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Singular products and universality in higher-derivative conformal theory

Yuri Makeenko

NRC "Kurchatov Institute" — ITEP,
Moscow, Russia

E-mail: makeenko@itep.ru

Abstract: I investigate universality of the two-dimensional higher-derivative conformal theory using the method of singular products. The previous results for the central charge at one loop are confirmed for the quartic and six-derivative actions.

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1 Introduction

Corformal invariance of Polyakov's string formulation is widely used since 1980's (see [1]). The simplest observable for a closed string without external sources is then the string susceptibility index $\gamma_{\rm str}$ (also known as the gravity anomalous dimension) which is defined through the number of surfaces of large area A as

$$\left\langle \delta\left(\int\sqrt{g}-A\right)\right\rangle \propto A^{\gamma_{\rm str}-3}\,{\rm e}^{CA}. \tag{1.1}$$

Here the constant C in the exponent is not universal, i.e. depends on the regularization procedure applied to the string, but the pre-exponential is perfectly universal and describes a gravitational dressing of the unit operator.

The celebrated calculation of $\gamma_{\rm str}$ was performed by Knizhnik-Polyakov-Zamolodchikov [2] (KPZ) by fixing the light-cone gauge and by David [3], Distler-Kawai [4] (DDK) using the conformal gauge with the result

$$\gamma_{\rm str}=(1-h)\left[\frac{d-25-\sqrt{(25-d)(1-d)}}{12}\right]+2 \tag{1.2}$$

for a surface of genus h embedded in d Euclidean dimensions. Equation (1.2) extends the one-loop result [5–7] to all orders in $1/d$ (as $d \to -\infty$) and describes critical indices of the vast amount of models in Statistical Mechanics where the central charge $c = d < 1$. However, as is seen from eq. (1.2), $\gamma_{\rm str}$ is not real for $1 < d < 25$ as it should say for the QCD string in $d = 4$. This was referred to as the $d = 1$ barrier for the string existence.

The derivation of eq. (1.2) is based on the Liouville action which emerges from the Polyakov string after doing the path integral over all fields but the independent world-sheet metric tensor g_{ab}. The integration over the target-space coordinates X^μ is commonly done by the DeWitt-Seeley expansion of the heat kernel [8, 9]

$$\left\langle \omega \middle| \mathrm{e}^{\tau\Delta} \middle| \omega \right\rangle = \frac{1}{4\pi\tau} + \frac{1}{24\pi} R(\omega) + \frac{\tau}{120\pi}\left[\Delta R(\omega) + \frac{1}{2} R^2(\omega)\right] + \mathcal{O}(\tau^2), \tag{1.3}$$

where Δ denotes the two-dimensional Laplacian and R is the scalar curvature for the metric tensor g_{ab}. The conformal gauge is fixed by choosing

$$g_{ab} = \hat{g}_{ab}\,\mathrm{e}^{\varphi} \tag{1.4}$$

with $\hat{g}_{ab}$ being the background (also termed fiducial) metric tensor and φ being a dynamical variable often termed the Liouville field. Also the ghosts have to be added when fixing the gauge (1.4).

Equation (1.3) results after the path-integrating over X^μ in the piece of the *emergent* action for φ which is proportional to d. Truncating the DeWitt-Seeley expansion at the first four terms shown in eq. (1.3) and adding the contribution from the ghosts, we arrive at the following emergent action for the Polyakov string:

$$\mathcal{S}^{\rm Pol} = \frac{1}{16\pi b_0^2}\int \sqrt{\hat{g}}\left[\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + 2\mu_0^2\,\mathrm{e}^{\varphi} + \varepsilon\,\mathrm{e}^{-\varphi}(\hat{\Delta}\varphi)^2\right], \qquad b_0^2 = \frac{6}{26-d}. \tag{1.5}$$

Here the first two terms forming the Liouville action are familiar from the original work by Polyakov [10]. The remaining third term is familiar from the studied [11–13] of R^2 gravity in two dimensions. The constant $\varepsilon \propto \tau/\sqrt{\hat{g}}$ originates from two contributions. The first comes from the path integration over X^μ and is proportional to d. It is easily calculable from the last two terms shown in the DeWitt-Seeley expansion (1.3). The second comes from the path integral over the ghosts. I shall give more details on this issue in the next section.

An analogue of the emergent action (1.5) can be derived also for the Nambu-Goto string. Doing again the path integral over all fields but g_{ab}, we arrive in the conformal gauge (1.4) at the emergent action of the type (1.5) but with the additional term

$$\mathcal{S} = \frac{1}{16\pi b_0^2}\int \sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + 2\mu_0^2\,\mathrm{e}^{\varphi} + \varepsilon\,\mathrm{e}^{-\varphi}\left[(\hat{\Delta}\varphi)^2 - G\hat{g}^{ab}\,\partial_a\varphi\partial_b\varphi\hat{\Delta}\varphi\right]\right\}. \tag{1.6}$$

It is the most general diffeomorphism-invariant action with four derivatives. All other terms with four derivatives can be reduced to these two (modulo boundary terms in the case of an open string). The term with G does not appear for the Polyakov string. Its occurrence is specific [14] to the Nambu-Goto string. How to compute the value of G for the Nambu-Goto string will be outlined in section 2 and appendix A. I shall not concentrate however at that particular value of G and rather consider the action (1.6) as such.

Strictly speaking eq. (1.6), as derived by the path integration over all the fields but g_{ab}, holds only for flat backgrounds when $\hat{R}$ — the scalar curvature for the metric tensor $\hat{g}_{ab}$ — vanishes. As is well-known, under the decomposition (1.4) one has

$$\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} - \hat{\Delta}\varphi\right) \tag{1.7}$$

with $\hat{\Delta}$ being the Laplacian for the metric tensor $\hat{g}_{ab}$, which results in the appearance of the addition $\propto \varphi\hat{R}$ to the Liouville action and similar terms for the four-derivative action (1.6), causing a non-minimal interaction of φ with background gravity. These additional terms are crucial for the Weyl invariance of the four-derivative action (1.6) as well as for the derivation of the "improved" energy-momentum tensor in flat space. I shall return to this issue in subsection 3.1. The action (1.6) is thus conformal invariant in flat space in spite of the presence of the dimensionful parameters μ_0^2 and ε, so the methods of conformal field theory can be applied.

The parameter τ of the DeWitt-Seeley expansion (1.3) is actually Schwinger's proper-time regularization of ultraviolet divergences in the path integral. Thus ε being proportional to the target-space cutoff τ is negligibly small. For this reason the four-derivative terms in the action (1.6) are classically suppressed for smooth metrics as εR. However, the role of the parameter ε in the quantum case is twofold. Firstly, the quartic derivative regularizes divergences with ε playing the role of an ultraviolet worldsheet cutoff. Secondly, ε is simultaneously a coupling constant of the self-interaction of φ so uncertainties like $\varepsilon \times \varepsilon^{-1}$ appear in the perturbation theory. In other words, typical metrics essential in the path integral over g_{ab} are not smooth and have $R \sim \varepsilon^{-1}$. These uncertainties look like anomalies in quantum field theory and may affect the large-distance behavior of strings as argued in [14].

In particular, the string susceptibility index computed for the four-derivative action (1.6) with the one-loop accuracy equals [15, 16]

$$\gamma_{\rm str} = (h-1)\left(\frac{1}{b_0^2} - \frac{7}{6} - G + \mathcal{O}(b_0^2)\right) + 2, \qquad b_0^2 = \frac{6}{26-d} \tag{1.8}$$

for closed surfaces of genus h, showing for $G \neq 0$ a deviation from the one-loop result [5–7] for the Polyakov string for which $G = 0$.

Equation (1.8) was derived from the action (1.6) in three different ways. Firstly, the technique [2–4] of conformal field theory developed for the Liouville action was applied to compute the central charge of φ at one loop and, secondly, the results where confirmed [15] by a direct computation of the one-loop diagrams of quantum field theory, which describe the renormalization of the propagator and the energy-momentum tensor. The third method has been recently proposed [16] as a pragmatic mixture of the two, accounting for the quantum equation of motion. It has also reproduced eq. (1.8) via emerging singular products.

We see from eq. (1.8) that the string susceptibility for the four-derivative action (1.5) coincides at one loop with the one for the Liouville action. A natural question is as to whether this holds to all loops? Another natural question is what about a more general action of the form

$$S^{\rm gen}[\varphi] = -\frac{1}{16\pi b_0^2}\int \sqrt{\hat{g}}\varphi\hat{\Delta}F(-\varepsilon\,{\rm e}^{-\varphi}\hat{\Delta})\varphi, \qquad F(0) = 1 \tag{1.9}$$

which differs from the action (1.5) by the terms of order ε^2 and higher. The action (1.9) has been discussed recently in [14, 17]. It arises by covariantizing a free higher-derivative action which is quadratic in φ and therefore modifies the propagator. The function F depends on the applied regularization and has been computed [18] to all orders in ε. The action (1.9) is a part of the most general higher-derivative action generated by yet higher-order terms of the DeWitt-Seeley expansion of the heat kernel after the path-integration over X^μ and the ghosts in the Polyakov string formulation. The difference between (1.9) and the most general action shows up already at the order ε^2, where the term $\varepsilon^2 R\Delta R$ is captured by (1.9) but the term $\varepsilon^2 R^3$ is not.

This Paper is addressed to the question of the universality, i.e. an independence of the central charge and γ_{str} on the precise form of the higher-derivative emergent action. In analyzing this I found it most useful to apply the method based on the singular products which has been recently developed in [16]. After a brief reviewing of the subject in section 2, I concentrate on the universality for higher-derivative actions. It is demonstrated in sections 3 and 4 by showing that the contribution to the central charge of φ from the four-derivative R^2 and six-derivative $R\Delta R$ or R^3 terms vanishes, so only the usual one coming from the quadratic part remains. How the universality works for the action (1.9) is illustrated by an example in section 5. I also apply the method of singular products to confirm eq. (1.8) by computing in subsection 3.3 the central charge of φ at one loop via the variation of the energy-momentum tensor under an infinitesimal conformal transformation. Appendix A is devoted to the computation of G in eq. (1.6) as emerging from the Nambu-Goto string. Numerous formulas for the singular products are derived in appendix B.

2 Preliminaries and the setup

Let us begin with reminding the relation between the Polyakov and Nambu-Goto string formulations. The action of the Polyakov string

$$S = \frac{K_0}{2} \int \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X, \tag{2.1}$$

where $K_0 = 1/2\pi\alpha'_0$ stands for the bare string tension, is quadratic in X^μ that makes it easy to integrate it out in the path integral. The world-sheet metric tensor g_{ab} is an independent field in the path integral. The Nambu-Goto action of the bosonic string is the area of the string worldsheet. It is highly nonlinear in X^μ but can be made quadratic introducing the (imaginary) Lagrange multiplier λ^{ab} and an independent metric tensor g_{ab} as

$$S_{\rm NG} = K_0 \int \sqrt{\det\left(\partial_a X \cdot \partial_b X\right)} = K_0 \int \left[\sqrt{g} + \frac{1}{2}\lambda^{ab}(\partial_a X \cdot \partial_b X - g_{ab})\right]. \tag{2.2}$$

In both cases it is convenient to diagonalize g_{ab}, choosing the conformal gauge (1.4). This procedure adds ghosts which are the same for both string formulations.

In the classical limit when $\alpha'_0 \to 0$ or $K_0 \to \infty$ we have

$$g^{\rm (cl)}_{ab} = \partial_a X \cdot \partial_b X, \tag{2.3}$$

i.e. it coincides with the induced metrics. Analogously $\lambda^{ab} = \sqrt{g} g^{ab}$ for the classical ground state. The substitution of (2.3) into the action (2.1) then reproduces the Nambu-Goto action (2.2). It was also demonstrated [19] that both string formulations give the same results at one loop providing the zeta-function regularization is used. The general argument in favor of the equivalence of the two string formulations is based [1] on the fact that λ^{ab} is localized at the distances of the order of the ultraviolet (UV) cutoff and does not propagate to macroscopic distances. I shall return soon to this argument.

A separate remark is required about the stability of the classical ground state (2.3). As shown in [20], it is stable only for $d \leq 2$. For $d > 2$ the mean-field ground state is stable instead for which $g_{ab} = \bar{\rho} g^{(\mathrm{cl})}_{ab}$ and $\lambda^{ab} = \bar{\lambda} \sqrt{g} g^{ab}$ with certain d-dependent values of $\bar{\rho} > 1$ and $\bar{\lambda} < 1$ well-defined for $d > 2$. The continuum limit is reached in the scaling regime when the UV cutoff is going away. Then $\bar{\rho} \to \infty$ forcing the stringy continuum limit to be Lilliputian [21]. I shall not touch upon that issue in this Paper which deals with the one-loop approximation justified by $d \to -\infty$ when the ground state is classical.

As is already mentioned, the main reason to introduce the Lagrange multiplier for the Nambu-Goto string was to path integrate over X^μ (as well as over the ghosts) to obtain the emergent action for the fields φ and λ^{ab}. The arising determinants diverge and have to be regularized. What regularization to use for this purpose is a matter of personal taste. I prefer to use the covariant Pauli-Villars regularization which was first introduced for the ghosts [22]. To regularize X^μ we then introduce the massive regulators Y^μ which obey wrong statistics and add to (2.2) the regulator action

$$S_{\mathrm{reg}} = \frac{K_0}{2} \int \left(\lambda^{ab} \partial_a Y \cdot \partial_b Y + M^2 \sqrt{\hat{g}}\, \mathrm{e}^{\varphi}\, Y^2 \right). \tag{2.4}$$

Now every loop of the regulator field Y^μ brings the minus sign to compensate divergences coming from X^μ.

Actually, we have to have [23] two such regulators of mass squared M^2 with wrong statistics which can be viewed as anticommuting Grassmann variables Y^μ and $\bar{Y}^\mu$ and one regulator Z^μ of mass squared $2M^2$ with normal statistics to regularize all the divergences including the ones in tadpole diagrams. But for the purposes of computing the finite parts (like anomalies) only one regulator will be enough because the contributions of the two others are canceled being independent of the masses.

For the covariant Pauli-Villars regularization we can apply the standard methods of quantum field theory. It is seen from eqs. (2.2) and (2.4) how the vertices of the interaction of the fields φ and λ^{ab} with X^μ and the regulators arise.[1] Feynman's diagrammatic technique can be used for the calulation of the emergent action. Also Noether's theorems apply to the system of X^μ plus the regulators. In particular, the total energy-momentum tensor can be derived, which is conserved and traceless thanks to the classical equations of motion. The covariant Pauli-Villars regulators thus preserve conformal symmetry in spite of they are massive. For this reason the emergent action will be conformal invariant which is crucial for what follows. The coeffients of the Taylor series of the function F in eq. (1.9)

[1]For the Polyakov string we substitute $\lambda^{ab} = \sqrt{\hat{g}} \hat{g}^{ab}$ so only the interaction between φ and the regulators remains.

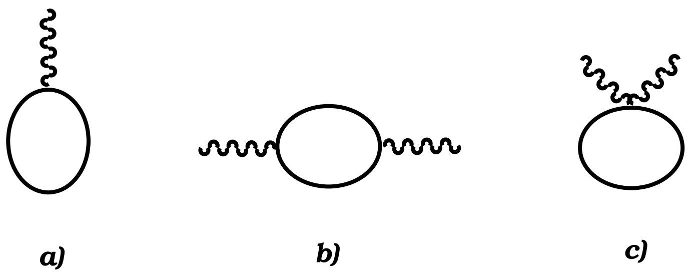

Figure 1. Diagrams contributing to the emergent action to quadratic order in φ or λ^{ab} represented by the wavy lines. The solid line represents the loop of either X^μ or the regulators.

and other terms in the emergent action depend on the regularization applied. For the proper-time and Pauli-Villars regularizations they are related by simple formulas as shown in [18] (appendix A).

2.1 The emergent action

The fields X^μ enter the action (2.2) quadratically (the same for the ghost fields in the ghost action) so they can be integrated out. In [14] the action, governing fluctuations of φ and λ^{ab}, that emerges from the Nambu-Goto string was analyzed by the use of the covariant Pauli-Villars regularization. I briefly reiew in this subsection the result and repeat the derivation by using the proper-time regularization in appendix A.

The diagrams which contribute after the path integration over X^μ and its regulators to the emergent action to quadratic order in φ or λ^{ab} are depicted in figure 1. The wavy lines represent either φ or λ^{ab} while the solid line represents the loop of either X^μ or its regulators. Covariantizing, we arrive at the following emergent action of the Nambu-Goto string:

$$\begin{aligned}
&\mathcal{S}_X[\varphi,\lambda^{ab}] \\
&\quad = \frac{d}{2}\int\left[-\frac{\sqrt{\hat g}\,\mathrm{e}^{\varphi}\Lambda^2}{\sqrt{\det\lambda^{ab}}}+\frac{1}{48\pi}\left(\sqrt{\hat g}\varphi\hat\Delta\varphi+\lambda^{ab}\hat g_{ab}\hat\Delta\varphi+2\lambda^{ab}\nabla_a\partial_b\varphi\right)+\frac{\sqrt{\hat g}\,\mathrm{e}^{-\varphi}}{160\pi M^2}(\hat\Delta\varphi)^2\right] \\
&\qquad +\mathcal{O}(M^{-4}), \qquad\qquad \Lambda^2=\frac{M^2}{2\pi}\log 2
\end{aligned} \tag{2.5}$$

with ∇_a being the covariant derivative for g_{ab} given by eq. (1.4). Expanding $1/\sqrt{\det\lambda^{ab}}$ in the fluctuating part $\delta\lambda^{ab}=\lambda^{ab}-\sqrt{\hat g}\hat g^{ab}$ [cf. eq. (A.10)], we see from the action (2.5) that $\delta\lambda$'s have the mass squared $\propto\tau^{-1}$ and are therefore localized at the distances $\sim\sqrt{\tau}$.

The contribution from the ghosts associated with fixing the conformal gauge is just the same as for the Polyakov string

$$\mathcal{S}_{\rm gh}[\varphi]=\int\sqrt{\hat g}\,\mathrm{e}^{\varphi}\left[\Lambda^2-\frac{13}{48\pi}\varphi\Delta\varphi-\frac{11}{160\pi M^2}(\Delta\varphi)^2\right]+\mathcal{O}(M^{-4}). \tag{2.6}$$

The fist two terms on the right-hand side are well-known and the third one has been recently computed [18].

Path-integrating over λ^{ab} by the saddle-point method as described in appendix A, using the identity (A.14) and dropping the terms $\mathcal{O}(M^{-4})$, we reproduce the four-derivative action (1.6).

2.2 "Improved" energy-momentum tensor

The T_{zz} component of the energy-momentum tensor associated with the action (1.6) reads [15]

$$
\begin{aligned}
-4b_0^2 T_{zz} = (\partial\varphi)^2 - 2\varepsilon\partial\varphi\partial\Delta\varphi - 2\partial^2(\varphi - \varepsilon\Delta\varphi) - G\varepsilon(\partial\varphi)^2\Delta\varphi + 4G\varepsilon\partial\varphi\partial(\,\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) \\
- 4G\varepsilon\partial^2(\,\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) + G\varepsilon\partial(\partial\varphi\Delta\varphi) + G\varepsilon\frac{1}{\bar\partial}\partial^2(\bar\partial\varphi\Delta\varphi),
\end{aligned}
\tag{2.7}
$$

where $\Delta = 4\,\mathrm{e}^{-\varphi}\partial\bar\partial$ when using the conformal coordinates z and $\bar z$ for a flat metric tensor $\hat g_{ab}$. We have used here the notation $\partial \equiv \partial/\partial z$ and $\bar\partial \equiv \partial/\partial\bar z$. Notice the nonlocality of the last term in (2.7) which is inherited from a nonlocality of the covariant generalization of the action (1.6). It is the presence of this nonlocal term which plays a crucial role in the computation of the addition $6G$ to the central charge at one loop.

It is important that the energy-momentum tensor (2.7) is "improved" á la Callan-Coleman-Jackiw [24, 25]. It is conserved and traceless owing to the classical equation of motion despite ε is dimensionful. This is a consequence of diffeomorphism invariance of the action (1.6) which thus possesses conformal symmetry at least at the classical level.

The action (1.6) generates the vertex

$$
\langle\varphi(k)\varphi(p)\varphi(q)\rangle_{\text{truncated}} = \frac{\varepsilon}{8\pi b_0^2}\left(k^2p^2 + k^2q^2 + p^2q^2\right)\delta^{(2)}(k+p+q) \tag{2.8}
$$

which depend on momenta, so some diagrams diverge inspite of the k^4 in the propagator. This produces generically quadratic divergences in (tadpole) diagrams of perturbation theory.

To regularize the divergences we implement the covariant Pauli-Villars regularization [22], adding to (1.6) the following action for the regulator field Y:

$$
\mathcal{S}^{(Y)} = \frac{1}{16\pi b_0^2}\int\sqrt{\hat g}\left\{\hat g^{ab}\partial_a Y\partial_b Y + M^2\,\mathrm{e}^{\varphi}Y^2 + \varepsilon\,\mathrm{e}^{-\varphi}[(\hat\Delta Y)^2 - G\varepsilon\hat g^{ab}\partial_a Y\partial_b Y\hat\Delta\varphi]\right\}. \tag{2.9}
$$

It has a very large mass M and obeys wrong statistics to produce the minus sign for every loop, regularizing divergences coming from the loops of φ. I use in eq. (2.9) the same letter Y as for the regulator Y^μ in eq. (2.4) but this should not cause any problems.

Once again, the introduction of one regulator is not enough to regularize all the divergences. Some logarithmic divergences still remain. The correct procedure is to introduce two regulators of mass squared M^2 with wrong statistics, which can be represented via anticommuting Grassmann variables, and one regulator of mass squared $2M^2$ with normal statistics. Then all diagrams including quadratically divergent tadpoles will be regularized. However, for the purposes of computing final parts one regulator Y would be enough because the contributions of the two others are canceled being independent on the masses.

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The contribution of the regulators to the T_{zz} component of the energy-momentum tensor for the action (2.9) reads

$$-4b_0^2 T_{zz}^{(Y)} = \partial Y \partial Y - 2\varepsilon \partial Y \partial \Delta Y - G\varepsilon \partial Y \partial Y \Delta\varphi + 4G\varepsilon \partial\varphi \partial(\,\mathrm{e}^{-\varphi}\partial Y \bar\partial Y) - 4G\varepsilon \partial^2(\,\mathrm{e}^{-\varphi}\partial Y \bar\partial Y). \tag{2.10}$$

The total one is the sum of (2.7) and (2.10). The total energy-momentum tensor is conserved and traceless thanks to the classical equations of motion for φ and Y. Thus the Pauli-Villars regulators are classically conformal fields in spite of they are massive. For this reason the effective action which emerges after the path-integrating over the regulators will be conformal invariant. To the quartic order in the derivatives it will be again of the type in eq. (1.6) but with renormalized parameters.

In the infrared limit the effective action, governing smooth fluctuations of φ, becomes the Liouville action and the effective energy-momentum tensor is quadratic

$$T_{zz}^{(\mathrm{eff})} = \frac{1}{2b^2}\left(q\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\right). \tag{2.11}$$

Here b^2 is the renormalization of φ, i.e. the change $b_0^2 \to b^2$ in the action (1.6) and q enters the renormalization of T_{zz}. The arguments are similar to David-Distler-Kawai (DDK) [3, 4]. In the usual case of the Liouville action where $\varepsilon = 0$ in (1.6) they obey the DDK equation

$$\frac{6q^2}{b^2} + 1 = \frac{6}{b_0^2}, \tag{2.12}$$

where the left-hand side is the central charge of φ. For the Polyakov string eq. (2.12) represents the vanishing of the total central charge.

An analogous computation of the central charge of φ for the higher-derivative action (1.6) at one loop results in [15, 16]

$$c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6G\left(1 - 2\int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}\right) + \mathcal{O}(b_0^2). \tag{2.13}$$

The logarithmic divergence on the right-hand side cancels with the one in the string susceptibility, so it is finite and given by eq. (1.8). Both additional finite and divergent parts come from the nonlocal (last) term in (2.7).

The emergence of the logarithmic divergence is due to subtleties in the realization of conformal symmetry generated by the energy-momentum tensor (2.7) which is classically not a primary conformal field. Under the infinitesimal conformal transformation

$$\delta_\xi \varphi = \xi' + \xi\partial\varphi \tag{2.14}$$

it changes as

$$\begin{aligned}\delta_\xi T_{zz}^{(\varphi)} = {}& \frac{1}{2b_0^2}\xi''' + 2\xi' T_{zz} + \xi\partial T_{zz} + \frac{1}{b_0^2} G\varepsilon\,\mathrm{e}^{-\varphi}\Big\{\xi''''\bar\partial\varphi + \xi'''(\partial\bar\partial\varphi - 3\partial\varphi\bar\partial\varphi) \\ & + \xi''\Big[2\bar\partial\varphi(\partial\varphi)^2 - \partial\varphi\partial\bar\partial\varphi - \bar\partial\varphi\partial^2\varphi\Big] - \mathrm{e}^{\varphi}\frac{1}{\bar\partial}[\xi''\partial(\,\mathrm{e}^{-\varphi}\bar\partial\varphi\partial\bar\partial\varphi)]\Big\},\end{aligned} \tag{2.15}$$

while the usual definition of the central charge c relies on the transformation law

$$\delta_\xi T_{zz} = \frac{c}{12}\xi''' + 2\xi' T_{zz} + \xi\partial T_{zz} \tag{2.16}$$

prescribed for the conserved tensorial field (a descendant of the unit operator).

The difference between the right-hand sides of eqs. (2.15) and (2.16) is due to the presence in the action (1.6) at $G \neq 0$ of the additional term involving the structure $g^{ab}\partial_a\varphi\partial_b\varphi$ which is scalar but not primary and transforms under (2.14) as

$$\delta_\xi \left(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi\right) = \xi\partial\left(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi\right) + \xi''\,\mathrm{e}^{-\varphi}\bar\partial\varphi. \tag{2.17}$$

Additional terms do not appear for $G=0$ because the scalar curvature $R=-4\,\mathrm{e}^{-\varphi}\partial\bar\partial\varphi$ is a primary scalar. Those also do not appear for $T^{(Y)}_{zz}$ given by eq. (2.10) which obeys eq. (2.16) with $c=0$.

Averaging (2.15) over φ, we get at one loop the following ξ''' term:

$$\langle\delta_\xi T_{zz}(0)\rangle = \xi'''(0)\left(\frac{1}{2b_0^2} - G\int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}\right), \tag{2.18}$$

reproducing the logarithmic divergence in eq. (2.13). The logarithmic divergence would appear neither in the central charge nor in the string susceptibility within the operator formalism where the operators are normal-ordered.[2] The conformal transformation for such a case of the most general action $S[\varphi]$ which is not quadratic in the fields and whose energy-momentum tensor does not obey eq. (2.16) can be generated by

$$\hat\delta_\xi \equiv \int_{D_1}\left(\xi'\frac{\delta}{\delta\varphi} + \xi\partial\varphi\frac{\delta}{\delta\varphi}\right) \overset{\text{w.s.}}{=} \int_{C_1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\xi(z)T_{zz}(z), \tag{2.19}$$

where the domain D_1 includes the singularities of $\xi(z)$ leaving outside possible singularities of the function $X(\omega_i)$ on which $\hat\delta_\xi$ acts and C_1 bounds D_1. The second equality in (2.19) is understood in the weak sense, i.e. under path integrals. In proving the equivalence of the two forms we have integrated the total derivative

$$\bar\partial T_{zz} = -\pi\partial\frac{\delta S}{\delta\varphi} + \pi\partial\varphi\frac{\delta S}{\delta\varphi} \tag{2.20}$$

and used the (quantum) equation of motion

$$\frac{\delta S}{\delta\varphi} \overset{\text{w.s.}}{=} \frac{\delta}{\delta\varphi}. \tag{2.21}$$

[2]In the Euclidean path-integral formalism the normal product is defined by subtracting all lower correlators from an operator. For example

$$:\varphi(\omega)^3 := \varphi(\omega)^3 - 3\varphi(\omega)^2\langle\varphi(\omega)\rangle - 3\varphi(\omega)\left\langle\varphi(\omega)^2\right\rangle - \left\langle\varphi(\omega)^3\right\rangle$$

and we have

$$\frac{\delta}{\delta\varphi(z)}:\varphi(\omega)^3 := 3\delta^{(2)}(z-\omega):\varphi(\omega)^2:, \qquad :\varphi(\omega)^2 := \varphi(\omega)^2 - 2\varphi(\omega)\langle\varphi(\omega)\rangle - \left\langle\varphi(\omega)^2\right\rangle.$$

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Actually, the form of $\hat{\delta}_\xi$ in the middle of eq. (2.19) is primary. Its advantage over the standard one on the right is that it takes into account a tremendous cancellation of the diagrams in the quantum case, while there are subtleties associated with singular products.

In the quantum case we have also an additional effect of the regulator

$$\left\langle \hat{\delta}_\xi X(\omega_i) \right\rangle = \left\langle \int_{D_1} \mathrm{d}^2 z \left(\xi'(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial\varphi(z)\frac{\delta}{\delta\varphi(z)} + \xi(z)\partial Y(z)\frac{\delta}{\delta Y(z)} \right) X(\omega_i) \right\rangle . \tag{2.22}$$

Averaging over the regulators as already discussed, we arrive at the effective action and the effective energy-momentum tensor (2.11), describing the infrared limit. Equation (2.22) is then substituted by

$$\left\langle \hat{\delta}_\xi X(\omega_i) \right\rangle = \left\langle \int_{D_1} \mathrm{d}^2 z \left(q\xi'(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial\varphi(z)\frac{\delta}{\delta\varphi(z)} \right) X(\omega_i) \right\rangle . \tag{2.23}$$

It was shown in ref. [16] how to reproduce

$$\hat{\delta}_\xi \, \mathrm{e}^{\varphi(\omega)} \overset{\text{w.s.}}{=} (q-b^2)\xi'(\omega)\,\mathrm{e}^{\varphi(\omega)} + \xi(\omega)\partial\varphi(\omega)\,\mathrm{e}^{\varphi(\omega)} \tag{2.24}$$

for the quadratic action by (2.23) via the singular products listed in appendix B.

3 Central charge via the singular products

3.1 General action and energy-momentum tensor

For the $2N$-th order in derivatives term in the action (1.9)

$$S^{(\varphi,2N)} = \frac{1}{16\pi b_0^2}\int \sqrt{g}\varphi(-\Delta)^N\varphi \tag{3.1}$$

the "improved" energy-momentum tensor reads[3]

$$T_{zz}^{(\varphi,2N)} = \frac{1}{4b_0^2}(-1)^N\left[\sum_{k=0}^{N-1}(\partial\Delta^k\varphi)(\partial\Delta^{N-k-1}\varphi) - 2\partial^2\Delta^{N-1}\varphi\right]. \tag{3.2}$$

For $N=1,2,3$ we write explicitly

$$T_{zz}^{(\varphi,2)} = -\frac{1}{4b_0^2}\left[(\partial\varphi)^2 - 2\partial^2\varphi\right], \tag{3.3a}$$

$$T_{zz}^{(\varphi,4)} = \frac{1}{2b_0^2}\left[(\partial\varphi)(\partial\Delta\varphi) - \partial^2\Delta\varphi\right], \tag{3.3b}$$

$$T_{zz}^{(\varphi,6)} = -\frac{1}{4b_0^2}\left[2(\partial\varphi)(\partial\Delta^2\varphi) + (\partial\Delta\varphi)^2 - 2\partial^2\Delta^2\varphi\right]. \tag{3.3c}$$

They obey the conservation law

$$\bar{\partial}T_{zz}^{(\varphi,2N)} = 0 \tag{3.4}$$

[3] These energy-momentum tensors are "improved" [24, 25] and therefore traceless. They differ for this reason from the ones in ref. [26]. Our "improvement" procedure also differs from the one [27] in the free case.

thanks to the classical equations of motion

$$-\Delta\varphi = 0, \tag{3.5a}$$

$$\Delta^2\varphi - \frac{1}{2}(\Delta\varphi)^2 = 0, \tag{3.5b}$$

$$-\Delta^3\varphi + (\Delta\varphi)\Delta^2\varphi = 0 \tag{3.5c}$$

demonstrating tracelessness of $T_{ab}^{(\varphi,2N)}$.

For the regulator Y we have analogously

$$S^{(Y,2N)} = \frac{1}{16\pi b_0^2}\int \sqrt{g}Y(-\Delta)^N Y \tag{3.6}$$

and

$$T_{zz}^{(Y,2N)} = \frac{1}{4b_0^2}(-1)^N \sum_{k=0}^{N-1}\left[(\partial\Delta^k Y)(\partial\Delta^{N-k-1}Y)\right]. \tag{3.7}$$

The sum $T_{zz}^{(2N)} + T_{zz}^{(Y,2N)}$ is traceless thanks to the classical equation of motion for φ.

This is a general property because in the conformal gauge (1.4) we have

$$T_a^a \equiv \hat{g}^{ab}\frac{\delta\mathcal{S}[g]}{\delta\hat{g}^{ab}} = -\frac{\delta\mathcal{S}[g]}{\delta\varphi}. \tag{3.8}$$

The left-hand side of eq. (3.8) represents the trace of the "improved" energy-momentum tensor while the right-hand side represents the classical equation of motion for φ. Thus the tracelessness of the "improved" energy-momentum tensor in the two-dimensional theory invariant under diffeomorphisms is guaranteed by the classical equation of motion for φ.

3.2 Quadratic action

Under the infinitesimal conformal transformation generated by (2.19) we have

$$\begin{aligned}\left\langle \hat{\delta}_\xi T_{zz}^{(\varphi,2)}(\omega)\right\rangle &= \frac{1}{2b^2}\int \mathrm{d}^2z\langle q^2\xi'''(z) + \xi'(z)\partial^2\varphi(z)\varphi(\omega) + \xi(z)\partial^3\varphi(z)\varphi(\omega)\rangle\delta^{(2)}(z-\omega)\\ &= \frac{\xi'''(\omega)}{2}\left(\frac{q^2}{b^2} + H_{2,1} - H_{3,1}\right) = \frac{\xi'''(\omega)}{2}\left(\frac{q^2}{b^2} + \frac{1}{3} - \frac{1}{6}\right)\\ &= \xi'''(\omega)\left(\frac{q^2}{2b^2} + \frac{1}{12}\right),\end{aligned} \tag{3.9}$$

where we have used eq. (B.1). Here $1/12$ corresponds to the usual quantum addition 1 to the central charge. The right-hand side of eq. (3.9) reproduces the left-hand side of the DDK formula (2.12).

Analogously for the massive conformal fields we obtain

$$\begin{aligned}\left\langle \hat{\delta}_\xi T_{zz}^{(Y,2)}(\omega)\right\rangle &= \frac{1}{2b^2}\int \mathrm{d}^2z\langle \xi'(z)\partial^2 Y(z)Y(\omega) + \xi(z)\partial^3 Y(z)Y(\omega)\rangle\delta_M^{(2)}(z-\omega)\\ &= \frac{\xi'''(\omega)}{2}\left(J_{2,1} - J_{3,1}\right) = \frac{\xi'''(\omega)}{2}\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{\xi'''(\omega)}{12}\end{aligned} \tag{3.10}$$

where we have used eq. (B.10). The plus sign is for normal statistics. For the regulators with the wrong statistics the sign changes for minus. This explicitly shows how the regulators compensate the quantum part of the central charge of φ so the total one equals the classical value and illustrates the statement that we can account for the effect of the regulators by $q \neq 1$ which emerges after path integration over the regulators.

Notice that the propagators in eqs. (3.9) and (3.10) are *exact* accounting for the interaction between φ and the regulators. This is why renormalized b^2 cancels.

3.3 Quartic action

Let us now add to the quadratic action the εR^2 term which changes the propagator as

$$G_\varepsilon(k) = \frac{1}{k^2+\varepsilon k^4}, \qquad \delta^{(2)}_\varepsilon(k) = \frac{1}{1+\varepsilon k^2} \tag{3.11}$$

and introduces a nontrivial self-interaction of φ. The computation of $\delta T^{(\varphi,4)}_{zz}$ is a bit lengthy but easily doable with Mathematica. Equation (3.9) remains unchanged while with the one-loop accuracy we find

$$\begin{aligned}\varepsilon\left\langle \hat\delta_\xi T^{(\varphi,4)}_{zz}(\omega)\right\rangle &= \frac{1}{b_0^2}\int d^2z \Big\langle 2\varepsilon\xi'''(z)[\partial\bar\partial\varphi(z)-\partial\bar\partial\varphi(\omega)]\varphi(\omega) \\ &+[2\varepsilon\xi''(z)\partial^2\bar\partial\varphi(z)-6\varepsilon\xi'(z)\partial^3\bar\partial\varphi(z)-4\varepsilon\xi(z)\partial^4\bar\partial\varphi(z)]\varphi(\omega)\Big\rangle\,\delta^{(2)}_\varepsilon(z-\omega) \\ &= \frac{\xi'''(\omega)}{4}\left(-2J_{0,1}+2J_{1,1}+6J_{2,1}-4J_{3,1}\right) = \frac{\xi'''(\omega)}{4}\left(-2\cdot 2+2\cdot 1+6\frac{2}{3}-4\frac{1}{2}\right)=0\,,\end{aligned} \tag{3.12}$$

where we have used eq. (B.6) and dropped the logarithmic divergence as prescribed by the normal ordering. Thus the central charge of φ coincides at one loop with that for the quadratic action.

For the regulators we obtain similarly

$$\left\langle \hat\delta_\xi T^{(Y,2)}_{zz}(\omega)\right\rangle = \frac{1}{b_0^2}\int d^2z \left\langle \tfrac12\xi'(z)\partial^2Y(z)Y(\omega)+\tfrac12\xi(z)\partial^3Y(z)Y(\omega)\right\rangle \delta^{(2)}_{\varepsilon,M}(z-\omega)=0 \tag{3.13}$$

in view of eq. (B.14) and

$$\begin{aligned}&\varepsilon\left\langle \hat\delta_\xi T^{(Y,4)}_{zz}(\omega)\right\rangle \\ &= \frac{1}{b_0^2}\int d^2z\Big\langle\big[-2\varepsilon\xi''(z)\partial^2\bar\partial Y(z)-6\varepsilon\xi'(z)\partial^3\bar\partial Y(z)-4\varepsilon\xi(z)\partial^4\bar\partial Y(z)\big]Y(\omega)\Big\rangle\,\delta^{(2)}_{\varepsilon,M}(z-\omega) \\ &= \frac14\left(-2P_{1,1}+6P_{2,1}-4P_{3,1}\right)\xi'''(\omega) = \frac14\left(-2\cdot1+6\frac56-4\frac23\right)\xi'''(\omega)=\frac{1}{12}\xi'''(\omega)\end{aligned} \tag{3.14}$$

at one loop in view of eq. (B.15). We see now the same compensation of the quantum part in the central charge of φ by the regulators as for the quadratic action although in a slightly different way.

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We can also perform the computation for the part of T_{zz} in eq. (2.7) which involves G. For the polynomial in derivatives terms we obtain for the regularization (3.11)

$$\begin{aligned}
&\Big\langle \hat\delta_\xi \frac{1}{b_0^2} G\varepsilon \Big[(\partial\varphi)^2 \partial\bar\partial\varphi - \partial\varphi\partial(\,\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) + \partial^2(\,\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) - \partial(\partial\varphi\,\mathrm{e}^{-\varphi}\partial\bar\partial\varphi)\Big]\Big\rangle \\
&= \frac{1}{b_0^2} G\varepsilon \int \mathrm{d}^2 z \,\Big\langle \xi'''(z)\big[-\partial\bar\partial\varphi(z)\varphi(\omega) - 3\partial\varphi(\omega)\bar\partial\varphi(\omega)\big] - 2\xi''(z)\partial^2\bar\partial\varphi(z)\varphi(\omega)\Big\rangle\, \delta_\varepsilon^{(2)}(z-\omega) \\
&= -\frac{3}{2} G\xi'''(\omega) \int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}. \qquad (3.15)
\end{aligned}$$

For the nonlocal term in (2.7) we analogously find

$$\begin{aligned}
\Big\langle \hat\delta_\xi \Big(-\frac{1}{b_0^2} G\varepsilon \frac{1}{\bar\partial} \partial^2 (\bar\partial\varphi\,\mathrm{e}^{-\varphi}\partial\bar\partial\varphi)\Big)\Big\rangle &= -\frac{1}{b_0^2} G\varepsilon \int \mathrm{d}^2 z\, \xi'''(z) \,\Big\langle \partial\bar\partial\varphi(z)\varphi(\omega) + \partial\bar\partial\varphi(\omega)\varphi(\omega)\Big\rangle \\
&\times \delta_\varepsilon^{(2)}(z-\omega) = \frac{1}{2} G\xi'''(\omega) + \frac{1}{2} G\xi'''(\omega) \int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}. \qquad (3.16)
\end{aligned}$$

In contrast to the average of the nonlocal (last) term in the classical formula (2.15) now a nonvanishing finite contribution arises.

The sum of (3.9), (3.12), (3.15) and (3.16) precisely reproduces the central charge (2.13) at one loop.

For a future use I present also the exact formula for the conformal variation of the normal-ordered $T_{zz}^{(\varphi,4)}$

$$\begin{aligned}
\Big\langle \hat\delta_\xi T_{zz}^{(\varphi,4)}(\omega)\Big\rangle &= 2\frac{1}{b^2} \int \mathrm{d}^2 z \Big\langle \mathrm{e}^{-\varphi(\omega)} \big\{ -\xi'''(z)\partial\bar\partial\varphi(z) + \xi''(z)\partial\varphi(z)\partial\bar\partial\varphi(\omega) \\
&+ \xi'(z)\big[2\partial^3\bar\partial\varphi(z) - \partial^2\varphi(z)\partial\bar\partial\varphi(\omega) - \partial\bar\partial\varphi(z)(\partial\varphi(\omega))^2 - \partial\varphi(z)\partial\varphi(\omega)\partial\bar\partial\varphi(\omega)\big] \\
&+ \xi(z)\big[\partial^4\bar\partial\varphi(z) - \partial^3\varphi(z)\partial\bar\partial\varphi(\omega) - \partial^2\bar\partial\varphi(z)(\partial\varphi(\omega))^2 + \partial\varphi(z)\partial^2\varphi(\omega)\partial\bar\partial\varphi(\omega)\big]\big\}\Big\rangle \delta_\varepsilon^{(2)}(z-\omega). \qquad (3.17)
\end{aligned}$$

Equation (3.12) is its expansion to quadratic order in φ. Equation (3.17) can be possibly useful to show, manipulating with the derivatives, that its right hand side in fact vanishing like (3.12). That would prove the universality to all loops.

3.4 Six-derivative action

Under the infinitesimal conformal transformation we have for the six-derivative action at one loop the following variation of (3.3c):

$$\begin{aligned}
\Big\langle \hat\delta_\xi T_{zz}^{(\varphi,6)}(\omega)\Big\rangle = \frac{1}{b_0^2} \int \mathrm{d}^2 z \Big\langle \Big[&-8\xi^{(4)}(z)\partial\bar\partial^2\varphi(z) - 8\xi'''(z)\partial^2\bar\partial^2\varphi(z) + 16\xi''(z)\partial^3\bar\partial^2\varphi(z) \\
&+ 48\xi'(z)\partial^4\bar\partial^2\varphi(z) + 24\xi(z)\partial^5\bar\partial^2\varphi(z)\Big]\varphi(\omega)\Big\rangle\, \delta_\varepsilon^{(2)}(z-\omega). \qquad (3.18)
\end{aligned}$$

If we consider the six-derivative action

$$S[\varphi] = S^{(\varphi,2)} + 2\varepsilon S^{(\varphi,4)} + \varepsilon^2 S^{(\varphi,6)} \qquad (3.19)$$

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the propagator will be of the form (B.4) with $m = 2$. Substituting in (3.9), (3.12), (3.18) we find

$$\left\langle \hat{\delta}_\xi T_{zz}^{(\varphi,2)}(\omega)\right\rangle = \frac{\xi'''(\omega)}{2}\left(\frac{q^2}{b^2}+H_{2,2}-H_{3,2}\right) = \frac{\xi'''(\omega)}{2}\left(\frac{q^2}{b^2}+\frac{3}{10}-\frac{2}{15}\right)$$
$$= \xi'''(\omega)\left(\frac{q^2}{2b^2}+\frac{1}{12}\right), \tag{3.20}$$

$$2\varepsilon\left\langle \hat{\delta}_\xi T_{zz}^{(\varphi,4)}(\omega)\right\rangle = 2\frac{\xi'''(\omega)}{4}\left(-2J_{0,2}+2J_{1,2}+6J_{2,2}-4J_{3,2}\right)$$
$$= 2\frac{\xi'''(\omega)}{4}\left(-2\frac{2}{3}+2\frac{1}{3}+6\frac{1}{5}-4\frac{2}{15}\right) = 0, \tag{3.21}$$

$$\varepsilon^2\left\langle \hat{\delta}_\xi T_{zz}^{(\varphi,6)}(\omega)\right\rangle = \xi'''(\omega)\left(-\frac{1}{2}Q_{0,2}-Q_{1,2}+3Q_{2,2}-\frac{3}{2}Q_{3,2}\right)$$
$$= \xi'''(\omega)\left(-\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{3}+3\frac{3}{10}-\frac{3}{2}\cdot\frac{4}{15}\right) = 0. \tag{3.22}$$

We have thus obtained the same result as for the quadratic and quartic actions.

One more argument in favor of the universality, i.e. independence of the central charge of φ on the form of the action, are the identities

$$H_{2,m} - H_{3,m} = \frac{1}{6}, \tag{3.23a}$$
$$-2J_{0,m}+2J_{1,m}+6J_{2,m}-4J_{3,m} = 0, \tag{3.23b}$$
$$-Q_{0,m}-Q_{1,m}+3Q_{2,m}-\frac{3}{2}Q_{3,m} = 0 \tag{3.23c}$$

satisfied for the propagator (B.4) associated with the regularization by yet higher derivatives.

The formula analogous to (3.18) can be derived also for the regulators

$$\left\langle \hat{\delta}_\xi T_{zz}^{(Y,6)}(\omega)\right\rangle = \frac{1}{b_0^2}\int \mathrm{d}^2 z\left\langle\Big[8\xi'''(z)(\partial^2\bar{\partial}^2Y(z)-\partial^2\bar{\partial}^2Y(\omega))+32\xi''(z)\partial^3\bar{\partial}^2Y(z)\right.$$
$$\left.+48\xi'(z)\partial^4\bar{\partial}^2Y(z)+24\xi(z)\partial^5\bar{\partial}^2Y(z)\Big]Y(\omega)\right\rangle\delta^{(2)}_{\varepsilon,M}(z-\omega). \tag{3.24}$$

The massive analogue of eq. (B.9) involves the terms $\propto M/\sqrt{\varepsilon}$ which make analisis more complicated.

4 Universality of the six-derivative action

4.1 $R\Delta R$

The most general six-derivative action of the form (1.9) reads

$$S[\varphi] = S^{(\varphi,2)} + (\varepsilon + a^2)S^{(\varphi,4)} + \varepsilon a^2 S^{(\varphi,6)} \tag{4.1}$$

involving the term $R\Delta R$. The action (3.19) corresponds to $a^2 = \varepsilon$. The six-derivative action (4.1) is associated to the regularization

$$G_{\varepsilon,a}(k) = \frac{1}{k^2(1+\varepsilon k^2)(1+a^2k^2)^{m-1}}, \qquad \delta^{(2)}_{\varepsilon,a}(k) = \frac{1}{(1+\varepsilon k^2)(1+a^2k^2)^{m-1}} \tag{4.2}$$

with $m = 2$.

For an arbitrary ratio a^2/ε we write

$$\frac{1}{b_0^2}\int \mathrm{d}^2 z f(z)\left\langle \partial^n\varphi(z)\varphi(\omega)\right\rangle \delta^{(2)}_{\varepsilon,a}(z-\omega) = (-1)^n H_{n.m}\left(\frac{a^2}{\varepsilon}\right)\partial^n f(\omega), \tag{4.3a}$$

$$\frac{1}{b_0^2}\int \mathrm{d}^2 z f(z)\left\langle -4\varepsilon\partial^{n+1}\bar\partial\varphi(z)\varphi(\omega)\right\rangle \delta^{(2)}_{\varepsilon,a}(z-\omega) = (-1)^n J_{n,m}\left(\frac{a^2}{\varepsilon}\right)\partial^n f(\omega), \tag{4.3b}$$

$$\frac{1}{b_0^2}\int \mathrm{d}^2 z f(z)\left\langle 16\varepsilon a^2\partial^{n+2}\bar\partial^2\varphi(z)\varphi(\omega)\right\rangle \delta^{(2)}_{\varepsilon,a}(z-\omega) = (-1)^n Q_{n,m}\left(\frac{a^2}{\varepsilon}\right)\partial^n f(\omega), \tag{4.3c}$$

where the functions on the right-hand side are like

$$J_{1,2}\left(\frac{a^2}{\varepsilon}\right) = \frac{\varepsilon\left(a^4-\varepsilon^2-2a^2\varepsilon\log\left(\frac{a^2}{\varepsilon}\right)\right)}{(a^2-\varepsilon)^3}. \tag{4.4}$$

They obey the identities

$$H_{2,m}\left(\frac{a^2}{\varepsilon}\right) - H_{3,m}\left(\frac{a^2}{\varepsilon}\right) = \frac{1}{6}, \tag{4.5a}$$

$$-2J_{0,m}\left(\frac{a^2}{\varepsilon}\right) + 2J_{1,m}\left(\frac{a^2}{\varepsilon}\right) + 6J_{2,m}\left(\frac{a^2}{\varepsilon}\right) - 4J_{3,m}\left(\frac{a^2}{\varepsilon}\right) = 0, \tag{4.5b}$$

$$-Q_{0,m}\left(\frac{a^2}{\varepsilon}\right) - Q_{1,m}\left(\frac{a^2}{\varepsilon}\right) + 3Q_{2,m}\left(\frac{a^2}{\varepsilon}\right) - \frac{3}{2}Q_{3,m}\left(\frac{a^2}{\varepsilon}\right) = 0, \tag{4.5c}$$

generalizing (3.23) to $a^2 \neq \varepsilon$ and illustrating the universality at one loop.

4.2 R^3

The consideration of the contribution to the central charge of φ from the R^3 term in the emergent action, which is not of the type shown in eq. (1.9), is pretty much similar. The difference between the R^3 term and the $R\Delta R$ term considered in the previous subsection is that its expansion in φ starts from φ^3 rather than φ^2. Then only the piece of the contribution of R^3 to the "improved" energy momentum tensor of the form

$$\widetilde{T}^{(\varphi)}_{zz} = \frac{12}{b_0^2}\partial^2(\partial\bar\partial\varphi)^2 + \mathcal{O}(\varphi^3) \tag{4.6}$$

which is linked to the shift (1.7) may give a nonvanishing result at one loop. The piece like φ^3 which yields φ^2 after acting by the variational derivative $\delta/\delta\varphi$ from the term with ξ' in (2.19) vanishes after the averaging because the variation of a normal product is again a normal product (see the footnote[2]).

The variation of (4.6) under the infinitesimal conformal transformation (2.19) is easily calculable

$$\begin{aligned}\varepsilon^2\left\langle\hat\delta_\xi\widetilde{T}^{(\varphi)}_{zz}(\omega)\right\rangle &= \frac{12}{b_0^2}\int \mathrm{d}^2 z\left\langle\left[2\varepsilon^2\xi'''(z)\partial^2\bar\partial^2\varphi(z) + 2\varepsilon^2\xi''(z)\partial^3\bar\partial^2\varphi(z)\right]\varphi(\omega)\right\rangle\delta^{(2)}_{\varepsilon,a}(z-\omega)\\ &= \frac{3}{2}\xi'''(\omega)\left[Q_{0,m}(\frac{a^2}{\varepsilon}) + Q_{1,m}(\frac{a^2}{\varepsilon})\right] = 0. \end{aligned}\tag{4.7}$$

It vanishes thus supporting the universality.

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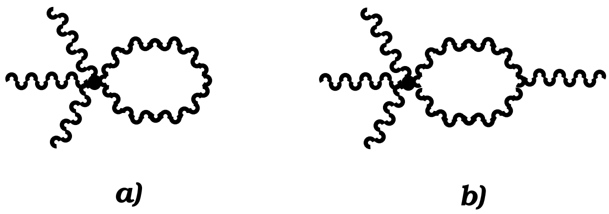

Figure 2. One-loop renormalization of e^{φ} whose position is depicted by the dot. The wavy lines represent φ.

From the above analysis of the contribution of the term R^3 to the central charge of φ at one loop it becomes clear that most of yet higher-derivative terms in the effective action do not contribute for the trivial reason — too many φ's. In addition to the action (1.9) only the terms $\varepsilon^{n+2} R\Delta^k R\Delta^{n-k} R$ with $n \geq 1$, $0 \leq k \leq n$ are to be analyzed like the R^3 term above.

5 Discussion

The above computations confirm the result (2.13) for the one-loop central charge of φ for the action (1.6). They are also useful for studying the universality of the higher-derivative actions. We have explicitly shown at one loop the universality of the central charge for the quartic and six-derivative actions.

How the universality works for the action (1.9) is easily seen in the one-loop computation of the renormalization of e^{φ} given by the diagrams in figure 2. The diagram in figure 2*a* is obviously universal with logarithmic accuracy. The diagram in figure 2*b* involves the propagator $k^2 F(\varepsilon k^2)$ and the triple vertex which comes from the variation of (1.9) with respect to φ resulting in $\varepsilon k^4 F'(\varepsilon k^2)$. Its contribution [14]

$$\text{Figure 2b} = \frac{\mathrm{e}^{\varphi}}{2} \times 8\pi b_0^2 \varphi \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\varepsilon k^4 F'(\varepsilon k^2)}{[k^2 F(\varepsilon k^2)]^2} = \frac{1}{F(0)}\, \mathrm{e}^{\varphi} b_0^2 \varphi = \mathrm{e}^{\varphi} b_0^2 \varphi \tag{5.1}$$

does not depend on the choice of the function F.

In fact my motivation to analyze the singular products was to go beyond the one-loop approximation for the central charge of φ. This may be doable by the described method of singular products if to prove the vanishing of $\left\langle \hat{\delta}_\xi T_{zz}^{(\varphi,2N)} \right\rangle$ to all loops like it was done for $N = 2, 3$ at one loop. Then we may expect the exact central charge to be

$$c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6Gq. \tag{5.2}$$

I hope to return to this issue elsewhere.

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A Derivation of emergent action for the Nambu-Goto string

The path integration over X^μ for the Nambu-Goto action (2.2) can be performed using the DeWitt-Seeley expansion of the operator $(\sqrt{g})^{-1}\partial_a\lambda^{ab}\partial_b$. Splitting the operator into the gravitational and "electromagnetic" parts, we have

$$\mathcal{O} \equiv \frac{1}{\sqrt{g}}\partial_a\lambda^{ab}\partial_b = h^{ab}\partial_a\partial_b + A^a\partial_a, \qquad h^{ab} = \frac{\mathrm{e}^{-\varphi}}{\sqrt{\hat{g}}}\lambda^{ab}, \quad A^b = \frac{\mathrm{e}^{-\varphi}}{\sqrt{\hat{g}}}\partial_a\lambda^{ab}. \tag{A.1}$$

Using the results known from [8, 9] (see [28] for a review), we write the expansion

$$\left\langle \left| \mathrm{e}^{\tau\mathcal{O}} \right| \right\rangle = \frac{1}{4\pi\tau} + \frac{1}{4\pi}\left(\frac{1}{6}R + E\right) + \frac{\tau}{120\pi}\left(\Delta R + \frac{1}{2}R^2\right) + \mathcal{O}(\tau^2), \tag{A.2}$$

where R and Δ are the scalar curvature and the Laplacian for the metric tensor h_{ab} defined in eq. (A.1). We have dropped here the term $\propto \tau A^2$ because $A \sim \tau$ with τ being the proper-time UV cutoff, as it will be momentarily seen.

To find the emergent action for the fluctuating fields φ and $\delta\lambda^{ab} = \lambda^{ab} - \sqrt{\hat{g}}\hat{g}^{ab}$, we use

$$h^{ab} = \hat{g}^{ab} + \delta h^{ab}, \qquad \delta h^{ab} = -\hat{g}^{ab}\varphi + \frac{\delta\lambda^{ab}}{\sqrt{\hat{g}}} \tag{A.3}$$

or

$$\delta h^{z\bar{z}} = 2\delta\lambda^{z\bar{z}} - 2\varphi, \quad \delta h^{zz} = 2\delta\lambda^{zz}, \quad \delta h^{\bar{z}\bar{z}} = 2\delta\lambda^{\bar{z}\bar{z}} \tag{A.4}$$

in the complex coordinates z and $\bar{z}$. To quadratc order in the fluctuations, we can use the formulas in an inertial frame

$$R = -\partial_a\partial_b h^{ab} + h_{ab}h^{cd}\partial_c\partial_d h^{ab} = -\partial^2\delta h^{zz} - \bar{\partial}^2\delta h^{\bar{z}\bar{z}} + 2\partial\bar{\partial}\delta h^{z\bar{z}}, \tag{A.5}$$

$$E = -\frac{1}{2}\left(\partial_a A^a - \partial_a\partial_b h^{ab} + \frac{1}{2}\hat{g}_{ab}\hat{\Delta}h^{ab}\right) \tag{A.6}$$

which yields

$$\frac{1}{6}R + E = -\frac{1}{3}\left(\partial^2\delta\lambda^{zz} + \bar{\partial}^2\delta\lambda^{\bar{z}\bar{z}} + 4\partial\bar{\partial}\delta\lambda^{z\bar{z}} + 2\partial\bar{\partial}\varphi\right). \tag{A.7}$$

Using the fact that the variational derivative of the emergent action with respect to φ is equal to (A.2) and covariantizing, we find the following contribution from X^μ to the emergent action:

$$\mathcal{S}_X[\varphi,\lambda^{ab}] = \frac{d}{2}\int\left[-\frac{\mathrm{e}^{\varphi}}{4\pi\tau\sqrt{\det\lambda^{ab}}} + \frac{1}{12\pi}\left(\varphi\partial\bar{\partial}\varphi + 4\lambda^{z\bar{z}}\partial\bar{\partial}\varphi + \lambda^{zz}\nabla\partial\varphi + \lambda^{\bar{z}\bar{z}}\bar{\nabla}\bar{\partial}\varphi\right)\right.$$
$$\left. + \frac{\tau}{15\pi}\,\mathrm{e}^{-\varphi}(\partial\bar{\partial}\varphi)^2\right] + \mathcal{O}(\tau^2). \tag{A.8}$$

Here $\nabla = \partial - \partial\varphi$ is the covariant derivative in the conformal gauge. Equations (2.5) and (A.8) perfectly agree.[4]

[4] The last term here is 2/3 of the one for the Pauli-Villars regularization as is prescribed by Apppendix A of [18].

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To describe fluctuations, we expand about $\lambda^{ab} = \sqrt{\hat g}\hat g^{ab}$ when

$$\lambda^{z\bar z} = 1 + \delta\lambda^{z\bar z}, \qquad \lambda^{zz} = \delta\lambda^{zz}, \qquad \lambda^{\bar z\bar z} = \delta\lambda^{\bar z\bar z} \tag{A.9}$$

to get

$$\frac{1}{\sqrt{\det \lambda^{ab}}} = 1 - \delta\lambda^{z\bar z} + (\delta\lambda^{z\bar z})^2 + \frac{1}{2}\delta\lambda^{zz}\delta\lambda^{\bar z\bar z} + \mathcal{O}(\delta\lambda)^3. \tag{A.10}$$

The integrals over $\delta\lambda^{zz}$ and $\delta\lambda^{\bar z\bar z}$ have then saddle points at

$$\delta\lambda^{zz} = \frac{2}{3}\tau\,\mathrm{e}^{-\varphi}\bar\nabla\bar\partial\varphi, \qquad \delta\lambda^{\bar z\bar z} = \frac{2}{3}\tau\,\mathrm{e}^{-\varphi}\nabla\partial\varphi \tag{A.11}$$

which demonstrates that $\delta\lambda \sim \tau$, justifying the expansion in $\delta\lambda$ and the saddle point.

It is slightly different with $\delta\lambda^{z\bar z}$ because of the linear term in eq. (A.10). It simply renormalizes the bare string tension in the classical part of the Nambu-Goto action (the last term on the right-hand side of eq. (2.2)). We thus have at the saddle point

$$\delta\lambda^{z\bar z} = \tau\left(\frac{2}{3}\,\mathrm{e}^{-\varphi}\partial\bar\partial\varphi - \frac{1}{2d\alpha'_R}\right), \tag{A.12}$$

where the second term in the brackets can be omitted for finite α'_R as $\tau \to 0$.

Inserting the saddle-point values (A.11), (A.12) into the action (A.8), we find

$$\mathcal{S}_X[\varphi] = \frac{d}{2}\int\left\{-\frac{\mathrm{e}^{\varphi}}{4\pi\tau} + \frac{1}{12\pi}\varphi\partial\bar\partial\varphi + \tau\,\mathrm{e}^{-\varphi}\left[\frac{8}{45\pi}(\partial\bar\partial\varphi)^2 + \frac{1}{18\pi}\nabla\partial\varphi\bar\nabla\bar\partial\varphi\right]\right\} + \mathcal{O}(\tau^2). \tag{A.13}$$

The final step to obtain the quartic in derivatives part of the action (1.6) is to use the identity

$$\mathrm{e}^{-\varphi}\nabla\partial\varphi\bar\nabla\bar\partial\varphi = \mathrm{e}^{-\varphi}\left[(\partial\bar\partial\varphi)^2 + \partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right] + \partial\left(\mathrm{e}^{-\varphi}\partial\varphi\bar\nabla\bar\partial\varphi\right) - \bar\partial(\mathrm{e}^{-\varphi}\partial\varphi\partial\bar\partial\varphi), \tag{A.14}$$

to get

$$\mathcal{S}_X[\varphi] = \frac{d}{2}\int\left\{-\frac{\mathrm{e}^{\varphi}}{4\pi\tau} + \frac{1}{12\pi}\varphi\partial\bar\partial\varphi + \tau\,\mathrm{e}^{-\varphi}\left[\frac{7}{30\pi}(\partial\bar\partial\varphi)^2 + \frac{1}{18\pi}\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right]\right\} + \mathcal{O}(\tau^2). \tag{A.15}$$

Summing (A.15) with the contribution from the ghosts (see the footnote[4])

$$\mathcal{S}_{\mathrm{gh}}[\varphi] = \int\left[\frac{\mathrm{e}^{\varphi}}{4\pi\tau} - \frac{13}{12\pi}\varphi\partial\bar\partial\varphi - \frac{11\tau}{15\pi}\,\mathrm{e}^{-\varphi}(\partial\bar\partial\varphi)^2\right] + \mathcal{O}(\tau^2), \tag{A.16}$$

we obtain eq. (1.6) previously derived [14] for the Pauli-Villars regularization.

B List of formulas for the singular products

The simplest singular product

$$\int \mathrm{d}^2 z\,\xi(z)\,\langle\partial^n\varphi(z)\varphi(0)\rangle\,\delta^{(2)}(z) = (-1)^n\frac{2}{n(n+1)}\xi^{(n)}(0) \tag{B.1}$$

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emerges in a free CFT, where the propagator is

$$\langle\varphi(z)\varphi(0)\rangle = 8\pi G_0(z), \qquad G_0(z) = -\frac{1}{2\pi}\log\left(\sqrt{z\bar{z}}\mu\right) \tag{B.2}$$

and μ represents an infrared cutoff. Equation (B.1) can be derived by the formulas

$$\delta^{(2)}(z) = \bar{\partial}\frac{1}{\pi z}, \qquad \frac{1}{z^n}\bar{\partial}\frac{1}{z} = (-1)^n\frac{1}{(n+1)!}\partial^n\bar{\partial}\frac{1}{z}. \tag{B.3}$$

Equation (B.1) can be alternatively derived introducing the regularization by ε via higher derivatives. In momentum space we define

$$G_\varepsilon(k) = \frac{1}{k^2(1+\varepsilon k^2)^m}, \qquad \delta^{(2)}_\varepsilon(k) = \frac{1}{(1+\varepsilon k^2)^m}. \tag{B.4}$$

We then have

$$\begin{aligned}\frac{1}{b_0^2}\int \mathrm{d}^2z f(z)\,\langle\partial^n\varphi(z)\varphi(\omega)\rangle\,\delta^{(2)}_\varepsilon(z-\omega) &= (-1)^n H_{n.m}\partial^n f(\omega),\\ H_{n.m} &= \frac{2^{2m}\Gamma(m+1/2)\Gamma(n+m)}{\sqrt{\pi}\,n\,\Gamma(n+2m)}\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}\frac{1}{b_0^2}\int \mathrm{d}^2z f(z)\left\langle -4\varepsilon\partial^{n+1}\bar{\partial}\varphi(z)\varphi(\omega)\right\rangle\delta^{(2)}_\varepsilon(z-\omega) &= (-1)^n J_{n.m}\partial^n f(\omega),\\ J_{n,m} = \frac{n}{2m-1}H_{n,m} &= \frac{2^{2m-1}\Gamma(m-1/2)\Gamma(n+m)}{\sqrt{\pi}\Gamma(n+2m)}.\end{aligned} \tag{B.6}$$

For $m = 1$ this gives

$$H_{n,1} = \frac{2}{n(n+1)}, \qquad J_{n,1} = \frac{2}{(n+1)}, \tag{B.7}$$

reproducing eq. (B.1).

For the six-derivative action we shall also need

$$\frac{1}{b_0^2}\int \mathrm{d}^2z f(z)\left\langle 16\varepsilon^2\partial^{n+2}\bar{\partial}^2\varphi(z)\varphi(\omega)\right\rangle\delta^{(2)}_\varepsilon(z-\omega) = (-1)^n Q_{n.m}\partial^n f(\omega) \tag{B.8}$$

with

$$\begin{aligned}Q_{0,m} = Q_{1,m} &= \frac{1}{(m-1)(2m-1)}, & Q_{2,m} &= \frac{3(m+1)}{2(m-1)(4m^2-1)},\\ Q_{3,m} &= \frac{(m+2)}{(m-1)(4m^2-1)}, & Q_{n,2} &= \frac{2(n+1)}{(n^2+5n+6)}.\end{aligned} \tag{B.9}$$

In the case of the massive conformal fields (regulators) we have

$$\int \mathrm{d}^2z\,\xi(z)\,\langle\partial^n Y(z)Y(0)\rangle\,\delta^{(2)}_M(z) = (-1)^n J_{n,1}\xi^{(n)}(0) \tag{B.10}$$

for the free massive propagator

$$\langle Y(z)Y(0)\rangle = 8\pi G_M(z) = 4K_0\left(M\sqrt{z\bar{z}}\right), \qquad G_M(k) = \frac{1}{k^2+M^2} \tag{B.11}$$

and an ad hoc regularization of the delta function by the large-M limit of

$$\delta^{(2)}_M(z) = \frac{M^2}{2\pi} K_0\left(M\sqrt{z\bar{z}}\right), \qquad \delta^{(2)}_M(k) = \frac{M^2}{k^2+M^2} \tag{B.12}$$

what is natural if Y is the Pauli-Villars regulator. The same $J_{n,1}$ as shown in eq. (B.7) appears in eq. (B.10).

For the quartic derivative we define

$$\langle Y(-k)Y(k)\rangle = 8\pi b_0^2 G_{\varepsilon,M}, \quad G_{\varepsilon,M} = \frac{(k^2+M^2)^{m-1}}{(k^2+M^2+\varepsilon k^4)^m}, \qquad \delta^{(2)}_{\varepsilon,M} = \frac{(k^2+M^2)^m}{(k^2+M^2+\varepsilon k^4)^m} \tag{B.13}$$

to reproduce (B.4) as $M \to 0$. In the opposite limit $M \to \infty$ we find

$$8\pi \int \mathrm{d}^2 z f(z) \partial^n G_{\varepsilon,M}(z-\omega)\delta^{(2)}_{\varepsilon,M}(z-\omega) \overset{M\to\infty}{\to} 0, \tag{B.14}$$

$$\begin{aligned} &8\pi \int \mathrm{d}^2 z f(z)(-4\varepsilon\partial^{n+1}\bar\partial) G_{\varepsilon,M}(z-\omega)\delta^{(2)}_{\varepsilon,M}(z-\omega) \overset{M\to\infty}{\to} (-1)^n P_{n,m}\partial^n f(\omega), \\ &P_{0,m} = P_{1,m} = \frac{1}{(2m-1)}, \quad P_{2,m} = \frac{(2m+3)}{2(4m^2-1)}, \quad P_{3,m} = \frac{2}{(4m^2-1)}. \end{aligned} \tag{B.15}$$

Note added in the proof. Equation (5.2) has been derived recently in [29].

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