Exact QFT duals of AdS black holes

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ABSTRACT: We construct large N saddle points of the matrix model for the $\mathcal{N} = 4$ Yang-Mills index dual to the BPS black holes in $AdS_5 \times S^5$, in two different setups. When the two complex chemical potentials for the angular momenta are collinear, we find linear eigenvalue distributions which solve the large N saddle point equation. When the chemical potentials are not collinear, we find novel solutions given by areal eigenvalue distributions after slightly reformulating the saddle point problem. We also construct a class of multi-cut saddle points, showing that they sometimes admit nontrivial filling fractions. As a byproduct, we find that the Bethe ansatz equation emerges from our saddle point equation.

Contents

1 Introduction

We want to better understand the microscopic black hole physics in AdS/CFT [1–3]. For quantitative studies, BPS black holes [4–7] are ideal objects. In 2005, [8, 9] defined the indices of 4d SCFTs and explored their large N behaviors. It took some time to understand how to see the black holes from this index [10–12]. In 4d $\mathcal{N} = 4$ Yang-Mills theory, we study the black holes in $AdS_5 \times S^5$. For this problem, one should study an apparently complicated large N matrix integral (see section 2 for more precise statements),

$$Z(\delta_I, \sigma, \tau) \sim \frac{1}{N!} \prod_{a=1}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} du_a \cdot \prod_{a \neq b} \frac{\prod_{I=1}^{3} \Gamma(\delta_I + u_{ab}, \sigma, \tau)}{\Gamma(u_{ab}, \sigma, \tau)} \ , \quad u_{ab} \equiv u_a - u_b \,, \qquad (1.1)$$

where $\sum_I \delta_I = \sigma + \tau$ (mod $\mathbb{Z}$). $\Gamma(z, \sigma, \tau)$ is the elliptic gamma function. In this paper, we construct its exact large N saddle points. Here we sketch their important features.

The integration contours of the N eigenvalues u_a are real circles, $u_a \sim u_a + 1$. One has to set the chemical potentials δ_I, σ, τ complex. (See [13, 14] to understand why.) The saddle points for u_a's are also complex. At our saddle point, the eigenvalues are typically distributed *areally*, in a 2 dimensional region on the complex u plane. We have not heard of areal distributions in holomorphic matrix models: one finds linear eigenvalue distributions called 'cuts'. Our claim is true only after carefully setting up the saddle point problem as we explain shortly.

We explain our basic saddle point which corresponds to the black hole solutions of [7]. The saddle point depends on two parameters σ, τ of the elliptic gamma function, related to the chemical potentials for the spatial rotations. u_a are distributed uniformly inside a parallelogram, whose two edge vectors are given by the complex numbers σ, τ. See figure 1a. This is typically a 2d distribution when σ, τ are not collinear. When σ, τ are

(a) Uniform parallelogram distribution on the complex u plane (left); linear degeneration at $\frac{\sigma}{\tau} \to$ real $\neq 1$ (middle); linear degeneration at $\frac{\sigma}{\tau} \to 1$ (right).

(b) Eigenvalue density $\rho(x)$ at $\frac{\sigma}{\tau} = \frac{1}{2}$ (middle case of figure 1a).

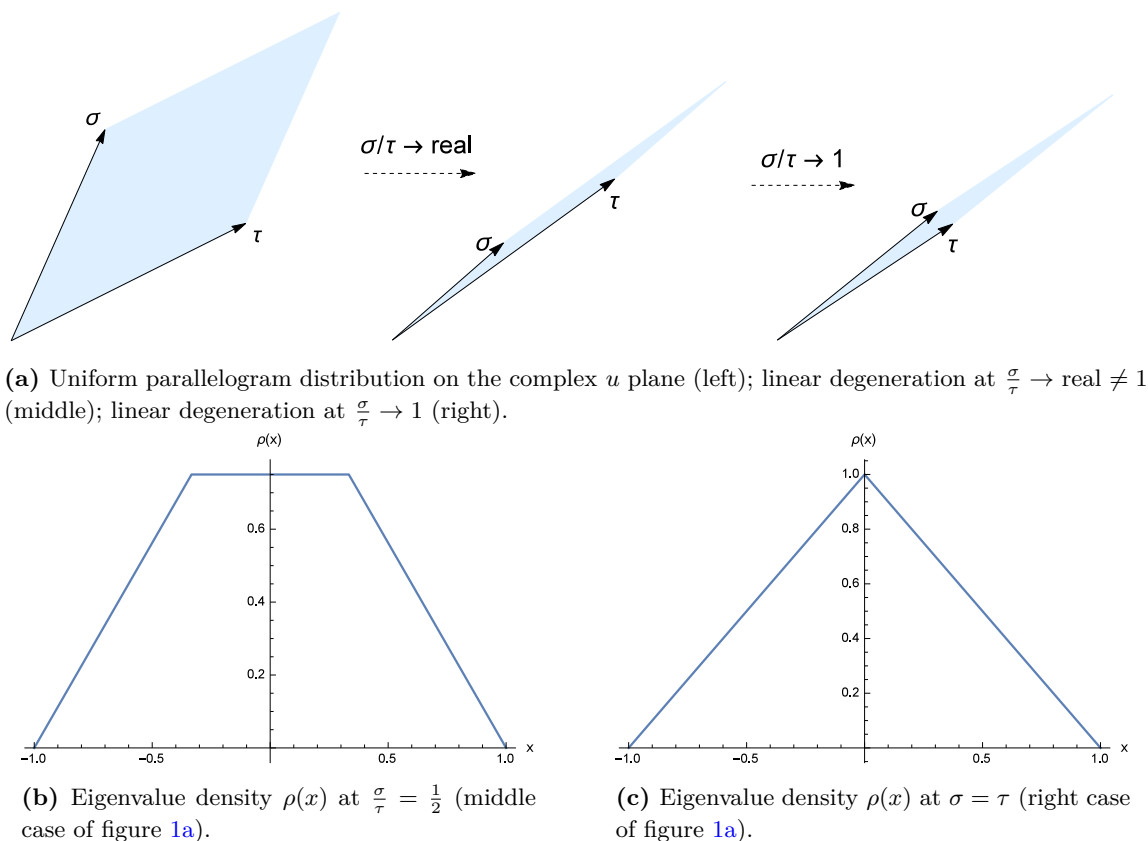**(c)** Eigenvalue density $\rho(x)$ at $\sigma = \tau$ (right case of figure 1a).

Figure 1. Illustrating the uniform parallelogram distribution, and how it degenerates to linear cuts. In figures (b) and (c), $x \in (-1, 1)$ parametrizes the linear cut.

collinear, the parallelogram degenerates to a line. In this limit, the density function on the line is given by a trapezoid: see figure 1b. When $\sigma = \tau$, corresponding to equal angular momenta $J_1 = J_2$, the linear density function is triangular: see figure 1c. These linear density functions were discovered recently in the small [14] and large [15] black hole limits, using different approaches. Our parallelogram ansatz was inferred by guessing how to naturally get the trapezoid limit.

For general non-collinear σ and τ, this distribution solves the following saddle point equation. We first note the integral identity (whose proof is reviewed in section 2)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d^N u \cdot \frac{1}{N!} \prod_{a<b} (1 - e^{2\pi i \kappa u_{ab}})(1 - e^{-2\pi i \kappa u_{ab}}) \cdot f(u) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d^N u \cdot \prod_{a<b} (1 - e^{2\pi i \kappa u_{ab}}) \cdot f(u), \quad (1.2)$$

which holds for any constant κ and any permutation-invariant function $f(u)$. For $\kappa = 1$, the factor $\frac{1}{N!} \prod_{a<b} (1 - e^{2\pi i u_{ab}})(1 - e^{-2\pi i u_{ab}})$ is the Haar measure of the unitary matrix integral and this identity leads to the so-called Molien-Weyl formula [16]. In our problem, we shall write (1.1) in a way that it looks like the left hand side of (1.2) at $\kappa = \frac{1}{\sigma}$ or $\frac{1}{\tau}$. One can set up the saddle point problem using either the left or right hand sides: integrals are the same, but the saddle point problems are slightly different. Our parallelogram solves the saddle point problem defined using the right hand side. See section 2 for the precise setup

of this problem.[1] As for the saddle point equation of the left hand side, presumably more complicated saddles exist with same physical properties, but we have nothing concrete to say at generic non-collinear σ, τ.

In the collinear limit, $\frac{\sigma}{\tau} \to$ real, the parallelogram degenerates to linear distributions. In this case, our ansatz solves both saddle point equations, defined using the left and right hand sides of (1.2). We explain how the saddle point equation of the left hand side is solved in section 3. Furthermore, at $\sigma = \tau$, we show that the saddle point equation of the left hand side of (1.2) is related to the Bethe ansatz equation for this index [18, 19], once employing our ansatz.

Our basic saddle points account for the physics of the 4-parameter BPS black hole solutions in $AdS_5 \times S^5$ [7], carrying three angular momenta in S^5 and two in AdS_5.

We also find multi-cut solutions. For technical reasons, we only discuss the case with collinear σ, τ. We find K-cut saddles given by K parallelograms degenerated as figure 1. When $\sigma = \tau$, they are related to some Bethe roots of [20], labeled by two integers K, r. Our K-cut solutions are related to the Bethe roots at $r = 0$, and sometimes admit generalization with unequal filling fractions. The solutions at $r \neq 0$ may be generated using the methods of appendix A. (At $\sigma \neq \tau$, the extra parameter r is further refined to two integers r, s.)

This paper is organized as follows. In section 2, we derive the parallelogram saddle point. In section 3, we discuss alternative derivation of the saddles at collinear σ, τ. Section 3.1 explains a relation to the Bethe ansatz at $\sigma = \tau$. In section 4, we derive the $K \geq 2$ cut saddle points. Section 5 concludes with comments and future directions. Appendix A discusses the (r, s)-refinements.

2 Basic setup and the areal saddle points

To highlight the key ideas, we first discuss general matrix models satisfying certain conditions and derive the saddle points given by areal distribution. We shall later show that the matrix model for the $\mathcal{N} = 4$ index belongs to this class. Consider the U(N) unitary matrix U diagonalized to $U = \text{diag}(e^{2\pi i u_1}, \cdots, e^{2\pi i u_N})$. We consider the following 'matrix integral'

$$Z = \frac{1}{N!} \prod_{a=1}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} du_a \cdot \prod_{a \neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\tau}}\right) \cdot \exp\left[-\sum_{a \neq b}(V_\sigma(u_{ab}) + V_\tau(u_{ab}))\right], \qquad (2.1)$$

where $u_{ab} \equiv u_a - u_b$. The potential consists of two terms, $V_\sigma(u)$ and $V_\tau(u)$, which we take to be holomorphic and periodic in two different directions given by the complex numbers σ, τ:

$$V_\sigma(u + \sigma) = V_\sigma(u), \quad V_\tau(u + \tau) = V_\tau(u). \qquad (2.2)$$

The functions V_σ, V_τ are assumed to contain no singularities such as poles or branch points in the region of u_a's to be specified below.

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[1]It is not uncommon to rewrite integrals using the symmetry of the integrand to find simplified saddle point solutions. For instance, this idea was used in the S^3 partition functions of large N quiver SCFTs [17].

We would like to apply (1.2) at $\kappa = \frac{1}{\tau}$ to this integral. Before this, we review how to prove this identity. First note that

$$\frac{1}{N!}\prod_{a<b}(1-e^{x_a-x_b})(1-e^{x_b-x_a})=\frac{1}{N!}\prod_{a<b}(e^{x_b}-e^{x_a})(e^{-x_b}-e^{-x_a})\equiv\frac{1}{N!}\prod_{a<b}(\lambda_b-\lambda_a)(\lambda_b^{-1}-\lambda_a^{-1})\,,$$
$$(2.3)$$

where $\lambda_a \equiv e^{x_a} \equiv e^{2\pi i \kappa u_a}$. We then recall the following formula for the Vandermonde matrix:

$$\prod_{a<b}(\lambda_b-\lambda_a)=\det(\lambda_a^{b-1})=\sum_{\rho\in S_N}(-1)^{\epsilon(\rho)}\prod_{a=1}^{N}(\lambda_{\rho(a)})^{a-1}\,,\tag{2.4}$$

and similar formula for $\prod_{a<b}(\lambda_b^{-1} - \lambda_a^{-1})$. S_N is the permutation group, and $\epsilon(\rho)$ is the signature of its element ρ. Applying these formulae, (2.3) can be written as

$$\frac{1}{N!}\sum_{\rho,\sigma\in S_N}(-1)^{\epsilon(\rho)}(-1)^{\epsilon(\sigma)}\prod_{a=1}^{N}(\lambda_{\rho(a)})^{a-1}(\lambda_{\sigma(a)}^{-1})^{a-1}\,.\tag{2.5}$$

At fixed σ, one can relabel ρ as $\rho \cdot \sigma$ and write

$$\frac{1}{N!}\sum_{\sigma}\sum_{\rho}(-1)^{\epsilon(\rho)}\prod_{a=1}^{N}(\lambda_{\rho(\sigma(a))})^{a-1}(\lambda_{\sigma(a)}^{-1})^{a-1}=\frac{1}{N!}\sum_{\sigma}\left[\prod_{a}(\lambda_{\sigma(a)})^{-(a-1)}\right]\cdot\left[\prod_{a<b}(\lambda_{\sigma(b)}-\lambda_{\sigma(a)})\right]$$
$$=\frac{1}{N!}\sum_{\sigma}\prod_{a<b}(1-\lambda_{\sigma(a)}\lambda_{\sigma(b)}^{-1})\,.\tag{2.6}$$

Now consider the integral (1.2) with the Haar measure like factor rewritten as the second line of (2.6). This is given by the sum of $N!$ integrals labeled by $\sigma \in S_N$, divided by $N!$. Since $f(u)$ and the integral domain $-\frac{1}{2} < u_a < \frac{1}{2}$ are all invariant under the permutations, all $N!$ integrals yield same values. Therefore one proves the identity (1.2).

So our matrix integral can be rewritten as

$$Z=\prod_{a=1}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}}du_a\cdot\prod_{a<b}\left(1-e^{\frac{2\pi i u_{ab}}{\tau}}\right)\cdot\exp\left[-\sum_{a\neq b}(V_\sigma(u_{ab})+V_\tau(u_{ab}))\right]\,.\tag{2.7}$$

The half-Haar measure like factor can also be exponentiated and contribute to the potential! as

$$-V\leftarrow\sum_{a<b}\log\left(1-e^{\frac{2\pi i u_{ab}}{\tau}}\right)\,.\tag{2.8}$$

One can regard it as modifying the τ-periodic potential V_τ. We write Z as

$$Z=\prod_{a=1}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}}du_a\cdot\exp\left[-\sum_{a\neq b}(V_\tau^{(\mathrm{sgn}(a-b))}(u_{ab})+V_\sigma(u_{ab}))\right]\,,\tag{2.9}$$

where

$$V_\tau^{(+)}(u)\equiv V_\tau(u)\,,\quad V_\tau^{(-)}(u)\equiv V_\tau(u)-\log\left(1-e^{\frac{2\pi i u}{\tau}}\right)\,.\tag{2.10}$$

If the extra term in $V^{(-)}$ does not contain branch points in the region of our interest, $V^{(\pm)}$ will also be holomorphic.

We shall find a large N saddle point of (2.9), and deform the integral contour to reach this saddle point. The integral domain $u_a \in \left(-\frac{1}{2}, \frac{1}{2} \right)$ has a boundary. (In our Yang-Mills matrix model, the original problem is not on an interval due to the $u_a \sim u_a + 1$ periods, but the integrand after applying (1.2) is not periodic.) Knowledgeable readers may be concerned that, when the integral contour has a boundary, the saddle point problem is well-posed only if the integrand vanishes at the boundary. (More formally, this is the condition for the Picard-Lefschetz theory of saddle point approximation to be applicable.) We can rephrase the saddle point approximation of our integral in the standard fashion, along a noncompact contour with the integrand vanishing at infinity, as follows. Relabeling the integral variables as

$$u_a = \frac{1}{2} \tanh x_a \,, \tag{2.11}$$

the domain $\left(-\frac{1}{2}, \frac{1}{2} \right)$ for u_a maps to the real axis $(-\infty, \infty)$ for x_a. The saddle point problem can then be phrased in terms of x_a, in which case one finds the extra contribution

$$-V(x) \leftarrow \sum_{a=1}^{N} \log \left(\frac{1}{2} \text{sech}^2 x_a \right) \tag{2.12}$$

added to the eigenvalue potential due to the Jacobian factor. Taking derivative of the net potential with x_a, the force from the original potential is at N^1 order, while that from the added potential (2.12) is at the subleading N^0 order and is thus negligible. (The Jacobian factor may well matter when computing subleading corrections in $1/N$.) Therefore, even if one considers a holomorphic integral along compact intervals, the saddle points of the original potential ignoring (2.12) are relevant for the large N saddle point approximation.

Since the additional potential (2.12) induced by relabelling $u_a \to x_a$ does not affect the leading large N saddle points, we keep working with u_a. In the large N limit, consider the uniform distribution of N eigenvalues on a parallelogram in the u space. In the continuum description, the eigenvalues label a can be replaced by a tuple of continuous parameters $a \to (x, y)$ where $-\frac{1}{2} < x, y < \frac{1}{2}$. Our uniform parallelogram ansatz is given by

$$u(x, y) = x\sigma + y\tau \,, \quad \rho(x, y) = 1 \,. \tag{2.13}$$

$\rho(x, y)$ is the uniform areal density function satisfying $\int dx dy \rho(x, y) = 1$. Since the integrand of (2.7) breaks Weyl invariance, we order u_a's as follows. Here we assume $\text{Im}(\frac{\sigma}{\tau}) > 0$, and order the N eigenvalues in a way that $x_a > x_b$ if $a < b$. Then (2.8) has no branch points in the region $u_a = x_a\sigma + y_a\tau$, $x_a, y_a \in (-\frac{1}{2}, \frac{1}{2})$ satisfying $x_a > x_b$ if $a < b$, since

$$\left| e^{\frac{2\pi i u_{ab}}{\tau}} \right| = \left| e^{\frac{2\pi i \sigma x_{ab}}{\tau}} \right| < 1 \tag{2.14}$$

for $a < b$. This means that we can stay in the principal branch of log for the configuration (2.13) and its small deformations, implying that $V_\sigma, V_\tau^{(\pm)}$ is holomorphic in this region.[2] We now show that (2.13) is a large N saddle point of the integral (2.9). We

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[2]One may think that the new V_τ defined piecewise by $V_\tau^{(\pm)}(x\sigma + y\tau)$ has a step function singularity at $x = 0$. But in the fine-grained discrete picture, we can assume that no two eigenvalues have precisely same values of x, with minimal differences at order $N^{-\frac{1}{2}}$.

consider the following force acting on the eigenvalue u_a:

$$\sum_{b(\neq a)}\left[\frac{\partial}{\partial u_a}V_\sigma(u_{ab}) + \frac{\partial}{\partial u_a}V_\tau^{\text{sgn}(a-b)}(u_{ab})\right] = -\sum_{b(\neq a)}\left[\frac{1}{\sigma}\frac{\partial}{\partial x_b}V_\sigma(u_{ab}) + \frac{1}{\tau}\frac{\partial}{\partial y_b}V_\tau^{\text{sgn}(a-b)}(u_{ab})\right].$$

We used $\frac{\partial}{\partial u_a} \sim -\frac{\partial}{\partial u_b}$ and also the fact that $u_b = x_b\sigma + y_b\tau$ derivatives can be replaced by either $\frac{1}{\sigma}\partial_{x_b}$ or $\frac{1}{\tau}\partial_{y_b}$. In the large N continuum limit, the sum over b is replaced by an integral over x_b, y_b with the areal eigenvalue density $\rho(x,y) = 1$:

$$-N\int_0^1 dx_2 dy_2 \left[\frac{1}{\sigma}\partial_{x_2}V_\sigma(u_{12}) + \frac{1}{\tau}\partial_{y_2}V_\tau^{\text{sgn}(x_2-x_1)}(u_{12})\right] . \tag{2.15}$$

Both terms in the integrand separately integrate to zero. For the first term, one finds

$$\int_0^1 dx_2 \partial_{x_2}V_\sigma(u_{12}) = V_\sigma(u_1 - \tau y_2 - \sigma) - V_\sigma(u_1 - \tau y_2) = 0 \tag{2.16}$$

because V_σ is periodic in σ shift. As for the second term, one integrates over y_2 first and similarly finds zero, from the periodicity of $V_\tau^{\text{sgn}(x_2-x_1)}$. This statement is invalid if branch points are inside the integral domain. They are absent partly by assumption on V_τ, and also because the integrand contains a half-Haar-like measure only. If one starts from the saddle point problem with (2.1), one indeed finds a nonzero force from the terms with branch points.

We found a class of matrix models which admits the uniform parallelogram solution: (1) The eigenvalue potential decomposes to two functions V_σ, V_τ which have their respective periods; (2) The potentials do not have branch points in the parallelogram domain (2.13); (3) The matrix integrand is changed to contain a half-Haar-like measure, using the identity (1.2).

Now we show that the matrix model for the index of the $\mathcal{N} = 4$ Yang-Mills theory belongs to this class, which will be proving that our parallelogram distribution is a saddle point. The index is defined by the trace over the Hilbert space of the radially quantized CFT [8, 9]:

$$Z(\Delta_I, \omega_i) \equiv \text{Tr}\left[(-1)^F e^{-\sum_{I=1}^3 Q_I\Delta_I}e^{-\sum_{i=1}^2 J_i\omega_i}\right] \tag{2.17}$$

where Q_I, J_i are $\text{U}(1)^3 \subset \text{SO}(6)$ R-charges and $\text{U}(1)^2 \subset \text{SO}(4)$ angular momenta. Q_I, J_i are half-integrally quantized for spinors. Z is defined on a 4-parameter space of chemical potentials, $\sum_{I=1}^3 \Delta_I - \sum_{i=1}^2 \omega_i = 0$, apparently mod $4\pi i\mathbb{Z}$ at this moment from the definition (2.17). The chemical potentials should satisfy $\text{Re}(\Delta_I) > 0$ and $\text{Re}(\omega_i) > 0$ for the trace to be well-defined. The matrix integral for the index is given by (1.1), which we repeat here:

$$Z(\delta_I, \sigma, \tau) = \frac{1}{N!}\prod_{a=1}^N \int_{-\frac{1}{2}}^{\frac{1}{2}} du_a \cdot \prod_{a\neq b}\frac{\prod_{I=1}^3 \Gamma(\delta_I + u_{ab}, \sigma, \tau)}{\Gamma(u_{ab}, \sigma, \tau)} \cdot \left[\text{U}(1)^N \text{ part}\right] . \tag{2.18}$$

The 'U$(1)^N$ part' is the contribution from N diagonal matrix components of fields, which is independent of the integral variable u_a and makes an $\mathcal{O}(N^1)$ contribution to the free

energy. As we are interested in the leading (nonzero) free energy of order N^2, we shall not distinguish the two expressions (1.1) and (2.18) in this paper. The parameters δ_I, σ, τ are defined by

$$\Delta_I = -2\pi i \delta_I \,, \quad \omega_1 = -2\pi i \sigma \,, \quad \omega_2 = -2\pi i \tau \,. \tag{2.19}$$

The elliptic gamma function is defined by

$$\Gamma(z,\sigma,\tau) \equiv \prod_{m,n=0}^{\infty} \frac{1 - e^{-2\pi iz}e^{2\pi i((m+1)\sigma+(n+1)\tau)}}{1 - e^{2\pi iz}e^{2\pi i(m\sigma+n\tau)}} \,. \tag{2.20}$$

It satisfies $\Gamma(z,\sigma,\tau) = \Gamma(z+1,\sigma,\tau) = \Gamma(z,\sigma+1,\tau) = \Gamma(z,\sigma,\tau+1)$. (Other properties of Γ will be presented below when necessary.) So the chemical potentials δ_I, σ, τ all have period 1 in the expression (2.18). They define an index on the following 4-parameter space:

$$\sum_{I=1}^{3} \delta_I = \sigma + \tau \quad \mod \mathbb{Z} \,. \tag{2.21}$$

They should further satisfy $\mathrm{Im}(\delta_I) > 0$, $\mathrm{Im}(\sigma) > 0$, $\mathrm{Im}(\tau) > 0$.

We first set a convenient parametrization of δ_I's, for given σ,τ. We set $\delta_I = -a_I + b_I(\sigma+\tau)$ with real coefficients a_I, b_I. From (2.21), the coefficients should satisfy $a_1 + a_2 + a_3 \in \mathbb{Z}$ and $b_1 + b_2 + b_3 = 1$. We shall first fix the ranges of a_I, b_I. From $\mathrm{Im}(\delta_I) > 0$ and $\delta_1 + \delta_2 + \delta_3 = \sigma + \tau \mod \mathbb{Z}$, one finds $0 < \mathrm{Im}(\delta_I) < \mathrm{Im}(\sigma+\tau)$. So all b_I's are in the range $b_I \in (0,1)$. The ranges of a_I are fixed partly by convention, using the periodicities of δ_I. Making shifts $\delta_I \to \delta_I + n_I$ with integral n_I's, each a_I can be put in a certain interval of length 1. For instance, one can set $a_I \in (0,1)$ for all $I = 1,2,3$. With this choice, one finds that $0 < a_1 + a_2 + a_3 < 3$, so we should take either $a_1 + a_2 + a_3 = 1$ or 2.[3] In the former case, we have $\delta_1 + \delta_2 + \delta_3 - \sigma - \tau = -1$ with $a_I \in (0,1)$. In the latter case, it is more convenient to redefine a_I's by shifting all of them by -1, so that $a_I \in (-1,0)$ and $a_1 + a_2 + a_3 = -1$. In this convention, we take $\delta_1 + \delta_2 + \delta_3 - \sigma - \tau = +1$ with $a_I \in (-1,0)$. To summarize, possible δ_I's belong to one of the following two cases:

$$\delta_1 + \delta_2 + \delta_3 = \sigma + \tau \mp 1, \quad \delta_I = -a_I + b_I(\sigma+\tau), \quad \pm a_I \in (0,1), \quad b_I \in (0,1). \tag{2.22}$$

In the unrefined index with all equal R-charges, we set $b_I = \frac{1}{3}$ and $a_I = \pm\frac{1}{3}$ for all I, respectively. The two cases define mutually complex conjugate regions in the chemical potential space, in which the real parts of all δ_I, σ, τ are sign-flipped. In other words, if the complex numbers (δ_I, σ, τ) belong to the upper case, $(-\delta_I^*, -\sigma^*, -\tau^*)$ belong to the lower case.

Now consider the following two identities of $\Gamma(z,\sigma,\tau)$ [21, 22]

$$\Gamma(z,\sigma,\tau) = e^{-\pi i Q_+(z,\sigma,\tau)} \Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) \Gamma\left(\frac{-z-1}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)$$

$$\Gamma(z,\sigma,\tau) = e^{-\pi i Q_-(z,\sigma,\tau)} \Gamma\left(-\frac{z}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) \Gamma\left(\frac{z-1}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) \tag{2.23}$$

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[3]Here and below, we shall be loose about the boundary values, e.g. not sharply distinguishing $a_I \in (0,1)$ and $a_I \in [0,1]$. But of course the latter is correct and there are two more isolated choices $a_1 + a_2 + a_3 = 0$ or 3. These two cases are equivalent, corresponding to all a_I being 0. But this point is again equivalent to a special point of (2.22), e.g. the upper case with $a_1 = a_2 = 0$, $a_3 = 1$.

where

$$Q_\pm = \frac{z^3}{3\sigma\tau} - \frac{\sigma + \tau \mp 1}{2\sigma\tau}z^2 + \frac{\sigma^2 + \tau^2 + 3\sigma\tau \mp 3\sigma \mp 3\tau + 1}{6\sigma\tau}z + \frac{1}{12}(\sigma + \tau \mp 1)\left(\frac{1}{\sigma} + \frac{1}{\tau} \mp 1\right).$$
(2.24)

These are part of the $SL(3, \mathbb{Z})$ transformation identities. One identity can be obtained from another by complex-conjugating an identity and regarding $(-z^*, -\tau^*, -\sigma^*)$ as the new (z, σ, τ) parameters. We shall use these identities to rewrite the integrands of the matrix model at $\text{Im}(\frac{\sigma}{\tau}) > 0$, in which case all the modular parameters have positive imaginary parts. For $\text{Im}(\frac{\sigma}{\tau}) < 0$, we can use the identities with the role of σ, τ flipped and proceed similarly. The limit of collinear σ and τ, i.e. $\text{Im}(\frac{\sigma}{\tau}) \to 0$, is singular with each Γ function. However, the collinear limit can be smoothly taken after all the calculation is done. This limit will be discussed in section 3.

It is convenient to use the first/second line of (2.23) for the two cases of (2.22) with upper/lower signs, respectively. The integral for the upper case of (2.22) can be written as

$$Z = \exp\left[-\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}\right] \frac{1}{N!} \int \prod_{a=1}^{N} du_a \cdot \exp\left[-\sum_{a \neq b}\left(V_\sigma(u_{ab}) + \tilde{V}_\tau(u_{ab})\right)\right]$$
(2.25)

where

$$-V_\sigma(u) \equiv \frac{1}{2}\sum_{I=1}^{3} \log\Gamma\left(-\frac{\delta_I + u + 1}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) - \frac{1}{2}\log\Gamma\left(-\frac{u+1}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) + (u \to -u)$$

$$-\tilde{V}_\tau(u) \equiv \frac{1}{2}\sum_{I=1}^{3} \log\Gamma\left(\frac{\delta_I + u}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) - \frac{1}{2}\log\Gamma\left(\frac{u}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + (u \to -u),$$
(2.26)

and for the lower case of (2.22) can be written as

$$Z = \exp\left[-\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}\right] \frac{1}{N!} \int \prod_{a=1}^{N} du_a \cdot \exp\left[-\sum_{a \neq b}\left(\tilde{V}_\sigma(u_{ab}) + V_\tau(u_{ab})\right)\right]$$
(2.27)

where

$$-\tilde{V}_\sigma(u) \equiv \frac{1}{2}\sum_{I=1}^{3} \log\Gamma\left(-\frac{\delta_I + u}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) - \frac{1}{2}\log\Gamma\left(-\frac{u}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right) + (u \to -u)$$

$$-V_\tau(u) \equiv \frac{1}{2}\sum_{I=1}^{3} \log\Gamma\left(\frac{\delta_I + u - 1}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) - \frac{1}{2}\log\Gamma\left(\frac{u-1}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + (u \to -u).$$
(2.28)

Here we averaged over the contributions from positive/negative roots to have $V_{\sigma,\tau}, \tilde{V}_{\sigma,\tau}$ to be even functions of u. The prefactor $\exp\left[-\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}\right]$ is obtained by collecting all four $e^{-\pi i Q_\pm}$ factors of the identity. Namely, one obtains

$$-\pi i\left[\sum_{I=1}^{3} Q_\pm(\delta_I + u) - Q_\pm(u)\right] + (u \to -u) = -\frac{2\pi i \delta_1 \delta_2 \delta_3}{\sigma\tau} - \frac{2\pi i u^2 \Delta_\pm}{\sigma\tau}$$
(2.29)

$$-\pi i \frac{2\Delta_\pm^3 + 3\Delta_\pm^2(\sigma + \tau \mp 1) + \Delta_\pm\left(\sigma^2 + \tau^2 + 3\sigma\tau \mp 3(\sigma + \tau) + 1 - 6(\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1)\right)}{3\sigma\tau},$$

where $\Delta_\pm \equiv \sum_{I=1}^{3} \delta_I - \sigma - \tau \pm 1$. Since $\Delta_\pm = 0$ in the two cases of (2.22), respectively, this factor simplifies to a constant $-\frac{2\pi i \delta_1 \delta_2 \delta_3}{\sigma \tau}$ in both cases. Then we can take $\frac{N^2 - N}{2} \approx \frac{N^2}{2}$ such factors out of the integral to obtain the prefactor of (2.25).

Since the analysis is completely the same in the two cases of (2.22), here we only discuss the case with upper signs. Apart from the prefactor, (2.25) takes the form of (2.1), as we explain now. First of all, the Γ functions appearing in V_σ and $\tilde{V}_\tau$ are periodic in shifting u by σ and τ, respectively. So if one does not cross the branch cuts for $\log \Gamma$ as one parallel shifts u by σ and τ, one would find $V_\sigma(u + \sigma) = V_\sigma(u)$, $\tilde{V}_\tau(u + \tau) = \tilde{V}_\tau(u)$. We will show that V_σ has no branch points in the parallelogram region defined by $u_a = \sigma x_a + \tau y_a$ with $x_a, y_a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and that all log functions of the form $\log(1 - x)$ can be defined by Taylor expansion $-\sum_{n=1}^{\infty} \frac{x^n}{n}$. On the other hand, $\tilde{V}_\tau$ will have branch points/cuts from a Haar-measure-like factor which looks like the left hand side of (1.2) with $\kappa = \frac{1}{\tau}$. After one proves these two assertions above, one can separate the dangerous Haar-measure-like factor and define $V_\tau(u)$ by

$$e^{-\sum_{a \neq b} \tilde{V}_\tau(u_{ab})} = \prod_{a \neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\tau}}\right) \cdot e^{-\sum_{a \neq b} V_\tau(u_{ab})}, \qquad (2.30)$$

and apply (1.2) to derive a matrix model of the form of (2.7). Then from our general discussion earlier, the parallelogram saddle point is trivially derived.

Now we only need to prove the key assertion, about the absence of branch points for V_σ and also for V_τ defined by (2.30), in the region $u_a = x_a \sigma + y_a \tau$, $u_b = x_b \sigma + y_b \tau$ with $x_a, x_b, y_a, y_b \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. We shall prove this when certain conditions for δ_I, σ, τ are met. Namely, we shall be deriving the 'basic parallelogram saddle point' in a certain region of the parameter space. In appendix A, we derive more general saddle points, virtually for any value of δ_I, σ, τ.

We first show that $\tilde{V}_\tau$ contains branch points only from the Haar measure like factor

$$\prod_{a \neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\tau}}\right). \qquad (2.31)$$

It suffices to show that $\Gamma\left(\frac{\delta_I + u_{12}}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)$ and $\Gamma\left(\frac{u_{12}}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)$ has no zeros or poles when u_1, u_2 are both in the parallelogram, except for the zeros from (2.31). We first consider the function

$$\Gamma\left(\frac{\delta + u_{12}}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right) = \prod_{m,n=0}^{\infty} \frac{1 - e^{2\pi i \left(-\frac{\delta + u_{12}}{\tau} - \frac{m+1}{\tau} + \frac{(n+1)\sigma}{\tau}\right)}}{1 - e^{2\pi i \left(\frac{\delta + u_{12}}{\tau} - \frac{m}{\tau} + \frac{n\sigma}{\tau}\right)}}, \qquad (2.32)$$

where $\delta \equiv -a + b(\sigma + \tau)$ with $a, b \in (0, 1)$ is either $\delta_{1,2,3}$. If the exponential factors appearing in the denominator always have their absolute values smaller than 1 for any m, n, the denominator will not have poles. This is true if the following imaginary part is always positive:

$$\text{Im}\left[\frac{\delta + u_{12}}{\tau} - \frac{m}{\tau} + \frac{n\sigma}{\tau}\right] = \text{Im}\left(-\frac{1}{\tau}\right)[m + a + \mathcal{N}(n + b + x_{12})] \qquad (2.33)$$

where $\mathcal{N} \equiv \frac{\text{Im}\left(\frac{\sigma}{\tau}\right)}{\text{Im}\left(-\frac{1}{\tau}\right)}$ is a positive number. One finds

$$m + a + \mathcal{N}(n + b + x_{12}) \geq a - \mathcal{N}(1 - b), \qquad (2.34)$$

where the inequality is saturated at $m = n = 0$ and $x_{12} = -1$. Therefore, this quantity remains positive when $\mathcal{N} < \frac{a}{1-b}$. This is one of the conditions that we shall assume for the chemical potentials. Now considering the numerator of (2.32), we similarly check if the imaginary part of the exponent is always positive. The relevant imaginary part is

$$\text{Im}\left[\frac{-\delta - u_{12}}{\tau} - \frac{m+1}{\tau} + \frac{(n+1)\sigma}{\tau}\right] = \text{Im}\left(-\frac{1}{\tau}\right)[m+1-a+\mathcal{N}(n+1-b-x_{12})] \geq 1 - a - b\mathcal{N}, \qquad (2.35)$$

where the inequality is saturated at $m = n = 0$ and $x_{12} = 1$. So we find that the imaginary part is positive when $\mathcal{N} < \frac{1-a}{b}$, which we also assume. So for

$$\mathcal{N} \equiv \frac{\text{Im}\left(\frac{\sigma}{\tau}\right)}{\text{Im}\left(-\frac{1}{\tau}\right)} < \min\left[\frac{a_I}{1-b_I}, \frac{1-a_I}{b_I}\right], \qquad (2.36)$$

$\log\Gamma\left(\frac{\delta_I + u_{12}}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)$ is holomorphic in our u_a domain. We can make a similar analysis for

$$\Gamma\left(\frac{u_{12}}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)^{-1} = \prod_{m,n=0}^{\infty} \frac{1 - e^{2\pi i\left(\frac{u_{12}}{\tau} - \frac{m}{\tau} + \frac{n\sigma}{\tau}\right)}}{1 - e^{2\pi i\left(-\frac{u_{12}}{\tau} - \frac{m+1}{\tau} + \frac{(n+1)\sigma}{\tau}\right)}}. \qquad (2.37)$$

This is actually repeating the studies of imaginary parts above, at $\delta = 0$. As for the denominator, one similarly finds its log is holomorphic at no extra condition. Log of the numerator at given m, n will be holomorphic if $m + \mathcal{N}(n + x_{12})$ is positive for all $-1 < x_{12} < 1$. (We ignore the edges at $x_{12} = \pm 1$, in that no eigenvalues will sharply assume the edge values in the fine-grained picture. Anyway, such edge factors will make measure 0 contributions in the large N limit.) This condition is always met except when $m = n = 0$.[4] In other words, the only factor in the numerator whose log fails to be holomorphic is collected as (2.31). So, as asserted, the integrand containing $\tilde{V}_\tau$ of (2.26) can be written as (2.30), where V_τ is holomorphic if the conditions (2.36) are met.

The branch cuts/points of V_σ can be studied in a completely analogous way. After the analysis of the poles/zeros of the functions $\Gamma\left(-\frac{\delta + u_{12} + 1}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)$ and $\Gamma\left(-\frac{u_{12} + 1}{\sigma}, -\frac{1}{\sigma}, -\frac{\tau}{\sigma}\right)$, one finds that there are no branch cuts if we assume

$$\frac{\text{Im}\left(-\frac{\tau}{\sigma}\right)}{\text{Im}\left(-\frac{1}{\sigma}\right)} < \min\left[\frac{1 - a_I}{1 - b_I}, \frac{a_I}{b_I}\right]. \qquad (2.38)$$

This follows by repeating the analysis of the previous paragraph.

To summarize, the Yang-Mills matrix model in the sector

$$\delta_1 + \delta_2 + \delta_3 = \sigma + \tau - 1, \quad \delta_I = -a_I + b_I(\sigma + \tau) \text{ with } a_I, b_I \in (0, 1) \qquad (2.39)$$

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[4]One may think an extra condition $\mathcal{N} < 1$ is required for $(m, n) = (1, 0)$, but this is always implied by (2.36).

at $\text{Im}(\frac{\sigma}{\tau}) > 0$ can be written as

$$Z = \exp\left[-\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}\right] \int \prod_{a=1}^{N} du_a \cdot \frac{1}{N!} \prod_{a\neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\tau}}\right) \cdot \exp\left[-\sum_{a\neq b} (V_\sigma(u_{ab}) + V_\tau(u_{ab}))\right],$$
(2.40)

which is basically (2.1) multiplied by a prefactor. V_σ, V_τ satisfy all the required conditions (periods, holomorphicity) supposing that the chemical potentials satisfy the conditions (2.36) and (2.38), which we write more intrinsically as

$$\text{Im}\left(\frac{\sigma - \delta_I}{\tau}\right) < 0, \ \ \text{Im}\left(\frac{1 + \delta_I}{\tau}\right) < 0, \ \ \text{Im}\left(\frac{1 - \tau + \delta_I}{\sigma}\right) < 0, \ \ \text{Im}\left(\frac{\delta_I}{\sigma}\right) > 0. \ \ (2.41)$$

In the same sector (2.39) at $\text{Im}(\frac{\sigma}{\tau}) < 0$ (i.e. $\text{Im}\left(\frac{\tau}{\sigma}\right) > 0$), one rewrites the matrix model by using a modular identity which exchanges the role of σ, τ, arriving at a matrix model of the form (2.40) with σ, τ flipped. The corresponding V_σ, V_τ again satisfy periodicities and holomorphy when the $\sigma \leftrightarrow \tau$ flipped version of (2.41) is met. So applying the identity (1.2) with either $\kappa = \frac{1}{\tau}$ or $\frac{1}{\sigma}$ to set up the saddle point problem, one finds the uniform parallelogram saddle point with the two edges given by σ, τ.

The complex conjugate sector

$$\delta_1 + \delta_2 + \delta_3 = \sigma + \tau + 1, \ \ \delta_I = -a_I + b_I(\sigma + \tau) \ \ \text{with} \ \ -a_I, b_I \in (0, 1) \ \ (2.42)$$

can be studied similarly, by using the second modular transformation of (2.23). If $\text{Im}\left(\frac{\sigma}{\tau}\right) > 0$, one obtains

$$Z = \exp\left[-\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}\right] \int \prod_{a=1}^{N} du_a \cdot \frac{1}{N!} \prod_{a\neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\sigma}}\right) \cdot \exp\left[-\sum_{a\neq b} (V_\sigma(u_{ab}) + V_\tau(u_{ab}))\right],$$
(2.43)

where the potentials V_σ, V_τ are periodic and holomorphic if

$$\frac{\text{Im}\left(\frac{\sigma}{\tau}\right)}{\text{Im}\left(-\frac{1}{\tau}\right)} < \min\left[\frac{1 - |a_I|}{1 - b_I}, \frac{|a_I|}{b_I}\right], \ \ \frac{\text{Im}\left(\frac{-\tau}{\sigma}\right)}{\text{Im}\left(-\frac{1}{\sigma}\right)} < \min\left[\frac{|a_I|}{1 - b_I}, \frac{1 - |a_I|}{b_I}\right] \ \ (2.44)$$

or more intrinsically

$$\text{Im}\left(\frac{\delta_I - \tau}{\sigma}\right) < 0, \ \ \text{Im}\left(\frac{1 - \delta_I}{\sigma}\right) < 0, \ \ \text{Im}\left(\frac{1 + \sigma - \delta_I}{\tau}\right) < 0, \ \ \text{Im}\left(\frac{\delta_I}{\tau}\right) < 0. \ \ (2.45)$$

Similar matrix model can be derived at $\text{Im}(\frac{\sigma}{\tau}) < 0$, with the roles of σ, τ flipped. So again we find the saddle given by the uniform parallelogram distribution with edges σ, τ.

We also consider the large N free energy at our parallelogram saddle point

$$\log Z = -\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau} - N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 dx_2 dy_2 \left[V_\sigma(\sigma x_{12} + \tau y_{12}) + V_\tau(\sigma x_{12} + \tau y_{12})\right] \ (2.46)$$

in all cases summarized above, where one of V_σ, V_τ includes the half-Haar-like measure, which we called $V_\sigma^{(\pm)}$ or $V_\tau^{(\pm)}$ in (2.10). We first evaluate

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \left[V_\sigma(u_{12}) + V_\tau(u_{12})\right] \ \ (2.47)$$

at fixed u_2. As for the integral of V_σ, we compute the integrals of the form

$$-\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \log\left(1 - f(u_2, y_1)e^{\pm 2\pi i x_{12}}\right),$$ (2.48)

where $|f| < 1$ in the whole integration domain. This can be computed by Taylor-expanding the integrand, since the integral domain is within the radius of convergence:

$$\sum_{n=1}^{\infty} \frac{f(u_2, y_1)^n}{n} \int_0^1 dx_1 e^{\pm 2\pi i x_{12}} = 0.$$ (2.49)

Similarly, the log terms contained in V_τ trivially integrates to zero for the same reason, by Taylor-expanding and integrating over y_1 first. So the contribution from the second term of (2.46) vanishes. The large N free energy of our parallelogram saddle point is thus given by

$$\log Z = -\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma \tau} = \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2\omega_1 \omega_2},$$ (2.50)

in both sectors with $\sum_I \delta_I = \sigma + \tau \mp 1$ (or $\sum_I \Delta_I = \omega_1 + \omega_2 \pm 2\pi i$). This completely accounts for the entropy function of the BPS black holes in $AdS_5 \times S^5$ discovered in [23]. The two sectors of (2.22) with upper/lower signs provide saddles in the mutually complex conjugate regions. This paired structure plays an important role in realizing the macroscopic index with oscillating signs upon Legendre transformation to microcanonical ensemble [13].

The conditions (2.41) or (2.45) are nontrivial if σ, τ are not collinear. Let us first show that these conditions are trivially met in the collinear limit, $\frac{\sigma}{\tau} = $ real. For simplicity, here we just discuss the first case (2.41). At $\frac{\sigma}{\tau} = $ real, these conditions reduce to

$$\text{Im}\left(\frac{\delta_I}{\tau}\right) > 0, \quad \text{Im}\left(\frac{1 + \delta_I}{\tau}\right) < 0,$$ (2.51)

which together with $\text{Im}(\tau) > 0$ demand $0 < a_I < 1$. This is the condition already met in the upper sector of (2.22), basically set by the periodic shifts of δ_I's. Therefore, the condition (2.41) is always met in the collinear limit.

If σ, τ are non-collinear, (2.41) or (2.45) nontrivially constrain the parameters δ_I, σ, τ for our basic parallelogram to solve the saddle point equation. We shall analyze in appendix A what kind of constraints are imposed by these conditions. These conditions are very reminiscent of the stability conditions of the Euclidean black hole solutions against D3-brane instantons wrapping $S^3 \subset S^5$ and $S^1 \subset S^3 \subset AdS_5$ [24]. Their contributions to the partition function are given by $Z \leftarrow e^{iS_{D3}}$ with $S_{D3} = \pm 2\pi N \frac{\delta_I}{\sigma}$ or $\pm 2\pi N \frac{\delta_I}{\tau}$ in the two cases of (2.22). Our conditions imply their stability conditions $\text{Im}(S_{D3}) > 0$, which makes our results consistent with the gravity analysis. Our conditions are stronger than their stability conditions, leading us to conjecture that there are more stability constraints from other instantons not discussed in [24]. Since our conditions (2.41), (2.45) come from the requirement that no branch points of the potential are included in the parallelogram, it would be interesting to establish a direct connection between the forces generated by the branch points and the gravitational instability. It would be interesting to study the

parameter regime outside the conditions (2.41) or (2.45), especially the physics of the corresponding black holes. For instance, the thermodynamic instability of small spinning black holes have been already discussed in [14].

We can also construct generalized parallelogram saddles with the edge vectors given by $(\sigma + r, \tau + s)$, where $r, s \in \mathbb{Z}$. To discuss these generalized solutions, we set the convention for δ_I's so that one of the two conditions (2.22) is met with σ, τ replaced by $\sigma + r, \tau + s$. Then repeating the calculus of this section, one finds that the parallelogram ansatz with edges given by $\sigma + r, \tau + s$ solves the saddle point equation if (2.41) or (2.45) is met after replacing σ, τ by $\sigma + r, \tau + s$. In appendix A, we show that (2.41) or (2.45) is always met by many choices of r, s, at least for typical choice of σ, τ which satisfies $\frac{\sigma}{\tau} \neq$ real and $\frac{\text{Im}(\sigma)}{\text{Im}(\tau)} \neq$ rational. Following [24], we interpret our r, s as labelling multiple Euclidean solutions which map to the same Lorentzian solution once we decompactify the temporal circle.

3   Alternative derivation in the collinear limit

Our analysis so far demands that σ, τ are not collinear, at least at the intermediate steps. Otherwise the modular parameters $\frac{\sigma}{\tau}$, $-\frac{\tau}{\sigma}$ in the identity (2.23) become real and make individual Γ functions ill defined. However, the full integrand is smooth at real $\frac{\sigma}{\tau}$. So even if we are interested in the case with collinear σ, τ, we may slightly deform them with small $\text{Im}(\frac{\sigma}{\tau})$ and remove this regulator after the calculations. This way, one obtains linear cut distributions for collinear σ, τ. As shown in figure 1, the eigenvalue distribution along the linear cut is no longer uniform. We can parametrize the eigenvalues as $u(x) = \frac{\sigma+\tau}{2}x$ with $x \in (-1, 1)$ in the collinear limit. Defining real $R \equiv \frac{\sigma}{\tau}$, the linear eigenvalue density $\rho(x)$ is given by

$$\rho(x) = \frac{1}{1 - \left(\frac{1-R}{1+R}\right)^2} \left(1 - \frac{1}{2}\left|x + \frac{1-R}{1+R}\right| - \frac{1}{2}\left|x - \frac{1-R}{1+R}\right| \right) . \tag{3.1}$$

$\rho(x)$ satisfies $\int_{-1}^{1} dx\rho(x) = 1$, and the eigenvalue sum is replaced by $\sum_a \to N\int_{-1}^{1} dx\rho(x)$. As shown in figure 1b, $\rho(x)$ has a trapezoid shape. We hope this shape is intuitively visible from the degeneration shown in figure 1a. This $\rho(x)$ has been already derived in the small black hole limit at real $\frac{\sigma}{\tau}$ [14], using the standard large N matrix model techniques. In the sector given by the upper signs of (2.22), the small black hole limit is given by

$$\sigma \to \frac{1}{2} + \frac{i\gamma}{2\pi}, \quad \tau \to \frac{1}{2} - \frac{i\gamma}{2\pi} \tag{3.2}$$

in the notation of [14]. (The imaginary part of $\sigma + \tau$ approaches zero because small black hole limit is a kind of high temperature limit.) When $\sigma = \tau$, or $R = 1$ ($J_1 = J_2$), (3.1) becomes triangular. The triangular distribution was found in the small black hole limit [14], and also from the subleading correction to the Cardy limit for large black holes [15].

Here we emphasize that the linear distributions at collinear σ, τ were found in special cases without changing the saddle point problem using the identity (1.2) [14, 15]. In fact, one can make an alternative derivation of our linear distributions at collinear σ, τ, without changing the integrand using (1.2) and also without using the modular identity (2.23).

For simplicity, we only consider the upper case of (2.22) in this section. The 2-body eigenvalue potential (including the Haar measure) is given by

$$-V(u) = \frac{1}{2}\sum_{I=1}^{3}\log\Gamma(\delta_I + u; \sigma, \tau) - \frac{1}{2}\log\Gamma(u; \sigma, \tau) + (u \to -u) \ . \tag{3.3}$$

As before, there is an issue of how we define log functions concerning the choice of branch sheets on the u space. In the full discrete setup, they only affect $2\pi i\mathbb{Z}$ constants of $\log Z$ and is thus irrelevant. In the continuum limit, the most natural and useful setup is to define the log functions to be smooth on the eigenvalue cut of our ansatz, so that their derivatives (force) are well defined everywhere. As we explain in a moment, with a careful definition of our ansatz, this will be possible everywhere except when two eigenvalues approach each other, where one finds the usual repulsive singularity. We know how to continue the log functions across such repulsive singularities, by making a principal-valued definition of log functions. Therefore, we shall be able to define the log functions based on continuity on the eigenvalue cut.

Now we discuss our ansatz for collinear σ, τ in more detail, in the original saddle point problem without using the identity (1.2). First of all, it is still convenient to parametrize the eigenvalues u_a's using two parameters, $u(x,y) = x\sigma + y\tau$ with $x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\rho(x,y) = 1$. This is no longer a regular parallelogram, but we can label the eigenvalues this way. A point on the physical eigenvalue cut in the u space is mapped to a segment in the square region $x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. $\log Z$ is given by a double sum $\sum_{a,b=1}^{N}$, which in the large N continuum limit is replaced by the double integral

$$\log Z = -N^2 \int dx_1 dy_1 \int dx_2 dy_2 V(u_1 - u_2) \ . \tag{3.4}$$

One finds $u = x\sigma + y\tau = \tau(Rx + y)$ with a real R. Regarding x and $\tilde{y} \equiv y + Rx$ as the integral variables, one can first trivially integrate over x at fixed $\tilde{y}$. The allowed ranges of x at given $\tilde{y}$ determine the linear eigenvalue density $\rho(\tilde{y})$. For instance, if $R < 1$, they are given by

$$
\begin{aligned}
0 < \tilde{y} < R &: 0 < x < \frac{\tilde{y}}{R} \;\to\; \rho(\tilde{y}) = \frac{\tilde{y}}{R} \\
R < \tilde{y} < 1 &: 0 < x < 1 \;\to\; \rho(\tilde{y}) = 1 \\
1 < \tilde{y} < 1 + R &: \frac{\tilde{y} - 1}{R} < x < 1 \;\to\; \rho(\tilde{y}) = 1 - \frac{\tilde{y} - 1}{R}
\end{aligned}
\tag{3.5}
$$

which is (3.1) upon reparametrizing $\tilde{y}$. Here we emphasize that our collinear ansatz with the original potential (without using (1.2)) will demand a small refinement to make it a saddle point. Namely, we take the precise ansatz to be

$$u(x,y) = e^{-i\epsilon}(x\sigma + y\tau), \;\; \rho(x,y) = 1 \;\; \left(-\frac{1}{2} < x, y < \frac{1}{2}\right) \tag{3.6}$$

with 'infinitesimal' $\epsilon > 0$. During most of the calculus, we can simply turn off $\epsilon = 0$. Below we shall keep small $\epsilon > 0$ only when necessary.

Employing the ansatz $u_{1,2} = \sigma x_{1,2} + \tau y_{1,2} + \mathcal{O}(\epsilon)$, it will be useful to rewrite the u_1 integral of $\log \Gamma(z \pm u_{12})$ as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \log \Gamma(z \pm u_{12}, \sigma, \tau) \tag{3.7}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \sum_{m,n=0}^{\infty} \left[-\log(1 - e^{2\pi i(\pm u_1 + m\sigma + n\tau \mp u_2 + z)}) + \log(1 - e^{2\pi i(\mp u_1 + m\sigma + n\tau + \sigma + \tau - z \pm u_2)}) \right]$$

$$= \int_{-\frac{1}{2}}^{\infty} dx dy \left[-\log(1 - e^{2\pi i(u \mp u_2 + z)}) + \log(1 - e^{2\pi i(u + \sigma + \tau - z \pm u_2)}) \right] \ .$$

On the last line, we defined $u = \sigma x + \tau y$ as $u \equiv \pm u_1 + m\sigma + n\tau$, with $x \in \left(-\frac{1}{2} + m, \frac{1}{2} + m\right)$, $y \in \left(-\frac{1}{2} + n, \frac{1}{2} + n\right)$, and patched the infinitely many integrals labeled by m, n into a single integral over $x, y \in \left(-\frac{1}{2}, \infty\right)$ (at $\epsilon \to 0$). This integral is a useful object for a couple of reasons. Firstly, if we further integrate once more with x_2, y_2, it will give the contribution of each supermultiplet to the free energy $\log Z$. Also, taking u_2 derivative, one obtains a force acting on the eigenvalue located at u_2. The natural and useful convention for the log functions on the last line is to define them as a continuous function of $x, y \in \left(-\frac{1}{2}, \infty\right)$, except for the principal-valued singularity when $z = 0$ and $u \mp u_2 = 0$. We shall call this the Haar measure singularity below. We now show that such continuous definitions of log are possible.

At large x, y, we naturally stay on a branch of log function which yields $\log 1 = 0$ at $x, y \to \infty$, and attempt to define the log function as a continuous function in the whole x, y domain, if necessary by moving on to different branch sheets. Although we employ a 2 dimensional parametrization, the eigenvalues are on a linear cut (labeled by $Rx + y$) so that there will be no ambiguities in 1 dimension to make such a continuous extension except at the Haar measure singularity. We shall establish the details of this continuation below.

Firstly, when $z = \delta_I$, $u \mp u_2 + \delta_I$ and $u + \sigma + \tau - \delta_I \pm u_2$ have positive imaginary part for large x, y, in which case the two log functions on the last line of (3.7) are defined on the standard branch with $\log 1 = 0$. As long as their imaginary parts are positive, the log function can be defined by the standard Taylor expansion $\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$. We study when this definition has to be modified by analytic continuation, by the imaginary parts changing sign. For the first log on the last line of (3.7), the sign changes at the following line on the x, y space:

$$0 = \text{Im}(u \mp u_2 + \delta_I) = \text{Im}(\tau) \left[R(x + b_I \mp x_2) + (y + b_I \mp y_2) \right] . \tag{3.8}$$

(When $z = \delta_I$, the refined definition (3.6) plays no role.) On this line, its real part

$$\text{Re}(u \mp u_2 + \delta_I) = -a_I + \text{Re}(\tau) \left[R(x + b_I \mp x_2) + (y + b_I \mp y_2) \right] = -a_I \tag{3.9}$$

is in the range $-1 < -a_I < 0$. In particular, the branch point is never crossed when analytic continuation is needed to define this log. So when $z = \delta_I$, the first log function of (3.7) in the region $\text{Im}(u \mp u_2 + \delta_I) < 0$ is continuously extended using the formula

$$\log(1 - e^{2\pi i z}) = \log(1 - e^{-2\pi i z}) + 2\pi i z + \pi i \quad \text{if } \text{Im}(z) < 0 \text{ and } \text{Re}(z) \in (-1, 0) . \tag{3.10}$$

The log on the right hand side is defined by Taylor expansion. Similarly, as for the second log of (3.7), we regard $-\delta_I$ as $-\delta_I - 1$ in the infinity branch, which is just a phase convention for the fugacity. Then one finds $\text{Im}(u + \sigma + \tau \pm u_2 - \delta_I - 1) = 0$ at

$$0 = \text{Im}(\tau) \left[R(x - b_I \pm x_2 + 1) + (y - b_I \pm y_2 + 1) \right] . \tag{3.11}$$

Then its real part is given by

$$a_I - 1 + \text{Re}(\tau) \left[R(x - b_I \pm x_2 + 1) + (y - b_I \pm y_2 + 1) \right] = a_I - 1 \tag{3.12}$$

which is in the range $-1 < a_I - 1 < 0$. So we can again use (3.10) to define this log by analytic continuation, after replacing $-\delta_I$ by $-\delta_I - 1$.

When $z = 0$, only the first log of (3.7) can hit the branch point, while the second log can always be defined by Taylor expansion. For the first log, the branch with $\log 1 = 0$ chosen at infinity extends smoothly to the region $\text{Im}(u \mp u_2) > 0$, whose boundary is the line $\text{Im}(u \mp u_2) = 0$. Had we defined our ansatz as $u(x, y) = \sigma x + \tau y$ rather than (3.6), the real and imaginary parts of $u \mp u_2$ become zero at the same point,

$$Rx + y = \pm(Rx_2 + y_2), \tag{3.13}$$

meaning that the branch point is on the eigenvalue cut. Here our refined definition (3.6) has a finite effect even at $\epsilon \to 0^+$. Since $u \equiv \pm u_{12} + m\sigma + n\tau$ with $m, n \geq 0$, one finds

$$u \mp u_2 = \tau \left[\pm e^{-i\epsilon}(Rx_{12} + y_{12}) + mR + n \right]. \tag{3.14}$$

Namely, due to $\epsilon > 0$ in our ansatz, the real and imaginary parts of $u \mp u_2$ do not simultaneously vanish unless $Rx_{12} + y_{12} = 0$ and $m = n = 0$. The last point is the Haar measure singularity. Our ϵ deforms other branch point singularities slightly away from the eigenvalue cut.

Let us explain this in detail. On the quadrant defined by $x, y \in \left(-\frac{1}{2}, \infty \right)$, the line after which analytic continuation is needed is

$$0 = \text{Im}(u \mp u_2) = \text{Im}(\tau) \left[\pm \cos \epsilon (Rx_{12} + y_{12}) + mR + n \right] \mp \text{Re}(\tau) \sin \epsilon (Rx_{12} + y_{12})$$
$$\approx \text{Im}(\tau) \left[\pm (Rx_{12} + y_{12}) + mR + n \right] \mp \epsilon \text{Re}(\tau)(R_{12} + y_{12}) \tag{3.15}$$

up to linear order in ϵ. The leading $\mathcal{O}(\epsilon^0)$ shape of these lines $Rx_1 + y_1 = Rx_2 + y_2 \mp (Rm + n)$, or $Rx + y = \pm(Rx_2 + y_2)$, are shown on the (x, y) space in figure 2. On this line, the real part of $u \mp u_2$ is given by

$$\text{Re}(u \mp u_2) = \text{Re}(\tau) \left[\pm \cos \epsilon (Rx_{12} + y_{12}) + mR + n \right] \pm \text{Im}(\tau) \sin \epsilon (Rx_{12} + y_{12})$$
$$\approx \pm \epsilon \left(\frac{\text{Re}(\tau)^2}{\text{Im}(\tau)} + \text{Im}(\tau) \right) (Rx_{12} + y_{12}) \tag{3.16}$$

again up to $\mathcal{O}(\epsilon^1)$, where we inserted the condition $\text{Im}(u \mp u_2) = 0$. Therefore, unless $Rx_{12} + y_{12} = 0$ (i.e. $u_1 = u_2$), a small real part of $u \mp u_2$ is generated by ϵ on the line (3.15), represented by the red/blue/orange/magenta parts of the line in figure 2.
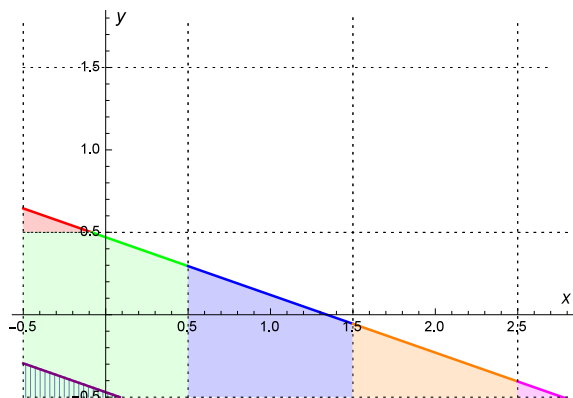
Figure 2. Singular points of the potential for the naive ansatz at $\epsilon = 0$, from $u = -u_2$ (purple line) and $u = u_2$ (red/green/blue/orange/magenta). Each square region bounded by dotted lines is a fundamental region of (x_1, y_1), whose integrand is the log potential at certain (m, n). Each colored line segment represents a point on the eigenvalue cut. Green/purple lines are the Haar measure singularities, which remain after the ansatz deformation by ϵ. Other singularities (red, blue, orange, magenta) are lifted after the deformation. ($R \equiv \frac{\sigma}{\tau} = .35$, $Rx_2 + y_2 = .47$).

We explain the situation of figure 2 in more detail. For illustration, we chose $R = 0.35$ and $Rx_2 + y_2 = .47 > 0$. The two lines with green/purple colors in the region $-\frac{1}{2} < x, y < \frac{1}{2}$ represent the Haar measure singularities, for which $\mathrm{Re}(u \mp u_2) = 0$ exactly. These are the only branch points on the eigenvalue cut at nonzero ϵ, for which we shall review in a moment how the functions are extended across the singularity (in a standard manner). The regions requiring continuations across these lines are also shown as shades with the same colors. Other line segments in red/blue/orange/magenta colors for $\mathrm{Im}(u - u_2) = 0$, are for the line $Rx_1 + y_1 \approx (Rx_2 + y_2) - (Rm + n)$ with $(m, n) = (0, 1), (1, 0), (2, 0), (3, 0)$ respectively at $\mathcal{O}(\epsilon^0)$. For these, note that $Rx_{12} + y_{12} = -(Rm + n) < 0$. So from (3.16), all parts of the line (3.15) except for the Haar measure singularities (green/purple) have small negative $\mathrm{Re}(u - u_2) \sim \mathcal{O}(\epsilon^1) < 0$. Since $-1 < \mathrm{Re}(u - u_2) < 0$, one can apply the analytic continuation (3.10) across these lines to the regions shaded with the same colors.

So except for the Haar measure contributions, which is the integral (3.7) in the domain $x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ (the square region including green/purple segments in figure 2), the log functions in (3.7) are defined either by the Taylor expansion or the continuation (3.10). The Haar measure contribution

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, dy \left[\log\left(1 - e^{2\pi i(\sigma(x-x_2)+\tau(y-y_2))}\right) + \log\left(1 - e^{2\pi i(\sigma(x+x_2)+\tau(y+y_2))}\right) \right] \tag{3.17}$$

has to be treated as the principal-valued integral. One way of doing this calculus is to eliminate the ε neighborhood of the singularity, and send $\varepsilon \to 0^+$ after the calculation. Another equivalent way is to average over the $\pm i\varepsilon$ deformations of the integration contour. This amounts to averaging over the integral done with the analytic continuation (3.10) and another integral with alternative continuation

$$\log(1 - e^{2\pi i z}) = \log(1 - e^{-2\pi i z}) + 2\pi i z - \pi i \quad \text{if} \ \ \mathrm{Im}(z) < 0 \ \text{and} \ \ \mathrm{Re}(z) \in (0, 1) \ . \tag{3.18}$$

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The two calculations differ by whether one uses $\pm\pi i$ on the last terms of (3.10) and (3.18), in the region shaded with green/purple colors in figure 2. So they are just different by integrating constants over these regions. If one averages over the two, the integrations of $\pm\pi i$ cancels. Therefore, (3.17) computed using (3.10) and (3.18) prescriptions are related to the principal-valued integration of (3.17) by having the following additional constants, respectively:

$$\pm\pi i\left[\text{area(green shaded region)} + \text{area(purple shaded region)}\right] = \pm\pi i . \tag{3.19}$$

At the last step we used the fact that the sum of the areas of the two regions is equal to the area of the square $-\frac{1}{2} < x, y < \frac{1}{2}$, which is 1. Therefore, even for the Haar measure integral we may employ a unified prescription to analytically continue with (3.10), and add a trivial constant $-\pi i$ to get the principal-valued integral for the Haar measure potential. In fact for most purposes, we can ignore this constant $-\pi i$. For the force calculation, this constant factor does not matter. Also, for the free energy calculation, this constant will provide an extra imaginary constant $-\frac{\pi i (N^2 - N)}{2} \sim -\frac{\pi i N^2}{2}$ to the free energy $\log Z$. This factor provides an overall sign factor (or phase factor) for $Z \leftarrow (-1)^{\frac{N^2 - N}{2}}$, but otherwise does not affect the large N thermodynamics. So we ignore this constant term from now and proceed by universally employing the analytic continuation (3.10) for (3.7).

We have set all the rules of calculus in the collinear case. In a moment we will show that the force vanishes, and then compute the free energy. Before the calculations, we pause to interpret what small $\epsilon > 0$ may mean. Certainly ϵ is part of our ansatz. In our leading large N calculus, only the sign of ϵ will matter. Our ansatz (3.6) is a saddle point for $\epsilon > 0$, but not for $\epsilon < 0$. Regarding it literally as an infinitesimal parameter appears to be unrealistic, since it measures the distance of a potential singularity from the eigenvalue configuration and it cannot happen in the discrete calculus that the saddle point is infinitesimally away from a point where the force diverges. So we interpret infinitesimal ϵ as emerging from the large N continuum limit. For instance, if one can make a subleading calculus in $\frac{1}{N}$, it may be related to N by $\epsilon \sim \frac{1}{N^\alpha}$ with a positive number α. (We expect $\alpha < 1$, for ϵ to be larger than the minimal eigenvalue separation $\sim \frac{1}{N}$.) Of course if one can actually do a subleading calculus, the saddle will be more complicated than (3.6). Our (3.6) merely prescribes how the branch point is avoided at large N. Here we note that a similar ϵ deformation was needed to get the saddle point of this index in the Cardy limit [15]. We expect their ϵ should be interpreted similarly, as a small number related to the large charges.

More physically, the singularities which are ϵ-distance away from our ansatz come from the gaugino operators dressed by derivatives. In the notation of [25], the gaugino 'letter' $(\partial_{+\dot{+}})^p (\partial_{+\dot{-}})^q \bar{\lambda}_{\pm}$ in the a'th row and b'th column of the $N \times N$ matrix is weighted by the following effective fugacity factor in the matrix integral:

$$e^{2\pi i u_{ab}} \cdot e^{2\pi i \sigma\left(\frac{1}{2} \pm \frac{1}{2} + p\right)} e^{2\pi i \tau\left(\frac{1}{2} \mp \frac{1}{2} + q\right)} . \tag{3.20}$$

This is a product of the 'color fugacity' factor $e^{2\pi i u_{ab}}$ which is not a physical fugacity, and the rest which is the physical fugacity. Since u_a in our ansatz is on the straight line interval

$(-\frac{\sigma+\tau}{2}, \frac{\sigma+\tau}{2})$ in the $\epsilon = 0$ limit, u_{ab} is on the interval $(-(\sigma+\tau), \sigma+\tau)$. Therefore, although the physical fugacity has its absolute value smaller than 1, the factor $e^{2\pi i u_{ab}}$ may be larger than 1 and make (3.20) close to 1 for certain u_a, u_b. If this happens, the potential $V(u_{ab})$ will almost diverge. The integrand of the vector multiplet can be simplified as

$$\prod_{a \neq b} \Gamma(u_{ab}, \sigma, \tau)^{-1} = \exp\left[-\sum_{a \neq b}\sum_{p=1}^{\infty}\frac{1}{p}e^{2\pi i p u_{ab}}\left(1 + \sum_{m=1}^{\infty}e^{2\pi i p m \sigma} + \sum_{n=1}^{\infty}e^{2\pi i p n \tau}\right)\right]. \quad (3.21)$$

The term 1 in the exponent comes from the Haar measure. The terms $e^{2\pi i p m \sigma}$ and $e^{2\pi i p n \tau}$ come from the letters $(\partial_{+\dot{+}})^{m-1}\bar{\lambda}_{\dot{+}}$ and $(\partial_{+\dot{-}})^{n-1}\bar{\lambda}_{\dot{-}}$, respectively. They are responsible for the divergences which are ϵ-distance away from the red/blue/orange/magenta lines of figure 2, labeled by either $(m, 0)$ or $(0, n)$ with $m, n \neq 0$. So our saddle point configuration is very close to the point where these charged operators become massless. The fact that these fermionic operators are very light (with mass at order ϵ) at our saddle point may provide important clues on the microstates of the dual black holes or further generalizations to hairy black holes. Interestingly, a Fermi surface model for these black holes has been proposed in [26, 27], precisely based on using the gaugino letters discussed above. Although the simplest operators of [26] acquire nonzero anomalous dimensions above the BPS bound [27], minor corrections to their ansatz may be relevant for better understanding the microstates of the BPS black holes. We hope our findings provide helpful insights.

We now compute the force and free energy. To compute both quantities, we first study

$$-N^2\int_{-\frac{1}{2}}^{\frac{1}{2}}dx_1 dy_1 V(u_{12}) = \frac{N^2}{2}\sum_{\pm}\int_{-\frac{1}{2}}^{\infty}dxdy\left[\log\left(1 - e^{2\pi i(u \mp u_2)}\right) - \log\left(1 - e^{2\pi i(u \pm u_2 + \sigma + \tau)}\right)\right.$$
$$\left. - \sum_{I=1}^{3}\left(\log\left(1 - e^{2\pi i(u \mp u_2 + \delta_I)}\right) - \log\left(1 - e^{2\pi i(u \pm u_2 + \sigma + \tau - \delta_I - 1)}\right)\right)\right] \quad (3.22)$$

where we inserted (3.3) and (3.7). Whenever the analytic continuation has to be made for the log functions, one uses (3.10) with the exponents as specified in the formula (including the -1 term on the last term). The continuation formula (3.10) is a special case of the identities for the polylog functions $\text{Li}_s(e^{2\pi i z})$. This function is defined by Taylor expansion

$$\text{Li}_s\left(e^{2\pi i z}\right) = \sum_{n=1}^{\infty}\frac{e^{2\pi i n z}}{n^s} \quad (3.23)$$

when $\text{Im}(z) > 0$, and by the analytic continuation

$$\text{Li}_s\left(e^{2\pi i z}\right) = -(-1)^s\text{Li}_s\left(e^{-2\pi i z}\right) - \frac{(2\pi i)^s}{s!}B_s(z+1), \quad (3.24)$$

when $\text{Im}(z) < 0$ and $-1 < \text{Re}(z) < 0$. $B_s(z)$ are the Bernoulli polynomials. Eq. (3.10) is a special case of (3.24) at $s = 1$, with $\text{Li}_1(e^{2\pi i z}) = -\log(1 - e^{2\pi i z})$ and $B_1(z) = z - \frac{1}{2}$. These formulas are relevant for computing (3.22) since $\frac{\partial}{\partial z}\text{Li}_s(e^{2\pi i z}) = 2\pi i \text{Li}_{s-1}(e^{2\pi i z})$. After

integrating twice with x and y, one obtains

$$-N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 V(u_{12}) = -\frac{N^2}{8\pi^2 \sigma\tau} \sum_{\pm} \left[- \mathrm{Li}_3\left(e^{2\pi i(\mp u_2 - \frac{\sigma+\tau}{2})}\right) + \mathrm{Li}_3\left(e^{2\pi i(\frac{\sigma+\tau}{2} \pm u_2)}\right) \right. \quad (3.25)$$

$$\left. + \sum_{I=1}^{3} \left(\mathrm{Li}_3\left(e^{2\pi i(\delta_I \mp u_2 - \frac{\sigma+\tau}{2})}\right) - \mathrm{Li}_3\left(e^{2\pi i(\frac{\sigma+\tau}{2} \pm u_2 - \delta_I - 1)}\right) \right) \right].$$

Any Li_3 functions are defined by the right hand side of (3.24) at $s = 3$ when they need analytic continuations, since they are obtained by integrating (3.10). On the first line, the second term is always defined by Taylor expansion while the first term needs to be defined by the right hand side of (3.24). On the second line, it is always that one of the two terms is defined by Taylor expansion while the other term is defined by the right hand side of (3.24). In both cases, one universally obtains the following expression:

$$-N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 V(u_{12})$$

$$= -\frac{\pi i N^2}{6\sigma\tau} \sum_{\pm} \left[\sum_{I=1}^{3} B_3\left(\delta_I \mp u_2 - \frac{\sigma+\tau}{2} + 1\right) - B_3\left(\mp u_2 - \frac{\sigma+\tau}{2} + 1\right) \right]$$

$$= \frac{\pi i N^2}{6\sigma\tau} \sum_{\pm} \left[\sum_{I} \left(\left(\frac{\sigma+\tau}{2} \pm u_2 - \delta_I - \frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{\sigma+\tau}{2} \pm u_2 - \delta_I - \frac{1}{2}\right) \right) \right.$$

$$\left. - \left(\frac{\sigma+\tau}{2} \pm u_2 - \frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{\sigma+\tau}{2} \pm u_2 - \frac{1}{2}\right) \right]$$

$$= -\frac{\pi i N^2}{\sigma\tau} \left[\delta_1 \delta_2 \delta_3 + \frac{\Delta^3}{12} - \frac{\Delta}{4}\left(\frac{1}{3} + 2\delta_1\delta_2 + 2\delta_2\delta_3 + 2\delta_3\delta_1 - \delta_1^2 - \delta_2^2 - \delta_3^2\right) + \Delta u_2^2 \right]$$

$$= -\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}, \quad (3.26)$$

where $\Delta \equiv \sum_I \delta_I - \sigma - \tau + 1$. We used

$$B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z \;\; \rightarrow \;\; B_3(z+1) = \left(z + \frac{1}{2}\right)^3 - \frac{1}{4}\left(z + \frac{1}{2}\right). \quad (3.27)$$

on the second line, and $\Delta = 0$ on the fourth line for the upper case of (2.22).

Now one can immediately compute the force, by taking the u_2 derivative of (3.26). Since the final expression contains no u_2 dependence, one finds that

$$-\frac{\partial}{\partial u_2} \int dx_1 dy_1 V(u_1 - u_2) = 0, \quad (3.28)$$

proving that our ansatz solves the saddle point equation. Eq. (3.26) fails to be u_2-independent if one uses the ansatz (3.6) with $\epsilon < 0$. We can also compute the saddle point free energy, by integrating (3.26) once more in x_2, y_2:

$$\log Z = -N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_2 dy_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 V(u_1 - u_2) = -\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma\tau}. \quad (3.29)$$

This again agrees with the free energy of the BPS black holes in $AdS_5 \times S^5$ [23].

3.1 A relation to the Bethe ansatz equation

We would like to provide an interpretation of the vanishing of the force that we have just proven. We first consider

$$\frac{\partial}{\partial u_2} \int dx_1 dy_1 \log \Gamma(z \pm u_{12}, \sigma, \tau)$$
$$= \mp \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\infty} dy dx \frac{\partial}{\partial x} \left[\text{Li}_1 \left(e^{2\pi i(u \mp u_2 + z)} \right) + \text{Li}_1 \left(e^{2\pi i(u + \sigma + \tau - z - 1 \pm u_2)} \right) \right]$$

where Li_1 is defined using analytic continuation if necessary, and we replaced $\frac{\partial}{\partial u_2}$ by $\mp \frac{1}{\sigma} \frac{\partial}{\partial x}$ and $\pm \frac{1}{\sigma} \frac{\partial}{\partial x}$ for the first and the second term. Instead of double-integrating these to Li_2 functions, similar to what we did so far in this section, we integrate with x only to obtain

$$\frac{\partial}{\partial u_2} \int dx_1 dy_1 \log \Gamma(z \pm u_{12}, \sigma, \tau)$$
$$= \pm \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\infty} dy \left[\text{Li}_1 \left(e^{2\pi i \left(y\tau \mp u_2 + z - \frac{\sigma}{2} \right)} \right) - \text{Li}_1 \left(e^{2\pi i \left(y\tau + \sigma + \tau - z - 1 \pm u_2 - \frac{\sigma}{2} \right)} \right) \right] .$$

This can be understood as

$$\mp \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \sum_{n=0}^{\infty} \left[-\log \left(1 - e^{2\pi i \left(\pm u_{12} + z - \frac{\sigma}{2} + n\tau \right)} \right) + \log \left(1 - e^{2\pi i \left(\mp u_{12} + \frac{\sigma}{2} - z - 1 + (n+1)\tau \right)} \right) \right] \tag{3.30}$$

where $u_1 \equiv y_1 \tau$. (This definition of u_1 will be assumed below when accompanied only by single integration $\int dy_1$.) The -1 shift in the exponent of the second log is again a helpful convention if this log is defined by the analytic continuation (3.10). Formally, we can write this as

$$\mp \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \sum_{n=0}^{\infty} \log \left[\left(1 - e^{2\pi i \left(z - \frac{\sigma}{2} \pm u_{12} \right)} e^{2\pi i n\tau} \right) \left(1 - e^{-2\pi i \left(z + 1 - \frac{\sigma}{2} \pm u_{12} \right)} e^{2\pi i (n+1)\tau} \right) \right]$$
$$` \equiv ' \mp \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log \theta \left(z - \frac{\sigma}{2} \pm u_{12}, \tau \right) , \tag{3.31}$$

where $\theta(z, \tau)$ is the 'q-theta function' with $q \equiv e^{2\pi i \tau}$ defined by

$$\theta(z, \tau) \equiv \prod_{n=0}^{\infty} (1 - e^{2\pi i z} e^{2\pi i n\tau})(1 - e^{-2\pi i(z+1)} e^{2\pi i(n+1)\tau})$$
$$\sim \prod_{n=0}^{\infty} (1 - e^{2\pi i z} e^{2\pi i n\tau})(1 - e^{-2\pi i z} e^{2\pi i(n+1)\tau}) . \tag{3.32}$$

(We shall often write it as $\theta(z)$ if no confusions are expected.) Let us explain the meaning of this calculus. Had one been sloppy about taking log of functions, an apparently similar result could have been obtained by a much neater calculation,

$$\frac{\partial}{\partial u_2} \int dx_1 dy_1 \log \Gamma(z \pm u_{12}, \sigma, \tau) = -\frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 dx_1 \frac{\partial}{\partial x_1} \log \Gamma(z \pm u_{12}, \sigma, \tau) \tag{3.33}$$
$$= -\frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log \frac{\Gamma \left(z \pm u_{12} \pm \frac{\sigma}{2}, \sigma, \tau \right)}{\Gamma \left(z \pm u_{12} \mp \frac{\sigma}{2}, \sigma, \tau \right)} = \mp \frac{1}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log \theta \left(z - \frac{\sigma}{2} \pm u_{12}, \tau \right)$$

if one can apply the identity

$$\frac{\Gamma(z+\sigma,\sigma,\tau)}{\Gamma(z,\sigma,\tau)} = \theta(z,\tau) \tag{3.34}$$

inside the log. However, it is obscure what it means to apply an identity inside the log, in particular if some log functions are defined by analytic continuations. The precise meaning of the last expression of (3.33) is given by the first line of (3.31), with continuations (3.10) understood. We have already specified the correct branch sheet of each log function. With these understood, we study the force F defined by

$$F \equiv 2N \int dx_1 dy_1 \partial_{u_2} V(u_1 - u_2) = \frac{2N}{\sigma} \int dx_1 dy_1 \partial_{x_2} V(u_1 - u_2)$$
$$= -\frac{2N}{\sigma} \int dx_1 dy_1 \partial_{x_1} V(u_1 - u_2) . \tag{3.35}$$

Applying (3.31), one obtains

$$F = \frac{N}{\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log \left[\frac{\theta\left(-u_{12} - \frac{\sigma}{2}, \tau\right)}{\theta\left(u_{12} - \frac{\sigma}{2}, \tau\right)} \prod_I \frac{\theta\left(\delta_I + u_{12} - \frac{\sigma}{2}, \tau\right)}{\theta\left(\delta_I - u_{12} - \frac{\sigma}{2}, \tau\right)} \right] \tag{3.36}$$

where $u_1 \equiv y_1 \tau$, and all $\log \theta$ functions are understood in the sense of (3.31).

Let us study the following function

$$f(u) \equiv \frac{\theta\left(-u - \frac{\sigma}{2}, \tau\right)}{\theta\left(u - \frac{\sigma}{2}, \tau\right)} \prod_I \frac{\theta\left(\delta_I + u - \frac{\sigma}{2}, \tau\right)}{\theta\left(\delta_I - u - \frac{\sigma}{2}, \tau\right)} \tag{3.37}$$

in more detail, which appears inside the log in (3.36). We first study its properties in the usual manner, without worrying about taking the log, to first get intuitions. We shall then make all the calculations rigorously inside the log. Using

$$\theta(z+\tau,\tau) = -e^{-2\pi i z}\theta(z,\tau), \quad \theta(z-\tau,\tau) = -e^{2\pi i z}e^{-2\pi i \tau}\theta(z,\tau), \tag{3.38}$$

one finds

$$f(u+\tau) = \frac{-e^{-2\pi i \tau}e^{-2\pi i\left(u+\frac{\sigma}{2}\right)}\theta\left(-u-\frac{\sigma}{2},\tau\right)}{-e^{-2\pi i\left(u-\frac{\sigma}{2}\right)}\theta\left(u-\frac{\sigma}{2},\tau\right)} \prod_I \frac{-e^{-2\pi i\left(u+\delta_I-\frac{\sigma}{2}\right)}\theta\left(\delta_I+u-\frac{\sigma}{2},\tau\right)}{-e^{-2\pi i \tau}e^{2\pi i\left(\delta_I-u-\frac{\sigma}{2}\right)}\theta\left(\delta_I-u-\frac{\sigma}{2},\tau\right)}$$
$$= e^{-4\pi i\left(\sum_I \delta_I - \sigma - \tau\right)} \cdot \frac{\theta\left(-u-\frac{\sigma}{2},\tau\right)}{\theta\left(u-\frac{\sigma}{2},\tau\right)} \prod_I \frac{\theta\left(\delta_I+u-\frac{\sigma}{2},\tau\right)}{\theta\left(\delta_I-u-\frac{\sigma}{2},\tau\right)} = f(u), \tag{3.39}$$

upon using $\sum_I \delta_I - \sigma - \tau \in \mathbb{Z}$. So the function $f(u_{12})$ appearing in the log in (3.36) is double-periodic, $u_{12} + 1 \sim u_{12} + \tau \sim u_{12}$. Of course after taking the log, $\log f$ is periodic in both directions up to $2\pi i \mathbb{Z}$ which we shall clarify shortly.

More specifically, we consider the case with $\sigma = \tau$. Then the function f can be written as

$$f(u) = \frac{\theta\left(-u-\frac{\tau}{2},\tau\right)}{\theta\left(u-\frac{\tau}{2},\tau\right)} \prod_I \frac{\theta\left(\delta_I + u - \frac{\tau}{2},\tau\right)}{\theta\left(\delta_I - u - \frac{\tau}{2},\tau\right)} \tag{3.40}$$

$$= \frac{-e^{2\pi i\left(\frac{\tau}{2}-u\right)}e^{-2\pi i\tau}\theta\left(-u+\frac{\tau}{2},\tau\right)}{\theta\left(u-\frac{\tau}{2},\tau\right)} \prod_I \frac{\theta\left(\delta_I + u - \frac{\tau}{2},\tau\right)}{-e^{2\pi i\left(\frac{\tau}{2}-u+\delta_I\right)}e^{-2\pi i\tau}\theta\left(\delta_I - u + \frac{\tau}{2},\tau\right)}$$

$$= e^{2\pi i\left(2u-\sum_I \delta_I + \tau\right)} \cdot \left(-e^{2\pi i\left(\frac{\tau}{2}-u\right)}\right) \prod_I \frac{\theta\left(\delta_I + u - \frac{\tau}{2},\tau\right)}{\theta\left(\delta_I - u + \frac{\tau}{2},\tau\right)}$$

$$= -e^{2\pi i\left(u-\frac{\tau}{2}\right)} \prod_I \frac{\theta\left(\delta_I + u - \frac{\tau}{2},\tau\right)}{\theta\left(\delta_I - u + \frac{\tau}{2},\tau\right)}$$

where we used (3.38) on the second line, $\theta(-z,\tau) = -e^{-2\pi i z}\theta(z,\tau)$ and $\sum_I \delta_I = 2\tau$ (mod $\mathbb{Z}$) on the third line, again without worrying about taking log. Note that $u - \frac{\tau}{2}$ in the last expression is given by $u_1 - \left(u_2 + \frac{\tau}{2}\right)$, where $u_1 = y_1\tau$ is on the segment $\left(-\frac{\tau}{2},\frac{\tau}{2}\right)$ and $u_2' \equiv u_2 + \frac{\tau}{2}$ is on the segment $\left(-\frac{\sigma}{2},\frac{\sigma+2\tau}{2}\right) \to \left(-\frac{\tau}{2},\frac{3\tau}{2}\right)$ at $\sigma = \tau$. If $u_2' \in \left(-\frac{\tau}{2},\frac{\tau}{2}\right)$, then u_2' is in the same range as u_1. If $u_2' \in \left(\frac{\tau}{2},\frac{3\tau}{2}\right)$, we can use the τ shift invariance of $f(u)$ to replace all u_2' arguments by $u_2' - \tau \in \left(-\frac{\tau}{2},\frac{\tau}{2}\right)$. So let us define

$$u_2^{\text{Bethe}} = \begin{cases} u_2' & \text{for } u_2 \in \left(-\frac{\tau}{2},\frac{\tau}{2}\right) \\ u_2' - \tau & \text{for } u_2 \in \left(\frac{\tau}{2},\frac{3\tau}{2}\right) \end{cases} \tag{3.41}$$

satisfying $u_2^{\text{Bethe}} \in \left(-\frac{\tau}{2},\frac{\tau}{2}\right)$. Then one can write

$$f(u_1 - u_2) = f\left(u_1 - u_2^{\text{Bethe}} + \frac{\tau}{2}\right) = -e^{2\pi i(u_1 - u_2^{\text{Bethe}})} \prod_I \frac{\theta(\delta_I + u_1 - u_2^{\text{Bethe}},\tau)}{\theta(\delta_I - u_1 + u_2^{\text{Bethe}},\tau)}. \tag{3.42}$$

Here we note that the last expression for f is same as the function $Q(u)$ appearing in the Bethe ansatz equation of [18, 19], defined by

$$Q(u) = -e^{6\pi i u}\frac{\theta(\delta_1 - u)\theta(\delta_2 - u)\theta(\delta_3 - u - 2\tau)}{\theta(\delta_1 + u)\theta(\delta_2 + u)\theta(\delta_3 + u - 2\tau)} = -e^{-2\pi i u}\prod_{I=1}^3 \frac{\theta(\delta_I - u)}{\theta(\delta_I + u)} \tag{3.43}$$

where we used $\theta(z-2\tau,\tau) = e^{4\pi i z}e^{-6\pi i\tau}\theta(z,\tau)$. Namely, $f\left(u_1 - u_2^{\text{Bethe}} + \frac{\tau}{2}\right) = Q(u_2^{\text{Bethe}} - u_1)$.

With these understood, now we review the Bethe ansatz equation. Suppose we have N variables u_a $(a = 1,\cdots,N)$ given by $u_a = -\frac{\tau}{2} + \frac{a\tau}{N}$. In the continuum limit, they are distributed uniformly on the interval $\left(-\frac{\tau}{2},\frac{\tau}{2}\right)$. They satisfy the following Bethe ansatz equation

$$1 = Q_a(\{u\}) \equiv \prod_{b(\neq a)} Q(u_a - u_b) \tag{3.44}$$

with $Q(u)$ given by (3.43). This is the Bethe root of [20] at $K = 1$, $r = 0$. The continuum version of this equation is

$$0 = \sum_{b(\neq a)} \log Q(u_a - u_b) \to N\int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log Q(u_2^{\text{Bethe}} - u_1) = N\int_{-\frac{1}{2}}^{\frac{1}{2}} dy_1 \log f\left(u_1 - u_2^{\text{Bethe}} + \frac{\tau}{2}\right). \tag{3.45}$$

We renamed the continuum variables $u_b \to u_1 \equiv y_1\tau$ and $u_a \to u_2^{\mathrm{Bethe}}$, both in the range $\left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$. This is precisely the vanishing condition of the force (3.36) of our interest. So we have shown that the saddle point equation at $\sigma = \tau$ is equivalent to the log of the Bethe ansatz equation within our ansatz. In this viewpoint, the uniform Bethe root on a segment is obtained by projecting (partially summing over) the uniform parallelogram distribution along one direction. What is unclear at this stage is the $2\pi i\mathbb{Z}$ ambiguities when applying identities inside the log, but it still establishes the relation solidly. The Bethe ansatz equation is usually discussed without taking log, and all the subtleties of $2\pi i\mathbb{Z}$ in our calculus are collected to the question of what $\log 1$ is on the left hand side of (3.44). In the remaining part of this section, we want to address what this constant is within our setup.

It suffices to reconsider all the theta function identities used to establish (3.42), (3.43), rigorously stating their log versions with our conventions. We have used the $\pm\tau$ shift identities (3.38) and the inversion identity $\theta(-z, \tau) = -e^{-2\pi i z}\theta(z, \tau)$ during the derivation. Consider the log of the function $f(u_{12})$ defined by (3.37), where $u_1 = y_1\tau$ and $u_2 = \sigma x_2 + \tau y_2$. Note that

$$
\begin{aligned}
&\log\theta\left(-u_{12} - \frac{\tau}{2}, \tau\right) \\
&= \sum_{n=0}^{\infty}\left(\log\left(1 - e^{2\pi i\left(-u_{12} - \frac{\tau}{2}\right)}e^{2\pi i n\tau}\right) + \log\left(1 - e^{-2\pi i\left(-u_{12} - \frac{\tau}{2} + 1\right)}e^{2\pi i(n+1)\tau}\right)\right) \\
&= \log\left(1 - e^{2\pi i\left(-u_{12} - \frac{\tau}{2}\right)}\right) - \log\left(1 - e^{2\pi i\left(u_{12} + \frac{\tau}{2} - 1\right)}\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\log\left(1 - e^{2\pi i\left(-u_{12} + \frac{\tau}{2}\right)}e^{2\pi i n\tau}\right) + \log\left(1 - e^{-2\pi i\left(-u_{12} + \frac{\tau}{2} + 1\right)}e^{2\pi i(n+1)\tau}\right)\right) \\
&= -2\pi i\left(u_{12} + \frac{\tau}{2}\right) + \pi i + \log\theta\left(-u_{12} + \frac{\tau}{2}, \tau\right),
\end{aligned}
\tag{3.46}
$$

where one of the two logarithms in the second line is defined by the Taylor series and the other is defined by an analytic continuation. In any of the two cases, we can use the continuation formula which yields the same result given by the last line. In a similar way, we get

$$
\begin{aligned}
&\log\theta\left(\delta_I - u_{12} - \frac{\tau}{2}, \tau\right) \\
&= \sum_{n=0}^{\infty}\left(\log\left(1 - e^{2\pi i\left(\delta_I - u_{12} - \frac{\tau}{2}\right)}e^{2\pi i n\tau}\right) + \log\left(1 - e^{-2\pi i\left(\delta_I - u_{12} - \frac{\tau}{2} + 1\right)}e^{2\pi i(n+1)\tau}\right)\right) \\
&= \log\left(1 - e^{2\pi i\left(\delta_I - u_{12} - \frac{\tau}{2}\right)}\right) - \log\left(1 - e^{2\pi i\left(-\delta_I + u_{12} + \frac{\tau}{2} - 1\right)}\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\log\left(1 - e^{2\pi i\left(\delta_I - u_{12} + \frac{\tau}{2}\right)}e^{2\pi i n\tau}\right) + \log\left(1 - e^{-2\pi i\left(\delta_I - u_{12} + \frac{\tau}{2} + 1\right)}e^{2\pi i(n+1)\tau}\right)\right) \\
&= 2\pi i\left(\delta_I - u_{12} - \frac{\tau}{2}\right) + \pi i + \log\theta\left(\delta_I - u_{12} + \frac{\tau}{2}, \tau\right).
\end{aligned}
\tag{3.47}
$$

Lastly, let us consider

$$
\begin{aligned}
\log\theta\left(-u_{12} + \frac{\tau}{2}, \tau\right) &= -\log\left(1 - e^{2\pi i\left(u_{12} - \frac{\tau}{2} - 1\right)}\right) + \log\left(1 - e^{-2\pi i\left(u_{12} - \frac{\tau}{2}\right)}\right) \\
&\quad + \sum_{n=0}^{\infty}\log\left(1 - e^{2\pi i\left(u_{12} - \frac{\tau}{2} - 1\right)}e^{2\pi i n\tau}\right)\left(1 - e^{-2\pi i\left(u_{12} - \frac{\tau}{2}\right)}e^{2\pi i(n+1)\tau}\right).
\end{aligned}
\tag{3.48}
$$

The first logarithm in the last line has to be defined by analytic continuation when

$$\text{Im}\left(u_{12} - \frac{\tau}{2} - 1 + n\tau\right) = \text{Im}\,\tau\left(n - \frac{1}{2} + y_{12} - x_2\right) < 0 \Leftrightarrow n < x_2 - y_{12} + \frac{1}{2}. \qquad (3.49)$$

The number of such non-negative integers n is $M \equiv \max\{\lceil x_2 - y_{12} + \frac{1}{2}\rceil, 0\}$. Similarly, the second logarithm in the last line is defined by analytic continuation when

$$\text{Im}\left(-u_{12} + \frac{\tau}{2}(n+1)\tau\right) = \text{Im}\,\tau\left(n + \frac{3}{2} - y_{12} + x_2\right) < 0 \Leftrightarrow n < y_{12} - x_2 - \frac{3}{2}. \qquad (3.50)$$

There is no such non-negative integer n, since the $y_{12} - x_2 < \frac{3}{2}$. Note that

$$\sum_{n=0}^{\infty} \log(1 - e^{2\pi i(u_{12} - \frac{\tau}{2} - 1)}e^{2\pi in\tau}) = -2\pi iM + \sum_{n=0}^{\infty} \log(1 - e^{2\pi i(u_{12} - \frac{\tau}{2})}e^{2\pi in\tau}) \qquad (3.51a)$$

$$\sum_{n=0}^{\infty} \log(1 - e^{-2\pi i(u_{12} - \frac{\tau}{2})}e^{2\pi i(n+1)\tau}) = \sum_{n=0}^{\infty} \log(1 - e^{-2\pi i(u_{12} - \frac{\tau}{2} + 1)}e^{2\pi i(n+1)\tau}), \qquad (3.51b)$$

by the analytic continuation formula. We can explicitly write the non-negative integer M as

$$M = \begin{cases} \lceil x_2 - y_{12} + \frac{1}{2}\rceil & \text{if} \quad x_2 - y_{12} \geq -\frac{1}{2} \\ 0 & \text{if} \quad -\frac{3}{2} \leq x_2 - y_{12} < -\frac{1}{2} \end{cases}. \qquad (3.52)$$

Thus, we can write

$$\log\theta\left(-u_{12} + \frac{\tau}{2}, \tau\right) = -2\pi i\left(u_{12} - \frac{\tau}{2}\right) + \pi i - 2\pi iM + \log\theta\left(u_{12} - \frac{\tau}{2}, \tau\right) \qquad (3.53)$$

Combining (3.46), (3.47), (3.53), we obtain

$$\log f(u_{12}) = \log\theta\left(-u_{12} - \frac{\tau}{2}, \tau\right) - \log\theta\left(u_{12} - \frac{\tau}{2}, \tau\right)$$

$$+ \sum_{I=1}^{3} \log\theta\left(\delta_I + u_{12} - \frac{\tau}{2}, \tau\right) - \log\theta\left(\delta_I - u_{12} - \frac{\tau}{2}, \tau\right) \qquad (3.54)$$

$$= 2\pi i\left(u_{12} - \frac{\tau}{2} + \frac{1}{2} - M\right) + \sum_{I=1}^{3} \log\left(\delta_I + u_{12} - \frac{\tau}{2}, \tau\right) - \log\theta\left(\delta_I - u_{12} + \frac{\tau}{2}, \tau\right),$$

where we have used $\sum_{I=1}^{3} \delta_I = 2\tau - 1$ in the second equality.

Now recall the definition of $u_2' \equiv u_2 + \frac{\tau}{2} \in \left(-\frac{\tau}{2}, \frac{3\tau}{2}\right)$ and u_2^{Bethe} (3.41). Then one obtains

$$\log f(u_{12}) = \log f\left(u_1 - u_2^{\text{Bethe}} + \frac{\tau}{2}\right)$$

$$= 2\pi i\left(u_1 - u_2^{\text{Bethe}} + \frac{1}{2} - M\right) + \sum_{I=1}^{3} \log\theta\left(\delta_I + u_1 - u_2^{\text{Bethe}}, \tau\right) \qquad (3.55)$$

$$- \log\theta\left(\delta_I - u_1 + u_2^{\text{Bethe}}, \tau\right)$$

when $u_2' \in \left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$ and

$$
\begin{aligned}
\log f(u_{12}) &= \log f\left(u_1 - u_2^{\text{Bethe}} + \frac{3\tau}{2}\right) \\
&= 2\pi i \left(u_1 - u_2^{\text{Bethe}} + \tau + \frac{1}{2} - M\right) \\
&\quad + \sum_{I=1}^{3} \log \theta\left(\delta_I + u_1 - u_2^{\text{Bethe}} + \tau, \tau\right) - \log \theta\left(\delta_I - u_1 + u_2^{\text{Bethe}} - \tau, \tau\right) \quad (3.56) \\
&= 2\pi i \left(u_1 - u_2^{\text{Bethe}} - \frac{1}{2} - M\right) + \sum_{I=1}^{3} \log \theta\left(\delta_I + u_1 - u_2^{\text{Bethe}}, \tau\right) \\
&\quad - \log \theta\left(\delta_I - u_1 + u_2^{\text{Bethe}}, \tau\right)
\end{aligned}
$$

when $u_2' \in \left(\frac{\tau}{2}, \frac{3\tau}{2}\right)$. All the log functions are defined using our convention (3.30). This is the precise meaning of the log of (3.42), where u_1, u_2^{Bethe} are defined to be on the segment $\left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$.

Finally, again applying (3.47), one finds

$$
\begin{aligned}
\log f(u_{12}) = &-6\pi i(u_1 - u_2^{\text{Bethe}}) + 2\pi i\left(\frac{1}{2} - M\right) + \log \theta(\delta_1 + u_1 - u_2^{\text{Bethe}}) \\
&- \log \theta(\delta_1 - u_1 + u_2^{\text{Bethe}}) + \log \theta(\delta_2 + u_1 - u_2^{\text{Bethe}}) - \log \theta(\delta_2 - u_1 + u_2^{\text{Bethe}}) \\
&+ \log \theta(\delta_3 + u_1 - u_2^{\text{Bethe}} - 2\tau) - \log \theta(\delta_3 - u_1 + u_2^{\text{Bethe}} - 2\tau)
\end{aligned}
$$

when $u_2' \in \left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$, and

$$
\begin{aligned}
\log f(u_{12}) = &-6\pi i(u_1 - u_2^{\text{Bethe}}) - 2\pi i\left(\frac{1}{2} + M\right) + \log \theta(\delta_1 + u_1 - u_2^{\text{Bethe}}) \\
&- \log \theta(\delta_1 - u_1 + u_2^{\text{Bethe}}) + \log \theta(\delta_2 + u_1 - u_2^{\text{Bethe}}) - \log \theta(\delta_2 - u_1 + u_2^{\text{Bethe}}) \\
&+ \log \theta(\delta_3 + u_1 - u_2^{\text{Bethe}} - 2\tau) - \log \theta(\delta_3 - u_1 + u_2^{\text{Bethe}} - 2\tau)
\end{aligned}
$$

when $u_2' \in \left(\frac{\tau}{2}, \frac{3\tau}{2}\right)$. These are what we precisely mean by $\log Q(u_2^{\text{Bethe}} - u_1)$ in (3.45).

Note that the significance of the ϵ deformation (3.6) became obscure in our final expression for the Bethe ansatz equation. For instance, had one chosen the wrong sign for $\epsilon < 0$, one would have got $-\pi i$ on the last line of (3.46) instead of $+\pi i$. This would have affected the final expression for $\log f$ only by an extra constant of the form $2\pi i\mathbb{Z}$, whose exponentiation yields the same Bethe equation. So we find that the map of the saddle point to the Bethe root is at best one-sided, e.g. the Bethe roots being unable to detect the sign of ϵ. Recall that this deformation was needed because otherwise there are eigenvalues u_1 and u_2 which differ by $u_{12} = \tau\mathbb{Z} \neq 0$ that makes some gaugino operators massless. This was explained in the paragraph containing (3.20). To repeat the explanation at $\sigma = \tau$, the eigenvalues lie on an interval $(-\tau, \tau)$. So there exists a pair u_1 to any u_2 which hits one of the singularities $u_{12} = \pm\tau$ which demands regularized definition of the ansatz. On the other hand, the Bethe root obtained (in our viewpoint) by projecting the parallelogram along x direction is distributed on the reduced interval $\left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$, causing no divergence

problems. So it is natural that the Bethe ansatz equation does not detect the subtle details of the true saddle point such as the sign of ϵ.

Similarly, for a more general case of collinear σ and τ with $\sigma/\tau \in \mathbb{Q}$, the relation between our ansatz and the Bethe ansatz equation studied in [45] can be shown as follows. Since $\sigma/\tau \in \mathbb{Q}$, one can define $\sigma \equiv a\omega, \tau \equiv b\omega$ where (a, b) are co-prime integers. The force equation (3.36) can be rewritten as follows.

$$F \propto \sum_{i=0}^{b-1} \int_0^1 dy_1 \log \left[\frac{\theta\left(u_2 + i\omega + \omega y_1 - \frac{\sigma+\tau}{2}, b\omega\right)}{\theta\left(i\omega + \omega y_1 - u_2 - \frac{\sigma+\tau}{2}, b\omega\right)} \prod_{I=1}^3 \frac{\theta\left(\delta_I + i\omega + \omega y_1 - u_2 - \frac{\sigma+\tau}{2}, b\omega\right)}{\theta\left(\delta_I + u_2 + i\omega + \omega y_1 - \frac{\sigma+\tau}{2}, b\omega\right)} \right].\tag{3.57}$$

Therefore,

$$F \propto \int_0^1 dy_1 \log \left[\frac{\theta\left(u_2 - \omega y_1 + \omega - \frac{\sigma+\tau}{2}, \omega\right)}{\theta\left(\omega y_1 - u_2 - \frac{\sigma+\tau}{2}, \omega\right)} \prod_{I=1}^3 \frac{\theta\left(\delta_I + \omega y_1 - u_2 - \frac{\sigma+\tau}{2}, \omega\right)}{\theta\left(\delta_I + u_2 - \omega y_1 + \omega - \frac{\sigma+\tau}{2}, \omega\right)} \right], \quad (3.58)$$

where we use $\prod_{i=0}^{b-1} \theta(x + i\omega, b\omega) = \theta(x, \omega)$. We further define n_1, n_2 and x as follows

$$\frac{\sigma+\tau}{2} - \omega - u_2 = (n_1 - x)\omega, \qquad \frac{\sigma+\tau}{2} + u_2 = (n_2 + x)\omega. \tag{3.59}$$

Note that n_1 and n_2 are integers and $x \in [0, 1)$. Finally, let $\omega y_1 - \omega x = u'_{12}$, so that

$$F \propto \int_0^1 dy_1 \log \left[\frac{\theta(u'_{21} - n_1\omega, \omega)}{\theta(u'_{12} - n_2\omega, \omega)} \prod_{I=1}^3 \frac{\theta(\delta_I + u'_{12} - n_2\omega, \omega)}{\theta(\delta_I + u'_{21} - n_1\omega, \omega)} \right] \tag{3.60}$$

Using equation (3.38), one can show that

$$F \propto \int_0^1 dy_1 \log \left[-e^{-6\pi i u'_{12}} \frac{\theta(\delta_1 + u'_{12}, \omega)}{\theta(\delta_1 - u'_{12}, \omega)} \frac{\theta(\delta_2 + u'_{12}, \omega)}{\theta(\delta_2 - u'_{12}, \omega)} \frac{\theta(-\delta_1 - \delta_2 + u'_{12}, \omega)}{\theta(-\delta_1 - \delta_2 - u'_{12}, \omega)} \right] = 0, \quad (3.61)$$

which is the continuum limit of the Bethe ansatz equation.

Note that the 'off-shell' integrand of our matrix model and that of the Bethe ansatz approach are different. In this subsection, we found a relation between their on-shell conditions, at least for particular saddle points. In this sense, our approach is also related to the elliptic extension approach of [46, 47]. In [46], the on-shell physical quantities of their saddle points were shown to agree with those of the corresponding Bethe roots. So in a broader sense, all three approaches yield the same on-shell results. Also, the elliptic extension approach is similar to our approach in that both use the periodic nature of the integrand.

4 Multi-cut saddle points

In this section, we construct multi-cut saddle points. For a technical reason, we only consider the case with collinear σ, τ.[5] Our K-cut ansatz is roughly given by

$$u_A(x, y) \equiv \frac{A}{K} + \sigma x + \tau y \tag{4.1}$$

_____

[5]For non-collinear σ, τ, they do not satisfy the saddle point equation. Also, we could not find a modification of the saddle point problem like section 2 which makes them saddle points. Unlike the single-parallelogram ansatz, we find that after $\mathrm{SL}(3, \mathbb{Z})$ modular transformation there are extra poles included in the parallelogram as well as the zeros from the Haar measure, causing more complications. We feel that this is related to the gravitational stability issue of [24].

with $x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $A = 0, 1, \cdots, K-1$ labels the K groups of eigenvalues forming K cuts. The 'ϵ deformations' of this ansatz will be specified below, depending on the values of chemical potentials. Although these are linear cuts, we again make a 2-parameter labelling of eigenvalues with uniform 2d distributions. Each cut contains $\frac{N}{K}$ eigenvalues, at equal filling fraction, so the density function is given by $\rho_A(x, y) = \frac{1}{K}$.

The large N free energy $\log Z$ of (4.1) in the continuum limit is given by

$$-\frac{N^2}{K} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_2 dy_2 \sum_{A=0}^{K-1} V\left(u_{12} + \frac{A}{K}\right) \equiv -\frac{N^2}{K} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_2 dy_2 V_K(u_{12}),$$
(4.2)

where V is given by (3.3), and $V_K(u) \equiv \sum_{A=0}^{K-1} V(u + \frac{A}{K})$. The force on the eigenvalue at u_2 in the $A = 0$ cut is given by

$$F = 2\frac{N}{K} \int dx_1 dy_1 \frac{\partial}{\partial u_2} V_K(u_{12})$$
(4.3)

after plugging in our ansatz. If this F is zero, the forces on eigenvalues in the other cuts also vanish by cyclicity. $F = 0$ can be shown by confirming that

$$\int dx_1 dy_1 V_K(u_{12})$$
(4.4)

is independent of u_2. We start by noting that $V_K(u)$ can be written in terms of

$$\log \Gamma(z, \sigma, \tau) + \Gamma\left(z + \frac{1}{K}, \sigma, \tau\right) + \cdots + \log \Gamma\left(z + \frac{K-1}{K}\right) = \log \Gamma(Kz, K\sigma, K\tau) \ . \quad (4.5)$$

So the calculation of (4.4) can be done in a manner similar to the case with $K = 1$, by replacing parameters by K times them. Here we have to be careful about the ranges of the imaginary parts of δ_I, σ, τ, since multiplying them by K takes them away from our convention (2.22). To be definite, we consider δ_I's in the upper case of (2.22), satisfying $\sum_{I=1}^{3} \delta_I = \sigma + \tau - 1$. Recall that such δ_I's were parametrized as $\delta_I = -a_I + b_I(\sigma + \tau)$, with $0 < a_I < 1$, $0 < b_I < 1$ satisfying $\sum_I a_I = \sum_I b_I = 1$. Then $K\delta_I$'s and $K\sigma$, $K\tau$ satisfy

$$K\delta_I = K\sigma + K\tau - K \ . \quad (4.6)$$

Here we define $\{Ka_I\} \equiv Ka_I - \lfloor Ka_I \rfloor$, which measures the fractional part of $Ka_I \in [0, 1)$. Then one finds $\sum_{I=1}^{3} \{Ka_I\} \in [0, 3)$, which has to be an integer since $\sum_I a_I = 1$. The case with $\sum_{I=1}^{3} \{Ka_I\} = 0$ is very exceptional, which can be met only if all three Ka_I are integers (because we should have $\{Ka_I\} = 0$ for all I's). This is possible for fine-tuned choices of a_I's, e.g. at $a_1 = a_2 = a_3 = \frac{1}{3}$ and $K = 3$ or $a_1 = a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$ and $K = 4$, etc. We shall understand these special cases with small deformations, so that they satisfy either $\sum_{I=1}^{3} \{Ka_I\} = 1$ or 2. For instance, for $a_1 = a_2 = a_3 = \frac{1}{3}$ and $K = 3$, slightly reducing a_1, a_2 and slightly increasing a_3 will make $\{3a_1\} = \{3a_2\} \lesssim 1$ and $0 < \{3a_3\} \ll 1$, making them satisfy $\sum_I \{3a_I\} = 2$. (Slightly reducing a_1 and slightly increasing a_2, a_3 will yield $\sum_I \{3a_I\} = 1$.) With these understood, we define new parameters as $\sigma' \equiv K\sigma$, $\tau' \equiv K\tau$ and

$$\delta_I' = \begin{cases} -\{Ka_I\} + b_I(\sigma' + \tau') & \text{if } \sum_{I=1}^{3} \{Ka_I\} = 1 \\ 1 - \{Ka_I\} + b_I(\sigma' + \tau') & \text{if } \sum_{I=1}^{3} \{Ka_I\} = 2 \end{cases} .$$
(4.7)

In the upper and lower cases, δ_I' and σ', τ' satisfy $\sum_I \delta_I' - \sigma' - \tau' = \mp 1$, respectively.

To compute (4.4), we should compute

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \log \Gamma(Kz \pm Ku_{12}, K\sigma, K\tau) \tag{4.8}$$

at $z = \delta_I$ or $z = 0$, which can again be done by integrating two log functions over $x, y \in \left(-\frac{1}{2}, \infty\right)$ after a manipulation similar to (3.7). In fact one can recycle the calculation at $K = 1$ as follows. At the infinity branch where log functions can be defined by Taylor expansion, we regard $K\delta_I$ as δ_I' by trivial period shifts valid at infinity. So the parameters $\delta_I', \sigma', \tau'$ belong to one of the two cases of (2.22). Also, in the ansatz for the first cut $A = 0$, $u' \equiv Ku$ given in terms of σ', τ' takes precisely the same form as the single-cut configuration. Therefore, the calculations for the single-cut can be literally repeated here. As for the three terms at $z = \delta_I$, the continuous extensions of log functions can be made similarly, using (3.10). As for the terms at $z = 0$, we need to refine the ansatz with ϵ like (3.6), depending on which condition of (2.22) is met by δ_I'. In the upper case of (2.22) (or (4.7)), the ansatz deformation is precisely given in the same direction as (3.6), i.e.

$$u_A(x, y) = \frac{A}{K} + e^{-i\epsilon}(\sigma x + \tau y), \quad \epsilon > 0 . \tag{4.9}$$

This deformation yields u_2-independent (4.4), implying $F = 0$. In the lower case of (2.22) or (4.7), this is the 'conjugate sector' so the ansatz deformation guaranteeing $F = 0$ is given by

$$u_A(x, y) = \frac{A}{K} + e^{+i\epsilon}(\sigma x + \tau y) . \tag{4.10}$$

In both cases, the integral (4.4) is given by

$$-\frac{N^2}{K} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 V_K(u_{12}) = -\frac{\pi i N^2 \delta_1' \delta_2' \delta_3'}{K\sigma'\tau'} \tag{4.11}$$

by repeating (3.26). This is independent of u_2, which proves $F = 0$. Integrating (4.11) once more in x_2, y_2, one obtains the following large N free energy

$$\log Z = -\frac{\pi i N^2 \delta_1' \delta_2' \delta_3'}{K\sigma'\tau'} . \tag{4.12}$$

Inserting the values of primed variables, one obtains

$$\log Z = \begin{cases} -\dfrac{\pi i N^2 \prod_{I=1}^3 \left(\delta_I + \frac{\lfloor Ka_I \rfloor}{K}\right)}{\sigma\tau} & \text{for the upper case of (4.7)} \\[4mm] -\dfrac{\pi i N^2 \prod_{I=1}^3 \left(\delta_I + \frac{\lfloor Ka_I \rfloor}{K} + \frac{1}{K}\right)}{\sigma\tau} & \text{for the lower case of (4.7)} \end{cases} . \tag{4.13}$$

Note again that δ_I's are in the sector defined by the upper signs of (2.22). Also, the cases with all Ka_I's being integral should be understood with care, by slightly moving them away from the integral values as illustrated above (4.7).

The entropy can be obtained by Legendre transforming the entropy function at fixed charges:

$$S(Q_I, J_i; \delta_I, \sigma, \tau) = \log Z(\delta_I, \sigma, \tau) - 2\pi i \sum_I \delta_I Q_I - 2\pi i \sigma J_1 - 2\pi i \tau J_2 . \tag{4.14}$$

After extremizing this function with δ_I, σ, τ subject to the constraint $\sum_I \delta_I - \sigma - \tau = -1$, one takes the real part $\text{Re}(S)$ to get the entropy [11, 13, 28]. This computation can be done easily using the universal form (4.12), by noting that primed variables δ_I', σ', τ' are K times δ_I, σ, τ apart from real constant shifts. Namely, the entropy function is given by

$$S = -\frac{\pi i N^2 \delta_1' \delta_2' \delta_3'}{K \sigma' \tau'} - 2\pi i \sum_I \left(\delta_I' + \cdots\right) \frac{Q_I}{K} - 2\pi i \sigma' \frac{J_1}{K} - 2\pi i \tau' \frac{J_2}{K}, \qquad (4.15)$$

where $\cdots$ are real constants which only affect S by an imaginary constant. To compute $\text{Re}(S)$, we can ignore them. So without these terms, the Legendre transformation takes completely same form as that at $K = 1$, with replacements $N^2, Q_I, J_i \to \frac{N^2}{K}, \frac{Q_I}{K}, \frac{J_i}{K}$. The entropy at $K = 1$ is homogeneous degree 1 in scaling N^2 and all the other extensive quantities with a same factor (which is a basic property of AdS$_5$ black holes). Therefore, the entropy of the K-cut saddles is related to that of the basic saddle at $K = 1$ by $S_K(Q_I, J_i) = \frac{1}{K} S_{K=1}(Q_I, J_i)$. So they would be subdominant saddle points, compared to the basic saddle at $K = 1$, of the Euclidean quantum gravity. Ref. [24] suggested the gravity duals of these saddles at general K as $\mathbb{Z}_K$ quotients of the analytically continued Euclidean saddle for the Lorentzian black hole at $K = 1$.

When $\sigma = \tau$, the final results for $\log Z$ are same as the K-cut Bethe roots of [20] at $r = 0$, although the eigenvalue configurations are different. Our linear density function is triangular on each cut, which is a segment $(-\tau, \tau)$. On the other hand, each cut in the Bethe root is a uniform distribution along a segment $\left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$. Similar to our analysis of section 3, we expect that integrating the force function F over either x or y first will project our saddle point equation to the Bethe ansatz equation. We shall not study the details here.

We comment that when the complex chemical potentials are in a particular regime, we find a one parameter generalization of the K-cut solution when $K \equiv 2m$ is even. To understand this, let us again start from the following ϵ-deformed ansatz with a free parameter $\nu \in [0, 1]$

$$u(x, y) = \begin{cases} \frac{A}{K} + e^{-i\epsilon}(\sigma x + \tau y) , \ \rho(x) = \frac{2\nu}{K} & \text{if } A = \text{even} \\ \frac{A}{K} + e^{-i\epsilon}(\sigma x + \tau y) , \ \rho(x) = \frac{2(1-\nu)}{K} & \text{if } A = \text{odd} \end{cases} . \qquad (4.16)$$

Namely, there are $\frac{2N\nu}{K}$ eigenvalues in each cut at even $A = 0, 2, \cdots, 2m - 2$, and $\frac{2N(1-\nu)}{K}$ eigenvalues at odd $A = 1, 3, \cdots, 2m - 1$. The force is given by the u_2 derivative of

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \sum_{l=0}^{m-1} \left[\nu V\left(u_{12} + \frac{2l}{K}\right) + (1-\nu) V\left(u_{12} + \frac{2l+1}{K}\right)\right]$$

$$= \int dx_1 dy_1 \left[(1-\nu) \sum_{A=0}^{K-1} V\left(u_{12} + \frac{A}{K}\right) + (1-2\nu) \sum_{l=0}^{m-1} V\left(u_{12} + \frac{l}{m}\right)\right]$$

$$= \int dx_1 dy_1 \left[(1-\nu) V_K(u_{12}) + (1-2\nu) V_m(u_{12})\right] \qquad (4.17)$$

when u_2 is on the $A = 0$ cut. If this vanishes for arbitrary $\nu \in [0, 1]$, then the forces acting on eigenvalues on different cuts also vanish. Now repeating the discussions below (4.8),

one finds that the ϵ-deformed ansatz (4.16) makes $\int dx_1 dy_1 V_K(u_{12})$ to be u_2-independent if $\epsilon > 0$ and $\sum_I \{Ka_I\} = 1$, or if $\epsilon < 0$ and $\sum_I \{Ka_I\} = 2$. Similarly, $\int dx_1 dy_1 V_m(u_{12})$ would be u_2-independent if $\epsilon > 0$ and $\sum_I \{ma_I\} = 1$, or $\epsilon < 0$ and $\sum_I \{ma_I\} = 2$. Therefore, if $\sum_I \{Ka_I\}$ and $\sum_I \{ma_I\} = \sum_I \{\frac{K}{2}a_I\}$ have same value between 1 and 2, one can set the sign of ϵ so that both terms on the last line of (4.17) separately vanish. So in this case, we have constructed a saddle point which admits a nontrivial filling fraction of eigenvalues. The free energy is given by

$$\log Z = -\frac{\pi i N^2}{\sigma \tau} \left[(1 - 2\nu)^2 \prod_{I=1}^{3} \left(\delta_I + \frac{\lfloor ma_I \rfloor}{m} \right) + 4\nu(1-\nu) \prod_{I=1}^{3} \left(\delta_I + \frac{\lfloor Ka_I \rfloor}{K} \right) \right] \quad (4.18)$$

if $\sum_I \{Ka_I\} = \sum_I \{ma_I\} = 1$, and

$$\log Z = -\frac{\pi i N^2}{\sigma \tau} \left[(1-2\nu)^2 \prod_{I=1}^{3} \left(\delta_I + \frac{\lfloor ma_I \rfloor}{m} + \frac{1}{m} \right) + 4\nu(1-\nu) \prod_{I=1}^{3} \left(\delta_I + \frac{\lfloor Ka_I \rfloor}{K} + \frac{1}{K} \right) \right] \tag{4.19}$$

if $\sum_I \{Ka_I\} = \sum_I \{ma_I\} = 2$.

To be concrete, we consider the case with $K = 2$. a_I's would satisfy the condition $\sum_I \{2a_I\} = \sum_I \{a_I\} = 1$ if $a_1, a_2 \in (0, \frac{1}{2})$ and $a_3 \in (\frac{1}{2}, 1)$. The fundamental domain of this solution is $\nu \in [0, \frac{1}{2}]$ since it has a symmetry under $\nu \to 1 - \nu$ combined with an overall shift of $u_a \to u_a + \frac{1}{2}$. It continuously interpolates the one-cut solution $(K, r) = (1, 0)$ at $\nu = 0$ and the two-cut solution $(K, r) = (2, 0)$ at $\nu = \frac{1}{2}$. The free energy is given by

$$\log Z = -\frac{\pi i N^2}{\sigma \tau} \left[(1 - 2\nu)^2 \delta_1 \delta_2 \delta_3 + 4\nu(1-\nu)\delta_1 \delta_2 \left(\delta_3 + \frac{1}{2} \right) \right]$$
$$= -\frac{\pi i N^2 \delta_1 \delta_2 \left(\delta_3 + 2\nu(1 - \nu) \right)}{\sigma \tau} . \tag{4.20}$$

Its Legendre transformation at fixed charges Q_I, J_i can be easily done by noting that

$$\log Z - 2\pi i \sum_I \delta_I Q_I - 2\pi i \sigma J_1 - 2\pi i \tau J_2$$
$$\sim (1 - 2\nu + 2\nu^2) \left[-\frac{\pi i N^2 \hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3}{\hat{\sigma} \hat{\tau}} - 2\pi i \sum_I \hat{\delta}_I Q_I - 2\pi i \hat{\sigma} J_1 - 2\pi i \hat{\tau} J_2 \right] \tag{4.21}$$

where $\sim$ holds up to an irrelevant imaginary constant, and $\hat{\delta}_I$, $\hat{\sigma}$, $\hat{\tau}$ defined by

$$\hat{\delta}_{1,2} \equiv \frac{\delta_{1,2}}{1 - 2\nu + 2\nu^2}, \quad \hat{\delta}_3 \equiv \frac{\delta_3 + 2\nu(1 - \nu)}{1 - 2\nu + 2\nu^2}, \quad \hat{\sigma} \equiv \frac{\sigma}{1 - 2\nu + 2\nu^2}, \quad \hat{\tau} \equiv \frac{\tau}{1 - 2\nu + 2\nu^2} \quad (4.22)$$

satisfy $\sum_I \hat{\delta}_I - \hat{\sigma} - \hat{\tau} = -1$. The extremization of the expression inside the square bracket is completely the same as the free energy at $K = 1$. So taking the real part of the extremized entropy function, the entropy of our new saddle point at filling fraction ν is given by

$$\mathrm{Re} S_{K=2,\nu}(Q_I, J_i) = (1 - 2\nu + 2\nu^2)\mathrm{Re} S_1(Q_I, J_i) . \tag{4.23}$$

It will be interesting to seek for the gravity duals of these solutions, for instance in Euclidean quantum gravity by generalizing [24].

5 Conclusion

In this paper, we found exact large N saddle points of the $\mathcal{N} = 4$ index which are dual to BPS black holes in $AdS_5 \times S^5$. We employed two different approaches. Firstly, when the complex chemical potentials σ, τ for the two angular momenta J_1, J_2 in AdS are non-collinear, we showed that a novel areal distribution of eigenvalues illustrated in figure 1 solves the saddle point equation defined after applying the integral identity (1.2). $SL(3, \mathbb{Z})$ modularity of the elliptic gamma function was used to prove this. Secondly, when σ, τ are collinear, we showed that linear distributions obtained by collapsing the areal distribution also solves the traditional saddle point equation. The saddle points we constructed precisely account for the entropies of the dual black holes [7]. In the remaining part of this section, we emphasize several subtle structures of our results, and also suggest possible future directions.

In the collinear case, we found that an '$i\epsilon$ type' deformation is needed to precisely define our large N saddle point ansatz. We interpreted that ϵ is related to small $\frac{1}{N}$. Without such a refined definition, the continuum eigenvalue distribution hits the singularity of the potential. Unlike the principal-valued integral which excludes the unphysical self-interaction from the Haar measure potential, this singularity comes from interactions of distinct eigenvalues so should be avoided in any sensible large N ansatz. In our leading large N calculus, only the sign of ϵ mattered.

More physically, these singularities very close to our saddle point configurations are where the matrix elements of the gaugino operators become massless. For some eigenvalue pairs, these operators have 'effective fugacities' greater than 1 which means these operators may condense. More generally, whenever we made analytic continuations of log functions in section 3, using (3.10), the corresponding operators could have condensed. Understanding their structures may shed more light on more general types of black holes, for instance related to the hairy AdS black holes where certain operators assume nonzero expectation values in the dual CFT. For instance, hairy black holes in AdS_5 and $AdS_5 \times S^5$ were constructed [29–32]. Also, it will be interesting to see a more direct connection between our light gaugino operators and the light near-horizon modes on the BPS black holes.

Perhaps as a related matter, we also discuss the integration contour and poles/residues. The subtleties summarized in the previous two paragraphs appear because the matrix integral contour is analytically continued. Since our final saddle point has many pairs of eigenvalues whose log potentials require analytic continuations beyond their radii of convergence, it is quite likely that the full contour deformation would cross the poles (bosonic branch points) of the integrand. When the contour crosses a pole, various terms can appear. The first term is the full N dimensional contour integral, where the contour passed through the pole. This is the term that we studied in this paper. On the other hand, when a contour crosses a pole, one also finds an extra term from its residue. One may replace $n \, (< N)$ of the N integral variables by their pole values, obtaining a term of the form

$$(\text{residue of } n \text{ dimensional integral}) \times (N - n \text{ dimensional contour integral}) \,. \tag{5.1}$$

We are tempted to interpret the first factor as the n dual giant gravitons [33, 34] formed outside the event horizon of a core black hole made of $N - n$ eigenvalues. This interpretation sounds heuristic to us for the following reasons. There are two types of giant gravitons [33, 34], which are D3-branes wrapping contractible S^3 cycles in either AdS_5 or S^5. The dual giant gravitons are pointlike in S^5, while occupying a spatial S^3 in AdS_5 at a fixed AdS radius. They are domain walls in AdS_5, reducing the RR 5-form flux inside it by 1 unit. So inside n dual giant gravitons, the core black hole will feel only $N - n$ units of RR-flux. Since the second factor of (5.1) takes the form of rank $N - n$ integral, it would yield a black hole like saddle point in certain U($N - n$) gauge theory, qualitatively agreeing well with the bulk picture. Also note that the first factor has definite values of n eigenvalues, which are often viewed as the radial locations of dual giant gravitons in AdS [8, 35]. Finally, there are studies on the color superconductivity using these branes in the gravity dual [36], which in the BPS sector should necessarily include the hairs carrying other global charges.

Even if this picture is correct, such configurations look somewhat different from the hairy black holes of [29–32] constructed with the condensation of the Kaluza-Klein gravitons within the gravity approximation. In the vacuum AdS_5, shrinking the dual giant graviton by reducing its energy makes the S^3 small, converging to the point-like graviton picture when the energy is small. The giant graviton expanded in S^5 can also shrink to the same point-like graviton, and the two brane descriptions provide complementary descriptions of the $\frac{1}{8}$-BPS sector [35, 37]. Once there is a core black hole at the center of AdS, giant gravitons are still contractible in S^5. But a dual giant graviton in this black hole background can 'shrink' only until its radial position reaches the event horizon. In fact we have made a provisional study of both types of giant graviton probes in the background of [4]. The behaviors of giant gravitons in S^5 are well connected to the point-like gravitons, and appear to exhibit features somewhat similar to the BPS hairy black holes reported in [32] constructed using the KK graviton modes. The dual giant graviton probes are somewhat trickier for us to interpret. In any case, we think there are many interesting questions in this direction.

In section 4, we constructed multi-cut solutions, whose physics is the same as the multi-cut Bethe roots of [20]. We also provided further generalizations of these multi-cut solutions with nontrivial filling fractions on the cuts. Within our ansatz, such generalized filling fractions were allowed only when the chemical potentials δ_I are in a particular regime. It will be interesting to find their gravity duals.

We also note that some of the large N techniques explored in this paper might find applications to study black holes in AdS_4/CFT_3, AdS_6/CFT_5 or AdS_7/CFT_6. These problems have been studied in [38–42] and [11, 43, 44], but we think we can do more interesting large N studies.

We finally remark that the treatment of our section 2, applying the identity (1.2) to slightly change the saddle point problem, might find useful applications in other matrix models. Although this technique is familiar in some branches of our community (e.g. enumerating BPS operators more efficiently via contour integral), we are not aware of this idea applied to construct large N saddle points. The solutions we got after this procedure

also look quite novel, in that we found areal distributions rather than traditional linear cut distributions. This approach might be helpful in other matrix model problems.

Acknowledgments

A Saddles from $(\sigma + r, \tau + s)$ parallelograms

In this appendix, we shall find more parallelogram saddle points, extending the ideas of section 2. We shall find saddles which take the form of $(\sigma_r, \tau_s) \equiv (\sigma+r, \tau+s)$-parallelograms for given δ_I, σ, τ where $r, s \in \mathbb{Z}$. For simplicity, we only consider the case with upper sign in (2.22), with $\sum_I \delta_I - \sigma - \tau = -1$. At given σ_r, τ_s, one can always find unique $n_I^{(r,s)} \in \mathbb{Z}$ so that $\delta_I^{(r,s)} \equiv \delta_I + n_I^{(r,s)}$ belong to one of the following two cases:

$$\delta_I^{(r,s)} = \delta_I + n_I^{(r,s)} = -a_I^{(r,s)} + b_I(\sigma_r + \tau_s)\,, \quad \pm a_I^{(r,s)} \in (0,1),\ b_I \in (0,1)\,,$$

$$\delta_1^{(r,s)} + \delta_2^{(r,s)} + \delta_3^{(r,s)} - \sigma_r - \tau_s = -\sum_{I=1}^{3} a_I^{(r,s)} = \mp 1\,, \qquad \sum_{I=1}^{3} b_I = 1\,. \tag{A.1}$$

Results in the two cases are related to each other by the transformation

$$(\delta_I^{(r,s)}, \sigma_r, \tau_s) \to (-(\delta_I^{(r,s)})^*, -\sigma_r^*, -\tau_s^*)\,. \tag{A.2}$$

Our ansatz for the large N distribution $u(x,y) = x\sigma_r + y\tau_s$ with uniform density for $0 < x, y < 1$ will satisfy the saddle point equation when

$$0 < \mathrm{Im}\left(\frac{\sigma_r}{\tau_s}\right) < \mathrm{Im}\left(\frac{\delta_I^{(r,s)}}{\tau_s}\right) < -\mathrm{Im}\left(\frac{1}{\tau_s}\right) \quad \& \quad 0 < \mathrm{Im}\left(\frac{\delta_I^{(r,s)}}{\sigma_r}\right) < \mathrm{Im}\left(\frac{\tau_s - 1}{\sigma_r}\right) \tag{A.3}$$

or

$$0 < \mathrm{Im}\left(\frac{\tau_s}{\sigma_r}\right) < \mathrm{Im}\left(\frac{\delta_I^{(r,s)}}{\sigma_r}\right) < -\mathrm{Im}\left(\frac{1}{\sigma_r}\right) \quad \& \quad 0 < \mathrm{Im}\left(\frac{\delta_I^{(r,s)}}{\tau_s}\right) < \mathrm{Im}\left(\frac{\sigma_r - 1}{\tau_s}\right)\,, \tag{A.4}$$

for the case with upper sign in (A.1). For the case with lower sign, the saddle point equation is solved when the condition obtained by applying the transformation (A.2) to (A.3), (A.4) is met. When one of the above inequalities is satisfied, (σ_r, τ_s)-saddle contributes to the large N free energy as following:

$$\log Z(\delta_I, \sigma, \tau) = -\frac{\pi i N^2 \delta_1^{(r,s)} \delta_2^{(r,s)} \delta_3^{(r,s)}}{\sigma_r \tau_s}\,. \tag{A.5}$$

One may take $\mathrm{Im}\left(\frac{\tau_s}{\sigma_r}\right) \to 0$ limit from either condition of the above. Then, since $0 < a_I < 1$, both conditions are trivially satisfied. Also, as all final quantities are regular in this limit, we can safely conclude that at $\mathrm{Im}\left(\frac{\tau_s}{\sigma_r}\right) = 0$, i.e. $\tau_s = k\sigma_r$ ($k \in \mathbb{R}$), the large N saddle point equation is always satisfied.

The condition '(A.3) or (A.4)' can be merged into the following set of inequalities:

$$
\begin{aligned}
-\min\left[\mathrm{Im}(\sigma), \mathrm{Im}(\tau)\right] \leq &-\min\left[\frac{a_I^{(r,s)}\mathrm{Im}(\sigma)}{1-b_I}, \frac{(1-a_I^{(r,s)})\mathrm{Im}(\sigma)}{b_I}, \frac{a_I^{(r,s)}\mathrm{Im}(\tau)}{b_I}, \frac{(1-a_I^{(r,s)})\mathrm{Im}(\tau)}{1-b_I}\right] \\
&< \mathrm{Im}(\sigma)\mathrm{Re}(\tau_s) - \mathrm{Im}(\tau)\mathrm{Re}(\sigma_r) \\
&< \min\left[\frac{a_I^{(r,s)}\mathrm{Im}(\tau)}{1-b_I}, \frac{(1-a_I^{(r,s)})\mathrm{Im}(\tau)}{b_I}, \frac{a_I^{(r,s)}\mathrm{Im}(\sigma)}{b_I}, \frac{(1-a_I^{(r,s)})\mathrm{Im}(\sigma)}{1-b_I}\right] \\
&\leq \min\left[\mathrm{Im}(\tau), \mathrm{Im}(\sigma)\right] .
\end{aligned}
\tag{A.6}
$$

These are the conditions to be met in the case with upper sign in (A.1). For the case with lower sign, the condition to be met is obtained by acting (A.2) on (A.6). Remarkably, the last condition takes the form of (A.6) with $a_I^{(r,s)}$ replaced by $\tilde{a}_I^{(r,s)} \equiv a_I^{(r,s)} + 1$. $a_I^{(r,s)}$ satisfy $\sum_I a_I^{(r,s)} = 1$, while $\tilde{a}_I^{(r,s)}$ satisfy $\sum_I \tilde{a}_I^{(r,s)} = 2$.

We want to find integers (r,s) which meet the above inequalities at $I = 1, 2, 3$, when δ_I, σ, τ are given. We have employed an algebraic procedure to solve (A.6) systematically. From (A.6) we can get geometric reasonings of our statements below. We shall only present the results.

i. $\frac{\tau}{\sigma} \in \mathbb{R}$: collinear case.

1) $\boldsymbol{\sigma = \tau}$. In this case, (A.6) is satisfied iff $r = s$. These correspond to the $(K, r) = (1, r)$ Bethe roots when $\sigma = \tau$. According to [24], the stable Euclidean black hole solution exists iff $r = s$ as we found.

2) $\boldsymbol{\tau = \frac{q}{p}\sigma}$ ($\boldsymbol{p, q}$ are coprime integers). In this case, (A.6) is satisfied iff $s = \frac{q}{p}r$. These correspond to the $(K, r) = (1, r)$ Bethe roots when $\tau = \frac{q}{p}\sigma$ [45]. These saddles are labelled by an integer l as $(r, s) = (pl, ql)$.

3) $\boldsymbol{\tau = k\sigma}$ ($\boldsymbol{k \in \mathbb{R}\backslash\mathbb{Q}}$). In this case, there are 'infinitely many' choices of (r, s) satisfying (A.6), but we cannot explicitly write down possible (r, s). They depend on $\delta_I, \mathrm{Im}(\sigma), \mathrm{Im}(\tau)$. (See the comments at the end of this appendix for the true meaning of 'infinitely many.')

ii. $\frac{\tau}{\sigma} \notin \mathbb{R}$: non-collinear case.

1) $\mathrm{Im}(\tau) = \frac{q}{p}\mathrm{Im}(\sigma)$ ($\boldsymbol{p, q}$ are coprime integers). In this case, depending on (σ, τ) the solutions may or may not exist. If there exist solutions, there are infinitely many. Given one solution (r_0, s_0) satisfying (A.6), all other solutions (r, s) are related to (r_0, s_0) by $(r, s) = (r_0 + pl, s_0 + ql)$ for some integer l. (However, not all values of l are allowed in general.) The case i-2 is the special case when $r_0 = s_0 = 0$ and (A.6) is satisfied for all integers l.

2) $\mathrm{Im}(\tau) = k\mathrm{Im}(\sigma)$ ($k \in \mathbb{R}\backslash\mathbb{Q}$). In this case, there exist 'infinitely many' choices of (r, s) satisfying (A.6) just as the case i-3. We cannot explicitly write down possible (r, s). They depend on $\delta_I, \mathrm{Im}(\sigma), \mathrm{Im}(\tau)$.

In summary, when $\frac{\mathrm{Im}(\tau)}{\mathrm{Im}(\sigma)} \in \mathbb{Q}$, there can be either 'infinitely many' saddle points or no saddle points. When $\frac{\mathrm{Im}(\tau)}{\mathrm{Im}(\sigma)} \notin \mathbb{Q}$, 'infinitely many' saddle points exist.

When we have multiple saddle points labelled by (r, s), we should sum over their contributions to the index. One can easily show that their leading large N entropies after the Legendre transformation are all the same. Namely, the real part of the following entropy function does not depend on the integer shifts $r, s, n_I^{(r,s)}$,

$$S = -\frac{\pi i N^2 \delta_1^{(r,s)} \delta_2^{(r,s)} \delta_3^{(r,s)}}{\sigma_r \tau_s} - 2\pi i \delta_I^{(r,s)} Q_I - 2\pi i \sigma_r J_1 - 2\pi i \tau_s J_2 + 2\pi i (n_I^{(r,s)} Q_I + r J_1 + s J_2),$$
(A.7)

since the dependence on $n_I^{(r,s)}$, r, s is collected to a pure imaginary constant (the last term).

It is known that there are finite numbers of Bethe roots at $K = 1$ [20], labeled by finitely many independent r's due to the symmetry $r \sim r + N$. This happens because N eigenvalues are exactly equal-spaced. It would be interesting to ask if our r, s enjoy similar symmetries. However, such a property is impossible to study in our large N continuum formalism. Therefore, when we say that we have found 'infinitely many' solutions for (r, s), this might imply finitely many solutions whose number scales with large N.

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