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A Goldstone theorem for continuous non-invertible symmetries

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Abstract: We study systems with an Adler-Bell-Jackiw anomaly in terms of non-invertible symmetry. We present a new kind of non-invertible charge defect where a key role is played by a local current operator localized on the defect. The charge defects are now labeled by elements of a continuous (1). We use this construction to prove an analogue of Goldstone's theorem for such non-invertible symmetries. We comment on possible applications to string theory.

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1 Introduction

Global symmetries and their associated conservation laws are one of the most fundamental tools that we have for the quantitative understanding of nature. One of the many uses of conventional global symmetries is to characterize phases of systems and their low-energy dynamics. To take one example, whenever a conventional continuous global symmetry is spontaneously broken, Goldstone's theorem guarantees that a gapless mode is present in the spectrum, and the low-energy dynamics is essentially completely characterized by the pattern of symmetry breaking.

Recently, our understanding of global symmetries has undergone something of a renaissance. The idea that the existence of conserved quantities can be recast in terms of the topological surface operators who count the charge has led to powerful generalizations of the very concept of "symmetry". Two such generalizations which will concern us in this paper are those of *higher-form symmetries* — i.e. the symmetries associated with the conservation of extended objects [1] — and *non-invertible symmetries*, i.e. symmetries for which the charges do not obey a simple group composition law [2–34]. See e.g. [35] for a recent pedagogical review of some of these developments.

In particular, consider a system (such as conventional QED with a single massless Dirac fermion) where a current j^A is nonconserved due to an Adler-Bell-Jackiw anomaly [36, 37], i.e.

$$d \star j^A = \frac{1}{4\pi^2} F \wedge F \tag{1.1}$$

where the operator $F \wedge F$ on the right-hand side is constructed from a *dynamical* (1) photon. It has recently been shown that though the naive symmetry is explicitly broken by the anomaly, this is not a "generic" kind of breaking; instead the system can be understood in terms of a novel kind of non-invertible symmetry [14, 15]. This precise characterization opens new doors for a non-perturbative understanding of systems exhibiting such an anomaly.

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In this work, we present a variation of this construction that permits us to prove a Goldstone theorem for such systems; i.e. we prove using Euclidean partition function techniques that if an operator $\mathcal{O}$ charged under the non-invertible symmetry has a non-zero vacuum expectation value, there must exist a gapless mode in the spectrum. As an application, we show how many of the massless fields of string theory can be understood as Goldstone modes of spontaneously broken non-invertible symmetries; this provides an alternative viewpoint for their masslessness.

Note added. In the last stages of preparation of this paper, [38] appeared, where a charge defect similar to our (2.10) (involving an extra scalar field on the defect) was constructed for different motivations.

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2 Goldstone's theorem for non-invertible symmetries

2.1 Goldstone's theorem in Euclidean formulation

To orient ourselves, we begin by reviewing a simple reformulation of the usual Goldstone theorem in the language of Euclidean path integrals. This argument was first given in [39] to prove a Goldstone theorem for higher-form symmetries.

Consider a Lorentz invariant quantum field theory with a (1) 0-form symmetry, with associated 1-form conserved current j. For simplicity we will restrict ourselves to the case of four spacetime dimensions. The Ward identity for the conserved current j in the presence of a charged operator $\mathcal{O}(x)$ with charge q is

$$d\star j(x)\mathcal{O}(0) = iq\mathcal{O}(0)\delta^{(4)}(x) \tag{2.1}$$

i.e. the current is not quite conserved in the presence of the charged operator. Let us now integrate both sides of this equation over a solid 4-ball of radius R centered at the origin, as in figure 1. We find the equation

$$\left(\int_{S^3(R)} \star j\right)\mathcal{O}(0) = iq\mathcal{O}(0) \tag{2.2}$$

where on the left hand side we have used Stokes theorem; the integral is taken over the boundary of the 4-ball. Finally, take the expectation value of both sides:

$$\left\langle \left(\int_{S^3(R)} \star j\right)\mathcal{O}(0)\right\rangle = iq\langle\mathcal{O}\rangle \tag{2.3}$$

Now, if we are in a phase where the symmetry is spontaneously broken, then $\langle\mathcal{O}\rangle$ is nonzero, and the integral on the left-hand side must be both nonzero and independent of the radius of the 3-sphere R. By spherical symmetry, we see that the correlation of the local operator j on the 3-sphere and $\mathcal{O}$ must therefore depend on R as

$$\langle j^i(x)\mathcal{O}(0)\rangle \sim iqn^iR^{-3} \tag{2.4}$$

where n^i is an outwardly pointing normal vector on the 3-sphere. The dependence on R is fixed by the requirement that the integral over the 3-sphere result in an R-independent

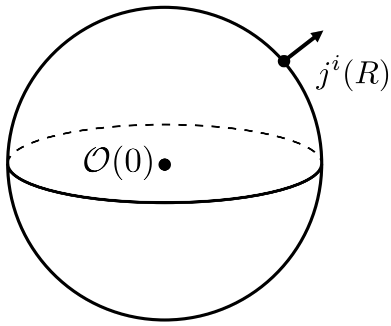

Figure 1. Charge operator defined on an S^3 of radius R wrapping operator $\mathcal{O}(0)$ at the origin.

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constant. Thus there is a power-law correlation in the theory.[1] We have shown the existence of at least one gapless excitation: this is the Goldstone mode.

We note that this is simply a reformulation of the usual Hamiltonian arguments; if we appropriately deform the S^3 and cut open the path integral then we obtain the commutators of j and $\mathcal{O}$ that are used in the standard proofs. This Euclidean formulation will be useful for the generalization that follows.

2.2 Axial symmetry defect operators

We now turn to the actual case of interest. In recent work [14, 15], it has been shown that models exhibiting the Adler-Bell-Jackiw anomaly actually are invariant under a novel kind of non-invertible symmetry. We briefly review some aspects of that discussion here.

Consider massless QED, defined by the action:

$$S[\psi,\bar{\psi},A] = \int d^4x \left(\frac{1}{4e^2}F^2 + i\bar{\psi}\left(\partial\!\!\!/ - iA\!\!\!/\right)\psi + \cdots\right) \tag{2.5}$$

The (1) gauge redundancy acts as

$$\psi \to e^{i\Lambda(x)}\psi \qquad A \to A + d\Lambda\,. \tag{2.6}$$

In this theory the axial current $j^\mu_A = \bar{\psi}\gamma^5\gamma^\mu\psi$ is not conserved. Instead, due to the ABJ anomaly it satisfies the following non-conservation equation [36, 37]:

$$d\star j^A = \frac{1}{4\pi^2}F\wedge F \tag{2.7}$$

We stress that the right-hand-side of this expression is a dynamical operator, and not a fixed external source (as would be the case for a 't Hooft anomaly). There will thus be dynamical violation of axial charge conservation, and we cannot construct a conserved charge in the conventional manner. There is a temptation to instead consider the following current:

$$\star j_A^{\text{not-gauge-inv}} = \star j_A - \frac{1}{4\pi^2}A\wedge dA\,. \tag{2.8}$$

[1]This argument is of course exactly the same as the one used to obtain the inverse square electric field of a point charge using Gauss's law in elementary electrodynamics.

This current is conserved, but as the notation suggests it is not gauge-invariant. Indeed it is not possible to construct a conserved gauge-invariant local current in this theory. However as the *integral* of the Chern-Simons term is gauge-invariant, one may try to construct a charge defect operator as follows:

$$\hat{U}_\alpha(\mathcal{M}_3) = \exp\left(i\frac{\alpha}{2}\int_{\mathcal{M}_3}\left(\star j_A - \frac{1}{4\pi^2}A\wedge dA\right)\right) \tag{2.9}$$

This is a topological operator under small deformations of $\mathcal{M}_3$, and thus is a candidate operator to define a conserved axial charge defect, which acts on an operator $\mathcal{O}$ with integer charge q as $\mathcal{O}\to e^{i\alpha q/2}\mathcal{O}$.

However for general $\mathcal{M}_3$ — in particular those with nontrivial 1-cycles — the expression (2.9) is not gauge-invariant under *large* gauge transformations. Such an invariance would require the coefficient of the Chern-Simons term to be quantized as $\alpha\in 2\pi\mathbb{Z}$, thus resulting in a trivial action on all operators. In [14, 15] this deficiency was remedied by adding extra degrees of freedom — a TQFT — living on $\mathcal{M}_3$. Their construction then permits the topological charge defect operator to be defined on any $\mathcal{M}_3$, provided that the rotation angle α is in $\mathbb{Q}/\mathbb{Z}$. Thus rather than having a continuous (1) symmetry as in the conventional case the defect operators are now labeled by $\mathbb{Q}/\mathbb{Z}$.

In this work however, we seek to generalize the Goldstone theorem described above, and it is important that we have some notion of a local conserved current. Thus we perform a slightly different construction. Consider the following charge defect operator, defined on 3-manifolds Σ:

$$U_\alpha(\mathcal{M}_3) = \int[\mathcal{D}\theta]\exp\left(i\frac{\alpha}{2}\int_{\mathcal{M}_3}\star\tilde{j}_A\right) \equiv \int[\mathcal{D}\theta]\exp\left(i\frac{\alpha}{2}\int_{\mathcal{M}_3}\left(\star j_A - \frac{1}{4\pi^2}(A-d\theta)\wedge dA\right)\right) \tag{2.10}$$

Here we have introduced a new degree of freedom; this is a compact scalar θ that lives only on the 3-manifold $\mathcal{M}_3$, and which transforms under the gauge redundancy (2.6) as $\theta\to\theta+\Lambda$. In the above expression we perform a path integral over θ. Thus the integrand $\star\tilde{j}_A$ is now locally gauge invariant, and there is no rationality constraint on the parameter α, so $e^{i\alpha}\in(1)$, and we have constructed a continuous (1) symmetry. We will show below it is non-invertible.

It should be noted that θ really has almost no dynamics; we will restrict attention only to the case where $\mathcal{M}_3$ is S^3, and in that case this construction can be viewed as a convenient way to extract the gauge-invariant information that is present in the more naive construction (2.9).

We now discuss some properties of the defect operator $U_\alpha(\mathcal{M}_3)$.

1. It is topological under small variations of $\mathcal{M}_3$. This can be seen heuristically by imagining some extension of θ off of the defect, and then noting that $d\star\tilde{j}^A = 0$ for any such extension. In appendix A we provide a somewhat more detailed explanation of this fact.

2. As usual for a 0-form defect operator, when the defect operator is collapsed onto an appropriately charged local operator it performs an axial rotation by an angle α, i.e.

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if $\mathcal{O}(x)$ carries axial charge q then

$$U_\alpha(S^3)\mathcal{O}(x) = e^{\frac{i\alpha q}{2}}\mathcal{O}(x) \tag{2.11}$$

if the S^3 in question is a small sphere wrapping x.

3. We may immediately imagine interesting phenomena whenever θ is allowed to have non-trivial winding, i.e. when (a) $\mathcal{M}_3$ is a manifold with non-trivial 1-cycles, or (b) in the presence of 't Hooft lines. We discuss some such properties — which do not affect our Goldstone proof — in a subsequent section.

2.3 Goldstone's theorem for non-invertible symmetry

We may now use this object to prove a variant of Goldstone's theorem by generalizing slightly the arguments used in section 2.1. We begin by considering the setup of (2.11), where U_α is defined on an S^3 of radius R. Now take a derivative with respect to α and set α to 0: we then find

$$\left\langle i\left(\int_{S^3(R)} \star\tilde{j}_A\right)\mathcal{O}(0)\right\rangle = iq\langle\mathcal{O}\rangle \tag{2.12}$$

Note that the expectation value on the left-hand side now involves the path integral over θ defined on the defect as well. $\tilde{j}_A$ is now defined only on the defect.

Now imagine that we are in a phase where $\langle\mathcal{O}\rangle \neq 0$, i.e. an operator charged under the non-invertible symmetry has a non-vanishing expectation value. Then, just as in (2.4), by spherical symmetry we see that the correlation function must depend on R in a very specific way:

$$\langle\tilde{j}^i_A(R)\mathcal{O}(0)\rangle \sim iqn^i R^{-3} \tag{2.13}$$

with n^i an outwards pointing normal vector on the S^3. We thus see that there is a non-trivial power-law correlation between an operator $\mathcal{O}$ in the bulk and a current $\tilde{j}_A$ defined on the defect. Thus there must exist at least one massless mode in the bulk, which is the Goldstone mode.

In this formulation the proof is essentially the same as in the conventional invertible symmetry case, except that the current $\tilde{j}_A$ is now defined only on the defect. Note that there is no way for the new degree of freedom θ living only on the defect to create such a correlation between bulk and defect operators unless the bulk is gapless.[2] This completes the proof.

The effective theory describing this low energy Goldstone mode (usually called an axion) is well understood. Denoting the axion by ϕ, the first few terms in the derivative expansion are:

$$S[A,\phi] = \int_{\mathbb{R}^4}\left(\frac{1}{2\gamma^2}d\phi\wedge\star d\phi + \frac{1}{2e^2}F\wedge\star F + ig\phi F\wedge F\right) \tag{2.14}$$

[2] It might be helpful to note that the expression above coincides with the *connected* correlator, as the expectation value of the defect-localized current *without* a local operator inserted is $\langle\tilde{j}_A\rangle = 0$, by the same argument as above with $q = 0$. Thus the connected correlator displays power-law behavior, requiring a gapless mode:

$$\langle\tilde{j}^A(R)\mathcal{O}(0)\rangle - \langle\tilde{j}^A(R)\rangle\langle\mathcal{O}\rangle \sim iqR^{-3}\,.$$

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It was shown in [14] that this theory does indeed exhibit the non-invertible symmetry if $g = \frac{1}{4\pi^2}$. It is easy to verify that the ϕ field saturates the Ward identity (2.13), where the charged operator $\mathcal{O}$ is realized as $\mathcal{O} \sim e^{iq\phi}$.

An example of a microscopic theory which realizes this phase is if one adds a complex scalar $\Phi(x)$ and a Yukawa coupling $h\Phi(x)\bar{\psi}\psi + h.c.$ to the Dirac action (2.2), and then arranges a potential $V(\Phi^\dagger\Phi)$ to condense the scalar Φ; $\phi(x)$ may then be viewed as the phase of $\Phi(x)$ (or equivalently the phase of the fermion condensate $\bar{\psi}\psi$).

Another example (discussed in [14]) is QCD with massless quarks, where the (non-invertible) axial symmetry is spontaneously broken, and indeed the anomaly famously provides the dominant channel for pion decay to two photons. We now see that the π_0 can be viewed as a Goldstone boson of this breaking; the effective Lagrangian takes the form above with ϕ identified with the π_0 (and with other interactions involving the photon A).

We make two statements about generalizations of this Goldstone theorem:

1. It may appear that the construction above requires access to a weakly coupled photon description of the theory. However the essential information of the ABJ anomaly can be phrased in a universal way[3] in terms of the combined symmetry structure of a conserved 2-form current J and the non-conserved axial current j^A:

$$d \star j^A = \frac{1}{4\pi^2} J \wedge J \qquad d \star J = 0 \tag{2.15}$$

   We believe the argument can be extended to any theory with this structure. The second equation implies that we can locally write $J = \frac{1}{2\pi} \star dA$ in terms of an *effective* photon A. A will not have a simple action, but we can nevertheless use it to construct the operator (2.10) on a topologically trivial S^3 and run the argument above.

2. Similar arguments can be made for a higher-form continuous non-invertible symmetry (e.g. as in [19, 20]) by changing the dimensionality of the defect manifold $\mathcal{M}_3$ and of the charged operator $\mathcal{O}$ (which will generically become an extended object), as was done in [39, 41] for (invertible) higher-form symmetries. The statement then is that if the charged object exhibits a perimeter law (i.e. its expectation value depends only locally on geometric data characterising its worldvolume), then there exists a gapless mode in the spectrum.

2.4 Behaviour in the non-zero flux sector

For the proof above we were only required to construct the defect operator U_α on an S^3 embedded inside $\mathbb{R}^4$. In the interest of completeness, we would like to highlight some properties of this defect on more general manifolds, particularly those with non-trivial homology groups.

Let us consider formulating our bulk theory on $S^1 \times S^1 \times S^2$. We are interested in studying a sector of the path integral where there is (1) electromagnetic flux $2\pi n$ on the

[3] See e.g. [40] where a similar characterization was used to understand finite-temperature physics from holography.

S^2, i.e.

$$\int_{S^2} F = 2\pi n \tag{2.16}$$

where $n \in \mathbb{Z}$.

Now let us place the defect operator $U_\alpha(\mathcal{M}_3)$ on an $\mathcal{M}_3 = S^1 \times S^2$, where it fills one of the two S^1's, i.e. we evaluate

$$\langle U_\alpha(\mathcal{M}_3)\rangle = \left\langle \exp\left(i\frac{\alpha}{2}\int_{\mathcal{M}_3}\left(\star j^A - \frac{1}{4\pi^2}(A-d\theta)\wedge dA\right)\right)\right\rangle . \tag{2.17}$$

We consider performing the path integral over θ first, keeping A fixed. Note that the dependence on θ is only through its characteristic class (i.e. the winding), and any other θ in the same class can be obtained by adding a topologically trivial θ to a fixed representative of its characteristic class. We have

$$\int_{\mathcal{M}_3} d\theta \wedge dA = 4\pi^2 wn \tag{2.18}$$

with $w \in \mathbb{Z}$ the winding of θ on the S^1 factor. Large gauge transformations act as

$$G : (a, w) \sim (a + n, w + n) \tag{2.19}$$

where a is the holonomy of A on the S^1. In this calculation we treat A as a background and we do not impose these identifications when summing over w; in a more complete treatment they will be taken into account when performing the full path integral over A (where they reduce the effective range of the holonomy).

The result of the path integral over θ, including the winding sectors, is therefore proportional to

$$\sum_{w\in\mathbb{Z}} \exp\left(i\frac{\alpha wn}{2}\right) . \tag{2.20}$$

Performing the sum over w, we find a delta function in αn that only has support if

$$\frac{\alpha n}{2} \in 2\pi\mathbb{Z} . \tag{2.21}$$

We thus see that for irrational $\frac{\alpha}{2\pi}$ our operator annihilates the path integral unless $n = 0$. This is a very non-invertible operator indeed. It would be interesting to further understand the properties of this defect and compare to the operator of [14, 15], which is defined for rational $\frac{\alpha}{2\pi}$.

3 Goldstone bosons and string theory

One of our original motivations for seeking an extension of Goldstone's theorem that applied to non-invertible symmetries was the observation that in string theory there are fields which appear to be exactly massless, even in non-supersymmetric configurations (as in [44–48], for instance). The masslessness of some of these fields is usually explained from the fact that they are connections for higher-form abelian gauge symmetries.

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This argument is somewhat unsatisfactory on its own; a gauge redundancy is after all a statement about a *description* of a system. We should instead seek an explanation in terms of realizations of (possibly approximate) *global* symmetries. Given our lack of understanding of the non-perturbative aspects of string theory, a systematic argument based on symmetry principles alone, valid for arbitrary compactifications, seems to be of some value.[4]

Consider, as an example, the operator measuring the D4 "Page" charge [50][5,6]

$$\hat{U}_\alpha(\Sigma_4) = \exp\left(2\pi i\alpha \int_{\Sigma_4} \tilde{F}_4 + A_1 \wedge H_3\right) , \tag{3.1}$$

where $\tilde{F}_4 \equiv dA_3 - A_1 \wedge H_3$, A_1 and A_3 are RR potentials, and $H \equiv dB_2$ is the NSNS field strength. Here Σ_4 is a 4-surface linking a 5-manifold X_5 where we have placed a D4 brane. We include no other background fields, and for avoidance of tadpoles we take the directions transverse to the D4 to be non-compact.

Importantly, we will first treat the D4 brane as an infinitely heavy object, so that we may consider it as a defect operator $D(X_5)$ in the effective supergravity field theory. We will eventually incorporate the fact that the tension is finite below.

In this setting — when the D4 branes are infinitely massive — $\hat{U}_\alpha(\Sigma_4)$ is topological, and would naively define a 5-form (1) symmetry. This is not so for a number of reasons. First, for generic α the resulting expression is not gauge invariant under large gauge transformations of A_1 if $H^1(\Sigma_4; \mathbb{R}) \neq 0$, and H_3 generic. We can fix these issues by taking Σ_4 to be S^4. We also note that the background we discuss does not source H_3, so we might be tempted to simply claim that $U_\alpha(\Sigma_4)$ is gauge invariant. But our goal is to argue that the presence of Goldstone bosons is robust, so we will not set $H_3 = 0$. Instead, we introduce an additional dynamical scalar on the charge defect, as above:

$$U_\alpha(\Sigma_4) = \exp\left(2\pi i\alpha \int_{\Sigma_4} \tilde{F}_4 + (A_1 - d\theta) \wedge H_3\right) , \tag{3.2}$$

with θ transforming as $\theta \to \theta + \lambda$ under a $A_1 \to A_1 + d\lambda$ gauge transformation. This defines a non-invertible symmetry analogous to (2.10).

The D4 brane is charged under $U_\alpha(\Sigma_4)$, so we may write:

$$U_\alpha(\Sigma_4) D(X_5) = e^{i\alpha} D(X_5) \tag{3.3}$$

Now the insertion of the defect operator $D(X_5)$, i.e. a D4 wrapping X_5, in the string theory partition function will give a contribution proportional to the volume of X_5, arising from the Dirac-Born-Infeld coupling on the brane worldvolume. In terms of standard language, it obeys a "perimeter law", so this nonzero expectation value implies a form of spontaneous symmetry breaking. We can, in particular, run a straightforward generalisation of the argument given above (see [39, 41] for details) that implies the existence of a Goldstone

[4] A different argument for the masslessness of the photon, based on swampland considerations, was given in [49].

[5] See [51] for other notions of charge.

[6] The integrand $\tilde{F}_4 + A_1 \wedge H_3$ can alternatively be written as dA_3, but the way we have written it makes it clearer that it is not gauge invariant under gauge transformations of A_1.

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mode. The minimal possibility saturating (3.3) is that we have a 5-form C_5 in the spectrum, with a coupling

$$\exp\left(i\int_{X_5} C_5\right) \tag{3.4}$$

to the D4 brane worldvolume. We identify this massless field as the ordinary RR 5-form field.

Clearly, this argument generalises with minor modifications to all branes in string theory, implying the exact masslessness of the fields that they are charged under.

So far we have worked in a semi-classical approximation to string theory, where the branes are infinitely massive objects. In actual string theory, this is not the case: branes are dynamical objects, and this means that there are no longer any exact global symmetries or completely topological objects [52–54]. Nevertheless it seems reasonable that a large mass for these objects should not affect questions in the extreme infrared, such as the masslessness of putative Goldstone modes. An argument to this effect for 1-form symmetries was given in [55]. In our context, the main physical effect is expected to be screening from "virtual" D4-branes. This makes the value of the measured Page charge depend on the distance R between the surface where we measure the charge and the brane, i.e. (3.3) becomes

$$U_\alpha(\Sigma_4)D(X_5) = e^{i\alpha f(R)}D(X_5) \tag{3.5}$$

for some nontrivial function of R. We are not aware of a systematic treatment of this effect for the case at hand, but related cases are analysed in detail in [56]. Extrapolating the main features of the cases discussed in that paper, we expect that the operator we have described above will be topological — i.e. $f(R\to\infty)$ approaches a constant — up to exponentially suppressed effects, for scales larger than the inverse of the brane tension.

This is good news: any small mass for the would be Goldstone would dramatically affect our argument for large enough radii (the relevant correlator would become exponentially suppressed), but it is precisely in this regime that the screening effect itself becomes exponentially suppressed. So we conclude that screening effects, while they do break the symmetry, cannot give a mass to the Goldstone boson. This is perhaps not much of a surprise: giving mass to a Goldstone boson for a higher form symmetry, since it is vector-like, is a discontinuous operation, so these Goldstone modes are much more robust than ordinary 0-form Goldstone scalars (which have the same degrees of freedom whether they are massive or not).

A summary of what we have just found is that the massless form fields in string theory are massless because branes are heavy. This motivates a final speculative comment: if we try to think what it would take to gap the spectrum of supergravity, by the arguments above it will necessarily involve making the branes tensionless. This resonates well with the idea that gapped phases of gravity should be thought of as phases where the metric vanishes. (We refer the reader to [57, 58] and the final comments in [59] for further references and discussion of such ideas.) Indeed recent work on emergent higher-form symmetries associated with the gravitational sector alone [60, 61] may eventually allow us to make such ideas concrete.

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A Topological invariance of defect operator

Here we discuss the topological properties of the new coupling added to the action of the defect operator in (2.10), i.e.

$$\exp\left(i\frac{\alpha}{8\pi^2}\int_{\mathcal{M}_3} d\theta\wedge F\right) \tag{A.1}$$

In particular, when placed in the path integral, is this invariant under deformations of $\mathcal{M}_3$?

Heuristically, if we extend θ off of the defect, then we can consider deforming $\mathcal{M}_3$ to a nearby surface $\mathcal{M}_3'$; comparing the integral over the two regions we then find we then find

$$\int_{\mathcal{M}_3} - \int_{\mathcal{M}_3'} (d\theta\wedge F) = \int_{\mathcal{M}_4} d(d\theta\wedge F) = 0 \tag{A.2}$$

where $\partial\mathcal{M}_4 = \mathcal{M}_3\cup\mathcal{M}_3'$.

We can present a slightly more abstract formulation of the argument which shows the relationship between the above relation and the 1-form symmetry. Consider coupling an external 2-form source b using the 1-form symmetry of QED:

$$S[\psi,\bar{\psi},A;b] = \int d^4x\left(\frac{1}{4e^2}F^2 + i\bar{\psi}\left(\partial\!\!\!/ - iA\!\!\!/\right)\psi + \frac{i}{2\pi}b\wedge F\right) \tag{A.3}$$

(Such a coupling is possible for any conserved 2-form current J, where in this particular case we have $J\equiv\frac{1}{2\pi}F$). As b couples to a conserved 2-form current, it is always true that

$$Z[b] = Z[b + d\Lambda] \tag{A.4}$$

for an arbitrary 1-form Λ.

Now we turn on the following source for b, parametrized by a choice of 3-surface $\mathcal{M}_3$:

$$b = b_{\mathcal{M}_3} = \frac{\alpha}{4\pi}d\left(\Theta_{\mathcal{M}_3}(x)d\theta(x)\right) \tag{A.5}$$

Here $\theta(x)$ is a scalar field defined on $\mathbb{R}^4$ which will eventually be restricted to the defect worldvolume $\mathcal{M}_3$. $\Theta_{\mathcal{M}_3}(x)$ is a scalar function with $\Theta_{\mathcal{M}_3}(x) = 0$ if x is "on one side" of $\mathcal{M}_3$ and $\Theta_{\mathcal{M}_3}(x) = 1$ if x is on the "other side" of $\mathcal{M}_3$. (E.g. if we were studying the simple case where $\mathcal{M}_3$ is the flat plane $x_0 = 0$, we would have $\Theta_{\mathcal{M}_3}(x) = \Theta(x_0)$ with Θ the usual Heaviside step function.) To make the manipulations well-defined, let us imagine that this interpolation from 0 to 1 takes place over a small length scale.

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Now as the choice for b in (A.5) is exact, the invariance (A.4) means that the partition function does not change under inserting this source. In particular it does not care about the choice of $\mathcal{M}_3$, independently of the value of θ, i.e.

$$Z[b_{\mathcal{M}_3}] = Z[b_{\mathcal{M}'_3}] \tag{A.6}$$

This invariance also does not care about details on how the Heaviside function above is regulated. However in the limit where the step function is made arbitrarily sharp, we find that $d\Theta_{\mathcal{M}_3}$ becomes a 1-form localized on the defect, and we have:

$$\frac{i}{2\pi}\int_{\mathbb{R}^4} b_{\mathcal{M}_3} \wedge F = \frac{i\alpha}{8\pi^2}\int_{\mathcal{M}_3} d\theta \wedge F \tag{A.7}$$

Thus we have constructed the coupling desired and confirmed that it is invariant under shifts of $\mathcal{M}_3$. At the final stage we see that it depends only on $\theta(x)$ on the defect. This argument is essentially the same as that leading to (A.2), but this presentation highlights the relationship between the topological character of the operator and the unbroken 1-form symmetry.

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