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Instanton solutions in open superstring field theory

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ABSTRACT: Open superstring field theory admits a “hybrid” formulation where $N = 1$ $D = 4$ supersymmetry is manifest for Calabi-Yau compactifications to four dimensions. Using this formulation, the half-BPS instanton solution of four-dimensional super-Yang-Mills can be easily generalized to the full open superstring field theory. In this paper, we compute the first stringy correction to the super-Yang-Mills instanton solution which involves turning on certain fields at the first massive level.

KEYWORDS: Solitons Monopoles and Instantons, String Field Theory, Superstrings and Heterotic Strings

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1 Introduction

Open superstring field theory (SFT) can be a powerful method to obtain non-perturbative information about superstring theory. The best example is tachyon condensation where tachyonic solutions to the SFT equations of motion have been shown to describe the decay of non-supersymmetric D-branes [1]. In addition to the non-supersymmetric tachyonic solutions, the equations of motion of SFT also contains spacetime supersymmetric solutions which can be studied. In particular, it would be interesting to use SFT to search for stringy generalizations of instanton solutions of super-Yang-Mills which give rise to non-perturbative effects in gauge theories. Note that instanton solutions in SFT have been studied in [2–5] in the RNS formalism starting from a system of $D3$ - and $D(-1)$ -branes. Here we will take a different approach, starting with the Yang-Mills instanton and calculating massive stringy corrections. It would be very interesting to compare these two approaches, although this is expected to be non-trivial since they correspond to perturbative expansions in opposite limits.

After compactifying to four dimensions, the massless states of the open superstring include $D = 4$ super-Yang-Mills fields and the instanton solution is obtained by requiring the four-dimensional Yang-Mills field-strength to be self-dual. However, it is difficult to

generalize the concept of self-dual field strength to SFT since the only gauge-invariant quantity that can be constructed from the string field Φ is $Q\Phi$, which vanishes on-shell. Nevertheless, one can instead define the four-dimensional instanton solution as a localized half-BPS solution to the equations of motion, i.e. a solution which is annihilated by half of the $N = 1$ $D = 4$ spacetime supersymmetries. In this paper, we will generalize this half-BPS definition of the instanton solution in SFT and will find the first stringy correction.

Since $D = 4$ spacetime supersymmetry plays an important role in our construction, we will use the manifestly $N = 1$ $D = 4$ super-Poincaré invariant action for SFT of [6], which is based on the $D = 4$ hybrid formalism of [7]. The massless contribution to this SFT action reproduces the usual $N = 1$ $D = 4$ superspace action for super-Yang-Mills theory, and the massive states are also described in terms of $N = 1$ $D = 4$ superfields. Although we will be unable to find an exact solution to the SFT equations of motion which generalizes the self-dual instanton, we will find the first stringy correction to the instanton solution which corresponds to turning on certain massive spin-2 and spin-0 fields.

In section 2 of this paper, we review self-dual instanton solutions in super-Yang-Mills theory. In section 3 we review the $D = 4$ hybrid formalism for the superstring and introduce the central problem of this paper — solving the half-BPS condition in open superstring field theory. Our strategy for tackling this problem is to write a series expansion for the string field and solve order by order in the expansion parameter, as explained in section 4. In section 5 we review the BPST instanton in super-Yang-Mills and generalize it to SFT using the star product. In section 6 we calculate the first stringy correction to the BPST instanton and show it corresponds to turning on certain off-shell massive fields by comparing with the vertex operators for the first massive level of the string.

2 Super-Yang-Mills instantons

$N = 1$ $D = 4$ super-Yang-Mills can be described by a vector superfield $V^i(x, \theta, \bar{\theta})$ where i is a gauge index and $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ are the usual $N = 1$ $D = 4$ superspace variables for $m = 0$ to 3 and $\alpha, \dot{\alpha} = 1$ to 2. We also define $V = V^i T^i$ where T^i are the generators of the gauge group. Supersymmetry transformations are generated by acting on V with

$$q_\alpha = \frac{\partial}{\partial \theta^\alpha} - \frac{i}{2} \left(\sigma^m \bar{\theta} \right)_\alpha \partial_m \tag{2.1}$$

and

$$\bar{q}^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \frac{i}{2} (\theta \bar{\sigma}^m)^{\dot{\alpha}} \partial_m. \tag{2.2}$$

where σ^m are the Pauli matrices and $\bar{\sigma}^0 = \sigma^0$, $\bar{\sigma}^i = -\sigma^i$ for $i = 1, 2, 3$.

We want to find field configurations that are preserved by half of this supersymmetry. Naively, this condition would be formulated as

$$(\epsilon q)V = 0, \tag{2.3}$$

where ϵ^α is a supersymmetry parameter and $\epsilon q = \epsilon^\alpha q_\alpha$.

However, that is not quite correct. We must keep in mind that the theory has a gauge symmetry, given by

$$\delta_{\text{gauge}}V = \mathcal{L}_{V/2} \left[\Lambda - \bar{\Lambda} - i \coth \left(\mathcal{L}_{V/2} \right) \left(\Lambda + \bar{\Lambda} \right) \right], \quad (2.4)$$

where Λ and $\bar{\Lambda}$ are chiral and antichiral, respectively, and \mathcal{L} is the Lie derivative:

$$\mathcal{L}_V(A) = [V, A] \quad (2.5)$$

So the correct condition is not that $(\epsilon q)V$ vanishes, but that it is pure gauge. In other words, V is preserved by half of the supersymmetry if there are Λ and $\bar{\Lambda}$ such that

$$(\epsilon q)V = -\delta_{\text{gauge}}V. \quad (2.6)$$

Of course, one could also consider states preserved by the antichiral supersymmetry, for which the analysis is similar.

To see what eq. (2.6) implies, we first write V in Wess-Zumino gauge and assume that the fermions vanish (since we are concerned with classical field configurations):

$$V = -\theta\sigma^m\bar{\theta}v_m + \frac{1}{2}\theta^2\bar{\theta}^2D. \quad (2.7)$$

Since $V^3 = 0$ in Wess-Zumino gauge, the gauge transformation of eq. (2.4) becomes

$$\delta_{\text{gauge}}V = \Lambda + \bar{\Lambda} + \frac{1}{2} [V, \Lambda - \bar{\Lambda}] + \frac{1}{12} [V, [V, \Lambda + \bar{\Lambda}]]. \quad (2.8)$$

The most general Λ can be written in components as

$$\Lambda = A(x) + \sqrt{2}\theta\psi(x) + \theta^2F(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\psi(x) + \frac{1}{2}\theta^2\bar{\theta}^2\Box A(x). \quad (2.9)$$

Similarly, the most general $\bar{\Lambda}$ is

$$\bar{\Lambda} = A^* + \sqrt{2}\bar{\theta}\bar{\psi} + \theta^2F^* - i\theta\sigma^m\bar{\theta}\partial_m A^* + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{2}\theta^2\bar{\theta}^2\Box A^*. \quad (2.10)$$

The gauge transformation is then

$$\begin{aligned} \delta_{\text{gauge}}V = & A + A^* + \sqrt{2}\theta\psi + \sqrt{2}\bar{\theta}\bar{\psi} + \theta^2F + \bar{\theta}^2F^* + \\ & \theta\sigma^m\bar{\theta} \left(i\partial_m(A - A^*) - \frac{1}{2} [v_m, A + A^*] \right) + \frac{i}{\sqrt{2}}\theta^2 \left(\bar{\theta}\bar{\sigma}^m\partial_m\psi + \frac{1}{2} [v_m, \psi\sigma^m\bar{\theta}] \right) + \\ & \frac{i}{\sqrt{2}}\bar{\theta}^2 \left(\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{2} [v_m, \bar{\psi}\bar{\sigma}^m\theta] \right) + \\ & \frac{1}{2}\theta^2\bar{\theta}^2 \left(\frac{1}{2}\Box(A + A^*) - \frac{i}{2} [v_m, \partial^m(A + A^*)] + \right. \\ & \left. v^2(A + A^*) - 2v^m(A + A^*)v_m + (A + A^*)v^2 \right). \end{aligned} \quad (2.11)$$

On the other hand, the supersymmetry transformation is

$$(\epsilon q)V = -\epsilon\sigma^m\bar{\theta}v_m + \epsilon\theta\bar{\theta}^2 D + \frac{i}{2}\epsilon\sigma^m\bar{\sigma}^n\theta\bar{\theta}^2\partial_m v_n \quad (2.12)$$

Plugging eqs. (2.11) and (2.12) into eq. (2.6) and splitting it in components, we find the conditions:

$$A + A^* = \psi^\alpha = F = F^* = \partial_m(A - A^*) = 0, \quad (2.13)$$

$$\bar{\psi}_{\dot{\alpha}} = \frac{1}{\sqrt{2}}(\epsilon\sigma^m)_{\dot{\alpha}} v_m, \quad (2.14)$$

$$i(\sigma^{mn})_{\beta}^{\alpha} F_{mn} = \delta_{\beta}^{\alpha} D \quad (2.15)$$

where $\sigma^{mn} = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)$. Since σ^{mn} is traceless, we conclude that

$$D = 0 \quad (2.16)$$

and

$$(\sigma^{mn})_{\beta}^{\alpha} F_{mn} = 0, \quad (2.17)$$

which tells us that the gauge field must be anti-self-dual.

3 String field theory instantons

We now want to generalize the Yang-Mills instanton solutions to SFT. One approach to this problem was developed in [2–5] by studying marginal deformations corresponding to the blow-up of a zero-size D(-1) brane on a background of D3-branes, and calculating the massless stringy corrections to the instanton profile. Here we will start with the super-Yang-Mills instanton profile and calculate the first stringy corrections using supersymmetry and the equations of motion. We will see that these two methods of computing stringy corrections correspond to opposite limits of small and large instantons.

To describe the SFT instantons, we'll use the hybrid formalism developed in [6] which describes a compactified superstring with manifest four dimensional super-Poincaré symmetry. We take the four-dimensional space to be Euclidean. Also, we will only consider states that are independent of the structure of the compactified manifold.

The hybrid formalism contains five free bosons (x^m, ρ) and eight free fermions $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}}, p^\alpha, \bar{p}^{\dot{\alpha}})$ with OPE's

$$\begin{aligned} x^m(z_1)x^n(z_2) &\sim -\log|z_1 - z_2|\delta^{mn}, & \rho(z_1)\rho(z_2) &\sim \log(z_1 - z_2) \\ p_\alpha(z_1)\theta^\beta(z_2) &\sim \delta_\alpha^\beta(z_1 - z_2)^{-1}, & \bar{p}_{\dot{\alpha}}(z_1)\theta^{\dot{\beta}}(z_2) &\sim \delta_{\dot{\alpha}}^{\dot{\beta}}(z_1 - z_2)^{-1} \end{aligned}$$

and a field theory for the six-dimensional compactification manifold which is described by the $\hat{c} = 3$ $N = 2$ superconformal generators $[T_C, G_C^+, G_C^-, J_C]$.

The $N = 4$ superconformal generators are defined in terms of these free fields by

$$T = \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}} + \frac{1}{2} \partial \rho \partial \rho - \frac{i}{2} \partial^2 \rho + T_C \quad (3.1)$$

$$G^+ = e^{i\rho} d^2 + G_C^+, \quad G^- = e^{-i\rho} \bar{d}^2 + G_C^- \quad (3.2)$$

$$\tilde{G}^+ = e^{-2i\rho + iH_C} \bar{d}^2 + e^{-i\rho + iH_c} (G_C^-) \quad (3.3)$$

$$\tilde{G}^- = e^{2i\rho - iH_C} d^2 + e^{i\rho - iH_c} (G_C^+) \quad (3.4)$$

$$J = i\partial\rho + iJ_C, \quad J^{++} = e^{-i\rho + iH_C}, \quad J^{--} = e^{+i\rho - iH_C} \quad (3.5)$$

where

$$d_\alpha = p_\alpha + \frac{i}{2} \bar{\theta}^{\dot{\alpha}} \partial x_{\alpha\dot{\alpha}} - \frac{1}{4} (\bar{\theta})^2 \partial \theta_\alpha + \frac{1}{8} \theta_\alpha \partial (\bar{\theta})^2 \quad (3.6)$$

$$\bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + \frac{i}{2} \theta^\alpha \partial x_{\alpha\dot{\alpha}} - \frac{1}{4} (\theta)^2 \partial \bar{\theta}_{\dot{\alpha}} + \frac{1}{8} \bar{\theta}_{\dot{\alpha}} \partial (\theta)^2 \quad (3.7)$$

$J_C = \partial H_C$, $x_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m x_m$, and the notation $e^{-i\rho + iH_c} (G_C^-)$ means the contour integral of $e^{-i\rho + iH_c}$ around G_C^- .

In principle, instantons in SFT can be found using the methods of the previous section and are defined to be field configurations whose supersymmetry variation is pure gauge:

$$(\epsilon q)V = -\delta_{\text{gauge}} V. \quad (3.8)$$

However, unlike the super-Yang-Mills case, there is no gauge choice where $V^3 = 0$. So the gauge transformation is now an infinite series in V :

$$\delta_{\text{gauge}} V = \Lambda + \bar{\Lambda} + \frac{1}{2} [V, \Lambda - \bar{\Lambda}] + \frac{1}{12} [V, [V, \Lambda + \bar{\Lambda}]] + \mathcal{O}(V^3), \quad (3.9)$$

where the products are to be interpreted as star (*) products, as defined in [8]. The gauge parameters Λ and $\bar{\Lambda}$ in eq. (3.9) have to satisfy

$$\tilde{G}^+(\Lambda) = 0 \quad (3.10)$$

and

$$G^+(\bar{\Lambda}) = 0, \quad (3.11)$$

where the notation $G^+(\bar{\Lambda})$ denotes the contour integral of G^+ around $\bar{\Lambda}$.

Although we will be unable to analytically solve eq. (3.8) because of the infinite number of terms, one can solve eq. (3.8) perturbatively by expanding around the linearized massless solutions to V , Λ and $\bar{\Lambda}$. In the rest of this paper, we will use this perturbative method to find the first stringy corrections to the super-Yang-Mills instanton solution.

4 Series expansion of the half-BPS condition

To perform a perturbative expansion, first write the string field V as a power series in some parameter λ :

$$V = \lambda V_0 + \lambda^2 V_1 + \lambda^3 V_2 + \dots \tag{4.1}$$

Similarly, write

$$\Lambda = \lambda \Lambda_0 + \lambda^2 \Lambda_1 + \lambda^3 \Lambda_2 + \dots \tag{4.2}$$

and

$$\bar{\Lambda} = \lambda \bar{\Lambda}_0 + \lambda^2 \bar{\Lambda}_1 + \lambda^3 \bar{\Lambda}_2 + \dots \tag{4.3}$$

We can then break up eq. (3.8) in powers of λ :

$$(\epsilon q)V_0 = \Lambda_0 + \bar{\Lambda}_0, \tag{4.4}$$

$$(\epsilon q)V_1 = \Lambda_1 + \bar{\Lambda}_1 + \frac{1}{2} [V_0, \Lambda_0 - \bar{\Lambda}_0], \tag{4.5}$$

and so on. The goal is to first determine V_0 and then recursively determine V_{i+1} from V_i .

Finding V_0 is relatively easy because eq. (4.4) is linear and thus the massless and massive states decouple. Actually, since we are working with off-shell string theory, the expressions “massless” and “massive” need some clarification. By “massless” we mean any state that depends only on the worldsheet zero modes of x^m , θ^α and $\bar{\theta}_{\dot{\alpha}}$ and not on their derivatives nor on p^α . That is, they are the states that would be massless if they were on-shell. All remaining states will be called “massive”.

Let us choose a gauge such that

$$V_0 = -\theta \sigma^m \bar{\theta} v_{0m} + \frac{1}{2} \theta^2 \bar{\theta}^2 D_0 + V_{0,\text{massive}}, \tag{4.6}$$

where $V_{0,\text{massive}}$ contains the massive components.

It is trivial to adapt the computations of section 2 to show that

$$D_0 = 0 \tag{4.7}$$

and

$$\sigma^{mn} (\partial_m v_{0n} - \partial_n v_{0m}) = 0. \tag{4.8}$$

Later we will describe an explicit solution of this condition for the case of a $U(2)$ gauge group.

What about the massive part? We will argue here that massive on-shell states cannot contribute to the instanton solution since they are not localized in spacetime. Note that in Euclidean space, the solution to the linearized equation of motion $(\delta^{mn} \partial_m \partial_n - M^2) \Phi = 0$ is

$$\Phi = \int d^4 k f(k) e^{k \cdot x} \delta(|k| - M). \tag{4.9}$$

Suppose $f(\hat{k})$ is non-zero for some \hat{k}^m satisfying $|\hat{k}| = M$. Then it is easy to see that $\Phi(x)$ diverges exponentially as e^{cM^2} in the direction $x^m = c \hat{k}^m$ for c large. So there are no localized solutions to $(\delta^{mn} \partial_m \partial_n - M^2) \Phi = 0$ in Euclidean space which can contribute to the instanton solution.

So the linearized condition of eq. (4.4) together with the requirement that the solution is localized in spacetime implies that

$$V_0 = -\theta\sigma^m\bar{\theta}v_{0m}, \tag{4.10}$$

where v_{0m} satisfies

$$\sigma^{mn}(\partial_m v_{0n} - \partial_n v_{0m}) = 0. \tag{4.11}$$

A specific choice for the compensating gauge transformation is

$$\Lambda_0 = 0, \tag{4.12}$$

and

$$\bar{\Lambda}_0 = -\left(\epsilon\sigma^m\bar{\theta}\right)v_{0m} \tag{4.13}$$

Equipped with this, we can now try to tackle eq. (4.5). We will find that due to the complexity of the star product, $V_0 * \bar{\Lambda}$ contains massive components even though V_0 and $\bar{\Lambda}$ contain only massless components. So unlike V_0 , V_1 will contain massive components.

5 BPST instanton

To gain intuition, it will be useful to consider an explicit super-Yang-Mills instanton solution and see how it can be extended to string field theory. For an $SU(2)$ gauge group, the $k = 1$ instanton can be written in a simple form which is the BPST instanton [9]:

$$v_m^i = 2\frac{\bar{\eta}_{mn}^i(x-x_0)^n}{(x-x_0)^2 + \rho^2}. \tag{5.1}$$

x_0^m and ρ are two fixed parameters representing the center and the size of the instanton, and $\bar{\eta}_{mn}^i$ are the 't Hooft symbols [10]:

$$\bar{\eta}_{imn} = \epsilon_{imn4} - \delta_{im}\delta_{n4} + \delta_{in}\delta_{m4} \tag{5.2}$$

where $i = 1, 2, 3$ is an $SU(2)$ index and $m, n = 1, 2, 3, 4$ are Euclidean spacetime indices. Note that the t'Hooft symbols are anti-self-dual in their Lorentz indices:

$$\bar{\eta}_{imn} = -\frac{1}{2}\epsilon_{mnpq}\bar{\eta}_{ipq}. \tag{5.3}$$

To show that eq. (5.1) does indeed represent an instanton, we have to show that the corresponding field strength,

$$F_{mn}^i = \partial_m v_n^i - \partial_n v_m^i + \epsilon^{ijk}v_m^j v_n^k, \tag{5.4}$$

is anti-self-dual. The derivative of v_m^i is

$$\partial_m v_n^i = 2\frac{\bar{\eta}_{inm}\left[(x-x_0)^2 + \rho^2\right] - 2\bar{\eta}_{inp}(x-x_0)^p(x-x_0)_m}{\left[(x-x_0)^2 + \rho^2\right]^2}, \tag{5.5}$$

whereas the quadratic part of the field strength is

$$\epsilon_{ijk} v_m^j v_n^i = 4 \frac{\epsilon_{ijk} \bar{\eta}_{jmp} \bar{\eta}_{knq} (x-x_0)^p (x-x_0)^q}{[(x-x_0)^2 + \rho^2]^2}. \quad (5.6)$$

Using the identity

$$\epsilon_{ijk} \bar{\eta}_{jmp} \bar{\eta}_{knq} = \delta_{mn} \bar{\eta}_{ipq} + \delta_{pq} \bar{\eta}_{imn} - \delta_{mq} \bar{\eta}_{ipn} - \delta_{pn} \bar{\eta}_{imq}, \quad (5.7)$$

we obtain

$$\epsilon_{ijk} v_m^j v_n^i = 4 \frac{\bar{\eta}_{imn} (x-x_0)^2 - \bar{\eta}_{ipn} (x-x_0)^p (x-x_0)_m - \bar{\eta}_{imp} (x-x_0)^p (x-x_0)_n}{[(x-x_0)^2 + \rho^2]^2}. \quad (5.8)$$

Then the field strength is

$$F_{mn}^i = -4 \frac{\bar{\eta}_{imn} \rho^2}{[(x-x_0)^2 + \rho^2]^2} \quad (5.9)$$

and the anti-self-duality of the field strength follows from the anti-self-duality of $\bar{\eta}_{imn}$.

To connect this with what we were doing in the previous section, let us expand v_m^i in powers of $1/\rho^2$:

$$v_m^i = \sum_{r=0}^{\infty} \frac{v_{rm}^i}{\rho^{2(r+1)}}, \quad (5.10)$$

where

$$v_{rm}^i = 2(-1)^r \bar{\eta}_{mn}^i (x-x_0)^n (x-x_0)^{2r}. \quad (5.11)$$

That is, we are working with a large instanton expansion which has the form of (4.1) where the expansion parameter is $\lambda = \frac{1}{\rho^2}$. This is different from [2–5] where they expand the solution in ρ corresponding to a small instanton expansion.

Now we would like to find the corrections to this instanton solution in superstring field theory — that is, a solution to eq. (2.6) where we take the massless part of the string field to have the form of the BPST instanton. Note that the gauge group should now be $U(2)$ rather than $SU(2)$. Since the star product is non-commutative, the massive levels of the string field will have singlet contributions, even if the massless level doesn't. We formally define the object v_m^i as

$$v_m^i = 2 \frac{\bar{\eta}_{mn}^i (x-x_0)^n}{(x-x_0)^2 + \rho^2}, \quad (5.12)$$

where the products are to be interpreted as \otimes products, and \otimes denotes the truncation of the star product to the massless level (see appendix A). To give meaning to division by x^m , we invoke the expansion in $1/\rho^2$ of eq. (5.10) where

$$v_{rm}^i = 2(-1)^r \bar{\eta}_{mn}^i (x-x_0)^n \otimes \left((x-x_0)^2 \right)^{r \otimes} \quad (5.13)$$

and $r \otimes$ in the exponent means we multiply r times with the \otimes product. Also, x^2 means the normal ordered product — that is, no contractions between the x 's.

As we showed in the last section, the massive part of V_0 is trivial. Thus we can take V_0 to be the linear part of the BPST instanton (we will fix the position to be at the origin):

$$V_0^0 = 0, \quad V_0^i = -2\theta \sigma^m \bar{\theta} \bar{\eta}_{mn}^i x^n \quad (5.14)$$

where V_0^0 denotes the U(1) component of the U(2) gauge group. We can also take the massless part of V_1 to be

$$V_{1,massless}^i = -\theta\sigma^m\bar{\theta}\bar{v}_{1m}^i = 2\theta\sigma^m\bar{\theta}\bar{\eta}_{mn}^i x^n \otimes x^2. \tag{5.15}$$

but this will have massive corrections.¹ Both the equations of motion and the half-BPS condition for the massless fields reduce, up to this order, to their form in super-Yang-Mills. Thus, it is immediate that (5.14), (5.15) are solutions. At higher orders, since the star product of massive fields can have massless components, we expect there will be both massive and massless corrections. That is, $V_{r,massless}^i = -\theta\sigma^m\bar{\theta}\bar{v}_{rm}^i$ is not a solution to the equations for $r \geq 2$.

In the next section we'll compute the first massive level of V_1 .

6 The first stringy correction

We now want to compute the first stringy correction to the BPST instanton, which is the first massive level of V_1 . To that end, we must solve the half-BPS equation at the quadratic level (i.e. $O(1/\rho^4)$ terms) which corresponds to eq. (4.5). Although V_0 , Λ_0 and $\bar{\Lambda}_0$ are massless, the star products in the commutator will contain massive terms which will imply that V_1 has a nontrivial massive part.

In addition to the expansion (4.1), we will also use level truncation. We define the n -th mass level of the string field as the component with conformal weight n at zero momentum. Note that each term in the perturbative expansion has, in principle, components at all mass levels. Here we truncate to the first level.

The first massive contribution to the half-BPS solution is not unique — if V_1 is a solution to eq. (4.5), then $V_1' = V_1 + q^2\Omega$ is also a solution for any field Ω since $(\epsilon q)q^2\Omega = 0$. We can ask if there is a half-BPS solution which also satisfies the SFT equations of motion.² The answer is yes, and we will show that the first massive contribution to this solution is unique if we also require that it vanishes at infinity, i.e. it is localized in spacetime like the Yang-Mills instanton. In other words, $q^2\Omega$ will be uniquely determined if we require that V_1' satisfies the SFT equations of motion and is localized in spacetime. This first stringy correction to the BPST instanton will be interpreted as turning on some off-shell massive fields. To see what kinds of fields these are, we will compare our result with the first massive states of the superstring which on-shell consists of a massive spin-two multiplet and two massive scalar multiplets [11, 12]. We will find that our solution corresponds to turning on a spin-two and a scalar field.

In subsection 6.1 we find the general solution to eq. (4.5) when V_0 is given by the linearized BPST solution. In subsection 6.2 we find the unique first massive contribution to

¹Note that the massless contribution to V_1 is ambiguous, since one can always shift V_1 by any linearized massless solution. One could try to remove this ambiguity by fixing the properties of the half-BPS solution, but we will not attempt this here.

²Note that the half-BPS solution is required to be annihilated by half of the N=1 d=4 supersymmetries, i.e. two supersymmetries. If one required that the half-BPS solution was annihilated by half of the N=4 d=4 supersymmetries, i.e. 8 supersymmetries, one expects that it would automatically satisfy the SFT equations of motion. We would like to thank discussions with Carlo Maccaferri and Jakub Vosmera on this point.

the half-BPS solution that also satisfies the equations of motion and vanishes at infinity. And in subsection 6.3 we compare our solution with the vertex operator for the first massive level.

6.1 Half-BPS solutions

In this section we will solve the equation

$$(\epsilon q)V_1 = \Lambda_1 + \bar{\Lambda}_1 + \frac{1}{2} [V_0, \Lambda_0 - \bar{\Lambda}_0] \tag{6.1}$$

given the linearized BPST solution for V_0

$$V_0^i = -\theta\sigma^m\bar{\theta}v_{0m}^i(x) = -2\theta\sigma^m\bar{\theta}\bar{\eta}_{mn}^i x^n \tag{6.2}$$

$$\bar{\Lambda}_0^i = -2\epsilon\sigma^m\bar{\theta}\bar{\eta}_{mn}^i x^n \tag{6.3}$$

$$\Lambda_0^i = 0 \tag{6.4}$$

As noted before, we can add any q^2 -exact term to a solution V_1 and still have a solution. For the massive part $V_{1,massive}$, we'll show that the solution for the first massive level is unique up to a q^2 -exact term, and up to gauge transformations. If, in addition to eq. (6.1), we require that the solution satisfies the equations of motion and vanishes at infinity, we will find that the q^2 -exact term is fixed uniquely. From now on, we'll use V_1 instead of $V_{1,massive}$ to denote the first massive level since we'll only deal with this level in this section. Similarly, Λ_1 and $\bar{\Lambda}_1$ will denote the first massive level of the gauge parameters.

First we have to calculate the commutator on the right-hand side of eq. (6.1), which involves evaluating a star product. Then we substitute the general form of the string field and gauge parameters at the first massive level. This is given by the fields that have conformal weight 1 at zero momentum, and which have mass squared $M^2 = 2$ when on-shell.

The commutator will contain terms that are not ϵq -exact, which must therefore be cancelled by the gauge terms. This will fix Λ_1 and $\bar{\Lambda}_1$, up to ϵq -exact gauge parameters (that is, up to $\epsilon q(\Lambda'_1 + \bar{\Lambda}'_1)$ where Λ'_1 and $\bar{\Lambda}'_1$ are gauge parameters). Once the gauge parameters are fixed, we have determined ϵqV_1 , which means we have determined V_1 up to a q^2 -exact term.

At the first massive level, the commutator $[V_0, \bar{\Lambda}_0]$ satisfies

$$\frac{1}{2} [V_0, \bar{\Lambda}_0] = \frac{1}{2} V_0^i * \bar{\Lambda}_0^j \{ \sigma^i, \sigma^j \} = V_0^i * \bar{\Lambda}_0^i \sigma^0 \tag{6.5}$$

where σ^0 is i times the 2×2 identity. This is because $(V_0^i * \bar{\Lambda}_0^j) = -(\bar{\Lambda}_0^j * V_0^i)$ at this level. Details on the calculation of the star product can be found in appendix A, and one finds at the first massive level that

$$V_0^i * \bar{\Lambda}_0^i = \partial\theta^\alpha V_\alpha + \partial\bar{\theta}^{\dot{\alpha}} \bar{V}_{\dot{\alpha}} + \Pi^m U_m \tag{6.6}$$

where

$$\Pi^m = \partial x^m - \frac{i}{2} \theta\sigma^m\partial\bar{\theta} - \frac{i}{2} \bar{\theta}\bar{\sigma}^m\partial\theta \tag{6.7}$$

and

$$V_\alpha = -\frac{2}{3\sqrt{3}} (\sigma^m \bar{\theta})_\alpha \epsilon \sigma^n \bar{\theta} v_{0m}^i \otimes v_{0n}^i \quad (6.8)$$

$$\bar{V}_\alpha = \frac{2}{3\sqrt{3}} \left((\theta \sigma^m)_{\dot{\alpha}} \epsilon \sigma^n \bar{\theta} - \theta \sigma^m \bar{\theta} (\epsilon \sigma^n)_{\dot{\alpha}} \right) v_{0m}^i \otimes v_{0n}^i (y) \quad (6.9)$$

$$U_m = \frac{4}{3\sqrt{3}} \theta \sigma^p \bar{\theta} \epsilon \sigma^q \bar{\theta} \left(\bar{\eta}_{pm}^i v_{0q}^i - v_{0p}^i \bar{\eta}_{qm}^i \right) \quad (6.10)$$

where $y^m = x^m + \frac{i}{2} \theta \sigma^m \bar{\theta}$.

The general form of the string field at the first massive level is [11]

$$\begin{aligned} V_1 = & d^\alpha W_\alpha (x, \theta, \bar{\theta}) + \bar{d}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} (x, \theta, \bar{\theta}) + \Pi^m W_m (x, \theta, \bar{\theta}) + \partial \theta^\alpha V_\alpha (x, \theta, \bar{\theta}) \\ & + \partial \bar{\theta}^{\dot{\alpha}} \bar{V}_{\dot{\alpha}} (x, \theta, \bar{\theta}) + i (\partial \rho - \partial H_C) B (x, \theta, \bar{\theta}) + (\partial H_C - 3 \partial \rho) C (x, \theta, \bar{\theta}) \end{aligned} \quad (6.11)$$

and the gauge transformations are

$$\begin{aligned} \Lambda_1 = & G^+ \left(e^{-i\rho} \left(\bar{d}^{\dot{\alpha}} \bar{E}_{\dot{\alpha}} (x, \theta, \bar{\theta}) + \partial \theta^\alpha B_\alpha (x, \theta, \bar{\theta}) \right) \right) \\ = & -2i \Pi_{\alpha\dot{\alpha}} D^\alpha \bar{E}^{\dot{\alpha}} + \bar{d}^{\dot{\alpha}} D^2 \bar{E}_{\dot{\alpha}} + d^\alpha \left(2B_\alpha - i \partial_{\alpha\dot{\alpha}} \bar{E}^{\dot{\alpha}} \right) + \partial \theta^\alpha D^2 B_\alpha \\ & + \partial \bar{\theta}^{\dot{\alpha}} \left(-i \partial_{\alpha\dot{\alpha}} B^\alpha + 4\bar{E}_{\dot{\alpha}} - \frac{1}{2} \partial^2 \bar{E}_{\dot{\alpha}} \right) + i \partial \rho \left(2D^\alpha B_\alpha - i \partial_{\alpha\dot{\alpha}} D^\alpha \bar{E}_{\dot{\alpha}} \right) \\ \bar{\Lambda}_1 = & \tilde{G}^+ \left(e^{2i\rho - iH_C} \left(d^\alpha E_\alpha (x, \theta, \bar{\theta}) + \partial \bar{\theta}^{\dot{\alpha}} \bar{B}_{\dot{\alpha}} (x, \theta, \bar{\theta}) \right) \right) \\ = & -2i \Pi_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} E^\alpha + d^\alpha \bar{D}^2 E_\alpha + \bar{d}^{\dot{\alpha}} \left(2\bar{B}_{\dot{\alpha}} - i \partial_{\alpha\dot{\alpha}} E^\alpha \right) + \partial \bar{\theta}^{\dot{\alpha}} \bar{D}^2 \bar{B}_{\dot{\alpha}} \\ & + \partial \theta^\alpha \left(-i \partial_{\alpha\dot{\alpha}} \bar{B}^{\dot{\alpha}} + 4E_\alpha - \frac{1}{2} \partial^2 E_\alpha \right) + i \partial (H_C - 2\rho) \left(2\bar{D}^{\dot{\alpha}} \bar{B}_{\dot{\alpha}} - i \partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} E_\alpha \right) \end{aligned} \quad (6.12)$$

where $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} \bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}$ and $\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \frac{i}{2} \theta^\alpha \partial_{\alpha\dot{\alpha}}$. We can use the gauge freedom to fix V_1 in the form

$$V_1 = \Pi^m W_m (x, \theta, \bar{\theta}) + i (\partial \rho - \partial H_C) B (x, \theta, \bar{\theta}) + (\partial H_C - 3 \partial \rho) C (x, \theta, \bar{\theta}) \quad (6.13)$$

Note that there are no terms proportional to d or \bar{d} in either V_1 or in the commutator. Fixing the d and \bar{d} terms to zero in (6.12), the gauge parameters take the form

$$\Lambda_1 = -\frac{1}{2} i \Pi_{\alpha\dot{\alpha}} D^\alpha \bar{E}^{\dot{\alpha}} + \frac{1}{4} \partial \theta^\alpha i \partial_{\alpha\dot{\alpha}} D^2 \bar{E}^{\dot{\alpha}} + \partial \bar{\theta}^{\dot{\alpha}} \left(\bar{E}_{\dot{\alpha}} - \frac{1}{8} \bar{D}^2 D^2 \bar{E}_{\dot{\alpha}} \right) - \frac{i}{4} \partial \rho D^\alpha \bar{D}^2 E_\alpha \quad (6.14)$$

$$\begin{aligned} \bar{\Lambda}_1 = & -\frac{1}{2} i \Pi_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} E^\alpha + \frac{1}{4} \partial \bar{\theta}^{\dot{\alpha}} i \partial_{\alpha\dot{\alpha}} \bar{D}^2 E^\alpha + \partial \theta^\alpha \left(E_\alpha - \frac{1}{8} D^2 \bar{D}^2 E_\alpha \right) \\ & - \frac{i}{4} \partial (H_C - 2\rho) \bar{D}^{\dot{\alpha}} D^2 \bar{E}_{\dot{\alpha}} \end{aligned} \quad (6.15)$$

where we have rescaled E_α and $\bar{E}_{\dot{\alpha}}$ by a factor of four for later convenience.

Equation (6.1) can be decomposed into five equations by separating the terms proportional to $\partial \theta^\alpha$, $\partial \bar{\theta}^{\dot{\alpha}}$, Π^m , $\partial \rho$ and ∂H_C . The first two will fix E_α and $\bar{E}_{\dot{\alpha}}$, up to ϵq -exact terms. The other equations, in turn, will determine V_1 .

The terms proportional to $\partial\theta^\alpha$ and $\partial\bar{\theta}^{\dot{\alpha}}$ give

$$V_\alpha = E_\alpha - \frac{1}{8}D^2\bar{D}^2E_\alpha + \frac{i}{4}\partial_{\alpha\dot{\alpha}}D^2\bar{E}^{\dot{\alpha}} \quad (6.16)$$

$$\bar{V}_{\dot{\alpha}} = \bar{E}_{\dot{\alpha}} - \frac{1}{8}\bar{D}^2D^2\bar{E}_{\dot{\alpha}} + \frac{i}{4}\partial_{\alpha\dot{\alpha}}\bar{D}^2E^\alpha \quad (6.17)$$

which are solved by

$$E_\alpha = V_\alpha + \frac{1}{8}D^2\bar{D}^2V_\alpha - \frac{i}{4}\partial_{\alpha\dot{\alpha}}D^2\bar{V}^{\dot{\alpha}} + E'_\alpha \quad (6.18)$$

$$\bar{E}_{\dot{\alpha}} = \bar{V}_{\dot{\alpha}} + \frac{1}{8}\bar{D}^2D^2\bar{V}_{\dot{\alpha}} - \frac{i}{4}\partial_{\alpha\dot{\alpha}}\bar{D}^2V^\alpha + \bar{E}'_{\dot{\alpha}} \quad (6.19)$$

where E'_α and $\bar{E}'_{\dot{\alpha}}$ are solutions to the homogeneous equations, i.e. they satisfy

$$E'_\alpha = -\frac{i}{2}\partial_{\alpha\dot{\alpha}}D^2\bar{E}'^{\dot{\alpha}}, \quad \bar{E}'_{\dot{\alpha}} = -\frac{i}{2}\partial_{\alpha\dot{\alpha}}\bar{D}^2E'^\alpha \quad (6.20)$$

$$(\partial^n\partial_n + 1)E'_\alpha = (\partial^n\partial_n + 1)\bar{E}'_{\dot{\alpha}} = 0. \quad (6.21)$$

As we'll see shortly, E'_α and $\bar{E}'_{\dot{\alpha}}$ are also ϵq -exact.

The terms proportional to Π^m give the equation

$$\epsilon q W_m + U_m = -\frac{i}{2}(\bar{\sigma}_m)^{\alpha\dot{\alpha}}D_\alpha\bar{E}_{\dot{\alpha}} - \frac{i}{2}(\bar{\sigma}_m)^{\alpha\dot{\alpha}}\bar{D}_{\dot{\alpha}}E_\alpha \quad (6.22)$$

Note that

$$\begin{aligned} \bar{\sigma}_m^{\dot{\alpha}\alpha}D_\alpha\bar{V}_{\dot{\alpha}} &= \frac{2}{3\sqrt{3}}\left[\left(-2\epsilon\sigma^q\bar{\theta}\delta_m^p + \epsilon\sigma^q\bar{\sigma}_m\sigma^p\bar{\theta}\right)v_{0p}^i \otimes v_{0q}^i(y) - 2i\theta\sigma^p\bar{\sigma}_m\sigma^n\bar{\theta}\epsilon\sigma^q\bar{\eta}_{pn}^i v_{0q}^i\right. \\ &\quad \left.+ 2i\theta\sigma^p\bar{\theta}\epsilon\sigma^q\bar{\sigma}_m\sigma^n v_{0p}^i\bar{\eta}_{qn}^i\right] \\ &= \frac{2}{3\sqrt{3}}\left[(-4\epsilon\sigma^q\bar{\theta}\delta^{mp} + \epsilon\sigma_m\bar{\theta}\delta^{pq})v_{0p}^i \otimes v_{0q}^i(y) + 4i\theta\sigma^p\bar{\theta}\epsilon\sigma^q\bar{\theta}\left(\bar{\eta}_{pm}^i v_{0q}^i - v_{0p}^i\bar{\eta}_{qm}^i\right)\right] \\ &= \frac{2}{3\sqrt{3}}(-4\epsilon\sigma^q\bar{\theta}\delta^{mp} + \epsilon\sigma_m\bar{\theta}\delta^{pq})v_{0p}^i \otimes v_{0q}^i(y) + 2iU_m \end{aligned} \quad (6.23)$$

In the second equality we used the fact that $\sigma^{mn}\bar{\eta}_{mn}^i = 0$ and $(\bar{\sigma}^{(m}\sigma^{n)})_{\dot{\beta}}^{\dot{\alpha}} = -2\delta^{mn}\delta_{\dot{\alpha}}^{\dot{\beta}}$. So eq. (6.22) becomes

$$\begin{aligned} \epsilon q W_m &= \frac{i}{3\sqrt{3}}\left(4\epsilon\sigma^q\bar{\theta}\delta^{mp} - \epsilon\sigma_m\bar{\theta}\delta^{pq}\right)v_{0p}^i \otimes v_{0q}^i(y) \\ &\quad - \frac{i}{2}\bar{\sigma}_m^{\alpha\dot{\alpha}}D_\alpha\left(\frac{1}{8}\bar{D}^2D^2\bar{V}_{\dot{\alpha}} - \frac{i}{4}\partial_{\beta\dot{\alpha}}\bar{D}^2V^\beta + \bar{E}'_{\dot{\alpha}}\right) - \frac{i}{2}(\bar{\sigma}_m)^{\alpha\dot{\alpha}}\bar{D}_{\dot{\alpha}}E_\alpha \end{aligned} \quad (6.24)$$

Finally, the equations for C and B are

$$\epsilon q(C + iB) = \frac{i}{4}D^\alpha\bar{D}^2E_\alpha \quad (6.25)$$

$$\epsilon q(C - iB) = -\frac{i}{4}\bar{D}^{\dot{\alpha}}D^2\bar{E}_{\dot{\alpha}} \quad (6.26)$$

In order for equations (6.24)–(6.26) to be consistent, we must verify that the right-hand sides are ϵq -exact. Note that any field of the form $\epsilon\psi(y, \bar{\theta})$ is ϵq -exact: $\epsilon\psi(y, \bar{\theta}) = \epsilon q(\theta\psi(y, \bar{\theta}))$. It's then easy to verify that V_α and $D^2\bar{V}_\alpha$ are ϵq -exact, as are any derivatives of these quantities, since q anti-commutes with D and \bar{D} . Thus, the equations are consistent as long as we require that $D_\alpha\bar{E}'_\alpha + \bar{D}_\alpha E'_\alpha$, $D^\alpha\bar{D}^2E'_\alpha$ and $\bar{D}^\alpha D^2\bar{E}'_\alpha$ are ϵq -exact.

One might suspect that the E'_α and \bar{E}'_α terms simply give a gauge transformation of V_1 . This is indeed the case, as we'll now show. First, note that requiring $D^\alpha\bar{D}^2E'_\alpha = \epsilon q(F)$ implies, by the first eq. in (6.20), that $\bar{E}'_\alpha = -\frac{1}{2}\epsilon q(\bar{D}_\alpha F) = \epsilon q(\bar{E}''_\alpha)$. Similarly, $\bar{D}^\alpha D^2\bar{E}'_\alpha = \epsilon q(\bar{F})$ implies $E'_\alpha = \epsilon q(E''_\alpha)$. Note that E''_α and \bar{E}''_α are only defined up to $q^2(\text{something})$. Also, they satisfy eqs. (6.20) and (6.21) up to $q^2(\text{something})$ — in other words, we can choose E''_α and \bar{E}''_α such that they satisfy those conditions. That means they only change Λ_1 and $\bar{\Lambda}_1$ by ϵq -exact gauge parameters, which in turn means that they only contribute pure gauge terms to V_1 .

Thus, from eqs. (6.24)–(6.26), we have uniquely determined $\epsilon q(V_1)$ up to a gauge transformation. This means we also have determined V_1 , up to a gauge transformation and up to $q^2(\text{something})$. The particular solution found by simply eliminating ϵq in eqs. (6.24)–(6.26) (after writing the right-hand sides as $\epsilon q(\text{something})$ in the way described in the previous paragraphs) is

$$\begin{aligned} \tilde{V}_1^0 &= \frac{2i}{3\sqrt{3}}\Pi^m \left[\left(2\theta\sigma^a\bar{\theta}\delta_m^p - \theta\sigma_m\bar{\theta}\delta^{pa} \right) v_{0p}^i \otimes v_{0q}^i(y) - 72\theta\sigma_m\bar{\theta} \right] \\ &\quad - \frac{8}{\sqrt{3}}\partial(5\rho - 3H_C)\theta\sigma^m\bar{\theta}y_m \\ V_1^i &= 0 \end{aligned} \tag{6.27}$$

where the x products, again, are normal ordered, i.e. no contractions.

6.2 Equations of motion

We now want to verify that we can find a half-BPS solution which satisfies the equation of motion, given, to quadratic order, by [6]

$$\tilde{G}^+(G^+(V_1)) = \frac{1}{2}\{G^+(V_0), \tilde{G}^+(V_0)\} \tag{6.28}$$

Although we will only solve for the first massive level, we can show that the equation of motion can be solved for all massive levels. There is a solution if the right-hand side is \tilde{G}^+G^+ -exact. Note that it is annihilated by both \tilde{G}^+ and G^+ . In other words, there is an obstruction to solving the equation if and only if the right hand side is a non-trivial state in the cohomology of \tilde{G}^+G^+ . This is equivalent to asking whether there is an on-shell massive state that can decay into two self-dual massless states. But V_0 is not an eigenstate of momentum and is a polynomial in x . In momentum space, it is therefore a linear combination of delta functions and derivatives of delta functions which vanishes at non-zero momentum.³ So there can be no such on-shell massive state and there is no obstruction to solving the equation of motion for V_1 at any mass level.

³We thank the referee for this useful comment.

The solution \tilde{V}_1 written above does not satisfy the equation of motion, but we can find a field V_1 that does, and which differs from \tilde{V}_1 by a q^2 -exact term. This solution V_1 is unique if we also require that it vanishes at infinity.

Calculating the anticommutator between $G^+(V_0)$ and $\tilde{G}^+(V_0)$ again involves evaluating a star product:

$$\frac{1}{2} \{G^+(V_0), \tilde{G}^+(V_0)\} = \frac{1}{2} G^+(V_0^i) * \tilde{G}^+(V_0^j) \{\sigma^i, \sigma^j\} = G^+(V_0^i) * \tilde{G}^+(V_0^i) \sigma^0 \quad (6.29)$$

where

$$\begin{aligned} G^+(V_0) &= e^{i\rho} \left(2d^\alpha D_\alpha - i\partial\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} D^\alpha \right) V_0, \\ \tilde{G}^+(V_0) &= e^{-2i\rho+iH_C} 2\bar{d}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} V_0, \end{aligned} \quad (6.30)$$

and we have used that $\partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} V_0 = 0$.

As before, let's separate the terms proportional to Π^m , d_α , $\bar{d}_{\dot{\alpha}}$, $\partial\rho$ and ∂H_C . Note that acting with G^+ or \tilde{G}^+ on the equation gives zero, so not all terms are independent. For example, we don't need to check the terms proportional to $\partial\theta^\alpha$ and $\partial\bar{\theta}^{\dot{\alpha}}$. Details on the star product can again be found in appendix A, and on the right-hand side of (6.28) we find (times an overall factor of $e^{-i\rho+iH_C}$):

- Terms proportional to Π^m :

$$\begin{aligned} & -\frac{16i}{3\sqrt{3}} \bar{\sigma}_m^{\dot{\alpha}\alpha} D_\alpha V_0^i \otimes \bar{D}_{\dot{\alpha}} V_0^i - \frac{8}{3\sqrt{3}} \left(\partial_m \bar{D}^{\dot{\alpha}} D_\alpha V_0^i \otimes D^\alpha \bar{D}_{\dot{\alpha}} V_0^i \right. \\ & \left. - \bar{D}^{\dot{\alpha}} D_\alpha V_0^i \otimes \partial_m D^\alpha \bar{D}_{\dot{\alpha}} V_0^i + i\partial_{\alpha\dot{\alpha}} D^\alpha V_0^i \otimes \partial_m \bar{D}^{\dot{\alpha}} V_0^i \right) \\ & = \frac{16i}{3\sqrt{3}} \left(2\theta\sigma^q \bar{\theta} \delta_m^p - \theta\sigma_m \bar{\theta} \delta^{pq} \right) v_{0p}^i \otimes v_{0q}^i(y) + \frac{192i}{\sqrt{3}} \theta\sigma_m \bar{\theta} \end{aligned} \quad (6.31)$$

- Terms proportional to d^α :

$$-\frac{16}{3\sqrt{3}} \bar{D}_{\dot{\alpha}} D_\alpha V_0^i \otimes \bar{D}^{\dot{\alpha}} V_0^i = -\frac{16}{3\sqrt{3}} \theta_\alpha v_{0p}^i \otimes v_0^{ip}(y) \quad (6.32)$$

- Terms proportional to $\bar{d}^{\dot{\alpha}}$:

$$-\frac{16}{3\sqrt{3}} D_\alpha \bar{D}_{\dot{\alpha}} V_0^i \otimes D^\alpha V_0^i = -\frac{16}{3\sqrt{3}} \left(-\bar{\theta}_{\dot{\alpha}} v_{0p}^i \otimes v_0^{ip}(y) + 2i\bar{\theta}^2 (\theta\sigma^m)_{\dot{\alpha}} \bar{\eta}_{mn}^i v^{\text{in}}(y) \right) \quad (6.33)$$

- Terms proportional to $\partial(H_C - 3\rho)$:

$$\begin{aligned} & \frac{8i}{3\sqrt{3}} \left(\bar{D}^{\dot{\alpha}} D_\alpha V_0^i \otimes D^\alpha \bar{D}_{\dot{\alpha}} V_0^i + i\partial_{\alpha\dot{\alpha}} D^\alpha V_0^i \otimes \bar{D}^{\dot{\alpha}} V_0^i \right) \\ & = -\frac{16i}{3\sqrt{3}} v_{0p}^i \otimes v_0^{ip}(y) - \frac{64}{\sqrt{3}} \theta\sigma^m \bar{\theta} y_m \end{aligned} \quad (6.34)$$

And on the left-hand side of (6.28) we find:

- Terms proportional to Π^m :

$$\begin{aligned}
 & - \left\{ \bar{D}^2, D^2 \right\} W_m + 8W_m - 4\partial^n \partial_n W_m - 8\partial_m B - 12(\bar{\sigma}_m)^{\dot{\alpha}\alpha} \left[\bar{D}_{\dot{\alpha}}, D_\alpha \right] C \quad (6.35) \\
 & = \frac{16i}{3\sqrt{3}} \left(2\theta\sigma^q \bar{\theta} \delta_m^p - \theta\sigma_m \bar{\theta} \delta^{pq} \right) v_{0p}^i \otimes v_{0q}^i(y) + \frac{192i}{\sqrt{3}} \theta\sigma_m \bar{\theta} + \frac{128}{\sqrt{3}} y_m
 \end{aligned}$$

- Terms proportional to d^α :

$$\begin{aligned}
 & 4i(\sigma^m)_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} W_m - 24\partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} C + D_\alpha \bar{D}^2 (18iC + 2B) \quad (6.36) \\
 & = -\frac{16}{3\sqrt{3}} \theta_\alpha v_{0p}^i \otimes v_0^{ip}(y)
 \end{aligned}$$

- Terms proportional to $\bar{d}^{\dot{\alpha}}$:

$$\begin{aligned}
 & 4i(\sigma^m)_{\alpha\dot{\alpha}} D^\alpha W_m + 24\partial_{\alpha\dot{\alpha}} D^\alpha C - \bar{D}_{\dot{\alpha}} D^2 (18iC - 2B) \quad (6.37) \\
 & = -\frac{16}{3\sqrt{3}} \left(-\bar{\theta}_{\dot{\alpha}} v_{0p}^i \otimes v_0^{ip}(y) + 2\bar{\theta}^2 (\theta\sigma^m)_{\dot{\alpha}} \bar{\eta}_{mn}^i v^{\text{in}}(y) \right) - \frac{256}{\sqrt{3}} \bar{\theta}_{\dot{\alpha}}
 \end{aligned}$$

- Terms proportional to $i\partial(\rho - H_C)$:

$$\begin{aligned}
 & -4\partial^m W_m - 8B - \frac{1}{2} \left\{ \bar{D}^2, D^2 \right\} B + 3\partial_{\alpha\dot{\alpha}} \left[\bar{D}^{\dot{\alpha}}, D^\alpha \right] C \quad (6.38) \\
 & = -\frac{128}{\sqrt{3}}
 \end{aligned}$$

- Terms proportional to $\partial(H_C - 3\rho)$:

$$\begin{aligned}
 & -2\bar{\sigma}_m^{\dot{\alpha}\alpha} \left[\bar{D}_{\dot{\alpha}}, D_\alpha \right] W^m - 8C + \frac{11}{2} \left\{ \bar{D}^2, D^2 \right\} C + 16\partial_m \partial^m C + \partial_{\alpha\dot{\alpha}} \left[\bar{D}^{\dot{\alpha}}, D^\alpha \right] B \quad (6.39) \\
 & = -\frac{32i}{3\sqrt{3}} v_{0p}^i \otimes v_0^{ip}(y) - \frac{64}{\sqrt{3}} \theta\sigma^m \bar{\theta} y_m + \frac{128i}{\sqrt{3}}
 \end{aligned}$$

We conclude that \tilde{V}_1 does not satisfy the equations of motion. However, the following V_1 can be defined which differs from \tilde{V}_1 by a q^2 -exact term and which satisfies the equations of motion:

$$\begin{aligned}
 V_1^0 & = \tilde{V}_1^0 - \frac{64}{\sqrt{3}} \Pi^m y_m - \frac{8i}{\sqrt{3}} (\partial H_C - 3\partial\rho) y^2 + \frac{64}{\sqrt{3}} i (\partial\rho - \partial H_C) \quad (6.40) \\
 & = \frac{2i}{3\sqrt{3}} \Pi^m \left[\left(2\theta\sigma^q \bar{\theta} \delta_m^p - \theta\sigma_m \bar{\theta} \delta^{pq} \right) v_{0p}^i \otimes v_{0q}^i(y) - 72\theta\sigma_m \bar{\theta} + 96iy_m \right] \\
 & \quad + \frac{8}{\sqrt{3}} \left(\theta\sigma^m \bar{\theta} y_m - iy^2 \right) (\partial H_C - 3\partial\rho) + \frac{16}{\sqrt{3}} \left(i\theta\sigma^m \bar{\theta} y_m + 4 \right) i (\partial\rho - \partial H_C) \\
 V_1^i & = 0
 \end{aligned}$$

We can still add to V_1 a term $q^2\Phi$ where Φ satisfies $\tilde{G}^+(G^+(\Phi)) = 0$, i.e. Φ describes an on-shell massive field. But as explained below eq. (4.9), any non-zero on-shell massive field will diverge exponentially when $x \rightarrow \infty$. So in order to have an instanton solution that vanishes at infinity, we should set Φ to zero. Note that the stringy contribution V_1 of eq. (6.40) naively diverges quadratically in x^m when $x \rightarrow \infty$, but this is expected since it was found using an expansion in $\frac{1}{\rho^2}$. After summing over all orders in $\frac{1}{\rho^2}$, one expects that the stringy contribution will vanish at infinity like the super-Yang-Mills instanton. However, adding an on-shell massive contribution to V_1 would diverge exponentially when $x \rightarrow \infty$ and could never cancel after summing over all orders in $\frac{1}{\rho^2}$. We therefore have a unique solution.

6.3 Meaning of the massive fields

The first massive level of the on-shell string field describes two massive scalar multiplets and a massive spin-two multiplet. The on-shell field can be gauge-fixed to the form

$$\Pi^m \left(\hat{W}_m + 2i\partial_m (F - \bar{F}) \right) - \frac{1}{8} (\partial\rho - \partial H_C) [D^2, \bar{D}^2] (F + \bar{F}) + (\partial H_C - 3\partial\rho) (F + \bar{F}) \quad (6.41)$$

where F and \bar{F} are the chiral and antichiral superfields for the scalar multiplets, and \hat{W}_m is the superfield for the spin-two multiplet. In components,

$$F = X(y) + \theta^\alpha \xi_\alpha(y) + (\theta)^2 Y(y), \quad \bar{F} = \bar{X}(\bar{y}) + \bar{\theta}^{\dot{\alpha}} \bar{\xi}_{\dot{\alpha}}(\bar{y}) + (\bar{\theta})^2 \bar{Y}(\bar{y}) \quad (6.42)$$

and

$$\begin{aligned} \hat{W}_m = & C_m + i\theta\chi_m - i\bar{\theta}\bar{\chi}_m + \theta\sigma^n\bar{\theta} \left(v_{mn} + \frac{1}{4}\epsilon_{mnpq}\partial^p C^q \right) \\ & + i\theta\bar{\theta}\bar{\theta}\bar{\lambda}_m - i\bar{\theta}\theta\theta\lambda_m - \theta\bar{\theta}\bar{\theta}\bar{\theta} \frac{1}{16}\partial^n\partial_n C_m \end{aligned} \quad (6.43)$$

where X, \bar{X}, Y and \bar{Y} are massive real bosons, ξ_α and $\bar{\xi}_{\dot{\alpha}}$ are massive spinors, v_{mn} is a massive symmetric traceless tensor, C_m is a massive vector, and χ and λ are massive spin-3/2 fermions [11, 12].

This is to be compared to our result of eq. (6.40). Since V_1 is not on-shell, it doesn't take the form (6.41). Nevertheless, we can still identify the components of V_1 with certain components of F, \bar{F} and \hat{W}_m . Comparing eq. (6.42) and (6.43) with eq. (6.40), one finds that the non-zero component fields are

$$X^0(y) = -\frac{8i}{\sqrt{3}}y^2 + \frac{16i}{\sqrt{3}}, \quad v_{mn}^0 = v_{0(m}^i \otimes v_{0n)}^i - 2v_{0p}^i \otimes v_0^{ip} \delta_{mn} \quad (6.44)$$

where $v_{0(m}^i \otimes v_{0n)}^i = v_{0m}^i \otimes v_{0n}^i + v_{0n}^i \otimes v_{0m}^i$. We can therefore interpret the correction to the BPST instanton as turning on the U(1) component of a massive scalar field and a massive spin-two field.

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A Star products

In this appendix we detail the calculation of the relevant star products. We'll use conformal mappings to the upper half complex plane (see [13]). The prescription we use is as follows: given a basis e^r of the string field space, the star product of two arbitrary string fields A and B is

$$A * B = \sum_r \left(e_r^C, A, B \right) e_r \tag{A.1}$$

where C denotes conjugate states. These are defined by

$$\left(e_r^C, e_s \right) = \delta_{rs} \tag{A.2}$$

where

$$(A, B) = \langle (I(z) \circ A(0)) B(0) \rangle_{\text{UHP}} \tag{A.3}$$

The right-hand side is a correlator in the upper half plane, and $I(z)$ is the conformal transformation

$$I(z) = -\frac{1}{z} \tag{A.4}$$

The coefficient is given by the correlator

$$(A, B, C) = \langle f_{-1} \circ A(0) f_0 \circ B(0) f_1 \circ C(0) \rangle_{\text{UHP}} \tag{A.5}$$

with the conformal transformations

$$f_n(z) = \tan \left[\frac{n\pi}{3} + \frac{2}{3} \arctan z \right] \tag{A.6}$$

The correlators are given by

$$\left\langle e^{ik_1x}(z_1) e^{ik_2x}(z_2) \right\rangle = i(2\pi)^4 \delta^4(k_1 + k_2) |z_1 - z_2|^{-k_1 \cdot k_2} \tag{A.7}$$

$$\left\langle \theta^2 \bar{\theta}^2 e^{-i\rho + iH_C} \right\rangle = 1 \tag{A.8}$$

A.1 Massless level

We will denote the massless level of the star product by \otimes . In the $p\theta$ sector, there are no non-trivial massless contributions. For example, it is immediate that

$$\theta^\alpha \otimes \theta^\beta = \theta^\alpha \theta^\beta \tag{A.9}$$

For the x^m , however, there are non-trivial (higher derivative/higher order in α') terms, even at the massless level. Computing the product of two functions of x^m is slightly tricky because x^m itself is not a well-behaved operator in the worldsheet theory. Rather, the actual operators are worldsheet derivatives of x^m and plane waves $e^{ik \cdot x}$. The solution is to assume that the functions of x^m we are interested in admit Fourier transform representations and to compute the star product of functions x^m in terms of the products of exponentials, that can be obtained by the usual methods. That is,

$$F(x) * G(x) = \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \tilde{F}(k_1) \tilde{G}(k_2) e^{ik_1 x} * e^{ik_2 x} \quad (\text{A.10})$$

We need then to calculate the product $e^{ik_1 x} * e^{ik_2 x}$, which amounts to calculating the coefficient $(e^{-ikx}, e^{ik_1 x}, e^{ik_2 x})$:

$$\begin{aligned} (e^{-ikx}, e^{ik_1 x}, e^{ik_2 x}) &= \left(\frac{8}{3}\right)^{-(k^2+k_2^2)/2} \left(\frac{2}{3}\right)^{-k_1^2/2} \langle e^{-ikx(-\sqrt{3})} e^{ik_1 x(0)} e^{ik_2 x(\sqrt{3})} \rangle \\ &= i(2\pi)^4 2^{-2k^2+2k_1 \cdot k_2} \sqrt{3}^{k^2+k_1 \cdot k_2} \delta^4(k_1 + k_2 - k) \end{aligned} \quad (\text{A.11})$$

So the \otimes product is

$$e^{ik_1 x} \otimes e^{ik_2 x} = e^{\ln 2[-2(k_1+k_2)^2+2k_1 \cdot k_2] + \ln \sqrt{3}[(k_1+k_2)^2+k_1 \cdot k_2]} e^{i(k_1+k_2)x} \quad (\text{A.12})$$

We conclude that

$$\begin{aligned} F(x) \otimes G(x) &= \\ \exp \left[-2 \ln 2 \left(\frac{\partial_F^2}{\partial x^2} + \frac{\partial_G^2}{\partial x^2} + \frac{\partial_F \partial_G}{\partial x^2} \right) - \ln \sqrt{3} \left(\frac{\partial_F^2}{\partial x^2} + \frac{\partial_G^2}{\partial x^2} + 3 \frac{\partial_F \partial_G}{\partial x^2} \right) \right] F(x) G(x) \end{aligned} \quad (\text{A.13})$$

A.2 First massive level

The star product between two fields $F(x, \theta, \bar{\theta})$, $G(x, \theta, \bar{\theta})$ will contain terms proportional to $\partial\theta^\alpha$, $\partial\bar{\theta}^{\dot{\alpha}}$ and Π^m . The conjugate states are proportional to d_α , $\bar{d}_{\dot{\alpha}}$ and Π_m , respectively. Thus, we need to calculate the coefficients $(d_\alpha V(x, \theta, \bar{\theta}), F, G)$, $(\bar{d}_{\dot{\alpha}} V(x, \theta, \bar{\theta}), F, G)$ and $(\Pi^m V(x, \theta, \bar{\theta}), F, G)$. The relevant conformal transformations are

$$f \circ (d_\alpha V(x, \theta, \bar{\theta})) (z) = f'(z) d_\alpha V(f(z)) + \frac{f''(z)}{2f'(z)} D_\alpha V \quad (\text{A.14})$$

$$f \circ (\bar{d}_{\dot{\alpha}} V(x, \theta, \bar{\theta})) (z) = f'(z) \bar{d}_{\dot{\alpha}} V(f(z)) + \frac{f''(z)}{2f'(z)} \bar{D}_{\dot{\alpha}} V \quad (\text{A.15})$$

$$f \circ (\Pi^m V(x, \theta, \bar{\theta})) (z) = f'(z) \Pi^m V(f(z)) + \frac{f''(z)}{2f'(z)} \partial^m V \quad (\text{A.16})$$

For the first coefficient we obtain

$$\begin{aligned} (d_\alpha V(x, \theta, \bar{\theta}), F, G) &= \frac{8}{3} \langle d_\alpha V(-\sqrt{3}) F(0) G(\sqrt{3}) \rangle - \frac{2}{\sqrt{3}} \langle D_\alpha V(-\sqrt{3}) F(0) G(\sqrt{3}) \rangle \\ &= -\frac{2}{3\sqrt{3}} (-1)^V \left(\langle V(-\sqrt{3}) D_\alpha F(0) G(\sqrt{3}) \rangle - (-1)^F \langle V(-\sqrt{3}) F(0) D_\alpha G(\sqrt{3}) \rangle \right) \end{aligned} \quad (\text{A.17})$$

where $(-1)^V$ is 1 if V is commuting and -1 if anticommuting. Note that $(-1)^V \partial \theta^\alpha W$ is the conjugate of $d_\alpha V$ if W is the conjugate of V . We conclude that

$$F * G|_{\partial\theta} = \frac{C}{2} \partial \theta^\alpha (-D_\alpha F \otimes G + F \otimes D_\alpha G) \quad (\text{A.18})$$

where $|_{\partial\theta}$ denotes the terms in the star product proportional to $\partial \theta^\alpha$, and $C = \frac{4}{3\sqrt{3}}$. A similar calculation yields the terms proportional to $\partial \bar{\theta}^{\dot{\alpha}}$ and Π^m :

$$(F * G)_{M^2=2} = \frac{C}{2} \left[\partial \theta^\alpha (-D_\alpha F \otimes G + F \otimes D_\alpha G) + \partial \bar{\theta}^{\dot{\alpha}} (-\bar{D}_{\dot{\alpha}} F \otimes G + F \otimes \bar{D}_{\dot{\alpha}} G) \right. \\ \left. + \Pi^m (-\partial_m F \otimes G + F \otimes \partial_m G) \right] \quad (\text{A.19})$$

A.3 Products involving d and $\partial\theta$

To calculate the equations of motion, we also need to evaluate products of the form $d_\alpha F * \bar{d}_{\dot{\alpha}} G$ and $\bar{d}_{\dot{\alpha}} F * \partial \bar{\theta}^{\dot{\beta}} G$. At the massless level, we have

$$(V(x, \theta, \bar{\theta}), d_\alpha F, \bar{d}_{\dot{\alpha}} G) = \frac{16}{9} \langle V(-\sqrt{3}) d_\alpha F(0) \bar{d}_{\dot{\alpha}} G(\sqrt{3}) \rangle \\ + \frac{4}{3\sqrt{3}} \langle V(-\sqrt{3}) d_\alpha F(0) \bar{D}_{\dot{\alpha}} G(\sqrt{3}) \rangle \quad (\text{A.20})$$

$$(V(x, \theta, \bar{\theta}), \bar{d}_{\dot{\alpha}} F, \partial \bar{\theta}^{\dot{\beta}} G) = \frac{16}{27} \delta_{\dot{\alpha}}^{\dot{\beta}} \langle V(-\sqrt{3}) F(0) G(\sqrt{3}) \rangle \quad (\text{A.21})$$

Therefore

$$d_\alpha F \otimes \bar{d}_{\dot{\alpha}} G = C^2 \left(\bar{D}_{\dot{\alpha}} F \otimes D_\alpha G - \frac{1}{2} (-1)^F D_\alpha \bar{D}_{\dot{\alpha}} F \otimes G \right. \\ \left. + \frac{1}{2} (-1)^F F \otimes \bar{D}_{\dot{\alpha}} D_\alpha G - \frac{1}{4} D_\alpha F \otimes \bar{D}_{\dot{\alpha}} G \right) \quad (\text{A.22})$$

$$\bar{d}_{\dot{\alpha}} F \otimes \partial \bar{\theta}^{\dot{\beta}} G = C^2 \delta_{\dot{\alpha}}^{\dot{\beta}} F \otimes G \quad (\text{A.23})$$

Now we move on to the first massive level. To get the terms proportional to Π^m , we need the coefficient

$$\left(\Pi^m V(x, \theta, \bar{\theta}), d_\alpha F, \bar{d}_{\dot{\alpha}} G \right) = C^2 \left(\left(\Pi^m V, \bar{D}_{\dot{\alpha}} F, D_\alpha G \right) \right. \\ \left. - \frac{1}{2} (-1)^F \left(\Pi^m V, D_\alpha \bar{D}_{\dot{\alpha}} F, G \right) + \frac{1}{2} (-1)^F \left(\Pi^m V, F, \bar{D}_{\dot{\alpha}} D_\alpha G \right) \right. \\ \left. - \frac{1}{4} \left(\Pi^m V, D_\alpha F, \bar{D}_{\dot{\alpha}} G \right) \right) + (-1)^F C^3 \sigma_{\alpha\dot{\alpha}}^m(V, F, G) \quad (\text{A.24})$$

where the last term comes from the $O(1/(z_{12}z_{13}z_{23}))$ term in the OPE $\bar{d}_{\dot{\alpha}}(z_1) d_\alpha(z_2) \Pi^m(z_3)$. For $\bar{d}_{\dot{\alpha}} F * \partial \bar{\theta}^{\dot{\beta}} G$ we have simply

$$\left(\Pi^m V, \bar{d}_{\dot{\alpha}} F, \partial \bar{\theta}^{\dot{\beta}} G \right) = C^2 \delta_{\dot{\alpha}}^{\dot{\beta}} (\Pi^m V, F, G) \quad (\text{A.25})$$

We conclude that

$$d_\alpha F * \bar{d}_{\dot{\alpha}} G|_{\Pi} = C^2 \left(\bar{D}_{\dot{\alpha}} F * D_\alpha G - \frac{1}{2} (-1)^F D_\alpha \bar{D}_{\dot{\alpha}} F * G \right. \\ \left. + \frac{1}{2} (-1)^F F * \bar{D}_{\dot{\alpha}} D_\alpha G - \frac{1}{4} D_\alpha F * \bar{D}_{\dot{\alpha}} G \right) \Big|_{\Pi} + (-1)^F i C^3 \Pi_{\alpha\dot{\alpha}}^m F \otimes G \\ \bar{d}_{\dot{\alpha}} F * \partial \bar{\theta}^{\dot{\beta}} G|_{\Pi} = C^2 \delta_{\dot{\alpha}}^{\dot{\beta}} F * G|_{\Pi} \quad (\text{A.26})$$

To obtain the terms proportional to d and \bar{d} , we need the following coefficients:

$$\begin{aligned} & \left(\partial\theta^\beta V(x, \theta, \bar{\theta}), d_\alpha F, \bar{d}_{\dot{\alpha}} G \right) = \tag{A.27} \\ & - (-1)^V C^3 \delta_\alpha^\beta \left((-1)^F \langle V(-\sqrt{3}) \bar{D}_{\dot{\alpha}} F(0) G(\sqrt{3}) \rangle + \frac{1}{2} \langle V(-\sqrt{3}) F(0) \bar{D}_{\dot{\alpha}} G(\sqrt{3}) \rangle \right) \end{aligned}$$

$$\begin{aligned} & \left(\partial\bar{\theta}^{\dot{\beta}} V(x, \theta, \bar{\theta}), d_\alpha F, \bar{d}_{\dot{\alpha}} G \right) = \tag{A.28} \\ & - (-1)^V C^3 \delta_{\dot{\alpha}}^{\dot{\beta}} \left(\langle V(-\sqrt{3}) F(0) D_\alpha G(\sqrt{3}) \rangle + \frac{1}{2} (-1)^F \langle V(-\sqrt{3}) D_\alpha F(0) G(\sqrt{3}) \rangle \right) \end{aligned}$$

$$\left(\partial\theta^\beta V, \bar{d}_{\dot{\alpha}} F, \partial\bar{\theta}^{\dot{\beta}} G \right) = \left(\partial\bar{\theta}^{\dot{\gamma}} V, \bar{d}_{\dot{\alpha}} F, \partial\bar{\theta}^{\dot{\beta}} G \right) = 0 \tag{A.29}$$

Therefore

$$d_\alpha F * \bar{d}_{\dot{\alpha}} G|_d = -C^3 d_\alpha \left((-1)^F \bar{D}_{\dot{\alpha}} F \otimes G + \frac{1}{2} F \otimes \bar{D}_{\dot{\alpha}} G \right) \tag{A.30}$$

$$d_\alpha F * \bar{d}_{\dot{\alpha}} G|_{\bar{d}} = -C^3 \bar{d}_{\dot{\alpha}} \left(F \otimes D_\alpha G + \frac{1}{2} (-1)^F D_\alpha F \otimes G \right) \tag{A.31}$$

$$\bar{d}_{\dot{\alpha}} F * \partial\bar{\theta}^{\dot{\beta}} G|_d = \bar{d}_{\dot{\alpha}} F * \partial\bar{\theta}^{\dot{\beta}} G|_{\bar{d}} = 0 \tag{A.32}$$

A.4 Products involving ρ and H_C

The last product we need for the equations of motion is $e^{i\rho} * e^{-2i\rho+iH_C}$. We will need the following conformal transformations

$$f \circ e^{ni\rho}(z) = (f'(z))^{-(n^2+n)/2} e^{ni\rho}(f(z)) \tag{A.33}$$

$$f \circ e^{niH_C}(z) = (f'(z))^{3(n^2-n)/2} e^{niH_C}(f(z)) \tag{A.34}$$

$$f \circ \partial\rho(z) = f'(z) \partial\rho(f(z)) + \frac{if''(z)}{2f'(z)} \tag{A.35}$$

$$f \circ \partial H_C(z) = f'(z) \partial H_C(f(z)) + \frac{3if''(z)}{2f'(z)} \tag{A.36}$$

The relevant coefficients are

$$\left(1, e^{i\rho}, e^{-2i\rho+iH_C} \right) = C^{-2} \tag{A.37}$$

$$\left(\partial\rho, e^{i\rho}, e^{-2i\rho+iH_C} \right) = -\frac{3i}{2} C^{-1} \tag{A.38}$$

$$\left(\partial H_C, e^{i\rho}, e^{-2i\rho+iH_C} \right) = -\frac{3i}{2} C^{-1} \tag{A.39}$$

where the conjugates of 1, ∂H_C and $\partial\rho$ are, respectively, $e^{-i\rho+iH_C}$, $-\partial H_C e^{-i\rho+iH_C}/3$ and $\partial\rho e^{-i\rho+iH_C}$. Thus, the product we are looking for is (truncated to the first massive level)

$$e^{i\rho} * e^{-2i\rho+iH_C} = C^{-2} e^{-i\rho+iH_C} + \frac{i}{2} C^{-1} (\partial H_C - 3\partial\rho) e^{-i\rho+iH_C} \tag{A.40}$$

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