

# Integrable supersymmetric deformations of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$

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**ABSTRACT:** We construct a family of type IIB string backgrounds that are deformations of  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  with a “squashed”  $\text{AdS}_3 \times \text{S}^3$  metric supported by a combination of NSNS and RR fluxes. They have global  $\text{SU}(1,1) \times \text{SU}(2)$  symmetry, regular curvature, constant dilaton and preserve 8 supercharges. Upon compactification to 4 dimensions they reduce to  $\mathcal{N} = 2$  supersymmetric  $\text{AdS}_2 \times \text{S}^2$  solutions with electric and magnetic Maxwell fluxes. These type IIB supergravity solutions can be found from the undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background by a combination of T-dualities and S-duality. In contrast to T-duality, S-duality transformations of a type IIB supergravity background do not generally preserve the classical integrability of the corresponding Green-Schwarz superstring sigma model. Nevertheless, we show that integrability is preserved in the present case. Indeed, we find that these backgrounds can be obtained, up to T-dualities, from an integrable inhomogeneous Yang-Baxter deformation (with unimodular Drinfel’d-Jimbo R-matrix) of the original  $\text{AdS}_3 \times \text{S}^3$  supercoset model.

**KEYWORDS:** AdS-CFT Correspondence, Integrable Field Theories, Sigma Models, String Duality

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## 1 Introduction

Exact solutions of classical string theory that have a direct target space interpretation (i.e. are described by conformal sigma models) are rare and hard to find. For most relevant leading-order solutions, such as non-supersymmetric 4d black holes, their exact form is not known. Additional global symmetries (in particular, supersymmetry) are important to have some control over deformations induced by  $\alpha'$ -corrections. An even smaller subclass of string sigma models are integrable (and thus have, in principle, a solvable string spectrum).

Given an integrable background that has some isometries one can generate new integrable solutions with more parameters by T-dualities (for some examples see, e.g., [1–8]). On the other hand, S-duality maps one type IIB supergravity solution into another, but is not a symmetry of the classical string theory (it does not act on the string worldsheet)

and thus, in contrast to T-duality, does not, in general, “commute” with  $\alpha'$ -corrections or integrability.

Supersymmetric integrable string backgrounds like  $\text{AdS}_n \times \text{S}^n \times \text{T}^{10-2n}$  that appear as near-horizon limits of brane configurations are of particular interest. It is important to study closely related solutions with non-trivial parameters that are also supersymmetric and integrable. Having extra parameters may help clarify the structure of the underlying integrable S-matrix and “resolve” special limits.

Below we will present a new 8-parameter class of deformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  type IIB backgrounds supported by a combination of homogeneous NSNS and RR fluxes. They have global  $\text{SU}(1,1) \times \text{SU}(2)$  symmetry, regular curvature, constant dilaton and preserve  $\frac{1}{4}$  of maximal 10d supersymmetry. As type IIB supergravity solutions their existence may not be surprising — they can be obtained from undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  (supported by RR 3-form flux) by a combination of T-dualities and S-dualities. What is non-trivial is that the corresponding Green-Schwarz (GS) superstring sigma model will be also integrable.

As the relation between the undeformed and deformed backgrounds will involve S-duality, integrability will not simply follow from the known integrability of the original  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  model (see [9–11] and references there). The proof of integrability will be based on the key observation that a particular subclass of backgrounds (from which the others can be obtained by just T-dualities) correspond to a Yang-Baxter (YB) deformed supercoset model [12] with a particular Drinfel’d-Jimbo R-matrix [13, 14]. Being an inhomogeneous YB deformation it will not simply be equivalent to a T-duality transformation of the original background. Also, in contrast to some familiar examples of YB or  $\eta$ -deformations (see, e.g., [15, 16] for a review) that have few manifest symmetries, singularities and solve the generalized supergravity equations [17–19], here the resulting background will share the key features of the undeformed  $\text{AdS}_3 \times \text{S}^3$  — manifest non-abelian isometry, supersymmetry, regular curvature, constant dilaton and, most importantly, will solve the standard type IIB supergravity equations, i.e. will represent a consistent string model.

Let us recall that the standard  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background can be supported by a mix of NSNS and RR 3-form fluxes. In the pure NSNS case the worldsheet theory is a supersymmetric extension of the  $\text{SL}(2, \mathbb{R}) \times \text{SU}(2)$  WZW theory (and thus admits a local NSR description and is solvable by 2d CFT methods). The model with non-zero RR flux has a local GS description and its integrability follows from its construction as a sigma model on the semi-symmetric supercoset  $\text{G}/\text{H}$  with  $\text{G} = \text{PSU}(1,1|2) \times \text{PSU}(1,1|2)$  and  $\text{H} = \text{SU}(1,1) \times \text{SU}(2)$  [9, 20–24].

We will be interested in “warped” or “squashed” deformations of  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  (depending on a continuous deformation parameter  $\kappa$ ) which preserve only half of the global symmetries,<sup>1</sup> i.e.  $\text{SU}(1,1) \cong \text{SL}(2, \mathbb{R})$  and  $\text{SU}(2)$ . One way to obtain such deformed  $\text{AdS}_{3\kappa}$  and  $\text{S}^3_\kappa$  geometries is to apply TsT transformations involving a particular abelian isometry as well as one extra torus direction.<sup>2</sup> Applying this TsT transformation at the level

<sup>1</sup>The isometry group of undeformed  $\text{AdS}_3$  is  $\text{SU}(1,1) \times \text{SU}(1,1)$ , while the one of  $\text{S}^3$  is  $\text{SU}(2) \times \text{SU}(2)$ .

<sup>2</sup>Let us mention that marginal NSNS “JJ” deformations [25, 26] of the  $\text{SL}(2, \mathbb{R}) \times \text{SU}(2)$  WZW model generated by T-dualities were discussed, e.g., in [27, 28]. We also note that the “squashed”  $\text{S}^3$  sigma model was first shown to be integrable in [29]. This model with WZ term added is not conformal (has 2-parameter RG flow in [30–32]) and is also integrable [33, 34].

of the GS model generates integrable embeddings into type IIB string theory [35, 36].<sup>3</sup> For  $\text{AdS}_{3\kappa} \times \text{S}^3 \times \text{T}^4$  and  $\text{AdS}_3 \times \text{S}_{\kappa}^3 \times \text{T}^4$  the corresponding supergravity backgrounds preserve 8 supersymmetries, while combining the two TsT transformations leads to  $\text{AdS}_{3\kappa} \times \text{S}_{\kappa'}^3 \times \text{T}^4$  that should break all supersymmetries [35].

Here we will find a different integrable embedding of the  $\text{AdS}_{3\kappa} \times \text{S}_{\kappa}^3 \times \text{T}^4$  metric (with equal deformation parameters  $\kappa$ ) into type IIB supergravity that will preserve 8 supersymmetries and will be supported by a 7-parameter family of homogeneous NSNS and RR 3-form and 5-form fluxes.<sup>4</sup> As mentioned above, it will not be related to the undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  just by T-dualities and thus its integrability will be non-trivial.

From a broader perspective, our results are of interest in the context of the following questions:

- (i) Given an integrable bosonic sigma model, when is its embedding into superstring theory, with non-zero RR fluxes required for conformality, also integrable?<sup>5</sup> It appears that a sufficient condition for a positive answer is the preservation of a sufficient amount of target space supersymmetry:<sup>6</sup> it may promote the classical integrability of the bosonic sector (the existence of Lax pair) to the full GS model. While a general proof of this is not known, our family of backgrounds provides a new explicit example of this connection (complementing the familiar ones discussed in [42–45]).
- (ii) When do S-duality transformations of a background accidentally preserve the integrability of the corresponding type IIB GS sigma model? Here we find a new non-trivial example of this in addition to the familiar S-dual undeformed backgrounds  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  with NSNS vs RR 3-form fluxes and also to the “Jordanian” YB deformation ones discussed in [46, 47].

The structure of the rest of this paper is as follows. We start in section 2 with the simplest example of the deformed  $\text{AdS}_{3\kappa} \times \text{S}_{\kappa}^3$  background supported by a combination of NSNS and RR 3-form fluxes obtained by a TsT transformation in the two Hopf fibres of the undeformed  $\text{AdS}_3 \times \text{S}^3$ . We shall then describe the type IIB supergravity solutions where the deformed  $\text{AdS}_{3\kappa} \times \text{S}_{\kappa}^3 \times \text{T}^4$  metric is supported by a 7-parameter family of NSNS  $H_3$  and RR  $F_3$  and  $F_5$  fluxes (and constant scalars). We shall explain how these backgrounds can be obtained from the standard undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  solution with  $F_3$  flux by

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<sup>3</sup>T-duality along Hopf fibres was originally discussed in a similar context in [37].

<sup>4</sup>For examples when the same deformed metric can be supported by different combinations of fluxes see, e.g., [38].

<sup>5</sup>If a GS sigma model is conformal (or at least scale invariant), hence it has  $\varkappa$ -symmetry [18, 19], one might think that  $\varkappa$ -symmetry implies integrability in the fermionic sector as well. This need not, however, be true as the  $\varkappa$ -symmetry is a gauge redundancy — fixing it will not, in general, leave a symmetry relating bosons and fermions.

<sup>6</sup>This is not a necessary condition: there are examples of integrable bosonic models that can be embedded into integrable superstring sigma models without any target space supersymmetry. These include the  $\eta$  — [39] and  $\lambda$  — [40] deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring, as well as the 3-parameter  $\gamma$ -deformed background [6, 7, 41]. Even in these examples, the global supersymmetry of the undeformed theory is not completely lost however, since it effectively becomes “hidden” in the deformed one.

U-duality, i.e. by a combination of TsT and S-duality transformations. We shall highlight some special “seed” solutions, in particular, the one that has only non-vanishing  $F_5$  flux.

Next, in section 3 we will show that our  $\text{AdS}_{3\kappa} \times \text{S}^3_\kappa \times \text{T}^4$  solutions admit 8 Killing spinors, i.e. preserves  $\frac{1}{4}$  of maximal 10d supersymmetry. Section 4 will be devoted to demonstrating the integrability of the GS sigma model corresponding to a non-trivial representative of the deformed backgrounds (from which all others can be obtained just by T-dualities). We shall consider a particular YB deformation of  $\frac{\text{PSU}(1,1|2) \times \text{PSU}(1,1|2)}{\text{SU}(1,1) \times \text{SU}(2)}$  supercoset model based on the Drinfel’d-Jimbo R-matrix built from a Cartan-Weyl basis with all fermionic simple roots. In this case the solution of the modified classical Yang-Baxter equation is unimodular [13, 14] and thus should lead [48] to backgrounds that solve the standard supergravity equations (rather than the generalized ones [17, 18]).

In section 5 we first consider the analytically-continued family of solutions with  $\kappa = i\tilde{\kappa}$  in which the  $\text{AdS}_3$  part is written as a Hopf fibration over the Minkowski-signature  $\text{AdS}_2$  space. The resulting type IIB background is shown to interpolate between  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  (for  $\tilde{\kappa} = 0$ ) and  $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$  (for  $\tilde{\kappa} = 1$ ), with the latter being supported by the  $F_5$  flux only. Dimensionally reducing on the two Hopf fibres for any value of  $\tilde{\kappa}$  we get a family of 4d supergravity solutions with  $\text{AdS}_2 \times \text{S}^2$  metric supported by a combination of equal-charge electric and magnetic Maxwell fluxes that are familiar near-horizon limits of a family of  $\mathcal{N} = 2, d = 4$  BPS black holes (with constant scalars). We shall also discuss special limits of our family of solutions including a Schrödinger background corresponding to a particular Jordanian limit of the YB deformation with Drinfel’d-Jimbo R-matrix. We also construct the pp-wave limit, which describes the quadratic approximation to the BMN-expanded superstring action and we comment on the corresponding dispersion relation and tree-level bosonic S-matrix for the corresponding string fluctuations.

Some concluding remarks will be made in section 6. Appendix A contains details of the proof of supersymmetry in section 3 and also a construction of Killing spinors in the pp-wave limit. The background of an inhomogeneous YB deformation constructed using a non-unimodular Drinfel’d-Jimbo R matrix built from a distinguished Cartan-Weyl basis is presented in appendix B. In contrast to the background discussed in section 4 it only solves a set of generalised supergravity equations of motion. We also include some related integrable deformations with non-constant dilaton constructed using other TsT transformations in appendix C.

## 2 Deformed $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ backgrounds

In this section we will construct a class of  $\text{M}^6 \times \text{T}^4$  type IIB supergravity solutions where the metric of  $\text{M}^6 = \text{AdS}_{3\kappa} \times \text{S}^3_\kappa$  will be that of “squashed” or deformed  $\text{AdS}_3 \times \text{S}^3$  with the same deformation parameter  $\kappa$  in the two factors. These backgrounds will have regular curvature, constant dilaton, homogeneous NSNS and RR 3-form and 5-form fluxes and global  $\text{SU}(1,1) \times \text{SU}(2)$  symmetry. They will preserve 8 supercharges (see section 3). Moreover, as in the undeformed  $\text{AdS}_3 \times \text{S}^3$  case, the corresponding GS superstring sigma model will be integrable (see section 4).

## 2.1 Motivation

To recall, the metric of  $\text{AdS}_3 \times \text{S}^3$  space in global coordinates is<sup>7</sup>

$$ds^2 = -(1 + \rho^2)dt^2 + \frac{d\rho^2}{1 + \rho^2} + \rho^2 d\psi^2 + (1 - r^2)d\varphi^2 + \frac{dr^2}{1 - r^2} + r^2 d\phi^2 . \quad (2.1)$$

Its product with  $\text{T}^4$  can be embedded into 10d type IIB supergravity by adding the NSNS 3-form flux

$$H_3 = d\hat{B}, \quad \hat{B} = \rho^2 dt \wedge d\psi + r^2 d\varphi \wedge d\phi . \quad (2.2)$$

This geometry arises as the near horizon limit of the F1-NS5 solution (with equal charges). The same metric can also be supported by the RR 3-form flux  $F_3$  (corresponding to the near-horizon limit of the D1-D5 solution). In view of the  $\text{SL}(2, \mathbb{R})$  symmetry of the type IIB equations one can also consider a 1-parameter ( $|q| \leq 1$ ) family of mixed flux backgrounds (with constant dilaton  $\Phi = \Phi_0$ )

$$H_3 = \sqrt{1 - q^2} d\hat{B}, \quad F_3 = e^{-\Phi_0} q d\hat{B} . \quad (2.3)$$

Three more parameters can be added by applying special  $\text{O}(d, d)$  or TsT transformations in the 4-torus directions.<sup>8</sup> The resulting four-parameter family of supergravity backgrounds preserves 16 supersymmetries. The corresponding GS superstring model is integrable [9, 24, 49].

Our aim will be to find a more general class of type IIB backgrounds that preserve half of the maximal 16 supersymmetries and are still integrable. Furthermore, we will find that they have regular curvature, homogeneous fluxes<sup>9</sup> and constant dilaton and RR scalar.

To motivate their construction let us first review the symmetries of the  $\text{AdS}_3 \times \text{S}^3$  background. The isometry algebra of  $\text{AdS}_3$  is  $\mathfrak{su}(1, 1)_L \oplus \mathfrak{su}(1, 1)_R$ , while that of  $\text{S}^3$  is  $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ .<sup>10</sup> Including fermions this is promoted to the superisometry algebra  $\mathfrak{psu}(1, 1|2)_L \oplus \mathfrak{psu}(1, 1|2)_R$ . Four of the Killing spinors are associated to  $\mathfrak{su}(1, 1)_L \oplus \mathfrak{su}(2)_L$  and the other four are associated to  $\mathfrak{su}(1, 1)_R \oplus \mathfrak{su}(2)_R$ . When embedded into 10d supergravity with the  $\text{T}^4$  factor the supersymmetries are doubled and the corresponding GS superstring sigma model has 16 supersymmetries.

It is then natural to expect that backgrounds with 8 supersymmetries can be obtained by deforming either  $\mathfrak{psu}(1, 1|2)_L$  or  $\mathfrak{psu}(1, 1|2)_R$  in the superisometry algebra, while preserving the other copy. One way to achieve this, which has the additional advantage of preserving integrability, is to apply a TsT transformation in the abelian isometries of one copy of  $\mathfrak{psu}(1, 1|2)$ . The Cartan subalgebra is  $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$  for  $\text{AdS}_3$  and  $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$

<sup>7</sup>We shall always use the string-frame metric and we mostly omit the overall factor of string tension.

<sup>8</sup>We shall refer to all transformations of the form “T-duality —  $\text{GL}(d)$  coordinate redefinition — T-duality” as TsT transformations. This includes both the usual TsT transformations where s stands for shift and the TrT transformations of [11] where r stands for rotation.

<sup>9</sup>By homogeneous flux we mean that the corresponding tensor has constant tangent space components and thus automatically satisfies some of the field equations (namely the ones analogous to the Maxwell equations).

<sup>10</sup>We use the labels  $L$  and  $R$  to distinguish the two copies of the algebra.

for  $S^3$ . Without loss of generality, we may apply the TsT transformation in the isometries associated to the left copy  $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_L$ .

Let us start with the background (2.1)–(2.3) with pure RR 3-form flux ( $q = 1$ ) and with  $x_r$ ,  $r = 1, \dots, 4$ , coordinates on  $T^4$ . The TsT transformation in the left Cartan directions (i.e. in the combinations  $t + \psi$  and  $\varphi + \phi$  of coordinates in (2.1)) with parameter  $\kappa$  then produces the “warped”  $\text{AdS}_{3\kappa}$  and “squashed”  $S^3_\kappa$  metrics

$$ds^2 = -(1 + \rho^2)dt^2 + \frac{d\rho^2}{1 + \rho^2} + \rho^2 d\psi^2 - \kappa^2((1 + \rho^2)dt - \rho^2 d\psi)^2 + (1 - r^2)d\varphi^2 + \frac{dr^2}{1 - r^2} + r^2 d\phi^2 + \kappa^2((1 - r^2)d\varphi + r^2 d\phi)^2 + dx_r dx_r, \quad (2.4)$$

and the following 3-form fluxes (with the dilaton remaining constant)

$$H_3 = \kappa \sqrt{1 + \kappa^2} d\check{B}, \quad F_3 = e^{-\Phi_0} \sqrt{1 + \kappa^2} d\hat{B}. \quad (2.5)$$

Here  $\hat{B}$  is as defined in (2.2) (i.e. is given by the sum of  $\text{AdS}_3$  and  $S^3$  parts) while  $\check{B}$  has instead a product structure “mixing”  $\text{AdS}_3$  and  $S^3$  coordinates

$$\check{B} = [(1 + \rho^2)dt - \rho^2 d\psi] \wedge [(1 - r^2)d\varphi + r^2 d\phi]. \quad (2.6)$$

This background preserves the  $\mathfrak{psu}(1, 1|2)_R$  superisometries.<sup>11</sup>

It is useful to write this background in a different coordinate system where it has the form of a deformation of the Hopf fibrations of  $\text{AdS}_3$  (over euclidean  $\text{AdS}_2$  or 2d hyperbolic space  $H^2$ ) and  $S^3$  (over  $S^2$ ). Introducing the coordinates  $(\zeta_1, \zeta_2, \sigma)$  for  $\text{AdS}_3$  and  $(\xi_1, \xi_2, \theta)$  for  $S^3$  as

$$\begin{aligned} t &= \frac{\zeta_1 - \zeta_2}{2}, & \psi &= \frac{\zeta_1 + \zeta_2}{2}, & \rho &= \sinh \frac{\sigma}{2}, \\ \varphi &= \frac{\xi_1 - \xi_2}{2}, & \phi &= \frac{\xi_1 + \xi_2}{2}, & r &= \sin \frac{\theta}{2}, \end{aligned} \quad (2.7)$$

the deformed metric (2.4) takes the form

$$ds^2 = \frac{1}{4}(\sinh^2 \sigma d\zeta_2^2 + d\sigma^2 - (1 + \kappa^2)(d\zeta_1 - \cosh \sigma d\zeta_2)^2) + \frac{1}{4}(\sin^2 \theta d\xi_2^2 + d\theta^2 + (1 + \kappa^2)(d\xi_1 - \cos \theta d\xi_2)^2) + dx_r dx_r. \quad (2.8)$$

Writing the metric as a fibration over  $H^2 \times S^2$  we see that the deformation simply rescales the fibres by  $1 + \kappa^2$ .<sup>12</sup> The auxiliary 2-forms in (2.2) and (2.6) and their corresponding field strengths are then

$$\begin{aligned} \hat{B} &= \frac{1}{2} \left( \sinh^2 \frac{\sigma}{2} d\zeta_1 \wedge d\zeta_2 + \sin^2 \frac{\theta}{2} d\xi_1 \wedge d\xi_2 \right), \\ \check{B} &= \frac{1}{4} (d\zeta_1 - \cosh \sigma d\zeta_2) \wedge (d\xi_1 - \cos \theta d\xi_2), \\ d\hat{B} &= \frac{1}{4} [\sinh \sigma d\zeta_1 \wedge d\zeta_2 \wedge d\sigma + \sin \theta d\xi_1 \wedge d\xi_2 \wedge d\theta], \\ d\check{B} &= \frac{1}{4} [\sinh \sigma d\zeta_2 \wedge d\sigma \wedge (d\xi_1 - \cos \theta d\xi_2) + \sin \theta (d\zeta_1 - \cosh \sigma d\zeta_2) \wedge d\xi_2 \wedge d\theta]. \end{aligned} \quad (2.9)$$

<sup>11</sup>A similar TsT transformation in the right Cartan directions, which would preserve  $\mathfrak{psu}(1, 1|2)_L$ , gives rise to the same background with  $t \rightarrow -t$  and  $\varphi \rightarrow -\varphi$ .

<sup>12</sup>In the terminology of [50] this deformation of  $\text{AdS}_3$  is also known as “time-like squashed”  $\text{AdS}_3$ .

Introducing the 1-form basis

$$\begin{aligned} e^0 &= \frac{1}{2}\sqrt{1+\kappa^2} (d\xi_1 - \cosh \sigma d\xi_2), & e^1 &= \frac{1}{2} \sinh \sigma d\xi_2, & e^2 &= \frac{1}{2} d\sigma, \\ e^3 &= \frac{1}{2}\sqrt{1+\kappa^2} (d\xi_1 - \cos \theta d\xi_2), & e^4 &= \frac{1}{2} \sin \theta d\xi_2, & e^5 &= \frac{1}{2} d\theta, \end{aligned} \tag{2.10}$$

we can write the metric (2.8) and the homogeneous 3-form fluxes (2.9) as

$$\begin{aligned} ds^2 &= -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2, \\ d\hat{B} &= \frac{2}{\sqrt{1+\kappa^2}} (e^0 \wedge e^1 \wedge e^2 + e^3 \wedge e^4 \wedge e^5), \\ d\check{B} &= \frac{2}{\sqrt{1+\kappa^2}} (e^1 \wedge e^2 \wedge e^3 + e^0 \wedge e^4 \wedge e^5). \end{aligned} \tag{2.11}$$

More general type IIB supergravity solutions can be obtained by applying additional U-duality transformations (including  $SL(2, \mathbb{R})$  S-duality and TsT in the 4-torus directions). In the next section 2.2 we will show how a more general family of flux backgrounds supporting the same deformed metric (2.8) can be constructed directly as type IIB supergravity solution.

## 2.2 Type IIB supergravity solutions with $AdS_{3\kappa} \times S_\kappa^3$ metric

The bosonic part of the type IIB supergravity action may be written as

$$\begin{aligned} S_{10} &= \int d^{10}x \sqrt{-G} \left( e^{-2\Phi} \left( R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_1|^2 - \frac{1}{2} |F_3|^2 - \frac{1}{4} |F_5|^2 \right) \\ &\quad + \frac{1}{2} \int F_5 \wedge F_3 \wedge B_2, \end{aligned} \tag{2.12}$$

where  $H_3 = dB_2$  and  $F_1, F_3, F_5$  are the RR field strengths,

$$F_n = dC_{n-1} + H_3 \wedge C_{n-3}, \tag{2.13}$$

where  $C_0, C_2, C_4$  are the RR potentials. We use the notation  $|F_n|^2 = \frac{1}{n!} F_{\mu_1 \dots \mu_n} F^{\mu_1 \dots \mu_n}$ .<sup>13</sup>

We will consider homogeneous flux backgrounds (meaning that their covariant derivatives vanish) and also assume that the dilaton and the RR scalar have constant values. Then the corresponding 10d supergravity equations simplify to<sup>14</sup>

$$\begin{aligned} R - \frac{1}{2} |H_3|^2 &= 0, \\ R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - \frac{1}{4} e^{2\Phi_0} \left( F_{\mu\rho\sigma} F_\nu{}^{\rho\sigma} + \frac{1}{4!} F_{\mu\rho\sigma\tau\nu} F^{\rho\sigma\tau\nu} - G_{\mu\nu} |F_3|^2 \right) &= 0, \\ F_{\mu\nu\rho\sigma\tau} H^{\rho\sigma\tau} &= 0, \quad F_{\mu\nu\rho\sigma\tau} F^{\rho\sigma\tau} = 0, \quad F_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0. \end{aligned} \tag{2.14}$$

Imposing the self-duality of the RR 5-form gives  $|F_5|^2 = 0$  and then the trace of (2.14) implies that  $|H_3|^2 = e^{2\Phi_0} |F_3|^2$ .

<sup>13</sup>As usual, we relax the self-duality constraint on  $F_5$ , which is to be imposed at the level of equations of motion.

<sup>14</sup>Whenever the indices are written explicitly we remove the form degree index.

Now let us assume that the 10d metric and  $F_5$  are of the  $M^6 \times T^4$  factorized form

$$\begin{aligned}
 ds^2 &= G_{\mu\nu} dX^\mu dX^\nu = g_{mn}(x^k) dx^m dx^n + e^{A(x^k)} dx_r dx_r, \\
 F_5 &= \sum_{i=1}^3 F_3^{(i)} \wedge J_2^{(i)}, \\
 J_2^{(1)} &= dx_6 \wedge dx_7 - dx_8 \wedge dx_9, & J_2^{(2)} &= dx_6 \wedge dx_8 + dx_7 \wedge dx_9, \\
 J_2^{(3)} &= dx_6 \wedge dx_9 - dx_7 \wedge dx_8,
 \end{aligned}
 \tag{2.15}$$

where  $m, n, k = 0, \dots, 5$  and  $r = 6, 7, 8, 9$  and we have defined the three orthogonal self-dual 2-forms  $J_2^{(i)}$  on the torus  $T^4$ . We shall assume that the five 3-forms  $H_3$ ,  $F_3$  and  $F_3^{(i)}$  have only  $M^6$  components. The 10d self-duality of  $F_5$  implies that  $F_3^{(i)}$  are self-dual on  $M^6$  and hence  $|F_3^{(i)}|^2 = 0$ .

If we relax the self-duality condition on  $F_3^{(i)}$ , then the bosonic part of the 6d supergravity action, corresponding to (2.12) upon dimensionally reducing on the 4-torus reads (see, e.g., [51])

$$\begin{aligned}
 S &= \int d^6x \sqrt{-g} \left( e^{-2(\Phi-A)} \left( R + 4(\partial\Phi)^2 - 4\nabla^2 A - 5(\partial A)^2 - \frac{1}{2}|H_3|^2 \right) \right. \\
 &\quad \left. - \frac{1}{2}e^{2A}|F_3|^2 - \frac{1}{2}e^{2A} \sum_{i=1}^3 |F_3^{(i)}|^2 \right).
 \end{aligned}
 \tag{2.16}$$

The variation over  $\Phi$  and  $A$  and setting the scalar  $A = 0$  implies

$$|F_3|^2 + \sum_{i=1}^3 |F_3^{(i)}|^2 = 0,
 \tag{2.17}$$

which, of course, also follows directly from the 10d equations in (2.14). Together with the self-duality constraints we are then left with the following equations

$$R = |H_3|^2 = |F_3|^2 = |F_3^{(i)}|^2 = H_3 \cdot F_3 = H_3 \cdot F_3^{(i)} = F_3 \cdot F_3^{(i)} = 0,
 \tag{2.18}$$

$$R_{mn} - \frac{1}{4}H_{mkl}H_n{}^{kl} - \frac{1}{4}e^{2\Phi_0} \left( F_{mkl}F_n{}^{kl} + \sum_{i=1}^3 (F^{(i)})_{mkl}(F^{(i)})_n{}^{kl} \right) = 0.
 \tag{2.19}$$

For constant scalars the five 3-forms  $H_3$ ,  $F_3$ ,  $F_3^{(i)}$  enter the 6d action (2.16) on an equal footing, i.e. there is an  $SO(5)$  symmetry relating them, which is implied by U-duality.

Let us now take the  $M^6$  metric to be given by the deformed  $AdS_{3\kappa} \times S_\kappa^3$  metric in (2.4) or (2.8), which has the vanishing 6d Ricci scalar as required by (2.18). The basic 3-forms  $d\hat{B}$  and  $d\check{B}$  in (2.2) and (2.6) that enter the simplest example of the background supporting  $AdS_{3\kappa} \times S_\kappa^3$  satisfy

$$|d\hat{B}|^2 = |d\check{B}|^2 = d\hat{B} \cdot d\check{B} = 0.
 \tag{2.20}$$

This implies that if we take the five 3-forms  $H_3, F_3, F_3^{(i)}$  to be given by linear combinations of  $d\hat{B}$  and  $d\check{B}$  in (2.11) as

$$\mathbf{F}_3 \equiv (H_3, F_3, F_3^{(1)}, F_3^{(2)}, F_3^{(3)}) = \mathbf{z}_1 d\hat{B} + \mathbf{z}_2 d\check{B},
 \tag{2.21}$$

where  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are constant 5-vectors, then all the equations in (2.18) will be automatically satisfied. Explicitly, we shall use the following ansatz<sup>15</sup>

$$\begin{aligned}
 H_3 &= s_1 d\hat{B} + s_2 d\check{B}, & F_3 &= y_1 d\hat{B} + y_2 d\check{B}, \\
 F_5 &= (y_3 d\hat{B} + y_4 d\check{B}) \wedge J_2^{(1)} + (y_5 d\hat{B} + y_6 d\check{B}) \wedge J_2^{(2)} + (y_7 d\hat{B} + y_8 d\check{B}) \wedge J_2^{(3)},
 \end{aligned}
 \tag{2.22}$$

where  $s_1, s_2$  and  $y_1, \dots, y_8$  are ten real parameters, which are the components of  $\mathbf{z}_1$  and  $\mathbf{z}_2$  in (2.21)

$$\mathbf{z}_1 = (s_1, y_1, y_3, y_5, y_7), \quad \mathbf{z}_2 = (s_2, y_2, y_4, y_6, y_8). \tag{2.23}$$

Starting with the  $\text{AdS}_{3\kappa} \times \text{S}_\kappa^3$  metric (2.8) and (2.11) and the fluxes in (2.21) and (2.22) we conclude that the supergravity equations (2.14) or (2.18) and (2.19) are satisfied provided that the ten constants in (2.23) are subject to the following constraints

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = 0, \quad \|\mathbf{z}_1\|^2 = 1 + \kappa^2, \quad \|\mathbf{z}_2\|^2 = \kappa^2(1 + \kappa^2). \tag{2.24}$$

The special background in (2.5) is obviously a particular solution of (2.22) and (2.24) with (for  $\Phi_0 = 0$ )

$$\mathbf{z}_1 = (0, \sqrt{1 + \kappa^2}, 0, 0, 0), \quad \mathbf{z}_2 = (\kappa\sqrt{1 + \kappa^2}, 0, 0, 0, 0). \tag{2.25}$$

Let us now discuss the meaning and consequences of the constraints (2.24).

### 2.2.1 U-duality transformations

The U-duality group of 6d maximal supergravity is  $\text{Spin}(5, 5)$  with maximal compact subgroup  $\text{Spin}(5)$ , which is locally isomorphic to  $\text{SO}(5)$ . Correspondingly, the equations (2.24) are invariant under simultaneously rotating the two vectors  $\mathbf{z}_1$  and  $\mathbf{z}_2$  by  $\mathcal{R} \in \text{SO}(5)$ . This has a natural geometric interpretation in terms of T-dualities and rotations in the 4-torus directions (or TsT transformations), and S-duality rotations.

An example of a TsT transformation involving a pair of coordinates  $x_r$  and  $x_s$  of the torus is to first apply T-duality  $x_r \rightarrow \tilde{x}_r$ , then rotate (with parameter  $\beta$ )

$$\tilde{x}_r \rightarrow \tilde{x}_r \cos \beta - x_s \sin \beta, \quad x_s \rightarrow x_s \cos \beta + \tilde{x}_r \sin \beta, \tag{2.26}$$

and finally T-dualise back  $\tilde{x}_r \rightarrow x_r$ . It is easy to see that TsT in  $x_6$  and  $x_7$  (or  $x_8$  and  $x_9$ ) results in a rotation in the  $(y_1, y_3)$  and  $(y_2, y_4)$  planes of the 10-parameter space in (2.23). Similarly, TsT in  $x_6$  and  $x_8$  (or  $x_7$  and  $x_9$ ) results in a rotation in the  $(y_1, y_5)$  and  $(y_2, y_6)$  planes, while TsT in  $x_6$  and  $x_9$  (or  $x_7$  and  $x_8$ ) gives a rotation in the  $(y_1, y_7)$  and  $(y_2, y_8)$  planes.

S-duality transformations (with  $\Phi_0 = 0$  and  $C_0 = 0$ ) leave the string-frame metric invariant and just rotate the NSNS  $H_3$  and RR  $F_3$  forms into each other so that their

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<sup>15</sup>In what follows we will set  $\Phi_0 = 0$  (the dependence on the constant factors  $e^{-\Phi_0}$  in the RR fluxes can easily be restored). Let us also mention that  $F_3 = dC_2$  (we set  $C_0 = 0$ ) and that  $F_5$  may be written as  $F_5 = dC'_4$  since the additional term in (2.13)  $H_3 \wedge C_2 \sim \frac{1}{2}d(\hat{B} \wedge \hat{B}) + \frac{1}{2}d(\check{B} \wedge \check{B})$  is a total derivative. We will always discard total derivative terms in the NSNS and RR potentials.

coefficients in (2.21) and (2.22) change as

$$\begin{aligned} s_1 &= s_1 \cos \alpha + y_1 \sin \alpha, & s_2 &= s_2 \cos \alpha + y_2 \sin \alpha, \\ y_1 &= y_1 \cos \alpha - s_1 \sin \alpha, & y_2 &= y_2 \cos \alpha - s_2 \sin \alpha. \end{aligned} \tag{2.27}$$

This is a simultaneous rotation in the  $(s_1, y_1)$  and  $(s_2, y_2)$  planes of the parameter space with angle  $\alpha$ . More general  $SO(5)$  rotations are then obtained by combining the above TsT and S-duality transformations.

### 2.2.2 Seed solutions

In the simplest case of  $\kappa = 0$  (corresponding to undeformed  $AdS_3 \times S^3$  metric) the third equation in (2.24) implies that  $\mathbf{z}_2 = 0$ , which means that the fluxes in (2.22) do not depend on  $d\hat{B}$ . The coefficients of  $d\hat{B}$  then satisfy  $\|\mathbf{z}_1\| = 1$  such that there are four independent parameters (parametrising a 4-sphere). Equivalently, the general solution is in one to one correspondence with rotations  $\mathcal{R} \in SO(5)/SO(4)$

$$\mathbf{z}_1 = \mathcal{R}\mathbf{v}_1, \tag{2.28}$$

where  $\mathbf{v}_1$  is a fixed unit 5-vector that is invariant under an  $SO(4)$  subgroup of  $SO(5)$ . We can choose  $\mathbf{v}_1$  to represent a simple “seed” solution, corresponding, e.g., to pure NSNS or pure RR flux. The most general background (with four parameters) is then obtained by applying the rotation  $\mathcal{R}$ , which can be decomposed into an S-duality rotation (producing the mixed flux background) and three additional TsT transformations in the torus directions as described in section 2.2.1.

For  $\kappa \neq 0$  we have ten parameters obeying three equations in (2.24), thus leaving seven free parameters. The space of solutions is now in one to one correspondence with rotations  $\mathcal{R} \in SO(5)/SO(3)$  and the most general solution to (2.24) can be parametrised as

$$\mathbf{z}_1 = \sqrt{1 + \kappa^2} \mathcal{R}\mathbf{v}_1, \quad \mathbf{z}_2 = \kappa \sqrt{1 + \kappa^2} \mathcal{R}\mathbf{v}_2, \tag{2.29}$$

where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are two orthogonal unit-norm 5-vectors, which are invariant under an  $SO(3)$  subgroup of  $SO(5)$ . Note that the orthogonality constraint forbids pure NSNS solutions for  $\kappa \neq 0$ .<sup>16</sup>

Examples of simple seed solutions include (we indicate only the non-zero fluxes)

- (i)  $H_3 \neq 0, F_3 \neq 0$  (e.g., the background (2.4) and (2.5) or (2.25));
- (ii)  $F_3 \neq 0, F_5 \neq 0$ ;
- (iii)  $H_3 \neq 0, F_5 \neq 0$  (related to (i) by double T-duality);
- (iv)  $F_5 \neq 0$  (related to (ii) by double T-duality).

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<sup>16</sup>In the pure NSNS case we have  $y_k = 0$  in (2.22), but then  $\mathbf{z}_1 \cdot \mathbf{z}_2 = 0$  in (2.24) implies  $s_1 s_2 = 0$  which is inconsistent with the remaining two equations if  $\kappa \neq 0$  and  $\kappa \neq i$ .

Additional parameters may be turned on through combinations of S-duality and TsT transformations in the torus directions. For instance, starting from case (i) with  $\mathbf{v}_1 = (1, 0, 0, 0, 0)$  and  $\mathbf{v}_2 = (0, 1, 0, 0, 0)$  we can do three TsT transformations in the torus directions; this leaves  $\mathbf{z}_1 = \mathbf{v}_1$  invariant, while introducing three new parameters in  $\mathbf{z}_2$ . Then using combinations of S-duality and TsT transformations we can add four more parameters, thereby generating the full seven-parameter family of solutions.

As discussed in section 2.1, the deformation parameter  $\kappa$  in the metric (2.4) or (2.8) can be turned on by starting with the undeformed  $\text{AdS}_3 \times \text{S}^3$  metric (2.1) and applying a TsT transformation in the left Cartan directions ( $\zeta_1$  and  $\xi_1$ ). Therefore, starting with the seed solution of (2.24) with  $\kappa = 0$  and pure RR 3-form flux the full seven-parameter family of solutions can be obtained by first turning on  $\kappa$  using a TsT transformation in the left Cartan directions.

### 2.2.3 TsT in left Cartan directions

To complete our discussion of symmetry transformations and the constraints (2.24) let us give details of the TsT transformation in the left Cartan directions.

We shall use the Hopf fibration parametrisation (2.7) and consider the following TsT transformation in  $\zeta_1$  and  $\xi_1$  with a parameter  $\gamma$

$$T : \xi_1 \rightarrow \tilde{\xi}_1, \quad \zeta_1 \rightarrow \zeta_1 + \gamma \tilde{\xi}_1, \quad T : \tilde{\xi}_1 \rightarrow \tilde{\xi}_1 \equiv \xi_1. \quad (2.30)$$

Up to a coordinate redefinition and a total derivative  $B$ -field, this takes the metric (2.8) and fluxes (2.22) with parameters  $(\kappa, s_a, y_k)$  to the same metric and fluxes with new parameters  $(\hat{\kappa}, \hat{s}_a, \hat{y}_k)$  where

$$\begin{aligned} \hat{s}_1 &= s_1, & \hat{s}_2 &= \frac{4(\hat{\kappa}^2 - \kappa^2) - \gamma s_2(1 + \hat{\kappa}^2)}{\gamma(1 + \kappa^2)}, \\ \hat{y}_k &= \frac{4y_k - \gamma s_1 y_{k+1}}{\sqrt{(4 - \gamma s_2)^2 - \gamma^2(1 + \kappa^2)^2}}, & \hat{y}_{k+1} &= \frac{4y_{k+1} - \gamma s_1 y_k}{\sqrt{(4 - \gamma s_2)^2 - \gamma^2(1 + \kappa^2)^2}}, \quad k = 1, 3, 5, 7, \end{aligned} \quad (2.31)$$

and  $\gamma = \gamma(\kappa, \hat{\kappa}, s_1, s_2)$  is given by

$$\gamma = \frac{-4s_2(1 + \hat{\kappa}^2) + 4\sqrt{1 + \kappa^2}\sqrt{(1 + \kappa^2)(1 + \hat{\kappa}^2)(\hat{\kappa}^2 - \kappa^2) - s_1^2(\hat{\kappa}^2 - \kappa^2) + s_2^2(1 + \hat{\kappa}^2)}}{(1 + \kappa^2)^2(1 + \hat{\kappa}^2) - s_1^2(1 + \kappa^2) - s_2^2(1 + \hat{\kappa}^2)}. \quad (2.32)$$

Note that, since TsT transformations map supergravity solutions into supergravity solutions, the constraints (2.24) are still obeyed by the transformed coefficients  $(\hat{\kappa}, \hat{s}_a, \hat{y}_k)$ .

If we start with undeformed background ( $\kappa = 0, \|\mathbf{z}_1\| = 1, \mathbf{z}_2 = 0$ ), the TsT transformation in the left Cartan directions then gives a background with non-zero  $\hat{\kappa}$ . The determinant of the coordinate redefinition needed to bring the metric into the standard deformed form (2.8) is  $D = \frac{(1 + \hat{\kappa}^2)(1 - s_1^2)}{1 + \hat{\kappa}^2 - s_1^2}$ . In the pure NSNS case we have  $\mathbf{z}_1 = (\pm 1, 0, \dots, 0)$ , i.e.,  $s_1^2 = 1$ , which means that the determinant vanishes, hence it is not a good starting

point,<sup>17</sup> but all other cases are.<sup>18</sup> From (2.31) and (2.32) for  $\kappa = 0$  we have that

$$\begin{aligned} \hat{s}_1 &= s_1, & \hat{s}_2 &= \hat{\kappa} \sqrt{1 + \hat{\kappa}^2 - s_1^2}, & \gamma &= \frac{4\hat{\kappa}}{\sqrt{1 + \hat{\kappa}^2 - s_1^2}}, \\ \hat{y}_k &= \frac{\sqrt{1 + \hat{\kappa}^2 - s_1^2}}{\sqrt{1 - s_1^2}} y_k, & \hat{y}_{k+1} &= -\frac{\hat{\kappa} s_1}{\sqrt{1 - s_1^2}} y_k, & k &= 1, 3, 5, 7. \end{aligned} \tag{2.33}$$

The corresponding background has four independent parameters in addition to  $\hat{\kappa}$ , which can be taken to be  $s_1$  and  $y_k$  ( $k = 1, 3, 5, 7$ ) subject to  $y_1^2 + y_3^2 + y_5^2 + y_7^2 = 1 - s_1^2$ . Setting  $\hat{s}_1 = \sqrt{1 - q^2}$  and introducing an auxiliary 4-vector  $\hat{\mathbf{u}} = (\hat{y}_1, \hat{y}_3, \hat{y}_5, \hat{y}_7)$  the resulting coefficients (2.23) of the supergravity background can be written as

$$\hat{\mathbf{z}}_1 = \left( \sqrt{1 - q^2}, \hat{\mathbf{u}} \right), \quad \hat{\mathbf{z}}_2 = \left( \hat{\kappa} \sqrt{q^2 + \hat{\kappa}^2}, -\frac{\hat{\kappa} \sqrt{1 - q^2}}{\sqrt{q^2 + \hat{\kappa}^2}} \hat{\mathbf{u}} \right), \quad \|\hat{\mathbf{u}}\|^2 = q^2 + \hat{\kappa}^2. \tag{2.34}$$

This is the most general deformed background that can be obtained from the undeformed one by TsT transformations alone. Since the undeformed string sigma model is integrable, and TsT preserves integrability, the same applies to this background as well. In section 4 we will prove the classical integrability of the string sigma model for the full seven-parameter family of solutions.

An example of a solution corresponding to (2.34) is found by starting from the undeformed  $\text{AdS}_3 \times \text{S}^3$  background supported by mixed flux (2.3) with one free parameter  $|q| \leq 1$ , i.e. with  $\mathbf{z}_1 = (\sqrt{1 - q^2}, q, 0, 0, 0)$  and  $\mathbf{z}_2 = 0$  (cf. (2.22) and (2.23)). Then, assuming  $q \neq 0$ , the above TsT transformation gives a deformed  $\text{AdS}_{3\hat{\kappa}} \times \text{S}_{\hat{\kappa}}^3$  background with  $\gamma = \frac{4\hat{\kappa}}{\sqrt{q^2 + \hat{\kappa}^2}}$  and

$$\hat{\mathbf{z}}_1 = \left( \sqrt{1 - q^2}, \sqrt{q^2 + \hat{\kappa}^2}, 0, 0, 0 \right), \quad \hat{\mathbf{z}}_2 = \left( \hat{\kappa} \sqrt{q^2 + \hat{\kappa}^2}, -\hat{\kappa} \sqrt{1 - q^2}, 0, 0, 0 \right), \tag{2.35}$$

i.e. with only the  $H_3$  and  $F_3$  fluxes non-vanishing.

### 3 Supersymmetry

The undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  solution preserves  $\frac{1}{2}$  of maximal 10d supersymmetry, i.e. has 16 supercharges. Let us now show that the family of deformed backgrounds (2.4), (2.22)–(2.24) preserves  $\frac{1}{4}$  of maximal supersymmetry, i.e. admits 8 Killing spinors.

This may be at first surprising. Indeed, in general, T-dualities in 4-torus directions and S-duality should preserve supersymmetry. However, the T-duality along Hopf fibres (which in the present case is responsible for introducing the deformation parameter  $\kappa$ ) may break supersymmetry of the supergravity background [37, 52].<sup>19</sup>

<sup>17</sup>The TsT transformation of the undeformed pure NSNS solution is again the undeformed pure NSNS solution.

<sup>18</sup>Note that  $\|\mathbf{z}_1\| = 1$  imposes  $s_1^2 \leq 1$  and if  $\hat{\kappa}^2 > 0$  then the denominator in  $D$  never vanishes.

<sup>19</sup>The full supersymmetry may still be “hidden” at the level of full string theory (cf. [52]). This may be indeed related to the integrability of the underlying supercoset model discussed in section 4.

In general, there are two 10d type IIB supergravity Killing spinor equations, associated with the invariance of the gravitino  $\psi_\mu$  and the dilatino  $\lambda$  fields under the supersymmetry variations (see, e.g., [53–55])<sup>20</sup>

$$\delta\psi_\mu = D_\mu\epsilon = \left(\nabla_\mu + \frac{1}{8}H_{\mu a_1 a_2}\Gamma^{a_1 a_2}\sigma_3 + \mathcal{S}\Gamma_\mu\right)\epsilon = 0, \quad (3.1)$$

$$\begin{aligned} \delta\lambda &= \left[\Gamma^\mu\partial_\mu\Phi + \frac{1}{12}H_{a_1 a_2 a_3}\Gamma^{a_1 a_2 a_3}\sigma_3 + e^\Phi\left(-i\mathcal{H}_1\sigma_2 + \frac{1}{12}\mathcal{H}_3\sigma_1\right)\right]\epsilon = 0, \\ \nabla_\mu &= \partial_\mu + \frac{1}{4}\omega_\mu^{a_1 a_2}\Gamma_{a_1 a_2}, \quad \mathcal{S} \equiv -\frac{1}{8}e^\Phi\left(i\mathcal{H}_1\sigma_2 + \frac{1}{3!}\mathcal{H}_3\sigma_1 + i\frac{1}{2\cdot 5!}\mathcal{H}_5\sigma_2\right). \end{aligned} \quad (3.2)$$

Here  $\epsilon = (\epsilon^1, \epsilon^2)$  is a doublet of 32-component Majorana-Weyl spinors with  $\sigma_k$  being Pauli matrices acting on  $I = 1, 2$ .  $\Gamma^a$  are  $32 \times 32$  10d Dirac matrices,  $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$  and  $\mathcal{H}_m \equiv F_{a_1 \dots a_m}\Gamma^{a_1 \dots a_m}$ , with  $\Gamma^{a_1 \dots a_m} = \Gamma^{a_1} \dots \Gamma^{a_m}$ . We use greek letters for the spacetime indices and latin letters  $a, a_j$  for tangent space indices.

The background in (2.22) that we are interested in has constant dilaton and RR scalar (i.e.  $F_1 = 0$ ) and 6d self-dual  $H_3$  and  $F_3$ . More precisely, in the vielbein basis defined in (2.10), the fluxes in (2.22) are such that

$$\begin{aligned} \mathcal{H}_3 &= \frac{2}{\sqrt{1+\kappa^2}}(s_1\Gamma^{012} + s_2\Gamma^{123})(1 + \Gamma^{11}\Gamma^{6789}), \\ \mathcal{H}_3 &= \frac{2}{\sqrt{1+\kappa^2}}(y_1\Gamma^{012} + y_2\Gamma^{123})(1 + \Gamma^{11}\Gamma^{6789}), \\ \mathcal{H}_5 &= \frac{2}{\sqrt{1+\kappa^2}}(\Gamma^{012}I + \Gamma^{123}J)(1 + \Gamma^{11})(1 + \Gamma^{6789}), \end{aligned} \quad (3.3)$$

where  $\Gamma^{11} = \Gamma^{0123456789}$ ,  $I = y_3\Gamma^{67} + y_5\Gamma^{68} + y_7\Gamma^{69}$  and  $J = y_4\Gamma^{67} + y_6\Gamma^{68} + y_8\Gamma^{69}$ .

The equation  $\delta\lambda = 0$  is then trivially satisfied if we use the type IIB chirality condition  $\epsilon^I = \hat{\Gamma}\epsilon^I$ , where  $\hat{\Gamma} = \frac{1}{2}(1 - \Gamma^{11})$ , together with the condition  $\epsilon^I = \mathcal{P}\epsilon^I$ , where the projector  $\mathcal{P} = \frac{1}{2}(1 + \Gamma^{6789})$ .<sup>21</sup> This leaves  $2 \times 8 = 16$  independent components of  $\epsilon$ . It also ensures that the Killing spinors do not depend on the 4-torus directions 6, 7, 8, 9.

The equations  $D_\mu\epsilon = 0$  with the index  $\mu$  corresponding to the directions  $(\zeta_1, \sigma, \xi_1, \theta)$  in (2.8) are first order differential equations of the form  $\partial_\mu\epsilon = \Omega_\mu\epsilon$  with coefficients  $\Omega_\mu$  that do not explicitly depend on the coordinates, see appendix A. The compatibility conditions  $[\Omega_{\zeta_1}, \Omega_{\xi_1}]\epsilon = [\Omega_\sigma, \Omega_\theta]\epsilon = 0$  are immediately satisfied. The other compatibility conditions are satisfied in two cases. One option (A) is to choose the parameters of the fluxes in (2.23) so that

$$\text{A :} \quad \|\mathbf{z}_1\|^2 = (1 + \kappa^2)^2, \quad \mathbf{z}_2 = 0, \quad (3.4)$$

<sup>20</sup>Construction of Killing spinors on AdS spaces and spheres was discussed, e.g., in [37, 56].

<sup>21</sup>In the corresponding GS action the Killing spinors correspond to the fermionic isometries. The fermionic coordinates  $\vartheta$  are the negative chirality MW spinors:  $\Gamma^{11}\vartheta = -\vartheta$ . The RR flux bispinor  $\mathcal{S}$  enters the GS action as  $\bar{\vartheta}\Gamma_\mu\mathcal{S}\Gamma_\nu\vartheta$  where  $\bar{\vartheta} = \vartheta^t\mathcal{C}$  and the charge conjugation matrix is  $\mathcal{C} = i\sigma_2 \otimes \mathbb{1}_{16}$  so that  $\bar{\vartheta}\Gamma^{11} = +\bar{\vartheta}$ . The Dirac matrices are  $\Gamma^a = \begin{pmatrix} 0 & (\gamma^a)^{\alpha\beta} \\ (\gamma^a)_{\alpha\beta} & 0 \end{pmatrix}$ . We can then write  $\bar{\vartheta}\Gamma_\mu\mathcal{S}\Gamma_\nu\vartheta = \bar{\vartheta}\Gamma_\mu\hat{\Gamma}\mathcal{S}\Gamma_\nu\hat{\Gamma}\vartheta$ , with  $\hat{\Gamma} = \frac{1}{2}(1 - \Gamma^{11})$ .

and impose no condition on  $\epsilon$ . The other option (B) is to impose the two equations

$$\text{B :} \quad \|\mathbf{z}_1 \pm \mathbf{z}_2\|^2 = (1 + \kappa^2)^2, \quad (3.5)$$

together with an additional constraint  $\epsilon = \mathcal{Q}\epsilon$  (where the operator  $\mathcal{Q}$  depends on the coefficients) which halves the number of independent Killing spinors. In both cases the solution is then given by

$$\epsilon = \exp(\Omega_{\zeta_1}\zeta_1 + \Omega_\sigma\sigma + \Omega_{\xi_1}\xi_1 + \Omega_\theta\theta)\chi, \quad (3.6)$$

where the spinor  $\chi = \mathcal{Q}\chi$  ( $\mathcal{Q} = 1$  for case (A)) depends only on the coordinates  $\zeta_2$  and  $\xi_2$ . Further using that

$$[\Omega_{\zeta_1}, \Omega_{\zeta_2}]\epsilon = [\Omega_{\xi_1}, \Omega_{\xi_2}]\epsilon = [\Omega_{\zeta_1}, \Omega_{\xi_2}]\epsilon = [\Omega_{\xi_1}, \Omega_{\zeta_2}]\epsilon = [\Omega_\theta, \Omega_{\zeta_2}]\epsilon = [\Omega_\sigma, \Omega_{\xi_2}]\epsilon = 0, \quad (3.7)$$

the two remaining equations  $D_{\zeta_2}\epsilon = 0$  and  $D_{\xi_2}\epsilon = 0$  simplify to

$$\begin{aligned} \partial_{\zeta_2}\chi &= \hat{\Omega}_{\zeta_2}\chi, & \hat{\Omega}_{\zeta_2} &= \exp(-\Omega_\sigma\sigma)\Omega_{\zeta_2}\exp(\Omega_\sigma\sigma), \\ \partial_{\xi_2}\chi &= \hat{\Omega}_{\xi_2}\chi, & \hat{\Omega}_{\xi_2} &= \exp(-\Omega_\theta\theta)\Omega_{\xi_2}\exp(\Omega_\theta\theta). \end{aligned} \quad (3.8)$$

Let us recall that  $\chi$  depends on  $\zeta_2$  and  $\xi_2$ ,  $\Omega_\sigma$  and  $\Omega_\theta$  are constant,  $\Omega_{\zeta_2}$  depends on  $\sigma$  and  $\Omega_{\xi_2}$  depends on  $\theta$ . Therefore, for (3.8) to be satisfied, we need  $\hat{\Omega}_{\zeta_2}$  and  $\hat{\Omega}_{\xi_2}$  to be constants, i.e. without  $\sigma$  or  $\theta$  dependence. This is indeed the case if one further imposes

$$\|\mathbf{z}_1\|^2 = 1 + \kappa^2. \quad (3.9)$$

One then shows that  $[\hat{\Omega}_{\zeta_2}, \hat{\Omega}_{\xi_2}]\chi = 0$  and therefore  $\chi = \exp(\hat{\Omega}_{\zeta_2}\zeta_2 + \hat{\Omega}_{\xi_2}\xi_2)\chi_0$ , with  $\chi_0$  a constant vector obeying  $\chi_0 = \mathcal{Q}\chi_0$ .

For option (A) we have  $\mathcal{Q} = 1$  and the conditions (3.4) and (3.9) together imply  $\kappa = 0$ . In this case the background is maximally supersymmetric in 6d (with 16 Killing spinors), and corresponds to the undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ .

To have  $\kappa \neq 0$  one needs to consider the option B, with the additional projection  $\epsilon = \mathcal{Q}\epsilon$ . The conditions (3.5) and (3.9) together are equivalent to the constraints (2.24) following from the supergravity field equations. The resulting background thus preserves only half of the original 16 supersymmetries, i.e. admits 8 Killing spinors.

## 4 Classical integrability of the superstring sigma model

In section 2, starting from the  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background supported by mixed flux (2.3), we constructed the 8-parameter background (including  $\kappa$ ) with metric (2.8), constant dilaton and axion, and fluxes (2.9), (2.22) subject to the constraints (2.23), (2.24). This background preserves 8 supersymmetries (for  $\kappa \neq 0$ ) and can be generated by a combination of TsT transformations *and* S-dualities (i.e. not TsT transformations alone). While TsT transformations preserve the classical integrability of the corresponding string sigma model, a priori S-dualities do not. Our aim in this section is to show that the full 8-parameter background defines an integrable string sigma model.

To do this we recall that in a particular  $\varkappa$ -symmetry gauge the  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  GS string sigma model with pure RR flux can be written as a semi-symmetric space sigma model plus 4 compact bosons [20]. The relevant  $\mathbb{Z}_4$  supercoset is

$$\frac{\text{PSU}(1,1|2) \times \text{PSU}(1,1|2)}{\text{SU}(1,1) \times \text{SU}(2)}, \tag{4.1}$$

where the isotropy group  $\text{SU}(1,1) \times \text{SU}(2)$  is the diagonal bosonic subgroup. This model is known to be classically integrable [9]. It is also known that it admits integrable deformations known as YB deformations (see [16] and references there).

In general, the YB deformation [39, 57] depends on an antisymmetric linear operator  $R$ , known as an R-matrix, acting on the symmetry algebra  $\mathfrak{g}$  of the model. This operator solves the (modified) classical Yang-Baxter equation ((m)cYBe)

$$[RX, RY] - R[X, RY] - R[RX, Y] + c^2[X, Y] = 0, \quad X, Y \in \mathfrak{g}. \tag{4.2}$$

When this R-matrix satisfies a so-called unimodularity condition, that is when the trace of the structure constants of the dual Lie algebra  $\mathfrak{g}_R$  with Lie bracket  $[X, Y]_R = [X, RY] + [RX, Y]$  vanishes, the deformed string sigma model remains Weyl invariant, i.e. the background fields solve the supergravity equations [48] (see also [58, 59]).

The case of interest for us is  $c = i$ , i.e. the non-split mcYBe. For the supergroups relevant for constructing string sigma models, e.g.  $\text{PSU}(2,2|4)$  for  $\text{AdS}_5 \times \text{S}^5$  and  $\text{PSU}(1,1|2)$  for  $\text{AdS}_3 \times \text{S}^3$  and  $\text{AdS}_2 \times \text{S}^2$  backgrounds, there exist unimodular solutions to the mcYBe [13, 14]. Introducing Cartan generators  $H_i$  and positive and negative roots  $E_\alpha$  and  $F_\alpha$ , these are the so-called Drinfel'd-Jimbo (DJ) solutions ( $R(H_i) = 0$ ,  $R(E_\alpha) = iE_\alpha$ ,  $R(F_\alpha) = -iF_\alpha$ ) built from a Cartan-Weyl basis with all fermionic simple roots. The existence of these solutions is crucial for constructing YB deformations based on solutions to the non-split mcYBe that define Weyl invariant string sigma models [13].

Other Drinfel'd-Jimbo solutions exist based on different Dynkin diagrams that do not satisfy the unimodularity condition and are not Weyl invariant [13, 14, 17, 18]. Nevertheless, one can still fix a light-cone gauge and it seems reasonable to expect that the resulting models are quantum integrable. Indeed, for e.g.  $\text{AdS}_5 \times \text{S}^5$ , it is possible to conjecture an exact S-matrix for the corresponding 8+8 transverse degrees of freedom [60–63]. The different S-matrices coming from different Dynkin diagrams are related by fermionic twists [63] and share the same Bethe equations [64]. However, this does not necessarily imply that the string sigma models are fully equivalent. Indeed, the lack of Weyl invariance suggests that there may be issues with light-cone gauge-fixing beyond the classical level, e.g., leading to an anomaly in the full deformed global symmetry. Here we will always consider unimodular R-matrices so that the corresponding sigma model is Weyl invariant. An example of deformation based on a non-unimodular R-matrix is provided in appendix B.

For the  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background, the direct product structure of the superisometry group means that the model admits a richer space of deformations. In particular, an additional WZ term can be included in the coset sigma model corresponding to the mixed flux background [49], and the two copies of  $\text{PSU}(1,1|2)$  can be deformed with different strengths [65]. While the full space of YB deformations has not been fully understood,

a three-parameter model is known based on deforming the mixed flux model with the DJ R-matrix on each copy of  $\text{PSU}(1,1|2)$ . This generalises the bi-YB deformation of the PCM plus WZ term [66, 67] (see also [68, 69]) to  $\mathbb{Z}_4$  supercosets. The action of this three-parameter model is [12]

$$S_{\eta_L, \eta_R, k} = T \int d^2x \text{STr} \left( J_+ ((1 - (2P_2 + P_F)kW\Omega_-)d_- + P_FkW(1 - d_-)) \frac{1}{1 + \Omega_-d_-} J_- \right) - 4Tk \int d^3x \epsilon^{\mu\nu\rho} \text{STr} \left( \frac{2}{3} W J_\mu^{(2)} J_\nu^{(2)} J_\rho^{(2)} + W [J_\mu^{(1)}, J_\nu^{(3)}] J_\rho^{(2)} \right). \quad (4.3)$$

Here  $J_\pm = g^{-1}\partial_\pm g$  with  $g = \text{diag}(g_L, g_R) \in \text{PSU}(1,1|2) \times \text{PSU}(1,1|2)$  and  $\text{STr}$  is the supertrace, an invariant bilinear form on the Lie superalgebra  $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ .  $P_{0,1,2,3}$  denote the projectors onto the  $\mathbb{Z}_4$  grading of  $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ , see, e.g., [14],  $P_F = P_1 + P_3$  is the projector onto the fermionic part of the superalgebra and  $J_\pm^{(i)} = P_i J_\pm$ .  $W$  acts as  $W \text{diag}(X, Y) = \text{diag}(X, -Y)$  where  $\text{diag}(X, Y) \in \mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ , while the linear operators  $d_-$  and  $\Omega_-$  (that depends on the dressed R-matrix  $R_g = \text{Ad}_g^{-1} R \text{Ad}_g$  where  $\text{Ad}_g$  denotes the standard adjoint action and  $R$  is also assumed to satisfy  $R^3 = -R$ ) are defined as

$$d_- = 2P_2 + \frac{1}{1 - k^2} ((\lambda - k^2)P_1 - (1 + \lambda)kW P_3 - (\lambda + k^2)P_3 - (1 - \lambda)kW P_1), \quad (4.4)$$

$$\Omega_- = -\frac{\sqrt{(\mu - 1)(1 - k^2\mu)}}{1 + kW} R_g - kW \left( \frac{\mu - 1}{1 + kW} \right) R_g^2,$$

where

$$\lambda = \sqrt{\frac{(1 - k^2 - \eta_L^2)(1 - k^2 - \eta_R^2)}{1 - k^2}}, \quad \mu = 1 + \frac{1}{\lambda^2 + k^2} \text{diag}(\eta_L^2, \eta_R^2). \quad (4.5)$$

The string tension  $T$  enters (4.3) as an overall factor. The deformation parameters  $\eta_L$  and  $\eta_R$  are expected to be associated with a quantum group deformation of the left and right copies of  $\mathfrak{psu}(1,1|2)$ , with standard deformation parameters  $q_{L,R}$  (see, e.g., [16] and references there). In the case  $\eta_R = 0$ , i.e.  $q_R = 1$ , we give a conjecture for the deformation parameter  $q_L$  in terms of the remaining parameters in the action (4.3) in appendix B. The final parameter  $k$  is related to the presence of the Wess-Zumino term in the second line of (4.3), which although defined in 3d gives a local coupling in the 2d sigma model. As usual, since  $\text{PSU}(1,1|2)$  has an  $\text{SU}(2)$  subgroup, the corresponding level  $k = Tk$  is integer-quantized. In the mixed flux model without the quantum group deformation (i.e.  $\eta_L = \eta_R = 0$ ), the  $k = 0$  point corresponds to pure RR flux, while the  $k = 1$  point to pure NSNS flux. The Lax connection demonstrating that the action (4.3) defines a classically integrable sigma model is given in [12].

Denoting by  $P_L$  and  $P_R$  the projectors onto the left and right copies of  $\mathfrak{psu}(1,1|2)$ , if we set  $\eta_L = 0$  (respectively  $\eta_R = 0$ ) then  $P_L \Omega_- P_L = 0$  (respectively  $P_R \Omega_- P_R = 0$ ). This implies that the deformed action (4.4) is invariant under the global left action  $g \rightarrow g_0 g$  where  $g_0 = \text{diag}(g_{0L}, 1)$  (respectively  $g_0 = (1, g_{0R})$ ). Therefore, only the right (respectively

left) copy of  $\mathfrak{psu}(1,1|2)$  is deformed, hence half of the original 16 supersymmetries are preserved.<sup>22</sup>

This provides a natural candidate to connect with the backgrounds discussed in section 2. Without loss of generality, we focus on the case  $\eta_R = 0$  for which  $\lambda$  in (4.5) is  $\sqrt{1 - k^2 - \eta_L^2}$ . We find it useful to define

$$\kappa = \frac{\eta_L}{\sqrt{1 - \eta_L^2}} . \tag{4.6}$$

To ensure Weyl invariance we take  $R$  to be a DJ R-matrix built from a Cartan-Weyl basis with all fermionic simple roots. The associated supergravity background, which we call the DJ background, can be extracted following the methods described in [14] (see also [48]). Choosing an appropriate parametrisation for  $g$ , the metric is given by (2.4). The NSNS and RR fluxes are of the form (2.22) with coefficients<sup>23,24</sup>

$$\begin{aligned} \mathbf{z}_1 &= \left( \sqrt{1 + \kappa^2} \sqrt{1 - q^2}, -\sqrt{\frac{1 + \kappa^2}{q^2 + \kappa^2}} q^2, -\sqrt{\frac{1 + \kappa^2}{q^2 + \kappa^2}} q\kappa, 0, 0 \right) , \\ \mathbf{z}_2 &= \left( 0, -\sqrt{\frac{1 + \kappa^2}{q^2 + \kappa^2}} \kappa^2, \sqrt{\frac{1 + \kappa^2}{q^2 + \kappa^2}} q\kappa, 0, 0 \right) , \quad q = \sqrt{1 - k^2(1 + \kappa^2)} . \end{aligned} \tag{4.7}$$

Here  $q$  is related to  $k$  in (4.3) so the parameters are  $q \in [0, 1]$  and  $\kappa \in [0, \infty)$  with  $k = T\sqrt{1 + \kappa^2}\sqrt{1 - q^2}$  being the Wess-Zumino level. Notice that  $\kappa = 0$  gives the mixed flux background (2.3), while  $q = 1$  gives the RR deformation of [14] with equal deformation parameters.<sup>25</sup> For comparison, we provide in appendix B the background associated to a non-unimodular R-matrix, which only satisfies a set of generalised supergravity equations of motion.

At this point we recall that only a subset of the background fluxes (2.22)–(2.24) can be obtained from the undeformed mixed flux model by TsT transformations (2.34). This subset is guaranteed to be classically integrable. To generate the full set of background fluxes we also need to allow S-dualities. The key point is that the DJ background (4.7), which defines a classically integrable string sigma model by construction, is not of the form (2.34). It follows that starting from the DJ background and applying TsT transformations it is possible to generate the full set of background fluxes, thereby demonstrating the classical integrability of the string sigma model.

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<sup>22</sup>Recall that the DJ R-matrix only commutes with the action of the Cartan subgroup, hence will generically break or deform all of the associated supersymmetries.

<sup>23</sup>Note that to make this matching explicit we rescale  $T \rightarrow (1 + \kappa^2)T$  and the 4-torus coordinates  $x_r \rightarrow \frac{x_r}{\sqrt{1 + \kappa^2}}$ . Recall that the NSNS and RR fluxes scale as  $H_3 \sim T$ ,  $F_3 \sim T$  and  $F_5 \sim T^2$ . After this rescaling the WZ level is  $k = (1 + \kappa^2)Tk$ .

<sup>24</sup>To quadratic order in the fermions, the RR fluxes and the dilaton  $\Phi$  only appear in the GS action through the combination  $\mathcal{F} = e^\Phi F$ . To determine the dilaton we require that the supergravity equations of motion are satisfied. That this is possible is ensured by the unimodularity of the R-matrix. In this case, the dilaton indeed turns out to be constant.

<sup>25</sup>With rescaled tension and torus coordinates.

The DJ background (4.7) has one free parameter,  $q$ , in addition to  $\kappa$ . Let us now show that it is possible to generate six additional free parameters using only TsT transformations. As a first step it is useful to perform an  $SO(4)$  rotation to bring (4.7) to

$$\mathbf{z}_1 = \left( \sqrt{1 + \kappa^2} \sqrt{1 - q^2}, q \sqrt{1 + \kappa^2}, 0, 0, 0 \right), \quad \mathbf{z}_2 = \left( 0, 0, -\kappa \sqrt{1 + \kappa^2}, 0, 0 \right). \quad (4.8)$$

This can equivalently be obtained through an appropriately chosen TsT in the torus directions  $x^6$  and  $x^7$ . One can then do a rotation  $\mathcal{R} \in SO(3)/SO(2)$  that leaves  $\mathbf{z}_1$  invariant but introduces two new parameters in  $\mathbf{z}_2$  (this is equivalent to rotating the torus coordinates). Then, one does another rotation  $\mathcal{R} \in SO(4)/SO(3)$  introducing three new parameters in  $\mathbf{z}_1$  (equivalent to TsT transformations). At this point it is useful to write the resulting vectors as

$$\mathbf{z}_1 = \left( \sqrt{1 + \kappa^2} \sqrt{1 - q^2}, \mathbf{u}_1 \right), \quad \mathbf{z}_2 = (0, \mathbf{u}_2), \quad (4.9)$$

where the vectors  $\mathbf{u}_1 = (y_1, y_3, y_5, y_7)$  and  $\mathbf{u}_2 = (y_2, y_4, y_6, y_8)$  satisfy the constraints  $\|\mathbf{u}_1\|^2 = q^2(1 + \kappa^2)$ ,  $\|\mathbf{u}_2\|^2 = \kappa^2(1 + \kappa^2)$  and  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ . There are five independent parameters in addition to  $\kappa$  and  $q$ . Notice that the first components of  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are left invariant under the above rotations, which traduces the fact that we only do transformations in the torus directions, and no S-duality rotation. The last step consists of obtaining a non-trivial  $s_2$  (the first component in  $\mathbf{z}_2$ ). To achieve this we use a TsT in the left Cartan directions, as discussed in 2.2.3. This leads to

$$\hat{\mathbf{z}}_1 = \left( \sqrt{1 + \kappa^2} \sqrt{1 - q^2}, \hat{\mathbf{u}}_1 \right), \quad \hat{\mathbf{z}}_2 = \left( \sqrt{\hat{\kappa}^2 - \kappa^2} \sqrt{\hat{\kappa}^2 + q^2}, \hat{\mathbf{u}}_2 \right), \quad (4.10)$$

with the constraints  $\|\hat{\mathbf{u}}_1\|^2 = \hat{\kappa}^2 - \kappa^2 + q^2(1 + \kappa^2)$ ,  $\|\hat{\mathbf{u}}_2\|^2 = -q^2(\hat{\kappa}^2 - \kappa^2) + \hat{\kappa}^2(1 + \kappa^2)$  and  $\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 = -\sqrt{1 + \kappa^2} \sqrt{1 - q^2} \sqrt{\hat{\kappa}^2 - \kappa^2} \sqrt{\hat{\kappa}^2 + q^2}$ . This is equivalent to the supergravity constraints (2.24), except with  $\hat{\kappa}$  on the right-hand side.

## 5 Analytically-continued solution and special limits

Thus far we have considered the deformed  $AdS_3 \times S^3$  solution with metric (2.4) and fluxes (2.22)–(2.24). In particular, the metric on  $AdS_3$  is written in global coordinates and in (2.8) as a time-like fibration over  $H^2$ . Our main motivation for this is that this is the metric that follows from the  $\eta_L$  deformation of the  $\mathbb{Z}_4$  permutation supercoset with WZ term as discussed in section 4, thereby allowing us to prove classical integrability.

Here we consider different transformations of the deformed background that still give deformations of  $AdS_3 \times S^3$  defining integrable string sigma models, albeit in different patches of  $AdS_3$ . This includes the analytic continuation  $\kappa = i\tilde{\kappa}$ , which gives a deformation of  $AdS_3$  written as a space-like fibration over  $AdS_2$ , as well as different scaling limits that give deformations in Poincaré patch. We will also discuss the Schrödinger space-time and pp-wave limits of the background in section 2.

### 5.1 Analytically-continued background

Given the deformed  $\text{AdS}_3 \times \text{S}^3$  solution with metric (2.4) and fluxes (2.22)–(2.24) one may formally consider the case of  $\kappa^2 < 0$  or set  $\kappa = i\tilde{\kappa}$ . Then the metric (2.4) remains real, but the final condition in (2.24) implies that at least one of the parameters in  $\mathbf{z}_2$  in (2.23) should become imaginary (we will assume that  $\tilde{\kappa}^2 < 1$ ). Since  $\mathbf{z}_2$  is the coefficient of  $d\check{B}$  in (2.21), the only way to preserve the reality of fluxes is to set  $\mathbf{z}_2 = -i\check{\mathbf{z}}_2$  and compensate this by an analytic continuation of the coordinates so that  $\check{B}$  becomes imaginary. We will set

$$\kappa = i\tilde{\kappa}, \quad \mathbf{z}_2 = -i\check{\mathbf{z}}_2, \quad \psi = i\tilde{t}, \quad t = i\check{\psi}, \quad (5.1)$$

which leads to the following metric

$$ds^2 = (1 + \rho^2)d\tilde{\psi}^2 + \frac{d\rho^2}{1 + \rho^2} - \rho^2 d\tilde{t}^2 - \tilde{\kappa}^2((1 + \rho^2)d\tilde{\psi} - \rho^2 d\tilde{t})^2 \\ + (1 - r^2)d\varphi^2 + \frac{dr^2}{1 - r^2} + r^2 d\phi^2 - \tilde{\kappa}^2((1 - r^2)d\varphi + r^2 d\phi)^2 + dx_r dx_r. \quad (5.2)$$

Note that, setting  $\tilde{\kappa} = 0$ , the analytically-continued metric still describes  $\text{AdS}_3$ , however the coordinates  $\tilde{t}, \check{\psi}$  and  $\rho$  are no longer global coordinates.<sup>26</sup>

The corresponding metric in Hopf parametrization found from (2.7), (2.8) and (5.1) is ( $\zeta_1 = i\check{\zeta}_1, \zeta_2 = -i\check{\zeta}_2$ )<sup>27</sup>

$$ds^2 = \frac{1}{4}(-\sinh^2 \sigma d\check{\zeta}_2^2 + d\sigma^2 + (1 - \tilde{\kappa}^2)(d\check{\zeta}_1 + \cosh \sigma d\check{\zeta}_2)^2) \\ + \frac{1}{4}(\sin^2 \theta d\xi_2^2 + d\theta^2 + (1 - \tilde{\kappa}^2)(d\xi_1 - \cos \theta d\xi_2)^2) + dx_r dx_r, \quad (5.3)$$

The fluxes supporting this metric are given by

$$H_3 = s_1 d\hat{B} + \tilde{s}_2 d\check{B}, \quad F_3 = y_1 d\hat{B} + \tilde{y}_2 d\check{B}, \\ F_5 = (y_3 d\hat{B} + \tilde{y}_4 d\check{B}) \wedge J_2^{(1)} + (y_5 d\hat{B} + \tilde{y}_6 d\check{B}) \wedge J_2^{(2)} + (y_7 d\hat{B} + \tilde{y}_8 d\check{B}) \wedge J_2^{(3)}, \\ d\hat{B} = \frac{1}{4}[\sinh \sigma d\check{\zeta}_1 \wedge d\check{\zeta}_2 \wedge d\sigma + \sin \theta d\xi_1 \wedge d\xi_2 \wedge d\theta], \\ d\check{B} = \frac{1}{4}[-\sinh \sigma d\check{\zeta}_2 \wedge d\sigma \wedge (d\xi_1 - \cos \theta d\xi_2) + \sin \theta (d\check{\zeta}_1 + \cosh \sigma d\check{\zeta}_2) \wedge d\xi_2 \wedge d\theta], \quad (5.4)$$

where  $\mathbf{z}_2 d\check{B} = \check{\mathbf{z}}_2 d\check{B}$ . The supergravity equations then imply that  $\mathbf{z}_1 = (s_1, y_1, y_3, y_5, y_7)$  and  $\check{\mathbf{z}}_2 = (\tilde{s}_2, \tilde{y}_2, \tilde{y}_4, \tilde{y}_6, \tilde{y}_8)$  should satisfy (cf. (2.24))

$$\mathbf{z}_1 \cdot \check{\mathbf{z}}_2 = 0, \quad \|\mathbf{z}_1\|^2 = 1 - \tilde{\kappa}^2, \quad \|\check{\mathbf{z}}_2\|^2 = \tilde{\kappa}^2(1 - \tilde{\kappa}^2). \quad (5.5)$$

<sup>26</sup>For  $\tilde{\kappa} = 0$  this metric follows from  $ds_3^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2$ , with  $X_{-1} \pm X_1 = \sqrt{1 + \rho^2} e^{\pm\tilde{\psi}}$  and  $X_2 \pm X_0 = \rho e^{\pm\tilde{t}}$ . Here  $X_2^2 - X_0^2 > 0$  and  $X_{-1} > 0$ . The other inequalities  $X_{-1}^2 - X_1^2 > 1$  and  $(X_{-1}^2 - X_1^2) - (X_2^2 - X_0^2) > 0$  follow from the first one and the embedding constraint  $-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -1$ .

<sup>27</sup>Closely related backgrounds were considered in [35, 36]. There the deformed metric was generated by applying TsT transformations in  $\mathfrak{su}(1, 1)_L \oplus \mathfrak{u}(1)_{T^4}$  and  $\mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{T^4}$  where  $\mathfrak{u}(1)_{T^4}$  denotes shift isometries associated to the 4-torus. This gave the metric (5.2), also allowing for different deformation parameters in the  $\text{AdS}_3$  and  $\text{S}^3$  parts. However, in contrast to the case we are considering here, the NSNS and RR 3-forms there involved the torus directions. This means that in the reduction to 6d (considered in section 2.2) there are also non-zero 2-form field strengths. Additionally, in the construction of [35, 36], if  $\text{AdS}_3$  and  $\text{S}^3$  were both deformed no supersymmetries were preserved, again in contrast to the background discussed here.

While the AdS part of (2.8) is a time-like fibration over  $H^2$ , in the analytically-continued metric (5.3) it is a space-like fibration over  $AdS_2$  with Lorentzian signature. The metric (5.3) thus interpolates between  $AdS_3 \times S^3 \times T^4$  ( $\tilde{\kappa} = 0$ ) and  $AdS_2 \times S^2 \times T^6$  ( $\tilde{\kappa} = 1$ ). In order to preserve the non-degeneracy of the metric (5.3) in the limit  $\tilde{\kappa} = 1$  we also need to rescale the coordinates

$$\tilde{\zeta}_1 \rightarrow \epsilon^{-1} \tilde{\zeta}_1, \quad \xi_1 \rightarrow \epsilon^{-1} \xi_1, \quad \epsilon = \sqrt{1 - \tilde{\kappa}^2}. \quad (5.6)$$

It then follows from (5.4) that  $d\hat{B} \sim d\check{B} \sim \epsilon^{-1}$ . To get a consistent solution with non-singular fluxes from (5.3)–(5.5) we need to simultaneously rescale the parameters  $\mathbf{z}_1 \rightarrow \mathbf{z}'_1 = \epsilon^{-1} \mathbf{z}_1$ ,  $\mathbf{z}_2 \rightarrow \mathbf{z}'_2 = \epsilon^{-1} \mathbf{z}_2$  such that  $\mathbf{z}'_1$  and  $\mathbf{z}'_2$  become unit-normalised and orthogonal.

Explicitly, introducing the vielbein

$$\begin{aligned} e'_1 &= \frac{1}{2} d\tilde{\zeta}_1, & e'_0 &= \frac{1}{2} \sinh \sigma d\tilde{\zeta}_2, & e_2 &= \frac{1}{2} d\sigma, \\ e_3 &= \frac{1}{2} d\xi_1, & e_4 &= \frac{1}{2} \sinh \theta d\xi_2, & e_5 &= \frac{1}{2} d\theta, \end{aligned} \quad (5.7)$$

we may write the resulting  $AdS_2 \times S^2 \times T^6$  metric and fluxes as

$$\begin{aligned} ds^2 &= (-e_0'^2 + e_2'^2) + (e_4^2 + e_5^2) + (e_1'^2 + e_3^2 + dx^r dx^r), \\ H_3 &= s'_1 d\hat{B}' + s'_2 d\check{B}', & F_3 &= y'_1 d\hat{B}' + y'_2 d\check{B}', \\ F_5 &= (y'_3 d\hat{B}' + y'_4 d\check{B}') \wedge J_2^{(1)} + (y'_5 d\hat{B}' + y'_6 d\check{B}') \wedge J_2^{(2)} + (y'_7 d\hat{B}' + y'_8 d\check{B}') \wedge J_2^{(3)}, \\ d\hat{B}' &= 2(e'_0 \wedge e'_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_5), & d\check{B}' &= 2(-e'_0 \wedge e_2 \wedge e_3 + e'_1 \wedge e_4 \wedge e_5), \end{aligned} \quad (5.8)$$

where  $\mathbf{z}'_1 = (s'_1, y'_1, y'_3, y'_5, y'_7)$  and  $\mathbf{z}'_2 = (s'_2, y'_2, y'_4, y'_6, y'_8)$  satisfy

$$\mathbf{z}'_1 \cdot \mathbf{z}'_2 = 0, \quad \|\mathbf{z}'_1\|^2 = 1, \quad \|\mathbf{z}'_2\|^2 = 1. \quad (5.9)$$

If we choose  $s'_1 = s'_2 = y'_1 = y'_2 = 0$  such that only the  $F_5$  flux is non-zero then the remaining parameters should satisfy

$$y'_3 y'_4 + y'_5 y'_6 + y'_7 y'_8 = 0, \quad y_3'^2 + y_5'^2 + y_7'^2 = 1, \quad y_4'^2 + y_6'^2 + y_8'^2 = 1. \quad (5.10)$$

This gives a three-parameter family of  $AdS_2 \times S^2 \times T^6$  type IIB solutions supported by

$$F_5 \sim \text{Vol}(AdS_2) \wedge \text{Re } \Omega_3 + \text{Vol}(S^2) \wedge \text{Im } \Omega_3, \quad (5.11)$$

where  $\text{Vol}(AdS_2) = e_0 \wedge e_2$ ,  $\text{Vol}(S^2) = e_4 \wedge e_5$  and  $\Omega_3 = dw^1 \wedge dw^2 \wedge dw^3$  is a holomorphic three-form on the 6-torus.<sup>28</sup>

Let us note that starting with the background (5.3)–(5.5) for general values of the parameters and compactifying to 4 dimensions along the isometric directions  $\tilde{\zeta}_1$  and  $\xi_1$  as well as  $T^4$  gives a family of  $AdS_2 \times S^2$  solutions of  $d = 4$  supergravity only supported by several equal-charge electric and magnetic Maxwell fluxes. One pair of 4d abelian vector fields come from the KK reduction on the fibres and others come from the five 3-forms

<sup>28</sup>This corresponds to a near-horizon limit of the type IIB solution representing a  $\frac{1}{4}$ -supersymmetric intersection of four D3-branes [70, 71].

in the 6d action (2.16). The effective 4d Lagrangian is then  $L = R_4 - \frac{1}{4} \sum_k c_k F_{uv}^{(k)} F^{(k)uv}$  (cf. [37, 72, 73]). These  $\text{AdS}_2 \times \text{S}^2$  solutions may be viewed as the near-horizon limits of a family of  $d = 4$   $\mathcal{N} = 2$  supersymmetric extremal RN black holes (with constant scalar fields).<sup>29</sup>

As discussed in section 4, the background (2.8) and (2.22) corresponds to an integrable GS sigma model, and thus the same applies also to its analytic continuation (5.2) and (5.4). The above relation may therefore be interpreted as an integrable embedding of these  $\text{AdS}_2 \times \text{S}^2$  backgrounds into type IIB string theory.

## 5.2 Scaling backgrounds

Another transformation of interest is the scaling limit

$$\sigma \rightarrow \log \frac{2z}{\epsilon}, \quad \zeta_2 \rightarrow \epsilon \zeta_2, \quad \epsilon \rightarrow 0. \quad (5.12)$$

Taking this limit in the metric (2.8) and fluxes (2.9) we find

$$\begin{aligned} ds^2 &= \frac{1}{4} (z^2 d\zeta_2^2 + z^{-2} dz^2 - (1 + \kappa^2) (d\zeta_1 - z d\zeta_2)^2) \\ &\quad + \frac{1}{4} (\sin^2 \theta d\xi_2^2 + d\theta^2 + (1 + \kappa^2) (d\xi_1 - \cos \theta d\xi_2)^2). \\ d\hat{B} &= \frac{1}{4} [d\zeta_1 \wedge d\zeta_2 \wedge dz + \sin \theta d\xi_1 \wedge d\xi_2 \wedge d\theta], \\ d\check{B} &= \frac{1}{4} [d\zeta_2 \wedge dz \wedge (d\xi_1 - \cos \theta d\xi_2) + \sin \theta (d\zeta_1 - z d\zeta_2) \wedge d\xi_2 \wedge d\theta]. \end{aligned} \quad (5.13)$$

Since we are not taking any limit on  $\kappa$  this metric can be completed to supergravity solutions by the ansätze (2.22) subject to the constraints (2.24).

Setting  $\kappa = 0$  in the AdS part of the metric (5.13) we find

$$ds_{\text{AdS}}^2 = \frac{1}{4} (z^{-2} dz^2 + 2z d\zeta_1 d\zeta_2 - d\zeta_1^2), \quad (5.14)$$

which we recognise as the background of a pp-wave in  $\text{AdS}_3$  and is locally equivalent to  $\text{AdS}_3$  [76, 77].<sup>30</sup> Furthermore, as well as being related by the scaling limit (5.12), the metric (2.8) and fluxes (2.9) for general  $\kappa$  are also related to (5.13) by just the following local coordinate redefinition

$$\begin{aligned} \sinh \frac{\sigma}{2} &\rightarrow \sqrt{\frac{1 - 4z + z^2(4 + \zeta_2^2)}{8z}}, \quad \sin 2\zeta_2 \rightarrow -\frac{8z^2 \zeta_2 (1 - z^2(4 - \zeta_2^2))}{1 - 2z^2(4 - \zeta_2^2) + z^4(4 + \zeta_2^2)^2}, \\ \sin 2\zeta_1 &\rightarrow \frac{(1 - 2z^2(4 + 3\zeta_2^2) + z^4(4 + \zeta_2^2)^2) \sin 2\zeta_1 - 4z\zeta_2(1 - z^2(4 + \zeta_2^2)) \cos 2\zeta_1}{(1 - 4z + z^2(4 + \zeta_2^2))(1 + 4z + z^2(4 + \zeta_2^2))}. \end{aligned} \quad (5.15)$$

<sup>29</sup>In general, one can use U-duality to represent the BPS 4d black hole solution (with non-constant scalars) as having 5 charges all in the NSNS sector [74, 75] but such solutions need not uplift just to our effective background (5.8) or its 6d reduction corresponding to (2.16). The reason is that we have not considered TsT transformations that change the structure of the 6d metric, i.e. there is effectively more freedom from the 4d perspective. Thus these extra T-dualities may allow one to put the metric into the form in which the most general solution can be generated from the one with NSNS charges only.

<sup>30</sup>Starting from the metric of  $\text{AdS}_3$  in Poincaré patch  $ds^2 = \tilde{z}^{-2} (d\tilde{z}^2 + 2dudv)$  and using the change of coordinates  $\tilde{z} = z^{-\frac{1}{2}} \sec \frac{\mu\zeta_1}{2}$ ,  $u = \mu^{-1} \tan \frac{\mu\zeta_1}{2}$ ,  $v = \frac{1}{2} (\zeta_2 - \mu z^{-1} \tan \frac{\mu\zeta_1}{2})$ , we find the metric (5.14). For  $\mu = i$ , which still defines a real coordinate transformation, we find the space-like pp-wave in  $\text{AdS}_3$ , i.e. (5.14) with  $\zeta_1 = i\tilde{\zeta}_1$ ,  $\zeta_2 = -i\tilde{\zeta}_2$ .

This means that the scaling limit does not change the form of the YB deformed supercoset action (4.3) or the unimodular R-matrix — it only modifies the particular parametrisation of the supergroup-valued field  $g$  that is used to find the form of the background in local coordinates.<sup>31</sup>

We can also take the analogous limit to (5.12), i.e. with  $\tilde{\zeta}_2 \rightarrow \epsilon \tilde{\zeta}_2$ , in the analytically-continued metric (5.3) and fluxes (5.4). The resulting metric and fluxes are the same as (5.13) with  $\zeta_1 = i\tilde{\zeta}_1$ ,  $\zeta_2 = -i\tilde{\zeta}_2$ ,  $\kappa = i\tilde{\kappa}$  and  $\check{B} = i\check{\tilde{B}}$ , and this metric can be completed to supergravity solutions by the ansätze (5.4) subject to the constraints (5.5). Setting  $\tilde{\kappa} = 0$  we find the metric  $ds_{\text{AdS}}^2 = \frac{1}{4}(z^{-2}dz^2 + 2zd\tilde{\zeta}_1d\tilde{\zeta}_2 + d\tilde{\zeta}_1^2)$  (which is a limit of the 3d F1+ppwave background [78]).<sup>32</sup> Again this metric is locally equivalent to AdS<sub>3</sub> and, moreover, the deformed metric and fluxes are related to (5.3) and (5.4) by a local coordinate redefinition similar to (5.15). Similar deformations to this analytically-continued pp-wave background and their holographic interpretation have been studied, e.g., in [80, 81].

### 5.3 Special limit leading to Schrödinger background

There is a particular limit of the deformed background (2.8), (2.22) that gives rise to the metric which is a direct sum of that of the 3d Schrödinger space-time and undeformed S<sup>3</sup> (and 4-torus). Indeed, let us rescale

$$\kappa \rightarrow \epsilon\kappa, \quad \mathbf{z}_2 \rightarrow \epsilon\mathbf{z}_2, \quad (\zeta_1, \zeta_2) \rightarrow \epsilon(\zeta_1, \zeta_2), \quad \sigma \rightarrow \sigma - \log \epsilon^2, \quad (5.16)$$

and take  $\epsilon \rightarrow 0$ . Then the metric in (2.8) and the auxiliary fluxes in (2.9), (2.21) become<sup>33</sup>

$$\begin{aligned} ds^2 &= \frac{1}{4} \left( d\sigma^2 + e^\sigma d\zeta_1 d\zeta_2 - \frac{1}{4} \kappa^2 e^{2\sigma} d\zeta_2^2 \right) + \frac{1}{4} \left( \sin^2 \theta d\xi_2^2 + d\theta^2 + (d\xi_1 - \cos \theta d\xi_2)^2 \right) + dx_r dx_r, \\ \mathbf{F}_3 &= \mathbf{z}_1 d\hat{B} + \mathbf{z}_2 d\check{B} = \mathbf{z}_1 \left( \frac{1}{8} e^\sigma d\zeta_1 \wedge d\zeta_2 \wedge d\sigma + \frac{1}{4} \sin \theta d\xi_1 \wedge d\xi_2 \wedge d\theta \right) \\ &\quad + \mathbf{z}_2 \left( \frac{1}{8} e^\sigma d\zeta_2 \wedge d\sigma \wedge (d\xi_1 - \cos \theta d\xi_2) - \frac{1}{8} e^\sigma \sin \theta d\zeta_2 \wedge d\xi_2 \wedge d\theta \right). \end{aligned} \quad (5.17)$$

Here the first 3d factor of the metric is that of the 3d Schrödinger space-time (equivalent to  $\frac{dz^2}{z^2} + \frac{1}{4z^2} d\zeta_1 d\zeta_2 - \frac{\kappa^2}{16z^4} d\zeta_2^2$  where  $z = e^{-\sigma/2}$ ) which is a deformation of AdS<sub>3</sub> directly in the Poincaré patch.

The corresponding limit of the supergravity constraints (2.24) is

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = 0, \quad \|\mathbf{z}_1\|^2 = 1, \quad \|\mathbf{z}_2\|^2 = \kappa^2. \quad (5.18)$$

<sup>31</sup>Explicitly, considering just the AdS<sub>3</sub> sigma model, or equivalently the SU(1,1) PCM, if we parametrise  $g \in \text{SU}(1,1)$  as  $g = \exp\left(-\frac{i\zeta_2}{2}\sigma_3\right) \exp\left(\frac{\sigma}{2}\sigma_2\right) \exp\left(\frac{i\zeta_1}{2}\sigma_3\right)$  with  $R(X) = \frac{1}{2}(\text{tr}(\sigma_2 X)\sigma_1 - \text{tr}(\sigma_1 X)\sigma_2)$  we recover the AdS part of the metric (2.8) from the YB deformation of the PCM. Alternatively, if we parametrise  $g = \frac{1}{2\sqrt{2z}}((1+2z)\mathbf{1}_2 - z\zeta_2\sigma_1 - (1-2z)\sigma_2 - iz\zeta_2\sigma_3) \exp\left(\frac{i\zeta_1}{2}\sigma_3\right)$ , then using the same R-matrix we recover the AdS part of the metric (5.13).

<sup>32</sup>This is, in fact, how the AdS<sub>3</sub> metric appears in the near-horizon limit of the F1+NS5+pp-wave+KK-monopole sigma model providing a particular string embedding of 4d BPS black holes in [74, 79] (see eq. (17) in [79] and eq. (20) in [74]).

<sup>33</sup>The same background can be found by taking an analogous limit, i.e. with  $\mathbf{z}_2 \rightarrow -\epsilon\mathbf{z}_2$  and  $(\tilde{\zeta}_1, \tilde{\zeta}_2) \rightarrow \epsilon(\tilde{\zeta}_1, \tilde{\zeta}_2)$ , in the analytically-continued background (5.3)–(5.4). It can also be found from the scaled background (5.13) by setting  $\kappa \rightarrow \epsilon\kappa$ ,  $\mathbf{z}_2 \rightarrow \epsilon\mathbf{z}_2$ ,  $\zeta_1 \rightarrow \epsilon\zeta_1$ ,  $\zeta_2 \rightarrow \epsilon^{-1}\zeta_2$  and  $z \rightarrow \frac{1}{2}e^\sigma$  and taking  $\epsilon \rightarrow 0$ , or from its analytic continuation by an analogous limit (cf. [80, 81]).

Let us note that it is possible to set  $\kappa = 1$  by rescaling  $\zeta_1 \rightarrow \kappa\zeta_1$ ,  $\zeta_2 \rightarrow \kappa^{-1}\zeta_2$  and  $\mathbf{z}_2 \rightarrow \kappa\mathbf{z}_2$ , meaning that  $\kappa$  is not a genuine deformation parameter.

For the particular solution of (5.18)

$$\mathbf{z}_1 = \left( \sqrt{1-q^2}, q, 0, 0, 0 \right), \quad \mathbf{z}_2 = (0, 0, -\kappa, 0, 0), \quad (5.19)$$

this limit is related to a special Jordanian deformation limit of the DJ background (with parameters given by (4.8)). Taking the limit in the YB deformed supercoset action (4.3) is equivalent to using, instead of the DJ R-matrix, a fermionic extension of the Jordanian R-matrix in [47, 82]. Introducing the usual Cartan-Weyl basis of generators  $H_L$ ,  $E_L$  and  $F_L$  for  $\mathfrak{sl}(2; \mathbb{R})_L$  and similarly for  $\mathfrak{sl}(2; \mathbb{R})_R$ , the bosonic part of the R-matrix acts as  $R(F_L) = -H_L$ ,  $R(H_L) = 2E_L$ , and  $R(E_L) = 0$ . This R-matrix solves the cYBe and as such corresponds to a homogeneous YB deformation [83] of the  $\text{AdS}_3 \times \text{S}^3$  background. This implies that the deformed background (5.17) can also be found from  $\text{AdS}_3 \times \text{S}^3$  by a non-abelian duality in a particular subalgebra of  $\mathfrak{psu}(1, 1|2)$  [84, 85].

Another case of the background (5.17) can also be found by starting from the mixed flux  $\text{AdS}_3 \times \text{S}^3$  background (2.1), (2.3) in Poincaré coordinates, i.e. (5.17) with

$$\kappa = 0, \quad \mathbf{z}_1 = \left( \sqrt{1-q^2}, q, 0, 0, 0 \right), \quad \mathbf{z}_2 = \mathbf{0}, \quad (5.20)$$

and applying a special TsT transformation in null coordinate  $\zeta_1$  or null Melvin twist [86]: T-duality in  $\xi_1$ , shift  $\zeta_1 \rightarrow \zeta_1 + \frac{4\kappa}{q}\xi_1$  and T-duality back in  $\xi_1$ . This leads to the background (5.17) corresponding to the following solution to the constraints (5.18)

$$\mathbf{z}_1 = \left( \sqrt{1-q^2}, q, 0, 0, 0 \right), \quad \mathbf{z}_2 = \left( \kappa q, -\kappa\sqrt{1-q^2}, 0, 0, 0 \right). \quad (5.21)$$

Just as for the deformation of  $\text{AdS}_3 \times \text{S}^3$  in global coordinates discussed in section 2, the full space of solutions of (5.18) can be found by starting from the Jordanian solution (5.19) and using the above null Melvin twist and TsT transformations on the 4-torus. This demonstrates the classical integrability of the corresponding string sigma model. Equivalently, while the full space of solutions of (5.18) cannot be generated from  $\text{AdS}_3 \times \text{S}^3$  by just TsT transformations, it can if we also allow a non-abelian duality transformation. This is in contrast to the situation before taking the limit (5.16), where, as discussed in section 2.2, to get the most general background (2.8), (2.21) using duality transformations we also need to apply S-duality (see (2.27)).<sup>34</sup>

While the full space of solutions (5.18) can be generated using non-abelian duality and TsT transformations, this cannot be lifted to the F1-D1-NS5-D5 brane background [87] which, in the near-horizon limit, becomes the mixed flux  $\text{AdS}_3 \times \text{S}^3$  background (2.1), (2.3).<sup>35</sup> On the other hand, we can apply the null Melvin twist, TsT trans-

<sup>34</sup>Note that after taking limit we can still generate the full space of solutions of (5.18) using TsT transformations and S-duality, i.e., without non-abelian duality.

<sup>35</sup>Explicitly,  $J = H_R - H_L$ ,  $P_+ = E_R$  and  $P_- = E_L$  are the generators of the Lorentz algebra, and  $D = H_R + H_L$ ,  $K_- = F_R$  and  $K_+ = F_L$  generate dilatations and special conformal transformations respectively.

formations on the 4-torus and S-duality to the brane background. This generates a 8-parameter (including  $\kappa$ ) deformation, which in a modified near-horizon limit, where  $\kappa$  is also rescaled, gives the deformed  $\text{AdS}_3 \times \text{S}^3$  background (5.17).

#### 5.4 Plane-wave limit and light-cone S-matrix

The plane-wave analog of the  $\kappa$ -deformed  $\text{AdS}_3 \times \text{S}^3$  metric (2.4) is obtained by redefining the coordinates and taking the Penrose-type limit  $L \rightarrow \infty$

$$\begin{aligned} t &= \mu x^+ + \frac{x^-}{\mu L^2}, & \varphi &= \mu x^+ - \frac{x^-}{\mu L^2}, & \psi &= \psi' - \kappa^2 \mu x^+, & \phi &= \phi' - \kappa^2 \mu x^+, \\ \rho &= \frac{\sqrt{1 + \kappa^2}}{L} \rho', & r &= \frac{\sqrt{1 + \kappa^2}}{L} r', & L &\rightarrow \infty. \end{aligned} \quad (5.22)$$

We also should rescale the 4-torus coordinates  $x_r = \frac{\sqrt{1 + \kappa^2}}{L} x'_r$  and the overall factor of string tension  $T = \frac{L^2}{1 + \kappa^2} T'$  (cf. [88]).<sup>36</sup> Here  $\mu$  is an effective curvature scale parameter. As a result, we get the following pp-wave metric

$$\begin{aligned} ds^2 &= -4dx^+ dx^- - (1 + \kappa^2)^2 \mu^2 (dx^+)^2 (\rho'^2 + r'^2) \\ &\quad + d\rho'^2 + \rho'^2 d\psi'^2 + dr'^2 + r'^2 d\phi'^2 + dx'_r dx'_r. \end{aligned} \quad (5.23)$$

Note that the deformation parameter  $\kappa$  enters only through  $\hat{\mu} = (1 + \kappa^2)\mu$ , which (for  $\hat{\mu} \neq 0$ ) can be rescaled away by  $x^+ \rightarrow \hat{\mu}^{-1} x^+$ ,  $x^- \rightarrow \hat{\mu} x^-$ . Thus the metric (5.23) is equivalent to the pp-wave metric [89] found from the undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ .

In the limit (5.22) the auxiliary potentials in (2.2), (2.6) differ only by an exact 2-form

$$\begin{aligned} \hat{B}' &= \mu dx^+ \wedge (\rho'^2 d\psi' + r'^2 d\phi'), & \check{B}' &= -\frac{2}{1 + \kappa^2} dx^+ \wedge dx^- + \mu dx^+ \wedge (\rho'^2 d\psi' + r'^2 d\phi'), \\ d\hat{B}' &= d\check{B}' = 2\mu dx^+ \wedge (\rho' d\rho' \wedge d\psi' + r' dr' \wedge d\phi'), \end{aligned} \quad (5.24)$$

Note that in terms of the four cartesian coordinates  $z_i$  defined as  $z_1 + iz_2 = \rho e^{i\psi}$  and  $z_3 + iz_4 = r e^{i\phi}$ , the metric (5.23) and 3-form (5.24) may be written as

$$\begin{aligned} ds^2 &= -4dx^+ dx^- - (1 + \kappa^2)^2 \mu^2 (dx^+)^2 z_i^2 + dz_i^2 + dx_r'^2, \\ d\hat{B}' &= -2\mu dx^+ \wedge (dz_1 \wedge dz_2 + dz_3 \wedge dz_4). \end{aligned} \quad (5.25)$$

The flux background (2.22) supporting the metric (5.23) thus becomes

$$\begin{aligned} H_3 &= (s_1 + s_2) d\hat{B}', & F_3 &= (y_1 + y_2) d\hat{B}', \\ F_5 &= d\hat{B}' \wedge ((y_3 + y_4) J_2'^{(1)} + (y_5 + y_6) J_2'^{(2)} + (y_7 + y_8) J_2'^{(3)}), \end{aligned} \quad (5.26)$$

i.e.  $\mathbf{F}_3 = (\mathbf{z}_1 + \mathbf{z}_2) d\hat{B}'$ , where the only condition on the parameters is (cf. (2.21), (2.24))

$$\|\mathbf{z}_1 + \mathbf{z}_2\|^2 = (s_1 + s_2)^2 + (y_1 + y_2)^2 + (y_3 + y_4)^2 + (y_5 + y_6)^2 + (y_7 + y_8)^2 = (1 + \kappa^2)^2. \quad (5.27)$$

<sup>36</sup>To recall, if we restore the dependence on string tension  $T$  then the metric and fluxes scale with  $T$  as follows:  $ds^2 \sim T$ ,  $H_3 \sim T$ ,  $F_3 \sim T$ ,  $F_5 \sim T^2$ . We shall set  $T' = 1$  after taking the limit.

Thus the background in this limit only depends on the parameters  $\mathbf{z}_1$  and  $\mathbf{z}_2$  through  $\mathbf{z}_1 + \mathbf{z}_2$ . Therefore, two different instances of the original background (2.21) will have the same pp-wave limit if they have equal  $\mathbf{z}_1 + \mathbf{z}_2$ , but will differ at  $O(L^{-2})$ .

Starting with the superstring action corresponding to the deformed background (2.4), (2.22) and considering the BMN-type expansion [90] in a light-cone gauge as in the undeformed case [91, 92] at the leading (quadratic) order we find the string propagating in the pp-wave background (5.23), (5.26). The resulting 2d dispersion relation  $\omega(p)$  for the bosonic string fluctuations originating from the  $\text{AdS}_3 \times \text{S}^3$  part of the model is<sup>37</sup>

$$(\omega - \zeta\kappa^2)^2 = p^2 + 2\zeta(s_1 + s_2)p + (1 + \kappa^2)^2, \quad \zeta \in \{-1, +1\}. \quad (5.28)$$

This dispersion relation becomes linear for the special choice of  $s_1 + s_2$

$$s_1 + s_2 = \pm(1 + \kappa^2) \quad \Rightarrow \quad |\omega - \zeta\kappa^2| = |p \pm \zeta(1 + \kappa^2)|. \quad (5.29)$$

It follows from (5.27) that in this case the pp-wave background is pure NSNS, i.e.  $y_k + y_{k+1} = 0$  for  $k$  odd, which implies  $F_3 = F_5 = 0$ . However, if we set  $s_1 + s_2 = \pm(1 + \kappa^2)$  and  $y_k + y_{k+1} = 0$  in the original background (2.21) (before taking the limit (5.22)) then the conditions (2.24) imply that  $s_1 = \pm 1$ ,  $s_2 = \pm \kappa^2$  and  $\sum_k y_k^2 = \kappa^2$ . Hence, it is only pure NSNS, i.e.  $y_k = 0$  for all  $k$ , if  $\kappa = 0$ . This corresponds to undeformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  supported by  $H_3$  only. While all backgrounds with  $s_1 + s_2 = \pm(1 + \kappa^2)$  and  $y_k + y_{k+1} = 0$  have the same pure NSNS pp-wave limit, they will differ at the next (quartic) order in the BMN-type expansion.

Fixing a light-cone gauge in the  $x^+$  direction (the expansion is around the  $t = \varphi \sim \tau$  massive geodesic, cf. (5.22)) in the full GS action we can compute the corresponding 4-point tree-level S-matrix for the bosonic fluctuations. It is convenient to introduce the complex scalar fields

$$Z = \frac{z_2 - iz_1}{\sqrt{2}}, \quad \bar{Z} = \frac{z_2 + iz_1}{\sqrt{2}}, \quad Y = \frac{z_3 - iz_4}{\sqrt{2}}, \quad \bar{Y} = \frac{z_3 + iz_4}{\sqrt{2}}, \quad (5.30)$$

for the 2+2 transverse coordinates on  $\text{AdS}_3$  ( $Z$  and  $\bar{Z}$ ) and  $\text{S}^3$  ( $Y$  and  $\bar{Y}$ ) respectively (cf. (5.25)). Assuming that  $s_1 + s_2 = 1 + \kappa^2$  and  $\frac{\partial\omega}{\partial p}\Big|_{p_1} > 0$  and  $\frac{\partial\omega}{\partial p}\Big|_{p_2} < 0$ , we find the following generalization of the pure NSNS  $\text{AdS}_3 \times \text{S}^3$  S-matrix (cf. [91–93, 95]) to the case of  $\kappa \neq 0$ :<sup>38</sup>

- Left-Left sector ( $\zeta_1 = \zeta_2 = -1$ )

$$\begin{aligned} S_{ZZZZ} &= -S_{YYYY} = -\frac{1}{2}(p_1 + p_2) \frac{1 + \kappa^2 - p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 - p_2} + \kappa^2 \frac{1 + \kappa^2 - p_1}{1 + \kappa^2 - p_2} p_2, \\ S_{ZYZY} &= -S_{YZYZ} = \frac{1}{2}(p_1 - p_2) \frac{1 + \kappa^2 - p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 - p_2} - \kappa^2 \frac{1 + \kappa^2 - p_1}{1 + \kappa^2 - p_2} p_2, \end{aligned} \quad (5.31)$$

<sup>37</sup>As expected, setting  $\kappa = 0$  this dispersion relation matches the small-momentum limit of the conjectured exact dispersion relation for strings on  $\text{AdS}_3 \times \text{S}^3$  with mixed flux [92, 93]. For  $s_1 = q(1 + \kappa^2)$  and  $s_2 = 0$  it matches the dispersion relation of the three parameter deformation [94] with equal quantum group deformation parameters.

<sup>38</sup> $S$  here stands for the tree-level T-matrix and we omit an overall factor proportional to the effective coupling (inverse string tension).

- Right-Right sector ( $\varsigma_1 = \varsigma_2 = +1$ )

$$\begin{aligned}
 S_{\bar{Z}\bar{Z}\bar{Z}\bar{Z}} &= -S_{\bar{Y}\bar{Y}\bar{Y}\bar{Y}} = \frac{1}{2}(p_1 + p_2) \frac{1 + \kappa^2 + p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 + p_2} - \kappa^2 \frac{1 + \kappa^2 + p_1}{1 + \kappa^2 + p_2} p_2, \\
 S_{\bar{Z}\bar{Y}\bar{Z}\bar{Y}} &= -S_{\bar{Y}\bar{Z}\bar{Y}\bar{Z}} = -\frac{1}{2}(p_1 - p_2) \frac{1 + \kappa^2 + p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 + p_2} + \kappa^2 \frac{1 + \kappa^2 + p_1}{1 + \kappa^2 + p_2} p_2,
 \end{aligned}
 \tag{5.32}$$

- Left-Right sector ( $\varsigma_1 = -1, \varsigma_2 = +1$ )

$$\begin{aligned}
 S_{Z\bar{Z}\bar{Z}\bar{Z}} &= -S_{Y\bar{Y}\bar{Y}\bar{Y}} = \frac{1}{2}(p_1 - p_2) \frac{1 + \kappa^2 + p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 + p_2} + \kappa^2 \frac{1 + \kappa^2 - p_1}{1 + \kappa^2 + p_2} p_2, \\
 S_{Z\bar{Y}\bar{Z}\bar{Y}} &= -S_{Y\bar{Z}\bar{Y}\bar{Z}} = -\frac{1}{2}(p_1 + p_2) \frac{1 + \kappa^2 + p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 + p_2} - \kappa^2 \frac{1 + \kappa^2 - p_1}{1 + \kappa^2 + p_2} p_2,
 \end{aligned}
 \tag{5.33}$$

- Right-Left sector ( $\varsigma_1 = +1, \varsigma_2 = -1$ )

$$\begin{aligned}
 S_{\bar{Z}\bar{Z}\bar{Z}\bar{Z}} &= -S_{\bar{Y}\bar{Y}\bar{Y}\bar{Y}} - \frac{1}{2}(p_1 - p_2) \frac{1 + \kappa^2 - p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 - p_2} - \kappa^2 \frac{1 + \kappa^2 + p_1}{1 + \kappa^2 - p_2} p_2, \\
 S_{\bar{Z}\bar{Y}\bar{Z}\bar{Y}} &= -S_{\bar{Y}\bar{Z}\bar{Y}\bar{Z}} = \frac{1}{2}(p_1 + p_2) \frac{1 + \kappa^2 - p_2(s_1 - s_2 + \kappa^2)}{1 + \kappa^2 - p_2} + \kappa^2 \frac{1 + \kappa^2 + p_1}{1 + \kappa^2 - p_2} p_2.
 \end{aligned}
 \tag{5.34}$$

It would be interesting to find also the fermionic sectors of this  $\kappa$ -dependent S-matrix and check its consistency with integrability of the model, similarly to what was done for the Drinfel'd-Jimbo deformation without WZ term [96].

## 6 Concluding remarks

In this paper we have constructed a family of type IIB supergravity backgrounds that are deformations of the mixed flux  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background. The “squashed”  $\text{AdS}_3 \times \text{S}^3$  metric is naturally written in terms of Hopf fibrations and the deformed backgrounds have a number of key properties: (i) they are supersymmetric, preserving half the supersymmetry of the undeformed  $\text{AdS}_3 \times \text{S}^3$  background, (ii) they have trivial dilaton, (iii) they have regular curvature and (iv) the corresponding Green-Schwarz superstring sigma model is classically integrable. Given the global symmetries and the amount of supersymmetry that is preserved, and that the fluxes are homogeneous, one might suspect that these backgrounds may be stable under  $\alpha'$ -corrections (possibly up to redefinitions of the parameters).<sup>39</sup>

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<sup>39</sup>We should add, however, the following reservation. While the T-duality, in general, should “commute” with  $\alpha'$ -corrections (modulo possible deformations that should trivialise in half-maximally supersymmetric case) this does not a priori apply to S-duality which is not a world-sheet symmetry. Thus, while the original  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  background should be stable under  $\alpha'$ -corrections, backgrounds related to it by combinations of T- and S-dualities may not share this property. Still,  $\alpha'$ -corrections may be absent in the special case of homogeneous fluxes. For some discussions of the absence or presence of  $\alpha'$ -corrections to some half-maximally supersymmetric backgrounds with inhomogeneous fluxes see, e.g., [97, 98].

The family of backgrounds (2.8), (2.9), (2.22)–(2.24) are deformations of  $\text{AdS}_3 \times \text{S}^3$  with the AdS space written as a time-like fibration over  $\text{H}^2$ . The existence of this family may not be surprising since it can be found starting from the mixed flux background and T-duality and S-duality. However, it can also be found by applying TsT transformations to the one-parameter Yang-Baxter deformation (deforming one copy of  $\mathfrak{psu}(1, 1|2)$ ) of the mixed flux string sigma model using a particular Drinfel'd-Jimbo R-matrix. This latter construction demonstrates the classical integrability of the corresponding string sigma model. Let us emphasise that the DJ R-matrix used in the YB sigma model is the unique one on  $\mathfrak{psu}(1, 1|2)$  that is unimodular [13, 14], and it is associated to the Dynkin diagram with all fermionic simple roots. Crucially, this ensures that the corresponding background solves the standard type II supergravity equations rather than the generalised supergravity equations [18, 19].

As discussed in the Introduction, this is another example of S-duality unexpectedly preserving integrability, and is the first case that involves an inhomogeneous YB deformation so that the symmetry algebra is q-deformed. In contrast to the Jordanian examples [46, 47], here the classical integrability does not have an alternative explanation in terms of twists and worldsheet dualities.

One motivation for studying these backgrounds is to explore the relation between integrability of bosonic sigma models and their embeddings into string theory. In particular, given an integrable bosonic sigma model, when does there exist an embedding into supergravity (or generalised supergravity) such that the corresponding GS string sigma model is also integrable? Moreover, if we have such a setup, when does an S-duality transformation of type IIB background, which modifies the bosonic truncation of the worldsheet model (leading, in general, to different metric and B-field) preserve integrability? In all such examples that we know of, including the new ones presented here, the dilaton is constant and (setting  $\Phi = \Phi_0 = 0$ ) the metric is unchanged under the S-duality.

In section 5.1, we considered the analytic continuation (5.3)–(5.5) of the family of deformed backgrounds presented in section 2 (with  $\tilde{\kappa} = i\kappa$ ). In this case, the  $\text{AdS}_3$  geometry is written as a space-like fibration over  $\text{AdS}_2$ . The metric interpolates between  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  ( $\tilde{\kappa} = 0$ ) and  $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$  ( $\tilde{\kappa} = 1$ ). In the  $\tilde{\kappa} = 1$  limit, we recover the  $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$  background with 8 supersymmetries, which for a particular choice of parameters corresponds to the near-horizon limit of the  $\frac{1}{4}$ -supersymmetric intersection of four D3-branes. It would be interesting to explore if there is any link with the 8-vertex R-matrix of [99] that interpolates between the building blocks of the exact  $\text{AdS}_3$  and  $\text{AdS}_2$  S-matrices, describing the scattering of string excitations above the BMN vacuum.

It is an important open question as to whether we can find a brane interpretation of the two families of deformed backgrounds. We may expect to be in a more promising situation than the familiar  $\eta$ -deformation of  $\text{AdS}_5 \times \text{S}^5$  [13, 39] due to the special properties discussed above, in particular the trivial dilaton and supersymmetry. The holographic interpretation of the scaling (in the analytically-continued case) background and Jordanian limit, described in sections 5.2 and 5.3 respectively, has been studied, e.g., in [80, 81, 100]. In these limits, the deformed backgrounds are naturally written in  $\text{AdS}_3$  pp-wave and Poincaré coordinates, and can again be generated by T-dualities and S-dualities. It would be interesting to investigate if we can move away from these limits perturbatively, in partic-

ular in the case of the scaling background, which is “intermediate” between our deformed backgrounds and their Jordanian limit (e.g., the sphere geometry is still deformed).

A brane interpretation of the analytically-continued metric in section 5.1, albeit with different parameters in the AdS and S parts and with different supporting fluxes, has been explored in [35, 36]. The backgrounds discussed there are generated by TsT transformations and the deformed brane background appears as a limit of the T-dual of the D1-D5+pp wave+KK monopole background. The T-duality in direction transverse to KK monopole leads to “non-geometric” background.

A similar construction to that presented in this paper is also possible for the mixed flux  $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$  background [9, 49]. In this case the relevant supercoset is  $\frac{\text{D}(2,1;\alpha) \times \text{D}(2,1;\alpha)}{\text{SU}(1,1) \times \text{SU}(2) \times \text{SU}(2)}$  and we can again deform a single copy of  $\mathfrak{d}(2,1;\alpha)$  preserving half the supersymmetry. The superalgebra  $\mathfrak{d}(2,1;\alpha)$  admits a fermionic Dynkin diagram and the corresponding YB deformation should give a background solving the standard type II supergravity equations. The  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 1$  limits correspond to decompactifying one of the spheres, hence there should be an intersection with the deformed  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  backgrounds constructed here. Therefore, it is natural to expect that the deformed  $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$  backgrounds have some of the key properties discussed above and it would be interesting to investigate the extent to which this is the case.

Finally, it would also be interesting to study string propagation on these backgrounds. In section (5.4) we initiated the study of the near-BMN light-cone gauge S-matrix, focusing on the limit in which the excitations become massless. It would be interesting to move away from this limit and propose a conjecture for the exact S-matrix along the lines of [65, 93, 101]. It would also be instructive to study how classical solitonic string solutions, such as long strings, are affected by the deformation.

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## A Details on supersymmetry

Here we shall add the explicit form of the Killing spinor equations in section 3 and also demonstrate that the pp-wave background of section 5.4 preserves 16 supersymmetries.

The Killing spinor equations (3.1) take the following explicit form

$$\begin{aligned} \partial_\mu \epsilon &= \Omega_\mu \epsilon, \\ 4\Omega_{\zeta_1} &= - (1 + \kappa^2) 1_2 \otimes \Gamma^{12} - \sigma_3 \otimes (s_1 \Gamma^{12} + s_2 \Gamma^{45}) \\ &\quad + \sigma_1 \otimes (y_1 \Gamma^{12} + y_2 \Gamma^{45}) + i\sigma_2 \otimes (\Gamma^{12} I + \Gamma^{45} J), \end{aligned}$$

$$\begin{aligned}
 4\sqrt{1+\kappa^2}\Omega_\sigma &= (1+\kappa^2)1_2 \otimes \Gamma^{01} - \sigma_3 \otimes (s_1\Gamma^{01} - s_2\Gamma^{13}) \\
 &\quad + \sigma_1 \otimes (y_1\Gamma^{01} - y_2\Gamma^{13}) + i\sigma_2 \otimes (\Gamma^{01}I - \Gamma^{13}J), \\
 4\Omega_{\xi_1} &= -(1+\kappa^2)1_2 \otimes \Gamma^{45} - \sigma_3 \otimes (s_1\Gamma^{45} + s_2\Gamma^{12}) \\
 &\quad + \sigma_1 \otimes (y_1\Gamma^{45} + y_2\Gamma^{12}) + i\sigma_2 \otimes (\Gamma^{45}I + \Gamma^{12}J), \\
 4\sqrt{1+\kappa^2}\Omega_\theta &= (1+\kappa^2)1_2 \otimes \Gamma^{34} - \sigma_3 \otimes (s_1\Gamma^{34} + s_2\Gamma^{04}) \\
 &\quad + \sigma_1 \otimes (y_1\Gamma^{34} + y_2\Gamma^{04}) + i\sigma_2 \otimes (\Gamma^{34}I + \Gamma^{04}J), \\
 4\Omega_{\zeta_2} &= \cosh\sigma \left( -(1-\kappa^2)1_2 \otimes \Gamma^{12} + \sigma_3 \otimes (s_1\Gamma^{12} + s_2\Gamma^{45}) \right. \\
 &\quad \left. - \sigma_1 \otimes (y_1\Gamma^{12} + y_2\Gamma^{45}) - i\sigma_2 \otimes (\Gamma^{12}I + \Gamma^{45}J) \right) \\
 &\quad + \frac{\sinh\sigma}{\sqrt{1+\kappa^2}} \left( -(1+\kappa^2)1_2 \otimes \Gamma^{02} + \sigma_3 \otimes (s_1\Gamma^{02} - s_2\Gamma^{23}) \right. \\
 &\quad \left. - \sigma_1 \otimes (y_1\Gamma^{02} - y_2\Gamma^{23}) - i\sigma_2 \otimes (\Gamma^{02}I - \Gamma^{23}J) \right), \\
 4\Omega_{\xi_2} &= \cos\theta \left( -(1-\kappa^2)1_2 \otimes \Gamma^{45} + \sigma_3 \otimes (s_1\Gamma^{45} + s_2\Gamma^{12}) \right. \\
 &\quad \left. - \sigma_1 \otimes (y_1\Gamma^{45} + y_2\Gamma^{12}) - i\sigma_2 \otimes (\Gamma^{45}I + \Gamma^{12}J) \right) \\
 &\quad + \frac{\sin\theta}{\sqrt{1+\kappa^2}} \left( -(1+\kappa^2)1_2 \otimes \Gamma^{35} + \sigma_3 \otimes (s_1\Gamma^{35} + s_2\Gamma^{05}) \right. \\
 &\quad \left. - \sigma_1 \otimes (y_1\Gamma^{35} + y_2\Gamma^{05}) - i\sigma_2 \otimes (\Gamma^{35}I + \Gamma^{05}J) \right),
 \end{aligned} \tag{A.1}$$

where

$$I = y_3\Gamma^{67} + y_5\Gamma^{68} + y_7\Gamma^{69}, \quad J = y_4\Gamma^{67} + y_6\Gamma^{68} + y_8\Gamma^{69}, \quad \Gamma^{a_1 \dots a_n} \equiv \Gamma^{a_1} \dots \Gamma^{a_n}. \tag{A.2}$$

In the special limit corresponding to the pp-wave background discussed in section 5.4 the only non-zero components of the spin connection (with curved-space indices) for the metric (5.25) are

$$\omega_+^{-j} = -\omega_+^{j-} = \frac{1}{2}z_j\hat{\mu}^2, \quad \hat{\mu} = (1+\kappa^2)\mu. \tag{A.3}$$

For the fluxes in (5.26) the RR bispinor in (3.2) is given by

$$\mathcal{S} = -\frac{1}{8} \left( -2\mu\Gamma^+(\Gamma^{12} + \Gamma^{34}) \right) (\sigma_1(y_1 + y_2) + i\sigma_2(I + J)\mathcal{P}), \tag{A.4}$$

where  $\mathcal{P} = \frac{1}{2}(1 + \Gamma^{6789})$  is the projector involving 4-torus directions. The Killing spinors should not depend on the torus directions. From the identities  $\mathcal{P}\Gamma^j\mathcal{P} = 0$  for  $j = 6, 7, 8, 9$  it follows that this is automatically satisfied if  $\epsilon = \mathcal{P}\hat{\Gamma}\epsilon$ . This reduces the number of Killing spinors from 64 to 16.

Using that  $\Gamma^+ = -\frac{1}{2}\Gamma_-$  and  $(\Gamma^+)^2 = 0$  the remaining Killing spinor equations are

$$\left( \partial_+ + \frac{1}{4}z_j\hat{\mu}^2\Gamma_{-j} \right) \epsilon - \frac{1}{4}\mu(s_1 + s_2)(\Gamma^{12} + \Gamma^{34})\sigma_3\epsilon + \mathcal{S}\Gamma_+\epsilon = 0, \tag{A.5}$$

$$\partial_-\epsilon = 0, \tag{A.6}$$

$$\partial_j\epsilon - \frac{1}{4}\mu(s_1 + s_2)\Gamma^{12+}\Gamma_j\sigma_3\epsilon + \mathcal{S}\Gamma_j\epsilon = 0, \quad j = 1, 2, \tag{A.7}$$

$$\partial_j\epsilon - \frac{1}{4}\mu(s_1 + s_2)\Gamma^{34+}\Gamma_j\sigma_3\epsilon + \mathcal{S}\Gamma_j\epsilon = 0, \quad j = 3, 4. \tag{A.8}$$

Eq. (A.6) shows that  $\epsilon$  does not depend on  $x^-$ , while (A.7), (A.8) imply  $\partial_j \partial_k \epsilon = 0$  and hence  $\epsilon$  is at most linear in  $z^j$ . In fact, (A.7), (A.8) can be written as  $\partial_j \epsilon = \Omega_j \epsilon$  with  $\Omega_j \Omega_k = 0$ . The solution reads

$$\epsilon = (1 + z^j \Omega_j) \chi, \tag{A.9}$$

with spinor  $\chi = (\chi_1, \chi_2)$  only dependent on  $x^+$ . Eq. (A.5) then gives

$$(\partial_+ + \Omega_+) \chi = -z^j \left( [\Omega_+, \Omega_j] + \frac{1}{4} \hat{\mu}^2 \Gamma_{-j} \right) \chi = -\frac{1}{4} z^j \left( \hat{\mu}^2 - \|\mathbf{z}_1 + \mathbf{z}_2\|^2 \mu^2 \right) \Gamma_{-j} \chi. \tag{A.10}$$

Both sides here must vanish. One solution is that  $\Gamma_- \chi = 0$ , which gives 8 Killing spinors. The other is found when

$$\|\mathbf{z}_1 + \mathbf{z}_2\|^2 \mu^2 = \hat{\mu}^2, \tag{A.11}$$

which is the same as (5.27) that was found from the supergravity conditions (2.24). This gives another set of 8 Killing spinors. The pp-wave background is therefore maximally supersymmetric, admitting 16 Killing spinors.

## B Integrable YB deformation with non-unimodular Drinfel'd-Jimbo R-matrix

In this appendix we present the background corresponding to the Yang-Baxter deformed sigma model (4.3) with the non-unimodular DJ R-matrix built from a distinguished Cartan-Weyl basis.<sup>40</sup> Interpreting (4.3) as the GS superstring sigma model one finds that the NSNS and RR fluxes supporting the metric (2.4) are given by

$$\begin{aligned} H_3 &= \sqrt{1 + \kappa^2} \sqrt{1 - q^2} d\hat{B}, \\ \mathcal{F}_1 &= \frac{2\kappa q(q^2 - \kappa^2)}{(q^2 + \kappa^2)^{3/2}} \hat{\mathcal{F}}_1, & \hat{\mathcal{F}}_1 &\equiv e^0 + e^3, \\ \mathcal{F}_3 &= -\frac{\sqrt{1 + \kappa^2}(q^2 - \kappa^2)}{(q^2 + \kappa^2)^{3/2}} \hat{\mathcal{F}}_3 + \frac{4\kappa^2 q^2}{(q^2 + \kappa^2)^{3/2}} \hat{\mathcal{F}}_1 \wedge J_2^{(1)}, & \hat{\mathcal{F}}_3 &\equiv q^2 d\hat{B} - \kappa^2 d\check{B}, \\ \mathcal{F}_5 &= -\frac{2\kappa q \sqrt{1 + \kappa^2}}{(\kappa^2 + q^2)^{3/2}} \hat{\mathcal{F}}_3 \wedge J_2^{(1)} - \frac{\kappa q(q^2 - \kappa^2)}{(q^2 + \kappa^2)^{3/2}} (1 + \star) \hat{\mathcal{F}}_1 \wedge J_2^{(1)} \wedge J_2^{(1)}. \end{aligned} \tag{B.1}$$

Here  $\kappa$  and  $q$  are defined as in (4.6), (4.7) and  $\mathcal{F}_n$  is the analog of the combination  $e^\Phi F_n$  that appears in the GS action in the standard supergravity background. The forms  $e^a$ ,  $\hat{B}$ ,  $\check{B}$  and  $J_2^{(1)}$  were defined in (2.10), (2.2), (2.6) and (2.15) respectively. Note that in contrast to the supergravity background (4.7), here the RR 1-form is non-vanishing.

When  $q = 1$  one recovers the pure RR deformation of [14].<sup>41</sup> Setting  $\kappa = 0$  on the other hand gives the mixed flux background (2.3).

As expected from the non-unimodularity of the R-matrix, the background (B.1) does not solve the supergravity field equations for generic deformation parameters  $\kappa$  and  $q$ . Instead, it satisfied generalised supergravity equations, which can be viewed as a consequence

<sup>40</sup>The distinguished Dynkin diagram of  $\mathfrak{psu}(1, 1|2)$  is the one with two bosonic simple roots and one fermionic simple root.

<sup>41</sup>Recall that we rescale the string tension and the torus coordinates to match the deformed metric (2.4).

of the  $\varkappa$ -symmetry [19] or scale invariance of the GS sigma model [18]. There is no notion of dilaton scalar and hence it is not possible to extract standard RR field strengths  $F_n$  from  $\mathcal{F}_n$ .

The generalised supergravity equations of motion are satisfied for the following choices of Killing and “generalised dilaton” 1-forms (see [18])

$$I = \frac{2\kappa q \sqrt{1 + \kappa^2}}{(q^2 + \kappa^2)^{3/2}}(e^0 + e^3), \quad Z = -\sqrt{\frac{1 - q^2}{1 + \kappa^2}}I. \quad (\text{B.2})$$

Notice that when  $\kappa = 0$  or  $q = 0$  we have  $I = Z = 0$  and the background solves the standard supergravity equations of motion, with constant dilaton.  $\kappa = 0$  corresponds to the mixed flux background (2.3), while for  $q = 0$  the background (B.1) simplifies to (we set  $\Phi = \Phi_0 = 0$ )

$$H_3 = \sqrt{1 + \kappa^2}d\hat{B}, \quad F_3 = -\kappa\sqrt{1 + \kappa^2}d\check{B}, \quad (\text{B.3})$$

which is the same as the  $q = 0$  case of the supergravity DJ background (4.7).

This can be expected from analysing the action (4.3) and, in particular, the expression (4.4). Recalling that  $q = \sqrt{1 - k^2(1 + \kappa^2)}$ , it turns out that when  $q = 0$ , only the term proportional to  $R_g^2$  in  $\Omega_-$  survives. But all DJ R-matrices obey  $R^2(H_i) = 0$ ,  $R^2(E_\alpha) = -E_\alpha$ ,  $R^2(F_\alpha) = -F_\alpha$ . This no longer depends on the choice of Cartan-Weyl basis (provided the Cartan generators  $H_i$  are the same). Therefore, all YB deformations for DJ R-matrices, unimodular or otherwise, will give rise to the same supergravity background (B.3) when  $q = 0$ . This is a one-parameter deformation of the pure NSNS solution, and is the S-dual of the TsT-transformed background (2.5).

Let us note that the  $q = 0$  case also plays a special role in the bosonic truncation of the string worldsheet sigma model containing the parameters  $\kappa, q$  and string tension  $T$  (cf. (4.3)). Apart from  $\kappa = q = 0$  case, which corresponds to the  $SL(2, \mathbb{R}) \times SU(2)$  WZW model, this model is not conformal and  $q = 0$  corresponds to the fixed line of the RG flow [31]. The remaining parameters  $\kappa$  and overall scale  $T$  run, such that the WZ level  $k = \sqrt{1 + \kappa^2}T$  is an RG invariant.<sup>42</sup> Moreover, this line separates the two regions with different behaviours in the UV. A priori it is not clear why  $q = 0$  should correspond to a fixed line. However, one possible explanation comes if we recall that the RG invariants of the bosonic model are [30–32, 67]

$$k = \sqrt{1 + \kappa^2}\sqrt{1 - q^2}T, \quad \varrho = \frac{\kappa q \sqrt{1 + \kappa^2}\sqrt{1 - q^2}}{\kappa^2 + q^2}. \quad (\text{B.4})$$

The deformation parameter associated to the quantum group symmetry should be an RG invariant and for  $q = 1$  is expected to behave as  $\log q \propto \frac{\varrho}{k} = \frac{\kappa}{(1 + \kappa^2)T}$ , at least to leading order in the inverse string tension  $T^{-1}$  [68, 102]. A natural conjecture for general  $q$  to leading order in  $T^{-1}$  is then  $\log q \propto \frac{\varrho + O(\varrho^2)}{k}$ . Now setting  $q = 0$  we find that  $\log q = 0$ , indicating that the symmetry is not deformed and potentially explaining why  $q = 0$  corresponds to a fixed line of the RG flow.

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<sup>42</sup>Here the string tension  $T$  has been rescaled compared to the action (4.3),  $T \rightarrow (1 + \kappa^2)T$  — see footnote 23.

A final curious observation is that for  $q = 0$  the proportionality constant relating the volume form of the  $\text{AdS}_3$  or  $S^3$  part of the metric (2.4) and the  $\text{AdS}_3$  or  $S^3$  part of the 3-form flux  $H$  (B.3) is the same as at the WZW point  $\kappa = q = 0$ .

### C Examples of integrable deformations with non-constant dilaton

Further potentially interesting cases of integrable deformations can be constructed using other TsT transformations in different Cartan directions. As an example, let us start from the  $q = 0$  case of the DJ background (4.7) (with the metric and fluxes given in (2.4) and (2.22) with  $\hat{B}$  and  $\check{B}$  defined in (2.2) and (2.6))<sup>43</sup>

$$ds^2 = -(1 + \rho^2)dt^2 + \frac{d\rho^2}{1 + \rho^2} + \rho^2 d\psi^2 - \kappa^2((1 + \rho^2)dt - \rho^2 d\psi)^2 + (1 - r^2)d\varphi^2 + \frac{dr^2}{1 - r^2} + r^2 d\phi^2 + \kappa^2((1 - r^2)d\varphi + r^2 d\phi)^2 + dx_s dx_s, \quad (\text{C.1})$$

$$H_3 = \sqrt{1 + \kappa^2} d\hat{B}, \quad F_3 = -\kappa \sqrt{1 + \kappa^2} e^{-\Phi_0} d\check{B}. \quad (\text{C.2})$$

Here  $\kappa = 0$  corresponds to the pure NSNS background for which the bosonic part of the string action is described by the  $\text{SL}(2, \mathbb{R}) \times \text{SU}(2)$  WZW model. As is well known, marginal deformations of this model can be generated using TsT transformations in the two abelian isometry directions of  $S^3$  (or  $\text{AdS}_3$ ), corresponding to vectorial or axial gauging in the WZW model (see, e.g., [25, 26]). We can perform similar transformations also in the case of the background (C.1), (C.2) with  $\kappa \neq 0$ .

Let us consider a TsT transformation along the two sphere isometries: T-duality  $\phi \rightarrow \tilde{\phi}$ , shift  $\varphi \rightarrow \varphi + \sqrt{1 + \kappa^2} \gamma \tilde{\phi}$  and T-duality back  $\tilde{\phi} \rightarrow \phi$ . Similarly, in the AdS sector we may first T-dualise  $\psi \rightarrow \tilde{\psi}$ , then shift  $t \rightarrow t - \sqrt{1 + \kappa^2} \gamma \tilde{\psi}$  and finally T-dualise back  $\tilde{\psi} \rightarrow \psi$ . To simplify the resulting geometries it will also be convenient to rescale the isometric directions as  $(\varphi, \phi, t, \psi) \rightarrow 2(\varphi, \phi, t, \psi)$ . We then find the following metric

$$ds^2 = \frac{d\rho^2}{1 + \rho^2} + \frac{1}{h_\rho} (-(1 + \rho^2)dt^2 + \rho^2 d\psi^2 - \kappa^2((1 + \rho^2)dt - \rho^2 d\psi)^2) + \frac{dr^2}{1 - r^2} + \frac{1}{h_r} ((1 - r^2)d\varphi^2 + r^2 d\phi^2 + \kappa^2((1 - r^2)d\varphi + r^2 d\phi)^2) + dx_s dx_s, \quad (\text{C.3})$$

$$h_\rho = \frac{1}{16}((\gamma + 2)^2(1 + \rho^2) - (\gamma - 2)^2 \rho^2), \quad h_r = \frac{1}{16}((\gamma + 2)^2(1 - r^2) + (\gamma - 2)^2 r^2).$$

The  $H_3$  flux and the dilaton become

$$H_3 = \frac{1}{4} \sqrt{1 + \kappa^2} (\gamma^2 - 4) \left( \frac{\rho^2}{h_\rho^2} d\rho \wedge dt \wedge d\psi + \frac{r^2}{h_r^2} dr \wedge d\varphi \wedge d\phi \right), \quad (\text{C.4})$$

$$\Phi = \Phi_0 - \frac{1}{2} \log(h_r h_\rho), \quad (\text{C.5})$$

while the  $F_3$  flux remains unchanged, i.e. is the same as in (C.2), up to the rescaling of the isometric directions.

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<sup>43</sup>This also corresponds to the S-dual of (2.5) or the solution (2.25) to (2.24) with  $\mathbf{z}_1 = (\sqrt{1 + \kappa^2}, 0, 0, 0, 0)$  and  $\mathbf{z}_2 = (0, -\kappa \sqrt{1 + \kappa^2}, 0, 0, 0)$ .

Note that the  $H_3$  flux vanishes when  $\gamma = \pm 2$ . In the undeformed theory ( $\kappa = 0$ ) for the  $S^3$  part these two special values of  $\gamma$  correspond to the axially-gauged and the vectorially-gauged  $\frac{SU(2)}{U(1)} \times U(1)$  gauged WZW models respectively. For  $\kappa \neq 0$ , setting  $\gamma = 2$  we have  $h_\rho = 1 + \rho^2$  and  $h_r = 1 - r^2$  so that the metric in (C.3) becomes

$$ds^2 = -dt^2 + \frac{d\rho^2}{1 + \rho^2} + \frac{\rho^2}{1 + \rho^2} d\psi^2 - \frac{\kappa^2}{1 + \rho^2} ((1 + \rho^2)dt - \rho^2 d\psi)^2 + d\varphi^2 + \frac{dr^2}{1 - r^2} + \frac{r^2}{1 - r^2} d\phi^2 + \frac{\kappa^2}{1 - r^2} ((1 - r^2)d\varphi + r^2 d\phi)^2 + dx_s dx_s . \tag{C.6}$$

Its Ricci scalar is  $R = -4(1 + \kappa^2) \frac{r^2 + \rho^2}{(1 - r^2)(1 + \rho^2)}$ . The dilaton is  $\Phi = \Phi_0 - \frac{1}{2} \log((1 + \rho^2)(1 - r^2))$  and we also have the non-zero RR  $F_3$  flux in (C.2). For  $\gamma = -2$  we find the metric

$$ds^2 = \frac{1 + \rho^2}{\rho^2} dt^2 + \frac{d\rho^2}{1 + \rho^2} - d\psi^2 + \frac{\kappa^2}{\rho^2} ((1 + \rho^2)dt - \rho^2 d\psi)^2 + \frac{1 - r^2}{r^2} d\varphi^2 + \frac{dr^2}{1 - r^2} + d\phi^2 + \frac{\kappa^2}{r^2} ((1 - r^2)d\varphi + r^2 d\phi)^2 + dx_s dx_s , \tag{C.7}$$

with the Ricci scalar  $R = -4(1 + \kappa^2) \frac{r^2 + \rho^2}{r^2 \rho^2}$ . The dilaton is  $\Phi = \Phi_0 - \frac{1}{2} \log(-\rho^2 r^2)$  implying that  $\Phi_0$  should be shifted by an imaginary constant, which makes  $F_3$  in (C.2) imaginary. For  $\gamma \geq 0$  the function  $h_\rho$  is strictly positive, while for  $\gamma < 0$  it is negative if  $\rho^2 > -\frac{(2+\gamma)^2}{8\gamma}$ . This suggests either restricting  $\gamma$  to be positive or that the solution needs to be analytically continued for negative  $\gamma$ .

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