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't Hooft expansion of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ $\mathcal{N}=4$ SYM revisited

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Abstract: We study the 't Hooft expansion of $d=4$ $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with the gauge group $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$. We consider the $1/N_5$ expansion with fixed $g_s N_5$, where g_s denotes the string coupling of bulk type IIB string theory on $AdS_5 \times \mathbb{RP}^5$ and N_5 refers to the RR 5-form flux through $\mathbb{RP}^5$. N_5 differs from N due to a shift coming from the RR charge of O3-plane. As an example, we consider the $1/N_5$ expansion of the free energy of $\mathcal{N}=4$ SYM on S^4 and the 1/2 BPS circular Wilson loops in the fundamental representation of $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$. We find that the $1/N_5$ expansion is more "closed string like" than the ordinary $1/N$ expansion.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, Matrix Models

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Contents

1 Introduction

$d = 4$ $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with the gauge group SO(N) or Sp(N) is realized as a worldvolume theory on D3-branes in the presence of orientifold 3-plane, and it is holographically dual to the type IIB string theory on $AdS_5 \times \mathbb{RP}^5$ [1]. In [2, 3], the 1/2 BPS circular Wilson loops in SO(N) and Sp(N) $\mathcal{N} = 4$ SYM are studied in the $1/N$ expansion with fixed $g_s N$, where g_s denotes the string coupling of bulk type IIB string theory. Note that g_s is proportional to the square of the Yang-Mills coupling constant g^2_{YM}. In [4, 5], it is suggested that in the context of AdS/CFT correspondence, it is more natural to consider the $1/N_5$ expansion with fixed $g_s N_5$, where N_5 refers to the RR 5-form flux through $\mathbb{RP}^5$ in the bulk type IIB string theory on $AdS_5 \times \mathbb{RP}^5$

$$N_5 = \int_{\mathbb{RP}^5} \frac{G_5}{2\pi}. \tag{1.1}$$

As shown in [6–8], N_5 is shifted from N due to the RR charge of orientifold 3-plane

$$N_5 = \begin{cases} \frac{N}{2} - \frac{1}{4}, & \text{for } \text{SO}(N), \\ \frac{N}{2} + \frac{1}{4}, & \text{for } \text{Sp}(N). \end{cases} \tag{1.2}$$

Rather surprisingly, the $1/N_5$ expansion with fixed $g_s N_5$ has not been fully explored in the literature before, as far as we know. In this paper, we will study the $1/N_5$ expansion for the partition function of $\mathcal{N} = 4$ SYM on S^4 as well as the 1/2 BPS circular Wilson loops in the fundamental representation of SO(N) or Sp(N).[1] In the original $1/N$ expansion, both

[1] Note that SO($2n$) gauge theories and Sp($2n$) gauge theories are formally related by the replacement $n \to -n$ [9, 10].

even and odd powers of $1/N$ appear in the expansion of the 1/2 BPS Wilson loops [2, 3]. It turns out that the $1/N_5$ expansion is more "closed string like": in the $1/N_5$ expansion of the partition function, only the even powers of $1/N_5$ appear and in the $1/N_5$ expansion of 1/2 BPS Wilson loops only the odd powers of $1/N_5$ appear, except for a constant term. This is consistent with the general property of holography where D-branes/O-planes are replaced by a closed string background in the bulk gravitational picture.[2]

This paper is organized as follows. In section 2, we study the $1/N_5$ expansion of the partition function of $\mathcal{N} = 4$ SYM, which is inversely proportional to the volume of the gauge group $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$. We find that the volume of the gauge group is characterized by a universal function $V(N_5)$ for both $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, up to an overall factor $2^{\pm N_5}$. It turns out that the $1/N_5$ expansion of $\log V(N_5)$ contains only even powers of $1/N_5$. In section 3, we study the $1/N_5$ expansion of the 1/2 BPS circular Wilson loops of $\mathcal{N} = 4$ SYM in the fundamental representation of $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$. Finally, we conclude in section 4 with some discussions. In appendix A, we present a proof of the relation (3.4).

2 $1/N_5$ expansion of the volume of SO(N) and Sp(N)

In this section, we consider the $1/N_5$ expansion of the free energy of $\mathcal{N} = 4$ SYM on S^4 with the gauge group $G = \mathrm{SO}(N)$ or $G = \mathrm{Sp}(N)$. As shown by Pestun [12], the partition function of $\mathcal{N} = 4$ SYM on S^4 reduces to a Gaussian matrix model owing to the supersymmetric localization

$$Z_G = \frac{1}{\mathrm{vol}(G)} \int_{\mathrm{Lie}(G)} dM e^{-\frac{1}{2g_s} \mathrm{Tr}\, M^2}, \tag{2.1}$$

where the integral of M is over the Lie algebra of gauge group G. Since the integral of M is Gaussian, the g_s-dependence of Z_G is rather simple

$$Z_G = \frac{(2\pi g_s)^{\frac{1}{2}\dim G}}{\mathrm{vol}(G)}, \tag{2.2}$$

and Z_G is essentially determined by the volume of the gauge group G. Thus, in what follows we will consider the $1/N_5$ expansion of $\mathrm{vol}(G)$.

The volume of $\mathrm{SO}(N)$ is given by [13–15]

$$\mathrm{vol}[\mathrm{SO}(N)] = \frac{2^{N-\frac{1}{2}} \pi^{\frac{1}{4}N(N+1)}}{\prod_{k=1}^{N} \Gamma(k/2)} = \begin{cases} \dfrac{2^{\frac{1}{2}} (2\pi)^{n^2}}{(n-1)! \prod_{i=1}^{n-1} (2i-1)!}, & (N = 2n), \\ \dfrac{2^{n+\frac{1}{2}} (2\pi)^{n^2+n}}{\prod_{i=1}^{n} (2i-1)!}, & (N = 2n+1). \end{cases} \tag{2.3}$$

Our definition of the volume of $\mathrm{SO}(2n)$ is the same as that in [15], but the volume of $\mathrm{SO}(2n+1)$ differs from [15] by a factor of $(\pi/2)^{\frac{1}{4}}$. The volume of $\mathrm{Sp}(N)$ is given by [15]

$$\mathrm{vol}[\mathrm{Sp}(2n)] = \frac{2^{-n} (2\pi)^{n^2+n}}{\prod_{i=1}^{n} (2i-1)!}. \tag{2.4}$$

[2]The question of open versus closed string expansions for the expectation values of Wilson loops of $\mathcal{N} = 4$ SYM was addressed for $G = \mathrm{SU}(N)$ in [11].

From the definition of N_5 in (1.2), one can rewrite the above volumes in terms of N_5 as

$$\begin{aligned} \mathrm{vol}(G) &= 2^{\pm N_5} V(N_5), \\ V(N_5) &= \frac{2^{N_5}\pi^{(N_5+1/4)(N_5+3/4)} G_2(1/2)}{G_2\left(N_5+\frac{3}{4}\right) G_2\left(N_5+\frac{5}{4}\right)}, \end{aligned} \tag{2.5}$$

where the $\pm$ sign corresponds to $G = \mathrm{SO}(N)$ and $G = \mathrm{Sp}(N)$, respectively, and $G_2(z)$ denotes the Barnes G-function.

Now, let us consider the free energy coming from the volume of the gauge group G

$$-\log[\mathrm{vol}(G)] = \mp N_5 \log 2 - \log V(N_5). \tag{2.6}$$

The $1/N_5$ expansion of the Barnes G-function in (2.5) can be computed by integrating the asymptotic expansion of the Γ-function[3]

$$\log \Gamma(z+a) = (z+a-1/2)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=2}^{\infty} \frac{(-1)^k B_k(a)}{k(k-1)z^{k-1}}, \tag{2.7}$$

where $B_k(a)$ denotes the Bernoulli polynomial. After some algebra, we find

$$\begin{aligned} -\log V(N_5) = c_0 &+ N_5^2\left(-\frac{3}{2} + \log\frac{N_5}{\pi}\right) - \frac{5}{48}\log N_5 \\ &+ \sum_{g=2}^{\infty} N_5^{2-2g}\left[\frac{B_{2g}(1/4)}{2g(g-1)} - \frac{B_{2g-1}(1/4)}{4(g-1)(2g-1)}\right], \end{aligned} \tag{2.8}$$

where c_0 is some constant. As we mentioned in the Introduction, the $1/N_5$ expansion of $\log V(N_5)$ has only even powers of $1/N_5$.

This $1/N_5$ expansion (2.8) should be compared with the $1/N$ expansion appearing in the topological string. In the case of topological string, the natural expansion parameter is $1/N_{\mathrm{top}}$, where N_{top} is given by [15]

$$N_{\mathrm{top}} = N \mp 1. \tag{2.9}$$

Here the upper and lower sign correspond to $G = \mathrm{SO}(N)$ and $G = \mathrm{Sp}(N)$, respectively. The $1/N_{\mathrm{top}}$ expansion of the volume of G is computed in [15]

$$-\log[\mathrm{vol}(G)] = \frac{1}{2}\sum_g \left(\chi(\mathcal{M}_g) N_{\mathrm{top}}^{2-2g} \pm \chi(\mathcal{M}_g^1) N_{\mathrm{top}}^{1-2g}\right), \tag{2.10}$$

where $\chi(\mathcal{M}_g)$ and $\chi(\mathcal{M}_g^1)$ denote the Euler characteristic of the moduli space of Riemann surfaces of genus g with zero and one cross-cap, respectively [16]

$$\chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g-2)}, \quad \chi(\mathcal{M}_g^1) = \frac{2^{2g-1} B_{2g}(1/2)}{2g(2g-1)}. \tag{2.11}$$

One can see that the $1/N_{\mathrm{top}}$ expansion of $\mathrm{vol}(G)$ contains both even and odd powers of $1/N_{\mathrm{top}}$, while the $1/N_5$ expansion of $\mathrm{vol}(G)$ contains only even powers of $1/N_5$ except for

[3] See e.g. http://dlmf.nist.gov/5.11.E8.

the first term of (2.6). This difference comes from the different definition of N_5 in (1.2) and $N_{\rm top}$ in (2.9). As discussed in [15, 17], the shift of N in $N_{\rm top}$ (2.9) comes from the RR charge of topological O-plane, which differs from the RR charge of O3-plane in type IIB string theory. Thus the $1/N_{\rm top}$ expansion (2.10) cannot be applied to our case of $\mathcal{N}=4$ SYM. In the holographic duality between $\mathcal{N}=4$ SYM with the gauge group SO(N) or Sp(N) and the type IIB string theory on $AdS_5\times\mathbb{RP}^5$, we should use the $1/N_5$ expansion (2.8), instead of the $1/N_{\rm top}$ expansion (2.10).

3 1/2 BPS Wilson loops in the fundamental representation of SO(N) and Sp(N)

In this section, we consider the $1/N_5$ expansion of the 1/2 BPS circular Wilson loops in the fundamental representation of $G=\text{SO}(N)$ or $G=\text{Sp}(N)$. As shown in [12, 18, 19], the expectation value of the 1/2 BPS circular Wilson loop is given by the Gaussian matrix model

$$W_G=\Big\langle \text{Tr}_F\, e^M\Big\rangle, \tag{3.1}$$

where the expectation value is defined by the Gaussian measure (2.1). Note that in our definition of W_G we do not divide it by the dimension N of the fundamental representation. The Gaussian integral (3.1) can be evaluated by the method of orthogonal polynomials and the result is written in terms of the Laguerre polynomials [2]

$$\begin{aligned} W_{\text{SO}(2n)}&=2e^{\frac{1}{2}g_s}\sum_{i=0}^{n-1}L_{2i}(-g_s),\\ W_{\text{SO}(2n+1)}&=1+2e^{\frac{1}{2}g_s}\sum_{i=0}^{n-1}L_{2i+1}(-g_s),\\ W_{\text{Sp}(2n)}&=2e^{\frac{1}{2}g_s}\sum_{i=0}^{n-1}L_{2i+1}(-g_s). \end{aligned} \tag{3.2}$$

One can check that they are correctly normalized as

$$W_{\text{SO}(N)}\Big|_{g_s=0}=N,\quad W_{\text{Sp}(N)}\Big|_{g_s=0}=N. \tag{3.3}$$

As explained in appendix A, the derivative of W_G with respect to g_s has a simple form

$$\partial_{g_s}W_{\text{SO}(N)}=e^{\frac{1}{2}g_s}L^{(2)}_{N-2}(-g_s),\quad \partial_{g_s}W_{\text{Sp}(N)}=e^{\frac{1}{2}g_s}L^{(2)}_{N-1}(-g_s), \tag{3.4}$$

which are both written in terms of N_5 as

$$\partial_{g_s}W_G=e^{\frac{1}{2}g_s}L^{(2)}_{2N_5-\frac{3}{2}}(-g_s). \tag{3.5}$$

3.1 $1/N_5$ expansion of W_G

In this subsection, we consider the $1/N_5$ expansion of W_G with fixed 't Hooft parameter λ

$$\lambda=8g_sN_5. \tag{3.6}$$

To do this, it is useful to express W_G as a contour integral [20]. Let us first consider $W_{\mathrm{SO}(N)}$ for definiteness. Using the series expansion of the Laguerre polynomial

$$L_n^{(\alpha)}(-g_s) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i} \frac{g_s^i}{i!}, \tag{3.7}$$

$\partial_{g_s} W_{\mathrm{SO}(N)}$ in (3.4) is written as

$$\begin{aligned}
\partial_{g_s} W_{\mathrm{SO}(N)} &= e^{\frac{1}{2}g_s} \sum_{i=0}^{N-2} \binom{N}{N-2-i} \frac{g_s^i}{i!} \\
&= e^{\frac{1}{2}g_s} \sum_{i=0}^{N-2} \frac{N! g_s^i}{(N-2-i)!(i+2)!i!} \\
&= e^{\frac{1}{2}g_s} \oint \frac{dw}{2\pi \mathrm{i}} \sum_{i=0}^{N-2} \frac{N!}{(N-2-i)!(i+2)!} w^{i-1} \frac{g_s^i}{i! w^i} \\
&= e^{\frac{1}{2}g_s} \oint \frac{dw}{2\pi \mathrm{i}} \frac{(1+w)^N}{w^3} e^{\frac{g_s}{w}},
\end{aligned} \tag{3.8}$$

where the contour of w-integral is a circle surrounding $w = 0$ counterclockwise. By the change of variable $w = e^{2z} - 1$ we find

$$\begin{aligned}
\partial_{g_s} W_{\mathrm{SO}(N)} &= \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{4\sinh^3 z} e^{(2N-1)z + \frac{g_s}{2}\coth z} \\
&= \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{4\sinh^3 z} e^{4N_5 z + \frac{g_s}{2}\coth z},
\end{aligned} \tag{3.9}$$

where the contour of z-integral is around $z = 0$. For the $\mathrm{Sp}(N)$ case, one can show that $\partial_{g_s} W_{\mathrm{Sp}(N)}$ is also given by the same formula (3.9). Thus we find

$$\partial_{g_s} W_G = \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{4\sinh^3 z} e^{4N_5 z + \frac{g_s}{2}\coth z}. \tag{3.10}$$

Finally, integrating this expression with respect to g_s, we arrive at

$$W_G = \pm\frac{1}{2} + \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{\sinh z \sinh 2z} e^{4N_5 z + \frac{g_s}{2}\coth z}. \tag{3.11}$$

Here we have determined the integration constant by the normalization condition (3.3).

In order to study the 't Hooft expansion of W_G, it is convenient to further rewrite the second term of (3.11) as

$$\begin{aligned}
&\oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{\sinh z \sinh 2z} e^{4N_5 z + \frac{g_s}{2}\coth z} \\
&= \oint \frac{dz}{2\pi \mathrm{i}} \left(\frac{1}{2\sinh^2 z} - \frac{2\sinh^2 \frac{z}{2}}{\sinh z \sinh 2z} \right) e^{4N_5 z + \frac{g_s}{2}\coth z} \\
&= \oint \frac{dz}{2\pi \mathrm{i}} \left(-\frac{1}{g_s} e^{4N_5 z} \partial_z e^{\frac{g_s}{2}\coth z} \right) - \oint \frac{dz}{2\pi \mathrm{i}} \frac{2\sinh^2 \frac{z}{2}}{\sinh z \sinh 2z} e^{4N_5 z + \frac{g_s}{2}\coth z} \\
&= \frac{4N_5}{g_s} \oint \frac{dz}{2\pi \mathrm{i}} e^{4N_5 z + \frac{g_s}{2}\coth z} - \oint \frac{dz}{2\pi \mathrm{i}} \frac{2\sinh^2 \frac{z}{2}}{\sinh z \sinh 2z} e^{4N_5 z + \frac{g_s}{2}\coth z}.
\end{aligned} \tag{3.12}$$

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One can show that the first term of (3.12) is equal to the 1/2 BPS Wilson loop of $\mathrm{U}(2N_5)$ $\mathcal{N}=4$ SYM [19]

$$\frac{4N_5}{g_s}\oint\frac{dz}{2\pi\mathrm{i}}e^{4N_5 z+\frac{g_s}{2}\coth z}=e^{\frac{1}{2}g_s}L^{(1)}_{2N_5-1}(-g_s)=W_{\mathrm{U}(2N_5)}. \tag{3.13}$$

Let us consider the 't Hooft expansion of $W_{\mathrm{U}(2N_5)}$ in (3.13) following the approach of [20]. By rescaling $z\to g_s z$, $W_{\mathrm{U}(2N_5)}$ is written as

$$W_{\mathrm{U}(2N_5)}=4N_5\oint\frac{dz}{2\pi\mathrm{i}}e^{\frac{1}{2}(\lambda z+z^{-1})+\frac{g_s}{2}\coth g_s z-\frac{1}{2}z^{-1}}. \tag{3.14}$$

The first part of the exponential $e^{\frac{1}{2}(\lambda z+z^{-1})}$ is essentially the generating function of the modified Bessel function of the first kind $I_n(x)$

$$e^{\frac{1}{2}(\lambda z+z^{-1})}=\sum_{n\in\mathbb{Z}}\frac{\widehat{I}_n}{z^n}, \tag{3.15}$$

where $\widehat{I}_n$ is given by

$$\widehat{I}_n=\frac{I_n(\sqrt{\lambda})}{(\sqrt{\lambda})^n}. \tag{3.16}$$

The second part of the exponential in (3.14) can be expanded in g_s as

$$e^{\frac{g_s}{2}\coth g_s z-\frac{1}{2}z^{-1}}=1+\frac{z}{6}g_s^2+\left(\frac{z^2}{72}-\frac{z^3}{90}\right)g_s^4+\mathcal{O}(g_s^6). \tag{3.17}$$

Then, taking the residue at $z=0$ we find the small g_s expansion of $W_{\mathrm{U}(2N_5)}$

$$W_{\mathrm{U}(2N_5)}=\frac{\lambda}{2g_s}\left[\widehat{I}_1+\frac{\widehat{I}_2}{6}g_s^2+\left(\frac{\widehat{I}_3}{72}-\frac{\widehat{I}_4}{90}\right)g_s^4+\mathcal{O}(g_s^6)\right]. \tag{3.18}$$

Note that the small g_s expansion with fixed $\lambda=8g_sN_5$ is basically the same as the $1/N_5$ expansion since g_s and $1/N_5$ are related by

$$g_s=\frac{\lambda}{8N_5}. \tag{3.19}$$

Next consider the second term of (3.12), which we will denote by W_T

$$W_T=-\frac{g_s}{4}\oint\frac{dz}{2\pi\mathrm{i}}\frac{8\sinh^2\frac{g_s z}{2}}{\sinh g_s z\sinh 2g_s z}e^{\frac{1}{2}(\lambda z+z^{-1})+\frac{g_s}{2}\coth g_s z-\frac{1}{2}z^{-1}}. \tag{3.20}$$

Again, the first part of the exponential has the expansion (3.15) and the rest of the integrand can be expanded in g_s as

$$\begin{aligned}\frac{8\sinh^2\frac{g_s z}{2}}{\sinh g_s z\sinh 2g_s z}e^{\frac{g_s}{2}\coth g_s z-\frac{1}{2}z^{-1}}&=1+\left(\frac{z}{6}-\frac{3z^2}{4}\right)g_s^2\\&\quad+\left(\frac{z^2}{72}-\frac{49z^3}{360}+\frac{3z^4}{8}\right)g_s^4+\mathcal{O}(g_s^6).\end{aligned} \tag{3.21}$$

Taking the residue at $z = 0$, we find the small g_s expansion of W_T with fixed λ

$$W_T = -\frac{g_s}{4}\left[\widehat{I}_1 + \left(\frac{\widehat{I}_2}{6} - \frac{3\widehat{I}_3}{4}\right) g_s^2 + \left(\frac{\widehat{I}_3}{72} - \frac{49\widehat{I}_4}{360} + \frac{3\widehat{I}_5}{8}\right) g_s^4 + \mathcal{O}(g_s^6)\right]. \tag{3.22}$$

To summarize, we find that W_G is decomposed as

$$W_G = \pm\frac{1}{2} + W_{\mathrm{U}(2N_5)} + W_T, \tag{3.23}$$

and the last two terms are expanded as

$$\begin{aligned} W_{\mathrm{U}(2N_5)} &= \sum_{g=0}^{\infty} a_g(\lambda) g_s^{2g-1} = \sum_{g=0}^{\infty} a_g(\lambda)\left(\frac{\lambda}{8N_5}\right)^{2g-1}, \\ W_T &= \sum_{g=0}^{\infty} b_g(\lambda) g_s^{2g+1} = \sum_{g=0}^{\infty} b_g(\lambda)\left(\frac{\lambda}{8N_5}\right)^{2g+1}, \end{aligned} \tag{3.24}$$

where $a_g(\lambda)$ and $b_g(\lambda)$ are some functions of λ whose explicit forms can be found in (3.18) and (3.22). One can see that $W_{\mathrm{U}(2N_5)}$ and W_T are both expanded in $1/N_5$ with only odd powers of $1/N_5$.

3.2 Relation to the ordinary $1/N$ 't Hooft expansion

Let us compare our $1/N_5$ expansion of W_G with the ordinary $1/N$ expansion of W_G. For definiteness, we consider the $G = \mathrm{SO}(N)$ case. The $1/N$ expansion of $W_{\mathrm{SO}(N)}$ is studied in [2, 3] where the 't Hooft coupling λ' is defined as

$$\lambda' = 8g_s N. \tag{3.25}$$

To this end, it is convenient to start with the expression of $W_{\mathrm{SO}(N)}$ found in [2][4]

$$\begin{aligned} W_{\mathrm{SO}(N)} &= e^{\frac{1}{2}g_s} L_{N-1}^{(1)}(-g_s) - \frac{1}{2}\int_0^{g_s} dx\, e^{\frac{1}{2}x} L_{N-1}^{(1)}(-x) \\ &= W_{\mathrm{U}(N)}(g_s) - \frac{1}{2}\int_0^{g_s} dx\, W_{\mathrm{U}(N)}(x). \end{aligned} \tag{3.26}$$

From the known $1/N$ expansion of the 1/2 BPS Wilson loop in $\mathrm{U}(N)$ $\mathcal{N} = 4$ SYM [19], one can easily compute the $1/N$ expansion of $W_{\mathrm{SO}(N)}$

$$W_{\mathrm{SO}(N)} = \frac{1}{2} + \frac{2\sqrt{2}N}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'/2}) - \frac{1}{2} I_0(\sqrt{\lambda'/2}) + \frac{\lambda' I_2(\sqrt{\lambda'/2})}{96N} - \frac{\lambda'^{3/2} I_3(\sqrt{\lambda'/2})}{384\sqrt{2}N^2} + \mathcal{O}(N^{-3}). \tag{3.27}$$

One can check that the $1/N$ expansion in (3.27) and our $1/N_5$ expansion are related by the change of parameters $(\lambda', N) \to (\lambda, g_s)$

$$\lambda' = 2\lambda + 4g_s, \quad N = \frac{\lambda}{4g_s} + \frac{1}{2}. \tag{3.28}$$

[4] See also appendix A for a derivation of this expression.

Plugging this relation into (3.27) and expanding in g_s, we find

$$W_{\mathrm{SO}(N)} = \frac{1}{2} + \frac{\lambda}{2g_s}\left[\widehat{I}_1 + \frac{\widehat{I}_2}{6}g_s^2\right] - \frac{g_s}{4}\widehat{I}_1 + \mathcal{O}(g_s^3). \tag{3.29}$$

This agrees with our result of $1/N_5$ expansion (3.18) and (3.22) up to this order $\mathcal{O}(g_s^3)$, as expected.

Note that, in the original $1/N$ expansion (3.27) both even and odd powers of N^{-1} appear. On the other hand, in our case (3.23) only the odd powers of g_s arise, except for the constant term $\pm 1/2$ in (3.23). Although our decomposition (3.23) is similar to (3.26), we stress that they are different. In particular, our W_T is not equal to the second term of (3.26).

4 Conclusions and outlook

In this paper, we have studied the $1/N_5$ expansion of the volume of the gauge group G and the 1/2 BPS Wilson loops in the fundamental representation of G in $\mathcal{N} = 4$ SYM with $G = \mathrm{SO}(N)$ or $G = \mathrm{Sp}(N)$. Due to the shift of N coming from the RR charge of O3-plane (1.2), the $1/N_5$ expansion with fixed 't Hooft parameter $\lambda = 8g_sN_5$ is different from the ordinary $1/N$ expansion. We found that the $1/N_5$ expansion looks more "closed string like" than the ordinary $1/N$ expansion. For instance, we found that the $1/N_5$ expansion of the volume of G contains only the even powers of $1/N_5$, except for the first term $\mp N_5 \log 2$ in (2.6). This is different from the $1/N_{\mathrm{top}}$ expansion of $\mathrm{vol}(G)$ in topological string [15]. It would be interesting to find a mathematical meaning, if any, of the coefficient of N_5^{2-2g} in (2.8) as a certain quantity on the moduli space of Riemann surfaces of genus g.

We have also studied the $1/N_5$ expansion of the 1/2 BPS Wilson loop W_G in the fundamental representation of $G = \mathrm{SO}(N)$ or $G = \mathrm{Sp}(N)$. We found that W_G is decomposed as (3.23). Except for the constant term $\pm 1/2$ in (3.23), $W_{\mathrm{U}(2N_5)}$ and W_T are both expanded in $1/N_5$ with only odd powers of $1/N_5$. It is tempting to speculate that $W_{\mathrm{U}(2N_5)}$ and W_T correspond to the untwisted and the twisted sector of bulk type IIB string theory on $AdS_5 \times \mathbb{RP}^5$. It would be interesting to understand the bulk gravitational interpretation of the decomposition (3.23) more clearly.

It would be interesting to extend our analysis to more general observables in $\mathcal{N} = 4$ SYM with the gauge group $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$, such as an integrated four-point correlator [5] and the 1/2 BPS Wilson loop in the spinor representation of $\mathrm{SO}(N)$ [2, 3], to name a few. We leave this as an interesting future problem.

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A Proof of (3.4)

In this appendix, we present a proof of the relation (3.4). For definiteness we consider $W_{\mathrm{SO}(2n)}$. To this end, we can use the fact that the Laguerre polynomial is written as a

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matrix element of the harmonic oscillator (see e.g. [21])

$$\langle i|e^{\sqrt{g_s}(a+a^\dagger)}|j\rangle = \langle j|e^{\sqrt{g_s}(a+a^\dagger)}|i\rangle = \sqrt{\frac{i!}{j!}}g_s^{\frac{j-i}{2}} L_i^{(j-i)}(-g_s), \tag{A.1}$$

where

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0, \quad |k\rangle = \frac{(a^\dagger)^k}{\sqrt{k!}}|0\rangle. \tag{A.2}$$

Then $W_{\mathrm{SO}(2n)}$ in (3.2) is written as

$$\begin{aligned} W_{\mathrm{SO}(2n)} &= 2\sum_{i=0}^{n-1}\langle 2i|e^{\sqrt{g_s}(a+a^\dagger)}|2i\rangle \\ &= \sum_{k=0}^{2n-1}\left[1+(-1)^k\right]\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k\rangle. \end{aligned} \tag{A.3}$$

Thus $W_{\mathrm{SO}(2n)}$ is naturally decomposed as

$$W_{\mathrm{SO}(2n)} = W^+_{\mathrm{SO}(2n)} + W^-_{\mathrm{SO}(2n)}, \tag{A.4}$$

where

$$\begin{aligned} W^+_{\mathrm{SO}(2n)} &= \sum_{k=0}^{2n-1}\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k\rangle, \\ W^-_{\mathrm{SO}(2n)} &= \sum_{k=0}^{2n-1}(-1)^k\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k\rangle. \end{aligned} \tag{A.5}$$

Note that $W^+_{\mathrm{SO}(2n)}$ is equal to the Wilson loop of $\mathrm{U}(2n)$ $\mathcal{N}=4$ SYM [19]. The sum over k in $W^+_{\mathrm{SO}(2n)}$ can be simplified as

$$\begin{aligned} \sqrt{g_s}W^+_{\mathrm{SO}(2n)} &= \sum_{k=0}^{2n-1}\langle k|\left[a, e^{\sqrt{g_s}(a+a^\dagger)}\right]|k\rangle \\ &= \sum_{k=0}^{2n-1}\left[\sqrt{k+1}\langle k+1|e^{\sqrt{g_s}(a+a^\dagger)}|k\rangle - \sqrt{k}\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k-1\rangle\right] \\ &= \sqrt{2n}\langle 2n|e^{\sqrt{g_s}(a+a^\dagger)}|2n-1\rangle \\ &= \sqrt{g_s}e^{\frac{1}{2}g_s}L^{(1)}_{2n-1}(-g_s). \end{aligned} \tag{A.6}$$

In the last step we used (A.1). Thus we find

$$W^+_{\mathrm{SO}(2n)} = W_{\mathrm{U}(2n)} = e^{\frac{1}{2}g_s}L^{(1)}_{2n-1}(-g_s), \tag{A.7}$$

which agrees with the known result of $W_{\mathrm{U}(2n)}$ in [19].

Next, let us consider the g_s-derivative of $W^-_{\mathrm{SO}(2n)}$

$$\begin{aligned}
\partial_{g_s} W^-_{\mathrm{SO}(2n)} &= \frac{1}{2\sqrt{g_s}} \sum_{k=0}^{2n-1} (-1)^k \langle k| e^{\sqrt{g_s}(a+a^\dagger)} (a+a^\dagger)|k\rangle \\
&= \frac{1}{2\sqrt{g}} \sum_{k=0}^{2n-1} (-1)^k \Big[\sqrt{k}\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k-1\rangle + \sqrt{k+1}\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k+1\rangle\Big] \\
&= \frac{1}{2\sqrt{g_s}} \sum_{k=0}^{2n-1} \Big[(-1)^k\sqrt{k}\langle k|e^{\sqrt{g_s}(a+a^\dagger)}|k-1\rangle - (-1)^{k+1}\sqrt{k+1}\langle k+1|e^{\sqrt{g_s}(a+a^\dagger)}|k\rangle\Big] \\
&= -\frac{1}{2\sqrt{g_s}}\sqrt{2n}\langle 2n|e^{\sqrt{g_s}(a+a^\dagger)}|2n-1\rangle \\
&= -\frac{1}{2} e^{\frac{1}{2}g_s} L^{(1)}_{2n-1}(-g_s).
\end{aligned} \tag{A.8}$$

Finally, we find

$$\begin{aligned}
\partial_{g_s} W_{\mathrm{SO}(2n)} &= \partial_{g_s} W^+_{\mathrm{SO}(2n)} + \partial_{g_s} W^-_{\mathrm{SO}(2n)} \\
&= \partial_{g_s}\Big[e^{\frac{1}{2}g_s} L^{(1)}_{2n-1}(-g_s)\Big] - \frac{1}{2} e^{\frac{1}{2}g_s} L^{(1)}_{2n-1}(-g_s) \\
&= e^{\frac{1}{2}g_s} L^{(2)}_{2n-2}(-g_s).
\end{aligned} \tag{A.9}$$

This proves (3.4) for the SO($2n$) case. SO($2n+1$) and Sp(N) cases can be proved in a similar manner.

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