Supersymmetric dS$_4$ solutions in $D = 11$ supergravity

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ABSTRACT: The necessary and sufficient conditions for warped product dS$_4$ solutions in $D = 11$ supergravity to be supersymmetric are determined. The Killing spinor is associated with two possible stabilizer groups, SU(3) and G$_2$. We show that there are no solutions to the Killing Spinor equations in the G$_2$ stabilizer case. For the SU(3) stablilzer case, all of the conditions imposed from supersymmetry on the 4-form flux, and the geometry of the internal manifold, are determined in terms of SU(3) invariant spinor bilinears.

Contents

1 Introduction

De Sitter geometry is of particular interest in terms of string cosmology and also in the context of the holographic principle. De Sitter spacetime plays a central role in the understanding of our present universe. From the work of [1–3] it has been observed that our universe is asymptotically dS$_4$, corresponding to a very small positive cosmological constant. However, the observed value of the cosmological constant differs by many orders of magnitude from the vacuum energy density value predicted by quantum field theory [4, 5]. Moreover, in the context of string cosmology there are also difficulties in obtaining de Sitter

space via compactification from higher dimensions. In particular, there are no go-theorems proving that smooth warped de Sitter solutions with compact, without boundary, internal manifold cannot be found in ten- and eleven-dimensional supergravity [6–8]. Issues relating to quantum gravity in de Sitter space have been investigated in [9].

In terms of holography, the AdS/CFT correspondence relates string theory in Anti-de Sitter (AdS) space to conformal field theories (CFT) defined on an appropriate boundary [10]. This has been particularly useful in developing a deeper understanding of the microscopic nature of the entropy-area law [11, 12]. In spite of the considerable insights produced via the holographic principle, there are still many open issues in this area. Building from the AdS_3/CFT_2 correspondence proposed by Brown and Henneaux in [13], the relation between quantum gravity on de Sitter space and conformal field theory on a sphere, the so-called dS/CFT correspondence, was considered in [14–16]. However, our understanding of the conjectured dS/CFT correspondence is less complete than for the case of AdS/CFT for a number of reasons. Firstly, in contrast to AdS, there is a lack of de Sitter space solutions in string theory (or in any quantum gravity theory) in which the conjecture can be tested. Also, there are subtle issues with defining the dual CFT on the past and future spheres $\mathcal{I}^{\pm}$, relating to the causal structure of dS space. Nevertheless, the macroscopic entropy-area law applies to a very wide class of black holes, including asymptotically flat, asymptotically AdS, and also asymptotically dS cases. The universality of this law provides strong motivation for understanding de Sitter holography.

Motivated by this, it is of particular interest to systematically understand the different types of de Sitter solutions which are possible in D=10 and D=11 supergravity. Such a classification may provide interesting new applications of the dS/CFT correspondence. As it is possible to embed dS_n inside both $\mathbb{R}^{1,n}$ and AdS_{n+1} as a warped product geometry [17], it follows that the maximally supersymmetric $AdS_7 \times S^4$ solution, as well as $\mathbb{R}^{1,10}$, can both be regarded as examples of warped product dS_4 geometries. However, as we shall establish here, there is a much larger class of supersymmetric warped product dS_4 solutions in $D = 11$ supergravity than these two very special solutions, and this is also somewhat in contrast to the results of recent analysis of supersymmetric warped product dS_n geometries for $5 \leq n \leq 10$.

In terms of $D = 11$ supergravity, there has been recent progress in the classification of supersymmetric warped product dS_n geometries for $5 \leq n \leq 10$ [18]. There are a number of different possibilities:

- For $7 \leq n \leq 10$, the geometry is the maximally supersymmetric $\mathbb{R}^{1,10}$ solution with vanishing 4-form flux.

- For warped product dS_6 solutions, the solution is either the maximally supersymmetric $AdS_7 \times S^4$ solution, or $\mathbb{R}^{1,6} \times N$ where N is a hyper-Kähler 4-manifold.

- The warped product dS_5 solutions are all examples of generalized M5-brane solutions for which the transverse space is $\mathbb{R} \times N$, where N is a hyper-Kähler 4-manifold.

It is clear from this list that the possible warped product dS_n geometries for $5 \leq n \leq 10$ is very highly constrained. In addition, a similar recent analysis of warped product

dS$_n$ solutions in heterotic supergravity [19], including first order α' corrections, has also produced a rather restricted class of such solutions. In this case, for $n \geq 3$, the geometry is $\mathbb{R}^{1,n} \times M_{9-n}$, where M_{9-n} is a $(9-n)$-dimensional manifold. The dilaton depends only on the co-ordinates of M_{9-n}, and all p-form fields have components only along the M_{9-n} directions. The heterotic warped product dS$_2$ solutions are the direct product AdS$_3 \times M_7$ solutions which have been classified in [20]. Compared to these types of solutions, the conditions on supersymmetric warped product dS$_4$ solutions in $D = 11$ supergravity are rather weaker.

Motivated by these results, in this paper we obtain the necessary and sufficient conditions for warped product dS$_4$ solutions in $D = 11$ supergravity to preserve the minimal $N = 8$ supersymmetry. We find, on integrating the Killing spinor equations along the dS$_4$ directions, that all of the necessary and sufficient conditions for supersymmetry are encoded in a single gravitino-type equation, which is satisfied by a spinor ψ_+ whose components depend only on the co-ordinates of the internal space. We analyse the solutions of this equation using spinorial geometry techniques. This technique was introduced in [21] and consists of writing the Killing spinors in terms of multi-differential forms and, utilizing the gauge-covariance of the KSE, gauge transformations are then used to write the spinors in one of several simple canonical forms. The main outcome of this approach is a linear system which imposes conditions on the spin connection and the fluxes of the theory. This in turn can be used to obtain conditions on the geometry which are necessary and sufficient for supersymmetry. These techniques have been applied to classify a wide variety of supergravity solutions [22].

In the case of warped product dS$_4$ solutions, we state explicitly the Spin(7) gauge transformations which are used to write the spinor ψ_+ in canonical forms with stabilizer subgroups SU(3) and G$_2$. We then solve the linear system obtained from the Killing spinor equations. In particular, we show that the linear system implies that there are no Killing spinors for which the stabilizer of ψ_+ is G$_2$. For the case of SU(3) stabilizer subgroup, the Killing spinor equations determine all components of the 4-form flux in terms of the geometry of the internal manifold, and we present the geometric conditions and the components of the flux, written in a SU(3) covariant fashion. On considering these conditions, we note that the warped product dS$_4$ geometries are manifestly less restricted in terms of the geometric structure and the 4-form flux in comparison to the warped product dS$_n$ solutions for $5 \leq n \leq 10$. Our analysis does not utilize the global techniques developed for the investigation of supersymmetric black holes [23]; we consider only local properties of the Killing spinor equations. This avoids the no-go theorems which exclude warped product dS$_n$ solutions when the warp product and 4-form flux are smooth, and the internal manifold is smooth and compact without boundary.

The plan of this paper is as follows. In section 2 we summarize the bosonic field equations, Bianchi identities, and Killing spinor equations of $D = 11$ supergravity, we also describe the ansatz for the warped product dS$_4$ solutions, and present the reduction of the bosonic conditions to the internal manifold. In section 3, we derive several integrability conditions from the Killing spinor equations, and we demonstrate how some of these integrability conditions can be derived from others. In section 4, we explicitly integrate up the Killing spinor equations along the dS$_4$ directions, and show how the Killing spinor

equations reduce to a single gravitino-type equation for a spinor ψ_+ which depends only on the internal manifold co-ordinates. We also prove that the supersymmetric dS$_4$ warped product solutions preserve $N = 8n$ supersymmetries for $n = 1, 2, 3, 4$. In section 5 we utilize spinorial geometry techniques, and prove that the spinor ψ_+ can be written in a particularly simple canonical form on applying appropriate Spin(7) gauge transformations. Furthermore, we prove that such a spinor has stabilizer subgroup which is either SU(3) or G$_2$; in the SU(3) case we also consider several possible special sub-cases. In section 6, we present the SU(3) covariant conditions on the flux and geometry, obtained from the gravitino-type equation in the case of SU(3) stabilizer. In section 7 we also prove that there are no supersymmetric warped product dS$_4$ solutions for which the stabilizer subgroup of ψ_+ is G$_2$. We present our conclusions in section 8. In appendix A, we list some conventions. In appendix B we present some properties of the explicit representation of the Clifford algebras in terms of differential forms, as utilized in the spinorial geometry method. In appendix C we list some equations which are used in the process of integrating up the Killing spinor equation along the dS$_4$ directions in section 4. In appendix D we list the linear system of equations obtained from the gravitino-type equation for the cases of ψ_+ with SU(3) and G$_2$ stabilizer. In appendix E we state a number of useful identities which are used to covariantize the solutions of the linear systems in terms of gauge-invariant bilinears.

2 Bosonic field equations and KSE

In this section, we summarize the bosonic field equations and Killing spinor equations (KSE) of $D = 11$ supergravity, and describe the ansatz for the warped product dS$_4$ geometries. The bosonic fields of $D = 11$ supergravity consist of a metric g, and a 3-form gauge potential A with 4-form field strength $F = \mathrm{d}A$. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x \sqrt{-g} R - \frac{1}{2} F \wedge *F + \frac{1}{6} F \wedge F \wedge A \,, \tag{2.1}$$

where κ^2 is proportional to the gravitational coupling constant. The equations of motion are thus given by

$$R_{AB} - \frac{1}{2} R g_{AB} - \frac{1}{12} F_{AB_1B_2B_3} F_B{}^{B_1B_2B_3} + \frac{1}{96} g_{AB} F^2 = 0$$
$$\mathrm{d} * F - \frac{1}{2} F \wedge F = 0 \,. \tag{2.2}$$

By using

$$R = \frac{1}{144} F^2 \,, \tag{2.3}$$

the first equation in (2.2) becomes

$$R_{AB} - \frac{1}{12} \left(F_{AB_1B_2B_3} F_B{}^{B_1B_2B_3} - \frac{1}{12} g_{AB} F^2 \right) = 0. \tag{2.4}$$

The supercovariant derivative $\mathcal{D}_A$ is defined as

$$\mathcal{D}_M \equiv \nabla_M - \frac{1}{288} \left(\Gamma_M{}^{A_1A_2A_3A_4} - 8\delta_M^{A_1} \Gamma^{A_2A_3A_4} \right) F_{A_1A_2A_3A_3A_4} \,. \tag{2.5}$$

Bosonic solutions to the equations of motion that preserve at least one supersymmetry are those that admit at least one non-vanishing Killing spinor ϵ, which satisfies

$$\mathcal{D}_A \epsilon = 0 \,. \tag{2.6}$$

In order to analyse supersymmetric warped product dS_4 solutions, we shall split the $D = 11$ spacetime in a 4+7 fashion $\mathrm{d}s^2 = \mathrm{dS}_4 \times_w M_7$, where $\times_w$ denotes a warped product of dS_4 with an internal manifold M_7. In terms of the $D = 11$ frame, capital latin letters such as A,B denote $D = 11$ frame indices. These D=11 frame indices are split in a 4+7 fashion as follows: we use greek letters for dS_4 frame directions, and latin letters from the middle of the alphabet and onwards for M_7. Latin letters from the beginning of the alphabet denote M_7 spacetime indices. M_7 is equipped with local co-ordinates y^a, whereas dS_4 is equipped with local co-ordinates x^μ. For further details about the conventions used are set out in appendix A.

The warped dS_4 product metric g is

$$\mathrm{d}s^2 = A^2 \mathrm{d}s^2_{\mathrm{dS}_4} + \mathrm{d}s^2_{M_7} = \eta_{\mu\nu} \mathbf{e}^\mu \mathbf{e}^\nu + \delta_{ij} \mathbf{e}^i \mathbf{e}^j \,, \tag{2.7}$$

where the vielbein frame is defined as

$$\begin{cases} \mathbf{e}^\mu \equiv \dfrac{A}{\mathcal{R}} \mathrm{d}x^\mu \\ \mathbf{e}^i \equiv e^i_b \mathrm{d}y^b \end{cases} \tag{2.8}$$

with

$$\mathcal{R}(x) = \left(1 + \frac{1}{4} K x_\nu x^\nu \right) , \qquad x_\nu \equiv x^\alpha \eta_{\alpha\nu} \,. \tag{2.9}$$

The conformal factor A and the vielbein e^j_a depend only on y^a co-ordinates. The scalar K is constant and greater than zero.

We require that the field strength F must be invariant under the isometries of dS_4, hence it decomposes as follows:

$$F = c\,\mathrm{dvol}(\mathrm{dS}_4) + X \,, \tag{2.10}$$

where c is a constant due to the Bianchi identity and X is a closed 4-form on M_7 depending only on y^a co-ordinates. The gauge field equation (2.2) is equivalent to

$$d(A^4 \star_7 X) = cX \,. \tag{2.11}$$

It will be convenient to state the non-vanishing components of the spin-connection, and curvature components. The non-vanishing spin-connection components are

$$\begin{aligned} \Omega_{\mu,\nu\rho} &= \frac{K}{A} x_{[\nu} \eta_{\rho]\mu} \\ \Omega_{\mu,i\nu} &= -\frac{\nabla_i A}{A} \eta_{\mu\nu} \\ \Omega_{ijk} &= \Omega_{ijk}(M_7) \,, \end{aligned} \tag{2.12}$$

where on the l.h.s. Greek indices are frame indices on dS_4, and on the r.h.s. they are co-ordinate indices on dS_4. ∇_i denotes the Levi-Civita connection on M_7.

The non-vanishing Riemann tensor components are

$$R_{\mu\nu\alpha\beta} = (\eta_{\mu\alpha}\eta_{\beta\nu} - \eta_{\nu\alpha}\eta_{\beta\mu})\left(\frac{K}{A^2} - \frac{\nabla_i A \nabla^i A}{A^2}\right)$$

$$R_{i\alpha j\beta} = -\frac{1}{A}\nabla_i\nabla_j A\,\eta_{\alpha\beta}$$

$$R_{ijkl} = R_{ijkl}(M_7) \tag{2.13}$$

where on the l.h.s. Greek indices are frame indices on dS$_4$, and on the r.h.s. they are co-ordinate indices on dS$_4$. The Ricci curvature tensor components are

$$R_{\mu\nu} = \eta_{\mu\nu}\left(3A^{-2}K - A^{-1}\nabla_i\nabla^i A - 3A^{-2}\nabla_i A\nabla^i A\right)$$

$$R_{\mu i} = 0$$

$$R_{ij} = -4A^{-1}\nabla_i\nabla_j A + R_{ij}(M_7) \tag{2.14}$$

where on the l.h.s. Greek indices are frame indices on dS$_4$, and on the r.h.s. they are co-ordinate indices on dS$_4$.

The $(\mu\nu)$-component of the Einstein equations of motion (2.4), imply that

$$3KA^{-1} - \nabla_i\nabla^i A - 3A^{-1}\nabla_i A\nabla^i A + \frac{1}{3}c^2 A^{-7} + \frac{A}{144}X^2 = 0\,. \tag{2.15}$$

From the (ij)-component of the Einstein equation of motion (2.4) and the third equation in (2.14), one finds

$$R_{ij}(M_7) = 4A^{-1}\nabla_i\nabla_j A + \frac{1}{12}X_{ia_1 a_2 a_3}X_j{}^{a_1 a_2 a_3} + \frac{1}{6}c^2 A^{-8}\delta_{ij} - \frac{1}{144}X^2\delta_{ij}\,. \tag{2.16}$$

On taking the trace of (2.16), and using (2.11) and (2.15), we obtain

$$R(M_7) - 8A^{-1}\nabla_i\nabla^i A - 12A^{-2}\nabla_i A\nabla^i A + 12A^{-2}K + \frac{1}{6}c^2 A^{-8} - \frac{1}{144}X^2 = 0\,. \tag{2.17}$$

3 Integrability conditions from the KSE

In this section, we begin the analysis of the KSE by computing the integrability conditions associated with (2.6). These results will be particularly useful when we explicitly integrate up the KSE along the dS$_4$ directions in the next section. From (2.5), we find

$$\frac{\partial}{\partial x^\mu}\epsilon = \frac{1}{\mathcal{R}}\left(-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu} + \frac{1}{2}\nabla_k A\Gamma^k\Gamma_\mu + \frac{A}{288}\Gamma_\mu\slashed{X} - \frac{c}{6}A^{-3}\Gamma_\mu\tilde{\Gamma}^4\right)\epsilon \tag{3.1}$$

and

$$\frac{\partial}{\partial y^a}\epsilon = e_a^j\left(\frac{1}{288}\slashed{X}\Gamma_j + \frac{c}{12}A^{-4}\Gamma_j\tilde{\Gamma}^4 - \frac{1}{36}\slashed{X}_j - \frac{1}{4}\Omega_{j,lm}\Gamma^{lm}\right)\epsilon\,, \tag{3.2}$$

where

$$\tilde{\Gamma}^4 \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3\,. \tag{3.3}$$

We remark that (3.2) is equivalent to

$$\nabla_i\epsilon = \left(\frac{1}{288}\slashed{X}\Gamma_i + \frac{c}{12}A^{-4}\Gamma_i\tilde{\Gamma}^4 - \frac{1}{36}\slashed{X}_i\right)\epsilon\,, \tag{3.4}$$

where ∇_i denotes the Levi-Civita connection on M_7.

We use these expressions to derive several integrability conditions. First, from the integrability condition on dS$_4$ spacetime

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right)\epsilon = 0\,, \tag{3.5}$$

we get

$$\left(|\nabla A|^2 - K - \frac{c^2}{9}A^{-6} - \frac{A^2}{(144)^2}\slashed{X}^2 + \frac{2}{3}cA^{-3}\nabla_i A\Gamma^i\tilde{\Gamma}^4 - \frac{1}{18}A\nabla_i A\slashed{X}^i\right)\epsilon = 0\,. \tag{3.6}$$

On the other hand, from the integrability condition with one direction on dS$_4$ and the other on M_7, i.e.

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^a} - \frac{\partial}{\partial y^a}\frac{\partial}{\partial x^\mu}\right)\epsilon = 0 \tag{3.7}$$

we get

$$\begin{aligned}
\Bigg(&-\frac{1}{2}\nabla_i\nabla_k A\Gamma^k + \frac{A}{288}\nabla_i\slashed{X} + \frac{5}{6}\frac{A}{288}\left(\Gamma_{[il_1l_2}{}^{j_3j_4}\delta^{j_2}_{l_3}\delta^{j_1}_{l_4]}X_{j_1j_2j_3j_4}\,X^{l_1l_2l_3l_4}\right)\\
&+\frac{5}{6}\frac{A}{288}\left(\Gamma_{[i}X_{l_1l_2l_3l_4]}\,X^{l_1l_2l_3l_4}\right) + \frac{c}{864}A^{-3}(10\slashed{X}_i - \Gamma_i\slashed{X})\tilde{\Gamma}^4\\
&+\frac{A}{144}\Gamma_{l_1}{}^{j_3j_4}X_i{}^{l_1j_1j_2}X_{j_1j_2j_3j_4} + \frac{c}{2}A^{-4}\nabla_i A\,\tilde{\Gamma}^4 + \frac{1}{72}\nabla_k A\Gamma_{l_1l_2l_3i}\,X^{l_1l_2l_3k}\\
&-\frac{c}{12}A^{-4}\nabla_k A\Gamma^k{}_i\,\tilde{\Gamma}^4 + \frac{1}{12}\nabla_k A\Gamma^{mn}\,X_{imn}{}^k\Bigg)\epsilon = 0\,.
\end{aligned} \tag{3.8}$$

The integrability conditions (3.6) and (3.8) are, however, not independent; (3.6) is implied by (3.8). To see this, contract (3.8) with Γ^i, and using equation of motion (2.15) and the Bianchi Identity $dF = 0$, we are able to derive the integrability condition (3.6).

So far, we have analyzed the integrability conditions involving the dS$_4$ part of the covariant derivative (3.1). The integrability condition on M_7 given by

$$[\nabla_i, \nabla_j]\,\epsilon = \frac{1}{4}R_{ijmn}\Gamma^{mn}\epsilon\,, \tag{3.9}$$

is

$$\begin{aligned}
\frac{1}{4}R_{ijmn}\Gamma^{mn}\epsilon =\ &\Bigg[\frac{1}{288}\left(\nabla_i(\slashed{\Gamma}\slashed{X}_j) - \nabla_j(\slashed{\Gamma}\slashed{X}_i)\right) - \frac{c}{3}A^{-5}\left(\nabla_i A\Gamma_j - \nabla_j A\Gamma_i\right)\tilde{\Gamma}^4\\
&-\frac{1}{36}\left(\nabla_i\slashed{X}_j - \nabla_j\slashed{X}_i\right) + \frac{1}{288^2}\left(\slashed{\Gamma}\slashed{X}_j\slashed{\Gamma}\slashed{X}_i - \slashed{\Gamma}\slashed{X}_i\slashed{\Gamma}\slashed{X}_j\right)\\
&+\frac{1}{36^2}\left(\slashed{X}_j\slashed{X}_i - \slashed{X}_i\slashed{X}_j\right) + \frac{1}{288}\frac{c}{12}A^{-4}\left(\slashed{\Gamma}\slashed{X}_j\Gamma_i - \Gamma_i\slashed{\Gamma}\slashed{X}_j\right)\tilde{\Gamma}^4\\
&+\frac{1}{288}\frac{c}{12}A^{-4}\left(\Gamma_j\slashed{\Gamma}\slashed{X}_i - \slashed{\Gamma}\slashed{X}_i\Gamma_j\right)\tilde{\Gamma}^4 + \frac{1}{288}\frac{1}{36}\left(\slashed{\Gamma}\slashed{X}_i\slashed{X}_j - \slashed{X}_j\slashed{\Gamma}\slashed{X}_i\right)\\
&+\frac{1}{288}\frac{1}{36}\left(\slashed{X}_i\slashed{\Gamma}\slashed{X}_j - \slashed{\Gamma}\slashed{X}_j\slashed{X}_i\right) + \frac{c^2}{72}A^{-8}\Gamma_{ij}\\
&+\frac{c}{432}A^{-4}\left(\slashed{X}_i\Gamma_j - \Gamma_j\slashed{X}_i\right)\tilde{\Gamma}^4 + \frac{c}{432}A^{-4}\left(\Gamma_i\slashed{X}_j - \slashed{X}_j\Gamma_i\right)\tilde{\Gamma}^4\Bigg]\epsilon\,. \tag{3.10}
\end{aligned}$$

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In fact, (3.8) is implied by (3.10). To see this, contract (3.10) with Γ^j and use the Einstein equation (2.16), the Bianchi identity, $R_{l[ijk]} = 0$, and the condition $dX = 0$, as well as the gauge field equations (2.11). In particular:

- The condition $dX = 0$ is used to derive:

$$4\Gamma^{la_1a_2a_3}\nabla_l X_{ka_1a_2a_3} = \nabla_k \slashed{X}\,. \tag{3.11}$$

- The gauge field equation (2.11) implies that from

$$4A(\Gamma_i\nabla_j\slashed{X}^j - \nabla_j\slashed{X}^j\Gamma_i) + cA^{-3}(\Gamma_i\slashed{X} - \slashed{X}\Gamma_i)\tilde{\Gamma}^4$$
$$+ 16\nabla_j A(\Gamma_i\slashed{X}^j - \slashed{X}^j\Gamma_i) = 0\,, \tag{3.12}$$

  and from this condition, it follows that

$$A\Gamma_{ia_1a_2a_3}\nabla_k X^{ka_1a_2a_3} = -cA^{-3}\slashed{X}_i\tilde{\Gamma}^4 - 4\nabla_k A\Gamma_{ia_1a_2a_3}X^{ka_1a_2a_3}\,. \tag{3.13}$$

- The gauge field equation (2.11) also implies that

$$4A(\Gamma_i\nabla_j\slashed{X}^j + \nabla_j\slashed{X}^j\Gamma_i) + cA^{-3}(\Gamma_i\slashed{X} + \slashed{X}\Gamma_i)\tilde{\Gamma}^4$$
$$+ 16\nabla_j A(\Gamma_i\slashed{X}^j + \slashed{X}^j\Gamma_i) = 0\,, \tag{3.14}$$

  and from this condition, it follows that

$$A\Gamma^{ab}\nabla^k X_{kiab} = -\frac{1}{12}cA^{-3}\Gamma_i\slashed{X}\tilde{\Gamma}^4 + \frac{1}{3}cA^{-3}\slashed{X}_i\tilde{\Gamma}^4 - 4\nabla^k A\Gamma^{ab}X_{kiab}\,. \tag{3.15}$$

Hence, it follows that the integrability conditions (3.6) and (3.8) are both implied by (3.10), which is derived from the integrability condition of (3.4).

4 Integration of KSE

In this section, we will explicitly integrate the KSE along the dS_4 directions. In this analysis, we shall show that the KSE reduce to a single gravitino-type KSE acting on a spinor ψ which is independent of the dS_4 co-ordinates. To begin, we shall define a spinor Φ, as follows:

$$\Phi \equiv \frac{A}{288}\slashed{X}\epsilon - \frac{1}{2}\nabla_k A\Gamma^k\epsilon + acA^{-3}\tilde{\Gamma}^4\epsilon\,, \tag{4.1}$$

where a is a constant to be fixed. We have chosen the relative coefficients between $\slashed{X}$ and $d\slashed{A}$ in (4.1) motivated by the first two terms in (3.8). We shall show that one can choose the constant a, as well as other constants k_1, k_2, q_1, q_2, q_3, q_4, q_5 such that

$$\nabla_i\Phi + k_1[\text{Eq. (3.8)}] + k_2 A^{-1}\Gamma^i[\text{Eq. (3.6)}]$$
$$+ q_1\slashed{D}X_i\Phi + q_2\slashed{X}_i\Phi + q_3cA^{-4}\Gamma_i\tilde{\Gamma}^4\Phi + q_4A^{-1}\nabla_k A\Gamma_i\Gamma^k\Phi + q_5A^{-1}\nabla_i A\Phi = 0\,. \tag{4.2}$$

Details of this calculation are presented in appendix C. One finds that

$$k_1 = -1 \quad a = -\frac{1}{6} \quad q_1 = \frac{1}{288} \quad q_2 = -\frac{1}{36} \quad q_3 = -\frac{1}{12} \quad k_2 = q_4 = q_5 = 0\,. \tag{4.3}$$

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Given this choice of constants, the spinor Φ is

$$\Phi = \left(\frac{A}{288}\slashed{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4 \right)\epsilon\,, \qquad (4.4)$$

which satisfies the following equations

$$\frac{\partial}{\partial x^\mu}\Phi = \frac{1}{\mathcal{R}}\left[-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu} + \Gamma_\mu\left(\frac{1}{2}\nabla_k A\Gamma^k + \frac{A}{288}\slashed{X} + \frac{c}{6}A^{-3}\tilde{\Gamma}^4 \right) \right]\Phi \qquad (4.5)$$

$$\frac{\partial}{\partial y^a}\Phi = e_a^j\left(-\frac{1}{288}\slashed{\Gamma}\slashed{X}_j + \frac{c}{12}A^{-4}\Gamma_j\tilde{\Gamma}^4 + \frac{1}{36}\slashed{X}_j - \frac{1}{4}\Omega_{j,lm}\Gamma^{lm} \right)\Phi\,. \qquad (4.6)$$

These equations are similar, but not identical, to the original Killing spinor equations for ϵ (3.1)–(3.2). The differences are in terms of certain signs appearing in (4.5)–(4.6), which are flipped with respect to (3.1)–(3.2) — in (4.5) the second and the fourth term with respect to (3.1) and in (4.6) the first and the third term with respect to (3.2).

Equations (4.5) and (4.6), will be particularly useful in the process of integrating up the KSE along the dS$_4$ directions. By using (3.6), (4.5) becomes

$$\frac{\partial}{\partial x^\mu}\Phi = -\frac{K}{4\mathcal{R}}x^\alpha\Gamma_{\alpha\mu}\Phi - \frac{K}{4\mathcal{R}}\Gamma_\mu\epsilon\,. \qquad (4.7)$$

By using the definition of Φ (4.4), one can rewrite $\frac{\partial}{\partial x^\mu}\epsilon$ as

$$\frac{\partial}{\partial x^\mu}\epsilon = \frac{1}{\mathcal{R}}\left(-\frac{1}{4}Kx^\alpha\Gamma_{\alpha\mu}\epsilon + \Gamma_\mu\Phi \right)\,. \qquad (4.8)$$

Applying a second derivative $\frac{\partial}{\partial x^\nu}$ to (4.8), using (4.7) and finally exploiting (4.8) to cancel $\mathcal{R}^{-1}\Gamma_\mu\Phi$ terms, one gets a second order differential equation for ϵ, namely

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\epsilon + \frac{K}{4\mathcal{R}}\left(x_\mu\frac{\partial}{\partial x^\nu}\epsilon + x_\nu\frac{\partial}{\partial x^\mu}\epsilon \right) - \frac{K^2}{16\mathcal{R}^2}x_\mu x_\nu\epsilon + \frac{K}{4\mathcal{R}}\eta_{\mu\nu}\epsilon = 0\,. \qquad (4.9)$$

On defining η by

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}\eta\,, \qquad (4.10)$$

it is straightforward to see that (4.9) is equivalent to

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\eta = 0 \quad \Rightarrow \quad \eta = \psi + x^\lambda\tau_\lambda\,, \qquad (4.11)$$

and hence this equation can be integrated to find

$$\eta = \psi + x^\lambda\tau_\lambda\,, \qquad (4.12)$$

where ψ, τ_λ with $\lambda = 0, 1, 2, 3$ are Majorana spinors which do not depend on the x_μ co-ordinates.

Given this expression for ϵ, i.e.

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}(\psi + x^\lambda\tau_\lambda)\,, \qquad (4.13)$$

we substitute it into the KSEs (3.1) and (3.2). As the spinors ψ, τ_λ are independent of the dS$_4$ co-ordinates, on expanding (3.1) and (3.2) order-by-order in x_α, we find various conditions.

In particular, from the KSE along the dS$_4$ directions (3.1), the vanishing of x-independent terms imply that the Majorana spinors τ_μ are given in terms of ψ, as follows:

$$\tau_\mu = \Gamma_\mu \left(\frac{A}{288}\slashed{X} - \frac{1}{2}\nabla_k A \Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4 \right)\psi \,. \tag{4.14}$$

The vanishing of the terms that are linear in x_μ in (3.1) imply

$$\left(|\nabla A|^2 - K - \frac{c^2}{9}A^{-6} - \frac{A^2}{(144)^2}\slashed{X}^2 + \frac{2}{3}cA^{-3}\nabla_i A \Gamma^i \tilde{\Gamma}^4 - \frac{1}{18}A\nabla_i A \slashed{X}^i \right)\psi = 0 \,, \tag{4.15}$$

and we remark that this condition is equivalent to the integrability condition (3.6), but with ϵ replaced with ψ. The terms in (3.1) which are quadratic in x_μ vanish identically; this then exhausts the content of (3.1).

Next we consider the KSE along the seven-dimensional internal directions, (3.2). Again, we substitute in (4.13) and expand order-by-order in dS$_4$ co-ordinates. The vanishing of x-independent terms gives

$$\nabla_i \psi = \left(\frac{1}{288}\slashed{\Gamma X}_i + \frac{c}{12}A^{-4}\Gamma_i \tilde{\Gamma}^4 - \frac{1}{36}\slashed{X}_i \right)\psi \,. \tag{4.16}$$

The above equation (4.16) implies that ψ satisfies a gravitino KSE along the internal directions, which is identical to the condition (4.16) but with ϵ replaced with ψ.

From the terms in (3.2) which are linear in x_μ we obtain

$$\begin{aligned}
\Bigg[&\frac{A}{288}\nabla_i \slashed{X} - \frac{1}{2}\nabla_i \nabla_k A \Gamma^k + \frac{A}{1728}\Gamma_i X^2 + \frac{A}{864}\Gamma_{j_1 j_2 j_3}{}^{l_1 l_2}X_{ij_4 l_1 l_2}X^{j_1 j_2 j_3 j_4} \\
&- \frac{A}{432}\Gamma_{j_1}X_{ij_2 j_3 j_4}X^{j_1 j_2 j_3 j_4} - \frac{A}{576}\Gamma_{ij_1 j_2}{}^{l_1 l_2}X_{j_3 j_4 l_1 l_2}X^{j_1 j_2 j_3 j_4} \\
&- \frac{1}{864}cA^{-3}\Gamma_i \slashed{X}\tilde{\Gamma}^4 + \frac{5}{432}cA^{-3}\slashed{X}_i \tilde{\Gamma}^4 + \frac{A}{144}\Gamma^m{}_{ab}X_{impq}X^{pqab} \\
&+ \frac{1}{72}\nabla_k A \Gamma_{ij_1 j_2 j_3}X^{kj_1 j_2 j_3} + \frac{1}{12}\nabla_k A \Gamma^{ab}X_i{}^k{}_{ab} \\
&- \frac{c}{12}A^{-4}\nabla_k A \Gamma^k \Gamma_i \tilde{\Gamma}^4 + \frac{7}{12}cA^{-4}\nabla_i A \tilde{\Gamma}^4 \Bigg]\psi = 0
\end{aligned} \tag{4.17}$$

which is identical to the integrability condition (3.8), with ϵ replaced by ψ. This then exhausts the content of (3.2).

Hence, we have shown that the spinor ϵ is given by

$$\epsilon = \mathcal{R}^{-\frac{1}{2}}(\psi + x^\lambda \tau_\lambda) \,, \tag{4.18}$$

where

$$\tau_\lambda = \Gamma_\lambda \left(\frac{A}{288}\slashed{X} - \frac{1}{2}\nabla_k A \Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4 \right)\psi \,. \tag{4.19}$$

The Majorana spinor ψ is independent of the dS$_4$ co-ordinates, and satisfies (4.16). Furthermore, ψ must also satisfy the algebraic conditions (4.17) and (4.15). However, as we have shown in the previous section, the integrability conditions of (4.16), together with the bosonic field equations and Bianchi identities, imply that (4.17) holds. Furthermore, we have shown that (4.17) also implies (4.15). Hence, the necessary and sufficient conditions for supersymmetry are encoded in (4.16).

4.1 Counting the supersymmetries

Having determined that the necessary and sufficient conditions for supersymmetry are given by (4.16), we shall now count the number of solutions to this equation. In particular, if ψ satisfies (4.16), then so does $\Gamma_{\mu\nu}\psi$. We choose a null basis for the Majorana representation of Spin(10,1) and take the dS$_4$ frame directions to correspond with the $+, -, 1, \bar{1}$ directions, see appendix B. The frame directions associated with the internal manifold M_7 correspond to the $2, 3, 4, \bar{2}, \bar{3}, \bar{4}, \#$ directions.

With these conventions for the de Sitter and internal manifold frames, we define lightcone projection operators as

$$P_\pm \equiv \frac{1}{2}\left(\mathbb{I} \pm \Gamma_{+-}\right). \tag{4.20}$$

As the projection operator $P_\pm$ commutes with the supercovariant derivative (4.16), we then decompose the spinor ψ using the lightcone projectors and we define $\psi_\pm$ to be

$$\psi_\pm \equiv P_\pm \psi \quad \Rightarrow \quad \Gamma_\pm \psi_\pm = 0\,. \tag{4.21}$$

Without loss of generality, utilizing these projection operators, any supersymmetric solution must admit a positive chirality solution ψ_+ to (4.16). Given such a ψ_+ spinor, we can then define

$$\begin{aligned}
\tilde{\psi}_+ &\equiv i\Gamma_{1\bar{1}}\psi_+ \\
\psi_- &\equiv \Gamma_-(\Gamma_1 + \Gamma_{\bar{1}})\psi_+ \\
\tilde{\psi}_- &\equiv i\Gamma_-(\Gamma_1 - \Gamma_{\bar{1}})\psi_+\,.
\end{aligned} \tag{4.22}$$

$\tilde{\psi}_+$ is an additional positive chirality solution to (4.16), and $\{\psi_-, \tilde{\psi}_-\}$ are two negative chirality solutions to (4.16). $\{\psi_+, \tilde{\psi}_+, \psi_-, \tilde{\psi}_-\}$ are linearly independent, as by construction they are mutually orthogonal with respect to the Dirac inner product $\langle \,\cdot\,, \,\cdot\, \rangle$.

It would therefore appear, a priori, that the number of supersymmetries is $4n$. However, there are, in fact further additional spinors. To see this, note that (4.6) implies that

$$\check{\psi}_+ \equiv (\Gamma_1 + \Gamma_{\bar{1}})\left(\frac{A}{288}\slashed{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4\right)\psi_+\,, \tag{4.23}$$

is also a positive chirality solution of (4.16). Furthermore, it can be shown that $\{\psi_+, \tilde{\psi}_+, \check{\psi}\}$ are linearly independent. To see this, suppose that

$$\check{\psi}_+ = c_1\psi_+ + ic_2\Gamma_{1\bar{1}}\psi_+\,, \tag{4.24}$$

for real constants c_1, c_2. Acting on both sides of this condition with the operator $(\Gamma_1 + \Gamma_{\bar{1}})(\frac{A}{288}\cancel{X} - \frac{1}{2}\nabla_k A\Gamma^k - \frac{c}{6}A^{-3}\tilde{\Gamma}^4)$, and utilizing the integrability condition (4.15) to simplify the l.h.s., we find

$$-\frac{K}{2}\psi = (c_1^2 + c_2^2)\psi\,, \qquad (4.25)$$

where we have also used (4.24) to simplify the r.h.s. It is clear that this admits no solution, as $K > 0$. Hence, we find that we can construct four linearly independent positive chirality spinors which solve (4.16), corresponding to $\{\psi_+, \tilde{\psi}_+, \check{\psi}_+, \check{\tilde{\psi}}_+\}$, where $\check{\tilde{\psi}}_+ \equiv i\Gamma_{1\bar{1}}\check{\psi}$. There are also four corresponding negative chirality spinors given by $\{\psi_-, \tilde{\psi}_-, \check{\psi}_-, \check{\tilde{\psi}}_-\}$, where

$$\check{\psi}_- = \Gamma_-(\Gamma_1 + \Gamma_{\bar{1}})\check{\psi}_+\,, \qquad \check{\tilde{\psi}}_- = i\Gamma_{1\bar{1}}\check{\psi}_-\,. \qquad (4.26)$$

Hence we have constructed 8 linearly independent solutions to (4.16),

$$\left\{\psi_+, \tilde{\psi}_+, \check{\psi}_+, \check{\tilde{\psi}}_+, \psi_-, \tilde{\psi}_-, \check{\psi}_-, \check{\tilde{\psi}}_-\right\} \qquad (4.27)$$

and it follows that the number of supersymmetries for warped product dS_4 solutions is $8n$, $n = 1, 2, 3, 4$.

We remark that the existence of the additional spinors $\check{\psi}_\pm$, $\check{\tilde{\psi}}_\pm$ is somewhat analogous to results found in the analysis of near-horizon geometries of supersymmetric extremal black holes [23] and also for warped product AdS solutions [24]. In these cases, given a Killing spinor, one also finds that additional Killing spinors can be generated by the action of certain algebraic operators constructed out of the fluxes of the theory.

5 Spinorial geometry: canonical forms for ψ

In this section we shall use Spin(7) gauge transformations to bring the spinor ψ_+ to one of several simple canonical forms. We will describe the gauge transformations used to do this explicitly.

The most general form of a positive chirality Majorana spinor $\psi_+ \in \Delta_{32}$ can be expressed by using (B.7), i.e.

$$\psi_+ = w1 + \bar{w}e_{1234} + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3}$$
$$+ \Omega^q e_{1q} - \frac{1}{2!}\bar{\Omega}^q \varepsilon_q{}^{mn} e_{mn}\,, \qquad (5.1)$$

with $l, q, m, m = 2, 3, 4$. As the action of SU(N) on $\mathbb{C}P^{N-1}$ is transitive, one can apply a SU(3) gauge transformation in the 2,3,4 directions to set, without loss of generality, $\Omega^3 = \Omega^4 = 0$,[1] i.e.

$$\psi_+ = w1 + \bar{w}e_{1234} + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega e_{12} - \bar{\Omega}e_{34}\,. \qquad (5.2)$$

---

[1]Generally, the complex value Ω^2 can be set to be real with the same SU(3) transformation used to set $\Omega^3 = \Omega^4 = 0$. It does not happen in this specific case due to the fact that Ω^2 will be promoted to be complex value in the next gauge transformation.

To proceed further, we define T^1, T^2, T^3 as

$$T_1 \equiv \frac{1}{2}(\Gamma_{34} + \Gamma_{\bar{3}\bar{4}}) \quad T_2 \equiv \frac{i}{2}(\Gamma_{34} - \Gamma_{\bar{3}\bar{4}}) \quad T_3 \equiv T_1 T_2 = \frac{i}{2}(\Gamma_{3\bar{3}} + \Gamma_{4\bar{4}}). \tag{5.3}$$

It is straightforward to verify that T^i with $i = 1, 2, 3$, which satisfy the algebra of the imaginary unit quaternions, preserve the span of the following basis elements

$$v_1 \equiv (1 + e_{1234}) \qquad v_2 \equiv i(1 - e_{1234})$$
$$v_3 \equiv (e_{12} - e_{34}) \qquad v_4 \equiv i(e_{12} + e_{34}) \tag{5.4}$$

and we remark that the Spin(7) gauge transformation generated by the T_i is of the form $p^4 \mathrm{id} + p^i T_i$ where $(p^1, p^2, p^3, p^4) \in S^3$.

Then one can carry out a SO(2) gauge transformation generated by T_3 to set $w \in \mathbb{R}$. So far, the spinor ψ_+ can be written as

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega e_{12} - \bar{\Omega} e_{34}. \tag{5.5}$$

A SU(3) gauge transformation generated by $i(\Gamma_{2\bar{2}} - \frac{1}{2}\Gamma_{3\bar{3}} - \frac{1}{2}\Gamma_{4\bar{4}})$, which leaves $\{1, e_{1234}\}$ invariant, is then used to set $\Omega \in \mathbb{R}$, so

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^j e_j - \frac{1}{3!}(*\bar{\lambda})^{l_1 l_2 l_3} e_{l_1 l_2 l_3} + \Omega(e_{12} - e_{34}). \tag{5.6}$$

We next exploit a SO(2) transformation generated by T_1, acting on v_1 and v_3 to put $\Omega = 0$. Then, we make a further SU(3) gauge transformation along the $2, 3, 4$ directions to set $\lambda^3 = \lambda^4 = 0$ with $\lambda^2 \in \mathbb{R}$, i.e.

$$\psi_+ = w(1 + e_{1234}) + \lambda^1 e_1 + \bar{\lambda}^1 e_{234} + \lambda^2(e_2 - e_{134}). \tag{5.7}$$

In order to simplify further the spinor ψ_+, we shall introduce additional Spin(7) generators L_1, L_2, L_3 given by

$$L_1 \equiv \frac{1}{\sqrt{2}}\Gamma_\sharp(\Gamma_2 + \Gamma_{\bar{2}}), \quad L_2 \equiv \frac{i}{\sqrt{2}}\Gamma_\sharp(\Gamma_2 - \Gamma_{\bar{2}}), \quad L_3 \equiv L_1 L_2 = i\Gamma_{2\bar{2}}. \tag{5.8}$$

The L_j also satisfy the algebra of the imaginary unit quaternions, and commute with the T_i, and the Spin(7) gauge transformation generated by the L_j is of the form $q^4 \mathrm{id} + q^j L_j$ where $(q^1, q^2, q^3, q^4) \in S^3$. We shall then consider a generic gauge transformation generated by the T_i and L_j of the acting on the spinor (5.7) of the form

$$(p^4 \mathrm{id} + p^i T_i)(q^4 \mathrm{id} + q^j L_j)\psi_+. \tag{5.9}$$

We set $q^2 = q^3 = 0$ and $q^1 = \sin\sigma, q^4 = \cos\sigma$, such that

$$w\lambda^2 \cos 2\sigma + \frac{1}{2}\sin 2\sigma(\omega^2 - (\lambda^2)^2 - |\lambda^1|^2) = 0 \tag{5.10}$$

and

$$p^1 = \ell \mathrm{Re}(\lambda^1)\sin\sigma, \quad p^2 = -\ell \mathrm{Im}(\lambda^1)\sin\sigma,$$
$$p^3 = 0, \qquad\qquad p^4 = \ell(\omega\cos\sigma - \lambda^2\sin\sigma) \tag{5.11}$$

where the constant ℓ is chosen such that $(p^1, p^2, p^3, p^4) \in S^3$. With this choice of parameters, the gauge transformation given in (5.9) can be used to set $\lambda^2 = 0$ in (5.7), so the simplest canonical form for the spinor ψ_+ is given by

$$\psi_+ = w(1 + \mathrm{e}_{1234}) + \lambda \mathrm{e}_1 + \bar{\lambda} \mathrm{e}_{234} \qquad w \in \mathbb{R}, \ \lambda \in \mathbb{C}. \tag{5.12}$$

5.1 Stabilizer group of ψ_+

It is useful to consider the stabilizer subgroup of $\mathrm{Spin}(7)$ which leaves ψ_+ invariant. In particular, we must determine the generators $f^{ij}\Gamma_{ij}$, where $f^{ij} \in \mathbb{R}$ are antisymmetric in i, j, and satisfy

$$f^{ij}\Gamma_{ij}\psi_+ = 0 \qquad i, j = \#, \alpha, \bar{\alpha}. \tag{5.13}$$

The conditions obtained from (5.13) are

$$\begin{aligned}
2wf^{\alpha\beta} &= \sqrt{2}\bar{\lambda}f^{\#\bar{\rho}}\varepsilon_{\bar{\rho}}{}^{\alpha\beta} \\
2\lambda f^{\alpha\beta} &= \sqrt{2}wf^{\#\bar{\rho}}\varepsilon_{\bar{\rho}}{}^{\alpha\beta} \\
f^{\alpha}{}_{\alpha} &= 0.
\end{aligned} \tag{5.14}$$

Depending on w and λ, there are two possible different stabilizer subgroups:

(a) if $w^2 - |\lambda|^2 \neq 0$ then (5.14) implies that $f_{\alpha\beta} = 0$ and $f_{\#\alpha} = 0$, hence the stabilizer is $\mathrm{SU}(3)$.

(b) if $w^2 - |\lambda|^2 = 0$, the stabilizer is G_2.

In the $\mathrm{SU}(3)$ stabilized case it is particularly useful to consider the complex $\mathrm{SU}(3)$ invariant spinor bilinear scalar $\langle \psi_+, \Gamma_{\bar{1}}\psi_+ \rangle = 2\sqrt{2}w\lambda$. There are various different cases, corresponding to whether this scalar vanishes, or it does not vanish:

(i) $w \neq 0$, $\lambda = 0$,

(ii) $\lambda \neq 0$, $w = 0$,

(iii) $w \neq 0$, $\lambda \neq 0$.

In fact, it is straightforward to see that the spinors associated with cases (i) and (ii) above are related by a $\mathrm{Pin}(7)$ transformation. To see this, consider the spinor from case (ii),

$$\psi_+ = \lambda \mathrm{e}_1 + \bar{\lambda}\mathrm{e}_{234}. \tag{5.15}$$

The $\mathrm{Spin}(7)$ gauge transformation generated by L_3 produces a $\mathrm{SO}(2)$ which acts transitively on $\{e_1 + e_{234}, i(e_1 - e_{234})\}$, and hence without loss of generality we can set $\psi_+ = \lambda(e_1 + e_{234})$ for $\lambda \in \mathbb{R}$. Next, note that

$$\Gamma_{234}(e_1 + e_{234}) = -(1 + e_{1234}). \tag{5.16}$$

It therefore follows that the spinor ψ_+ in case (ii) is $\mathrm{Spin}(7)$ gauge-equivalent to a spinor which in turn is Pin-equivalent, with respect to $\Gamma_{234} \in \mathrm{Pin}(7)$, to the spinor in case (i). The

effect of the Γ_{234} transformation is to flip holomorphic with anti-holomorphic directions and to reflect along the $\#$ direction, namely

$$\alpha \to \bar{\alpha}, \qquad \# \to -\# . \tag{5.17}$$

It is therefore sufficient to consider spinors ψ_+ corresponding to the G_2 stabilizer case, and the two SU(3) stabilizer cases (i), (iii). Having determined the stabilizers associated with these three canonical types of spinors, we next proceed to obtain a linear system of equations by substituting these expressions for ψ_+ into (4.16). The linear system consists of relations between the flux and spin-connection, which when covariantized with respect to the appropriate stabilizer group, give rise to conditions on the flux X and the geometry of the internal manifold M_7. In the following sections, we shall present the covariant solution of the linear system for each of the stabilizer subgroups.

6   SU(3) invariant spinor

In this section, we solve the KSEs (4.16) when the stabilizer of ψ_+ is SU(3), corresponding to

$$\psi_+ = w(1 + \mathrm{e}_{1234}) + \lambda \mathrm{e}_1 + \bar{\lambda} \mathrm{e}_{234} \qquad w \in \mathbb{R}, \ \lambda \in \mathbb{C}, \tag{6.1}$$

for $w^2 - |\lambda|^2 \neq 0$. We begin by considering the case for which both w and λ are non-vanishing. Furthermore, we will write $\lambda = \rho e^{i\theta}$, where $\rho > 0$ and $\theta \in [0, 2\pi[$ are two real spacetime functions. The associated linear system and the components of the flux are presented in appendix D. The linear system is initially expressed non-covariantly in terms of SU(3)-components of the spin-connection and the fluxes, but, it can be rewritten in SU(3)-covariant form by using the SU(3) gauge invariant bilinears. In appendix E we set out the main relations which are used to write the relations in a manifestly SU(3) covariant fashion, in terms of the following SU(3) invariant bilinears:

$$\xi \equiv \mathrm{e}^{\#}, \qquad \omega \equiv -i\delta_{\alpha\bar{\beta}} \, \mathrm{e}^{\alpha} \wedge \mathrm{e}^{\bar{\beta}}, \qquad \chi \equiv \frac{1}{3!}\varepsilon_{\alpha\beta\gamma}\mathrm{e}^{\alpha} \wedge \mathrm{e}^{\beta} \wedge \mathrm{e}^{\gamma}. \tag{6.2}$$

The above forms are obtained from the following SU(3)-invariant spinor bilinears:

$$\langle \psi_+, \Gamma_a \psi_+ \rangle \mathrm{e}^a = -2(w^2 - |\lambda|^2)\xi \tag{6.3}$$

$$\frac{1}{2!}\langle \psi_+, \Gamma_{ab}\tilde{\Gamma}^4 \psi_+ \rangle \mathrm{e}^a \wedge \mathrm{e}^b = -2(w^2 - |\lambda|^2)\omega \tag{6.4}$$

$$\frac{1}{3!}\langle \psi_+, (\Gamma_1 + \Gamma_{\bar{1}})\Gamma_{abc}\psi_+ \rangle \mathrm{e}^a \wedge \mathrm{e}^b \wedge \mathrm{e}^c = 2iw(\bar{\lambda} - \lambda)\xi \wedge \omega$$
$$+ 4(w^2 - \lambda^2)\chi + 4(w^2 - \bar{\lambda}^2)\bar{\chi} \tag{6.5}$$

where $a, b, c = \alpha, \bar{\alpha}, \#$.

In constructing the solution to the linear system (D.2)–(D.13), it is convenient to make use of the two Lee forms built from χ, and ω, which are

$$Z_i \equiv \nabla^j \chi_{jkl}\bar{\chi}^{kl}{}_i, \qquad W_i \equiv \nabla^j \omega_{jk}\omega^k{}_i. \tag{6.6}$$

After some computation, the SU(3)-covariant conditions are as follows (here $i, j, k = \#, \alpha, \bar{\alpha}$ are frame indices on M_7):

$$\nabla^i \xi_i = -\frac{6\xi^j}{(w^2 - |\lambda|^2)^2}\Big[w(2w^2 + 3|\lambda|^2)(\mathrm{d}w)_j + (3w^2 + 2|\lambda|^2)\mathfrak{Re}(\lambda\mathrm{d}\bar{\lambda})_j\Big] \tag{6.7}$$

$$\nabla^i \xi^j \omega_{ij} = -cA^{-4} - 6w^2\frac{\xi^k \mathfrak{Im}(\lambda\mathrm{d}\bar{\lambda})_k}{w^4 - |\lambda|^4} \tag{6.8}$$

$$\chi_{ijk}(\mathcal{L}_\xi \bar{\chi})^{ijk} = -6\xi^l\Bigg[\bar{\lambda}\frac{(2w^4 + 2|\lambda|^4 + 11w^2|\lambda|^2)}{(w^2 + |\lambda|^2)(w^2 - |\lambda|^2)^2}(\mathrm{d}\lambda)_l$$
$$+ \lambda\frac{(7w^4 + 4|\lambda|^4 + 4w^2|\lambda|^2)}{(w^2 + |\lambda|^2)(w^2 - |\lambda|^2)^2}(\mathrm{d}\bar{\lambda})_l$$
$$+ 3w\frac{(2w^2 + 3|\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}w)_l\Bigg] - 4icA^{-4} \tag{6.9}$$

$$i\chi_{ijk}(\mathrm{d}\omega)^{ijk} = 9\sqrt{2}\xi^l\Bigg[\bar{\lambda}\frac{(9w^2 + |\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}w)_l + w\frac{(w^2 + 4|\lambda|^2)}{(w^2 - |\lambda|^2)^2}(\mathrm{d}\bar{\lambda})_l$$
$$+ 5\frac{\bar{\lambda}^2 w}{(w^2 - |\lambda|^2)^2}(\mathrm{d}\lambda)_l\Bigg] \tag{6.10}$$

$$(\mathrm{d}\xi)^{ij}\chi_{ijk} = \frac{\sqrt{2}}{(w^2 - |\lambda|^2)}\Bigg[w\frac{\bar{\lambda}}{\lambda}(\mathrm{d}\lambda)_k - w(\mathrm{d}\bar{\lambda})_k\Bigg] \tag{6.11}$$

$$Z = \frac{4}{w^2 - |\lambda|^2}\Bigg[w^2\mathrm{d}\log\rho + w^2 i_{\mathrm{d}\theta}w - \frac{|\lambda|^2}{w}\mathrm{d}w + \xi\left(\frac{|\lambda|^2}{w}(\mathrm{d}w)_j - w^2(\mathrm{d}\log\rho)_j\right)\xi^j\Bigg] \tag{6.12}$$

$$W = -\frac{1}{3}\frac{1}{w^2 - |\lambda|^2}\Bigg[\frac{1}{w}(w^2 - 4|\lambda|^2)\Big(\mathrm{d}w - \xi(\mathrm{d}w)_j\xi^j\Big)$$
$$+ (5w^2 + 4|\lambda|^2)\Big(\mathrm{d}\log\rho + i_{\mathrm{d}\theta}w - \xi(\mathrm{d}\log\rho)_j\xi^j\Big)\Bigg] \tag{6.13}$$

$$\mathcal{L}_\xi \xi = \frac{1}{3}\Bigg[\frac{1}{w}(7w^2 + 2|\lambda|^2)\Big(\mathrm{d}w - \xi(\mathrm{d}w)_j\xi^j\Big)$$
$$+ (w^2 + 2|\lambda|^2)\Big(\xi(\mathrm{d}\log\rho)_j\xi^j - \mathrm{d}\log\rho - i_{\mathrm{d}\theta}w\Big)\Bigg] \tag{6.14}$$

$$\chi_{ml[i}(\mathrm{d}\xi)^{mn}\bar{\chi}_{j]n}{}^l = \frac{1}{3}cA^{-4}\omega_{ij}\frac{(w^2 + |\lambda|^2)}{(w^2 - |\lambda|^2)} + 2\omega_{ij}\frac{w^2}{(w^4 - |\lambda|^4)}\xi^k\mathfrak{Im}(\lambda\mathrm{d}\bar{\lambda})_k \tag{6.15}$$

$$8|\lambda|^2(5w^2 + |\lambda|^2)\mathrm{d}w + 4w\mathrm{d}\rho^2(w^2 + 5|\lambda|^2) + 8w\rho^2 i_{\mathrm{d}\theta}\omega(w^2 - |\lambda|^2)$$
$$- \xi^i\big[8|\lambda|^2(5w^2 + |\lambda|^2)\mathrm{d}w + 4w\mathrm{d}\rho^2(w^2 + 5|\lambda|^2) + 8w\rho^2 i_{\mathrm{d}\theta}\omega(w^2 - |\lambda|^2)\big]_i = 0. \tag{6.16}$$

We also obtain a SU(3) invariant expression for the flux X. In general, any real 4-form on M_7 can be written as

$$X = \mathrm{e}^\# \wedge Y + \omega \wedge \sigma + \beta \wedge \chi + \bar{\beta} \wedge \bar{\chi} + X^{\mathrm{TT}} \tag{6.17}$$

where

- σ is a real two-form;

- β is a complex one-form, and $\bar{\beta}$ is its complex conjugate;

- Y is real 3-form;

- X^{TT} is the traceless (2,2)-part of the flux.

We remark that X^{TT} is the only part of the flux that is not fixed by the linear system. However, a traceless $(2,2)$ 4-form in 6 dimensions vanishes identically. To see this, note that X^{TT} is dual (in 6 dimensions) to a $(1,1)$ 2-form R, $R = *_6 X^{\mathrm{TT}}$. Furthermore, by definition

$$
\begin{aligned}
R_{\alpha\bar{\beta}} &= \frac{1}{4!}\varepsilon_{\alpha\bar{\beta}}{}^{b_1 b_2 b_3 b_4} X^{\mathrm{TT}}_{b_1 b_2 b_3 b_4} = \frac{1}{4}\varepsilon_{\alpha\bar{\beta}}{}^{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2} X^{\mathrm{TT}}_{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2} \\
&= \frac{i}{4}\epsilon_\alpha{}^{\bar{\nu}_1\bar{\nu}_2}\varepsilon_{\bar{\beta}}{}^{\mu_1\mu_2} X^{\mathrm{TT}}_{\mu_1\mu_2\bar{\nu}_1\bar{\nu}_2} = \frac{i3!}{4}\delta^{\beta\mu_1\mu_2}_{\alpha\nu_1\nu_2} X^{\mathrm{TT}}_{\mu_1\mu_2}{}^{\nu_1\nu_2} = 0
\end{aligned}
\tag{6.18}
$$

as the contribution from trace terms in the final term vanishes. Hence R vanishes identically, and so $X^{\mathrm{TT}} = 0$.

It follows that the flux can be written as

$$
X = \mathrm{e}^{\#} \wedge Y + \omega \wedge \sigma + \beta \wedge \chi + \bar{\beta} \wedge \bar{\chi}
\tag{6.19}
$$

where all of these terms are fixed by the Killing spinor equations. In particular, the components of the real 2-form σ and of the complex 1-form β are given by

$$
\begin{aligned}
\sigma_{ij} = &-\frac{(w^2 - |\lambda|^2)}{(w^2 + |\lambda|^2)}\left(\mathcal{L}_\xi\omega\right)_{ij} \\
&- 2\omega_{ij}\frac{\xi^k}{(w^2 - |\lambda|^2)(w^2 + |\lambda|^2)}\left[w(w^2 + 2|\lambda|^2)(\mathrm{d}w)_k + (2w^2 + |\lambda|^2)\mathfrak{Re}(\lambda\mathrm{d}\bar{\lambda})_k\right]
\end{aligned}
\tag{6.20}
$$

$$
\begin{aligned}
\beta_i = &-3\sqrt{2}\frac{\lambda w}{(w^2 - |\lambda|^2)}\left[\left(\mathcal{L}_\xi\xi\right)_i + i\left(\mathcal{L}_\xi\xi\right)_j\omega^j{}_i\right] + \frac{3}{2}\frac{(w^2 + |\lambda|^2)}{(w^2 - |\lambda|^2)}(\mathrm{d}\xi)^{kj}\bar{\chi}_{kji} \\
&+ i\frac{(w^4 + 4w^2|\lambda|^2 + |\lambda|^4)}{(w^2 - |\lambda|^2)(w^2 + |\lambda|^2)}\left(\mathcal{L}_\xi\omega\right)^{kj}\bar{\chi}_{kji} .
\end{aligned}
\tag{6.21}
$$

The real 3-form Y has components

$$
Y_{ijk} = q\omega_{[i}{}^l(\mathrm{d}\omega)_{jk]l} + (\omega \wedge V)_{ijk} + h\chi_{ijk} + \bar{h}\bar{\chi}_{ijk}
\tag{6.22}
$$

where q and h are functions, and V is a real one-form, given by:

$$
q = -3\frac{(w^2 - |\lambda|^2)}{(w^2 + |\lambda|^2)}
\tag{6.23}
$$

$$
V_i = \frac{(w^2 - |\lambda|^2)}{(w^2 + |\lambda|^2)}\omega_i{}^j(\mathcal{L}_\xi\xi)_j
\tag{6.24}
$$

$$
\begin{aligned}
h = &\frac{3\sqrt{2}}{2}\frac{\xi^k}{(w^2 - |\lambda|^2)}\left[\lambda(\mathrm{d}w)_k + \frac{w^3}{(w^2 + |\lambda|^2)}(\mathrm{d}\lambda)_k + \frac{w\lambda^2}{(w^2 + |\lambda|^2)}(\mathrm{d}\bar{\lambda})_k\right] \\
&- q\frac{i}{6}\bar{\chi}^{ijk}(\mathrm{d}\omega)_{ijk} .
\end{aligned}
\tag{6.25}
$$

6.1 SU(3) invariant spinor with $\lambda = 0$, $w \neq 0$

Next, we consider the special case of the SU(3) invariant spinor

$$\psi_+ = w(1 + e_{1234}) \qquad w \in \mathbb{R}, \quad w \neq 0. \tag{6.26}$$

The SU(3) covariant geometric conditions obtained from the linear system are:

$$(\mathrm{d}\omega)^{(3,0)} = (\mathrm{d}\omega)^{(0,3)} = 0 \tag{6.27}$$

$$\mathrm{d}(w^4\xi) = -\frac{c}{3}A^{-4}\omega w^4 \tag{6.28}$$

$$\bar{\chi}^{ijk}\left(\mathcal{L}_\xi\omega\right)_{jk} = 0 \tag{6.29}$$

$$Z_i = -20(w^{-1}\mathrm{d}w)_i + 20\xi_i\xi^k(w^{-1}\mathrm{d}w)_k \tag{6.30}$$

$$W_i = 8(w^{-1}\mathrm{d}w)_i - 8\xi_i\xi^k(w^{-1}\mathrm{d}w)_k \tag{6.31}$$

$$\mathfrak{Im}\left(\bar{\chi}^{ijk}(\mathcal{L}_\xi\chi)_{ijk}\right) - 4cA^{-4} = 0 \tag{6.32}$$

$$\nabla^i\xi_i = -12\xi^k w^{-1}(\mathrm{d}w)_k \tag{6.33}$$

$$\nabla^i\xi^j\omega_{ij} = -cA^{-4}. \tag{6.34}$$

The flux X can be expressed as

$$X = e^\# \wedge Y + \omega \wedge \sigma \tag{6.35}$$

with

$$\sigma = -w^{-2}\mathcal{L}_\xi(w^2\omega) \tag{6.36}$$

and

$$Y_{ijk} = -3\omega_{[i}{}^l(\mathrm{d}\omega)_{jk]l} + (\omega \wedge V)_{ijk} \tag{6.37}$$

where

$$V_i = \omega_i{}^j(\mathcal{L}_\xi\xi)_j. \tag{6.38}$$

7 G_2 invariant spinor

In this section, we shall consider the case when the stabilizer of ψ_+ is G_2, corresponding to the case

$$\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda} e_{234} \qquad w \in \mathbb{R}, \ \lambda \in \mathbb{C}, \tag{7.1}$$

with $w^2 = |\lambda|^2$. We shall show that this orbit admits no solutions to the Killing spinor equations and the bosonic field equations. To establish this result, we set $\lambda \equiv e^{i\zeta}w$, where ζ is a real function. The geometric conditions we obtained by solving the linear system are:

$$\mathrm{d}w = \mathrm{d}\zeta = 0 \tag{7.2}$$

$$\Omega_{\mu,\alpha}{}^\alpha = 0 \tag{7.3}$$

$$\Omega_{\#,\alpha}{}^\alpha = i\frac{c}{6}A^{-4} \tag{7.4}$$

$$\Omega_{\#,\alpha\beta}\varepsilon^{\alpha\beta\gamma} = \sqrt{2}\,e^{i\zeta}\,\Omega_{\#,\#}{}^\gamma \tag{7.5}$$

$$\Omega_{\bar{\mu},\alpha\beta}\varepsilon^{\alpha\beta\gamma} = \sqrt{2}\,e^{i\zeta}\,\Omega_{\bar{\mu},\#}{}^\gamma \tag{7.6}$$

$$2\Omega_{\bar{\mu}}{}^{\gamma\rho} - \sqrt{2}\,e^{-i\zeta}\varepsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha} + i\sqrt{2}\frac{c}{6}e^{-i\zeta}A^{-4}\varepsilon_{\bar{\mu}}{}^{\gamma\rho} = 0. \tag{7.7}$$

Furthermore, we find that all of the components of the flux X vanish,

$$X = 0.\tag{7.8}$$

As $X = 0$, the integrability condition $\Gamma^j [\nabla_i, \nabla_j] \psi_+$ from (3.8) implies that

$$\left[-\frac{1}{2}\nabla_i\nabla_k A\Gamma^k + \frac{c}{2}A^{-4}\nabla_i A\,\tilde{\Gamma}^4 - \frac{c}{12}A^{-4}\nabla_k A\Gamma^k{}_i\,\tilde{\Gamma}^4 \right]\psi_+ = 0.\tag{7.9}$$

Multiplying (7.9) by Γ_l, we find

$$\begin{aligned}\left[-\frac{1}{2}\nabla_i\nabla_k A(\delta_l^k + \Gamma_l{}^k) + \frac{c}{2}A^{-4}\nabla_i A\Gamma_l\,\tilde{\Gamma}^4 \right.\\ \left. -\frac{c}{12}A^{-4}\nabla_k A(\Gamma_l{}^k{}_i + \delta_l^k\Gamma_i - \delta_{li}\Gamma^k)\,\tilde{\Gamma}^4 \right]\psi_+ = 0.\end{aligned}\tag{7.10}$$

We next take the inner product of (7.10) with ψ_+, noting that the anti-hermitian terms vanish identically as ψ_+ is Majorana. The hermitian part gives

$$\langle\psi_+, \left[-\frac{1}{2}\nabla_i\nabla_l A - \frac{c}{12}A^{-4}\nabla_k A\Gamma_l{}^k{}_i\,\tilde{\Gamma}^4 \right]\psi_+\rangle = 0.\tag{7.11}$$

The symmetric part of (7.11) then gives

$$\nabla_i\nabla_l A||\psi_+||^2 = 0 \quad\Rightarrow\quad \nabla_i\nabla_l A = 0,\tag{7.12}$$

and the antisymmetric part of (7.11) implies

$$-\frac{c}{12}A^{-4}\nabla_k A\langle\psi_+, \Gamma_l{}^k{}_i\psi_+\rangle = 0.\tag{7.13}$$

The 3-form spinor bilinear in (7.13) is proportional to the G_2-invariant 3-form φ given by

$$\varphi = e^\# \wedge \omega - 2i\,\sqrt{2}\,\mathfrak{Im}(e^{i\theta}\chi),\tag{7.14}$$

$$\varphi_{\#\alpha\bar{\beta}} \equiv -i\delta_{\alpha\bar{\beta}} \qquad \varphi_{\alpha\beta\gamma} = -i\sqrt{2}\,e^{i\theta}\varepsilon_{\alpha\beta\gamma} = (\varphi_{\bar{\alpha}\bar{\beta}\bar{\gamma}})^*.\tag{7.15}$$

Hence (7.13) implies that

$$\varphi_{lik}\nabla^k A = 0\tag{7.16}$$

which in turn implies that $dA = 0$, so A is constant. However, from the Einstein field equation (2.15) we obtain

$$3KA^{-1} - \nabla_i\nabla^i A - 3A^{-1}\nabla_i A\nabla^i A + \frac{1}{3}c^2 A^{-7} + \frac{A}{144}X^2 = 0.\tag{7.17}$$

It is clear that this equation admits no solution in the case for which A is constant and $X = 0$, as the l.h.s. is strictly positive. Therefore, we conclude that there are no supersymmetric warped product dS_4 solutions for which the spinor ψ_+ is G_2 invariant.

8 Conclusion

We have obtained the necessary and sufficient conditions for warped product $dS_4 \times_w M_7$ solutions in $D = 11$ supergravity to preserve the minimal $N = 8$ supersymmetry. To do this, we first integrated explicitly the gravitino equation along the dS_4 directions. This reduces the conditions imposed by supersymmetry to a gravitino-type equation on M_7 acting on a Majorana spinor ψ_+, whose components depend only on the co-ordinates of M_7. Using spinorial geometry techniques, the spinor ψ_+ was then simplified to two possible canonical forms by $\mathrm{Spin}(7)$ gauge transformations. These two canonical forms have stabilizer subgroups corresponding to G_2 and $SU(3)$. In the G_2 case, we show that there is no solution to the Killing spinor equations. For the $SU(3)$ case we have determined the 4-form flux in terms of $SU(3)$ invariant geometric structures on M_7, as well as determining all of the conditions imposed on the geometry of M_7.

Having obtained these conditions for the $N = 8$ solutions, it would be interesting to further investigate the resulting (local) conditions on the geometry. It would also be interesting to consider the $N = 16$ case, as well as the $N = 24$ and $N = 32$ cases. In particular, for the latter two cases of $N = 24$ and $N = 32$ supersymmetry, it is possible to find further conditions on such solutions utilizing the homogeneity theorem analysis constructed in [25]. To proceed with this, suppose that there we have N linearly independent solutions $\{\psi^r : r = 1, \ldots, N\}$ for $N = 24$ or $N = 32$ to the gravitino equation (4.16). We then consider the integrability condition (4.15), which implies

$$\left(|\nabla A|^2 - K - \frac{c^2}{9} A^{-6} - \frac{A^2}{(144)^2} \slashed{X}^2 + \frac{2}{3} c A^{-3} \nabla_i A \Gamma^i \tilde{\Gamma}^4 - \frac{1}{18} A \nabla_i A \slashed{X}^i \right) \psi^s = 0 \,. \quad (8.1)$$

This implies that

$$\langle \psi^r, \Gamma_0 \tilde{\Gamma}^4 \left(|\nabla A|^2 - K - \frac{c^2}{9} A^{-6} - \frac{A^2}{(144)^2} \slashed{X}^2 \right.$$
$$\left. + \frac{2}{3} c A^{-3} \nabla_i A \Gamma^i \tilde{\Gamma}^4 - \frac{1}{18} A \nabla_i A \slashed{X}^i \right) \psi^s \rangle = 0 \quad (8.2)$$

and hence

$$c \langle \psi^r, \Gamma_0 \Gamma^i \psi^s \rangle \nabla_i A = 0 \,. \quad (8.3)$$

On defining vector fields $\Theta_i^{rs} = \langle \psi^r, \Gamma_0 \Gamma_i \psi^s \rangle$, this implies

$$c \mathcal{L}_{\Theta^{rs}} A = 0 \,. \quad (8.4)$$

For $N = 24$ and $N = 32$ solutions, it follows from the homogeneity theorem analysis of [25] that the Θ^{rs} span pointwise the tangent space of M_7, and hence

$$c \, dA = 0 \,. \quad (8.5)$$

If $c \neq 0$, then this implies that $dA = 0$. However, (7.17) implies that there are no solutions for which A is constant. Hence, for $N = 24$ or $N = 32$ solutions, we must take $c = 0$.

This determines all possible $N = 32$ warped product dS_4 solutions. From [26], where all maximally supersymmetric solutions in $D = 11$ supergravity were determined, the maximally supersymmetric solutions are $\mathbb{R}^{1,10}$ with $F = 0$; $\mathrm{AdS}_4 \times \mathrm{S}^7$ with 4-form F proportional to the volume form of AdS_4, $\mathrm{AdS}_7 \times \mathrm{S}^4$, with 4-form F proportional to the volume form of S^4, and a maximally supersymmetric plane wave solution which has $F \neq 0$, but $F^2 = 0$. In terms of possible $N = 32$ warped product dS_4 solutions, the condition $c = 0$ implies that $F^2 \geq 0$ with equality if and only if $F = 0$. Hence we exclude $\mathrm{AdS}_4 \times \mathrm{S}^7$ and the maximally supersymmetric plane wave as $N = 32$ warped product dS_4 solutions. It follows that the $N = 32$ warped product dS_4 solutions are $\mathbb{R}^{1,10}$ and $\mathrm{AdS}_7 \times \mathrm{S}^4$. In particular, it is possible to write both $\mathbb{R}^{1,4}$ and AdS_7 as warped product dS_4 geometries [17]. It would be interesting to further understand the possible $N = 16$ and $N = 24$ warped product dS_4 solutions, though the homogeneity theorem does not apply to the $N = 16$ solutions.

Acknowledgments

A Conventions

We use the mostly plus sign signature $\eta = \mathrm{diag}(-, +, \ldots, +)$. The gamma matrices satisfy

$$\{\Gamma_A, \Gamma_B\} = 2g_{AB}. \tag{A.1}$$

In these conventions, we take

$$\Gamma_{0123456789\#} = \mathbb{I}, \tag{A.2}$$

and consequently the following duality relation holds

$$\Gamma_{A_1 \ldots A_p} = (-1)^{\frac{(p+1)(p-2)}{2}} \frac{1}{(11-p)!} \varepsilon_{A_1 \ldots A_p}{}^{A_{p+1} \ldots A_{11}} \Gamma_{A_{p+1} \ldots A_{11}}, \tag{A.3}$$

where

$$\varepsilon_{0123456789\#} = +1. \tag{A.4}$$

The Hodge star of a p-form ω is defined by

$$*\omega_{A_1 \ldots A_{11-p}} = \frac{1}{p!} \varepsilon_{A_1 \ldots A_{11-p}}{}^{B_1 \ldots B_p} \omega_{B_1 \ldots B_p}. \tag{A.5}$$

For every k-form ω, one can define a Clifford algebra element $\psi\!\!\!/$ given by

$$\psi\!\!\!/ \equiv \omega_{A_1 \ldots A_k} \Gamma^{A_1 \ldots A_k}. \tag{A.6}$$

In addition, one can define

$$\psi\!\!\!/_C \equiv \omega_{C A_1 \ldots A_k} \Gamma^{A_1 \ldots A_k}, \quad \text{and} \quad \Gamma\!\!\!/\omega_C \equiv \Gamma_{C A_1 \ldots A_k} \omega^{A_1 \ldots A_k}. \tag{A.7}$$

B   Spinors from forms

The Majorana representation of $Spin(10,1)$ can be constructed from the $Spin(9,1)$ spinor representations and then adding the tenth gamma matrix $\Gamma_\#$. This construction is derived in an explicit representation, in terms of differential forms, in [27, 28]. We take the space U of 1-forms on $\mathbb{R}^5$, with basis $\{e_1, \ldots, e_5\}$. The space of Dirac spinors, $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$, is identified with the complexified space of multi-forms constructed from this basis. Δ_c is equipped with a canonical Euclidean Hermitian inner product $\langle \, \cdot \, , \, \cdot \, \rangle$

We then take the following representation for the gammma matrices:

$$\Gamma_0\eta = -e_5 \wedge \eta + i_{e_5}\eta \qquad \Gamma_5\eta = e_5 \wedge \eta + i_{e_5}\eta$$
$$\Gamma_i\eta = e_i \wedge \eta + i_{e_i}\eta \qquad\qquad i = 1, \ldots, 4$$
$$\Gamma_{i+5}\eta = i(e_i \wedge \eta - i_{e_i}\eta) \tag{B.1}$$

where $\eta \in \Delta_c$ and i_{e_i} is the inner derivative along the direction e_i. The tenth gamma matrix can be chosen as

$$\Gamma_\# = -\Gamma_{0123456789}. \tag{B.2}$$

One can verify that $\Gamma_\#^2 = \mathbb{I}$. The gamma matrices satisfy the Clifford Algebra, namely $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}\mathbb{I}$. The Hermitian inner product, acting only on 1-forms, is defined by

$$\langle z^a e_a, w^b e_b \rangle = (z^a)^* \eta_{ab} w^b, \tag{B.3}$$

and is then extended to the complexified space of multi-forms, Δ_c.

The gamma matrices are chosen such that Γ_0 is skew-hermitian and Γ_i, $i = 1, \ldots, 9$ are hermitian with respect to $\langle \, \cdot \, , \, \cdot \, \rangle$. The $\text{Spin}(10,1)$ invariant Dirac inner product is defined as

$$D(\eta, \theta) = \langle \Gamma_0\eta, \theta \rangle. \tag{B.4}$$

In eleven dimensions a spinor can be Majorana; the reality condition is

$$\eta^* = \Gamma_{6789}\eta, \tag{B.5}$$

where $C = \Gamma_{6789}$ is the charge conjugation matrix, and $C*$ commutes with the gamma matrices, i.e. $C*\Gamma_A = \Gamma_A C*$. The Dirac representation of $Spin(10,1)$ admits an oscillator basis as

$$\Gamma_- = \frac{1}{\sqrt{2}}(\Gamma_5 - \Gamma_0) = \sqrt{2}\, e_5 \wedge \quad \Gamma_\alpha = \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+5}) = \sqrt{2}\, e_\alpha \wedge$$
$$\Gamma_+ = \frac{1}{\sqrt{2}}(\Gamma_5 + \Gamma_0) = \sqrt{2}\, i_{e_5} \quad \Gamma_{\bar\alpha} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5}) = \sqrt{2}\, i_{e_\alpha} \tag{B.6}$$

and $\Gamma_\sharp$ defined as in (B.2). In this oscillator basis, the gamma matrices satisfy the Clifford Algebra, $\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}\mathbb{I}$, with non-vanishing components are $\eta_{+-} = \eta_{\sharp\sharp} = 1$, $\eta_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$.

We note that $(\Gamma_+)^\dagger = \Gamma_-$ and $(\Gamma_\alpha)^\dagger = \Gamma_{\bar\alpha}$; $(\Gamma^+, \Gamma^{\bar\alpha})$ act as creation operators on the Clifford vacuum represented by the 0-degree form 1, where $\Gamma^A = \eta^{AB}\Gamma_B$. A general spinor ϵ can be written as

$$\epsilon = \sum_{k=0}^{5} \frac{1}{k!} \phi_{\bar a_1 \ldots \bar a_k} \Gamma^{\bar a_1 \ldots \bar a_k} 1, \qquad \bar a = +, \bar\alpha. \tag{B.7}$$

C Derivation of equation (4.2)

Given the spinor Φ defined in (4.1), we consider (4.2). In particular we begin by examining the following terms:

$$\nabla_i \Phi + k_1 [\text{Eq. (3.8)}]\,, \tag{C.1}$$

where k_1, k_2 are some constants to be determined. To begin with, note that the terms which are linear in X are:

$$\begin{aligned}
(1 + k_1) \frac{A}{288} \nabla_i \slashed{X} &+ \frac{1}{576} \nabla_i A \slashed{X} + \frac{(12a + 1 - 4k_1)}{12} \frac{1}{288} cA^{-3} \Gamma_i \slashed{X} \tilde{\Gamma}^4 \\
&+ \frac{(5k_1 - 1 - 18a)}{432} cA^{-3} \slashed{X}_i \tilde{\Gamma}^4 - \frac{1}{576} \nabla^k A \Gamma_{ki} \slashed{X} + \frac{1}{48} \nabla_k A \Gamma^{k j_1 j_2 j_3} X_{i j_1 j_2 j_3} \\
&+ \frac{(3 + 4k_1)}{48} \nabla^k A \Gamma^{ab} X_{ikab} + \frac{k_1}{72} \nabla_k A \Gamma_{i j_1 j_2 j_3} X^{k j_1 j_2 j_3}\,.
\end{aligned} \tag{C.2}$$

In order to set to zero the term involving $\nabla_i \slashed{X}$, we set $k_1 = -1$. Having done so, we then consider imposing the condition

$$\begin{aligned}
\nabla_i \Phi - [\text{Eq. (3.8)}] &+ k_2 A^{-1} \Gamma^i [\text{Eq. (3.6)}] + q_1 \slashed{\Gamma} \slashed{X}_i \Phi + q_2 \slashed{X}_i \Phi \\
&+ q_3 cA^{-4} \Gamma_i \tilde{\Gamma}^4 \Phi + q_4 A^{-1} \nabla_k A \Gamma_i \Gamma^k \Phi + q_5 A^{-1} \nabla_i A \Phi = 0\,,
\end{aligned} \tag{C.3}$$

and compute all of the terms on the l.h.s., choosing the constants $a, k_2, q_1, q_2, q_3, q_4, q_5$ so that the identity above holds. The terms involving the quadratic contribution of X are

$$\begin{aligned}
A\Bigg[&\frac{(4k_2 + 1 - 288q_1)}{1152} \Gamma^{l_1 l_2}{}_{i j_1 j_2} X_{j_3 j_4 l_1 l_2} X^{j_1 j_2 j_3 j_4} \\
&+ \frac{(288q_1 - 72q_2 - 3)}{1728} \Gamma_{j_1 j_2 j_3}{}^{l_1 l_2} X_{i j_4 l_1 l_2} X^{j_1 j_2 j_3 j_4} \\
&+ \frac{(8k_2 - 288q_1 - 72q_2 - 1)}{576} \Gamma^{abm} X_{impq} X^{pq}{}_{ab} - \frac{(1 + 4k_2 - 288q_1)}{12} \frac{1}{288} \Gamma_i X^2 \\
&+ \frac{(3 - 288q_1 + 72q_2)}{864} \Gamma_{j_1} X_{i j_2 j_3 j_4} X^{j_1 j_2 j_3 j_4} \Bigg]\,.
\end{aligned} \tag{C.4}$$

The terms involving the linear contribution of X are

$$\begin{aligned}
\frac{(1 - 288q_1 + 2q_4 + 2q_5)}{576} \nabla_i A \slashed{X} &+ \frac{(12a + 5 + 12q_3 + 3456aq_1)}{12} \frac{1}{288} cA^{-3} \Gamma_i \slashed{X} \tilde{\Gamma}^4 \\
&+ \frac{(288q_1 - 2q_4 - 1)}{576} \nabla^k A \Gamma_{ki} \slashed{X} + \frac{(1 + 24q_2 - 96q_1)}{48} \nabla_k A \Gamma^{k j_1 j_2 j_3} X_{i j_1 j_2 j_3} \\
&+ \frac{(8k_2 - 1 - 288q_1 - 72q_2)}{48} \nabla^k A \Gamma^{ab} X_{ikab} - \frac{(1 + 4k_2 - 288q_1)}{72} \nabla_k A \Gamma_{i j_1 j_2 j_3} X^{k j_1 j_2 j_3} \\
&- \frac{(6 + 18a + 1728aq_1 - 432q_2 a)}{432} cA^{-3} \slashed{X}_i \tilde{\Gamma}^4\,.
\end{aligned} \tag{C.5}$$

The terms involving no contribution of X are

$$\frac{1}{24}(1 - 16k_2 - 24aq_4 + 12q_3)cA^{-4}\nabla^k A\Gamma_k\Gamma_i\tilde{\Gamma}^4$$

$$+ \frac{1}{12}(16k_2 - 7 - 36a - 12q_3 + 24aq_4 + 12aq_5)cA^{-4}\nabla_i A\tilde{\Gamma}^4 +$$

$$+ \left(k_2 - \frac{1}{2}q_4\right)A^{-1}|\nabla A|^2\Gamma_i - k_2 A^{-1}K\Gamma_i - \frac{1}{2}q_5 A^{-1}\nabla_k A\nabla_i A\Gamma^k$$

$$- \frac{1}{36}(3a + 4k_2 + 36q_3 a)c^2 A^{-7}\Gamma_i\,. \tag{C.6}$$

By requiring that all terms in the above expressions should vanish, we are able to determine the constant values, that are

$$a = -\frac{1}{6} \quad q_1 = \frac{1}{288} \quad q_2 = -\frac{1}{36} \quad q_3 = -\frac{1}{12} \quad k_2 = q_4 = q_5 = 0\,. \tag{C.7}$$

D KSE linear system — SU(3) stabilizer

The linear system associated to the KSEs (4.16), with the spinor given by

$$\psi_+ = w(1 + e_{1234}) + \lambda e_1 + \bar{\lambda}e_{234} \qquad w \in \mathbb{R},\ \lambda \in \mathbb{C} \tag{D.1}$$

is as follows:

$$\partial_\# w + \frac{w}{2}\Omega_{\#,\alpha}{}^\alpha + \frac{1}{24}wX_\alpha{}^\alpha{}_\beta{}^\beta - i\frac{c}{12}A^{-4}w - \frac{\sqrt{2}}{3}\bar{\lambda}X_{\#234} = 0 \tag{D.2}$$

$$\partial_\# \lambda + \frac{\lambda}{2}\Omega_{\#,\alpha}{}^\alpha - \frac{1}{24}\lambda X_\alpha{}^\alpha{}_\beta{}^\beta + \frac{\sqrt{2}}{3}wX_{\#234} - i\frac{c}{12}A^{-4}\lambda = 0 \tag{D.3}$$

$$w\Omega_{\#,\alpha\beta}\varepsilon^{\gamma\alpha\beta} - \sqrt{2}\lambda\Omega_{\#,\#}{}^\gamma + \frac{\sqrt{2}}{3}\lambda X_{\#\alpha}{}^{\alpha\gamma} + \frac{w}{3}X^\gamma{}_{234} = 0 \tag{D.4}$$

$$\bar{\lambda}\varepsilon^{\gamma\alpha\beta}\Omega_{\#\alpha\beta} - \sqrt{2}w\Omega_{\#,\#}{}^\gamma - \frac{w\sqrt{2}}{3}X_{\#\alpha}{}^{\alpha\gamma} - \frac{\bar{\lambda}}{3}X^\gamma{}_{234} = 0 \tag{D.5}$$

$$\partial_\mu w + \frac{1}{2}w\Omega_{\mu,\alpha}{}^\alpha - \frac{1}{4}wX_{\mu\#\alpha}{}^\alpha = 0 \tag{D.6}$$

$$\partial_\mu \lambda + \frac{1}{2}\lambda\Omega_{\mu,\alpha}{}^\alpha + \frac{1}{4}\lambda X_{\mu\#\alpha}{}^\alpha = 0 \tag{D.7}$$

$$\partial_\mu w - \frac{w}{2}\Omega_{\mu,\alpha}{}^\alpha - \frac{w}{12}X_{\#\mu\alpha}{}^\alpha - \frac{\sqrt{2}}{3}\lambda X_{\mu\bar{2}\bar{3}\bar{4}} = 0 \tag{D.8}$$

$$\partial_\mu \bar{\lambda} - \frac{\bar{\lambda}}{2}\Omega_{\mu,\alpha}{}^\alpha + \frac{1}{12}\bar{\lambda}X_{\#\mu\alpha}{}^\alpha + \frac{\sqrt{2}}{3}wX_{\mu\bar{2}\bar{3}\bar{4}} = 0 \tag{D.9}$$

$$w\Omega_{\bar{\mu},\alpha\beta}\varepsilon^{\alpha\beta\gamma} - \sqrt{2}\lambda\Omega_{\bar{\mu},\#}{}^\gamma - \frac{w}{2}\varepsilon^{\gamma\alpha\beta}X_{\bar{\mu}\#\alpha\beta} - \frac{w}{3}\varepsilon_{\bar{\mu}}{}^{\gamma\rho}X_{\#\rho\alpha}{}^\alpha + \frac{\sqrt{2}}{6}\lambda\varepsilon_{\bar{\mu}}{}^{\gamma\rho}X_{\rho\bar{2}\bar{3}\bar{4}} = 0 \tag{D.10}$$

$$w\Omega_{\bar{\mu},}{}^{\gamma\rho} - \frac{\sqrt{2}}{2}\bar{\lambda}\varepsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha} + \varepsilon_{\bar{\mu}}{}^{\gamma\rho}\left[\frac{w}{6}X_{\#\bar{2}\bar{3}\bar{4}} - \frac{\bar{\lambda}\sqrt{2}}{24}X_\alpha{}^\alpha{}_\beta{}^\beta\right.$$

$$\left. + i\frac{c\sqrt{2}}{12}\bar{\lambda}A^{-4}\right] - \frac{\sqrt{2}}{4}\bar{\lambda}\varepsilon^{\gamma\rho\alpha}X_{\bar{\mu}\alpha\beta}{}^\beta = 0 \tag{D.11}$$

$$\bar{\lambda}\Omega_{\bar{\mu},\alpha\beta}\varepsilon^{\alpha\beta\gamma} - \sqrt{2}w\Omega_{\bar{\mu},\#}{}^{\gamma} + \frac{\bar{\lambda}}{2}\varepsilon^{\gamma\alpha\beta}X_{\bar{\mu}\#\alpha\beta} + \frac{\bar{\lambda}}{3}\varepsilon_{\bar{\mu}}{}^{\gamma\rho}X_{\#\rho\alpha}{}^{\alpha} - \frac{\sqrt{2}}{6}w\varepsilon_{\bar{\mu}}{}^{\gamma\rho}X_{\rho\bar{2}\bar{3}\bar{4}} = 0 \quad \text{(D.12)}$$

$$\lambda\Omega_{\bar{\mu},}{}^{\gamma\rho} - \frac{\sqrt{2}}{2}w\varepsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha} + \varepsilon_{\bar{\mu}}{}^{\gamma\rho}\left[\frac{\sqrt{2}w}{24}X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}\right.$$
$$\left. +i\frac{c\sqrt{2}}{12}wA^{-4} - \frac{\lambda}{6}X_{\#\bar{2}\bar{3}\bar{4}}\right] + \frac{\sqrt{2}}{4}w\varepsilon^{\gamma\rho\alpha}X_{\bar{\mu}\alpha\beta}{}^{\beta} = 0. \quad \text{(D.13)}$$

D.1 Solution for $\lambda \neq 0$, $w \neq 0$

From the linear system (D.2)–(D.13), we find that the components of the flux are given by the following expressions

$$X_{\#\alpha}{}^{\alpha\gamma} = \frac{3}{w^2 - |\lambda|^2}\left[\sqrt{2}w\bar{\lambda}\varepsilon^{\gamma\alpha\beta}\Omega_{\#,\alpha\beta} - (w^2 + |\lambda|^2)\Omega_{\#,\#}{}^{\gamma}\right] \quad \text{(D.14)}$$

$$X_{\bar{\gamma}234} = \frac{3}{w^2 - |\lambda|^2}\left[2\sqrt{2}\lambda w\Omega_{\#,\#}{}^{\gamma} - (w^2 + |\lambda|^2)\Omega_{\#,\alpha\beta}\varepsilon^{\gamma\alpha\beta}\right] \quad \text{(D.15)}$$

$$X_{\#234} = \frac{1}{w^2 - |\lambda|^2}\left[-\frac{3}{\sqrt{2}}\lambda\partial_{\#}w - \frac{3}{\sqrt{2}}w\partial_{\#}\lambda - \frac{3}{\sqrt{2}}w\lambda\Omega_{\#,\alpha}{}^{\alpha} + i\frac{c}{2\sqrt{2}}A^{-4}\lambda w\right] \quad \text{(D.16)}$$

$$X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = \frac{1}{w^2 - |\lambda|^2}\left[-24w\partial_{\#}w - 24\bar{\lambda}\partial_{\#}\lambda\right.$$
$$\left. - 12(w^2 + |\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} + 2icA^{-4}(w^2 + |\lambda|^2)\right] \quad \text{(D.17)}$$

$$X_{\bar{\mu}\#\alpha\beta} = 2\Omega_{\bar{\mu},\alpha\beta} - \sqrt{2}\frac{\lambda}{w}\varepsilon_{\gamma\alpha\beta}\Omega_{\bar{\mu},\#}{}^{\gamma} + \delta_{\bar{\mu}[\alpha}\left(2\Omega_{\#,\#\beta]} - \sqrt{2}\frac{\lambda}{w}\varepsilon_{\beta]\delta\sigma}\Omega_{\#}{}^{\delta\sigma}\right) \quad \text{(D.18)}$$

$$X_{\bar{\mu}\alpha\beta}{}^{\beta} = \sqrt{2}\frac{w}{\bar{\lambda}}\varepsilon_{\alpha\gamma\rho}\Omega_{\bar{\mu},}{}^{\gamma\rho} - 2\Omega_{\bar{\mu},\#\alpha} + \frac{\delta_{\bar{\mu}\alpha}}{w^2 - |\lambda|^2}\left(3w\partial_{\#}w - \frac{w^2}{\bar{\lambda}}\partial_{\#}\bar{\lambda}\right.$$
$$\left. + 4\bar{\lambda}\partial_{\#}\lambda + \Omega_{\#,\alpha}{}^{\alpha}(3w^2 + 2|\lambda|^2) - i\frac{c}{6}A^{-4}(4|\lambda|^2 + w^2)\right). \quad \text{(D.19)}$$

From the linear system (D.2)–(D.13), we also find the following geometric conditions:

$$4(w^2 - |\lambda|^2)\partial_{\mu}w + 2w(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha}$$
$$- 3\sqrt{2}w^2\lambda\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} + 3w(w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \quad \text{(D.20)}$$

$$4(w^2 - |\lambda|^2)\partial_{\mu}\lambda + 2\lambda(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha}$$
$$+ 3\sqrt{2}w\lambda^2\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} - 3\lambda(w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \quad \text{(D.21)}$$

$$4(w^2 - |\lambda|^2)\partial_{\mu}w - 2w(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha}$$
$$+ \sqrt{2}\lambda(5w^2 + 4|\lambda|^2)\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} - w(w^2 + 17|\lambda|^2)\Omega_{\#,\#\mu} = 0 \quad \text{(D.22)}$$

$$4(w^2 - |\lambda|^2)\partial_{\mu}\bar{\lambda} - 2\bar{\lambda}(w^2 - |\lambda|^2)\Omega_{\mu,\alpha}{}^{\alpha}$$
$$- \sqrt{2}w(4w^2 + 5|\lambda|^2)\varepsilon_{\mu\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} + \bar{\lambda}(17w^2 + |\lambda|^2)\Omega_{\#,\#\mu} = 0 \quad \text{(D.23)}$$

$$2\bar{\lambda}w\varepsilon^{\gamma\alpha\beta}\Omega_{\bar{\mu},\alpha\beta} - \sqrt{2}(w^2 + |\lambda|^2)\Omega_{\bar{\mu},\#}{}^{\gamma}$$
$$- 2\bar{\lambda}w\varepsilon^{\gamma\beta}{}_{\bar{\mu}}\Omega_{\#,\#\beta} + \sqrt{2}(w^2 + |\lambda|^2)\Omega_{\#,\bar{\mu}}{}^{\gamma} = 0 \quad \text{(D.24)}$$

$$8(|\lambda|^2 + w^2)\Omega_{\bar{\mu}}{}^{\gamma\rho} - 8\sqrt{2}w\bar{\lambda}\varepsilon^{\gamma\rho\alpha}\Omega_{\bar{\mu},\#\alpha}$$
$$+ \sqrt{2}\varepsilon_{\bar{\mu}}{}^{\gamma\rho}\left[2w\bar{\lambda}\Omega_{\#,\alpha}{}^{\alpha} - 2w\partial_{\#}\bar{\lambda} - 2\bar{\lambda}\partial_{\#}w + icA^{-4}\bar{\lambda}w\right] = 0 \quad \text{(D.25)}$$

$$6(\bar{\lambda}\partial_{\#}\lambda - \lambda\partial_{\#}\bar{\lambda}) + 6(w^2 + |\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} - icA^{-4}(w^2 + |\lambda|^2) = 0 \tag{D.26}$$

$$2w(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\alpha\gamma} + \sqrt{2}\lambda(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\#}{}^{\beta}\varepsilon_{\alpha\beta\gamma}$$
$$- w\Omega_{\#,\#\gamma}(w^2 + 5|\lambda|^2) + \sqrt{2}\lambda(2w^2 + |\lambda|^2)\varepsilon_{\gamma\alpha\beta}\Omega_{\#,}{}^{\alpha\beta} = 0 \tag{D.27}$$

$$2\sqrt{2}w(w^2 - |\lambda|^2)\Omega^{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} - 4\bar{\lambda}(w^2 - |\lambda|^2)\Omega^{\alpha}{}_{,\#\alpha} - 30w\bar{\lambda}\partial_{\#}w$$
$$- 6w^2\partial_{\#}\bar{\lambda} - 24\bar{\lambda}^2\partial_{\#}\lambda - 6\bar{\lambda}(w^2 + 2|\lambda|^2)\Omega_{\#,\alpha}{}^{\alpha} + 3icA^{-4}\bar{\lambda}w^2 = 0 \tag{D.28}$$

$$(w^2 - |\lambda|^2)^2 (\Omega_{\mu,\#\bar{\alpha}} - \Omega_{\bar{\alpha},\#\mu}) + \delta_{\mu\bar{\alpha}}\Big[(3w^2 + 2|\lambda|^2)(\lambda\partial_{\#}\bar{\lambda} - \bar{\lambda}\partial_{\#}\lambda)$$

$$- 2(w^4 + 3w^2|\lambda|^2 + |\lambda|^4)\Omega_{\#,\alpha}{}^{\alpha} + i\frac{c}{3}A^{-4}(2w^4 + w^2|\lambda|^2 + 2|\lambda|^4)\Big] = 0. \tag{D.29}$$

D.2 Solution for $\lambda = 0$, $w \neq 0$

We next present the components of the flux and the geometric conditions associated to the KSEs (4.16), with the spinor given by

$$\psi_+ = w(1 + e_{1234}) \qquad w \in \mathbb{R}. \tag{D.30}$$

We find that the components of the flux are given by the following expressions

$$X_{\#234} = X_{\bar{\mu}234} = 0 \tag{D.31}$$

$$X_{\#\alpha}{}^{\alpha\gamma} = -3\Omega_{\#,\#}{}^{\gamma} \tag{D.32}$$

$$X_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = -24w^{-1}\partial_{\#}w \tag{D.33}$$

$$X_{\bar{\mu}\#\alpha\beta} = 2\Omega_{\bar{\mu},\alpha\beta} + 2\delta_{\bar{\mu}[\alpha}\Omega_{\#,\#\beta]} \tag{D.34}$$

$$X_{\bar{\mu}\nu\beta}{}^{\beta} = (\Omega_{\bar{\mu},\#\nu} + \Omega_{\nu,\#\bar{\mu}}) + 4\delta_{\bar{\mu}\nu}w^{-1}\partial_{\#}w. \tag{D.35}$$

Furthermore, we find that the geometric conditions are given by the following expressions

$$\Omega_{\#,\alpha\beta}\varepsilon^{\gamma\alpha\beta} = 0 \tag{D.36}$$

$$\Omega_{\#,\#\mu} = -4w^{-1}\partial_{\mu}w \tag{D.37}$$

$$\Omega_{\mu,\alpha}{}^{\alpha} = 4w^{-1}\partial_{\mu}w \tag{D.38}$$

$$\Omega_{\mu,\alpha\beta} = 0 \tag{D.39}$$

$$\Omega_{\mu,\#\beta} = 0 \tag{D.40}$$

$$\Omega^{\alpha}{}_{\alpha\beta} = -2\frac{\partial_{\beta}w}{w} \tag{D.41}$$

$$\Omega^{\alpha}{}_{\#\alpha} - 6\frac{\partial_{\#}w}{w} - \frac{i}{2}cA^{-4} = 0 \tag{D.42}$$

$$(\Omega_{\bar{\alpha},\#\beta} - \Omega_{\beta,\#\bar{\alpha}}) - i\frac{c}{3}A^{-4}\delta_{\bar{\alpha}\beta} = 0 \tag{D.43}$$

$$\Omega_{\#,\alpha}{}^{\alpha} = i\frac{c}{6}A^{-4}. \tag{D.44}$$

E Covariant relations

In this appendix, we present the main relations used to covariantize the linear system. These expressions relate spin connection terms to SU(3)-covariant terms involving the SU(3) invariant 1-forms ξ, ω and χ, their Lie derivatives with respect to ξ, and also the Lee forms W and Z:

$$\nabla_{\#}\xi_\mu = (\mathcal{L}_\xi\xi)_\mu = -\Omega_{\#,\#\mu} \tag{E.1}$$

$$\nabla_{\bar{\alpha}}\xi_\beta = -\Omega_{\bar{\alpha},\#\beta} \tag{E.2}$$

$$\nabla_{\#}\omega_{\alpha\beta} = 2i\Omega_{\#,\alpha\beta} \tag{E.3}$$

$$W_\alpha = -\Omega_{\#,\#\alpha} - 2\Omega^\beta{}_{\beta\alpha} \tag{E.4}$$

$$W_{\bar{\alpha}} = -\Omega_{\#,\#\bar{\alpha}} - 2\Omega^{\bar{\beta}}{}_{\bar{\beta}\bar{\alpha}} \tag{E.5}$$

$$Z_{\bar{\rho}} = 2\Omega_{\#,\#\bar{\rho}} + 2\Omega_{\bar{\rho},\gamma}{}^\gamma + 2\Omega^{\bar{\gamma}}{}_{\bar{\gamma}\bar{\rho}} \tag{E.6}$$

$$Z_\rho = 2\Omega_{\#,\#\rho} + 2\Omega_{\rho,\bar{\gamma}}{}^{\bar{\gamma}} + 2\Omega^\gamma{}_{\gamma\rho} \tag{E.7}$$

$$(\mathcal{L}_\xi\omega)_{\alpha\beta} = 2i\Omega_{\#,\alpha\beta} + i(\mathrm{d}\xi)_{\alpha\beta} \tag{E.8}$$

$$(\mathcal{L}_\xi\omega)_{\bar{\alpha}\bar{\beta}} = -2i\Omega_{\#,\bar{\alpha}\bar{\beta}} - i(\mathrm{d}\xi)_{\bar{\alpha}\bar{\beta}} \tag{E.9}$$

$$(\mathcal{L}_\xi\omega)_{\alpha\bar{\beta}} = i(\Omega_{\bar{\beta},\#\alpha} + \Omega_{\alpha,\#\bar{\beta}}) \tag{E.10}$$

$$(\mathcal{L}_\xi\chi)_{\alpha\beta\gamma} = 3\Omega_{\#,[\gamma}{}^\lambda\varepsilon_{\alpha\beta]\lambda} - 3\Omega_{[\gamma,\#}{}^\lambda\varepsilon_{\alpha\beta]\lambda} = (\Omega_{\#,\lambda}{}^\lambda - \Omega_{\lambda,\#}{}^\lambda)\varepsilon_{\alpha\beta\gamma} \tag{E.11}$$

$$(\mathcal{L}_\xi\chi)_{\alpha\beta\bar{\gamma}} = (\Omega_{\#,\bar{\gamma}}{}^\lambda - \Omega_{\bar{\gamma},\#}{}^\lambda)\varepsilon_{\lambda\alpha\beta} \tag{E.12}$$

$$\mathcal{L}_\xi\bar{\chi} = (\mathcal{L}_\xi\chi)^* . \tag{E.13}$$

The spin-connection components are rewritten in terms of those covariant quantities as

$$\Omega_{\mu,\alpha}{}^\alpha = -\frac{1}{2}\left((\mathcal{L}_\xi\xi)_\mu + W_\mu + Z_\mu\right) \tag{E.14}$$

$$\Omega_{\#,\#\mu} = -\nabla_{\#}\xi_\mu = -i\nabla_{\#}\omega_{\#\mu} = -(\mathcal{L}_\xi\xi)_\mu \tag{E.15}$$

$$\Omega_{\#,\alpha\beta} = -\frac{i}{2}\nabla_{\#}\omega_{\alpha\beta} = -\frac{1}{2}\left(i(\mathcal{L}_\xi\omega)_{\alpha\beta} + (\mathrm{d}\xi)_{\alpha\beta}\right) \tag{E.16}$$

$$\Omega_{\bar{\alpha},\beta\gamma} = -\frac{i}{2}(\mathrm{d}\omega)_{\bar{\alpha}\beta\gamma} \tag{E.17}$$

$$\Omega_{\bar{\alpha},\#\bar{\beta}} = \frac{1}{2}\left(i(\mathcal{L}_\xi\omega)_{\bar{\alpha}\bar{\beta}} - (\mathrm{d}\xi)_{\bar{\alpha}\bar{\beta}} - \varepsilon_{\bar{\beta}}{}^{\gamma\rho}(\mathcal{L}_\xi\chi)_{\gamma\rho\bar{\alpha}}\right) \tag{E.18}$$

$$\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \frac{i}{2}\nabla_{\bar{\alpha}}\omega_{\bar{\beta}\bar{\gamma}} \tag{E.19}$$

$$\varepsilon_{\alpha\beta\gamma}\nabla^\alpha\omega^{\beta\gamma} = \frac{1}{3}\varepsilon_{\alpha\beta\gamma}(\mathrm{d}\omega)^{\alpha\beta\gamma} \tag{E.20}$$

$$\Omega_{\bar{\alpha},\#\beta} = -\nabla_{\bar{\alpha}}\xi_\beta = -i\nabla_{\bar{\alpha}}\omega_{\#\beta} \tag{E.21}$$

$$\Omega_{\#,\rho}{}^\rho = -\frac{1}{6}\varepsilon_{\alpha\beta\gamma}(\mathcal{L}_\xi\bar{\chi})^{\alpha\beta\gamma} + \nabla^\lambda\xi_\lambda \tag{E.22}$$

$$\Omega^\beta{}_{\beta\alpha} = \frac{1}{2}\left((\mathcal{L}_\xi\xi)_\alpha - W_\alpha\right) \tag{E.23}$$

$$\Omega^{[\alpha}{}_{\#}{}^{\beta]} = -\frac{1}{2}(\mathrm{d}\xi)^{\alpha\beta} \tag{E.24}$$

$$\Omega^\alpha{}_{\#\alpha} = -\nabla^\alpha\xi_\alpha = -i\nabla^\alpha\omega_{\#\alpha} . \tag{E.25}$$

References

[1] SUPERNOVA SEARCH TEAM collaboration, *The High Z supernova search: Measuring cosmic deceleration and global curvature of the universe using type-IA supernovae*, *Astrophys. J.* **507** (1998) 46 [`astro-ph/9805200`] [INSPIRE].

[2] SUPERNOVA SEARCH TEAM collaboration, *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009 [`astro-ph/9805201`] [INSPIRE].

[3] SUPERNOVA COSMOLOGY PROJECT collaboration, *Measurements of Ω and Λ from 42 high redshift supernovae*, *Astrophys. J.* **517** (1999) 565 [`astro-ph/9812133`] [INSPIRE].

[4] S. Weinberg, *The Cosmological Constant Problem*, *Rev. Mod. Phys.* **61** (1989) 1 [INSPIRE].

[5] A.D. Linde, *Is the Lee constant a cosmological constant?*, *JETP Lett.* **19** (1974) 183 [INSPIRE].

[6] G.W. Gibbons, *Aspects of Supergravity Theories*, in *Supersymmetry, Supergravity and Related Topics*, F. del Aguila, J.A. de Azcárraga and L.E. Ibanez eds., pp. 346–351, World Scientific (1985).

[7] B. de Wit, D.J. Smit and N.D. Hari Dass, *Residual Supersymmetry of Compactified $D = 10$ Supergravity*, *Nucl. Phys. B* **283** (1987) 165 [INSPIRE].

[8] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys. A* **16** (2001) 822 [`hep-th/0007018`] [INSPIRE].

[9] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, (2001) [`hep-th/0106109`] [INSPIRE].

[10] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [`hep-th/9711200`] [INSPIRE].

[11] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333 [INSPIRE].

[12] S.W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199 [*Erratum ibid.* **46** (1976) 206] [INSPIRE].

[13] J.D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207 [INSPIRE].

[14] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [`astro-ph/0210603`] [INSPIRE].

[15] A. Strominger, *The dS/CFT correspondence*, *JHEP* **10** (2001) 034 [`hep-th/0106113`] [INSPIRE].

[16] A. Strominger, *Inflation and the dS/CFT correspondence*, *JHEP* **11** (2001) 049 [`hep-th/0110087`] [INSPIRE].

[17] U. Gran, J.B. Gutowski and G. Papadopoulos, *On supersymmetric Anti-de-Sitter, de-Sitter and Minkowski flux backgrounds*, *Class. Quant. Grav.* **35** (2018) 065016 [`arXiv:1607.00191`] [ɪɴSPIRE].

[18] D. Farotti and J.B. Gutowski, *Supersymmetric dS $_n$ solutions for $n \geq 5$ in $D = 11$ supergravity*, *J. Phys. A* **55** (2022) 375401 [`arXiv:2204.11903`] [ɪɴSPIRE].

[19] D. Farotti, *Heterotic de-Sitter Solutions*, `arXiv:2206.05190` [ɪɴSPIRE].

[20] S.W. Beck, J.B. Gutowski and G. Papadopoulos, *Geometry and supersymmetry of heterotic warped flux AdS backgrounds*, *JHEP* **07** (2015) 152 [`arXiv:1505.01693`] [ɪɴSPIRE].

[21] J. Gillard, U. Gran and G. Papadopoulos, *The Spinorial geometry of supersymmetric backgrounds*, *Class. Quant. Grav.* **22** (2005) 1033 [`hep-th/0410155`] [ɪɴSPIRE].

[22] U. Gran, J. Gutowski and G. Papadopoulos, *Classification, geometry and applications of supersymmetric backgrounds*, *Phys. Rept.* **794** (2019) 1 [`arXiv:1808.07879`] [ɪɴSPIRE].

[23] J. Gutowski and G. Papadopoulos, *Index theory and dynamical symmetry enhancement of M-horizons*, *JHEP* **05** (2013) 088 [`arXiv:1303.0869`] [ɪɴSPIRE].

[24] J.B. Gutowski and G. Papadopoulos, *Supersymmetry of AdS and flat backgrounds in M-theory*, *JHEP* **02** (2015) 145 [`arXiv:1407.5652`] [ɪɴSPIRE].

[25] J. Figueroa-O'Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds*, *JHEP* **10** (2012) 014 [`arXiv:1208.0553`] [ɪɴSPIRE].

[26] J.M. Figueroa-O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities*, *JHEP* **03** (2003) 048 [`hep-th/0211089`] [ɪɴSPIRE].

[27] U. Gran, G. Papadopoulos and D. Roest, *Systematics of M-theory spinorial geometry*, *Class. Quant. Grav.* **22** (2005) 2701 [`hep-th/0503046`] [ɪɴSPIRE].

[28] U. Gran, J. Gutowski and G. Papadopoulos, *The Spinorial geometry of supersymmetric IIB backgrounds*, *Class. Quant. Grav.* **22** (2005) 2453 [`hep-th/0501177`] [ɪɴSPIRE].