

Macaulay matrix for Feynman integrals: linear relations and intersection numbers

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ABSTRACT: We elaborate on the connection between Gel'fand-Kapranov-Zelevinsky systems, de Rham theory for twisted cohomology groups, and Pfaffian equations for Feynman Integrals. We propose a novel, more efficient algorithm to compute Macaulay matrices, which are used to derive Pfaffian systems of differential equations. The Pfaffian matrices are then employed to obtain linear relations for \mathcal{A} -hypergeometric (Euler) integrals and Feynman integrals, through recurrence relations and through projections by intersection numbers.

KEYWORDS: Differential and Algebraic Geometry, Scattering Amplitudes

ARXIV EPRINT: [2204.12983](https://arxiv.org/abs/2204.12983)

Contents

1	Introduction	1
1.1	Background	1
1.2	Macaulay matrix for GKZ system	3
2	GKZ hypergeometric systems	4
2.1	Integral representation of GKZ-hypergeometric system	5
2.2	GKZ \mathcal{D} -modules and de Rham cohomology	6
2.3	Representing integrals inside \mathcal{D}_N -modules	9
2.4	Homogeneity and integrand rescaling	10
3	Pfaffian systems	11
3.1	From \mathcal{D}_N -module to DEQ system	12
3.2	Basis change	13
3.3	Basis change without derivatives	14
4	Constructing Pfaffian systems from Macaulay matrices	15
4.1	From Pfaffian to Macaulay matrix	15
4.2	From Macaulay matrix to Pfaffian	17
4.3	Discussion	18
4.4	Examples	20
5	Macaulay matrix method and generalized Feynman integrals	24
5.1	Massless one-loop diagrams	25
5.2	Example: One-loop box	26
5.3	Example: One-loop pentagon	28
6	Linear relations for generalized Feynman integrals	31
6.1	Recurrence relations for GKZ systems	32
6.2	Recurrence relations for generalized Feynman integrals	33
6.3	Example: One-loop bubble	34
7	Decomposition via cohomology intersection numbers	37
7.1	Example: One-loop box	39
8	Conclusion	41
A	Further details on the homogeneity property	42
B	Holonomic \mathcal{D}-modules in a nutshell	44
B.1	Holonomic ideals	44
B.2	Standard monomials	45
B.3	Proof of theorem 3.1	47
B.4	Integration and restriction	48

1 Introduction

1.1 Background

Within the perturbative approach to quantum as well as classical field theory, the evaluation of multi-loop Feynman integrals is necessary for the determination of scattering amplitudes and related quantities (see [1] for a recent review). The by-now standard evaluation techniques of Feynman integrals (in momentum-space representation) exploit loop-momentum shift invariance to establish *integration-by-parts* (IBP) relations [2, 3] among integrals whose integrands are built from products of the same set of denominators (and scalar products), but raised to different powers. IBP identities play a crucial role in the calculation of multi-loop integrals because they can be used to identify a minimal set of elements, dubbed *master integrals* (MIs), that can be used as a basis for the decomposition of multi-loop amplitudes. At the same time, IBP relations can be exploited to build systems of equations solved by the MIs: differential equations [4–13], dimensional recurrence relations [14, 15], and finite difference equations [3, 16] are *linear relations* emerging from the IBP reduction of integrals which spring from the action of special (polynomial and differential) operators on the integrands of the MIs. Solving such equations amounts to the actual determination of the MIs themselves, as an alternative to direct integration.

The derivation of IBP-decomposition formulas requires the solution of a system of linear relations, generated by imposing that integrals of a total differential (w.r.t. integration variables) vanish on the integration boundary [3, 17]. For multi-loop, multi-scale scattering amplitudes, solving the system of IBP relations may, however, represent a formidable task. This problem has motivated the development of important techniques based on reconstruction of rational functions using finite fields [18–21].

Already during the early developments of S -matrix theory, it was recognized that topology and cohomological methods offer a connection between analytic properties of Feynman integrals and the geometry of their singularity structure, which emerge from the varieties associated to their graph polynomials. In more recent studies, *intersection numbers* for twisted de Rham cohomology groups [22–33] have been exploited to uncover the vector space structure of Feynman integrals and to derive novel algorithms for the direct *projection* of multi-loop integral onto MIs [34–38] (see also [39–42]). Important developments have been carried out in [43–49] and example of applications to non-linear relations can be found in [50, 51]. See [1, 42, 52, 53], for recent reviews.

Within this approach, the number of MIs corresponds naturally to the dimension of the vector space of Feynman integrals, and can be related to topological quantities such as the dimension of the cohomology groups, the number of certain critical points [34, 54], Euler characteristics [37, 55–58], as well as to the dimension of quotient rings of polynomials for zero dimensional ideals (Shape lemma) [38]. Linear relations (for integrals with shifted indices), Pfaffian systems of differential equations, finite difference equations, as well as quadratic relations, such as Riemann twisted period relations, can be derived by means of intersection numbers for differential (twisted) forms.

The pivotal role that twisted de Rham theory seems to have in controlling the algebra of Feynman integrals, and more generally of Euler integrals, has been stimulating us to

elaborate on the isomorphism between cohomology groups, whose elements are differential forms, and \mathcal{D} -modules, whose elements are partial differential operators in a Weyl algebra.

Within momentum-space representation, differential operators acting on multi-loop integrands and integrals are used to establish IBP relations, system of partial differential equations (SPDE) for MIs, and Lorentz invariance identities [9]. In the former type of relations, differentiation is carried out w.r.t. the integration momenta, while for the latter two ones, w.r.t. the momenta of external particles and masses. Also, within parametric representations of Feynman integrals, it is possible to study the action of differential operators acting on multi-fold integrands and integrals, distinguishing between the cases for which the differentiation is carried out w.r.t. integration or external variables.

In [57, 58], it was shown that linear relations between Feynman integrals having shifted indices, similar to IBPs, can arise from the action of parametric annihilators of the integrand, within the Lee-Pomeransky (LP) representation. The algebraic properties of these special partial differential operators, w.r.t. the integration variables, are derived by studying the ideals of parametric annihilators within the language of \mathcal{D} -modules. Remarkably, the number of master integrals, known to be finite [59], was found to be identical to the Euler characteristic of the complement of the hypersurface determined by the LP polynomial [57, 58].

While the previous study focused on \mathcal{D} -module theory for differential operators in the *internal* variables, in this work, we investigate the properties of differential operators in the *external* variables. In particular, we exploit the properties of GKZ-hypergeometric systems, introduced by Gel'fand, Kapranov, Zelevinsky [60], also known as \mathcal{A} -hypergeometric systems, to derive Pfaffian equations for the generators of the corresponding \mathcal{D} -module. As established in [61], GKZ systems provide a dictionary between Euler integrals and differential operators, which was later made fully algorithmic in [62]. Feynman integrals can be considered as special cases of Euler integrals, therefore the Pfaffian matrices derived in the context of \mathcal{D} -module theory correspond to the matrices of the SPDE satisfied by the MIs [63].

The connection between Feynman integrals and the generalized hypergeometric functions was first proposed by Regge [64] and it was better understood with the knowledge of the Feynman integrals satisfying a holonomic differential equation [65], where the singularities of the differential equation were governed by the Landau singularities. The relation between Feynman integrals and hypergeometric system were studied in [66–72] (and reference therein). The isomorphism between the GKZ system and the Feynman integral was established in [63] and later realized within the LP representation in [73, 74].

Within the proposed approach, many problems, including the derivation of differential equations, are translated into a ring theoretic computation where we may utilize various notions of computational ring theory such as *Gröbner bases*. As Gröbner basis calculations may be computationally expensive, we propose a different approach based on the *Macaulay matrix*, which is of central interest in this work. Usually, relations among integrals are employed to derive systems of partial differential equations. Instead, in the current work, we reverse the perspective, and show how Pfaffian matrices, built from Maculay matrices, can be used to derive linear relations for integrals. Moreover, through the *secondary equation* introduced in [33], we exploit the role of Pfaffian matrices for building *cohomology*

intersection matrices to be used in the *master decomposition formula* presented in [34–37, 39, 43], for the direct integral decomposition.

1.2 Macaulay matrix for GKZ system

GKZ systems can be rewritten into systems of differential equations dubbed Pfaffian systems. By extension, the same holds true for Feynman integrals. A Pfaffian system for a holonomic function f in variables $z = (z_1, \dots, z_N)$ is of the form

$$\partial_i F = P_i \cdot F, \quad F = (s_1 f, \dots, s_r f), \quad i = 1, \dots, N, \quad (1.1)$$

where $\partial_i := \partial/\partial z_i$, s_i are differential operators acting on f , and P_i are $r \times r$ matrices with entries being rational functions of z . The P_i are called *Pfaffian matrices* (or simply *Pfaffians*) in this paper. They satisfy the integrability condition

$$\partial_j P_i + P_i \cdot P_j = \partial_i P_j + P_j \cdot P_i. \quad (1.2)$$

Note that a Gröbner basis for the annihilating ideal \mathcal{I} of the function f gives a Pfaffian system. Conversely, a Pfaffian system, for instance obtained by Macaulay matrices, gives a Gröbner basis for the annihilating ideal: we have $\partial_i e - P_i e \equiv 0 \pmod{\mathcal{I}}$, $e := (s_1, \dots, s_r)$. Thus, computing the Macaulay matrix naturally leads to a Gröbner basis, as alternative to the Buchberger algorithm.

The Macaulay matrix is a generalization of the Sylvester matrix. The method was regarded as less efficient than the Buchberger algorithm in the ring of polynomials until J.C.Faugère proposed the algorithm F4 in 1999 [75]. F4 constructs Gröbner bases in the ring of polynomials by a variation of Macaulay matrix together with symbolic preprocessing of S -pairs. This algorithm is extremely efficient when combined with linear algebra optimization methods, for which reason the Macaulay matrix method is currently attracting more attention.¹

There are several advantages of utilizing Macaulay matrix in the ring of differential operators. One is that once we construct a relevant Macaulay matrix, we do not need expensive computation in a non-commutative ring. Moreover, in applications to GKZ-hypergeometric systems, standard monomials can be obtained *without* computing Gröbner bases in the ring of differential operators [77]. Hence, we may assume that a set of standard monomials is given before constructing a Macaulay matrix. These advantages were utilized in [78], where they used a Macaulay matrix to evaluate normalizing constants for a statistical distribution by the holonomic gradient method. This calculation was not possible with the original holonomic gradient method which utilizes Gröbner bases in the ring of differential operators (see e.g. [79]).

In this article, we propose a method for constructing *smaller Macaulay* matrices in the ring of differential operators compared to [78]. Using smaller Macaulay matrices, one can construct larger Gröbner bases or Pfaffian systems. By means of smaller Macaulay matrices, we provide new methods for computing recurrence relations, which may be called IBP

¹Note that there is no unique definition of a Macaulay matrix: by a Macaulay matrix, we usually mean variations of the matrix used by Macaulay in [76].

identities for GKZ-hypergeometric systems. In this setting, Feynman integrals are viewed as Mellin transforms of a certain (graph) polynomial \mathcal{G} in Schwinger parameters, where the coefficients in front of the monomials of \mathcal{G} are lifted to independent, indeterminate variables $z = (z_1, \dots, z_N)$ [1, 63, 70, 74, 80–83].

The main results of our investigation can be summarised in three points:

- We obtain Pfaffian systems of partial differential equations for GKZ-hypergeometric functions, proposing a novel and more efficient algorithm to obtain it from Macaulay matrices, therefore improving the algorithm of [78].
- We use the Pfaffians generated with the Macaulay matrices to obtain linear (contiguity) relations for the GKZ systems, using the matrix factorial, within the holonomic gradient method [84];
- We exploit the rational solution of the secondary equation for intersection matrices, controlled by the Pfaffians [33], to derive a novel version of the integral decomposition formula [34–37] based on intersection theory.

Our results apply to Feynman integrals, and, more generally to GKZ-hypergeometric functions.

Our presentation is organized as follows. In section 2, we review basic notions of the GKZ hypergeometric systems and their Euler integral representation. In section 3, we discuss Pfaffian systems of differential equations, which are intimately related to GKZ systems. We present the Macaulay matrix algorithm, based only on linear algebra, to compute Pfaffian matrices in section 4. We show its application to examples of differential equations for Feynman integrals in section 5. In section 6, we show how Pfaffians can be used to derive linear relations for GKZ systems, similar to IBP identities for Feynman integrals. Finally, in section 7, we present the integral decomposition via intersection numbers, using Pfaffians to compute the required intersection matrices.

All algorithms in this paper are implemented in the computer algebra system RISA/ASIR [85], MAPLE [86] and MATHEMATICA [87] with FINITEFLOW [88], while the calculations involving Feynman integrals are checked with LITERED [89, 90]. Programs used in this paper and machine readable data can be obtainable from [91].

2 GKZ hypergeometric systems

In this section, we briefly review some basic properties of the GKZ-hypergeometric systems to fix our notation. Section 2.1 introduces a particular integral representation related to the GKZ systems we work with, and section 2.2 covers its relation to the algebraic de Rham cohomology groups. In section 2.3, we describe how to represent a cohomology class by an element of Weyl algebra. Finally, in section 2.4, we discuss the homogeneity property of GKZ systems, which allows us to reduce the number of independent variables.

2.1 Integral representation of GKZ-hypergeometric system

In this work, we consider Euler integrals of the form

$$f_{\Gamma}(z) = \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{dx}{x}, \quad \frac{dx}{x} := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}. \quad (2.1)$$

Here Γ is a *twisted cycle*,² $\beta = (\beta_0, \dots, \beta_n) \in \mathbb{C}^{n+1}$ are complex parameters, and $g(z; x)$ is a Laurent polynomial in x

$$g(z; x) = \sum_{i=1}^N z_i x^{\alpha_i}. \quad (2.2)$$

The monomials above are written in multivariate exponent notation: given an integer vector $\alpha_i \in \mathbb{Z}^n$ we set

$$x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}, \quad (2.3)$$

where $\alpha_{i,j}$ stands for the j -th component of the vector α_i . Crucially, in (2.2) we regard each coefficient z_i as an *independent* variable of $f_{\Gamma}(z)$.

Let us construct the $(n+1) \times N$ matrix

$$A = \begin{pmatrix} a_1 & \cdots & a_N \end{pmatrix}, \quad (2.4)$$

whose columns a_i are built from the monomial exponents α_i as $a_i := (1, \alpha_i)$, with the assumption that $\text{Span}\{a_1, \dots, a_N\} = \mathbb{Z}^{n+1}$. Moreover, we introduce the (left) kernel of A , defined as,

$$\text{Ker}(A) = \left\{ u = (u_1, \dots, u_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N u_j a_j = \mathbf{0} \right\}. \quad (2.5)$$

Then, by using A and β as input, we build the following set of differential operators:

$$E_j = \sum_{i=1}^N a_{j,i} z_i \frac{\partial}{\partial z_i} - \beta_j, \quad j = 1, \dots, n+1 \quad (2.6)$$

$$\square_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial z_i} \right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial z_i} \right)^{-u_i}, \quad \forall u \in \text{Ker}(A). \quad (2.7)$$

The function $f_{\Gamma}(z)$, defined in (2.1), satisfies the system of partial differential equations (PDE)

$$E_j f_{\Gamma}(z) = 0, \quad (2.8)$$

$$\square_u f_{\Gamma}(z) = 0, \quad (2.9)$$

therefore it is dubbed an \mathcal{A} -hypergeometric function [60].

²A twisted cycle is an integration contour with no boundary, along which the branch of the integrand is specified. For details, see [27, Chapter 3].

2.2 GKZ \mathcal{D} -modules and de Rham cohomology

The operators in (2.6)–(2.7) can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \dots, z_N] \langle \partial_1, \dots, \partial_N \rangle, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, z_j] = \delta_{ij}. \quad (2.10)$$

In multivariate exponent notation, the elements of \mathcal{D}_N take the form $\sum_{k \in K} h_k(z) \partial^k$ for some finite collection of sets $K = \{K_i \in \mathbb{N}_0^N\}_i$, where the $h_k(z)$ are polynomials in z with complex coefficients. The symbol ∂_i is an alias of $\frac{\partial}{\partial z_i}$.

We introduce the *GKZ system* as the left \mathcal{D}_N -module $\mathcal{D}_N/H_A(\beta)$, where $H_A(\beta)$ is the left ideal generated by E_j and \square_u ,

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \text{Ker}(A)} \mathcal{D}_N \cdot \square_u. \quad (2.11)$$

Further details on \mathcal{D} -modules theory can be found in the appendix B.

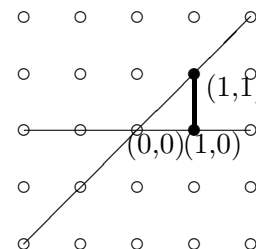
Let us list a few important properties of GKZ systems and their relations to de Rham cohomology groups, which are expressed through the following theorems and propositions.

□ First, we recall a theorem on the number of solutions to GKZ systems. Let Δ_A denote the convex polytope spanned by the columns of A . We say β is *non-resonant* when it does not belong to any set of the form $\text{span}_{\mathbb{C}}\{a_i \mid a_i \in F\} + \mathbb{Z}^{n+1}$ where F is a facet of Δ_A .

For example, if we take a 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the polytope Δ_A is a segment, and there are exactly two facets, respectively indicated by the thick segment and the black dots in the nearby picture. Each facet defines a linear subspace on which β is resonant.



Theorem 2.1 ([92]).

1. $H_A(\beta)$ is a holonomic ideal.³
2. When β is non-resonant, the holonomic rank r of $H_A(\beta)$ is given by the volume⁴

$$r = n! \cdot \text{vol}(\Delta_A). \quad (2.12)$$

The holonomic rank equals the number of independent solutions to the system of PDEs (2.6)–(2.7) at a generic point $z \in \mathbb{C}^N$. The first statement ensures that the rank is finite, while

³For the definition of a holonomic ideal, see appendix B or p. 31 of [93].

⁴vol stands for the Lebesgue measure and can be calculated with software such as Polymake [94]. The holonomic rank is the number of standard monomials of $\mathcal{R}H_A(\beta)$ (see appendix B).

the second statement gives an exact formula for computing it in terms of combinatorial data.

□ Next, letting \mathbb{G}_m (resp. \mathbb{A}) stand for the complex torus (resp. complex Affine line) equipped with the Zariski topology⁵ and

$$X := \{(z, x) \in \mathbb{A}^N \times (\mathbb{G}_m)^n \mid g(z; x) \neq 0\}, \quad Y := \mathbb{A}^N, \quad (2.13)$$

we denote by $\pi : X \rightarrow Y$ the natural projection from the space of GKZ and integration variables to the space of GKZ variables only.

Setting $\mathcal{O}(X) := \mathbb{C}[z_1, \dots, z_N, x_1^{\pm 1}, \dots, x_n^{\pm 1}, \frac{1}{g}]$, we define an action of \mathcal{D}_N on $f = f(z, x) \in \mathcal{O}(X)$ by

$$\frac{\partial}{\partial z_i} \bullet f = \frac{\partial f}{\partial z_i} + \beta_0 \left(\frac{1}{g(z; x)} \frac{\partial g(z; x)}{\partial z_i} \right) f, \quad (2.14)$$

$$\frac{\partial}{\partial x_i} \bullet f = \frac{\partial f}{\partial x_i} + \beta_0 \left(\frac{1}{g(z; x)} \frac{\partial g(z; x)}{\partial x_i} \right) f - \beta_i \frac{f}{x_i}. \quad (2.15)$$

The symbol $\mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}$ denotes the left \mathcal{D}_N -module $\mathcal{O}(X)$ endowed with this action. Formally, we have the identities

$$\frac{\partial}{\partial z_i} \left(g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} f \right) = g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} \left(\frac{\partial}{\partial z_i} \bullet f \right), \quad (2.16)$$

$$\frac{\partial}{\partial x_i} \left(g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} f \right) = g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} \left(\frac{\partial}{\partial x_i} \bullet f \right). \quad (2.17)$$

The direct image \mathcal{D} -module $\int_{\pi} \mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}$ is defined canonically as in appendix B.

Theorem 2.2 ([61]). *Suppose that β is non-resonant. Then there is a canonical isomorphism of left \mathcal{D}_N -modules*

$$\mathcal{D}_N/H_A(\beta) \simeq \int_{\pi} \mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}. \quad (2.18)$$

Let us make this isomorphism explicit [95]. We let

$$\Omega_{X/Y}^k = \bigoplus_{J \subset \{1, \dots, n\}, |J|=k} \mathcal{O}(X) dx^J \quad (2.19)$$

represent the space of relative k -forms⁶ onto which we act with the covariant derivative in integration variables

$$\nabla_x = d_x + \beta_0 \frac{d_x g(z; x)}{g(z; x)} \wedge - \sum_{i=1}^n \beta_i \frac{dx_i}{x_i} \wedge. \quad (2.20)$$

⁵As a set, the torus \mathbb{G}_m (resp. the complex Affine line \mathbb{A}) is equivalent to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (resp. \mathbb{C}).

⁶ X/Y does not denote a quotient space. Rather, it is a symbol for relative differential forms, by which we mean that we do not consider differential forms dz^J where $z \in Y$.

We then obtain a chain complex

$$\dots \xrightarrow{\nabla_x} \Omega_{X/Y}^k \xrightarrow{\nabla_x} \Omega_{X/Y}^{k+1} \xrightarrow{\nabla_x} \dots \quad (2.21)$$

The k -th *relative de Rham cohomology group* is defined as follows:

$$\mathbb{H}^k := \text{Ker}(\nabla_x : \Omega_{X/Y}^k \rightarrow \Omega_{X/Y}^{k+1}) / \text{Im}(\nabla_x : \Omega_{X/Y}^{k-1} \rightarrow \Omega_{X/Y}^k). \quad (2.22)$$

It can be shown that the direct image \mathcal{D} -module $\int_{\pi} \mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}$ is isomorphic to the n -th relative de Rham cohomology group \mathbb{H}^n , for which reason the latter is a left \mathcal{D}_N -module by theorem 2.2. In fact, theorem 2.2 can be rephrased as

Proposition 2.3. *Suppose that β is non-resonant. Then there is a unique isomorphism of \mathcal{D}_N -modules*

$$\mathcal{D}_N/H_A(\beta) \simeq \mathbb{H}^n \quad (2.23)$$

such that $[1] \in \mathcal{D}_N/H_A(\beta)$ is sent to $\left[\frac{dx}{x}\right] \in \mathbb{H}^n$.

A consequence of Proposition 2.3, which will be essential for our application of \mathcal{D}_N -module theory to Feynman integrals, is the following: given a cohomology class $[\omega(z)] \in \mathbb{H}^n$, there exists a differential operator $P \in \mathcal{D}_N$, which is unique modulo $H_A(\beta)$, such that

$$P \left[\frac{dx}{x}\right] = [\omega(z)]. \quad (2.24)$$

The partial differential operators ∂_i in P act on a cohomology class $[\omega(z)] \in \mathbb{H}^n$ via

$$\partial_i \bullet [\omega(z)] = \left[\partial_i \omega(z) + \beta_0 \frac{x^{\alpha_i}}{g(z; x)} \omega(z) \right]. \quad (2.25)$$

The action (2.25) comes from differentiation under the integral sign:

$$\frac{\partial}{\partial z_i} \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} \omega(z) = \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n} \partial_i \bullet \omega(z). \quad (2.26)$$

Since a Feynman integral can be represented by a cohomology class [34], we may equally well consider the operator P as representing that integral. An algorithm for computing P was developed in [62] and will be outlined in the following section. Moreover, in the view of the relation of GKZ-systems and Feynman integrals (to be elaborated on in section 5), we observe that the finiteness of the rank, established by the first statement of Theorem 2.1, can be related to the finiteness of the number of master integrals [59]. The formula for its evaluation, given in the second statement of Theorem 2.1, offers an alternative way of determining $\dim(\mathbb{H}^n)$, which is ordinarily computed in terms of Betti numbers, by counting the number of certain critical points, or by Euler characteristics — all of which are related to the number of master integrals.

2.3 Representing integrals inside \mathcal{D}_N -modules

Let us define the following family of differential forms:

$$\omega_q = g(z; x)^{-q_0} x^{q'} \frac{dx}{x}, \quad q = (q_0, q') \in \mathbb{Z} \times \mathbb{Z}^n. \quad (2.27)$$

Pairing ω_q with a cycle Γ yields the integral

$$\langle \omega_q \rangle_\Gamma = \int_\Gamma g(z; x)^{\beta_0 - q_0} x_1^{-\beta_1 + q'_1} \dots x_n^{-\beta_n + q'_n} \frac{dx}{x}. \quad (2.28)$$

We observe that differentiating the integral w.r.t. z_i shifts the q vector by $+a_i$ (defined after (2.4)):

$$\partial_i \langle \omega_q \rangle_\Gamma = (\beta_0 - q_0) \langle \omega_{q+a_i} \rangle_\Gamma. \quad (2.29)$$

It is also possible to construct an operator which does the opposite, i.e. shifting q by $-a_i$. In [96] it was shown that there exists a so-called step-up (or creation) operator U_i and polynomial $b_i(\beta)$ such that

$$U_i \partial_i - b_i(\beta) = 0 \quad \text{mod} \quad H_A(\beta). \quad (2.30)$$

The operator U_i then acts on the integral (2.28) via

$$U_i \langle \omega_q \rangle_\Gamma = b_i(\beta - a_i) \langle \omega_{q-a_i} \rangle_\Gamma. \quad (2.31)$$

The polynomial $b_i(\beta)$ is called the b -function [96]. Its computation is algorithmic and is implemented in computer algebra packages such as `mt_gkz.rr` [62].

Now, assume an integral of the form (2.28) is given for some choice of the integer vector q . Since the a_i 's span \mathbb{Z}^{n+1} by assumption, we may write

$$q = \sum_{i=1}^N r_i a_i. \quad (2.32)$$

The coefficients $r_i \in \mathbb{Z}$ allow us to balance the shifts by $\pm a_i$ in the q of $\langle \omega_q \rangle_\Gamma$ via equations (2.29) and (2.31). We may therefore construct the operator

$$P(q) = \prod_{r_i < 0} U_i^{-r_i} \prod_{r_i > 0} \frac{1}{B(\beta) B'(\beta)} \partial^{r_i}, \quad (2.33)$$

with the property that

$$P(q) \langle \omega_0 \rangle_\Gamma = \langle \omega_q \rangle_\Gamma, \quad \omega_0 = \frac{dx}{x}. \quad (2.34)$$

The two functions $B(\beta)$ and $B'(\beta)$ are built from the prefactors on the RHSs of (2.29) and (2.31), and their explicit expressions can be found in [62]. The operator $P(q)$ will be used extensively in our applications to Feynman integrals, in which case the vector q will be related to the propagator powers and space-time dimension of a given integral. Let us observe that the vector (r_1, \dots, r_N) in (2.32) is not unique, because we can shift it by adding a term like $\sum_i c_i u_i$ for some choice of $c_i \in \mathbb{Z}$, and $u_i \in \mathbb{Z}^N$ span $\text{Ker}(A)$. This freedom can be exploited to simplify the expression of the operators $P(q)$ appearing in (2.34), and for practical purposes, we find it convenient to choose $r_i > 0$ for all i , in order to avoid the appearance of U_i in (2.34), because it may contain monomials in ∂_i of large degree.

2.4 Homogeneity and integrand rescaling

Under rescaling of the z_i variables, Euler integrals scale as

$$f_{\Gamma}(t^{a_1} z_1, \dots, t^{a_N} z_N) = t_0^{\beta_0} t_1^{\beta_1} \cdots t_n^{\beta_n} f_{\Gamma}(z), \quad (2.35)$$

where we used the multivariate exponent notation from (2.3) on the l.h.s. Indeed, (2.35) is equivalent to homogeneity equations

$$E_j f_{\Gamma} = 0, \quad j = 1, \dots, n + 1. \quad (2.36)$$

This property can be exploited to work on a simpler set of integrals obtained from the original definition by *freezing* $(n + 1)$ out of N variables z_i to e.g. 1. The residual functional dependence is then given by $N - (n + 1)$ ratios of z_i -variables. More details can be found in appendix A.

Example 2.4. Consider the following Euler integral:

$$f_{\Gamma}(z) = \int_{\Gamma} (z_1 + z_2 x_1 + z_3 x_2 + z_4 x_1 x_2)^{\beta_0} x_1^{-\beta_1} x_2^{-\beta_2} \frac{dx}{x}, \quad (2.37)$$

which, according to eq. (2.1), corresponds to a 3×4 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (2.38)$$

The homogeneity property from (2.35) involves three rescaling parameters $\{t_0, t_1, t_2\}$:

$$f_{\Gamma}(t_0 z_1, t_0 t_1 z_2, t_0 t_2 z_3, t_0 t_1 t_2 z_4) = t_0^{\beta_0} t_1^{\beta_1} t_2^{\beta_2} f_{\Gamma}(z_1, z_2, z_3, z_4). \quad (2.39)$$

To derive this relation we need to rescale integration variables as $x_1 \mapsto x_1/t_1$ and $x_2 \mapsto x_2/t_2$. The homogeneity property allows us to factorize the dependence on z_1, z_2 , and z_3 by the following choice of the rescaling parameters t_i :

$$t_0 = 1/z_1, \quad t_1 = z_1/z_2, \quad t_2 = z_1/z_3. \quad (2.40)$$

The original Euler integral from (2.37) then becomes

$$f_{\Gamma}(z_1, z_2, z_3, z_4) = z_1^{-\beta_0 + \beta_1 + \beta_2} z_2^{-\beta_1} z_3^{-\beta_2} f_{\Gamma}(1, 1, 1, w), \quad w = \frac{z_1 z_4}{z_2 z_3}, \quad (2.41)$$

and so we have effectively rescaled away $(n + 1)$ of the z_i variables.

Finally, we show the differential equation obeyed by the rescaled Euler integral $f_{\Gamma}(1, 1, 1, z)$. The generator (2.7) of the original GKZ system with the A -matrix shown in (2.38) reads as

$$\square_u f_{\Gamma} = (\partial_1 \partial_4 - \partial_2 \partial_3) f_{\Gamma}(z_1, z_2, z_3, z_4) = 0. \quad (2.42)$$

Upon substitution of the representation (2.41) on the r.h.s., we can determine what the two terms in \square_u map to when changing variables to w :

$$\partial_1 \partial_4 \longrightarrow -w^{-1}(\beta_0 - \beta_1 - \beta_2 - w \partial_w)(w \partial_w), \tag{2.43}$$

$$\partial_2 \partial_3 \longrightarrow (\beta_1 + w \partial_w)(\beta_2 + w \partial_w). \tag{2.44}$$

These expressions can also be derived from Proposition A.1. Thus the original equation (2.42) becomes

$$\left[w(1-w) \partial_w^2 + (1 - \beta_0 + \beta_1 + \beta_2 - (1 + \beta_1 + \beta_2)w) \partial_w - \beta_1 \beta_2 \right] f_\Gamma(1, 1, 1, w) = 0, \tag{2.45}$$

which we recognize as the differential equation for the Gauß hypergeometric function.

This concludes our short overview of the basic properties of GKZ systems and Euler integrals. In the following section, we describe how to obtain a Pfaffian system of first-order PDEs given a basis of Euler integrals.

3 Pfaffian systems

In this section, we introduce notation related to Pfaffian systems to be used in the rest of this work. In section 3.1, we recall the passage from a holonomic \mathcal{D}_N -module to a Pfaffian system, while in section 3.2 and section 3.3 we present two methods for basis change.

For any positive integer N , we write $\mathbb{C}(z) = \mathbb{C}(z_1, \dots, z_N)$ for the field of rational functions in variables z_1, \dots, z_N . Let $\mathcal{R} := \mathbb{C}(z_1, \dots, z_N) \otimes_{\mathbb{C}[z_1, \dots, z_N]} \mathcal{D}_N$ be the rational Weyl algebra;⁷ note that \mathcal{R} also has the structure of a non-commutative ring. Any element $Q \in \mathcal{R}$ can be uniquely written in the so-called normally ordered form with all the partial derivatives commuted to the right:

$$Q = \sum_k q_k \cdot \partial^k, \quad q_k \in \mathbb{C}(z). \tag{3.1}$$

As we review in appendix B, given a Gröbner basis G of a left ideal \mathcal{I} in \mathcal{R} , we denote by⁸ Std the standard monomials w.r.t. G and a term order \prec . For our purposes it is enough to consider a zero-dimensional left ideal \mathcal{I} generated by a finitely many elements:

$$\mathcal{I} := \langle h_1, h_2, \dots, h_d \rangle, \quad h_i \in \mathcal{R}. \tag{3.2}$$

The ideal \mathcal{I} is zero-dimensional iff the corresponding set of standard monomials Std is finite (see [97, §6]).

⁷The subscript of the tensor product \otimes denotes the ring over which the product is defined. For example, it implies the identity: $r \otimes p Q = r p \otimes Q$, for $r \in \mathbb{C}(z)$, $p \in \mathbb{C}[z]$, and $Q \in \mathcal{D}_N$.

⁸When considering Std as a basis, we instead write $e^{(\text{Std})}$.

3.1 From \mathcal{D}_N -module to DEQ system

We consider a left \mathcal{R} -module M which is finite dimensional over $\mathbb{C}(z)$. Such a left \mathcal{R} -module arises as an extension $\mathcal{R} \otimes_{\mathcal{D}_N} M'$ for some other holonomic \mathcal{D}_N -module M' . When $\{u_1, \dots, u_r\} \subset M$ is a $\mathbb{C}(z)$ -basis, one can find an $r \times r$ matrix $P_j(z)$ with entries in $\mathbb{C}(z)$ such that

$$\partial_j(u_1, \dots, u_r)^T = P_j(z) \cdot (u_1, \dots, u_r)^T. \tag{3.3}$$

We call $P_j(z)$ the Pfaffian matrix in direction z_j . Let us now consider the GKZ \mathcal{D}_N -module. Namely, we consider a left \mathcal{R} -module $M = \mathcal{R} \otimes_{\mathcal{D}_N} (\mathcal{D}_N/H_A(\beta))$, where $H_A(\beta)$ is as in (2.11). In view of Proposition 2.3, each u_i corresponds to a cohomology class $[\omega_i] \in \mathbb{H}^n$. Integrating (3.3) over a cycle Γ , we obtain an identity

$$\partial_j(\langle \omega_1 \rangle_\Gamma, \dots, \langle \omega_r \rangle_\Gamma)^T = P_j(z) \cdot (\langle \omega_1 \rangle_\Gamma, \dots, \langle \omega_r \rangle_\Gamma)^T, \tag{3.4}$$

which is precisely the system of differential equations that the master Euler integrals are subject to. A standard method of computing the Pfaffian matrix $P_j(z)$ is to use a Gröbner basis of \mathcal{R} ([97, Chapter 6]). Unfortunately, this approach is far from effective. We develop a different approach based on the Macaulay matrix in section 4.

Our construction of Pfaffian matrices by the Macaulay matrix method, to be given in section 4, will assume that a set of standard monomials S for GKZ ideal $\mathcal{R}H_A(\beta)$ is given. Note that the set S gives rise to a $\mathbb{C}(z)$ -basis of M . Although one usually computes a Gröbner basis of the ideal $\mathcal{R}H_A(\beta)$ to find the set S , we claim that the set S can be found by computing a Gröbner basis of an ideal in a commutative subring of \mathcal{R} . This reduces the cost of computation in a significant way.

For this purpose, we fix a term order \prec on the commutative ring $\mathbb{C}[\partial_1, \dots, \partial_N]$. It naturally induces a term order \prec on the ring \mathcal{R} because a term order on the ring \mathcal{R} is a total order on the set of monomials ∂^k (cf. [97, §6.1]). We denote by $\text{in}_\prec(\mathcal{I}_\square)$ the ideal of $\mathbb{C}[\partial_1, \dots, \partial_N]$ generated by initial terms of $\mathcal{I}_\square = \langle \square_u \rangle$ by the order \prec , where the box operators are given in (2.7). Any generator of $\text{in}_\prec(\mathcal{I}_\square)$ can be written as $\partial^k = \prod_{i=1}^N \partial_i^{k_i}$. Let us write θ_i for the Euler operator $z_i \partial_i$ and consider a subring of \mathcal{R} generated by $\theta_1, \dots, \theta_N$ over \mathbb{C} . We will denote this ring by $\mathbb{C}[\theta]$, which is clearly a commutative ring. By the correspondence $\theta^k \leftrightarrow \partial^k$, we equip a term order \prec on the ring $\mathbb{C}[\theta]$ induced from \prec . We then define the *distraction* of the element ∂^k ([93, p.68]) by

$$\prod_{i=1}^N \theta_i(\theta_i - 1) \cdots (\theta_i - k_i + 1). \tag{3.5}$$

We consider the ideal \mathcal{I}' of the ring $\mathbb{C}[\theta]$ generated by the distractions of the generators of $\text{in}_\prec(\mathcal{I}_\square)$ and the E_1, \dots, E_{n+1} from (2.6). The following theorem is essential whose proof is based on a technique called *Gröbner deformation* as we will see in appendix B.

Theorem 3.1. *The set of standard monomials of \mathcal{I}' coincides with that of GKZ ideal $\mathcal{R}H_A(\beta)$ when β is generic. More concretely, if $\{\theta^{a_1}, \dots, \theta^{a_r}\}$ is a set of standard monomials of \mathcal{I}' , the set $\{\partial^{a_1}, \dots, \partial^{a_r}\}$ is a set of standard monomials of $\mathcal{R}H_A(\beta)$.*

The ideal \mathcal{I}_\square is known as *toric ideal*. Computation of a Gröbner basis of such an ideal is effective as it is generated by binomials. Thus, the computation of a set of generators of the ideal \mathcal{I}' is efficient. Running Buchberger’s algorithm for the commutative ideal \mathcal{I}' , we obtain a set of standard monomials of GKZ ideal $\mathcal{RH}_A(\beta)$ by Theorem 3.1. We emphasize that the process explained above does not involve any computation in the non-commutative ring \mathcal{R} . Finally, we remark that the process above is implemented as a function `cbase_by_euler` in the Risa/Asir package `mt_gkz.rr`.

Example 3.2. *Let us consider a matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Let \prec be the lexicographic order such that $\partial_1 \succ \cdots \succ \partial_4$. The ideal $\text{in}_\prec(\mathcal{I}_\square)$ is generated by monomials

$$\partial_2 \partial_4^2, \partial_1 \partial_4, \partial_1 \partial_3^2, \partial_1^2 \partial_3.$$

The distractions of them are

$$\theta_2 \theta_4 (\theta_4 - 1), \theta_1 \theta_4, \theta_1 \theta_3 (\theta_3 - 1), \theta_1 (\theta_1 - 1) \theta_3.$$

We compute the Gröbner basis of an ideal generated by these distractions and $\theta_1 + \theta_2 + \theta_3 + \theta_4 - b_1$ and $\theta_2 + 3\theta_3 + 4\theta_4 - b_2$ with respect to \succ . The set of the standard monomials is $\{\theta_3, \theta_4^2, \theta_4, 1\}$. Thus, the set of the standard monomials of GKZ ideal $\mathcal{RH}_A(\beta)$ is given by

$$\{\partial_3, \partial_4^2, \partial_4, 1\}.$$

3.2 Basis change

Assume that we know Pfaffians $P_i^{(\text{Std})}$ in a basis of standard monomials $e^{(\text{Std})}$. In this section we show how to gauge transform the system into an arbitrary basis e .

Consider an ideal \mathcal{I} over a rational Weyl algebra \mathcal{R} . Let a basis $e = (e_1, \dots, e_r)^T \subset \mathcal{R}/\mathcal{I}$ be given. In the following, the symbol \circ denotes multiplication inside of \mathcal{R} , i.e. we write $\partial_i \circ h$ when $h \in \mathcal{R}$. We introduce \circ to distinguish composition in the ring from the action $\partial_i H$ when, say, H is a matrix with entries in the field $\mathbb{C}(z)$.

We seek the Pfaffian matrices $P_i \in \mathbb{C}(z)^{r \times r}$ satisfying the differential equation for the basis e :

$$\partial_i \circ e = P_i \cdot e. \tag{3.6}$$

Assume that we know $P_i^{(\text{Std})} \in \mathbb{C}(z)^{r \times r}$ in

$$\partial_i \circ e^{(\text{Std})} = P_i^{(\text{Std})} \cdot e^{(\text{Std})}, \tag{3.7}$$

where $e^{(\text{Std})} = \{\partial^\alpha\} \subset \mathcal{R}/\mathcal{I}$ are the standard monomials. If we could find a matrix $G \in \mathbb{C}(z)^{r \times r}$ relating the two bases,

$$e = G \cdot e^{(\text{Std})}, \tag{3.8}$$

then we could relate the Pfaffian matrices in different bases via a gauge transformation:

$$P_i = \left(\partial_i G + G \cdot P_i^{(\text{Std})} \right) \cdot G^{-1}. \tag{3.9}$$

Note that G is invertible by assumption, since both $e^{(\text{Std})}$ and e are bases.

By (3.9), our problem is reduced to finding the transformation matrix G . To this end, we begin by expanding e in terms of $e^{(\text{Std})}$:

$$e = \sum_{k \in K} G_k^{(1)} \cdot \left(\partial^k \circ e^{(\text{Std})} \right), \quad K \subset \mathbb{N}_0^N, \tag{3.10}$$

for some matrix $G_k^{(1)} \in \mathbb{C}(z)^{r \times r}$. Here we relied on the fact that any monomial of e is divisible by an entry of $e^{(\text{Std})}$. Next we transform each $\partial^k \circ e^{(\text{Std})}$ into the form $G_k^{(2)} \cdot e^{(\text{Std})}$ via repeated application of (3.7). Let us illustrate the last step with an example.

Example 3.3. Given $k = \{1, 1, 0, \dots, 0\}$, we can rewrite $\partial^k \circ e^{(\text{Std})}$ as

$$\partial_1 \circ \partial_2 \circ e^{(\text{Std})} = \partial_1 \circ \left(P_2^{(\text{Std})} \cdot e^{(\text{Std})} \right) \tag{3.11}$$

$$= \partial_1 P_2^{(\text{Std})} \cdot e^{(\text{Std})} + P_2^{(\text{Std})} \cdot \left(\partial_1 \circ e^{(\text{Std})} \right) \tag{3.12}$$

$$= \partial_1 P_2^{(\text{Std})} \cdot e^{(\text{Std})} + P_2^{(\text{Std})} \cdot P_1^{(s)} \cdot e^{(\text{Std})}. \tag{3.13}$$

We hence have that $G_k^{(2)} = \partial_1 P_2^{(\text{Std})} + P_2^{(\text{Std})} \cdot P_1^{(\text{Std})}$.

We conclude that the gauge transformation matrix G decomposes as follows:

$$G = \sum_{k \in K} G_k^{(1)} \cdot G_k^{(2)}, \tag{3.14}$$

where each $G_k^{(2)}$ is built from the matrix products between Pfaffians and derivatives thereof. Thus, equations (3.9), (3.14) constitute one way to translate the Pfaffian matrices between different bases.

3.3 Basis change without derivatives

The matrix $G_k^{(2)}$ appearing in the gauge transformation matrix (3.14) involves derivatives of Pfaffian matrices. If we were to work over a finite field, there would be no z_i variables with respect to which we could take the derivative. This situation is amended, in the GKZ case, by the following proposition.

Proposition 3.4. *The derivative of a Pfaffian matrix can be computed solely via addition and matrix multiplication from*

$$(\partial_i P_j)(\beta) = (P_j(\beta - a_i) - P_j(\beta)) P_i(\beta). \tag{3.15}$$

Proof. Let

$$F(\beta) := \frac{1}{\Gamma(\beta_0 + 1)} (s_1 f_\Gamma(\beta), \dots, s_r f_\Gamma(\beta)) \tag{3.16}$$

be a solution to a GKZ Pfaffian system given some monomial basis $s_i = \partial^{k_i}$, $k_i \in \mathbb{N}_0^N$, for which $s_i \partial_j = \partial_j s_i$. The following relations hold true:

$$\partial_j F(\beta) = P_j(\beta)F(\beta), \tag{3.17}$$

$$\partial_j F(\beta) = F(\beta - a_j). \tag{3.18}$$

Differentiating (3.17) w.r.t. z_i , we get

$$\partial_i \partial_j F = (\partial_i P_j)F + P_j(\partial_i F) \tag{3.19}$$

$$= (\partial_i P_j)F + P_j P_i F, \tag{3.20}$$

where we omitted the argument β for clarity. On the other hand, differentiating (3.18) we get

$$\partial_i \partial_j F(\beta) = \partial_i F(\beta - a_j) \tag{3.21}$$

$$= F(\beta - a_j - a_i) \tag{3.22}$$

$$= P_j(\beta - a_i)P_i(\beta)F(\beta), \tag{3.23}$$

where we applied the identity $P_k(\beta)F(\beta) = F(\beta - a_k)$ twice in the last step. The proposition follows upon equating (3.20) and (3.23) and isolating $(\partial_i P_j)(\beta)$. \square

Pfaffian systems introduced above are systems of *linear* partial differential equations (SPDE), satisfied by the solutions of a given GKZ system. As we will see later on, these equations are extremely useful in physical applications. Next we present an efficient way to calculate the Pfaffian systems, essentially via linear algebra.

4 Constructing Pfaffian systems from Macaulay matrices

In this section we describe a method for building the Pfaffian systems defined in eq. (3.3). The method amounts to first building an auxiliary matrix M called the Macaulay matrix, and then solving a special system of linear equations. In section 4.1 we derive the Macaulay matrix (4.5) and the linear system (4.12), (4.13) that it satisfies. In section 4.2 we then present Algorithm 1 for calculation of Pfaffian systems. In section 4.3 we give several remarks about the algorithm and its efficiency. We close this section with several examples in section 4.4, showcasing the steps and runtime statistics of the algorithm in practice.

4.1 From Pfaffian to Macaulay matrix

We present how the Macaulay matrix arises from a Pfaffian system in the basis of standard monomials. Since we will focus our discussion on the case of GKZ systems later on, based on the comments at the end of section 3.1 we may safely assume that

$$\text{a set of standard monomials } \text{Std} := \{\partial^k\} \text{ is given,}$$

and that its size equals the holonomic rank $|\text{Std}| = r$, defined in (2.12). We remind that ∂^k denotes a monomial in derivatives, while ∂_i denotes a single derivative w.r.t. z_i .

Now, the Pfaffian matrix P_i in the direction z_i , defined in (3.3), is specified by the following relation:

$$\partial_i \circ \text{Std}_\alpha = \sum_{\beta} (P_i)_{\alpha\beta} \cdot \text{Std}_\beta \quad \text{in } \mathcal{R}/\mathcal{I}, \quad (4.1)$$

where \circ denotes composition in a ring \mathcal{R} , $\alpha \in \{1, \dots, |\text{Std}|\}$ and \mathcal{I} is a general holonomic ideal for now. Using the expression (3.2) for \mathcal{I} , we carry over the relation (4.1) from the quotient \mathcal{R}/\mathcal{I} to the whole ring \mathcal{R} as

$$\partial_i \circ \text{Std}_\alpha = \sum_j Q_{\alpha j} \circ h_j + \sum_{\beta} (P_i)_{\alpha\beta} \cdot \text{Std}_\beta \quad \text{in } \mathcal{R}, \quad (4.2)$$

where each $Q_{\alpha j}$ is an element of \mathcal{R} and $j \in \{1, \dots, d\}$ labels the ideal generators h_j from (3.2). Now we bring the operators $Q_{\alpha j}$ into normal ordered form as in (3.1):

$$Q_{\alpha j} = \sum_k q_{\alpha j k} \cdot \partial^k, \quad \partial^k \in \text{Der}, \quad (4.3)$$

where $q_{\alpha j k} \in \mathbb{C}(z)^{|\text{Std}| \times d \times |\text{Der}|}$ is a newly introduced rank 3 tensor,⁹ and we denote by Der the set of all partial derivatives ∂^k appearing on the r.h.s.

Now we are ready to introduce the main player of this section: the Macaulay matrix. It arises due to the normal ordered form of the first sum in (4.2). Substitution of the $Q_{\alpha j}$ operators from (4.3) back into (4.2) results in the action of $\partial^k \in \text{Der}$ on the ideal generators h_j , which we bring to normal ordered form as follows:

$$\partial^k \circ h_j = \sum_L M_{kjL} \cdot \partial^L, \quad \partial^L \in \text{Mons}. \quad (4.4)$$

Here we introduced a rank 3 tensor $M_{kjL} \in \mathbb{C}(z)^{|\text{Der}| \times d \times |\text{Mons}|}$ and a set Mons , which collects the partial derivatives ∂^L emerging on the r.h.s. Upon concatenation¹⁰ of the first two indices k and j , the rank 3 tensor M_{kjL} turns into the Macaulay matrix

$$M_{RL} := M_{(kj)L} \in \mathbb{C}(z)^{(|\text{Der}| \cdot d) \times |\text{Mons}|}, \quad (4.5)$$

where the combined index $R := (kj)$ runs over a product of sets $R \in \text{Der} \times \{1, \dots, d\}$. Plugging equations (4.3), (4.4), (4.5) back into the main relation (4.2) for the Pfaffian, we observe that the first term can be written as a product of matrices:

$$\sum_j Q_{\alpha j} \circ h_j = \sum_{RL} q_{\alpha R} M_{RL} \cdot \partial^L \equiv (C \cdot M \cdot \text{Mons})_\alpha, \quad (4.6)$$

where $C \in \mathbb{C}(z)^{|\text{Std}| \times (|\text{Der}| \cdot d)}$ stands for the coefficient matrix $C_{\alpha R} := q_{\alpha(jk)}$ of (4.3), and we regard the set Mons as a column vector.

The set Mons , containing all monomials in ∂_i , can be partitioned into two disjoint sets consisting of standard and, what we call, exterior monomials:

$$\text{Mons} = \text{Ext} \sqcup \text{Std}, \quad \text{Ext} := \text{Mons} \setminus \text{Std}. \quad (4.7)$$

⁹The notation $a_{nm} \in \mathbb{C}(z)^{N \times M}$ means that the indices range over the sets: $n \in \{1, \dots, N\}$ and $m \in \{1, \dots, M\}$.

¹⁰In MATHEMATICA, this operation can be done with `Flatten[#, {{1, 2}, {3}}]&`.

This partition naturally induces a column partition of the Macaulay matrix:

$$M = (M_{\text{Ext}} | M_{\text{Std}}), \tag{4.8}$$

with the two column blocks $M_{\text{Ext}} \in \mathbb{C}(z)^{(|\text{Der}| \cdot d) \times |\text{Ext}|}$ and $M_{\text{Std}} \in \mathbb{C}(z)^{(|\text{Der}| \cdot d) \times |\text{Std}|}$. Similarly, the l.h.s. of (4.2) can be decomposed as¹¹

$$\partial_i \circ \text{Std} =: C' \cdot \text{Mons} \tag{4.9}$$

$$=: C'_{\text{Ext}} \cdot \text{Ext} + C'_{\text{Std}} \cdot \text{Std}, \tag{4.10}$$

where we introduced the coefficient matrix $C' \in \mathbb{C}(z)^{|\text{Std}| \times |\text{Mons}|}$ in addition to its column blocks $C'_{\text{Ext}} \in \mathbb{C}(z)^{|\text{Std}| \times |\text{Ext}|}$ and $C'_{\text{Std}} \in \mathbb{C}(z)^{|\text{Std}| \times |\text{Std}|}$, such that $C' = (C'_{\text{Ext}} | C'_{\text{Std}})$ in accordance with (4.8).

Finally substituting equations (4.6), (4.8), (4.10) into the main relation (4.2), we arrive at the following matrix relation:

$$(C'_{\text{Ext}} - C \cdot M_{\text{Ext}}) \cdot \text{Ext} + (C'_{\text{Std}} - C \cdot M_{\text{Std}}) \cdot \text{Std} = P_i \cdot \text{Std}. \tag{4.11}$$

Due to the linear independence of Ext and Std, this is equivalent to a system

$$\boxed{C'_{\text{Ext}} - C \cdot M_{\text{Ext}} = 0,} \tag{4.12}$$

$$\boxed{C'_{\text{Std}} - C \cdot M_{\text{Std}} = P_i.} \tag{4.13}$$

This system relates the Macaulay matrix M to the Pfaffian P_i in the direction z_i , and is central to our computational method presented next.

4.2 From Macaulay matrix to Pfaffian

Now we reverse the logic in the derivation above: although we started from (4.2), we would like to regard it as the goal. We view (4.2) and its matrix version (4.11) as a system of linear equations for the unknown matrix of coefficients C . Given the Macaulay matrix M , the idea is to solve first the system (4.12), and then substitute the solution into (4.13) in order to obtain the desired Pfaffian matrix P_i .

Let us now turn this idea into a practical algorithm. Say we would like to compute P_i in direction z_i given a zero-dimensional ideal \mathcal{I} . First, given a $D \in \mathbb{Z}_{\geq 0}$ we construct the set of all derivatives whose degree is bounded by D :

$$\text{Der}_D := \{\partial^k\}_{|k| \leq D}. \tag{4.14}$$

Next, similarly to (4.4), we write Mons_D for the set of monomials in derivatives appearing in the normal ordered form of $\{\partial^k \circ h_j\}_{\partial^k \in \text{Der}_D, j=1, \dots, d}$. We have

$$\{\partial^k \circ h_j\}_{\partial^k \in \text{Der}_D, j=1, \dots, d} = (M_D)_{(kj)L} \cdot \partial^L \tag{4.15}$$

$$\equiv M_D \cdot \text{Mons}_D, \tag{4.16}$$

where we call M_D the Macaulay matrix of degree D (compare to (4.5)). We adjust the maximal degree D and the sets Der_D and Mons_D until the linear system (4.12) can be resolved w.r.t. the unknown matrix C . Finally, the desired Pfaffian follows from using C in (4.13). We summarize this procedure in Algorithm 1.

¹¹Note that C' , C'_{Ext} and C'_{Std} depend on the direction z_i , since these matrices are born from the expression $\partial_i \circ \text{Std}$.

Algorithm 1 : Pfaffian matrix by Macaulay matrix method

Input: Standard monomials Std, generators h_1, h_2, \dots of a zero-dimensional ideal $\mathcal{I} \subset \mathcal{R}$, direction i .

Output: Pfaffian matrix P_i .

- 1: $D := 0$;
- 2: $M := M_0 = (M_{\text{Ext}} | M_{\text{Std}})$;
- 3: $\text{Mons} := \text{Mons}_0$;
- 4: **while** ($\partial_i \circ \text{Std}$ cannot be written as $C' \cdot \text{Mons}$) or ($C'_{\text{Ext}} = C \cdot M_{\text{Ext}}$ is not solvable w.r.t. C) **do**
- 5: $D ++$;
- 6: $M \leftarrow M_D$;
- 7: $\text{Mons} \leftarrow \text{Mons}_D$;
- 8: Solve $\partial_i \circ \text{Std} = C' \cdot \text{Mons}$ and $C'_{\text{Ext}} = C \cdot M_{\text{Ext}}$;
- 9: Return $P_i := C'_{\text{Std}} - C \cdot M_{\text{Std}}$;

Proposition 4.1. *Algorithm 1 outputs the Pfaffian matrix P_i in a finite number of steps.*

Proof. Since the left ideal \mathcal{I} is zero-dimensional, there exists a matrix P_i with entries in $\mathbb{C}(z)$ such that $\partial_i \circ \text{Std} - P_i \cdot \text{Std} \equiv 0 \pmod{\mathcal{I}}$ (see e.g. [97, Chap 6]). This means that the l.h.s. can be expressed in terms of the generators $\{h_1, \dots, h_d\}$ inside the ring \mathcal{R} , as in (4.2). Let N_0 be the maximal degree of the monomials $\partial^k \in \text{Der}$ from (4.3). Then Algorithm 1 outputs the Pfaffian matrix P_i until the degree parameter D reaches N_0 . \square

4.3 Discussion

Some comments about Algorithm 1 are in order.

Other directions. The condition on line 4 can be extended by checking $\partial_i \circ \text{Std}$ for all $i = 1, \dots, N$ to generate a Macaulay matrix M_D large enough for calculating Pfaffian matrices in all directions $\{P_i\}_{i=1, \dots, N}$.

Rank of M_{Ext} . The Macaulay matrix M_{Ext} produced by the algorithm may not be of full rank. This makes the matrices $M_{\text{Ext}}, M_{\text{Std}}$ smaller and therefore the whole algorithm more efficient than the algorithm of [78].

Degree of freedom. The unknown coefficient matrix C , which solves the system (4.12), can be chosen up to the left nullspace of the Macaulay matrix M . Indeed, suppose that both $C = C^{(1)}$ and $C = C^{(2)}$ solve the system (4.12). Clearly, we have $(C^{(1)} - C^{(2)}) \cdot M_{\text{Ext}} = 0$. Uniqueness of the Pfaffian matrix alongside (4.13) imply $(C^{(1)} - C^{(2)}) \cdot M_{\text{Std}} = 0$ as well. Therefore, we deduce

$$(C^{(1)} - C^{(2)}) \cdot M = 0, \tag{4.17}$$

according to the partition (4.8) of the Macaulay matrix.

Solvability condition. The system (4.12) has a solution when the rows of the C'_1 matrix lie in the row space of the M_{Ext} block of the Macaulay matrix. This can be formulated as a condition on the rank of the following matrix:

$$\text{rank} \begin{bmatrix} M_{\text{Ext}} \\ C'_{\text{Ext}} \end{bmatrix} = \text{rank}[M_{\text{Ext}}]. \quad (4.18)$$

Probabilistic method. Checking the solvability condition (4.18) can be computationally demanding. In order to reduce the computation time and memory usage we may employ the so-called probabilistic method. In the GKZ case, it amounts to setting the all the variables z_i and parameters β (see (2.1)) to numbers in \mathbb{Q} or a finite field \mathbb{F}_p , which greatly simplifies the row reduction required for the solvability check.

Smaller Macaulay matrix. Once the condition (4.18) is proved (almost surely, if one employs the probabilistic method), we further simplify the system (4.12) by selecting a subset of rows of the Macaulay matrix necessary for solving the system. In practice, we perform such a selection using the probabilistic method together with a determination of the left nullspace of the matrix on the l.h.s. of (4.18).

Finite fields. For fixed values of the variables z_i and the exponents β , numerical values of the Pfaffian matrix P_i can be efficiently obtained via the Macaulay matrix method. Note that one only needs numerical values of P_i to obtain the values of normalizing constants for a certain class of statistical distributions, see [78]. In our applications, however, we would like to (re)construct the full rational dependence of P_i on all the variables z_i and β . When it becomes too demanding to solve the system (4.12) analytically, the probabilistic approach can be used in tandem with rational function reconstruction over finite fields \mathbb{F}_p . Indeed, using modular arithmetic (over prime numbers) and the Chinese remainder theorem, the rational entries of $P_i \in \mathbb{C}(z)^{r \times r}$ can be efficiently obtained by means of functional reconstruction algorithms for multivariate, rational functions, which were developed in the context of scattering amplitudes in [19]. This algorithm reconstructs analytic expressions for polynomials and rational functions via repeated numerical evaluation over finite fields by means of iterative application of Newton's polynomial representation and Thiele's interpolation formula. The reconstruction algorithm is implemented in the public code FINTEFLOW [21], which uses so-called dataflow graphs for building numerical algorithms over finite fields.

Tests. Once we reconstruct the Pfaffian matrices from their numerical values over the finite fields, we check correctness as follows. Recall that Pfaffian equations (4.1) can be regarded as a Gröbner basis in \mathcal{R} (see the discussion below eq. (1.2)). Therefore, if the solution rank of the left ideal (3.2) is known to be r , the correct $r \times r$ Pfaffian matrices should satisfy the integrability condition (1.2), and should also reduce the generators h_i of the ideal (3.2) to 0 via the Pfaffian equations (4.1).

Further improvements. Let us comment on potential ways to improve the efficiency of the Algorithm 1. By virtue of (4.16), to generate the Macaulay matrix of degree D we are instructed to act with Der_D , the set of all derivatives of degree $\leq D$ (see (4.14)), on the generators of ideal. The set Der_D might be larger than necessary, so one way to improve the Algorithm 1 is to find a smaller set of derivatives needed for the construction of the Macaulay matrices. Another important direction of study is the prediction of singularities for the Pfaffian matrix in question, since such knowledge would greatly simplify the solution of (4.12) using rational reconstruction.

4.4 Examples

Example 4.2. Let us apply the Macaulay matrix method to compute the Pfaffian system for the Appell function $F_1(\alpha, \beta, \beta', \gamma; z_1, z_2)$.

Setup. Consider the following integral representation:

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma; z_1, z_2) = \int_{\Delta^2} (1-x_1-x_2)^{\gamma-\beta-\beta'-1} (1-z_1x_1-z_2x_2)^{-\alpha} x_1^\beta x_2^{\beta'} \frac{dx_1 dx_2}{x_1 x_2} \tag{4.19}$$

$$= \frac{\Gamma(1-\gamma+\alpha+\beta+\beta')}{\Gamma(1-\gamma+\beta+\beta')\Gamma(\alpha)} \int_{(0,+\infty)\times\Delta^2} g(z; x)^{\gamma-\alpha-\beta-\beta'-1} x_0^\alpha x_1^\beta x_2^{\beta'} \frac{dx_0 dx_1 dx_2}{x_0 x_1 x_2}, \tag{4.20}$$

where¹²

$$g(z; x) = 1 - x_1 - x_2 - x_0(1 - z_1x_1 - z_2x_2), \tag{4.21}$$

and Δ^2 is the two-dimensional simplex $\Delta^2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2, 1 - x_1 - x_2 \geq 0\}$. The ideal of the corresponding GKZ system is generated by three independent differential operators $\mathcal{I} = \langle h_1, h_2, h_3 \rangle$. However, to compute the Pfaffian matrices with generic parameters $\{\alpha, \beta, \beta', \gamma\}$ and $\{z_1, z_2\}$, it is enough to consider only the following pair:

$$\begin{aligned} h_1 &= p_2 \partial_1 \partial_2 + p_3 \partial_2^2 + p_4 \partial_1 + p_5 \partial_2 + p_6, \\ h_2 &= q_2 \partial_1 \partial_2 + q_4 \partial_1 + q_5 \partial_2, \end{aligned} \tag{4.22}$$

where the abbreviated coefficients are

$$\begin{aligned} p_2 &= z_1(1 - z_2), & p_3 &= z_2(1 - z_2), & p_4 &= -\beta' z_1, \\ p_5 &= \gamma - z_2(\alpha + \beta' + 1), & p_6 &= -\alpha\beta', \\ q_2 &= z_1 - z_2, & q_4 &= -\beta', & q_5 &= \beta. \end{aligned} \tag{4.23}$$

Basis. The GKZ system in question has rank 3. The column vector of standard monomials reads

$$\text{Std} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 1 \end{pmatrix}. \tag{4.24}$$

¹²Although this polynomial is not written in the standard form (2.2), it can be shown that F_1 does indeed satisfy a GKZ system.

Macaulay matrix. For brevity, let us only consider the Pfaffian matrix P_1 in the direction z_1 . According to (4.1), our task is to express

$$\partial_1 \circ \text{Std} = \begin{pmatrix} \partial_1^2 \\ \partial_1 \partial_2 \\ \partial_1 \end{pmatrix} \quad (4.25)$$

as a linear combination of Std over the rational function field $\mathbb{Q}(\alpha, \beta, \beta', \gamma)(z_1, z_2)$.

Following the steps of Algorithm 1, we build the Macaulay matrix of degree D (4.16) as follows. First we act with $\text{Der} = \{1, \partial_1, \partial_2\}$, the list of all possible derivatives of degree $D \leq 1$, on the two generators $\{h_1, h_2\}$ from (4.22). Then we bring the ensuing expressions into normally ordered form (see (3.1)) and rearrange the result into a matrix. The rows of the matrix are labeled by the list of operators $\{h_1, \partial_1 h_1, \partial_2 h_2, h_2, \partial_1 h_2, \partial_2 h_2\}$, while the columns correspond to the monomials Mons appearing in the normally ordered form of these operators. Explicitly, we get the Macaulay matrix of degree¹³ $D = 1$

$$\begin{array}{c} h_1 \\ \partial_1 h_1 \\ \partial_2 h_1 \\ h_2 \\ \partial_1 h_2 \\ \partial_2 h_2 \end{array} \begin{bmatrix} \partial_2 \partial_1^2 & \partial_2^2 \partial_1 & \partial_2^3 & \partial_1^2 & \partial_2 \partial_1 & \partial_2^2 & \partial_1 & \partial_2 & 1 \\ 0 & 0 & 0 & 0 & p_2 & p_3 & p_4 & p_5 & p_6 \\ p_2 & p_3 & 0 & p_4 & p_5 + p_{2,1} & p_{3,1} & p_6 + p_{4,1} & p_{5,1} & p_{6,1} \\ 0 & p_2 & p_3 & 0 & p_4 + p_{2,1} & p_5 + p_{3,2} & p_{4,2} & p_6 + p_{5,2} & p_{6,2} \\ 0 & 0 & 0 & 0 & q_2 & 0 & q_4 & q_5 & 0 \\ q_2 & 0 & 0 & q_4 & q_5 + q_{2,1} & 0 & q_{4,1} & q_{5,1} & 0 \\ 0 & q_2 & 0 & 0 & q_4 + q_{2,2} & q_5 & q_{4,2} & q_{5,2} & 0 \end{bmatrix}, \quad (4.26)$$

where $p_{i,j} = \partial_{z_j} p_i$ and $q_{i,j} = \partial_{z_j} q_i$. As dictated by (4.8), we decompose the Macaulay matrix into two blocks: the second block M_{Std} has columns labeled by Std, and the first one M_{Ext} by the rest of the derivatives

$$\text{Ext} = \text{Mons} \setminus \text{Std} = (\partial_2 \partial_1^2, \partial_2^2 \partial_1, \partial_2^3, \partial_1^2, \partial_2 \partial_1, \partial_2^2)^T. \quad (4.27)$$

Explicitly, we obtain the blocks

$$M_{\text{Ext}} = \begin{pmatrix} 0 & 0 & 0 & 0 & z_2^- z_1 & z_2 z_2^- \\ z_2^- z_1 & z_2 z_2^- & 0 & -\beta' z_1 & (-\beta' - \alpha - 2)z_2 + \gamma + 1 & 0 \\ 0 & z_2^- z_1 & z_2 z_2^- & 0 & -\beta' z_1 + z_2^- & (-\beta' - \alpha - 3)z_2 + \gamma + 1 \\ 0 & 0 & 0 & 0 & z_1 - z_2 & 0 \\ z_1 - z_2 & 0 & 0 & -\beta' & \beta + 1 & 0 \\ 0 & z_1 - z_2 & 0 & 0 & -\beta' - 1 & \beta \end{pmatrix}, \quad (4.28)$$

¹³Had we used all three generators of the ideal, it would have been enough to consider the degree $D = 0$ Macaulay matrix. However, the shortened setup (4.22) considered here provides a better showcase for the steps of Algorithm 1.

where $z_2^- := 1 - z_2$, and

$$M_{\text{Std}} = \begin{pmatrix} -\beta' z_1 & (-\beta' - \alpha - 1)z_2 + \gamma & -\alpha\beta' \\ (-\alpha - 1)\beta' & 0 & 0 \\ 0 & (-\alpha - 1)(\beta' + 1) & 0 \\ -\beta' & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.29}$$

Pfaffian matrix. We now turn to the linear system (4.12), (4.13). The coefficient matrices C'_{Ext} and C'_{Std} , defined in (4.10), are obtained by expressing (4.25) in terms of the sets (4.24) and (4.27):

$$\partial_1 \circ \text{Std} = \begin{pmatrix} \partial_1^2 \\ \partial_1 \partial_2 \\ \partial_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_2 \partial_1^2 \\ \partial_2^2 \partial_1 \\ \partial_2^3 \\ \partial_1^2 \\ \partial_2 \partial_1 \\ \partial_2^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \\ 1 \end{pmatrix} \equiv C'_{\text{Ext}} \cdot \text{Ext} + C'_{\text{Std}} \cdot \text{Std}. \tag{4.30}$$

Let $C = (c_{ij} | 1 \leq i \leq 3, 1 \leq j \leq 6)$ be a matrix of unknowns. We require that C satisfies the system (4.2). In this case,

$$C \cdot \begin{pmatrix} h_1 \\ \partial_1 h_1 \\ \partial_2 h_1 \\ h_2 \\ \partial_1 h_2 \\ \partial_2 h_2 \end{pmatrix} + P_1 \cdot \text{Std} = \partial_1 \circ \text{Std}. \tag{4.31}$$

Upon collecting terms proportional to Ext, we arrive at the system (4.12) to be solved with respect to C . By the rank condition (4.18), we note that the degree $D = 1$ Macaulay matrix (4.26) does indeed lead to a solution. Once C is determined, (4.13) yields the Pfaffian matrix

$$P_1 = \begin{pmatrix} \frac{(-(\alpha + \beta - \beta' + 1)z_1 + \gamma - \beta')z_2 + (\alpha + \beta + 1)z_1^2 - \gamma z_1}{(1 - z_1)z_1(z_1 - z_2)} & \frac{-\beta z_2^2 + \beta z_2}{(1 - z_1)z_1(z_1 - z_2)} & \frac{\beta \alpha}{(1 - z_1)z_1} \\ \frac{\beta'}{z_1 - z_2} & \frac{-\beta}{z_1 - z_2} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.32}$$

■

Example 4.3. Consider the matrix

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The solution rank of this GKZ system is $r = 13$, and a standard basis obtained via the method of [77] reads:

$$\text{Std} = (\partial_9 \partial_{11}^2, \partial_9^2, \partial_{10}^2, \partial_8 \partial_{11}, \partial_9 \partial_{11}, \partial_{10} \partial_{11}, \partial_{11}^2, \partial_7, \partial_8, \partial_9, \partial_{10}, \partial_{11}, 1)^T.$$

The Macaulay matrix M_{Ext} of degree $D = 2$ is a 189×113 matrix of rank 113 (over $\mathbb{Z}/65537\mathbb{Z}$). The Macaulay matrix is obtained in **0.558 sec.**¹⁴ It took **0.603 sec** to obtain the numerical Pfaffian matrix in the direction z_6 for fixed numerical values of all z variables and β parameters. ■

Example 4.4. Consider the matrix

$$A_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

whose solution rank is $r = 33$. Our choice of standard basis is

$$\begin{aligned} \text{Std} = & (\partial_{15}^3, \partial_5^3, \partial_5 \partial_6 \partial_{11}, \partial_6^2 \partial_{11}, \partial_3 \partial_6^2, \partial_4 \partial_6^2, \partial_6^3, \partial_{11} \partial_{15}, \partial_{14} \partial_{15}, \partial_{15}^2, \partial_4^2, \\ & \partial_5 \partial_{11}, \partial_2 \partial_5, \partial_3 \partial_5, \partial_4 \partial_5, \partial_5^2, \partial_6 \partial_{11}, \partial_2 \partial_6, \partial_6 \partial_{14}, \partial_3 \partial_6, \partial_6 \partial_{15}, \\ & \partial_4 \partial_6, \partial_5 \partial_6, \partial_6^2, \partial_{11}, \partial_2, \partial_{14}, \partial_3, \partial_{15}, \partial_4, \partial_5, \partial_6, 1)^T. \end{aligned}$$

Using the homogeneity property discussed in section 2.4, we may fix the following variables: $z_1 = z_7 = z_8 = z_9 = z_{10} = z_{12} = z_{13} = 1$. The block M_{Ext} of the Macaulay matrix of degree $D = 2$ turns out to be a sparse 945×958 matrix, whose rank is 534 by the probabilistic method. Therefore, there are many rows that are not needed for solving the system (4.12). Again, using the probabilistic method over the finite field $\mathbb{Z}/65537\mathbb{Z}$, we determine a maximal set of independent rows of the matrix M_{Ext} . We find that a 534×958 matrix N_{Ext} is enough to solve a smaller version of the system (4.12), namely $C'_{\text{Ext}} = C \cdot N_{\text{Ext}}$. As was mentioned in the Smaller Macaulay Matrix paragraph of section 4.3, we can further reduce this new system as follows: since there are exactly 27 independent row vectors in C'_{Ext} , we may choose only a subset of the row vectors in N_{Ext} , whose span includes the independent row vectors of C'_{Ext} .

The Macaulay matrix is then obtained in **15.549 sec.** It took **0.66 sec** to calculate the Pfaffian matrix in the z_6 direction for fixed numerical values of variables z and parameters β . We leave the problem of full functional reconstruction of the Pfaffian matrices of this example for future work. ■

¹⁴Our implementation is in the RISA/ASIR system and the language C. It is executed on a machine with an Intel Core i7-10700K processor of 3.8GHz and 16Gb of RAM. We do not use multicore functionality. All timings are taken on this machine unless specified otherwise.

We have proposed an efficient Macaulay matrix method to construct Pfaffian systems. It can be applied to relatively large systems which are not feasible by Gröbner basis methods in the ring of differential operators. The latter methods are usually feasible for systems of up to, approximately, rank 10.

5 Macaulay matrix method and generalized Feynman integrals

The relation between Feynman integrals and GKZ-hypergeometric functions has been studied in [63, 68, 83, 98, 99]. Here, we show applications of GKZ systems combined with Macaulay matrices to Feynman integrals.

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2 \cdot \mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$. Moreover, fix the exponents of the Euler integral (2.1) to

$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) - (d_0/2, \nu_1, \dots, \nu_n). \tag{5.1}$$

We define a *generalized Feynman integral* as

$$I(d_0, \nu; z) := c(d_0, \nu) f_\Gamma(\beta), \tag{5.2}$$

where

$$f_\Gamma(\beta) := \int_\Gamma \mathcal{G}(z; x)^{\epsilon-d_0/2} x_1^{\nu_1+\epsilon\delta} \dots x_n^{\nu_n+\epsilon\delta} \frac{dx}{x}, \tag{5.3}$$

$$c(d_0, \nu) := \frac{\Gamma(d_0/2 - \epsilon)}{\Gamma((L+1)(d_0/2 - \epsilon) - |\nu| - n\epsilon\delta) \prod_{i=1}^n \Gamma(\nu_i + \epsilon\delta)}, \quad |\nu| := \nu_1 + \dots + \nu_n. \tag{5.4}$$

The polynomial \mathcal{G} takes the form

$$\mathcal{G}(z; x) = \sum_{i=1}^N z_i x^{\alpha_i}, \quad \alpha_i \in \mathbb{N}_0^n, \tag{5.5}$$

and we set $a_i := (1, \alpha_i) \in \mathbb{N}_0^{n+1}$ as usual. The introduction of the δ parameter in (5.1) follows from the analytic regularization of [100, 101]

Using the notation of equations (2.27) and (2.28), we can equally well define the generalized Feynman integral as a pairing between a twisted cycle Γ and a twisted form

$$\omega_{d_0/2, \nu} := c(d_0, \nu) \times \mathcal{G}(z; x)^{-d_0/2} x^\nu \frac{dx}{x}. \tag{5.6}$$

In particular, we have

$$I(d_0, \nu; z) = \langle \omega_{d_0/2, \nu} \rangle_\Gamma. \tag{5.7}$$

The generalized Feynman integral reduces to the Lee-Pomeransky (LP) representation [54] of an L -loop integral in $d = d_0 - 2\epsilon$ dimensions, with propagator powers ν_i when:

$$\Gamma = (0, +\infty)^n, \quad \delta \rightarrow 0, \tag{5.8}$$

and

$$z_i \in \mathbb{N} \vee z_i \in \text{kinematic variables } (m_i^2, p_i^2, p_i \cdot p_j), \tag{5.9}$$

where m_i and p_i stand for masses and external momenta, so that the polynomial \mathcal{G} becomes the LP polynomial built from Symanzik (or graph) polynomials \mathcal{U}, \mathcal{F} , as $\mathcal{G} = \mathcal{U} + \mathcal{F}$.¹⁵

Let us observe that within the LP polynomial $\mathcal{G}(z; x)$, each z_i may not necessarily be independent from each other. This is different from the polynomial $g(z; x)$ appearing in the Euler integral $f_\Gamma(z)$, for which each z_i is considered independent. Their independence ensures a non-degenerate correspondence between the monomials in x_i variables and the partial differential operators in the z_i variables — a crucial property to establish the isomorphism between twisted de Rham cohomology group and \mathcal{D} -modules. This observation implies that Feynman integrals are *restrictions* of GKZ integrals, obtained from the latter by choosing suitable values of the variables z_i [70, 74, 80–83]. We note that the z_i limits in (5.9) are generically not smooth at the level of the Pfaffian system for a GFI. In a forthcoming publication [105], we show that it is nevertheless possible to construct the Pfaffian system with all z_i variables identified with their proper kinematic counterparts.

5.1 Massless one-loop diagrams

Let us consider a massless one-loop diagram G with n external legs and n internal edges. We use the symbol p_i for each external momentum and x_i for the Schwinger parameter of the i -th edge. The first and the second Symanzik polynomials read

$$\mathcal{U}_G = x_1 + \dots + x_n, \tag{5.10}$$

$$\mathcal{F}_G = \sum_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2), \dots, (n-1,n), (1,n)}} \sigma_{ij} x_i x_j, \tag{5.11}$$

where we set $\sigma_{ij} := -(p_i + \dots + p_{j-1})^2$. Since the Lee-Pomeransky polynomial $\mathcal{G} = \mathcal{U}_G + \mathcal{F}_G$ consists of $\binom{n}{2}$ terms, it gives rise to an $(n+1) \times \binom{n}{2}$ A -matrix. In view of the discussion of section 2.4, the number of reduced variables is

$$\binom{n}{2} - n - 1 = \frac{n(n-3)}{2}, \tag{5.12}$$

which coincides with the number of independent Mandelstam variables s_{ij} (as was also noted in [70]). In sum, we conclude that the master integrals for G are precisely subject to the GKZ system of PDEs.

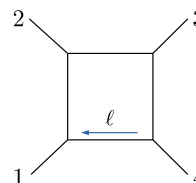
Moreover, the discussion above can be easily extended to the cases when external masses $m_k^2 = p_k^2$ are non-zero but treated as independent variables. For example, in the case of the one-mass n -gon diagram, all external masses are zero but one, say $p_k^2 \neq 0$, therefore a new term $-p_k^2 x_k x_{k+1}$ is added to the second Symanzik polynomial (5.11). The number of reduced variables, in this case, reads $\frac{n(n-3)}{2} + 1$ which is equal to $\{\sigma_{ij}\} \cup \{p_k^2\}$.

¹⁵The limit $\delta \rightarrow 0$ of a generalized Feynman integral may yield to ill-defined expressions of the form $\frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_i)}$, when $\exists \nu_i \in \mathbb{Z}_{\leq 0}$. In these cases, we adopt the replacement $\frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_i)} \rightarrow \partial_i^{-\nu_i} (\delta_D(x_i))$, with $\delta_D(\bullet)$ being the Dirac delta function, as proposed in [102–104].

5.2 Example: One-loop box

In the following two sections, we illustrate the Macaulay matrix method for obtaining Pfaffian systems for Feynman integrals. We have chosen examples where the number of GKZ variables correspond to number of independent kinematic variables. The general case will be investigated in [105].

Setup. Let us first derive the Pfaffian system for the one-loop massless box diagram. This example was studied in Lee-Pomeransky representation using twisted cohomology in [43]. The kinematics are



$$\sum_{i=1}^4 p_i = 0, \quad p_1^2 = \dots = p_4^2 = 0, \quad s = 2p_1 \cdot p_2, \quad t = 2p_2 \cdot p_3. \quad (5.13)$$

The inverse propagators in momentum space are given by¹⁶

$$D_1 = -\ell^2, \quad D_2 = -(\ell - p_1)^2, \quad D_3 = -(\ell - p_1 - p_2)^2, \quad D_4 = -(\ell - p_1 - p_2 - p_3)^2. \quad (5.14)$$

The corresponding generalized Feynman integral reads

$$I(d_0, \nu; z) = c(d_0, \nu) (-s)^{d_0/2 - \epsilon - |\nu| - 4\epsilon\delta} \int_{\Gamma} \mathcal{G}(z; x)^{\epsilon - d_0/2} x_1^{\nu_1 + \epsilon\delta} \dots x_4^{\nu_4 + \epsilon\delta} \frac{dx}{x}, \quad (5.15)$$

where

$$\mathcal{G}(z; x) = \sum_{i=1}^4 z_i x_i + z_5 x_1 x_3 + z_6 x_2 x_4, \quad \text{with} \quad A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \quad (5.16)$$

The s -dependent prefactor in (5.15) comes from rescaling integration variables as $x_i \rightarrow x_i/(-s)$. Within the LP-representation, the coefficients z_i in (5.16) take the explicit form

$$z_1 = \dots = z_5 = 1, \quad z_6 = \frac{t}{s} =: z, \quad (5.17)$$

leaving just one external kinematic variable z . This counting is compatible with the number of GKZ variables after using the homogeneity property discussed in section 2.4. Indeed, out of the $N = 6$ variables z_i appearing in the generalized Feynman integral, $n + 1 = 4 + 1 = 5$ of them can be rescaled, thereby yielding $z_1 = \dots = z_5 = 1$ and $z_6 = z$.

Basis. Following [11], we choose the canonical basis

$$e = (-s)^\epsilon \left(z(-s)I(4, 0, 1, 0, 2), (-s)I(4, 1, 0, 2, 0), \epsilon z(-s)^2 I(4, 1, 1, 1, 1) \right) \quad (5.18)$$

corresponding to the cohomology basis (see (5.6))

$$e^{(\text{dR})} = \Lambda \cdot \left(\frac{x_4}{x_1 x_3 \mathcal{G}(z; x)^2} dx, \frac{x_3}{x_2 x_4 \mathcal{G}(z; x)^2} dx, \frac{1}{\mathcal{G}(z; x)^2} dx \right), \quad (5.19)$$

¹⁶Here $\ell^2 = \ell_0^2 - \ell_1^2 - \dots - \ell_4^2$.

where Λ is a diagonal matrix containing prefactors:

$$\Lambda_{11} = \frac{(-s)^{-4\epsilon\delta} z \Gamma(2 - \epsilon)}{\Gamma(1 - 2\epsilon - 4\epsilon\delta) \Gamma(\epsilon\delta)^2 \Gamma(1 + \epsilon\delta) \Gamma(2 + \epsilon\delta)}, \quad (5.20a)$$

$$\Lambda_{22} = \frac{(-s)^{-4\epsilon\delta} \Gamma(2 - \epsilon)}{\Gamma(1 - 2\epsilon - 4\epsilon\delta) \Gamma(\epsilon\delta)^2 \Gamma(1 + \epsilon\delta) \Gamma(2 + \epsilon\delta)}, \quad (5.20b)$$

$$\Lambda_{33} = \frac{\epsilon(-s)^{-4\epsilon\delta} z \Gamma(2 - \epsilon)}{\Gamma(-2\epsilon - 4\epsilon\delta) \Gamma(1 + \epsilon\delta)^4}. \quad (5.20c)$$

Applying the correspondence in (2.34) (or rather (A.10), which employs homogeneity) to the basis $e^{(\text{dR})}$, we obtain the \mathcal{D} -module basis

$$e^{(\mathcal{D})} = \Lambda \cdot \left(\frac{\delta\epsilon - 1}{(\epsilon - 1)\epsilon} \partial - \frac{z}{(\epsilon - 1)\epsilon} \partial^2, \right. \\ \left. \frac{(1 - 3\delta)(4\delta - 1)\epsilon}{\epsilon - 1} + \frac{z(7\delta\epsilon - 2\epsilon - 1)}{(\epsilon - 1)\epsilon} \partial - \frac{z^2}{(\epsilon - 1)\epsilon} \partial^2, \right. \\ \left. \frac{4\delta\epsilon - \epsilon - 1}{(\epsilon - 1)\epsilon} \partial - \frac{z}{(\epsilon - 1)\epsilon} \partial^2 \right), \quad (5.21)$$

where $\partial := \partial_z$.

Macaulay and Pfaffian matrices. Using the Macaulay matrix method, we construct the Pfaffian system in the basis of standard monomials,

$$\partial \circ \begin{pmatrix} \partial^2 \\ \partial \\ 1 \end{pmatrix} = \begin{pmatrix} \partial^3 \\ \partial^2 \\ \partial \end{pmatrix} = P^{(\text{Std})} \cdot \begin{pmatrix} \partial^2 \\ \partial \\ 1 \end{pmatrix}. \quad (5.22)$$

Afterwards, we perform a gauge transformation to the basis $e^{(\mathcal{D})}$, so that P can be read off the equation $\partial \circ e^{(\mathcal{D})} = P \cdot e^{(\mathcal{D})}$.

The Macaulay data for the system (5.22) is found to be

$$M_{\text{Ext}} = \left(z^2(z + 1) \right), \quad (5.23a)$$

$$M_{\text{Std}}^T = \begin{pmatrix} z((6\delta + 2)\epsilon + z(6\delta\epsilon + \epsilon + 3) + 3) \\ (3\delta\epsilon + \epsilon + 1)^2 + z((9\delta + 2)\epsilon^2\delta + 6\delta\epsilon + \epsilon + 1) \\ (4\delta + 1)\epsilon^3\delta^2 \end{pmatrix}, \quad (5.23b)$$

$$C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C'_{\text{Ext}} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad C'_{\text{Std}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.23c)$$

$$\text{Mons} = \text{Ext} \sqcup \text{Std} = \{\partial^3\} \sqcup \{\partial^2, \partial, 1\}. \quad (5.23d)$$

Setting $C = (c_{11} \ c_{12} \ c_{13})$ and solving $C'_{\text{Ext}} = C \cdot M_{\text{Ext}}$, we obtain

$$C = \left(\frac{1}{z^2(z+1)} \ 0 \ 0 \right). \quad (5.24)$$

The Pfaffian matrix is then

$$P^{(\text{Std})} = C'_{\text{Std}} - C \cdot M_{\text{Std}} \tag{5.25}$$

$$= \begin{pmatrix} P_{11}^{(\text{Std})} & P_{12}^{(\text{Std})} & P_{13}^{(\text{Std})} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.26}$$

where

$$P_{11}^{(\text{Std})} = -\frac{\epsilon((z+1)6\delta + z + 2) + 3(z+1)}{z(z+1)}, \tag{5.27a}$$

$$P_{12}^{(\text{Std})} = -\frac{(3\delta\epsilon + \epsilon + 1)^2 + z((9\delta + 2)\epsilon^2\delta + 6\delta\epsilon + \epsilon + 1)}{z^2(z+1)}, \tag{5.27b}$$

$$P_{13}^{(\text{Std})} = -\frac{(4\delta + 1)\epsilon^3\delta^2}{z^2(z+1)}. \tag{5.27c}$$

Finally, according to the algorithm of section 3.2, it is possible to build a suitable gauge transformation matrix G such that

$$P = (\partial G + G \cdot P^{(\text{Std})}) \cdot G^{-1} \tag{5.28}$$

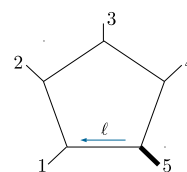
$$= \epsilon \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{z(z+1)} & \frac{2}{z+1} & -\frac{1}{z(z+1)} \end{pmatrix}, \tag{5.29}$$

where the limit $\delta \rightarrow 0$ has been taken. The matrix (5.29) is canonical and in agreement with LITERED. ■

5.3 Example: One-loop pentagon

Setup. We consider a one-loop massless pentagon integral with one massive leg. The kinematics are

$$\sum_{i=1}^5 p_i = 0, \quad p_1^2 = \dots = p_4^2 = 0, \quad p_5^2 := p^2, \quad s_{ij} := 2p_i \cdot p_j. \tag{5.30}$$



Note that the identity $(p_1 + p_2 + p_3 + p_4)^2 = (-p_5)^2 = p^2$ imposes a relation on the Mandelstam variables s_{ij} . The propagators are given by

$$\begin{aligned} D_1 &= -\ell^2, & D_2 &= -(\ell - p_1)^2, & D_3 &= -(\ell - p_1 - p_2)^2, \\ D_4 &= -(\ell - p_1 - p_2 - p_3)^2, & D_5 &= -(\ell - p_1 - p_2 - p_3 - p_4)^2. \end{aligned} \tag{5.31}$$

We can write the generalized Feynman integral as

$$I(d_0, \nu; z) = c(d_0, \nu) (-s_{12})^{d_0/2 - \epsilon - |\nu| - 5\epsilon\delta} \int_0^\infty \mathcal{G}(z; x)^{\epsilon - d_0/2} x_1^{\nu_1 + \epsilon\delta} \dots x_5^{\nu_5 + \epsilon\delta} \frac{dx}{x}, \tag{5.32}$$

with

$$\mathcal{G}(z; x) = \sum_{i=1}^5 z_i x_i + z_6 x_1 x_3 + z_7 x_1 x_4 + z_8 x_1 x_5 + z_9 x_2 x_4 + z_{10} x_2 x_5 + z_{11} x_3 x_5, \tag{5.33}$$

and the corresponding A matrix A_5 is given in Example 4.3.

The s_{12} -dependent prefactor in (5.32) comes from rescaling integration variables by $x_i \rightarrow x_i/(-s_{12})$. In the Lee-Pomeransky representation, the monomial coefficients are given by

$$\begin{aligned}
 z_1 &= \dots = z_6 = 1, \\
 z_7 &= 1 + y_2 + y_4, \\
 z_8 &= y_1, \\
 z_9 &= y_4, \\
 z_{10} &= -1 + y_1 - y_2 - y_3, \\
 z_{11} &= -1 + y_1 - y_2 - y_3 - y_4 - y_5,
 \end{aligned}
 \tag{5.34}$$

where the y_i are ratios of kinematic variables:

$$y_1 = \frac{p^2}{s_{12}}, \quad y_2 = \frac{s_{13}}{s_{12}}, \quad y_3 = \frac{s_{14}}{s_{12}}, \quad y_4 = \frac{s_{23}}{s_{12}}, \quad y_5 = \frac{s_{24}}{s_{12}}.
 \tag{5.35}$$

The Lee-Pomeransky polynomial hence contains 5 monomial coefficients different from unity. Similar to the previous example, we can obtain equally many non-unity GKZ variables via homogeneity. In particular, we start with $N = 11$ generic variables z_i , after which we rescale $n + 1 = 5 + 1 = 6$ GKZ variables as $z_1 = \dots = z_6 = 1$, leaving us with 5 variables z_7, \dots, z_{11} .

Basis. Solving an IBP system numerically, we can swiftly identify a set of master integrals. Simplifying notation to $I(d_0, \nu)$, we choose

$$\begin{aligned}
 e_1 &= (-s_{12})^\epsilon I(2, 0, 0, 1, 0, 1) & e_2 &= (-s_{12})^\epsilon I(2, 0, 1, 0, 0, 1) \\
 e_3 &= (-s_{12})^\epsilon I(2, 0, 1, 0, 1, 0) & e_4 &= (-s_{12})^\epsilon I(2, 1, 0, 0, 0, 1) \\
 e_5 &= (-s_{12})^\epsilon I(2, 1, 0, 0, 1, 0) & e_6 &= (-s_{12})^\epsilon I(2, 1, 0, 1, 0, 0) \\
 e_7 &= \epsilon(-s_{12})^\epsilon I(4, 1, 0, 1, 0, 1) & e_8 &= \epsilon(-s_{12})^{\epsilon+1} I(4, 0, 1, 1, 1, 1) \\
 e_9 &= \epsilon(2\epsilon - 1)(-s_{12})^\epsilon I(6, 1, 0, 1, 1, 1) & e_{10} &= \epsilon(-s_{12})^{\epsilon+1} I(4, 1, 1, 0, 1, 1) \\
 e_{11} &= \epsilon(-s_{12})^{\epsilon+1} I(4, 1, 1, 1, 0, 1) & e_{12} &= \epsilon(-s_{12})^{\epsilon+1} I(4, 1, 1, 1, 1, 0) \\
 e_{13} &= \epsilon^2(-s_{12})^{\epsilon+1} I(6, 1, 1, 1, 1, 1).
 \end{aligned}
 \tag{5.36}$$

The dimensions d_0 are chosen so as to have non-negative r -vectors, as per the ending remark of section 2.3. Using (2.34) (or formula (A.10) employing homogeneity), we find the \mathcal{D} -module basis corresponding to e . We write $e^{(\mathcal{D})} = \Lambda \cdot e'^{(\mathcal{D})}$ for a diagonal matrix Λ_{ij} containing prefactors,

$$\Lambda_{ii} := c(d_0^{(i)}, \nu^{(i)}) (-s_{12})^{d_0^{(i)}/2 - \epsilon - |\nu^{(i)}| - 5\epsilon\delta} \times \Lambda'_i,
 \tag{5.37}$$

where $(d_0^{(i)}, \nu^{(i)})$ refers to the indices of the i -th integral in (5.36) and¹⁷

$$\begin{aligned}
 \Lambda'_1 &= (-s_{12})^\epsilon \frac{1}{\epsilon}, & \Lambda'_2 &= (-s_{12})^\epsilon \frac{1}{\epsilon}, \\
 \Lambda'_3 &= (-s_{12})^\epsilon \frac{1}{\epsilon}, & \Lambda'_4 &= (-s_{12})^\epsilon \frac{1}{\epsilon}, \\
 \Lambda'_5 &= (-s_{12})^\epsilon \frac{1}{\epsilon}, & \Lambda'_6 &= (-s_{12})^\epsilon \frac{-1}{\epsilon}, \\
 \Lambda'_7 &= \epsilon(-s_{12})^\epsilon \frac{1}{(\epsilon-1)\epsilon}, & \Lambda'_8 &= \epsilon(-s_{12})^{\epsilon+1} \frac{1}{(\epsilon-1)\epsilon}, \\
 \Lambda'_9 &= \epsilon(2\epsilon-1)(-s_{12})^\epsilon \frac{-1}{(\epsilon-2)(\epsilon-1)\epsilon}, & \Lambda'_{10} &= \epsilon(-s_{12})^{\epsilon+1} \frac{1}{(\epsilon-1)\epsilon}, \\
 \Lambda'_{11} &= \epsilon(-s_{12})^{\epsilon+1} \frac{-1}{(\epsilon-1)\epsilon}, & \Lambda'_{12} &= \epsilon(-s_{12})^{\epsilon+1} \frac{-1}{(\epsilon-1)\epsilon}, \\
 \Lambda'_{13} &= \epsilon^2(-s_{12})^{\epsilon+1} \frac{-1}{(\epsilon-2)(\epsilon-1)\epsilon} & & .
 \end{aligned} \tag{5.38}$$

Moreover,

$$\begin{aligned}
 e_1^{(\mathcal{D})} &= \partial_{11}, \quad e_2^{(\mathcal{D})} = \partial_{10}, \quad e_3^{(\mathcal{D})} = \partial_9, \quad e_4^{(\mathcal{D})} = \partial_8, \quad e_5^{(\mathcal{D})} = \partial_7 \\
 e_6^{(\mathcal{D})} &= \epsilon(5\delta+1) + z_7\partial_7 + z_8\partial_8 + z_9\partial_9 + z_{10}\partial_{10} + z_{11}\partial_{11} \\
 e_7^{(\mathcal{D})} &= (4\delta\epsilon + \epsilon + 1)\partial_{11} + z_{11}\partial_{11}^2 + z_9\partial_9\partial_{11} + z_{10}\partial_{10}\partial_{11} \\
 e_8^{(\mathcal{D})} &= \partial_9\partial_{11} \\
 e_9^{(\mathcal{D})} &= \delta\epsilon(4\delta\epsilon+1)\partial_{11} + \delta z_{11}\epsilon\partial_{11}^2 + z_7(4\delta\epsilon + \epsilon + 1)\partial_7\partial_{11} + z_9(5\delta\epsilon + \epsilon + 2)\partial_9\partial_{11} + \delta z_{10}\epsilon\partial_{10}\partial_{11} \\
 &\quad + z_7z_{11}\partial_7\partial_{11}^2 + z_9z_{11}\partial_9\partial_{11}^2 + z_9^2\partial_9^2\partial_{11} + z_7z_9\partial_7\partial_9\partial_{11} + z_7z_{10}\partial_7\partial_{10}\partial_{11} + z_9z_{10}\partial_9\partial_{10}\partial_{11} \\
 e_{10}^{(\mathcal{D})} &= \partial_7\partial_{10} \\
 e_{11}^{(\mathcal{D})} &= (5\delta\epsilon + \epsilon + 1)\partial_{10} + z_{10}\partial_{10}^2 + z_7\partial_7\partial_{10} + z_8\partial_8\partial_{10} + z_9\partial_9\partial_{10} + z_{11}\partial_{11}\partial_{10} \\
 e_{12}^{(\mathcal{D})} &= (5\delta\epsilon + \epsilon + 1)\partial_9 + z_9\partial_9^2 + z_7\partial_7\partial_9 + z_8\partial_8\partial_9 + z_{10}\partial_{10}\partial_9 + z_{11}\partial_{11}\partial_9 \\
 e_{13}^{(\mathcal{D})} &= (4\delta\epsilon + \epsilon + 2)\partial_{11}\partial_9 + z_9\partial_{11}\partial_9^2 + z_{10}\partial_{10}\partial_{11}\partial_9 + z_{11}\partial_{11}^2\partial_9.
 \end{aligned} \tag{5.39}$$

Macaulay and Pfaffian matrices. The Macaulay matrix M_{Ext} has dimensions 189×113 , so in this case there are 113 exterior monomials Ext. The basis of 13 standard monomials is given by

$$e^{(\text{Std})} = (\partial_9\partial_{11}^2, \partial_9^2, \partial_{10}^2, \partial_8\partial_{11}, \partial_9\partial_{11}, \partial_{10}\partial_{11}, \partial_{11}^2, \partial_7, \partial_8, \partial_9, \partial_{10}, \partial_{11}, 1)^T \tag{5.40}$$

We identify 133 independent rows of M_{Ext} by row reducing it numerically. We therefore construct an unknown matrix C of dimensions 13×113 which must satisfy $C'_{\text{Ext},i} - C \cdot N_{\text{Ext}} =$

¹⁷The numerators in Λ'_i come from (5.36), while the denominators arise due to taking derivatives of \mathcal{G}^c when constructing P such that $P\langle\omega_0\rangle = \langle\omega_q\rangle$, in the notation of (2.34).

0, where N_{Ext} contains the 133 independent rows of M_{Ext} . This linear system can be solved in reasonable time on a laptop using FINITEFLOW [88]. The Pfaffian matrix in direction i then follows from $C'_{\text{Std},i} - C \cdot N_{\text{Std}} = P_i^{(\text{Std})}$, where N_{Std} is built from M_{Std} by taking the same rows as in N_{Ext} .

Using the basis change algorithm outlined in section 3.2, we perform a gauge transformation to relate $P^{(\text{Std})}$ to $P_i^{(e)}$, written in the basis e . At this stage, we may safely take the limit $\delta \rightarrow 0$. We have verified that the resulting Pfaffian matrices are in agreement with LITERED [89, 90]. ■

A similar analysis can be extended to diagrams with either more legs or more loops. For the purposes of this first work on this subject, we limited the application to one-loop integrals up to six external legs. In fact, Example 4.4 refers to the one-loop massless hexagon. Application to higher-loop cases will be discussed in future works.

6 Linear relations for generalized Feynman integrals

Let $e = (I_1, \dots, I_r)$ be a basis of master integrals, where each I_i is a generalized Feynman integral of the form (5.2), i.e depending on generic variables z_i . The vector e will be regarded as the column vector in the sequel; when the distinction between row and column vectors is clear from context, we will omit the transpose symbol \bullet^T .

Denote by $e' = (I'_1, \dots, I'_{r'})$ a set of integrals in the same family as e . In both e and e' , we replace the integrand factors $x^{\nu+\epsilon\delta}$ by $x_1^{\nu_1+\epsilon\delta_1} \dots x_n^{\nu_n+\epsilon\delta_n}$ for a set of new parameters δ_i .

Theorem 6.1.

1. *There exists a matrix $U \in \mathbb{Q}^{r' \times r}(z)$ such that*

$$e' = U \cdot e. \tag{6.1}$$

2. *Let $L(e') = (\ell_1 I'_1, \dots, \ell_{r'} I'_{r'})$, where the ℓ_i are differential operators w.r.t. z with rational function coefficients. There exists a matrix $V \in \mathbb{Q}^{r' \times r}(z)$ such that*

$$L(e') = V \cdot e. \tag{6.2}$$

Moreover, there are construction algorithms for U and V .

Remark 6.2. This theorem can be regarded as an analogue of IBP relations for generalized Feynman integrals. Recall the rank r for generalized Feynman integrals is possibly larger than for conventional Feynman integrals. However, once we set the GKZ variables z_i equal to their physically relevant values in e.g. $e' = U \cdot e$, some of the master integrals in e might vanish or become equal, in which case we arrive at a conventional IBP relation.

Although the theorem can be proved by a method analogous to Algorithm 1 of [62], here we present a different approach based on Pfaffian matrices and the matrix factorial [84]. We begin by constructing recurrence relations in the general framework of GKZ hypergeometric systems, after which we specialize to the case of generalized Feynman integrals.

6.1 Recurrence relations for GKZ systems

For convenience, in this section we will work with a rescaled version of the Euler integral (2.1):

$$f_{\Gamma}(z) \rightarrow f(\beta) := \frac{1}{\Gamma(\beta_0 + 1)} \int_{\Gamma} g(z; s)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{dx}{x}, \quad (6.3)$$

where β is assumed to be non-resonant. We have the relation $\partial_i f(\beta) = f(\beta - a_i)$ with this rescaling. Since z and Γ will stay fixed in what follows, we only emphasize the dependence on β in $f(\beta)$. The integrals in e and e' are constant multiples of $f(\beta)$ with

$$\beta = (\epsilon, -\delta_1 \epsilon, \dots, -\delta_n \epsilon) - (d_0/2, \nu). \quad (6.4)$$

Let $s = \{s_1, \dots, s_r\}$ be a basis of $\mathcal{D}_N/H_A(\beta)$ consisting of monomials in ∂_i . The vector $F(\beta) = (s_1 f(\beta), \dots, s_r f(\beta))$ is then a solution to a Pfaffian system. Now, because $\partial_i s_j = s_j \partial_i$ and $\partial_i f(\beta) = f(\beta - a_i)$ we have that

$$\partial_i F(\beta) = P_i(\beta) F(\beta) = F(\beta - a_i). \quad (6.5)$$

In other words, $P_i(\beta)$ yields a difference equation for $F(\beta)$. Moreover, since β is non-resonant, the matrix $P_i(\beta + a_i)$ is invertible, in which case we also have

$$Q_i(\beta) F(\beta) = F(\beta + a_i), \quad Q_i(\beta) := P_i(\beta + a_i)^{-1}. \quad (6.6)$$

To make the analogy clear between equations (6.5) and (6.6), let us introduce the operator ∂_i^{-1} acting as

$$\partial_i^{-1} F(\beta) = Q_i(\beta) F(\beta) = F(\beta + a_i). \quad (6.7)$$

We define the *falling matrix factorial* as

$$[P_i(\beta)]_j := P_i(\beta - (j - 1)a_i) \cdots P_i(\beta - a_i) P_i(\beta), \quad j > 0, \quad (6.8)$$

and the *rising matrix factorial* as

$$(Q_i(\beta))_j := Q_i(\beta + (j - 1)a_i) \cdots Q_i(\beta + a_i) Q_i(\beta), \quad j > 0. \quad (6.9)$$

These matrix products can be swiftly computed using rational reconstruction over finite fields.

When the subscript j is replaced by an integer vector $\kappa \in \mathbb{N}_0^N$, we extend the definition of matrix factorials¹⁸ to:

$$[P(\beta)]_{\kappa} := \prod_{i=1}^N \left[P_i \left(\beta - \sum_{j=i+1}^N \kappa_j a_j \right) \right]_{\kappa}, \quad (6.10)$$

$$(Q(\beta))_{\kappa} := \prod_{i=1}^N \left(Q_i \left(\beta - \sum_{j=i+1}^N \kappa_j a_j \right) \right)_{\kappa}. \quad (6.11)$$

¹⁸When $i = N$ in the product we use the convention $\sum_{j=N+1}^N \kappa_j a_j = 0$.

Lemma 6.3. For $\kappa \in \mathbb{N}_0^N$ we have the recurrence relations

$$F\left(\beta - \sum_{i=1}^N \kappa_i a_i\right) = [P_i(\beta)]_{\kappa} F(\beta) = \partial^{\kappa} F(\beta), \quad (6.12)$$

$$F\left(\beta + \sum_{i=1}^N \kappa_i a_i\right) = (Q_i(\beta))_{\kappa} F(\beta) = \partial^{-\kappa} F(\beta). \quad (6.13)$$

Proof. By induction, let us derive that

$$\partial_i^j F(\beta) = F(\beta - j a_i) = [P_i(\beta)]_j F(\beta). \quad (6.14)$$

The case $j = 1$ is shown in equation (6.5). Suppose that (6.14) holds for $j - 1$. Then we have

$$\partial_i \left(\partial_i^{j-1} F(\beta) \right) = \partial_i F(\beta - (j-1)a_i) \quad (6.14) \quad (6.15a)$$

$$= P_i(\beta - (j-1)a_i) F(\beta - (j-1)a_i) \quad (\text{Pfaffian equation}) \quad (6.15b)$$

$$= P_i(\beta - (j-1)a_i) [P_i(\beta)]_{j-1} F(\beta) \quad (6.14) \quad (6.15c)$$

$$= [P_i(\beta)]_j F(\beta) \quad (6.10). \quad (6.15d)$$

Applying (6.14) to each $(\kappa)_j$, we obtain the relation (6.12) for the falling matrix factorial. The case of the rising matrix factorial (6.13) is proved in a similar fashion. \square

Remark 6.4. Comparing $F((\beta - a_i) - a_j) = P_i(\beta - a_j) P_j(\beta) F(\beta)$ to $F((\beta - a_j) - a_i) = P_j(\beta - a_i) P_i(\beta) F(\beta)$, we note that the commutation relation $P_i(\beta - a_j) P_j(\beta) = P_j(\beta - a_i) P_i(\beta)$ holds. The explicit matrix factorial representation of (6.12) is hence not unique. The same statement holds for (6.13).

Observing that $(F(\beta))_i = s_i f(\beta)$, we can write $F(\beta)$ as

$$F(\beta) = (f(\beta - A \cdot k_1), \dots, f(\beta - A \cdot k_r)), \quad (6.16)$$

where the vectors $k_i \in \mathbb{N}_0^N$ are fixed according to $s_i = \partial^{k_i}$. Suppose that we want to obtain a recurrence relation for $f(\beta - A \cdot k_0)$ given some choice of $k_0 \in \mathbb{Z}^n$. Notice that $f(\beta - A \cdot k_0)$ is the first element of $F(\beta - A \cdot (k_0 - k_1))$. We propose to obtain the recurrence relation by Algorithm 2, whose correctness follows from Lemma 6.3. Note that the algorithm does not perform differentiation with respect to z_i , meaning that it can still produce recurrence relations when the z_i are fixed to numbers in the Pfaffian matrices.

6.2 Recurrence relations for generalized Feynman integrals

We can employ Algorithm 2 to find relations among generalized Feynman integrals by specializing β to (6.4). The matrices $Q_i = P_i^{-1}$ exist for this generic choice of ϵ and δ_i 's. In certain cases, we can even take a limit $\delta_i \rightarrow 0$ to obtain conventional IBP relations as we observe in the example below.

Algorithm 2 : Recurrence relation by Macaulay matrix

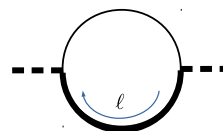
Input: Vector $k_0 \in \mathbb{Z}^N$, indeterminate vector β , monomial basis $s = \{s_1, \dots, s_r\}$ where $s_i = \partial^{k_i}$, $k_i \in \mathbb{N}_0^N$.

Output: Recurrence relation $f(\beta - A \cdot k_0) = \sum_{i=1}^r u_i(\beta) f(\beta - A \cdot k_i)$, $u_i \in \mathbb{Q}(\beta)$

- 1: Construct $P_i(\beta)$, $i = 1, \dots, r$, w.r.t. basis s by calling Algorithm 1
 - 2: Decompose $k_0 - k_1 = \kappa^+ - \kappa^- =: \kappa$ where $\kappa^\pm \in \mathbb{N}_0^N$.
 - 3: Compute the matrix factorial $[P(\beta - \sum_{\kappa_j < 0} \kappa_j a_j)]_{\kappa^+} (Q(\beta))_{\kappa^-} F(\beta)$
 - 4: Output the first element of step 3.
-

6.3 Example: One-loop bubble

Setup. We use Algorithm 2 to obtain a linear relation for the one-loop bubble integral with one massive propagator (see also Example 1.1 of [74]), whose denominators are defined as



$$D_1 = -\ell^2 + m^2, \quad D_2 = -(\ell + p)^2. \tag{6.17}$$

From the Lee-Pomeransky representation we read off

$$\mathcal{G}(z; x) = z_1 x_1 + z_2 x_2 + z_3 x_1 x_2 + z_4 x_1^2, \quad z = (1, 1, m^2 - p^2, m^2), \tag{6.18}$$

and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{6.19}$$

We pick master integrals $e = (I(4, 1, 1), I(4, 2, 0))$ and wish to decompose $e' = (I(4, 1, 2))$. For this diagram, the choice $\delta_1 = \delta_2 := \delta$ is allowed.

In order to apply Algorithm 2, we require k_i corresponding to a relevant set of standard monomials. We use auxiliary vectors k'_i and α such that k_i correspond to master integrals e . Their definitions are given by (6.28) and (6.29) in the proof of Theorem 6.1.

Step 0: Input. The master integrals correspond to the choice

$$k'_1 = (1, 1, 0, 0) \quad \text{s.t.} \quad A \cdot k'_1 = (2, 1, 1), \tag{6.20}$$

$$k'_2 = (2, 0, 0, 0) \quad \text{s.t.} \quad A \cdot k'_2 = (2, 2, 0). \tag{6.21}$$

Given that these k'_i only have positive entries, we have $\alpha = (0, 0, 0, 0)$ using (6.29). Then the $A \cdot \alpha$ terms drop, and we have

$$k_0 = (0, 1, 1, 0) \quad \text{s.t.} \quad A \cdot k_0 = (2, 1, 2), \tag{6.22}$$

$$\beta = \beta' \quad \text{with} \quad \beta' = (\epsilon, -\epsilon\delta, -\epsilon\delta), \tag{6.23}$$

using notation defined in (6.30). The vector k_0 represents the integral we want to reduce, $I(4, 1, 2)$.

Steps 1 and 2. Calling Algorithm 1, we obtain Pfaffian matrices P_i , $i = 1, \dots, 4$. Moreover, we decompose

$$k_0 - k_1 = (-1, 0, 1, 0) \tag{6.24a}$$

$$= (0, 0, 1, 0) - (1, 0, 0, 0) \tag{6.24b}$$

$$= \kappa_+ - \kappa_- := \kappa. \tag{6.24c}$$

Note that $k_1 = k'_1 + \alpha = k'_1$ in this case.

Step 3. Next, we are instructed to compute the matrix factorial corresponding to the operator $\partial_3^{+1}\partial_1^{-1}$, where the ∂_3 stems from κ_+ and the ∂_1 stems from κ_- . Explicitly, we calculate

$$P_3(\beta + a_1)Q_1(\beta). \tag{6.25}$$

Step 4: Output. The recurrence relations follow from the 1st element of the matrix factorial above, apart from an adjustment of Γ -prefactors. This adjustment is due to the constants c and $c^{(i)}$ in equations (6.32) and (6.33c).

Given all of the above, Algorithm 2 produces the result

$$\begin{aligned} I(4, 1, 2) &= \frac{(-1)((4\epsilon\delta^2 + (6\epsilon - 2)\delta + 2\epsilon - 1)p^2 + (5\epsilon\delta^2 + (6\epsilon - 1)\delta + 2\epsilon - 1)m^2)}{(\delta + 1)(\epsilon\delta + 1)(p^2 - m^2)^2} I(4, 1, 1) \\ &\quad + \frac{(-1)m^2(3\delta + 2)}{(\delta + 1)(p^2 - m^2)^2} I(4, 2, 0) \\ &= \frac{(1 - 2\epsilon)(p^2 + m^2)}{(p^2 - m^2)^2} I(4, 1, 1) - \frac{2m^2}{(p^2 - m^2)^2} I(4, 2, 0) \quad \text{as } \delta \rightarrow 0, \end{aligned} \tag{6.26}$$

in agreement with LITERED. ■

Critical case. We observe that our method may not give linear relations for some A and *non-generic* choices of ϵ and δ_i 's. Put $\delta_1 = \dots = \delta_n := \delta$ and take

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.27}$$

Define the linear form $b(\beta) := \beta_2 - \beta_3$, where β_j denotes the j -th component of β . We have $b(a_1) = b(a_2) = 0$, $b(a_3) = 1$, $b(a_4) = 2$. Then $b(\beta)$ is a primitive supporting function of the facet $\{a_1, a_2\}$ of the convex polytope Δ_A . It is known that the supporting function gives a denominator of ∂_i^{-1} modulo $H_A(\beta)$, $i = 3, 4$ [96]. In fact, we have

$$(-2z_3z_4\partial_4 - 4z_1z_4\partial_3 + (-2\beta_1 + \beta_2 + \beta_3)z_3)\partial_3 \equiv (-1)(\beta_2 - \beta_3)(2\beta_1 - \beta_2 - \beta_3) \pmod{H_A(\beta)},$$

and

$$(-2z_3z_4\partial_3 - z_3^2\partial_1 + (-2\beta_2 + 2\beta_3 + 2)z_4)\partial_4 \equiv (-1)(\beta_2 - \beta_3)(\beta_2 - \beta_3 - 1) \pmod{H_A(\beta)}.$$

Note that ∂_3 and ∂_4 stand for P_i . We have $b((\epsilon, -\epsilon\delta, -\epsilon\delta)) = 0$, for which reason Q_i , $i = 3, 4$ acquires a 0 in the denominator. Since Q_i acts similarly to ∂_i^{-1} , the recurrence relations involving Q_i do not exist. Moreover, $b(\beta) = 0$ has the effect of breaking several isomorphism theorems for \mathcal{D} -modules (see, e.g., [96], [93, §4.5] and [106] for recent references on isomorphism theorems of GKZ systems). ■

We close this section with proving Theorem 6.1.

Proof. (Theorem 6.1.) We begin by noting that the matrices Q_i exist since P_i are invertible for generic parameters δ_i .

Without loss of generality, suppose that $r' = 1$. Then $e' = (I_1')$ is given by an integral $I_1' = I(d_0, \nu)$ in the notation of section 5. We seek to reduce I_1' in terms of master integrals $e = (I_1, \dots, I_r)$, where we use the notation $I_i = I(d_0^{(i)}, \nu^{(i)})$.

Take vectors $k_i' \in \mathbb{Z}^N$ such that

$$A \cdot k_i' = (d_0^{(i)}/2, \nu^{(i)}). \tag{6.28}$$

We decompose k_i' into a sum of positive and negative parts: $k_i' = \kappa_i^+ - \kappa_i^-$ for $\kappa_i^\pm \in \mathbb{N}_0^N$. Then we may define the vector

$$\alpha = \left(\max_{1 \leq i \leq r'} \kappa_{i,1}^-, \dots, \max_{1 \leq i \leq r'} \kappa_{i,N}^- \right), \tag{6.29}$$

where $\kappa_{i,j}^-$ is the j -th component of κ_i^- . Put¹⁹ $k_i = k_i' + \alpha \in \mathbb{N}_0^N$ and take $k_0 \in \mathbb{Z}^N$ such that $A \cdot k_0 = A \cdot \alpha + (d_0/2, \nu)$. We apply Algorithm 2 using input k_0 , $\beta = \beta' + A \cdot \alpha$ for $\beta' = (\epsilon, -\epsilon\delta_1, \dots, -\epsilon\delta_n)$ and $s_i = \partial^{k_i}$. The output of the algorithm takes the form

$$f(\beta - A \cdot k_0) = \sum_{i=1}^{r'} u_i(\beta) f(\beta - A \cdot k_i), \quad u_i \in \mathbb{Q}(\beta). \tag{6.30}$$

Let us inspect the expressions for f on both sides of (6.30). On the l.h.s.,

$$f(\beta - A \cdot k_0) = f(\beta' - (d_0/2, \nu)) \tag{6.31}$$

$$= I(d_0, \nu) / c. \tag{6.32}$$

So we obtain the generalized Feynman integral I_1' apart from constant prefactors $c = c(d_0, \nu)$ defined in (5.4). On the r.h.s.,

$$f(\beta - A \cdot k_i) = f\left(\beta - (d_0^{(i)}/2, \nu^{(i)}) - A \cdot \alpha\right) \tag{6.33a}$$

$$= f\left(\beta' - (d_0^{(i)}/2, \nu^{(i)})\right) \tag{6.33b}$$

$$= I(d_0^{(i)}, \nu^{(i)}) / c^{(i)}. \tag{6.33c}$$

In other words, we obtain I_i apart from Γ -prefactors $c^{(i)} = c^{(i)}(d_0^{(i)}, \nu^{(i)})$. The coefficients $u_i(\beta)$ multiplied by Γ -prefactors give the matrix U . Thus, we conclude Statement 1. of Theorem 6.1.

Statement 2. can be proven by noting that ∂_i induces the parameter shift and applying Statement 1. □

¹⁹ α is constructed in order to have a common shift for all k_i' such that their entries become non-negative.

7 Decomposition via cohomology intersection numbers

Relations between Feynman integrals, equivalent to IBP identities, and, more generally, identities for Euler-Mellin integrals, equivalent to contiguity relations, can be derived by means of intersection theory for twisted de Rham cohomologies [34, 35, 37, 38]. According to the mentioned algorithm, the decomposition of any given integrals in terms of an independent basis of MIs can be obtained from the projection of the twisted differential form appearing in the integrand of the integral to decompose into a basis of differential forms that generate a de Rham twisted cohomology group, *via* intersection numbers.

For the case of generalized Feynman integrals (5.7), the covariant derivative (2.20) reads

$$\nabla_x = d_x + \epsilon \frac{d_x \mathcal{G}}{\mathcal{G}} \wedge + \epsilon \delta \sum_{i=1}^n \frac{dx_i}{x_i} \wedge, \tag{7.1}$$

and we denote the associated n -th *de Rham cohomology group* as \mathbb{H}^n (see also (2.22)). We can also introduce a *dual* covariant derivative $\nabla_x^\vee = \nabla_x|_{\epsilon \rightarrow -\epsilon}$ and let $\mathbb{H}^{n\vee}$ be the n -th (*dual*) *de Rham cohomology group* associated to it. The *cohomology intersection number*

$$\langle \bullet, \bullet \rangle_{ch} : \mathbb{H}^n \times \mathbb{H}^{n\vee} \rightarrow \mathbb{C}(z), \tag{7.2}$$

is a natural pairing between the elements of the two groups.

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$; the decomposition of any twisted form $\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$ can be obtained via cohomology intersection numbers according to the master decomposition formula [34, 35, 37, 38],

$$\varphi = \sum_{i=1}^r c_i e_i, \quad \text{with} \quad c_i := \sum_{j=1}^r \langle \varphi, h_j \rangle_{ch} \left(I_{ch}^{-1} \right)_{ji}, \quad \text{and} \quad (I_{ch})_{ij} := \langle e_i, h_j \rangle_{ch}. \tag{7.3}$$

This formula implies the decomposition of (generalized) Feynman integrals in terms of master integrals, upon the identification in (5.7).

Looking at (7.3) we infer that *two* distinct sets of intersection numbers are required, namely $\{\langle \varphi, h_i \rangle_{ch}\}_{i=1}^r$ and $\{(I_{ch})_{ij}\}_{i,j=1}^r$. Therefore, in order to apply the decomposition formula (7.3), it is required the determination of the matrix I_{ch} and of the vector $\langle \varphi, h_i \rangle_{ch}$.

According to the algorithm proposed in [62], given a basis $\{e_i\}_{i=1}^r$ for \mathbb{H}^n ($\{h_i\}_{i=1}^r$ for $\mathbb{H}^{n\vee}$) and $\{P_i\}_{i=1}^N$ the associated set of Pfaffian matrices (resp $\{P_i^\vee\}_{i=1}^N$), the cohomology intersection matrix $(I_{ch})_{ij} = \langle e_i, h_j \rangle_{ch}$ can be obtained as a *rational solution* of the so called *secondary equation*,

$$\partial_i I_{ch} = P_i \cdot I_{ch} + I_{ch} \cdot (P_i^\vee)^T, \quad i = 1, \dots, N; \tag{7.4}$$

which is a (system of partial) differential equation(s) for the cohomology intersection matrix I_{ch} , controlled by the Pfaffian matrices $\{P_i\}_{i=1}^N$ and its dual $\{P_i^\vee\}_{i=1}^N$. In particular, any non-zero rational solution of the secondary equation I is related to I_{ch} as

$$I_{ch} = \kappa_0 I \tag{7.5}$$

up to a constant κ_0 , corresponding to a boundary value, to be independently provided, see theorem 2.1 of [33].

Remarkably, also the vector $\langle \varphi, h_i \rangle_{ch}$ can be obtained by a second application of the same algorithm to a suitably chosen set of differential forms. For this purpose, it is sufficient to build the *auxiliary* basis,²⁰ $\{e_i^{\text{aux}}\}_{i=1}^r := \{e_i\}_{i=1}^{r-1} \cup \{\varphi\}$, as well as the *auxiliary* dual basis $\{h_i^{\text{aux}}\}_{i=1}^r := \{h_i\}_{i=1}^r$, and the corresponding Pfaffian matrices, say $\{P_i^{\text{aux}}\}_{i=1}^N$ and $\{P_i^{\vee \text{aux}}\}_{i=1}^N$, where, owing to the dual bases choice, $P_i^{\vee \text{aux}} \equiv P_i^{\vee}$, $\forall i \mid 1 \leq i \leq N$. Then, the cohomology intersection matrix for the auxiliary bases, I_{ch}^{aux} , whose elements are defined as $(I_{ch}^{\text{aux}})_{ij} = \langle e_i^{\text{aux}}, h_j^{\text{aux}} \rangle = \langle e_i^{\text{aux}}, h_j \rangle$, can be obtained as rational solution of this matrix differential equation,

$$\begin{aligned} \partial_i I_{ch}^{\text{aux}} &= P_i^{\text{aux}} \cdot I_{ch}^{\text{aux}} + I_{ch}^{\text{aux}} \cdot (P_i^{\vee \text{aux}})^T \\ &= P_i^{\text{aux}} \cdot I_{ch}^{\text{aux}} + I_{ch}^{\text{aux}} \cdot (P_i^{\vee})^T, \quad i = 1, \dots, N. \end{aligned} \tag{7.6}$$

The wanted vector of intersection numbers $\langle \varphi, h_j \rangle_{ch}$, can be read off the r^{th} -row of I_{ch}^{aux} , because

$$(I_{ch}^{\text{aux}})_{rj} = \langle \varphi, h_j^{\text{aux}} \rangle_{ch} = \langle \varphi, h_j \rangle_{ch}, \quad j = 1, \dots, r. \tag{7.7}$$

Actually the solution of the auxiliary secondary equation,

$$I_{ch}^{\text{aux}} = \kappa_0^{\text{aux}} I^{\text{aux}} \tag{7.8}$$

also requires the independent knowledge of the constant κ_0^{aux} .

The proposed twofold procedure, yields the determination of all the cohomology intersection numbers needed in (7.3), up to the knowledge of the ratio of two boundary constants $\kappa_0^{\text{aux}}/\kappa_0$.

Let us observe that the master decomposition formula (7.3) requires the knowledge of the inverse matrix I_{ch}^{-1} . By using the secondary equation (7.4), one can show that I_{ch}^{-1} can be directly determined as rational solution of the following matrix differential equation,

$$\partial_i (I_{ch}^{-1}) = -(P_i^{\vee})^T \cdot (I_{ch}^{-1}) - (I_{ch}^{-1}) \cdot P_i. \tag{7.9}$$

Determination of coefficients. Let us apply the master decomposition (7.3) to a vector formed by r differential forms $e_1, e_2, \dots, e_{r-1}, \varphi$, reading as,

$$\begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{pmatrix} = I_{ch}^{\text{aux}} \cdot I_{ch}^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{pmatrix}. \tag{7.10}$$

By construction, the product $I_{ch}^{\text{aux}} \cdot I_{ch}^{-1}$ has the following form,

$$I_{ch}^{\text{aux}} \cdot I_{ch}^{-1} = \left(\begin{array}{c|c} \mathbb{I}_{r-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array} \right), \tag{7.11}$$

²⁰The set $\{e_i\}_{i=1}^{r-1} \cup \{\varphi\}$ fails to be a basis when $\varphi = 0$. In the following discussion, we implicitly assume that the cohomology class φ is not zero and the set $\{e_i\}_{i=1}^{r-1} \cup \{\varphi\}$ is a basis. This can be achieved when φ is represented by a *monomial differential form* $\omega_{d_0/2, \nu}$.

where \mathbb{I}_{r-1} is the identity matrix in the $(r-1) \times (r-1)$ space, and the entries of the last row are the coefficients of the decomposition (7.3). On the other side, owing to the solution of the secondary equations, one has,

$$I_{ch}^{\text{aux}} \cdot I_{ch}^{-1} = \frac{\kappa_0^{\text{aux}}}{\kappa_0} I^{\text{aux}} \cdot I^{-1}. \tag{7.12}$$

This relation among matrices can be exploited to fix the value of the ratio $\kappa_0^{\text{aux}}/\kappa_0$. In fact, any of the elements on the diagonal, say $(I_{ch}^{\text{aux}} \cdot I_{ch}^{-1})_{kk}$, for any $k \in \{1, \dots, r-1\}$, amounts to 1, therefore,

$$\frac{\kappa_0^{\text{aux}}}{\kappa_0} (I^{\text{aux}} \cdot I^{-1})_{kk} = 1, \tag{7.13}$$

yielding the relation

$$\frac{\kappa_0}{\kappa_0^{\text{aux}}} = (I^{\text{aux}} \cdot I^{-1})_{kk}, \tag{7.14}$$

namely fixing the ratio of the boundary constants from one of the elements of the product of two known matrices. Finally, (7.10) becomes,

$$\begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{pmatrix} = \frac{1}{(I^{\text{aux}} \cdot I^{-1})_{kk}} I^{\text{aux}} I^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{pmatrix}, \tag{7.15}$$

hence the coefficients of decomposition of the differential form φ in (7.3) reads as,

$$\varphi = \sum_{i=1}^r c_i e_i, \quad \text{with} \quad c_i = \frac{1}{(I^{\text{aux}} \cdot I^{-1})_{kk}} (I^{\text{aux}} \cdot I^{-1})_{ri}, \tag{7.16}$$

for any chosen $k \in \{1, \dots, r-1\}$. This result bypasses the determination of the complete cohomology intersection matrices, I_{ch} and I_{ch}^{aux} , and relies only on the knowledge of the Pfaffian matrices, P and P^{aux} , which control the secondary equations. The determination of their *rational* solutions, I and I^{aux} , can be efficiently combined with the rational function reconstruction over finite fields, earlier discussed.

Let us remark that, if needed, the individual expressions of the normalization constants κ_0 , and κ_0^{aux} can be determined by means of [107, Theorem 8.1].

7.1 Example: One-loop box

We reconsider the example of section 5.2; we aim to decompose:

$$\epsilon(-s)^\epsilon (-s)^3 z I(4, 2, 1, 1, 1; z), \tag{7.17}$$

in terms of the basis given by (5.18). The associated differential forms are:

$$\varphi = \frac{\epsilon(-s)^{-4\epsilon\delta} z \Gamma(2-\epsilon)}{\Gamma(-1-2\epsilon-4\epsilon\delta) \Gamma(1+\epsilon\delta)^3 \Gamma(2+\delta\epsilon)} \frac{x_1}{\mathcal{G}(z; x)^2} dx, \tag{7.18}$$

and,

$$(e_1, e_2, e_3) = (5.19), \tag{7.19}$$

while we explicitly choose the basis of dual forms as:

$$(h_1, h_2, h_3) = (e_1, e_2, e_3) \Big|_{\epsilon \rightarrow -\epsilon} = (5.19) \Big|_{\epsilon \rightarrow -\epsilon}. \quad (7.20)$$

The Pfaffian matrix associated to (5.18) reads:

$$P = \begin{pmatrix} -\frac{\epsilon(\delta^2(12z+11)+7\delta(z+1)+z+1)}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & \frac{\delta^2\epsilon(\delta(z+2)+1)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ \frac{\delta^2\epsilon}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & -\frac{\delta^2\epsilon(\delta+2\delta z+z)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ -\frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)z(z+1)} & \frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)(z+1)} & -\frac{\epsilon(\delta^2(5z+7)+\delta(2z+5)+1)}{(3\delta+1)z(z+1)} \end{pmatrix}, \quad (7.21)$$

while its dual is:

$$P^\vee = P \Big|_{\epsilon \rightarrow -\epsilon}. \quad (7.22)$$

A rational solution of (7.4) obtained via [108] is

$$I = \begin{pmatrix} -\frac{(2\delta+1)(4\delta+1)}{\delta} & \delta & -2(\delta\epsilon-1) \\ \delta & -\frac{(2\delta+1)(4\delta+1)}{\delta} & -2(\delta\epsilon-1) \\ 2(\delta\epsilon+1) & 2(\delta\epsilon+1) & -\frac{4(10\delta^2+6\delta+1)(\delta\epsilon-1)(\delta\epsilon+1)}{\delta^3} \end{pmatrix}. \quad (7.23)$$

The auxiliary basis reads:

$$(e_1^{\text{aux}}, e_2^{\text{aux}}, e_3^{\text{aux}}) = (e_1, e_2, \varphi). \quad (7.24)$$

The expression for P^{aux} is too lengthy to be reported here. By solving (7.6), we obtain I^{aux} . The entries of its last row read:

$$(I^{\text{aux}})_{31} = \frac{2(3\delta\epsilon+\epsilon+1)((4\delta+2)\epsilon+1)((6\delta^2+5\delta+1)\epsilon+\delta z(2\delta\epsilon+\epsilon+1))}{\delta z(2\delta\epsilon+\epsilon+1)^2}, \quad (7.25a)$$

$$(I^{\text{aux}})_{32} = \frac{2((4\delta+2)\epsilon+1)(2(6\delta^2+5\delta+1)\epsilon^2+(5\delta+2)\epsilon+1)}{(2\delta\epsilon+\epsilon+1)^2}, \quad (7.25b)$$

$$(I^{\text{aux}})_{33} = -\frac{4(\delta\epsilon-1)((4\delta+2)\epsilon+1)}{\delta^3 z(2\delta\epsilon+\epsilon+1)^2} \left(\delta(6\delta^2+5\delta+1)\epsilon(3\delta\epsilon+\epsilon+1)+z(66\delta^4\epsilon^2+\delta^3\epsilon(91\epsilon+50)) \right. \\ \left. +\delta^2(47\epsilon^2+50\epsilon+10)+\delta(11\epsilon^2+17\epsilon+6)+(\epsilon+1)^2 \right). \quad (7.25c)$$

For the example under consideration, we find:

$$\frac{\kappa_0}{\kappa_0^{\text{aux}}} = 1. \quad (7.26)$$

All the cohomology intersection numbers needed for the decomposition formula (7.3) are therefore computed:

$$\langle \varphi, h_j \rangle_{ch} = (I_{ch}^{\text{aux}})_{3j}, \quad j = 1, \dots, 3; \quad (7.27a)$$

$$\langle e_i, h_j \rangle_{ch} = (I_{ch})_{ij}, \quad i, j = 1, \dots, 3, \quad (7.27b)$$

and, in the $\delta \rightarrow 0$ limit, the resulting decomposition in terms of master forms reads,

$$\varphi = -\frac{2\epsilon(2\epsilon+1)}{z(\epsilon+1)} \cdot e_1 + 0 \cdot e_2 + (2\epsilon+1) \cdot e_3, \quad (7.28)$$

which is in agreement with the IBP decomposition obtained with LITERED. To conclude, let us also observe that, the individual normalization constants $\kappa_0 = \kappa_0(\delta, \epsilon)$, can be determined by means of [107, Theorem 8.1], as

$$\kappa_0 = \frac{\sin^4(\delta\epsilon\pi) \sin(2(2\delta + 1)\epsilon\pi)}{\epsilon^2 2\delta(2\delta + 1)^2 (\epsilon^2 \delta^2 - 1) \pi^4 \sin(\pi\epsilon)}, \tag{7.29}$$

which, combined with (7.23), gives the cohomology intersection matrix I_{ch} . ■

8 Conclusion

In this work, we presented a simplified algorithm for the determination of Pfaffian matrices from Macaulay matrix, making use of the theory of \mathcal{D} -modules for Gel'fand-Kapranov-Zelevinsky systems.

Then, we introduced two algorithms for the derivation of *linear relations* for GKZ-hypergeometric integrals making use of Pfaffians: *i*) the first one uses the properties of the *matrix factorial*, within the holonomic gradient method; *ii*) the second one uses the direct integral decomposition *via* cohomology *intersection numbers* and the rational solution of the *secondary equation* for cohomology intersection matrices. In the latter case, we derived a novel variant of the master formula for the projection of differential forms onto a set of independent forms.

Our investigation exploited the *isomorphism* between de Rham twisted cohomology groups, whose elements are differential forms, and \mathcal{D} -modules, whose elements are partial differential operators in a Weyl algebra. Our results can be applied to GKZ-hypergeometric functions as well as to Feynman integrals, which can be considered *restrictions* of the former class of functions. Within our analyses the number of master integrals, corresponding to the dimension of the de Rham cohomology group, is related to the rank of the GKZ system, computed by polyhedra combinatorics in terms of volumes of polytopes. We showcased a few applications to simple mathematical functions and one-loop integrals.

Within the standard approaches, linear relations among integrals, such as contiguity relation for GKZ-hypergeometric functions and integration-by-parts identities for Feynman integrals, are employed to derive systems of partial differential equations for a chosen set of independent function. Instead, in the current work, we reversed the perspective, and showed how Pfaffian matrices for system of partial differential equations, built from Maculay matrices, can be used to derive linear relations for integrals.

We hope that our study could offer a novel, more complete view on integral relations which emerge from the application of differential operators acting both on internal and external variables, and could provide additional insights to the investigation of de Rham twisted co-homology theory in computational (quantum) field theory.

Acknowledgments

We wish to thank Tiziano Peraro for interesting discussions and for assistance with the use of FINITEFLOW.

We acknowledge the stimulating discussions stemming from the workshop *MathAmplitudes 2019: Intersection Theory & Feynman Integrals*, U. Padova, [MA2019].

F.G. is supported by Fondazione Cassa di Risparmio di Padova e Rovigo (CARIPARO). The work of M.K.M is supported by Fellini — Fellowship for Innovation at INFN funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754496. V.C. is supported by the *Diagrammalgebra* Stars Wild-Card Grant UNIPD. S.M. and N.T. are supported in part by the JST CREST Grant Number JP19209317. S.M. is supported by JSPS KAKENHI Grant Number 22K13930. N.T. is supported in part by JSPS KAKENHI Grant Number JP21K03270.

A Further details on the homogeneity property

Here we provide additional details and formulas for section 2.4. We begin by introducing simplified matrix notation to reduce the number of indices in the formulas below.

Firstly, we give a generalization of the multivariate exponent from (2.3). Given an $(n + 1) \times N$ matrix $A := (a_1, \dots, a_N)$ and a list of variables $t := (t_1, \dots, t_{n+1})$, we construct the following list of exponentials:²¹

$$t^A := (t^{a_1}, \dots, t^{a_N}). \tag{A.1}$$

This matrix exponentiation obeys the rule $(t^A)^B = t^{AB}$.

Secondly, we define an item-wise product of two lists having the same lengths. Given two lists t and s of lengths n , we define²²

$$s \times t := (s_1 t_1, \dots, s_n t_n). \tag{A.2}$$

Using the formulas above, we compactly rewrite the homogeneity property (2.35) of Euler integrals as follows:

$$f_\Gamma(t^A \times z) = t^\beta f_\Gamma(z). \tag{A.3}$$

Now we pick a set

$$\sigma \subset \{1, \dots, N\}, \quad \eta := \{1, \dots, N\} \setminus \sigma, \tag{A.4}$$

where σ is of size $|\sigma| = n + 1$, and η is its complement. Denote by A_σ the submatrix of matrix A constructed from the columns labeled by σ . Given a vector v , we likewise construct v_σ . Using this notation, we can separate the following matrix product into two terms:

$$Av = A_\sigma v_\sigma + A_\eta v_\eta. \tag{A.5}$$

We shall require σ to be chosen such that the A_σ submatrix is invertible, i.e. $\det(A_\sigma) \neq 0$.

²¹In MATHEMATICA this can be achieved by `Inner[Power, t, A, Times]`.

²²Using MATHEMATICA syntax, we can write `Inner[Times, s, t, List]`.

Now observe that for the special choice of the rescaling parameters $t = z_\sigma^{-A_\sigma^{-1}}$ in (A.3), we obtain the following factorized representation of a given Euler integral:

$$f_\Gamma(z) = z_\sigma^{-A_\sigma^{-1}\beta} f_\Gamma(z_\sigma^{-A_\sigma^{-1}A} \times z). \tag{A.6}$$

The key feature of the r.h.s. is that all arguments labeled by σ are set to 1:

$$\{z_\sigma^{-A_\sigma^{-1}A} \times z\}_j = \begin{cases} 1 & j \in \sigma. \\ z_\sigma^{-A_\sigma^{-1}a_j} z_j & j \in \eta. \end{cases} \tag{A.7}$$

This encourages us to define the rescaled variables

$$w := z_\sigma^{-A_\sigma^{-1}A_\eta} \times z_\eta, \tag{A.8}$$

as well as the rescaled Euler integral

$$g_\Gamma(w) := z_\sigma^{A_\sigma^{-1}\beta} f_\Gamma(z). \tag{A.9}$$

With this notation we are now ready to formulate the main proposition of this section.

Proposition A.1. *A generic partial derivative operator $\partial_z^v := \partial_{z_1}^{v_1} \partial_{z_2}^{v_2} \dots \partial_{z_N}^{v_N}$ acts on the Euler integral (A.6) as follows:*

$$\partial_z^v f_\Gamma(z) = z_\sigma^{-A_\sigma^{-1}(\beta + Av)} (-1)^{|v_\sigma|} \left(w^{-v_\eta} (A_\sigma^{-1}\beta + A_\sigma^{-1}A_\eta \theta_w)_{v_\sigma} [\theta_w]_{v_\eta} g_\Gamma(w) \right) \Big|_{w=(A.8)}, \tag{A.10}$$

where $|v_\sigma| := \sum_{i \in \sigma} v_i$.

Proof. Let us start with a few useful formulas. The splitting of the vector v into v_σ and v_η implies the corresponding factorization of the differential operator:

$$\partial_z^v = \partial_{z_\sigma}^{v_\sigma} \partial_{z_\eta}^{v_\eta}. \tag{A.11}$$

This operator can also be expressed in terms of the lowering factorial of an Euler operator like so:

$$\partial_z^v = z^{-v} [\theta_z]_v. \tag{A.12}$$

Finally, the raising and lowering factorials²³ are related as follows:

$$(a)_b = (-1)^{|b|} [-a]_b. \tag{A.13}$$

Using these formulas, it follows that the action of $\partial_{z_\eta}^{v_\eta}$ on the Euler integral (A.9) is

$$\partial_{z_\eta}^{v_\eta} f_\Gamma(z) = z_\sigma^{-A_\sigma^{-1}\beta} \partial_{z_\eta}^{v_\eta} g_\Gamma(z_\sigma^{-A_\sigma^{-1}A_\eta} \times z_\eta) = z_\sigma^{-A_\sigma^{-1}(\beta + A_\eta v_\eta)} \left\{ \partial_w^{v_\eta} g_\Gamma(w) \right\}, \tag{A.14}$$

²³Recall the definitions $(a)_b := a(a+1) \dots (a+b-1)$ and $[a]_b := a(a-1) \dots (a-b+1)$. In vector case, these definitions are applied component-wise: $(u)_v := (u_1)_{v_1} \dots (u_N)_{v_N}$.

where the curly brackets indicate the substitution $w = (A.8)$, once the expression inside of them was evaluated. Proceeding with the action of the σ -part of ∂_z^v , we have

$$\partial_{z_\sigma}^{v_\sigma} \partial_{z_\eta}^{v_\eta} f_\Gamma(z) = z_\sigma^{-v_\sigma} [\theta_{z_\sigma}]_{v_\sigma} \partial_{z_\eta}^{v_\eta} f_\Gamma(z) \tag{A.15}$$

$$= z_\sigma^{-v_\sigma} z_\sigma^{-A_\sigma^{-1}(\beta + A_\eta v_\eta)} [-A_\sigma^{-1}(\beta + A_\eta v_\eta) + \theta_{z_\sigma}]_{v_\sigma} \left\{ \partial_w^{v_\eta} g_\Gamma(w) \right\} \tag{A.16}$$

$$= z_\sigma^{-A_\sigma^{-1}(\beta + Av)} \left\{ [-A_\sigma^{-1}(\beta + A_\eta v_\eta) - A_\sigma^{-1} A_\eta \theta_w]_{v_\sigma} w^{-v_\eta} [\theta_w]_{v_\eta} g_\Gamma(w) \right\} \tag{A.17}$$

$$= z_\sigma^{-A_\sigma^{-1}(\beta + Av)} \left\{ w^{-v_\eta} [-A_\sigma^{-1}(\beta + A_\eta \theta_w)]_{v_\sigma} [\theta_w]_{v_\eta} g_\Gamma(w) \right\} \tag{A.18}$$

$$= z_\sigma^{-A_\sigma^{-1}(\beta + Av)} (-1)^{|v_\sigma|} \left\{ w^{-v_\eta} (A_\sigma^{-1}(\beta + A_\eta \theta_w))_{v_\sigma} [\theta_w]_{v_\eta} g_\Gamma(w) \right\}, \tag{A.19}$$

where $A_\eta \theta_w$ is a column vector, obtained from matrix multiplication of A_η and a column vector θ_w . \square

Let us close this discussion with one application of Proposition A.1, namely the action of a generic partial derivative operator on the cohomology class $[dx/x]$:

$$\partial_z^v \bullet \left[\frac{dx}{x} \right] \Big|_{z_i=1, i \in \sigma} = (-1)^{|v|} \left(w^{-v_\eta} (A_\sigma^{-1}(\beta + A_\eta \theta_w))_{v_\sigma} [\theta_w]_{v_\eta} \bullet \left[\frac{dx}{x} \right] \right) \Big|_{w=z_\eta}. \tag{A.20}$$

B Holonomic \mathcal{D} -modules in a nutshell

In this appendix, we summarize basics of the theory of holonomic systems utilized in our study of Feynman integrals. We cite introductory textbooks rather than the original papers or comprehensive textbooks.

B.1 Holonomic ideals

Let \mathcal{D} be the Weyl algebra with polynomial coefficients:

$$\mathcal{D} := \mathbb{C}\langle z_1, \dots, z_N, \partial_1, \dots, \partial_N \rangle = \mathbb{C}[z_1, \dots, z_N]\langle \partial_1, \dots, \partial_N \rangle \tag{B.1}$$

where the commutators are $[z_i, z_j] = 0$, $[\partial_i, \partial_j] = 1$, and $[\partial_i, z_j] = \delta_{ij}$. When we need to specify the number of variables, we also denote this algebra by \mathcal{D}_N . Any element $L \in \mathcal{D}$ can be written in the normally ordered form

$$L = \sum_{(p,q) \in E} c_{p,q} z^p \partial^q, \quad z^p = z_1^{p_1} \cdots z_N^{p_N}, \partial^q = \partial_1^{q_1} \cdots \partial_N^{q_N}, c_{p,q} \in \mathbb{C} \tag{B.2}$$

via the commutator relations. The principal symbol $\text{in}_{(\mathbf{0}, \mathbf{1})}(L)$ of L is defined as the sum of the highest order differential operators in L :

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(L) = \sum_{(p,q) \in E, |q| \text{ is max}} c_{p,q} z^p \xi^q \in \mathbb{C}[z, \xi] \tag{B.3}$$

where $|q| := q_1 + \cdots + q_N$, and the commutative variable ξ denotes the derivative ∂ when a given expression is written in the normally ordered form. We see that the principal symbol

is the sum of the highest weight terms with respect to the weight $\mathbf{0}$ for z and the weight $\mathbf{1} = (1, \dots, 1)$ for ∂ .

Let \mathcal{I} be a left ideal in \mathcal{D} . The left \mathcal{D} -module \mathcal{D}/\mathcal{I} is called *holonomic* when the dimension of the ideal of principal symbols

$$\text{in}_{(\mathbf{0},\mathbf{1})}(\mathcal{I}) = \mathbb{C}\{\text{in}_{(\mathbf{0},\mathbf{1})}(L) \mid L \in \mathcal{I}\} \tag{B.4}$$

equals N (see also [93, Th 1.4.12, p.32]). In this case, the left ideal \mathcal{I} is called a *holonomic ideal*. For general \mathcal{D} -modules, the definition of a holonomic \mathcal{D} -module requires the notion of filtration, which lies out of the scope of this short review. We refer the interested reader to [97, §6.7, §6.8] and references therein for further details.

Example B.1. Take $N = 2$ and let \mathcal{I} be a left ideal generated by z_1 and ∂_2 . The ideal of principal symbols is then generated by z_1 and ξ_2 . The dimension of the zero set of $z_1 = \xi_2 = 0$ in \mathbb{C}^4 is 2, and so we conclude that \mathcal{D}/\mathcal{I} is holonomic.

B.2 Standard monomials

Let \mathcal{R} be the rational Weyl algebra

$$\mathcal{R} := \mathbb{C}(z_1, \dots, z_N) \otimes_{\mathbb{C}[z_1, \dots, z_N]} \mathcal{D} = \mathbb{C}(z_1, \dots, z_N) \langle \partial_1, \dots, \partial_N \rangle. \tag{B.5}$$

When \mathcal{I} is a holonomic ideal in \mathcal{D} , the left ideal $\mathcal{R}\mathcal{I}$ of \mathcal{R} is a zero-dimensional ideal. In other words, the dimension of $\mathcal{R}/(\mathcal{R}\mathcal{I})$ as a vector space over the rational function field $\mathbb{C}(z_1, \dots, z_N)$ is finite. Similar to the polynomial case (B.2), any element L of the rational Weyl algebra \mathcal{R} can be brought into the normally ordered form

$$L = \sum_{q \in E} c_q(z) \partial^q, \quad c_q(z) \in \mathbb{C}(z_1, \dots, z_N). \tag{B.6}$$

Let \prec be a term order among the monomials in ∂_i 's. The largest monomial ∂^q in L (stripped of its rational function coefficient) is called the initial monomial $\text{in}_{\prec}(L)$ of L . Monomials in derivatives naturally form the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_N]$. A finite subset G of \mathcal{R} is called a Gröbner basis of $\mathcal{R}\mathcal{I}$ with respect to \prec when the initial monomial ideal

$$\text{in}_{\prec}(\mathcal{R}\mathcal{I}) = \mathbb{C}\{\text{in}_{\prec}(L) \mid L \in \mathcal{R}\mathcal{I}\} \subset \mathbb{C}[\partial_1, \dots, \partial_N] \tag{B.7}$$

is generated by $\text{in}_{\prec}(L)$, $L \in G$. A monomial ∂^α is called a *standard monomial* with respect to G when ∂^α does not belong to the initial monomial ideal ($\text{in}_{\prec}(\mathcal{R}\mathcal{I})$ in our case). Within the theory of Gröbner basis in \mathcal{R} , the zero-dimensionality of $\mathcal{R}\mathcal{I}$ is equivalent to the finiteness of the set of standard monomials. In the following, an element ∂^α of $\text{in}_{\prec}(\mathcal{R}\mathcal{I}) \subset \mathbb{C}[\partial_1, \dots, \partial_N]$ is also denoted by ξ^α to emphasize that it is an element of the polynomial ring. Let us now give a few examples of standard monomials for different ideals.

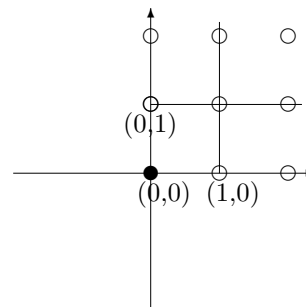
Example B.2. When $N = 2$ and $\mathcal{I} = \langle z_1, \partial_2 \rangle \subset \mathcal{D}$, we have $\mathcal{R}\mathcal{I} = \langle 1 \rangle$ because $\frac{1}{z_1} z_1 = 1$. Then the set of standard monomials is empty.

Example B.3. Put

$$L_1 = (z_1 z_2 - 1) \partial_1 + \beta_0 z_2 \tag{B.8}$$

$$L_2 = z_2(z_1 z_2 - 1) \partial_2 + (-\beta_1 + \beta_0) z_1 z_2 + \beta_1, \tag{B.9}$$

where β_0 and β_1 are complex numbers. These two operators annihilate the function $(1 - z_1 z_2)^{-\beta_0} z_2^{\beta_1}$. By a Gröbner basis computation in the Weyl algebra \mathcal{D} , we find that the ideal of principal symbols is generated by the two elements $\xi_1 - z_2^2 \xi_2$ and $z_1 \xi_1 - z_2 \xi_2$. Since it defines a 2-dimensional variety in \mathbb{C}^4 , we conclude that the module $\mathcal{D}/\langle L_1, L_2 \rangle$ is holonomic. The set $\{L_1, L_2\}$ is a Gröbner basis in \mathcal{R} with any term order \prec and the initial ideal is generated by ξ_1 and ξ_2 . The set of standard monomials is simply $\{\xi_1^0 \xi_2^0\} \equiv \{1\}$, corresponding to the point $(0, 0)$ in the figure to the right, where the axes represent powers of ξ_1, ξ_2 .



Theorem B.4. (see e.g. [97, §6.9]) *Let \mathcal{I} be a holonomic ideal in \mathcal{D} . Then $\mathcal{R}\mathcal{D}$ is a zero-dimensional ideal in \mathcal{R} . Conversely, let \mathcal{J} be a zero-dimensional ideal in \mathcal{R} . Then $\mathcal{D} \cap \mathcal{J}$ is a holonomic ideal.*

Note that the second construction $\mathcal{D} \cap \mathcal{J}$ from \mathcal{J} is a highly non-trivial. Algorithms realising such a construction are called Weyl closure algorithms.

We close this subsection with a note on the relation between Pfaffian equations and Gröbner bases. Let G be a reduced Gröbner basis of a zero dimensional ideal $\mathcal{R}\mathcal{I}$ in the rational Weyl algebra \mathcal{R} . Since it is zero-dimensional, the set of the standard monomials is a finite set. We denote it by $\{s_1, \dots, s_r\}$. Computing the normal form of $\partial_i s_j$ by the Gröbner basis, we obtain

$$\partial_i s_j = \sum_{k=1}^r P_{ij}^k s_k \quad \text{mod } \mathcal{R}G$$

The matrix $P_i = (P_{ij}^k)_{j,k}$ is the Pfaffian matrix for the operator ∂_i .

Conversely, assume that we are given a basis $S = \{s_1, \dots, s_r\}$ of $\mathcal{R}/\mathcal{R}\mathcal{I}$ as a vector space over $\mathbb{C}(z_1, \dots, z_N)$. Here the s_i 's are monomials. Let P_i be the Pfaffian matrix for ∂_i . Then we have $\partial_i s_j = \sum_{k=1}^r (P_i)_{jk} s_k$ modulo $\mathcal{R}\mathcal{I}$. We assume that there exists a term order \prec such that $s \prec t$ for all monomials $s \in S$ and $t \in S^c$ where S^c is the set of the monomials which do not belong to S . If such an order exists, it can be found by solving a system of linear inequalities [93, §2.1]. See also a note below. Then the set $G = \{g_{ij} := \partial_i s_j - \sum_{k=1}^r (P_i)_{jk} s_k \mid \partial_i s_j \notin S\}$ is a Gröbner basis with respect to the term order \prec , and the set S is the set of the standard monomials for the Gröbner basis. Let us prove it. Since $u \succeq 1$ for any monomial u , we have $ut \succeq t$ for any monomial t . Therefore, we have $ut \in S^c$ for $t \in S^c$ and any monomial u by the condition of the order. It implies that a Gröbner basis of the monomial ideal $\langle S^c \rangle$ is $\{\partial_i s_j \mid \partial_i s_j \notin S\}$, and S is the set of standard monomials for that basis. It follows from the condition on the order that we have $\text{in}_\prec(g_{ij}) = \partial_i s_j \notin S$ and $\text{in}_\prec(G) \subset \text{in}_\prec(\mathcal{R}\mathcal{I})$. Since $\dim \mathbb{C}[\partial_1, \dots, \partial_N] / \text{in}_\prec(G) = r$ by $\text{in}_\prec(G) = \langle S^c \rangle$, $\text{in}_\prec(G) = \text{in}_\prec(\mathcal{R}\mathcal{I})$ holds. Thus, G is a Gröbner basis for the order \prec .

Note that it is necessary to pose the condition on the order and the basis S . For example, consider $N = 1$ and $M = \mathcal{R}/\mathcal{R}(\partial_1 - z_1)$. $S = \{\partial_1\}$ is a vector space basis of M over $\mathbb{C}(z_1)$ and the Pfaffian system with respect to S is $\partial_1 - (z_1 + 1/z_1)$, because $\partial_1(\partial_1 - z_1) = \partial_1^2 - z_1\partial_1 - 1$ and $\frac{1}{z_1}\partial_1 - 1$ belong to the ideal $\mathcal{R}(\partial_1 - z_1)$. However, the reduced Gobner basis of the ideal for any term order is $\{\partial_1 - z_1\}$. In fact, the condition $\partial_1 \prec 1$ does not hold for any term order.

Finally, we explain a procedure to check if there exists a term order \prec such that $s \prec t$ for all monomials $s \in S$ and $t \in S^c$. Let $w \in \mathbb{R}_{\geq 0}^N$ be a weight vector. We solve a system of linear inequalities with respect to w

$$w \cdot \alpha \leq w \cdot \beta \quad \text{for all } s = \partial^\alpha \in S \text{ and } t = \partial^\beta \in S'$$

where S' is the set of the minimal generators of the monoid S^c . If the solution space is an N -dimensional cone, then take w from the interior of the cone and define the order by \prec_w , where we may take any tie breaker \prec . Here, we mean $\partial^a \prec_w \partial^b$ if and only if $w \cdot a < w \cdot b$ or $(w \cdot a = w \cdot b \text{ and } \partial^a \prec \partial^b)$. If the cone is empty, there does not exist such an order. When the cone has dimension less than N , take $w(1) \in \mathbb{R}_{\geq 0}^N$ from the relative interior of the cone and also take the set P of (α, β) 's such that $w(1) \cdot \alpha = w(1) \cdot \beta$. We solve a system of linear inequalities

$$w \cdot \alpha \leq w \cdot \beta \quad \text{for all } (\alpha, \beta) \in P.$$

Take $w(2) \in \mathbb{R}_{\geq 0}^N$ from the relative interior of the solution cone. If this $w(2)$ satisfies $w(2) \cdot \alpha = w(2) \cdot \beta$ for all $(\alpha, \beta) \in P$, there exists no term order we want. If not, take $\prec_{w(2)}$ as a tie breaker. We repeat this procedure. Since any term order can be expressed by a matrix of weight vectors [109], we can check if there exists a term order $s \prec t$ for all monomials $s \in S$ and $t \in S^c$ and we can construct it if it exists.

B.3 Proof of theorem 3.1

In this subsection, we prove Theorem 3.1. We follow the notation of section 3.1. Let J be an ideal of $\mathbb{C}[\theta]$. It naturally lifts to a left ideal of \mathcal{R} generated by the elements of J which we will denote by $\mathcal{R}J$. Recall that a term order \prec on the ring $\mathbb{C}[\theta]$ naturally corresponds to an order among monomials ∂^k by the correspondence $\theta^k \leftrightarrow \partial^k$. Thus, \prec induces a term order on the ring \mathcal{R} .

Lemma B.5. *The set of \prec -standard monomials of the ideal J coincides with that of $\mathcal{R}J$ through the correspondence $\theta^k \leftrightarrow \partial^k$.*

Proof. Let us recall an identity

$$z_i^k \partial_i^k = \theta_i(\theta_i - 1) \cdots (\theta_i - k + 1) \tag{B.10}$$

for any $k \in \mathbb{N}$. For any $k = (k_1, \dots, k_N) \in \mathbb{N}^N$, we set

$$[\theta]_k := \prod_{i=1}^N \theta_i(\theta_i - 1) \cdots (\theta_i - k_i + 1). \tag{B.11}$$

We claim that a \prec -Gröbner basis $G \subset J$ of J is again a \prec -Gröbner basis of $\mathcal{R}J$. Let us take $P, Q \in G$ and write them as

$$P = \underline{a\theta^k} + \dots, \quad Q = \underline{b\theta^\ell} + \dots \tag{B.12}$$

where $a, b \in \mathbb{C}$ and underlines indicate the leading terms. Regarding P and Q as elements of the ring \mathcal{R} , we obtain an expansion

$$P = \underline{az^k\partial^k} + \dots, \quad Q = \underline{bz^\ell\partial^\ell} + \dots. \tag{B.13}$$

We set $c_i = \max(k_i, \ell_i)$ and $c = (c_1, \dots, c_N)$. The S-pair of P and Q in the ring \mathcal{R} (cf. [97, §6.1]) is

$$bz^\ell\partial^{c-k}P - az^k\partial^{c-\ell}Q = z^{k+\ell-c}(b[\theta]_{c-k}P - a[\theta]_{c-\ell}Q). \tag{B.14}$$

The second factor clearly belongs to the ideal J and the S-pair of P and Q is reduced to zero by G in the ring \mathcal{R} . This shows that G is a \prec -Gröbner basis of $\mathcal{R}J$. Thus, the set of leading monomials of J is identical to that of $\mathcal{R}J$ through the correspondence $\theta^k \leftrightarrow \partial^k$. This proves the lemma. \square

The following theorem is known as *Gröbner deformation*.

Theorem B.6 (a simplified version of Theorem 3.1.3 of [93]). *For any parameter β generic enough, one has an identity*

$$\text{in}_\prec(\mathcal{R}H_A(\beta)) = \mathcal{R}\langle E_1, \dots, E_{n+1}, \text{in}_\prec(\mathcal{I}\square) \rangle. \tag{B.15}$$

Now, let us prove Theorem 3.1. Since z^k is invertible in the ring \mathcal{R} , Theorem B.6 shows that $\text{in}_\prec(\mathcal{R}H_A(\beta))$ is identical to the ideal $\mathcal{R}\mathcal{I}'$. Clearly, the right-hand side is a lift of an ideal \mathcal{I}' in $\mathbb{C}[\theta]$. By Lemma B.5, we can conclude that the set of standard monomials of \mathcal{I}' is identical to that of $\text{in}_\prec(\mathcal{R}H_A(\beta))$ through the correspondence $\theta^k \leftrightarrow \partial^k$.

B.4 Integration and restriction

Now we discuss two important properties of the holonomic \mathcal{D} -modules: the so-called integration and restriction constructions.

Let \mathcal{D}_{M+N} be the ring of differential operators in $M + N$ variables:

$$\mathbb{C}\langle x_1, \dots, x_M, z_1, \dots, z_N, \partial_{x_1}, \dots, \partial_{x_M}, \partial_{z_1}, \dots, \partial_{z_N} \rangle,$$

and \mathcal{I} be a left holonomic ideal of \mathcal{D}_{M+N} . We denote $\mathbb{C}\langle x_1, \dots, x_M, \partial_{x_1}, \dots, \partial_{x_M} \rangle$ by \mathcal{D}_M and $\mathbb{C}\langle z_1, \dots, z_N, \partial_{z_1}, \dots, \partial_{z_N} \rangle$ by \mathcal{D}_N .

The *integration* of the \mathcal{D}_{M+N} -module $L = \mathcal{D}_{M+N}/\mathcal{I}$ for the x variables is defined by

$$\frac{\mathcal{D}_{M+N}}{\sum_{i=1}^M \partial_{x_i} \mathcal{D}_{M+N}} \otimes_{\mathcal{D}_{M+N}} L = \frac{\mathcal{D}_{M+N}}{\mathcal{I} + \sum_{i=1}^M \partial_{x_i} \mathcal{D}_{M+N}}, \tag{B.16}$$

which is a left \mathcal{D}_N -module (see [93, §5.5] or [97, §6.10] for related algorithms and examples). The integration of a \mathcal{D} -module is an algebraic counterpart of the integration in calculus (for example, see [93, Th 5.5.1, p.227]). It also gives a different approach to the twisted

cohomology theory (see, e.g., [93, Th 5.5.11, p.233]). The integration module of L is sometimes denoted with an integral sign:

$$\int_{\pi} L, \tag{B.17}$$

where π denotes the natural projection from the (z, x) -space to the z -space.

The *restriction* of the \mathcal{D} -module L to a linear subspace $z_1 = c_1, \dots, z_N = c_N$, for some constant $c_i \in \mathbb{C}$, is defined by

$$\frac{\mathcal{D}_{M+N}}{\sum_{j=1}^n (z_j - c_j) \mathcal{D}_{M+N}} \otimes_{\mathcal{D}_{M+N}} L = \frac{\mathcal{D}_{M+N}}{\mathcal{I} + \sum_{j=1}^n (z_j - c_j) \mathcal{D}_{M+N}}. \tag{B.18}$$

It is an algebraic counterpart of the restriction of solutions of differential equations to the linear subspace (see [93, Prop. 5.2.14, p.207]).

Theorem B.7. (see e.g. [97, Th 6.10.8, p.303].) *If L is a holonomic \mathcal{D}_{M+N} -module, then the integration and the restriction defined above are holonomic \mathcal{D}_N -modules.*

Finally, let us note that the integration and the restriction constructions commute:

$$\frac{\mathcal{D}_n}{\sum_{j=1}^n (z_j - c_j) \mathcal{D}_n} \otimes_{\mathcal{D}_N} \left(\frac{\mathcal{D}_{M+N}}{\sum_{i=1}^M \partial_{x_i} \mathcal{D}_{M+N}} \otimes_{\mathcal{D}_{M+N}} L \right) \tag{B.19}$$

$$= \frac{\mathcal{D}_M}{\sum_{i=1}^M \partial_{x_i} \mathcal{D}_M} \otimes_{\mathcal{D}_M} \left(\frac{\mathcal{D}_{M+N}}{\sum_{j=1}^n (z_j - c_j) \mathcal{D}_{M+N}} \otimes_{\mathcal{D}_{M+N}} L \right) \tag{B.20}$$

$$= \frac{\mathcal{D}_{M+N}}{\mathcal{I} + \sum_{j=1}^n (z_j - c_j) \mathcal{D}_{M+N} + \sum_{i=1}^M \partial_{x_i} \mathcal{D}_{M+N}}. \tag{B.21}$$

Restrictions of generalized Feynman integrals will be discussed in the forthcoming paper [105].

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