2 D3-branes ending on it. However, it was argued convincingly by studying the effects of moving D3-segments off to infinity in [20] that these Nahm poles are necessary when more than one D3 branes terminate on a D5-brane. See also [1, 23].

Because there is no solution to Nahm’s equations satisfying the given boundary conditions, we conclude that supersymmetry must be broken in this configuration. Note that we did not have to impose the *s*-rule as a separate condition; it simply follows as a consequence of our analysis. We could also have analyzed this system directly, without using the complex gauge quotient; such a calculation will reach the same conclusion.

Our computation gives a new derivation of the *s*-rule, specifically in the D3-D5-NS5 duality frame. Previous derivations of the *s*-rule in other duality frames appeared in [24–27].

If the two D3-branes are allowed to end on two different NS5 branes then we no longer have a restriction from the *s*-rule, as in the brane configuration in figure 8.

Because we have NS5-branes on the right, we have the freedom to choose a convenient $G_{\mathbb{C}}$ gauge. Let us take

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2y} & 0 \\ 0 & -\frac{1}{2y} \end{pmatrix}, \tag{3.12}$$

so that

$$\mathcal{X} = \begin{pmatrix} a & \frac{1}{y} \\ by & a \end{pmatrix}. \tag{3.13}$$

Without loss of generality, we can locate the leftmost NS5-brane at $y = 1$. At this NS5-brane, there are bifundamental fields A, B . The equations they satisfy are

$$AB = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}, \tag{3.14}$$

$$BA = 0, \tag{3.15}$$

which fix

$$A = \begin{pmatrix} v \\ 0 \end{pmatrix}, \tag{3.16}$$

$$B = \left(0, \frac{1}{v}\right), \tag{3.17}$$

and $a = b = 0$. We may further fix v by a $G_{\mathbb{C}}$ transformation, or equivalently by solving the equation for X_6 and modding out by a real gauge transformation. The moduli space is zero-dimensional, but it is not trivial, which is consistent with the fact that the configuration in figure 8 satisfies the s -rule.

4 $3d \mathcal{N} = 2$ U(1) gauge theories

Now let us come to address theories of our main interest, namely $3d \mathcal{N} = 2$ theories. In this section we study U(1) gauge theories with N_f flavors, starting with $N_f = 1$ and then $N_f = 2$ and more general N_f . The moduli spaces for even these simple theories have a significant amount of structure, and we will see that our techniques recover all previously known results [8–10].

A typical example of a brane construction for this class of theories is illustrated in figure 1. Both of the brane orderings illustrated there should give rise to the same theory in the limit where all but the 2+1 dimensional dynamics is decoupled. For concreteness, let us consider the brane ordering illustrated on the right in figure 1. As in the $\mathcal{N} = 4$ case, the D3-brane contributes a vector field (which in the classical theory gives rise to the dual scalar φ), but because the NS5 and NS5' branes are rotated relative to each other, the D3 is only free to move in the shared direction X_9 . There are also fundamental quark multiplets localized at the D5'-branes.

The $\mathcal{N} = 2$ U(1) theories have moduli spaces with a Higgs branch and a Coulomb branch. On the Higgs branch, the fundamental quarks have expectation values. Because the gauge symmetry is broken by the quark vevs, this branch can be computed classically and has dimension $2N_f - 1$. The Coulomb branch, however, is potentially subject to quantum corrections. The goal of this section is to show that the S-dual Nahm analysis captures these quantum effects.

There is also an elaborate structure of deformations one can consider by giving masses to the quarks and turning on Fayet-Iliopoulos terms, which correspond in the brane construction to changing the positions of the 5-branes. These structures have been studied in earlier work [8–10], using mirror symmetry and some degree of educated guesswork.

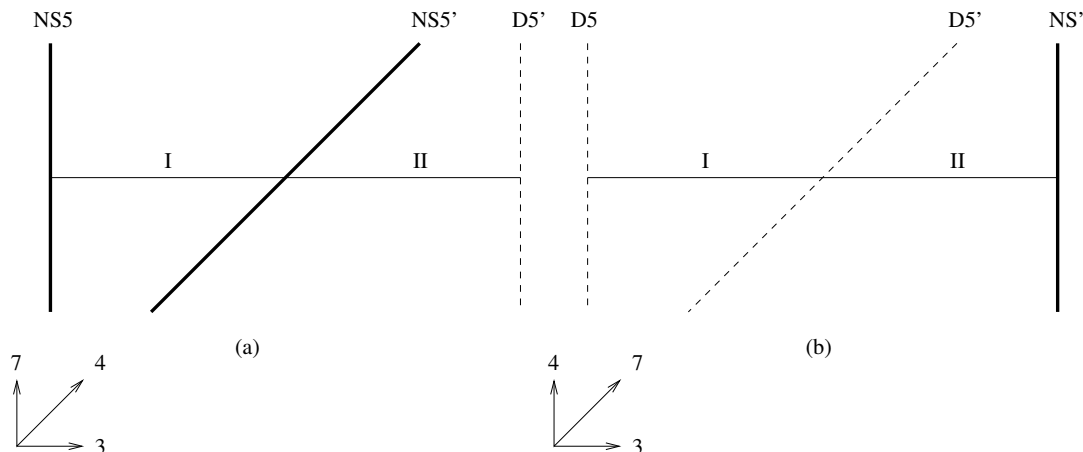


Figure 9. (a) Brane construction of $\mathcal{N} = 2$ $N_c = 1$ $N_f = 1$ theory and (b) its S-dual. We will refer to the original theory as the electric theory or the A-model and the S-dual as the magnetic theory or the B-model.

The new perspective we aim to present in this work is the observation that the analysis of supersymmetric field equations of the boundary/defect field theory in 3+1 dimensions provides a complementary *systematic* tool to study features such as the structure of moduli spaces. In the course of this analysis, we discover some interesting new features with regard to the action of S-duality on these branches.

4.1 $\mathcal{N} = 2$, $N_c = N_f = 1$

In this subsection, we will analyze the simplest non-trivial case of U(1) theory with $N_f = 1$ quarks (the case of $N_f = 0$ will be discussed from the mass deformation of the $N_f = 1$ theory). It turns out that this example is sufficient to illustrate some of the most important aspects of $\mathcal{N} = 2$ mirror symmetry. Let us re-draw figure 1 for the specific case of interest in figure 9(a). We will refer to the theory as depicted in figure 9(a) as the “electric” formulation of the gauge theory. This is the formulation where the gauge theory interpretation is simplest; there is a vector multiplet with U(1) gauge symmetry from the D3 extending between the NS5 and NS5’ branes, and the D5’ brane contributes one flavor. We will also consider the S-dual brane configuration, shown in 9(b), which we will sometimes call the “magnetic” formulation. In both cases, we divide the y -direction into two regions, which we label as region I and region II.

Let us analyze the moduli space of the electric configuration shown in figure 9(a). We start in region I of figure 9(a), and make the gauge choice $\mathcal{A} = 0$; the bulk equations (2.2)–(2.5) then imply that all the fields are constants. In region I, we have

$$\mathcal{Y}_I = a, \tag{4.1}$$

$$\mathcal{X}_I = 0, \tag{4.2}$$

$$X_{6I} = 0. \tag{4.3}$$

At the NS5', we impose the conditions

$$a = \mathcal{Y}_I = \mathcal{Y}_{II} = AB, \tag{4.4}$$

$$iX_{6I} = AA^* - BB^* = 0, \tag{4.5}$$

where A and B are the 1×1 bifundamentals living on the NS5'. We now use the complex gauge formalism to set $\mathcal{A} = 0$ and suppress the real equation (4.5). Furthermore, we must mod out by the gauge transformation at the NS5' interface,

$$A \rightarrow gA, \quad B \rightarrow g^{-1}B. \tag{4.6}$$

We see that there is a branch of moduli space which is one-complex-dimensional (parameterized by A , for example, with B fixed by a choice of gauge). The quarks have VEVs on this branch, which one would naturally call the Higgs branch of the gauge theory.

When $A = B = 0$, an additional branch of moduli space opens up because X_9 and the dual scalar can have expectation values. This is the classical Coulomb branch of the gauge theory, with complex structure $\mathbb{R} \times S^1$. Note that there are no branches with $A = 0$ and $B \neq 0$ (or vice versa) because they consist of unstable points of the complex gauge quotient.

Now, let us now analyze the same system from the point of view of the S-dual ‘‘magnetic’’ formulation, as shown in figure 9(b). In the S-dual, there are no gauge degrees of freedom in the 2+1 dimensional low energy effective theory. One of the points of considering the defect theory of the UV embedding of 2+1 theories is that the 3+1-dimensional defect theory is a gauge theory with a known Lagrangian, even if the 2+1-dimensional formulation is not.

To carry out this analysis, we start in region I of figure 9(b) and find

$$\mathcal{X}_I = a, \tag{4.7}$$

$$iX_{6I} = x_6, \tag{4.8}$$

$$\mathcal{Y}_I = 0. \tag{4.9}$$

Recall that a factor of i appears in front of X_{6I} because we are using the convention that the bosonic fields are anti-hermitian.

Next, at the D5' between regions I and II, we impose the condition

$$\mathcal{Y}_{II} = \mathcal{Y}_I + Q\tilde{Q}, \tag{4.10}$$

$$\mathcal{X}_I Q = 0, \tag{4.11}$$

$$\mathcal{X}_{II} Q = 0, \tag{4.12}$$

$$\tilde{Q}\mathcal{X}_I = 0, \tag{4.13}$$

$$\tilde{Q}\mathcal{X}_{II} = 0, \tag{4.14}$$

$$iX_{6II} = iX_{6I} + (|Q|^2 - |\tilde{Q}|^2), \tag{4.15}$$

where Q and \tilde{Q} are the quark fields associated with the D5' brane. Finally, the NS5' brane imposes the condition

$$\mathcal{Y}_{II} = 0. \tag{4.16}$$

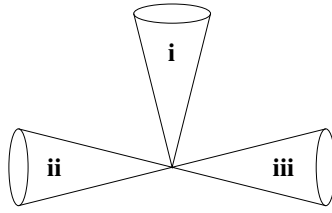


Figure 10. Quantum-corrected branch structure of the moduli space of the $\mathcal{N} = 2$ $N_c = 1$ $N_f = 1$ theory. The three branches of moduli space each have the same complex structure as \mathbb{C} ; they are drawn as cones to show that the branches intersect at a point.

Now, combining (4.9) and (4.10), we learn that

$$Q\tilde{Q} = 0. \quad (4.17)$$

Notice that we have no further constraint on the magnitudes of the scalars in the quark multiplets because the parameter x_6 in (4.8) is free to vary. This means we have as possibilities that either Q or \tilde{Q} vanishes, or that both are vanishing. If either one of Q or \tilde{Q} is non-vanishing, then one of (4.11)–(4.14) forces $a = 0$. On the other hand, should Q and \tilde{Q} simultaneously vanish, then there is no additional constraint on a . So, we have found three branches

- i** $Q = \tilde{Q} = 0$, a arbitrary
- ii** $a = \tilde{Q} = 0$, Q arbitrary
- iii** $a = Q = 0$, \tilde{Q} arbitrary

which we illustrate schematically in figure 10.

In the magnetic formulation, the gauge symmetry is completely broken on all branches of the moduli space. This suggests that the moduli space which we have computed using the S-dual is actually quantum mechanically exact. Indeed, this moduli space structure is consistent with a superpotential of the form

$$W = aQ\tilde{Q}. \quad (4.18)$$

This is precisely the superpotential obtained in (3.2) of [10]: $a(Q, \tilde{Q})$ can be identified with the meson (monopole) operators of the electric theory. We see that from purely semi-classical considerations, we have obtained the quantum-corrected branch structure predicted in figure 1 of [10] which we reproduce here. The classical Coulomb branch which had the complex structure $\mathbb{R} \times S^1$ splits into two separate branches as shown; under S-duality we see that the Coulomb branch has been mapped to Higgs branches **ii** and **iii** where Q or \tilde{Q} has an expectation value. The Higgs branch **i** parameterized by a , on the other hand, is in the Higgs phase on both sides of the S-duality. See also the bottom of figure 11 for an illustration of this point.

4.1.1 Complex mass deformation to $\mathcal{N} = 2, N_c = 1, N_f = 0$

We can add a complex mass deformation to this system (of the electric theory) to reduce the number of flavors to $N_f = 0$. We do this simply by following the preceding analysis but shift

$$\mathcal{Y}_{\text{II}} = m_c, \tag{4.19}$$

which gives the complex equation

$$Q\tilde{Q} = m_c. \tag{4.20}$$

This requires $\tilde{Q}, Q \neq 0$ which in turn forces $a = 0$. The moduli space is one-dimensional, and is T^*S^1 (the cylinder), which is expected for pure U(1) theory.

4.1.2 Real mass and FI deformations

There are two naturally real mass deformations we might consider in the U(1) theory with $N_f = 1$. They are to displace the D5'-brane in X_9 , which we identify as a real mass deformation, and to separate the NS5 and NS5' branes in the X_6 direction. Because the NS5 and NS5' are both extended in X_9 , the real mass deformation is trivial (this will not be the case if $N_f > 1$).

The real FI parameter, however, is not trivial. We can express this deformation by modifying (4.3) to $iX_{6,\text{I}} = \zeta_r$ so that (4.5) becomes

$$AA^* - BB^* = \zeta_r. \tag{4.21}$$

Then the fields A, B cannot both vanish. This lifts the classical Coulomb branch (X_9 and the dual scalar φ are both forced to vanish) and we are left with the Higgs branch.

In the S-dual, the corresponding deformation may be thought of as a real mass parameter arising from moving the D5-brane in the X_9 direction to $iX_{9,\text{I}} = m_r$ while keeping the D5' fixed at $X_9 = 0$. This forces $Q = \tilde{Q} = 0$, and we are left with only one 1-dimensional branch of moduli space.

The moduli space we find is in a certain sense “self-dual” under mirror symmetry, with the understanding that we need to trade an FI parameter for a mass parameter. The reason why the moduli space appears self-dual is easy to understand from the brane perspective. In the representation of figure 9, because of the mass deformation, the D3-brane is unable to intersect the 5-brane in the middle of the brane diagram. Therefore we are just left with a D3 brane stretched between a D5 and NS5' or between an NS5 and D5', which are simply exchanged by S-duality. The pattern of mass deformation is shown in figure 11. This is precisely the class of branches of moduli space which we highlighted at the end of table 1.

4.2 $\mathcal{N} = 2, N_c = 1, N_f = 2$

As our second example, we consider the case of U(1) theory with $N_f = 2$ quarks. This will turn out to be an instructive example highlighting many interesting features.

The brane construction realizing this field theory is shown in figure 12(a). The two D5'-branes give rise to two flavors and the relative orientation of the NS5 and NS5'-branes breaks the supersymmetry to $\mathcal{N} = 2$. We have chosen a convenient ordering of the branes

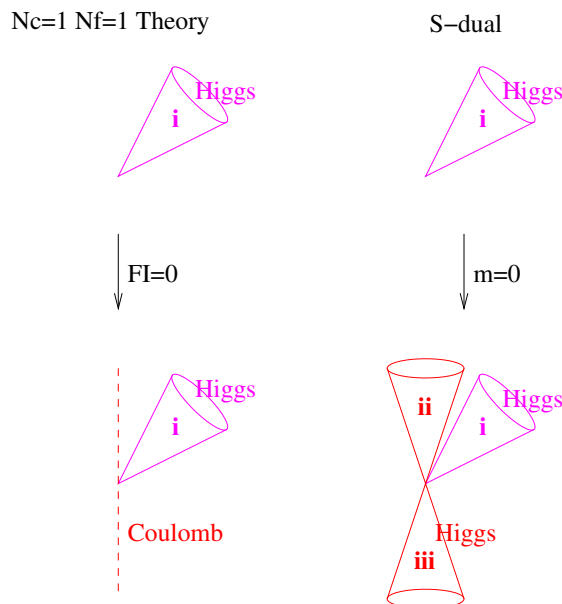


Figure 11. Branch structure of $\mathcal{N} = 2$, $N_c = 1$, $N_f = 1$ theory, deformed by an FI parameter, and its S-dual, deformed by a real mass parameter. The FI and mass deformations are S-dual. In the limit of vanishing FI parameter, one expects the Coulomb branch to open up, but its complex structure and its metric are subject to quantum corrections. In a given duality frame, the branches which are subject to quantum corrections are illustrated with a dotted line. The quantum corrected structure of the Coulomb branch can be inferred from the S-dual picture, where the gauge symmetry is broken.

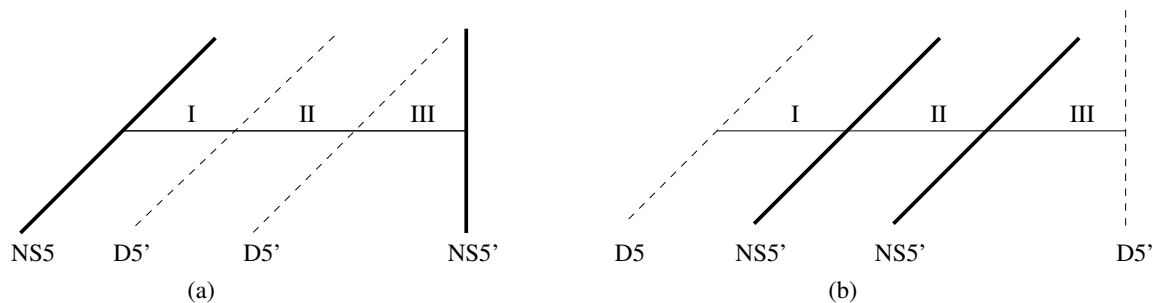


Figure 12. (a) Brane construction for $\mathcal{N} = 2$, $N_c = 1$ theory with 2 flavors and (b) the S-dual.

so that all the fields are Abelian. We leave it as a simple exercise to show that the analysis of our Nahm boundary conditions reproduces the result of the classical field theory computation. In particular, there is a $1d$ Coulomb branch (with complex structure $\mathbb{R} \times S^1$) and a $3d$ Higgs branch (with the complex structure of the conifold).

Because we are interested in the quantum-corrected Coulomb branch, we consider the S-dual, which is shown in figure 12(b). From our experience with the $N_f = 1$ example, our expectation is that the Coulomb branch will be mapped by S-duality to a branch where the gauge symmetry is completely broken, and then the semiclassical computation of the moduli space will be reliable.

The scalars may be taken to be

$$\mathcal{X}_I = a, \tag{4.22}$$

$$\mathcal{Y}_I = 0, \tag{4.23}$$

$$\mathcal{X}_{II} = b, \tag{4.24}$$

$$\mathcal{Y}_{II} = c, \tag{4.25}$$

$$\mathcal{X}_{III} = 0, \tag{4.26}$$

$$\mathcal{Y}_{III} = d. \tag{4.27}$$

We also have bifundamental matter A_1, B_1 at the I–II interface, and A_2, B_2 at the II–III interface. Because everything is Abelian, the constraint equations are rather simple:

$$A_1(\mathcal{X}_I - \mathcal{X}_{II}) = 0, \tag{4.28}$$

$$B_1(\mathcal{X}_I - \mathcal{X}_{II}) = 0, \tag{4.29}$$

$$A_2(\mathcal{X}_{II} - \mathcal{X}_{III}) = 0, \tag{4.30}$$

$$B_2(\mathcal{X}_{II} - \mathcal{X}_{III}) = 0, \tag{4.31}$$

and

$$\mathcal{Y}_I = A_1 B_1, \tag{4.32}$$

$$\mathcal{Y}_{II} = B_1 A_1 = A_2 B_2, \tag{4.33}$$

$$\mathcal{Y}_{III} = B_2 A_2. \tag{4.34}$$

These conditions actually set $\mathcal{Y} = 0$ in all three regions.

When the real deformations are not turned on, the complex gauge quotient works straightforwardly. There are two classes of stable solutions. The first class of solutions has $A_i = B_i = 0$. Then the scalars a and b are not fixed. In addition to these we have the scalar \mathcal{Z}_{II} which is free to vary. This branch is 3-dimensional. The second class of solutions has two branches where all the bulk scalars vanish. On the first branch the interface fields $B_1 = B_2 = 0$ but A_1, A_2 are nonzero, and on the second branch $A_1 = A_2 = 0$ while B_1, B_2 are nonzero. After modding out by the complex gauge symmetry we are left with two one-dimensional branches.

The natural picture for the fully quantum-corrected moduli space comes from combining the reliable parts of the analysis from both the electric and magnetic descriptions. In the electric defect system, we have a 3-dimensional branch with the complex structure of the conifold and with no gauge symmetry. There is also a 1-dimensional branch with unbroken gauge symmetry (so we don't trust the analysis) with complex structure $\mathbb{R} \times S^1$. On the magnetic side, we have a 3-dimensional mixed branch with unbroken $U(1)$ symmetry which we do not trust, and two 1-dimensional branches where the gauge symmetry is fully broken, which we do trust. So to construct the full moduli space we should take the 3-dimensional branch from the electric description and the 1-dimensional branches from the magnetic description.

To describe the case where real FI terms are turned on, we should restore the equations for X_6 . We have (including generic real FI terms, which correspond to real masses of the original theory)

$$iX_{6,I} = A_1 A_1^\dagger - B_1^\dagger B_1, \tag{4.35}$$

$$iX_{6,II} = A_1^\dagger A_1 - B_1 B_1^\dagger, \tag{4.36}$$

$$iX_{6,II} = A_2 A_2^\dagger - B_2^\dagger B_2 - \zeta_r, \tag{4.37}$$

$$iX_{6,III} = A_2^\dagger A_2 - B_2 B_2^\dagger - \zeta_r. \tag{4.38}$$

We immediately identify some branches, where only one of the A_i, B_i is non-vanishing:

- i** $A_2 = B_2 = A_1 = 0, B_1 \neq 0$. This requires $\zeta_r > 0$, and $\mathcal{X}_I = \mathcal{X}_{II} \neq 0$ for a $1d$ moduli space.
- ii** $A_1 = B_1 = A_2 = 0, B_2 \neq 0$. This requires $\zeta_r > 0$. Here $\mathcal{X}_I \neq 0$ but $\mathcal{X}_{II,III} = 0$.
- iii** $A_1 = B_1 = B_2 = 0, A_2 \neq 0$. This requires $\zeta_r < 0$. Here also $\mathcal{X}_I \neq 0$ but $\mathcal{X}_{II,III} = 0$.
- iv** $A_2 = B_2 = B_1 = 0, A_1 \neq 0$. This requires $\zeta_r < 0$. Here we have $\mathcal{X}_I = \mathcal{X}_{II} \neq 0$ for a $1d$ moduli space.

These branches of moduli space do not satisfy the real equations for nonzero ζ_r ; we excluded them from the analysis using $G_{\mathbb{C}}$ because they consist of unstable points.

The stable branches are those where two of the A, B fields are non-zero. This forces us to set $\mathcal{X} = 0$ and $\mathcal{Z} = 0$ in all regions. Explicitly, the four branches are

- v** $A_1 = A_2 = 0$ and $B_1, B_2 \neq 0$. This requires $\zeta_r = |B_1|^2 - |B_2|^2$.
- vi** $B_1 = B_2 = 0$ and $A_1, A_2 \neq 0$. This requires $\zeta_r = |A_2|^2 - |A_1|^2$.
- vii** $A_1 = B_2 = 0$ and $A_2, B_1 \neq 0$. This requires $\zeta_r = |B_1|^2 + |A_2|^2 > 0$.
- viii** $A_2 = B_1 = 0$ and $A_1, B_2 \neq 0$. This requires $\zeta_r = -|A_1|^2 - |B_2|^2 < 0$.

Some qualitative features of this analysis can be understood from the brane diagram. The brane configurations corresponding to the stable branches for the case with positive ζ_r are illustrated in figure 13. The basic picture is that the various branches intersect when the D3 intersects an NS5-brane. Locally, this intersection is identical to what was seen in the case of $N_c = 1$ and $N_f = 1$. We also see that branch **vii**, which is bounded on both sides by the NS5-branes, does not have an asymptotic region.

We summarize our findings for the moduli space of this system in figure 14. The analysis of S-dual we carried out first is summarized in column (d). The dependence on FI parameter ζ_r in this dual frame is to be mapped to the dependence on real mass of the $N_c = 1, N_f = 2$ theory of interest. For non-vanishing ζ_r , we find Higgs branches **i, ii, v, vi, and vii**, the complex structure of all of which are protected against quantum corrections. The Coulomb branch of the mass deformed $N_c = 1, N_f = 2$ theory, illustrated in column (c), receives quantum corrections which can be inferred from column (d) using S-duality.

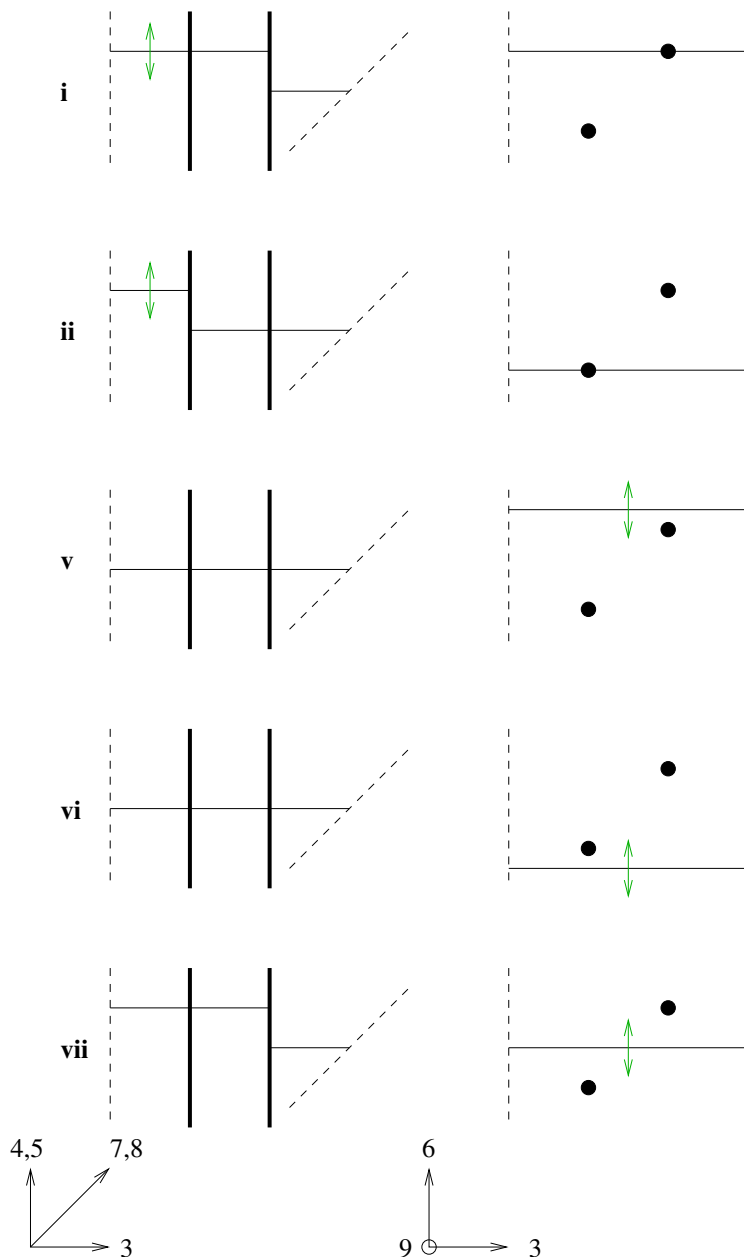


Figure 13. Brane configurations corresponding to stable branches **i**, **ii**, **v**, **vi**, and **vii** for the $\mathcal{N} = 2$ $N_c = 1$ $N_f = 2$ theory in the S-dual magnetic description with the real FI parameter (of the magnetic theory) $\zeta_r > 0$. The black dots indicate the NS5 branes. The green arrows indicate the unconstrained directions along the moduli space in the brane picture.

In addition, we studied the $N_c = 1$ $N_f = 2$ system directly in the presence of FI term and inferred the structure outlined in column (a).

One can also identify the operators which serves as a gauge invariant order parameters in each of these phases. On branches **i** and **ii**, traces of powers of \mathcal{X} at any point along regions I and II will serve that purpose. On the branches **v** and **vi**, one can consider the

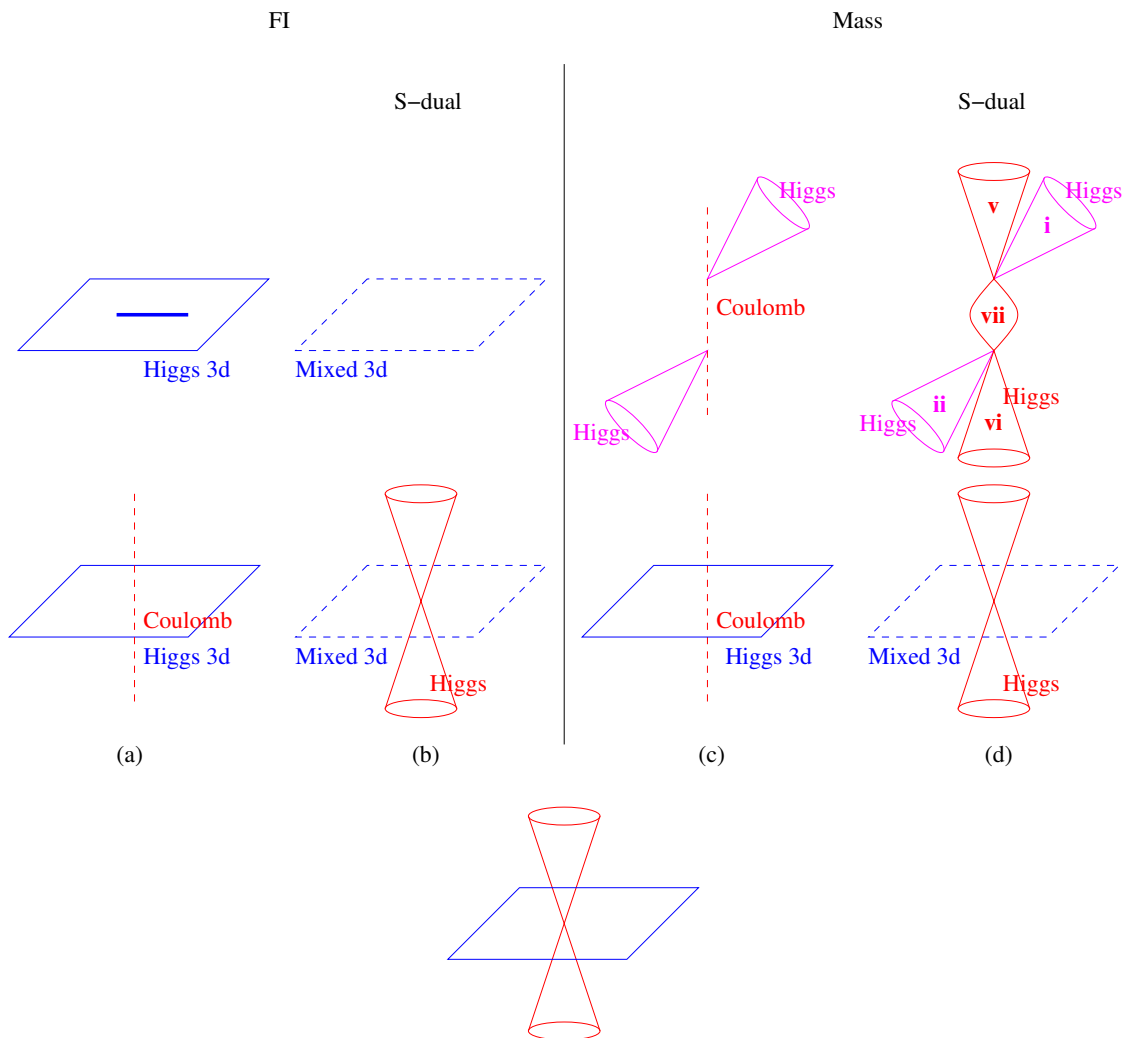


Figure 14. (a) The branch structure of $\mathcal{N} = 2$ $N_c = 1$ $N_f = 2$ theory deformed by an FI parameter, (b) its S-dual, (c) $\mathcal{N} = 2$ $N_c = 1$ $N_f = 2$ theory deformed by quark mass, and (d) its S-dual. The colors reflect the mapping of branches under S-duality. The bottom illustrates the fully-quantum-corrected moduli space in the undeformed limit. The Higgs branch is the conifold.

following set of chiral operators.

$$\mathcal{O}_v = e^{-\int_{\text{III}} \mathcal{A}} B_2 e^{-\int_{\text{II}} \mathcal{A}} B_1 e^{-\int_{\text{I}} \mathcal{A}}, \tag{4.39}$$

$$\mathcal{O}_{vi} = e^{\int_{\text{I}} \mathcal{A}} A_1 e^{\int_{\text{II}} \mathcal{A}} A_2 e^{\int_{\text{III}} \mathcal{A}}. \tag{4.40}$$

These operators are invariant under complexified gauge transformations. The Wilson line factors are necessary in order to make these operators gauge invariant, but it can be eliminated in the limit that the size of the regions I, II, III goes to zero. In that limit, the order parameters are simply

$$\mathcal{O}_v = B_2 B_1, \tag{4.41}$$

$$\mathcal{O}_{vi} = A_1 A_2. \tag{4.42}$$

The order parameter for the compact branch **vii** requires some additional care. Even in the zero interval limit, one can not construct an invariant combination of A_2 and B_1 without involving complex conjugation, e.g.

$$\mathcal{O}_{\mathbf{vii}} = A_2^\dagger B_1. \tag{4.43}$$

This operator is not invariant under the complexified gauge symmetry. This is consistent with the fact that the existence and the stability of this branch depended explicitly on a real datum, namely the positivity of ζ_r . Here we see one limitation of the complex gauge formalism. Some observables require partial gauge fixing to the real formalism where one can construct additional sets of operators invariant under the smaller gauge group.

This is also a useful point to comment on the status of the moduli-space metric. From the explicit solution to Nahm equations at our disposal, it is straightforward, although tedious, to compute the Manton metric [28]. With only $\mathcal{N} = 2$ supersymmetry, one expects generic quantum corrections. In cases such as the $N_c = 1$ and $N_f = 2$ examples under consideration, it was pointed out in [29] that the Kähler form is also protected from quantum corrections. Does this mean that one can extract the quantum exact metric by computing the Manton metric? A closer look into the argument going into the non-renormalization of the Kähler form in [29] relied on conformal invariance. In the analysis of the of the 3+1 defect system, there are features such as the size of the intervals which introduces a scale and the Manton metric will certainly depend on these parameters. Perhaps, for certain simple configurations, one can extract the quantum exact moduli space metric from the zero interval size limit of the Manton metric. These, however, correspond to the rich subject of massless monopoles, also known as clouds, reviewed in [22]. It would be interesting to work out specific criteria for when such a prescription to compute the metric succeeds or fails.

By combining the expectations based on $\zeta_r \rightarrow 0$ limit of columns (a) and (d), we infer the structure of the undeformed moduli space as intersections of a three dimensional Higgs branch and two Coulomb branches illustrated at the bottom of figure 14. We expect the Coulomb branch, illustrated by red cones, to consist of complex plane C for branches **v** and **vi**, and have the structure of \mathbb{CP}_1 for **vii**, consistent with the expectation of taking the subspace of resolved $\mathbb{C}^2/\mathbb{Z}_2$.

The Higgs branches **i** and **ii**, illustrated in column (d), become unstable and melt into the Coulomb branch (of theory B) if $\zeta_r = 0$. It is interesting to note, on the other hand, that these Higgs branches survive when both the FI term and the real mass are turned on as we alluded to earlier in section 2.2.

The FI-deformed moduli space in column (d) of figure 14 is identical in structure to moduli space illustrated in figure 2 of [10]. The main difference between the our result and the result of [10] is that we arrive at our conclusion via a strictly *classical* analysis of the S-dual description of figure 9(b).

It is straightforward to extend the analysis for $N_f = 1, 2$ for any N_f . When $N_f = 3$, for example, with no mass deformations, one expects a 1-dimensional Coulomb branch and a 5-dimensional Higgs branch. The branch structure for $N_f = 3$ as the real masses of the 3 flavors of quarks are varied is illustrated in figure 15. As shown in the figure, for each flavor

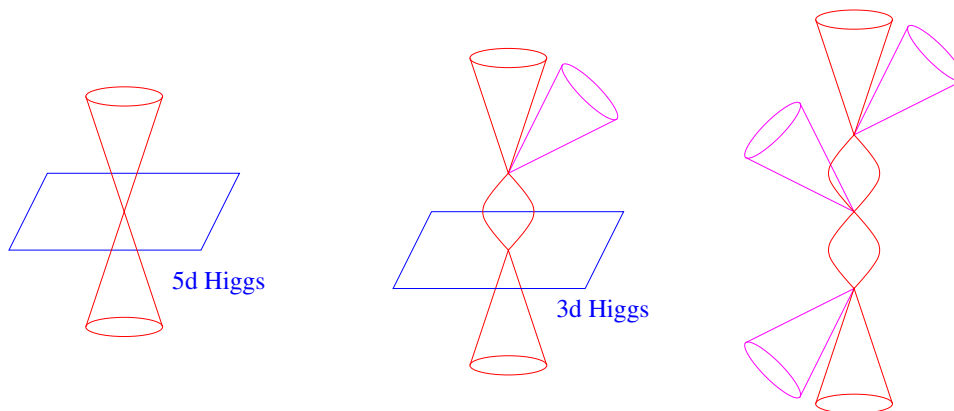


Figure 15. The moduli space of $\mathcal{N} = 2$ $N_c = 1$ $N_f = 3$ theory with three equal real quark masses, two equal real quark masses, and all distinct real quark masses.

that we add (with a generic real mass), the moduli space develops an additional compact branch and an additional noncompact branch. Some further remarks on the Coulomb branch for general N_f appear in appendix A.

4.2.1 Complex mass deformation

It is simple to deform the previous analysis by adding a complex mass (of the electric theory). We do the analysis in the magnetic theory, where it appears as a complex FI term. In the brane picture this means we displace one of the NS5'-branes in the $X_{7,8}$ directions. We can implement this by changing equations (4.33) and (4.34) to

$$\mathcal{Y}_{\text{II}} = B_1 A_1 = A_2 B_2 - \zeta_c, \tag{4.44}$$

$$\mathcal{Y}_{\text{III}} = B_2 A_2 - \zeta_c. \tag{4.45}$$

We are forced to set $b = c = d = 0$. So the equations are

$$A_2 B_2 = \zeta_c, \tag{4.46}$$

$$A_1 B_1 = 0, \tag{4.47}$$

$$a A_1 = a B_1 = 0. \tag{4.48}$$

We can use the complex gauge transformation to remove one degree of freedom in A_2, B_2 . We are left with three branches of moduli space, corresponding to $a \neq 0, A_1 = B_1 = 0$, or $a = 0, A_1 = 0, B_1 \neq 0$ or $a = 0, A_1 \neq 0, B_1 = 0$. It is natural to identify these three branches with the branches we obtained from directly analyzing the case $N_c = 1, N_f = 1$.

4.3 $\mathcal{N} = 2$ U(1) theory with hidden parameters

Some interesting phenomena can arise for brane configurations with arbitrary arrangements of 1/4 BPS 5-branes which are hard to understand from the point of view of 3d field theory. In particular, as the branes are reordered in the y direction, the low energy theory undergoes phase transitions which can change the geometry of the moduli space (in particular the dimensions of the branches can change).

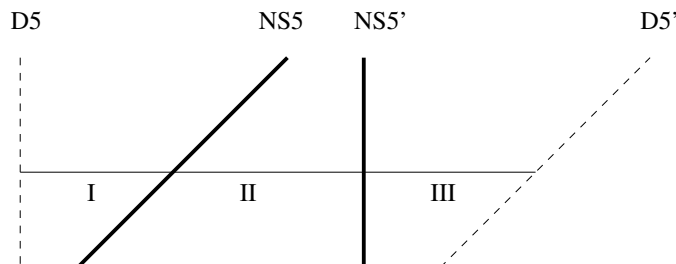


Figure 16. The brane configuration $D5-1D3(I)-NS5-1D3(II)-NS5'-1D3(III)-D5'$.

The y positions of the 5-branes should correspond to deformations by irrelevant operators from the point of view of the three-dimensional theory. However, because they change the vacuum structure of the theory, they are dangerously irrelevant. In [2] these 5-brane positions were called “hidden parameters” of the $3d$ theory, although of course they are not hidden in the $4d$ defect theory.

We will consider two Abelian examples which are related by varying the hidden parameters. Actually, these systems were considered previously in [30]. There the moduli spaces were constructed by making educated guesses for an effective superpotential. We will reproduce their results for the moduli spaces, but the point we wish to emphasize is that the moduli space is *derived* rather than guessed. Because our method is systematic, it can be generalized to more complicated systems where it may be hard to guess an appropriate effective superpotential.

4.3.1 $D5-1D3-NS5-1D3-NS5'-1D3-D5'$

Let us consider the brane configuration shown in figure 16.

The scalars may be taken to be

$$\mathcal{X}_I = a, \tag{4.49}$$

$$\mathcal{Y}_I = 0, \tag{4.50}$$

$$\mathcal{X}_{II} = b, \tag{4.51}$$

$$\mathcal{Y}_{II} = c, \tag{4.52}$$

$$\mathcal{X}_{III} = 0, \tag{4.53}$$

$$\mathcal{Y}_{III} = d. \tag{4.54}$$

The interfaces have matter fields A_1, B_1 at the NS5 and A_2, B_2 at the NS5'. They obey the equations

$$\mathcal{X}_I = A_1 B_1, \quad \mathcal{X}_{II} = B_1 A_1 \quad \implies \quad b = a, \tag{4.55}$$

$$\mathcal{Y}_{II} = A_2 B_2, \quad \mathcal{Y}_{III} = B_2 A_2 \quad \implies \quad d = c, \tag{4.56}$$

$$(\mathcal{Y}_I - \mathcal{Y}_{II})A_1 = 0, \tag{4.57}$$

$$(\mathcal{Y}_I - \mathcal{Y}_{II})B_1 = 0, \tag{4.58}$$

$$(\mathcal{X}_{II} - \mathcal{X}_{III})A_2 = 0, \tag{4.59}$$

$$(\mathcal{X}_{II} - \mathcal{X}_{III})B_2 = 0. \tag{4.60}$$

The last four equations can be simplified to

$$A_2 a = B_2 a = 0, \tag{4.61}$$

$$A_1 c = B_1 c = 0. \tag{4.62}$$

We also have

$$iX_{6,I} = A_1 A_1^\dagger - B_1^\dagger B_1, \tag{4.63}$$

$$iX_{6,II} = A_1^\dagger A_1 - B_1 B_1^\dagger, \tag{4.64}$$

$$iX_{6,II} = A_2 A_2^\dagger - B_2^\dagger B_2 - \zeta_r, \tag{4.65}$$

$$iX_{6,III} = A_2^\dagger A_2 - B_2 B_2^\dagger - \zeta_r. \tag{4.66}$$

We can distinguish several branches. First, there are two branches with the complex structure of \mathbb{C} :

- i** $a = A_1 B_1 \neq 0$ which forces $A_2 = B_2 = 0$ and $c = 0$. The relative magnitudes of A_1, B_1 are fixed by $|A_1|^2 - |B_1|^2 = -\zeta_r$.
- ii** $c = A_2 B_2 \neq 0$ which forces $A_1 = B_1 = 0$ and $a = 0$. The relative magnitudes of A_1, B_1 are fixed by $|A_2|^2 - |B_2|^2 = \zeta_r$.

We also have four branches with $a = c = 0$:

- iii** $A_1 = 0, B_2 = 0$ and $A_2 \neq 0, B_1 \neq 0$. This requires $\zeta_r = |B_1|^2 + |A_2|^2 > 0$.
- iv** $A_1 = 0, A_2 = 0$ and $B_1 \neq 0, B_2 \neq 0$. This requires $\zeta_r = |B_1|^2 - |B_2|^2$.
- v** $B_1 = 0, B_2 = 0$ and $A_1 \neq 0, A_2 \neq 0$. This requires $\zeta_r = |A_2|^2 - |A_1|^2$.
- vi** $B_1 = 0, A_2 = 0$ and $A_1 \neq 0, B_2 \neq 0$. This requires $\zeta_r = -|A_1|^2 - |B_2|^2 < 0$.

So there are a total of six 1-dimensional branches on which the gauge symmetry is completely broken. Four of the branches exist for any value of ζ_r , and they consist of semistable points. The third and sixth of the bulleted branches are unstable.

If we allow some gauge symmetry to be unbroken, we can set all the $A_i = 0$ and $B_i = 0$. This also sets $a = b = c = d = 0$. Ignoring quantum corrections, we have one classical dimension left in the moduli space from \mathcal{Z}_{II} . This branch only exists when $\zeta_r = 0$. Presumably for nonzero ζ_r it is transmuted into the third and sixth of the above branches, which only exist when $\zeta_r > 0$ or $\zeta_r < 0$.

We see that for any value of ζ_r , we have 5 one-dimensional branches of moduli space.

We can also analyze this system in the S-dual, or equivalently after having done Hanany-Witten transitions. The brane configuration is NS5—1D3(I)—D5—1D3(II)—D5'—1D3(III)—NS5'.

The branch structure is illustrated in figure 17. Note that even in the $\zeta_r \rightarrow 0$ limit, the Higgs branches **i** and **ii** do not disappear in this case.

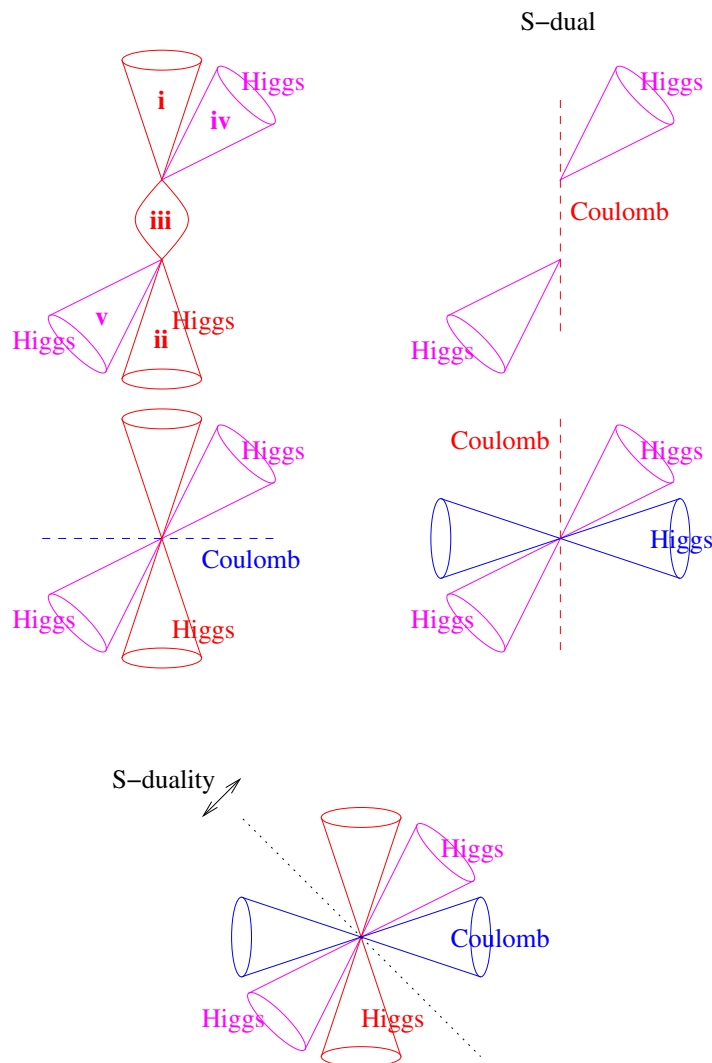


Figure 17. The branch structure of configuration D5—1D3(I)—NS5—1D3(II)—NS5'—1D3(III)—D5' with FI deformation, and its S-dual also with FI deformation. In this example, two of the Higgs branches are stable and survive the $\zeta_r \rightarrow 0$ limit.

4.3.2 D5—1D3—NS5'—1D3—NS5—1D3—D5'

Now let us consider the system where we have interchanged the position in y of the two NS branes, corresponding to a change of what were called “hidden parameters” by [2]. The brane configuration is shown in figure 18. As the 5-brane positions are varied, the moduli space changes qualitatively, signaling that there is a phase transition when the NS-brane ordering is changed.

We can choose the starting ansatz

$$\mathcal{X}_I = a, \tag{4.67}$$

$$\mathcal{Y}_I = 0, \tag{4.68}$$

$$\mathcal{X}_{II} = b, \tag{4.69}$$

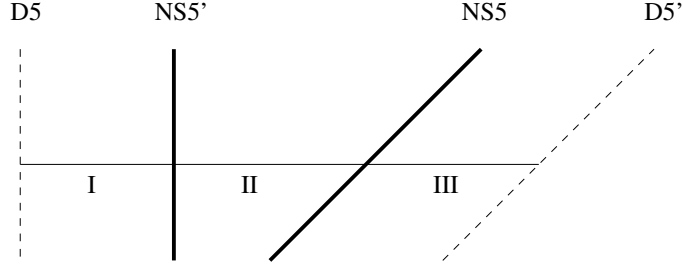


Figure 18. The brane configuration $D5-1D3(I)-NS5'-1D3(II)-NS5-1D3(III)-D5'$. It differs from the configuration in figure 16 by the ordering of the NS5 and NS5'.

$$\mathcal{Y}_{II} = c, \quad (4.70)$$

$$\mathcal{X}_{III} = 0, \quad (4.71)$$

$$\mathcal{Y}_{III} = d. \quad (4.72)$$

We introduce A_1, B_1 at the NS5' and A_2, B_2 at the NS5. The interface conditions are

$$\mathcal{Y}_I = A_1 B_1 = \mathcal{Y}_{II} \quad \Longrightarrow \quad A_1 B_1 = c = 0, \quad (4.73)$$

$$\mathcal{X}_{II} = A_2 B_2 = \mathcal{X}_{III} \quad \Longrightarrow \quad A_2 B_2 = b = 0, \quad (4.74)$$

and also

$$(\mathcal{X}_I - \mathcal{X}_{II})A_1 = (\mathcal{X}_I - \mathcal{X}_{II})B_1 = 0, \quad (4.75)$$

$$(\mathcal{Y}_{II} - \mathcal{Y}_{III})A_2 = (\mathcal{Y}_{II} - \mathcal{Y}_{III})B_2 = 0, \quad (4.76)$$

which imply

$$A_1 a = B_1 a = 0, \quad (4.77)$$

$$A_2 d = B_2 d = 0. \quad (4.78)$$

We also have

$$iX_{6,I} = A_1 A_1^\dagger - B_1^\dagger B_1, \quad (4.79)$$

$$iX_{6,II} = A_1^\dagger A_1 - B_1 B_1^\dagger, \quad (4.80)$$

$$iX_{6,II} = A_2 A_2^\dagger - B_2^\dagger B_2 - \zeta_r, \quad (4.81)$$

$$iX_{6,III} = A_2^\dagger A_2 - B_2 B_2^\dagger - \zeta_r. \quad (4.82)$$

Now we classify the branches of moduli space. First, suppose that either $a \neq 0$ or $d \neq 0$:

- i** $a \neq 0$ forces $A_1 = B_1 = 0$. We also have $A_2 B_2 = b = 0$ so either A_2 or B_2 is zero. Which one is nonvanishing is controlled by the sign of the real FI term, $|A_2|^2 - |B_2|^2 = \zeta_r$.
- ii** $d \neq 0$ gives a similar one-dimensional branch with A_1, B_1 nonzero and $|B_1|^2 - |A_1|^2 = \zeta_r$.

For either sign of ζ_r , this gives rise to two branches of moduli space. These branches are unstable, and do not exist for $\zeta_r = 0$.

In addition to these, there are some $1d$ branches with $a = d = 0$:

- viii** $A_1 = 0, B_2 = 0$ and $A_2 \neq 0, B_1 \neq 0$. This requires $\zeta_r = -|B_1|^2 - |A_2|^2 < 0$. This branch is unstable; if it is made physical by activating ζ_r , it has the complex structure of \mathbb{P}^1 .
- vii** $A_2 = 0, B_1 = 0$ and $A_1 \neq 0, B_2 \neq 0$. This requires $\zeta_r = |A_1|^2 + |B_2|^2 > 0$, and this branch is also unstable.
- iv** $A_1 = A_2 = 0$ and $B_1 \neq 0, B_2 \neq 0$. On this branch $|B_1|^2 - |B_2|^2 = \zeta_r$.
- v** $B_1 = B_2 = 0$ and $A_1 \neq 0, A_2 \neq 0$. On this branch $|A_1|^2 - |A_2|^2 = -\zeta_r$.

The last two of these branches are stable.

If we have $\zeta_r = 0$, we have solutions where all the A_i, B_j vanish and there is unbroken gauge symmetry. This moduli space is classically 3-dimensional, parameterized by a, d , and \mathcal{Z} . In fact, this space looks essentially like what was illustrated in figure 14.

This example shows something important — the moduli space is qualitatively different than when the NS5 and NS5'-branes are interchanged. This appears to be a phase transition from varying what Hanany and Witten (and Aharony and Hanany) called the “hidden parameters”.

We may also deform this system by a real mass by displacing the D5' brane in the X_9 direction. To make this explicit, we reintroduce the coordinate \mathcal{Z} (recall the discussion in section 2.3) and set

$$\mathcal{Z}_I = 0, \tag{4.83}$$

$$\mathcal{Z}_{II} = f, \tag{4.84}$$

$$\mathcal{Z}_{III} = m', \tag{4.85}$$

where m' is fixed. Then we have two choices, $f = 0$ or $f = m'$, for which $A_1 = B_1 = 0$ or $A_2 = B_2 = 0$, respectively. We see that the real mass lifts all but the first two bulleted branches (which are only present if $\zeta_r \neq 0$) and the $3d$ branch which has an unbroken $U(1)$ gauge symmetry.

The 3-dimensional branch we found has unbroken gauge symmetry, so we should analyze it in the S-dual configuration NS5—1D3(I)—D5'—1D3(II)—D5—1D3(III)—NS5' where the gauge symmetry will be broken. Starting with an ansatz satisfying the boundary conditions at the endpoints,

$$\mathcal{X}_I = 0, \tag{4.86}$$

$$\mathcal{Y}_I = a, \tag{4.87}$$

$$\mathcal{X}_{II} = b, \tag{4.88}$$

$$\mathcal{Y}_{II} = c, \tag{4.89}$$

$$\mathcal{X}_{III} = d, \tag{4.90}$$

$$\mathcal{Y}_{III} = 0, \tag{4.91}$$

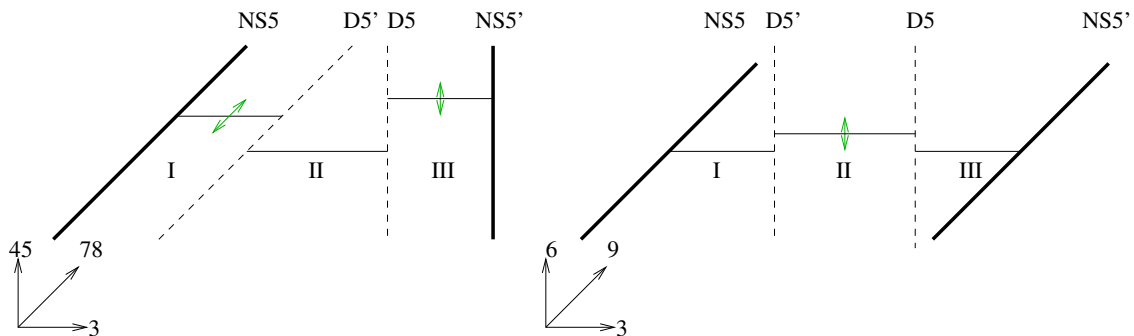


Figure 19. The brane representation of the 3 complex dimensional moduli space of the NS5—1D3(I)—D5'—1D3(II)—D5—1D3(III)—NS5' system.

and imposing the junction conditions at the D5' and D5, we obtain $b = c = 0$ and

$$Q_1 \tilde{Q}_1 = a, \quad (4.92)$$

$$Q_2 \tilde{Q}_2 = -d, \quad (4.93)$$

$$|Q_1|^2 + |Q_2|^2 - |\tilde{Q}_1|^2 - |\tilde{Q}_2|^2 = 0. \quad (4.94)$$

Because a and d are free to vary, we see that there is a branch with broken gauge symmetry where the Q and \tilde{Q} have only a D-term constraint. The resulting 3-dimensional space is the conifold. There is also a branch with all the $Q, \tilde{Q} = 0$ with a U(1) gauge symmetry. From the brane perspective the three dimensional moduli space can be visualized as illustrated in figure 19.

Note that in the original S-dual frame with unbroken gauge symmetry, the gauge symmetry is U(1). Naively we might have thought that this branch should have a direct product structure, with a 1-dimensional Coulomb part and a 2-dimensional Higgs part. However, the computation using the S-dual shows that the Coulomb and Higgs parts merge to form a 3-dimensional space which is not a direct product and instead has the complex structure of the conifold.

4.4 Vortices and skyrmions in defect theories

The three-dimensional $\mathcal{N} = 2$ field theories can have an interesting array of nonperturbative solutions, whose existence can serve as order parameters for the vacuum phases of a given theory. In particular, for U(1) theory with N_f flavors, at various points on the Higgs branch, it is possible to find ANO [31, 32] vortex solutions which are BPS. This was emphasized by [10]. With a real FI parameter turned on, the vortices have the asymptotic (in large r) behavior

$$Q \sim \sqrt{\zeta_r} e^{\pm i\phi}, \quad A_\phi \sim \pm \frac{1}{r}. \quad (4.95)$$

The full r -dependence of the vortex solutions can be found numerically. In the case of $\mathcal{N} = 2$, $N_c = 1$, and $N_f = 2$ illustrated in figure 14, these vortex solutions can appear in all the Higgs branches, namely **i**, **ii**, **v**, and **vi** of column (d), as well as the S-duals of **i** and **ii** in column (c). They could also appear in the $3d$ Higgs branch in column (a).

It was argued in [10] that the vortices are the mirror duals of the (Coulomb branch) monopole operators. In addition to the vortex solutions, on compact Higgs branch such as **vii** in column (d) of figure 14, one expects to find skyrmions [29]. The skyrmion solutions are nontopological solitons [33, 34].

We expect that similar nonperturbative solutions exist for our 3+1-dimensional defect theories — after all, they must appear in the limit where the defect theory becomes a 2+1-dimensional theory. However, there is a difference between the classic analysis and our defect systems, where the matter fields which acquire VEVs are only supported at the interfaces, while the gauge fields propagate in the 3+1-dimensional bulk. It would be interesting to construct these solutions explicitly. Similar systems have been considered in the context of high- T_c superconductivity, where the electromagnetic fields are allowed to propagate in the bulk of some material with a coupling to matter fields localized on planar defects. In this situation, vortex-like solutions called “pancake” vortices have been constructed; see [35] for a brief review with references to the earlier literature.

Such nonperturbative solutions, as understood from the point of view of the theory in 3+1 dimensions, might serve as a probe of the transition discussed in section 4.3, where some branches of moduli space change dimension as the NS5-brane and NS5'-brane cross in y . The transition is desingularized by turning on FI terms (displacing the NS branes in X_6). In this picture one expects to find vortices on the Higgs branch and skyrmions in the S-dual. As the FI terms are taken to zero, the vortices condense and a Coulomb branch opens up. In the example of section 4.3.1, the Coulomb branch is 1-dimensional, while in section 4.3.2 it is a 3-dimensional mixed Coulomb-Higgs branch which emerges. This mixed branch should also emerge from some kind of vortex condensation. It would be interesting if this difference were reflected in some kind of a change of the soliton solutions as the 5-brane positions are varied in the y coordinate.

5 $3d \mathcal{N} = 2$ U(2) gauge theories

We now turn to an analysis of brane configurations which are related to field theories with U(2) gauge symmetry. From the field theory point of view, new phenomena appear (compared to the Abelian case) because the non-Abelian theories can have instantons [5]. The instantons generate corrections to the superpotential which can change the complex structure of moduli space, in some cases lifting the vacuum completely.

In this section, we will see how these instanton effects are encoded in the 3+1-dimensional defect theory by using S-duality. In the magnetic formulation of a particular defect theory, the effect we need to include is the existence of Nahm poles at D5-brane boundaries. Because of the Nahm poles, the scalar fields of the bulk $\mathcal{N} = 4$ theory do not necessarily commute; this intrinsically non-Abelian behavior is what makes the naive geometric brane analysis invalid.

5.1 $\mathcal{N} = 2$, $N_c = N_f = 2$

The $\mathcal{N} = 2$ U(2) gauge theory with $N_f = 2$ is a rather rich system, and serves as a good illustration of the strengths and limitations of our methods. The cases with $N_f = 1$ and

$N_f = 0$ can be extracted from the two-flavor analysis by adding complex mass deformations.

First, let us recall some expectations from the direct field theory analysis for $N_c = 2$, $N_f = 2$. When no mass deformations are present, there should be a Coulomb branch with $U(1) \times U(1)$ gauge symmetry, a Higgs branch with the gauge symmetry fully broken, and some mixed branches with $U(1)$ gauge symmetry. We expect from the instanton-based computations that the Coulomb branch is 2 complex dimensional. We also have a Higgs branch which can be computed classically in the electric theory; its dimension is $2N_f N_c - N_c^2 = 4$.⁶

There should also be mixed branches, which are 4 complex dimensional. To see this, note that we can have expectation values for the 2 flavors of the form

$$Q_1 = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} q_2 \\ 0 \end{pmatrix}, \quad \tilde{Q}_1^T = \begin{pmatrix} \tilde{q}_1 \\ 0 \end{pmatrix}, \quad \tilde{Q}_2^T = \begin{pmatrix} \tilde{q}_2 \\ 0 \end{pmatrix}. \quad (5.1)$$

These expectation values break the gauge symmetry from $U(2)$ to $U(1)$. There are four complex degrees of freedom, one of which can be eliminated by a gauge transformation. In addition to the three Higgs moduli, we expect a one-dimensional Coulomb branch because of the unbroken $U(1)$ gauge symmetry, so the total dimension of the mixed branch is 4. For $U(1)$ theory the Coulomb branch actually consists of two separate branches, and correspondingly we expect to see two mixed branches in this case.

These features are summarized by an effective superpotential (obtained in [10] using considerations of holomorphy and symmetry)

$$W = v_+ v_- \det(M). \quad (5.2)$$

In this expression M is a 2×2 meson matrix while v_{\pm} are monopole operators. On the Coulomb branch, both $v_{\pm} \neq 0$ while $M_{ij} = 0$, while on the Higgs branch $v_{\pm} = 0$ and M is unconstrained. On the mixed branches only one of v_{\pm} is nonzero, while $\det M = 0$.

5.1.1 Analysis in the electric theory

The brane construction which defines the $U(2)$ theory with two flavors is NS5—2D3(I)—NS5'—2D3(II)—D5'—1D3(III)—D5', as shown in figure 20(a). We label the three regions as

$$\begin{aligned} 0 < y < y_1, & \quad (\text{Region I}) \\ y_1 < y < y_2, & \quad (\text{Region II}) \\ y_2 < y < y_3. & \quad (\text{Region III}) \end{aligned} \quad (5.3)$$

In this system, there is enough complex gauge symmetry to set $\mathcal{A} = 0$ in all three regions, which also makes the scalar fields piecewise constant. This choice does not completely fix the gauge; there is a residual rigid $U(2)_{\mathbb{C}}$ gauge transformation in region I.

⁶This Higgs branch is parameterized by four gauge invariant mesons, $Q_i \tilde{Q}_j$ with no constraints relating them. Had the gauge group been $SU(2)$ rather than $U(2)$, we would have had to include two baryonic operators, $B = Q_1 Q_2$ and $\tilde{B} = \tilde{Q}_1 \tilde{Q}_2$, with the constraint $\det(M) - B \tilde{B} = 0$, leaving a 5-dimensional moduli space.

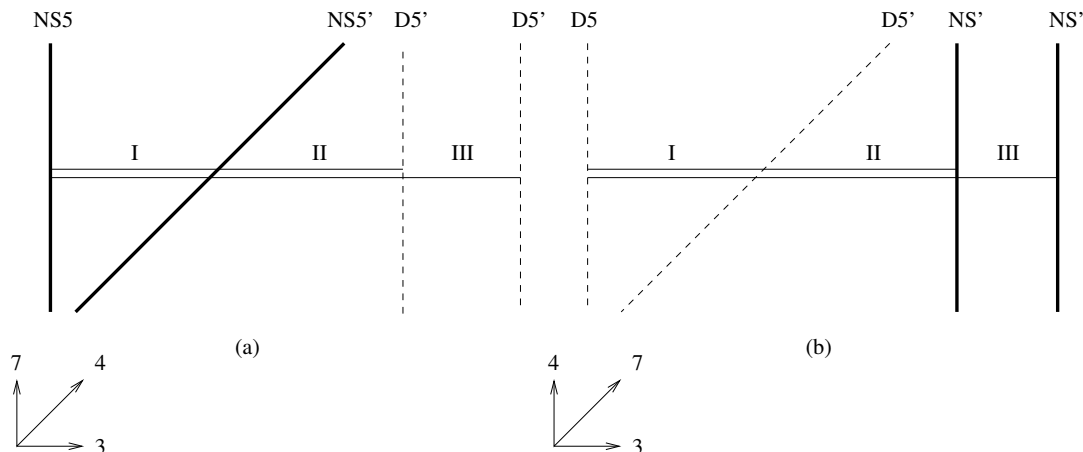


Figure 20. The defect system for $3d \mathcal{N} = 2$ SYM with $N_c = 2$, $N_f = 2$. It is shown in the defining electric representation (a) and the S-dual magnetic representation (b).

The analysis begins with the NS5 boundary condition at $y = 0$ and then proceeds from the left to the right. With all the mass deformations turned off, the NS5 boundary puts no constraint on \mathcal{Y}_I but sets

$$\mathcal{X}_I = 0. \tag{5.4}$$

At the I–II interface ($y = y_1$), we have bifundamentals A and B which are 2×2 matrices satisfying

$$\mathcal{Y}_I = AB, \quad \mathcal{Y}_{II} = BA, \tag{5.5}$$

but these equations are trivial because the \mathcal{Y} fields are otherwise unconstrained. There are also real moment map equations for X_6 :

$$AA^\dagger - B^\dagger B = 0, \tag{5.6}$$

$$A^\dagger A - BB^\dagger = \text{any}. \tag{5.7}$$

For the moment, we can suppress these equations by using the complexified gauge symmetry.

We also need to find \mathcal{X}_{II} to satisfy the equations $\mathcal{X}_I A = A \mathcal{X}_{II}$, etc. But these are automatically satisfied because of (5.4) combined with the fact that the $D5'$ ordinary Dirichlet boundary conditions on the right set

$$\mathcal{X}_{II} = 0. \tag{5.8}$$

Moreover, the ordinary Dirichlet boundary conditions put no constraint on \mathcal{Y}_{II} .

So the moduli space is given by A and B with no constraints other than that we must mod out by the $U(2)$ gauge symmetry; given A and B , $\mathcal{Y}_{I,II}$ are determined and give no additional moduli. This means there are 8 degrees of freedom with 4 gauge symmetries, so the Higgs branch is 4-dimensional.

In this analysis we have assumed that A and B are generic (so that, for example, a $G_{\mathbb{C}}$ transformation can set $A = \mathbb{I}$). However, if they have some vanishing eigenvalues, then

\mathcal{Y}_I will also have vanishing eigenvalues. When this happens there can be unbroken gauge symmetry and we should restore the field \mathcal{Z} , for which we have

$$\mathcal{Z}_I A = A \mathcal{Z}_{II}, \quad B \mathcal{Z}_I = \mathcal{Z}_{II} B. \tag{5.9}$$

The ordinary Coulomb branch arises when $A = B = 0$ (which satisfies the real equations). Then \mathcal{Z}_I gives two moduli (because it is arbitrary but we can diagonalize it by a gauge transformation) and we have a $2d$ Coulomb branch.

To see the mixed branch, we want to assume that \mathcal{Z}_I has one zero eigenvalue. Then we can do a gauge transformation to

$$\mathcal{Z}_I = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}, \tag{5.10}$$

which breaks the gauge symmetry to $U(1) \times U(1)$. To further break the gauge symmetry to just $U(1)$, we take

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ b_2 & 0 \end{pmatrix}. \tag{5.11}$$

Modding out by a complex $U(1)$ leaves us with the three-dimensional conifold. So the classical mixed branch moduli space is \mathbb{C}^* times the conifold.

We can also consider deforming by a real FI parameter. All the equations are the same except that one of the real equations is modified to

$$AA^\dagger - B^\dagger B = \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r \end{pmatrix}. \tag{5.12}$$

This completely lifts both the pure Coulomb branch and the mixed branch, but does not affect the pure Higgs branch, for which there was no constraint on A and B anyway.

5.1.2 Analysis in the S-dual

The Coulomb branch as described in section 5.1.1 preserves some unbroken gauge symmetry, and is therefore subject to instanton corrections. We can study the quantum-corrected Coulomb branch by considering the S-dual.

The S-dual brane configuration is $D5-2D3(I)-D5'-2D3(II)-NS5'-1D3(III)-NS5'$, divided into regions I, II, III as indicated. The configuration is shown pictorially in figure 20(b).

Because of the Nahm pole at the D5-brane boundary, we cannot set $\mathcal{A} = 0$. Instead, we need to make a choice of gauge consistent with the Nahm pole singularity. In region I, we have

$$\mathcal{X}_I = \begin{pmatrix} a & f_1(y) \\ b f_1(y)^{-1} & a \end{pmatrix}, \tag{5.13}$$

$$\mathcal{Y}_I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.14}$$

$$\mathcal{A}_I = \begin{pmatrix} f_2(y) & 0 \\ 0 & -f_2(y) \end{pmatrix}. \tag{5.15}$$

We use $G_{\mathbb{C}}$ to pick the form where $f_1(y)$ and $f_2(y)$ are a particular choice of Euler top functions [22]. We can choose them so that

$$f_1(y) = \frac{iDe^{i\phi}}{2} (\text{ns}_{\kappa}(Dy) + \text{ds}_{\kappa}(Dy)) \tag{5.16}$$

$$f_2(y) = -\frac{i f_1'(y)}{2 f_1(y)} = \frac{iD}{2} \text{cs}_{\kappa}(Dy) \tag{5.17}$$

with real parameters κ , D , and ϕ . The parameter κ is fixed at the I-II interface by requiring $f_1'(y_1) = 0$ which implies $f_2(y_1) = 0$ which conveniently sets $\mathcal{A}_I(y_1) = 0$. The complex parameter b is related to D and ϕ via

$$b = -\kappa(De^{i\phi})^2. \tag{5.18}$$

With these choices, \mathcal{X} has eigenvalues $a \pm \sqrt{b}$. Then, without any loss of generality, we can further set $f_1(y_1) = 1$ using complex gauge transformation to simplify the expressions which follow. We can then choose the gauge $\mathcal{A} = 0$ in regions II and III, so all the fields are piecewise constant in these regions.

The point of the preceding complicated analysis is that at $y = y_1$ we are free to simply set

$$\mathcal{X}_I(y_1) = \mathcal{X}_{II}(y_1) = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}. \tag{5.19}$$

To determine \mathcal{Y} in region II, we introduce the matter fields at the I-II interface, \tilde{Q} and Q . Because

$$\mathcal{X}_{II}(y_1)Q = \tilde{Q}\mathcal{X}_{II}(y_1) = 0, \tag{5.20}$$

we have to either set $\det(\mathcal{X}_I) = 0$ or else we have $Q = \tilde{Q} = 0$. It is useful to distinguish three separate cases:

1. $Q = \tilde{Q} = 0$, so there is no constraint on \mathcal{X} . Then we have \mathcal{X} with 2 distinct non-zero eigenvalues at generic points of moduli space. We can do a rigid $G_{\mathbb{C}}$ transformation to simplify the analysis:

$$\mathcal{X}_{II} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \tag{5.21}$$

unless the two eigenvalues are equal, in which case we can only put the matrix in Jordan normal form:

$$\mathcal{X}_{II} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}. \tag{5.22}$$

2. At least one of \tilde{Q} or Q is nonzero so $\det(\mathcal{X}_I) = 0$. The eigenvalues of \mathcal{X} are $2a$ and 0 , and suppose $a \neq 0$. Then we may do a rigid $G_{\mathbb{C}}$ transformation to

$$\mathcal{X}_{II} = \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.23}$$

3. At least one of \tilde{Q} or Q is nonzero and both eigenvalues of \mathcal{X} vanish. Then we put \mathcal{X} in Jordan normal form:

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.24}$$

The way to think about the G_{C} transformation is that we do a rigid transformation in both regions I and II; this changes the form of \mathcal{A} in region I so it is not solution-generating; it simply rewrites every solution in a more convenient form for doing the linear algebra calculations. The reason why we are free to do this rotation is that the $4d$ $U(2)$ gauge theory has an NS5-like boundary condition on the right. It should also be apparent that the only nontrivial effect of the Nahm pole is to force \mathcal{X}_{II} to take Jordan normal form if two of the eigenvalues are equal.

Now let us analyze each case, beginning with case 1. Because $Q = \tilde{Q} = 0$, we have $\mathcal{Y}_{\text{II}} = 0$. Using

$$\mathcal{Y}_{\text{II}} = AB, \tag{5.25}$$

$$\mathcal{Y}_{\text{III}} = BA = 0, \tag{5.26}$$

we see that $\mathcal{Y}_{\text{III}} = 0$ as well, and that either $A = 0$ or $B = 0$.

In the absence of real mass deformations, stability requires that in fact both $A = 0$ and $B = 0$. Then the gauge symmetry in region III between the NS5'-branes is unbroken and we do not strictly trust our classical solution. Nevertheless, if we compute the associated moduli space we find 4 complex degrees of freedom: 2 corresponding to a and b (or equivalently, the two eigenvalues of \mathcal{X}_{II}), and two corresponding to \mathcal{X}_{III} and \mathcal{Z}_{III} which are unconstrained if the interface matter is trivial.

Next, we proceed to case 2. The relations $\mathcal{X}_{\text{II}}Q = \tilde{Q}\mathcal{X}_{\text{II}} = 0$ imply that

$$Q = \begin{pmatrix} 0 \\ v_+ \end{pmatrix}, \quad \tilde{Q} = (0 \ v_-). \tag{5.27}$$

This implies

$$\mathcal{Y}_{\text{II}} = Q\tilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & v_+v_- \end{pmatrix} = AB, \tag{5.28}$$

but because

$$\mathcal{Y}_{\text{III}} = BA = 0, \tag{5.29}$$

we are forced to set

$$v_+v_- = 0. \tag{5.30}$$

This implies that at least one of A, B vanishes, and stability then requires that we have both $A = 0$ and $B = 0$.

The remaining analysis is similar to that of case 1, except that we can have either $v_+ \neq 0$ or $v_- \neq 0$. As in case 1, because both bifundamentals vanish, \mathcal{X}_{III} and \mathcal{Z}_{III} are free to vary. So we have two 4-dimensional branches, parameterized by a , the nonzero v_{\pm} , \mathcal{X}_{III} , and \mathcal{Z}_{III} , again with an unbroken $U(1)$ gauge symmetry.

Last but not least, we turn to case 3, where both eigenvalues of \mathcal{X}_{II} vanish. In this case we can have both Q, \tilde{Q} nonvanishing, and of the form

$$Q = \begin{pmatrix} v_+ \\ 0 \end{pmatrix}, \quad \tilde{Q} = (0 \ v_-). \quad (5.31)$$

Then we have

$$\mathcal{Y}_{\text{II}} = Q\tilde{Q} = \begin{pmatrix} 0 & v_+v_- \\ 0 & 0 \end{pmatrix} = AB, \quad (5.32)$$

which is compatible with $\mathcal{Y}_{\text{III}} = BA = 0$. The solution requires $A = pQ, B = p^{-1}\tilde{Q}$ and the parameter p can be thought of as being fixed by $G_{\mathbb{C}}$. So we are left with a two-dimensional branch, which is evidently semistable (and, as one can check, it exists for any value of the real FI parameter ζ_r). The expectation values of A and B break the $U(1)$ gauge symmetry in region III, so for this branch the classical analysis should be reliable. It is natural to identify this as the dual of the Coulomb branch in the original (electric) gauge theory.

Let us summarize our findings for $N_f = N_c = 2$ with no mass or FI deformations. We have a four-dimensional branch in case 1 which we identify as the Higgs branch of the electric theory. This branch has unbroken gauge symmetry in the magnetic formulation, but it can be computed reliably in the electric formulation. In case 2, there are two mixed Coulomb-Higgs branches which are 4-dimensional; for the mixed branches, there is unbroken gauge symmetry in both S-duality frames, so our analysis is not fully trustworthy. Finally, in case 3 there is a single 2-dimensional branch which we identify as the Coulomb branch; on this branch the analysis in the magnetic frame is reliable but not the electric analysis.

The counting of the branches and their dimensions match what one would have inferred from the superpotential (5.2). Our analysis might not seem terribly impressive, because the dimensions of the branches of moduli space are simply the classical ones, but we will now proceed to some more nontrivial examples by adding mass deformations.

5.1.3 Complex mass deformation to $\mathcal{N} = 2, N_c = 2, N_f = 1$

We may add a complex mass (of the electric theory) by moving one of the NS5' branes in the 78-directions. This will, for example, set

$$\mathcal{Y}_{\text{III}} = BA = c', \quad (5.33)$$

with $c' \neq 0$.

This criterion lifts the moduli spaces in case 1 and 3, both of which require $\mathcal{Y}_{\text{III}} = 0$. Case 2 survives, with the constraints

$$a \neq 0, \quad (5.34)$$

$$v_+v_- = c'. \quad (5.35)$$

We see that we no longer have one of the v_{\pm} vanishing, so the solution space has characteristics of both cases 2 and 3. The branch has the same complex structure as $(\mathbb{C}^*)^2$. Note

that we could have changed the normalization of v_{\pm} by a function of a without changing the complex structure.

Note also that because we can solve for B, A as $B \sim \tilde{Q}, A \sim Q$, with both B and A nonvanishing, this branch is semistable.

The complex structure is independent of the value parameter c' (as long as it is nonzero), so we should be able to take the limit $c' \rightarrow \infty$ which corresponds to completely removing one of the NS5'-branes.

To compare with the field theory, we should consider a superpotential with a mass deformation

$$W = v_+ v_- \det(M) + \mu M_{22}, \tag{5.36}$$

and one can show that the complex structure from this superpotential is also $(\mathbb{C}^*)^2$. In the classical field theory analysis, there is a 2-dimensional Coulomb branch and a 2-dimensional mixed Higgs-Coulomb branch, but the quantum-corrected analysis has only a single 2-dimensional branch. This moduli space has characteristics of the Coulomb branch (both monopole operators have nonzero expectation values) and of the Higgs or mixed branches (the mesons have expectation values), so one can think of it as a quantum mechanical merging of the Coulomb and mixed branches.

5.1.4 Real mass deformations

As in the Abelian theories, it is also interesting to consider real mass deformations. In the S-dual formulation, the (electric) real masses appear as real FI terms. In this situation, the defect analysis is especially useful — real masses break holomorphy, so the standard field theory arguments based on superpotentials are inapplicable. Moreover, in the S-dual formulation, the FI terms generically break the gauge symmetry completely, which is precisely the condition that we want to guarantee that our analysis is reliable.

When the real masses take generic values, a brane analysis described in [3, 4] claims that for $N_c = N_f = 2$, there should be six branches of moduli space, each of which is 2-dimensional. The counting is given by the following picture. In the magnetic description, one considers 2 D3-branes suspended between a D5 and a D5' brane. In between the D5 branes one places 2 NS5 (or NS5') branes, displaced in the X_6 -direction which is common to the D5 and D5'. This creates five slots in which the D3 branes can be inserted. One then adds the constraint that repulsive forces between the D3-branes prevent 2 D3's from being placed in the same slot or in adjacent slots. Then, according to this recipe, there are six distinct allowed configurations for the D3-branes. For general values of N_c and N_f , this counting argument gives

$$\binom{2N_f - N_c + 2}{N_c} \tag{5.37}$$

branches of moduli space.

We will now show that the Nahm computation reproduces this counting, although the way our method accounts for the six branches is slightly different from what is described in the literature. We do the analysis in the S-dual, and revisit the three cases in turn. The FI terms modify the real equations and relax the stability conditions.

In case 1, we obtained the constraint that either $A = 0$ or $B = 0$. Then the stability condition forced us to set both $A = B = 0$, but with an FI term we are no longer required to do so. Instead, we have the real equation.

$$AA^\dagger - B^\dagger B = \zeta_r, \tag{5.38}$$

and $A = 0$ requires $\zeta_r < 0$ while $B = 0$ requires $\zeta_r > 0$. Suppose that $\zeta_r > 0$ so that $B = 0$. Then we must have

$$A = \begin{pmatrix} v \\ 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{5.39}$$

and the relation $\mathcal{X}_{II}A = A\mathcal{X}_{III}$ implies that for each choice we have either $\mathcal{X}_{III} = \lambda_1$ or λ_2 . It might seem that these are two distinct branches, but really they are not — we can move between the two choices continuously by varying a and b and then doing a discrete gauge transformation to interchange the choices of A . (When the two eigenvalues are equal there is only one choice of A .) So we find that in case 1 there is one $2d$ branch of moduli space if $\zeta_r \neq 0$, which we can think of as the remnant of the $4d$ branch with unbroken gauge symmetry which exists for $\zeta_r = 0$.

In case 2, the analysis with $\zeta_r \neq 0$ is similar to case 1, except that the two forms of A (or B) in (5.39) are distinct. Combining this with the choice of whether v_+ or v_- is nonzero, we see that there are four 2-dimensional branches if $\zeta_r \neq 0$.

In case 3, the analysis with $\zeta_r = 0$ is unchanged when $\zeta_r \neq 0$, so this case contributes a single 2-dimensional branch. So to summarize, when ζ_r is nonzero, we see six 2-dimensional branches, one from case 1, four from case 2, and one from case 3.

For the sake of comparison, it is useful to revisit the naive geometric brane analysis. The brane configurations corresponding to the branches of the moduli space in cases 1, 2, and 3 for $\zeta_r > 0$ are shown in figure 21. In the figure, the Higgs branch moduli correspond to unbroken D3-branes, while the Coulomb-type moduli (shown by green arrows) correspond to broken D3-branes. In the counting argument of [3, 4], the configurations 2.ii and 2.iii have D3-branes in adjacent slots, and so do not give rise to branches of moduli space. On the other hand, in the S-dual Nahm analysis, we retained the cases 2.ii and 2.iii but excluded the diagrams from case 3, namely 3.ii and 3.iii.

The difference between the two ways of accounting for the branches of moduli space is partly an artifact of the way we organized the algebraic computation, but more importantly, is a consequence of the quantum merging of branches. For example, in 2.ii and 3.iii of figure 21, we could move one of the NS5'-branes to infinity, effectively reducing the system to $N_c = 2$ $N_f = 1$. The classical moduli spaces of 2.ii and 3.iii (or 2.iii and 3.ii) then merge quantum mechanically, following the analysis of section 5.1.3.

The larger point is that the geometric brane drawing, with D3-branes depicted as straight lines, is misleading because it does not accurately capture the non-Abelian nature of the defect theory. This is not just a matter of nomenclature; from the S-dual Nahm analysis we can determine the complex structure of the mass-deformed moduli space, including the loci of intersection of the branches. In general these loci as computed by the Nahm analysis will be different from what one might infer from the brane diagram.

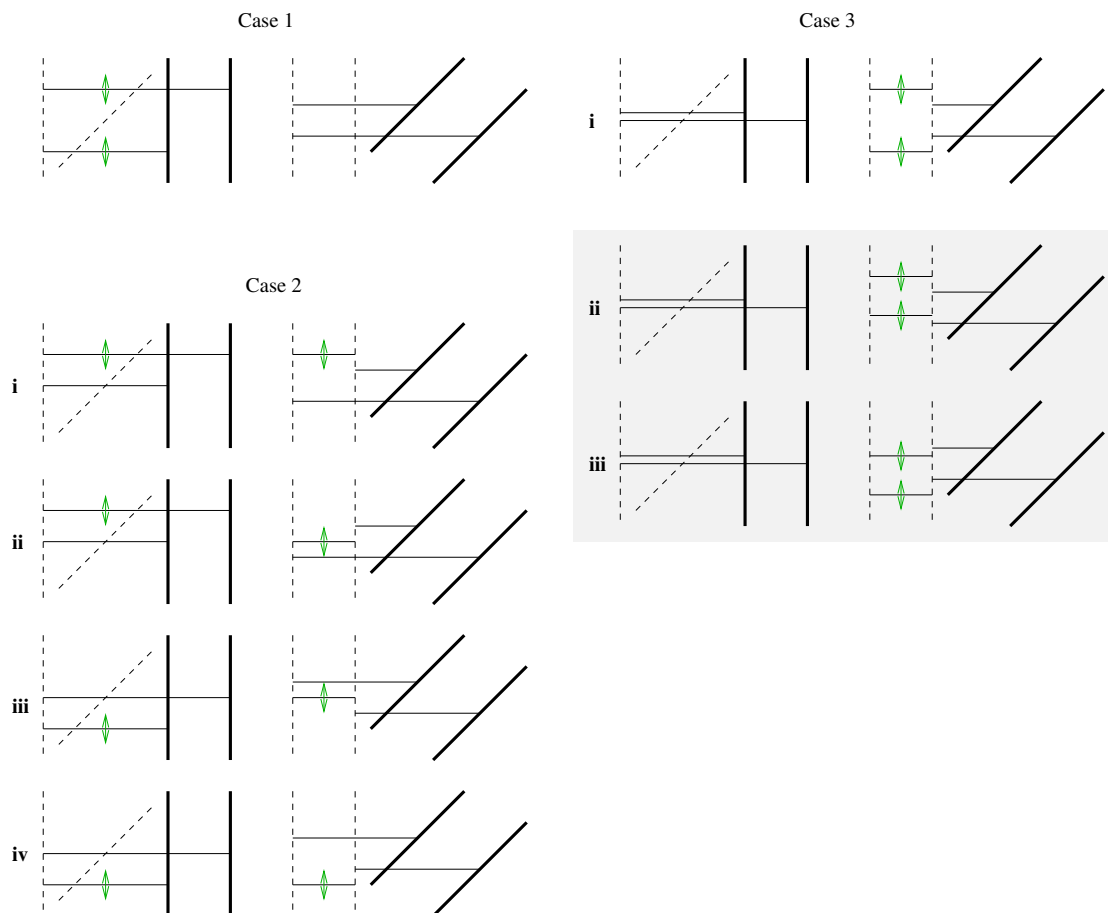


Figure 21. Brane configurations corresponding to one complex branch in case 1, four complex branches in case 2, and one complex branch in case 3. Each green arrow corresponds to one complex dimension of moduli. The branches 3.ii and 3.iii do not appear explicitly in our analysis. However, they can be understood as having merged with branches 2.iii and 2.ii respectively through the mechanism of Coulomb-Higgs merging, and therefore do not count as separate branches. This is exactly what is needed since in the counting of (5.37), branches 2.ii and 2.iii were not included but 3.ii and 3.iii were. That such a merging occurs is easy to see, by moving one of the NS5-branes in 3.ii and 2.iii to effectively reduce the system to $N_c = 2$ and $N_f = 1$.

5.1.5 Complex mass deformation to $\mathcal{N} = 2$, $N_c = 2$, $N_f = 0$

Next, suppose we turn on *two* complex masses by moving both NS5'-branes (in the magnetic picture). Then we have

$$\mathcal{Y}_{\text{II}} = \begin{pmatrix} c' & 0 \\ 0 & d' \end{pmatrix}, \tag{5.40}$$

but we also have to satisfy $\mathcal{Y}_{\text{II}} = Q\tilde{Q}$, and it is easy to check that this is impossible if both c' and d' are nonzero.

We could have obtained the same result by attempting to solve the system D5—2D3—D5' with poles at both the D5 and D5'. A careful analysis of the generalized Nahm equations shows that no solution exists.

This recovers the classic result that pure $\mathcal{N} = 2$ U(2) gauge theory in 3 dimensions has no supersymmetric vacuum [5].

5.2 $\mathcal{N} = 2, N_c = 2, N_f = 1$

It is also possible to do the U(2) with $N_f = 1$ analysis directly, in the brane ordering D5—2D3(I)—NS5'—2D3(II)—D5' (it is much more cumbersome to analyze this system if we do a Hanany-Witten transition to put the NS5' to the right of the D5').

We allow the gauge transformations to be discontinuous at the NS5', which we place at $y = 0$, and we place the D5 and D5' at $y = \pm 1$. Then we can set

$$\mathcal{A}_I = \begin{pmatrix} \frac{1}{2(y+1)} & 0 \\ 0 & -\frac{1}{2(y+1)} \end{pmatrix}, \tag{5.41}$$

$$\mathcal{A}_{II} = \begin{pmatrix} \frac{1}{2(1-y)} & 0 \\ 0 & -\frac{1}{2(1-y)} \end{pmatrix}, \tag{5.42}$$

and we will have a solution of the Nahm equations, with

$$\mathcal{Y}_I = 0, \tag{5.43}$$

$$\mathcal{X}_{II} = 0, \tag{5.44}$$

and

$$\mathcal{X}_I = \begin{pmatrix} a & 1/(y+1) \\ b(y+1) & a \end{pmatrix}, \tag{5.45}$$

$$\mathcal{Y}_{II} = \begin{pmatrix} c & 1/(1-y) \\ d(1-y) & c \end{pmatrix}. \tag{5.46}$$

In addition to these equations, we also have to satisfy $\mathcal{Y}_I = AB, \mathcal{Y}_{II} = BA$. This forces the characteristic polynomial of \mathcal{Y}_{II} to vanish, so we have to set $c = d = 0$.

We also have to satisfy $\mathcal{X}_I A = 0$ and $B \mathcal{X}_I = 0$. This requires that at least one eigenvalue of \mathcal{X}_I vanishes, so we set $b = a^2$.

Solving for A, B we find

$$A = \begin{pmatrix} 0 & s \\ 0 & -sa \end{pmatrix}, \tag{5.47}$$

$$B = \begin{pmatrix} -at & t \\ 0 & 0 \end{pmatrix}, \tag{5.48}$$

along with the constraint

$$-2ast = 1. \tag{5.49}$$

This is the quantum deformed moduli space with the complex structure $(\mathbb{C}^*)^2$.

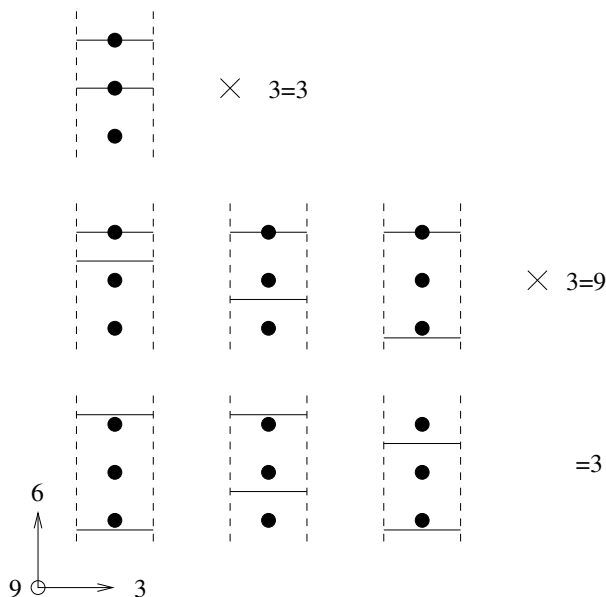


Figure 22. Enumeration of moduli space branches for the $\mathcal{N} = 2$, $N_c = 2$, $N_f = 3$ theory with real masses from the S-dual perspective.

5.3 $\mathcal{N} = 2$, $N_c = 2$, $N_f = 3$

We can apply our method to any number of flavors. For example we can analyze the situation with $N_f = 3$. The algebra becomes rather complicated, so we will simply state the results which an energetic reader will be able to confirm.

The branches of moduli space fall into three parts. There is an 8-dimensional part which is the ordinary Higgs branch and which can be computed classically, a 6-dimensional mixed branch with $U(1)$ gauge symmetry, and a 2-dimensional Coulomb branch which can be computed classically in the S-dual.

With generic real masses turned on, we find 15 distinct 2-dimensional branches, consistent with the old brane counting (5.37) for $N_c = 2$ and $N_f = 3$. An abbreviated version of figure 21 for this setup accounting for the 15 branches is illustrated in figure 22.

6 $U(3)$ examples

In this section, we will extend our analysis to the case where the gauge group is $U(3)$. The main novelty of theories with $N_c > 2$ is the fact that purely Coulomb branches are completely lifted by non-perturbative effects. There are, however, mixed branches with some Coulomb components.

6.1 $\mathcal{N} = 2$, $N_c = 3$, $N_f = 3$

We continue our analysis with the simplest example for a $U(3)$ gauge group.

Let us take a quick look at the classical moduli space of the field theory (in the electric description). There is a classical Coulomb branch with $U(1)^3$ gauge symmetry which is

3-dimensional and a classical (exact) Higgs branch which is $2N_f N_c - N_c^2 = 9$ -dimensional. There are also mixed branches with varying amounts of gauge symmetry. For example, we can choose quark expectation values which leave a $U(1)$ gauge symmetry. This mixed branch has a product structure with a Higgs part of dimension $\dim(U(3)/U(1)) = 8$ and a 1-dimensional Coulomb part, so it is 9-dimensional. There is another branch for which the quarks leave a $U(2)$ gauge symmetry unbroken, which is further broken to $U(1) \times U(1)$ on a partial Coulomb branch. The dimension of the Coulomb part is 2 and the Higgs part has dimension $\dim(U(3)/U(2)) = 5$. This mixed branch is 7-dimensional.

Let us compare this with the superpotential proposed by Aharony et al. [10]:

$$W = v_+ v_- \det(M), \tag{6.1}$$

where M is an $N_f \times N_f$ matrix. The F-term equations have three solution branches:

- $v_+ = v_- = 0$, M is unfixed — which is 9-dimensional.
- $v_+ = 0$ with $v_- \neq 0$ (or vice versa) with $\det(M) = 0$ — which is 9-dimensional
- $v_{\pm} \neq 0$ with $\delta \det(M) = 0$ — which is 7-dimensional.

Comparing to the classical moduli space, the simplest picture is that the effective superpotential is capturing the Higgs and mixed branches of the moduli space but not the pure Coulomb branch. We will see that the Nahm computation matches this field theory analysis, with the mixed branches but not the $3d$ Coulomb branch with $U(1)^3$ gauge symmetry.

Working in the magnetic formulation, the brane configuration is $D5-3D3(I)-D5'-3D3(II)-NS5'-2D3(III)-NS5'-1D3(IV)-NS5'$. The I-II interface supports fundamental quarks Q, \tilde{Q} . At the II-III interface, the $NS5'$ supports bifundamental fields A_1, B_1 and at the III-IV interface, the $NS5'$ supports bifundamentals A_2, B_2 . In the following analysis we disregard all the unstable branches.

As in the previous section, we can do a G_C transformation in regions II, III, IV to make \mathcal{X} constant and to put it in Jordan normal form. Then we distinguish three cases by the form of the eigenvalues of \mathcal{X}_{II} . The first case is when all three eigenvalues are distinct and nonzero. Then we have to set $\tilde{Q} = Q = 0$, and $\mathcal{Y} = 0$ everywhere. Neglecting unstable branches, we are forced to set A_1, B_1, A_2, B_2 all to zero. The resulting moduli space has unbroken gauge symmetry, so we don't trust the geometry, but we can compute the dimension of the moduli space anyway. There are 3 moduli corresponding to the eigenvalues of \mathcal{X}_{II} . In addition to these, between the $NS5'$ -branes there are three D3-brane segments, each of which contributes 2 complex moduli corresponding to their positions in \mathcal{X} and \mathcal{Z} . So there are a total of 9 complex moduli. It is natural to identify this as the Higgs branch of the electric theory.

The second case is when two eigenvalues are nonzero and one vanishes. In this case either Q or \tilde{Q} is nonzero. Again, $\mathcal{Y} = 0$ everywhere so there are 6 moduli from the D3 segments between the $NS5'$ -branes. Comparing with the previous case, we lose one modulus because one of the eigenvalues of \mathcal{X}_{II} was fixed to zero, but we gain one from the length of Q or \tilde{Q} (whichever is nonvanishing). So the total moduli space again is 9-dimensional,

with two branches. This appears to correspond to the mixed branch of the electric theory with U(1) gauge symmetry.

Next we consider the case where two eigenvalues of \mathcal{X}_{II} vanish, and take the G_C gauge $\mathcal{A} = 0$ in regions II, III, and IV. Then we can write

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

for which we have

$$Q^T = (0, v_+, 0), \tag{6.3}$$

$$\tilde{Q} = (0, 0, v_-), \tag{6.4}$$

and

$$\mathcal{Y}_{\text{II}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v_+v_- \\ 0 & 0 & 0 \end{pmatrix} = A_1 B_1. \tag{6.5}$$

The bifundamental fields also have to satisfy

$$\mathcal{X}_{\text{II}} A_1 = A_1 \mathcal{X}_{\text{III}}, \tag{6.6}$$

$$B_1 \mathcal{X}_{\text{II}} = \mathcal{X}_{\text{III}} B_1. \tag{6.7}$$

We identify two independent solutions of these conditions which break all the gauge symmetry:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & v_+ \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & v_- & 0 \\ 0 & 0 & v_- \end{pmatrix}, \tag{6.8}$$

and

$$A_1 = \begin{pmatrix} 0 & 0 \\ v_+ & 0 \\ 0 & v_+ \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & v_- \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.9}$$

with (in both cases)

$$\mathcal{X}_{\text{III}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{6.10}$$

$$\mathcal{Y}_{\text{III}} = \begin{pmatrix} 0 & v_+v_- \\ 0 & 0 \end{pmatrix}. \tag{6.11}$$

The rest of the analysis is identical to case 3 of $N_c = 2, N_f = 2$. We see that we have two 3-dimensional branches on which the gauge symmetry is completely broken. However, we need to be a bit careful about the action of G_C . We have U(2) complex gauge transformations which act as $A_1 \rightarrow A_1 g^{-1}, B_1 \rightarrow g B_1, A_2 \rightarrow g A_2, B_2 \rightarrow B_2 g^{-1}$, with

$$g = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}. \tag{6.12}$$

In evaluating the gauge quotient, we need to consider not just the regular gauge transformations but also the *closure* of the gauge orbit. It is not hard to see that the closure of the gauge orbit can map these points to

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & v_+ \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v_- \end{pmatrix}, \quad (6.13)$$

where the gauge symmetry is broken. At these points, new moduli appear. We have

$$\mathcal{X}_{\text{III}} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.14)$$

$$\mathcal{Y}_{\text{III}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.15)$$

where r is unfixed. Now, at the III–IV interface condition at least one of A_2, B_2 must vanish. If one of the two vanishes, we have an unstable branch; if both vanish then the U(1) gauge symmetry in region IV is unbroken. It is the latter case which is of interest to us. We see that there is an unbroken U(1) symmetry in region III and an unbroken U(1) in region IV. Associated with each unbroken U(1) are two complex moduli, from \mathcal{X} and \mathcal{Z} . The total dimension of this branch of moduli space is therefore 7 (2 from v_{\pm} , 1 from a , and 4 from the scalars between the NS5'-branes). This appears to correspond to the 7-dimensional mixed branch.

If we turn on real masses (which are FI terms in the magnetic formulation we are analyzing), when we solve the real equation we will find that we have 3 branches from case 3. This is because we can solve the real equation for all three forms of A_1, B_1 (up to G_C) with distinct solutions. The brane picture associated with each of these branches are illustrated in figure 23.

The total counting of the number of branches in this case comes out to

$$1 + 6 + 3 = 10, \quad (6.16)$$

which again is consistent with (5.37) for $N_c = 3$ and $N_f = 3$.

6.2 $\mathcal{N} = 2, N_c = 3, N_f = 2$ by complex mass deformation

Analogously to the $N_c = 2, N_f = 1$ case, we can extract the moduli space of $N_c = 3, N_f = 2$ by adding a complex mass deformation. This means in particular that we should set $\mathcal{Y}_{\text{IV}} = c' \neq 0$.

Classically, we would have the following expectations. There is a pure Higgs branch with dimension 3, and a Coulomb branch which is 3 dimensional. The classical geometry of the Coulomb branch is \mathbb{C}^3/\mathcal{W} where \mathcal{W} is the SU(3) Weyl chamber. We also have a mixed branch with an unbroken U(1) gauge symmetry and 3 Higgs moduli; this mixed branch is 4 dimensional. There is no separate mixed branch with U(1) \times U(1) gauge symmetry as it is part of the U(1)³ Coulomb branch.

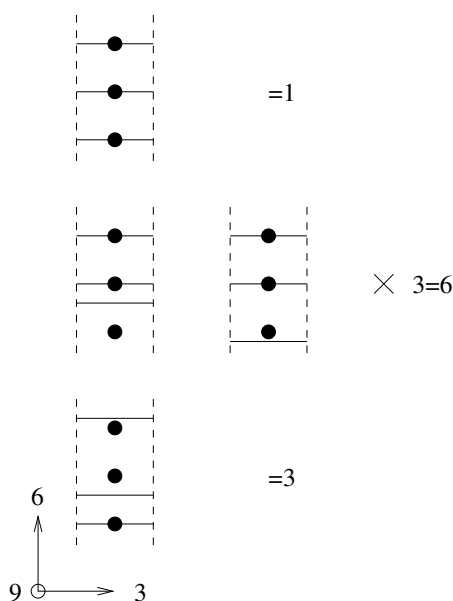


Figure 23. Enumeration of moduli space branches for the $\mathcal{N} = 2$ $N_c = 3$ $N_f = 3$ theory with real masses from the S-dual perspective.

Quantum mechanically we expect to see merging of the Higgs and Coulomb branches.

As in the case for $N_c = 2$, $N_f = 1$, the unlifted solutions appear when there is only one vanishing eigenvalue of \mathcal{X}_{II} , so we can write

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.17}$$

and with $Q^T = (0, 0, v_+)$, $\tilde{Q} = (0, 0, v_-)$, we have

$$\mathcal{Y}_{\text{II}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v_+v_- \end{pmatrix}. \tag{6.18}$$

If we require that $\mathcal{Y}_{\text{IV}} = c' \neq 0$, then we cannot eliminate the zero eigenvalue of \mathcal{X} until we reach the last NS5'. So we have to have

$$\mathcal{X}_{\text{III}} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.19}$$

$$\mathcal{Y}_{\text{III}} = \begin{pmatrix} 0 & 0 \\ 0 & v_+v_- \end{pmatrix}, \tag{6.20}$$

Moreover, we have enough gauge freedom to make \mathcal{X} and \mathcal{Y} constant in regions II, III, etc. Note that there are N parameters which are free to vary.

In region II we can organize the computation by the eigenvalue structure of \mathcal{X} ; this involves choosing a different gauge than the one used in (7.1):

1. All the eigenvalues of \mathcal{X}_{II} are distinct and nonzero:

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & & \\ & & \mu_3 & & \\ & & & \dots & \\ & & & & \mu_N \end{pmatrix}. \tag{7.2}$$

This choice forces

$$Q, \tilde{Q} = 0, \tag{7.3}$$

$$\mathcal{Y}_{\text{II}} = 0. \tag{7.4}$$

2. One eigenvalue of \mathcal{X}_{II} is zero; the rest are distinct and nonzero

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & & \\ & & \dots & & \\ & & & \mu_{N-1} & \\ & & & & 0 \end{pmatrix}. \tag{7.5}$$

This forces

$$Q^T = (0, \dots, 0, v_+), \tag{7.6}$$

$$\tilde{Q} = (0, \dots, 0, v_-), \tag{7.7}$$

$$\mathcal{Y}_{\text{II}} = \text{diag}(0, \dots, 0, v_+v_-). \tag{7.8}$$

3. Two eigenvalues of \mathcal{X}_{II} vanish and the rest are distinct and nonzero

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} \mu_1 & & & & \\ & \dots & & & \\ & & \mu_{N-2} & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}. \tag{7.9}$$

Then we have

$$Q^T = (0, \dots, 0, v_+, 0), \tag{7.10}$$

$$\tilde{Q} = (0, \dots, 0, 0, v_-), \tag{7.11}$$

and

$$\mathcal{Y}_{\text{II}} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \dots & \\ & & & 0 \ v_+ v_- \\ & & & & 0 \end{pmatrix}. \tag{7.12}$$

The other possibilities (for example, when more than two eigenvalues vanish, or when two eigenvalues coincide but don't vanish) can be smoothly related to these three cases, which are distinguished by the form of the fundamental quarks Q, \tilde{Q} . So it suffices to consider these three cases (other possibilities do not give distinct branches of moduli space).

Now we proceed to analyze the dimension of moduli space for these three cases. This calculation is not strictly reliable because there will be some unbroken gauge symmetry, but we can do it anyway and see what we get.

In case 1, we have $\mathcal{Y}_{\text{II}} = 0$. Stability requires that all the A_i, B_i supported at the NS5'-branes vanish. Then we have $N(N - 1)/2$ brane segments between the NS5'-branes which are free to move. Each brane segment contributes 2 complex moduli. In addition we have N moduli from the eigenvalues of \mathcal{X} . So the total dimension of moduli space is N^2 .

In case 2, because the eigenvalues of \mathcal{Y}_{II} have to vanish (from the coupling to $T[\text{SU}(N)]$), we must have $v_+ v_- = 0$. There is no gauge symmetry to fix v_{\pm} . We see that there are two branches of moduli space where either v_+ or v_- is zero. Because \mathcal{Y}_{II} vanishes completely, we again have to set all the A_i, B_i to zero. So (as in case 1) we get $N(N - 1)$ moduli from the brane segments which are free to move. We also have $N - 1$ eigenvalues of \mathcal{X} , and one modulus from v_{\pm} . The total dimension is N^2 . Note however that this is a distinct branch of moduli space from case 1 because the fundamentals Q, \tilde{Q} are not both zero.

In case 3, \mathcal{Y}_{II} does not vanish even though all its eigenvalues are zero, so it is no longer necessary for all the bifundamentals of $T[\text{SU}(N)]$ to vanish. Instead we can set $\mathcal{Y}_{\text{II}} = A_1 B_1$ with

$$A_1 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & v_+ \\ 0 & \dots & 0 & 0 \end{pmatrix}, \tag{7.13}$$

$$B_1 = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \dots & & & \dots \\ 0 & & & 0 \\ 0 & \dots & 0 & 0 \ v_- \end{pmatrix}. \tag{7.14}$$

Note that $B_1 A_1 = 0$ so that $\mathcal{Y}_{\text{III}} = 0$. This forces all the remaining A_i, B_i to vanish. I believe this is the only solution for the bifundamentals consistent with stability, up to gauge transformations.

Counting brane segments which are free to move between the NS5'-branes, we see that there are $N^2 - N - 2$ associated complex moduli. In addition to this, we have 2 moduli

from v_{\pm} and $N - 2$ eigenvalues of \mathcal{X}_{II} . The total dimension of this branch of moduli space is $N^2 - 2$.

These three branches can be compared to our field theory expectations. We expect an N_c^2 -dimensional Higgs branch. There should also be a $U(1)$ mixed branch with Coulomb dimension 1 and Higgs dimension $\dim(U(N)/U(1)) = N^2 - 1$ so the total dimension is N^2 . For $N \geq 2$ there is a mixed branch with $U(1)^2$ gauge symmetry with Coulomb dimension 2 and Higgs dimension $\dim(U(N)/U(2)) = N^2 - 4$, so the total dimension is $N^2 - 2$.

Crucially, these are the only three cases. The pure Coulomb branch (for $N_c > 2$) and most of the mixed branches, which are present classically, do not appear in the S-dual Nahm analysis. These additional branches appear to be lifted quantum mechanically, in accord with the superpotential of [10].

7.2 $N_f < N_c$

We can extract the dimensions of moduli space for $N_f < N_c$ by giving complex masses to the quarks. In the algebraic analysis this corresponds to fixing some of the eigenvalues of \mathcal{Y}_{II} to be nonzero.

For $N_f = N_c - 1$, we can start from $N_f = N_c$ and deform by a single complex mass. We see that cases 1 and 3 are immediately excluded, leaving only case 2. There are $N_c - 1$ moduli contained in \mathcal{X}_{II} and 1 modulus in \mathcal{Y}_{II} (and the quarks Q, \tilde{Q}). In addition there are $(N_c - 1)(N_c - 2)$ moduli from the D3-brane segments between the NS5'-branes which are allowed to move when the bifundamentals vanish. So the total dimension of moduli space is $N_c^2 - 2N_c + 2$.

For $N_f \leq N_c - 2$, however, we see immediately that there are no supersymmetric solutions, because \mathcal{Y}_{II} in the $N_f = N_c$ analysis can have at most one nonzero eigenvalue.

7.3 $N_f = N_c + 1$

The brane configuration is D5— N D3(I)—D5'— $N + 1$ D3(II)— $T[\text{SU}(N + 1), \text{NS5}']$. Again we have \mathcal{X}_{I} given by (7.1). However, moving into region II we have

$$\mathcal{X}_{\text{II}} = \begin{pmatrix} \mathcal{X}_{\text{I}} & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.15}$$

$$\mathcal{Y}_{\text{II}} = \begin{pmatrix} 0 & Q \\ \tilde{Q} & t \end{pmatrix}, \tag{7.16}$$

and the commutator $[\mathcal{X}, \mathcal{Y}] = 0$ implies $\mathcal{X}_{\text{II}}Q = \tilde{Q}\mathcal{X}_{\text{II}} = 0$.

Again, we divide the analysis into separate cases depending on the eigenvalue structure of \mathcal{X} , doing appropriate $GL(N, \mathbb{C})$ rotations to simplify the form of the fields to one of the following three possibilities:

1. All the eigenvalues of \mathcal{X}_I are distinct and nonzero, so

$$\mathcal{X}_{II} = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \dots & \\ & & & \mu_{N_c} \\ & & & & 0 \end{pmatrix}. \tag{7.17}$$

This forces $Q, \tilde{Q} = 0$.

2. One eigenvalue of \mathcal{X}_I zero and the rest are distinct and nonzero. Then \mathcal{X}_{II} has the form

$$\mathcal{X}_{II} = \begin{pmatrix} \mu_1 & & & \\ & \dots & & \\ & & \mu_{N_c-1} & \\ & & & 0 \\ & & & & 0 \end{pmatrix}. \tag{7.18}$$

Then \mathcal{Y}_{II} is

$$\mathcal{Y}_{II} = \begin{pmatrix} 0 & & & \\ & \dots & & \\ & & 0 & \\ & & & 0 & v_+ \\ & & & v_- & t \end{pmatrix}. \tag{7.19}$$

3. Two eigenvalues of \mathcal{X}_I vanish

$$\mathcal{X}_{II} = \begin{pmatrix} \mu_1 & & & & \\ & \dots & & & \\ & & \mu_{N_c-2} & & \\ & & & 0 & 1 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \tag{7.20}$$

$$\mathcal{Y}_{II} = \begin{pmatrix} 0 & & & \\ & \dots & & \\ & & 0 & 0 & v_+ \\ & & 0 & 0 & 0 \\ & & 0 & v_- & t \end{pmatrix}. \tag{7.21}$$

Case 1 here is much like Case 1 for $N_c = N_f$. We are forced to set $\mathcal{Y}_{II} = 0$ and then all the A_i, B_i vanish (for stability). The number of moduli from the D3 segments between NS5' branes is $N_f(N_f - 1)$. In addition there are $N_c = N_f - 1$ moduli from \mathcal{X} . So the total dimension of moduli space is $N_f^2 - 1$.

Case 2 is also similar to the analogous situation for $N_c = N_f$. The coupling to $T(\text{SU}(N_f))$ demands that the characteristic polynomial of \mathcal{Y}_{II} vanishes. This sets $t = 0$ and $v_+v_- = 0$. However unlike the $N_c = N_f$ analysis, it is no longer the case that \mathcal{Y}_{II} vanishes

completely, as one of v_+ or v_- is nonzero. Counting the number of brane segments which are free to move, the corresponding number of complex moduli is $(N_f + 1)(N_f - 2)$. In addition to this, we have $N_f - 1$ moduli from \mathcal{X} and v_{\pm} . The total dimension of this branch is $N_f^2 - 3$.

In case 3, vanishing of the characteristic polynomial of \mathcal{Y} implies $t = 0$ but places no constraint on v_{\pm} . We find that up to G_C , the first bifundamentals are

$$A_1 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & v_+ \\ 0 & \dots & 0 & 0 \\ 0 & \dots & v_- & 0 \end{pmatrix}, \tag{7.22}$$

$$B_1 = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \dots & & & \dots \\ 0 & & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{7.23}$$

This also implies

$$\mathcal{Y}_{\text{III}} = \begin{pmatrix} 0 & & & \\ & \dots & & \\ & & 0 & 0 \\ & & v_- & 0 \end{pmatrix}. \tag{7.24}$$

There are $N_f(N_f - 3)$ complex moduli from the D3 segments between NS5'-branes. In addition there are two moduli from v_{\pm} and $N_f - 3$ moduli from \mathcal{X} . So the total dimension of this branch of moduli space is $N_f^2 - 2N_f - 1$.

It is easy to verify that the dimensions of these branches match the classical expectations for the branches with no gauge symmetry, with U(1) gauge symmetry, and with U(1)² gauge symmetry. However the branches with more gauge symmetry do not arise in the Nahm analysis; presumably they are lifted by quantum effects.

8 Examples of quantum merging with one NS5

One of the interesting phenomena in $3d \mathcal{N} = 2$ theories is the quantum merging of Higgs and Coulomb branches. The prototypical example where this occurs is in the U(2) theory with one flavor. In that theory, classically there is a 2-dimensional Coulomb branch with U(1)² gauge symmetry and a 2-dimensional mixed Higgs-Coulomb branch with U(1) gauge symmetry (in this case there is no pure Higgs branch). The two branches intersect on a complex line. Quantum mechanically the Coulomb and mixed branches merge and there is only one $2d$ branch which is smooth. This merging can be understood by analyzing the field theory but is not manifest from considering the brane cartoons. In section 5 we described the quantum merging from the point of view of the 3+1-dimensional defect realization.

We would like to see more examples of this quantum merging, but the field theory analysis is more difficult when the theories become more complicated, as the effective

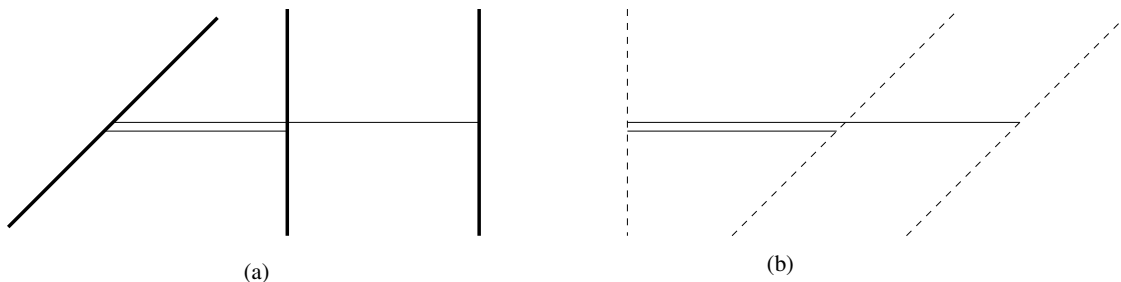


Figure 24. $U(2) \times U(1)$ quiver theory (a) and its S-dual, (b).

Lagrangians on moduli space are less constrained by symmetry as the field content becomes richer. However it turns out that the Nahm analysis is well-suited for constructing some of these examples.

The strategy for finding theories with quantum merging is to look for theories where the instanton effects are strongest; we might expect this to occur when the number of flavors is as small as possible without breaking supersymmetry. In the case with only a $U(2)$ gauge group, the pure $\mathcal{N} = 2$ theory broke supersymmetry because of instanton effects. However, by adding one flavor, the theory became supersymmetric with a quantum-merged moduli space. So what we will do is to look for other theories which break supersymmetry but which can be saved by adding a single flavor.

Consider the $U(2) \times U(1)$ quiver theory defined by the brane configuration $NS5-2D3-NS5'-1D3-NS5'$, shown in figure 24(a). From analyzing the S-dual, $D5-2D3-D5'-1D3-D5'$, shown in figure 24(b), it is clear that this theory has no supersymmetric vacuum. We can think of it in terms of a Nahm pole on the left and $D5'$ ordinary Dirichlet boundary conditions on the right. The Nahm pole from the $D5$ requires that \mathcal{X} is nonvanishing but the $D5'$ boundary conditions set $\mathcal{X} = 0$. So it is impossible to solve the Nahm equations and supersymmetry is broken.

We can add a flavor to the $U(2) \times U(1)$ theory by adding either a $D5$ or $D5'$ to make the configuration $NS5-2D3-D5/D5'-2D3-NS5'-1D3-NS5'$. The S-dual is $D5-2D3-NS5/NS5'-2D3-D5'-1D3-D5'$. The S-dual has no gauge symmetry in the $3d$ limit and so we expect the Nahm computation to determine the complex structure reliably. We will analyze the situation for both types of flavors.

8.1 $D5-2D3-NS5'-2D3-D5'-1D3-D5'$

Let us consider $D5-2D3(I)-NS5'-2D3(II)-D5'-1D3-D5'$, as illustrated in figure 25. It suffices to consider regions I and II. At the location of the $NS5'$, we can take the form of \mathcal{X} generated by the Nahm pole as

$$\mathcal{X}_I = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}. \tag{8.1}$$

Because of the ordinary Dirichlet boundary condition on the right, we have

$$\mathcal{X}_{II} = 0. \tag{8.2}$$

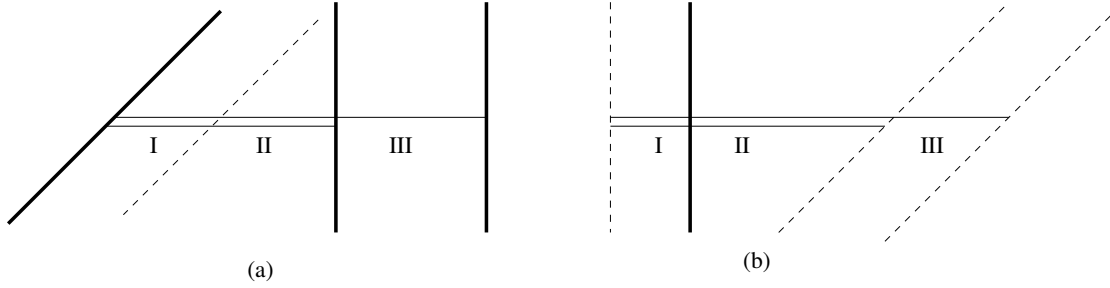


Figure 25. $U(1) \times U(2)$ quiver theory with one flavor added to the $U(2)$ gauge group (a) and its S-dual, (b).

We also have to satisfy

$$\mathcal{Y}_I = AB = 0, \quad (8.3)$$

$$\mathcal{Y}_{II} = BA = \text{any}, \quad (8.4)$$

and

$$\mathcal{X}_I A = A \mathcal{X}_{II} = 0, \quad (8.5)$$

$$B \mathcal{X}_I = \mathcal{X}_{II} B = 0. \quad (8.6)$$

One solution branch is given by setting the bifundamentals $A = B = 0$. Then a, b are free parameters. We have a 2-dimensional moduli space. This is natural to interpret as being parameterized by the D3-brane segments between the D5 and NS5' while everything else is fixed. So this is a sort of Higgs branch (of self-dual type).

If A or B is not strictly zero, then we have to set $b = a^2$ to find solutions. The solutions for the bifundamentals are most easily written as an outer product

$$A = \begin{pmatrix} 1 \\ -a \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix}, \quad (8.7)$$

$$B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix}, \quad (8.8)$$

with the constraint

$$x_1 x_3 + x_2 x_4 = 0. \quad (8.9)$$

This branch of moduli space is \mathbb{C} (parameterized by a) times the conifold. The dimension of moduli space comes out to what one expects based on counting degrees of freedom in a brane diagram, but the fact that the complex structure is that of a conifold would have required additional data not contained in the brane diagram. The Nahm analysis, however, contains this data. Perhaps it is also possible to arrive at the same conclusion from non-perturbative consideration of the field theory in the electric formalism.

8.2 D5—2D3—NS5—2D3—D5'—1D3—D5'

Next we consider the other type of flavor: D5—2D3(I)—NS5—2D3(II)—D5'—1D3—D5'. Again we have

$$\mathcal{X}_I = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}, \tag{8.10}$$

$$\mathcal{X}_{II} = 0, \tag{8.11}$$

but now because of the NS5 brane we have

$$\mathcal{X}_I = AB, \tag{8.12}$$

$$\mathcal{X}_{II} = BA. \tag{8.13}$$

These actually require the trace and determinant of \mathcal{X}_I to vanish, so we have to set $a = b = 0$. This fixes the form of A, B :

$$A = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}, \tag{8.14}$$

$$B = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}, \tag{8.15}$$

with the constraint

$$x_1x_3 + x_2x_4 = 1, \tag{8.16}$$

parameterizing a deformed conifold.

We still have to consider the \mathcal{Y} equations, $\mathcal{Y}_I = 0$, $A\mathcal{Y}_{II} = \mathcal{Y}_IA = 0$ and $\mathcal{Y}_{II}B = B\mathcal{Y}_I = 0$. These actually fix the form of \mathcal{Y}_{II} to

$$\mathcal{Y}_{II} = p \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix}, \tag{8.17}$$

with a free coefficient p .

So the total moduli space appears to be \mathbb{C} times the deformed conifold. There is only one branch, which we might be able to interpret as a quantum-merged branch of moduli space. Once again, the form of the complex structure would have been impossible to infer from inspection of the brane diagram.

8.3 $U(1)^2 \times U(2)$ example

Another interesting quiver theory is defined by the brane configuration NS5—1D3—NS5—2D3—NS5'—1D3—NS5'. We see that it has gauge group $U(1)^2 \times U(2)$.

It is best to analyze this theory in the S-dual frame where the gauge symmetry is broken. The S-dual is D5—1D3—D5—2D3—D5'—1D3—D5' or in other words, it is a D5 ordinary Dirichlet boundary condition for $U(2)$ gauge theory on the left and a D5' ordinary Dirichlet boundary condition on the right. The brane configuration is pictured in figure 26.

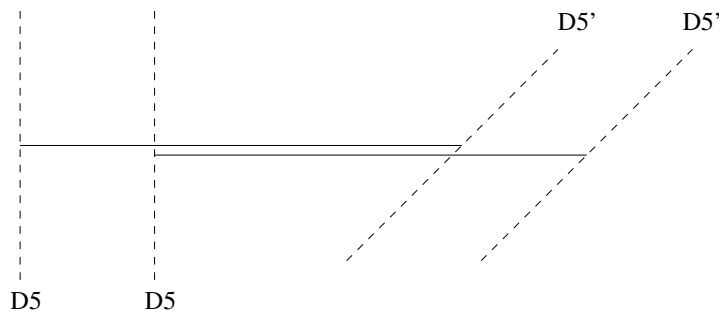


Figure 26. The S-dual of a $U(1) \times U(2) \times U(1)$ quiver theory. The brane configuration can be understood as $U(2)$ gauge theory with ordinary Dirichlet boundary conditions of D5 type on the left and $D5'$ type on the right.

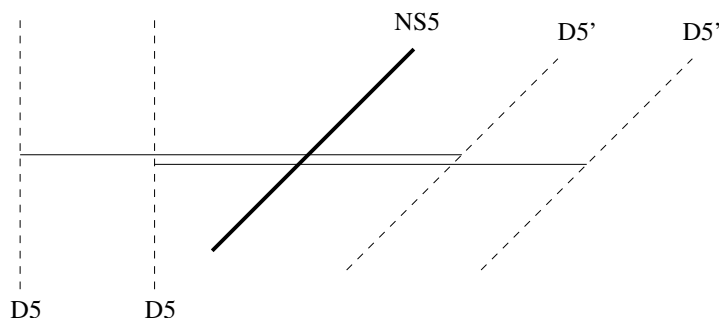


Figure 27. The S-dual of a $U(1) \times U(2) \times U(1)$ quiver theory with one flavor added to the $U(2)$ gauge group.

The Nahm equations force $\mathcal{X} = \mathcal{Y} = 0$ in all regions. The moduli space is still not quite trivial because we have to fix the gauge carefully. We are not allowed to fix $\mathcal{A} = 0$ but only $\mathcal{A} = \text{constant}$. This leaves a four-dimensional moduli space.

Now suppose we add a flavor by adding an NS5-brane to the S-dual, as in figure 27. So now we have D5 ordinary Dirichlet on the left and $D5'$ ordinary Dirichlet on the right. These boundary conditions set

$$\mathcal{X}_L = \text{any}, \tag{8.18}$$

$$\mathcal{X}_R = 0, \tag{8.19}$$

$$\mathcal{Y}_L = 0, \tag{8.20}$$

$$\mathcal{Y}_R = \text{any}. \tag{8.21}$$

The additional constraints we need to impose are

$$\mathcal{X}_L = AB, \tag{8.22}$$

$$\mathcal{X}_R = BA = 0, \tag{8.23}$$

$$A\mathcal{Y}_R = 0, \tag{8.24}$$

$$\mathcal{Y}_R B = 0. \tag{8.25}$$

If we write

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \tag{8.26}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \tag{8.27}$$

the constraints from $BA = 0$ are

$$a_1 b_1 + a_3 b_2 = 0, \tag{8.28}$$

$$a_2 b_1 + a_4 b_2 = 0, \tag{8.29}$$

$$a_3 b_1 + a_4 b_3 = 0, \tag{8.30}$$

$$a_3 b_2 + a_4 b_4 = 0. \tag{8.31}$$

It appears that there are four constraints, but actually only three of them are independent. In particular, when A, B are essentially generic, they can be written as

$$a_1 a_4 - a_2 a_3 = 0, \tag{8.32}$$

$$b_1 b_4 - b_2 b_3 = 0, \tag{8.33}$$

$$a_1 b_1 + a_3 b_2 = 0. \tag{8.34}$$

The constraints degenerate, of course, if either $A = 0$ or $B = 0$.

When $A \neq 0$ and $B \neq 0$, we have $\det A = 0$ and $\det B = 0$ so they both have a single zero eigenvector. By taking the outer product of these eigenvectors we can construct \mathcal{Y}_R up to an overall normalization. So \mathcal{Y}_R contributes one complex modulus. From A and B we get five complex moduli. Note that if we start by thinking about B as parameterizing a conifold, then A contains two moduli that are completely unconstrained. So the moduli space for this branch is the conifold times \mathbb{C}^3 .

Next, suppose that $A = 0$ and B is completely generic. This fixes $\mathcal{Y}_R = 0$ and we have a four-dimensional branch of moduli space which is just \mathbb{C}^4 . There is a second copy of this branch from interchanging A and B .

Or, if $A = 0$ and B has vanishing determinant (but is otherwise generic) then \mathcal{Y}_R contributes two moduli. So this branch is the conifold times \mathbb{C}^2 . This branch also has a second copy.

Finally, if $A = 0$ and $B = 0$ then \mathcal{Y}_R is completely free and there is a \mathbb{C}^4 branch of moduli space. We stress that the complex structure as determined by this analysis should be quantum exact, on all of the branches.

Suppose we give a complex mass to the quarks. In the S-dual picture this comes from adding a complex FI term from moving the NS5 in the 45 directions. We need to change the moment map to

$$\mathcal{X}_L = AB - m\mathbb{I}, \tag{8.35}$$

$$\mathcal{X}_R = BA - m\mathbb{I} = 0. \tag{8.36}$$

This sets $B = A^{-1}$. Because A, B are invertible, their determinants are nonzero and they have no vanishing eigenvalues. This forces $\mathcal{Y} = 0$.

Of course, adding a complex mass returns us to the unflavored case, so we just recovered the result that the moduli space is $4d$. Because the defining equation is $a_1 a_4 - a_2 a_3 \neq 0$ the moduli space is the group manifold $\text{GL}(2, \mathbb{C})$.

9 Discussion

In this article, we applied the junction and boundary conditions of $\mathcal{N} = 4$ SYM in 3+1 dimensions preserving 1/4 of the supersymmetries (which was worked out in [1]) to analyze the moduli space of a defect/impurity system on an interval. Such a theory naturally flows in the IR to an $\mathcal{N} = 2$ SYM in 2+1 dimensions. The main input is the boundary condition and the generalized Nahm equations reviewed in section 2 and appendix B. The main output is the mapping out of the moduli space of examples with varying degrees of sophistication in sections 4–8.

By analyzing several examples in detail, we illustrated how one can reliably extract the complex structure of the moduli space, from the classical analysis for the branch of moduli spaces where the gauge group is completely broken. On branches where some gauge symmetry remain unbroken, the classical analysis is subject to quantum corrections. Nonetheless, in many cases, one can reliably extract the quantum corrected moduli space classically by analyzing the S-dual defect/impurity system.

Using these techniques, we are able to reproduce all of the known moduli space structures for $U(1)$, and $U(2)$ theories with $\mathcal{N} = 2$ supersymmetry in 2+1 dimensions [8–10]. As one might expect with fewer supersymmetries, the structure of the moduli space for the $\mathcal{N} = 2$ models is far more intricate than their $\mathcal{N} = 4$ counterparts. We also found that some issues such as stability [16, 17] play a critical role in mapping out the moduli space of $\mathcal{N} = 2$ theories.

Despite these subtleties, the analysis of the boundary/impurity system eventually boils down to an algebraic analysis of solutions to the generalized Nahm equations with appropriate boundary/junction conditions. Combined with S-duality, this formalism appears to know about many subtle dynamical effects including the s -rule, splitting of the Coulomb branch into multiple branches, merging of various branches, etc. Some effects, such as that of instanton generated superpotential lifting certain branches of the moduli space, arise in non-Abelian examples. These effects can be accounted for classically in the S-dual frame, allowing one to infer the at least part of the structure of the exact moduli space with relative ease.

Perhaps the most striking observation is that intricate dynamics of non-perturbative effects such as generation of superpotentials, quantum merging, and the s -rule on the electric side is reproduced faithfully through the intricacy of non-commutativity of non-abelian field configurations.

The power of this approach over direct analysis in 2+1 dimensions is that in a number of examples, no guess-work regarding the form of the superpotential or the mirror description was necessary. From the point of view of the defect/impurity theory on an interval, the

theory as well as its S-dual is defined microscopically. Using this construction, one can specify, in the UV, a system which may not admit a Lagrangian description in the IR, and map out the moduli space.

One can think of the program being employed in this work in the same broad class as the approach to study field theories in 3+1 dimensions using M-theory [7, 36]. In both of these approaches, the field theory of interest is embedded in some UV framework, which enable one to apply broader set of analytical tools. The regime of validity and reliability of these tools is always an issue in interpreting the analysis for the original field theory system, but for certain classes of observables and features, one can take advantage of non-renormalization theorems. The generalized Nahm equations and the interface/junction conditions described in [1] are playing roles analogous to the holomorphic curves characterizing the M5-brane worldvolume in the approach of [7, 36]. One advantage of the generalized Nahm analysis is the fact that the UV completion is field theoretic and does not rely on the full machinery of string theory or M-theory.

These impurity systems, of course, are conveniently engineered as zero slope limit of brane constructions. We have focused mainly on the case where the $\mathcal{N} = 2$ system are constructed only using the NS5, D5, NS5', and D5' branes. The formalism can also be generalized to the (p, q) 5-brane boundary/junction conditions, a problem to which we hope to return in the future.

Acknowledgments

We would like to thank P. Argyres, T. Clark, D. Gaiotto, K. Intriligator, S. Khlebnikov, M. Kruczenski, G. Michalogiorgakis, N. Seiberg, B. Willett, E. Witten, D. Xie and K. Yonekura for discussion.

This work is supported in part by the DOE grant DE-FG02-95ER40896 (A.H.), by DOE grant DE-FG02-91ER40681 (P.O.) and partly by Princeton Center for Theoretical Science, by Institute for Advanced Study and by WPI program, MEXT, Japan (M.Y.).

This work was initiated at the SPOCK meeting (University of Cincinnati, 2011) and we would like to thank the organizers for the meeting. M.Y. would like to thank Aspen Center for Physics, Newton Institute (Cambridge), Simons Center (Stony Brook), YITP (Kyoto), and KITP (UCSB) for hospitality where part of this work has been performed. The content of this paper was presented at presentations by A.H. (Michigan, Dec. 2012, KITP Apr. 2014, Princeton June 2014), P.O. (Purdue, Nov. 2013, Madison Apr. 2014) and M.Y. (IAS, Apr. 2013, Mar. 2014; Kavli IPMU, Aug. 2013; Perimeter, Jan. 2014; KITP, Feb. 2014), and we thank the audiences of these talks for feedback.

A Coulomb branch of $\mathcal{N} = 4, 2$, $N_c = 1$ theories with N_f flavors

In this appendix, we review the complex geometry of the Coulomb branch of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories with general N_f .

A.1 Coulomb branch of $\mathcal{N} = 4$, $N_c = 1$, N_f theory

Let us revisit an old result, the Coulomb branch moduli space for $\mathcal{N} = \text{U}(1)$ gauge theory with N_f flavors. Because we are interested in the Coulomb branch, we work in the S-dual frame where the brane construction is $\text{D5} - \text{NS5} - \text{NS5} - \dots - \text{NS5} - \text{D5}$.

In this $\mathcal{N} = 4$ example, we can set $\mathcal{Y} = \mathcal{Z} = 0$ in all regions. As for \mathcal{X} , we have a sequence of relations at each NS5 interface,

$$A_1 B_1 = A_2 B_2 = \dots = A_{N_f} B_{N_f} \equiv x, \quad (\text{A.1})$$

when all the complex masses (of the electric theory) vanish. With the complex mass deformations turned on, we have instead

$$A_1 B_1 - m_1 = A_2 B_2 - m_2 = \dots = A_{N_f} B_{N_f} - m_{N_f} \equiv x. \quad (\text{A.2})$$

To analyze the moduli space, it is convenient to define the gauge invariant quantities

$$a \equiv \prod_i A_i, \quad (\text{A.3})$$

$$b \equiv \prod_i B_i, \quad (\text{A.4})$$

which satisfy the constraint equations

$$ab = \prod (x + m_i). \quad (\text{A.5})$$

We see that we have $N_f - 1$ complex parameters which act as deformations of the complex structure, and one ‘‘center of mass’’ of the complex deformations which acts trivially. The moduli space is evidently $\mathbb{C}^2 / \mathbb{Z}_{N_f}$ with deformations induced by the complex masses.

A.2 Coulomb branch of $\mathcal{N} = 2$, $N_c = 1$, N_f theory

To repeat the preceding computation for $\text{U}(1)$ $\mathcal{N} = 2$ theory, we need to rotate one of the D5-branes to $\text{D5}'$, so that the brane configuration is $\text{D5} - \text{NS5} - \text{NS5} - \dots - \text{NS5} - \text{D5}'$. The analysis is the same as before, except that we have to set

$$x = 0. \quad (\text{A.6})$$

The result is that the complex structure of the Coulomb branch is

$$ab = \prod (m_i). \quad (\text{A.7})$$

If any of the complex masses vanishes, we are simply left with

$$ab = 0, \quad (\text{A.8})$$

which is a $1d$ moduli space with two branches. If all the complex masses are nonzero, we have instead $ab = \text{const}$ which is simply a cylinder. This recovers the result that the Coulomb branch of $\text{U}(1)$ $\mathcal{N} = 2$ is $\mathbb{C} \oplus \mathbb{C}$ except when $N_f = 0$, for which it is $\mathbb{R} \times S^1$.

B Review of 1/4 BPS boundary conditions

In this section, we review the supersymmetry conditions for these defect systems. The derivations of these equations as well as detailed explanations may be found in our earlier paper. The spirit of the analysis follows the treatment of Gaiotto and Witten [20, 21].

There are two especially important classes of boundary conditions for this paper, which one can think of as corresponding to some number of D3-branes either ending on or intersecting a D5-brane or an NS5-brane.

At a D5-like interface with N D3-branes on each side, the interface conditions are

$$\Delta\mathcal{X}(y_0) = Q\tilde{Q}, \tag{B.1}$$

$$i\Delta X_6 = QQ^\dagger - \tilde{Q}^\dagger\tilde{Q}, \tag{B.2}$$

$$\mathcal{Y}Q = 0, \tag{B.3}$$

$$\tilde{Q}\mathcal{Y} = 0, \tag{B.4}$$

where the fields Q (\tilde{Q}) are complex scalars which transform in the fundamental (anti fundamental) representation of $U(N)$. These boundary conditions are derived from the effective theory of D3-branes intersecting a D5-brane, as discussed in [37, 38]. They are equivalent to the standard jumping equations used to find monopole solutions; see [19] for a review. For a D5'-brane interface, the analogous conditions are obtained by exchanging \mathcal{X} and \mathcal{Y} :

$$\Delta\mathcal{Y}(y_0) = Q\tilde{Q}, \tag{B.5}$$

$$i\Delta X_6 = QQ^\dagger - \tilde{Q}^\dagger\tilde{Q}, \tag{B.6}$$

$$\mathcal{X}Q = 0, \tag{B.7}$$

$$\tilde{Q}\mathcal{X} = 0. \tag{B.8}$$

These conditions can be generalized by displacing the D5-brane in the $X_{7,8,9}$ directions (or displacing D5'-branes in the $X_{4,5,9}$ directions). This will amount to shifting the scalars in the equations (B.3), (B.4), (B.7), (B.8) by appropriate constants proportional to the identity. In the field theory these deformations correspond to turning on masses for the quarks.

When the gauge groups on the two sides of the D5 interface are unequal, the conditions are a little more subtle, see [1] for details. It is possible to understand them by starting with the case where there are equal gauge groups on the two sides of the interface and then taking expectation values of the interface fields Q , \tilde{Q} to partially Higgs the gauge group on one side of the interface.

For an NS5 oriented along 012789 with N D3 ending from the left and M D3 ending from the right, we impose the junction conditions

$$\mathcal{X}_L = AB - \zeta_c \mathbb{I}_L, \tag{B.9}$$

$$\mathcal{X}_R = BA - \zeta_c \mathbb{I}_R, \tag{B.10}$$

$$iX_{6,L} = AA^\dagger - B^\dagger B - \zeta_r \mathbb{I}_L, \tag{B.11}$$

$$iX_{6,R} = A^\dagger A - BB^\dagger - \zeta_r \mathbb{I}_R, \tag{B.12}$$

$$\mathcal{Y}_L A = A \mathcal{Y}_R, \tag{B.13}$$

$$B \mathcal{Y}_L = \mathcal{Y}_R B. \tag{B.14}$$

We can also generalize these conditions for the case of NS5' brane junction oriented along 012459 by exchanging \mathcal{X} and \mathcal{Y} .

$$\mathcal{Y}_L = AB - \zeta_c \mathbb{I}_L, \tag{B.15}$$

$$\mathcal{Y}_R = BA - \zeta_c \mathbb{I}_R, \tag{B.16}$$

$$iX_{6,L} = AA^\dagger - B^\dagger B - \zeta_r \mathbb{I}_L, \tag{B.17}$$

$$iX_{6,R} = A^\dagger A - BB^\dagger - \zeta_r \mathbb{I}_R, \tag{B.18}$$

$$\mathcal{X}_L A = A \mathcal{X}_R, \tag{B.19}$$

$$B \mathcal{X}_L = \mathcal{X}_R B. \tag{B.20}$$

We have included FI deformations ζ_c and ζ_r . The complex FI term ζ_c corresponds to changing the positions of the NS5(NS5')-branes in the 45(78) directions, while the real FI term comes from moving the NS branes in the x^6 direction.

In addition to the simple D5 and NS5 interfaces, one can build more complicated composite boundaries from multiple 5-branes. Details about such composite boundaries may be found in [1, 20, 21].

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