Thermalization in large-N CFTs

Robin Karlsson, Andrei Parnachev and Petar Tadić

School of Mathematics, Trinity College Dublin,
Dublin 2, Ireland

E-mail: karlsson@maths.tcd.ie, parnachev@maths.tcd.ie,
tadicp@maths.tcd.ie

ABSTRACT: In d-dimensional CFTs with a large number of degrees of freedom an important set of operators consists of the stress tensor and its products, multi stress tensors. Thermalization of such operators, the equality between their expectation values in heavy states and at finite temperature, is equivalent to a universal behavior of their OPE coefficients with a pair of identical heavy operators. We verify this behavior in a number of examples which include holographic and free CFTs and provide a bootstrap argument for the general case. In a free CFT we check the thermalization of multi stress tensor operators directly and also confirm the equality between the contributions of multi stress tensors to heavy-heavy-light-light correlators and to the corresponding thermal light-light two-point functions by disentangling the contributions of other light operators. Unlike multi stress tensors, these light operators violate the Eigenstate Thermalization Hypothesis and do not thermalize.

Contents

1 Introduction and summary

Holography [1–3] provides us with a useful tool to study d-dimensional CFTs at large central charge C_T, especially when combined with modern CFT techniques (see e.g. [4–6] for reviews). One of the basic objects in this setup is a Witten diagram with a single graviton exchange which contributes to four-point functions. It can be decomposed into the conformal blocks of the stress-tensor and of the double-trace operators made out of external fields [7].

When a pair of the external operators denoted by $\mathcal{O}_H$ is taken to be heavy, with the conformal dimension $\Delta_H \sim C_T$, and the other pair denoted by $\mathcal{O}_L$ stays light, the resulting heavy-heavy-light-light (HHLL) correlator describes a light probe interacting with a heavy state. In this case, operators which are comprised out of many stress tensors (multi stress tensor operators) contribute, together with the multi-trace operators involving $\mathcal{O}_L$. As we review below, the OPE coefficients of the scalar operators with a (unit-normalized) multi stress tensor operator $T_{\tau,s}^k$, which contains k stress tensors and has twist τ and spin s, scale like $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\tau,s}^k} \sim \Delta^k / C_T^{k/2}$ for large Δ.

The contribution of a given multi stress tensor operator to the HHLL four-point function $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ can be compared to the contribution of the same operator to the corresponding two-point function at finite temperature[1] β^{-1}, $\langle \mathcal{O}_L \mathcal{O}_L \rangle_\beta$. In this paper we argue that they are the same in generic large-C_T CFTs. As we explain later, this means that OPE coefficients of $T_{\tau,s}^k$ with the two heavy operators $\mathcal{O}_H$, $\langle \mathcal{O}_H T_{\tau,s}^k \mathcal{O}_H \rangle$, are equal to their finite temperature expectation values, $\langle T_{\tau,s}^k \rangle_\beta$. The relation between the inverse temperature β and the conformal dimension Δ_H is set by considering the stress tensor $(k=1, \tau=d-2, s=2)$, but the equality between the thermal expectation values and the OPE coefficients for all other multi stress tensor operators is a nontrivial statement. We call it "the thermalization of the stress tensor sector".[2] It is directly related to the Eigenstate Thermalization Hypothesis (ETH) [26–30], as we review below. Hence, we argue that all multi stress tensor operators in the large-C_T CFTs satisfy the ETH. In $d=2$ the ETH and thermalization have been studied in e.g. [31–61].

---

[1]See [8–25] for some previous work on finite temperature conformal field theories in $d > 2$.

[2]We show this explicitly for certain primary heavy operators $\mathcal{O}_H$ in free CFTs. We also observe that other light operators do not satisfy the thermalization property that the stress tensor sector enjoys.

Here we want to address the $d > 2$ case. In holographic theories CFT and bootstrap techniques provide a lot of data which indicates that the thermalization of the stress tensor sector happens [62–73]. Some of the OPE coefficients in holographic CFTs were computed using two-point functions in a black hole background [63] — these are thermal correlators according to the standard holographic dictionary. It is also worth noting that the leading Δ behavior of the OPE coefficients in holographic models does not depend on the coefficients of the higher derivative terms in the bulk lagrangian [71] (this should not be confused with the universality of the OPE coefficients of the minimal-twist multi stress tensors [63]). Such a universality follows from the thermalization of the stress tensor sector as we discuss below.

A natural question is whether the thermalization of the stress tensor sector is just a property of holographic CFTs or if it holds more generally. In this paper we argue for the latter scenario. We compute the OPE coefficients (and the thermal expectation values) for a number of multi stress tensor operators in a free CFT and observe thermalization as well as universality of OPE coefficients. We also provide a bootstrap argument for all CFTs with a large central charge.

The rest of the paper is organized as follows. In section 2, we begin by considering the thermalization of multi stress tensor operators $T_{\tau,s}^k$. The heavy state we consider is created by a scalar operator $\mathcal{O}_H$ with dimension $\Delta_H \sim C_T$ and by thermalization of a multi stress tensor operator we mean[3]

$$\langle \mathcal{O}_H | T_{\tau,s}^k | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T_{\tau,s}^k} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T_{\tau,s}^k \rangle_\beta, \tag{1.1}$$

where the heavy state $|\mathcal{O}_H\rangle$ on the sphere of unit radius is created by the operator $\mathcal{O}_H$, $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\tau,s}^k}$ are the OPE coefficients of $T_{\tau,s}^k$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $|_{\Delta_H^k/C_T^{k/2}}$ means we keep only leading terms that scale like $\Delta_H^k/C_T^{k/2} \sim C_T^{k/2}$. In (1.1) $\langle T_{\tau,s}^k \rangle_\beta$ is the one-point function on the sphere at finite temperature β^{-1}. Note that the OPE coefficients involving the stress tensor are fixed by the Ward identity, and hence eq. (1.1) for the stress tensor establishes a relation between the temperature β^{-1} and Δ_H. By the large-C_T factorization,[4] the thermal one-point functions of multi stress tensors can be related to the thermal one-point function of the stress tensor itself. Explicitly,

$$\langle T_{\tau,s}^k \rangle_\beta = c_{\tau,s}^k (\langle T_{d-2,2}^1 \rangle_\beta)^k = c_{\tau,s}^k \left(\lambda_{\mathcal{O}_H \mathcal{O}_H T_{d-2,2}^1} \right)^k, \tag{1.2}$$

where $c_{\tau,s}^k$ are theory-independent coefficients that appear because of the index structure in $\langle T_{\tau,s}^k \rangle_\beta$. In the second equality in (1.2) we used (1.1) for the stress tensor. Note that (1.1) and (1.2) imply that the leading Δ_H behavior of the multi stress tensor OPE coefficients is universal, i.e. it does not depend on the theory.[5] We provide a bootstrap argument for

---

[3]Here we are suppressing the tensor structure. Note that all terms scale like $C_T^{k/2}$ which is consistent with $T_{\tau,s}^k$ being unit-normalized.

[4]See [74] for a general discussion of large-N factorization and [75, 76] and [8] for the discussion in the context of gauge theories and CFTs respectively. The factorization holds in adjoint models in the 't Hooft limit at finite temperature, but there are counterexamples, like e.g. a direct product of low-C_T CFTs. However the factorization of multi stress tensors would still apply in these models.

[5]This amounts to the large-C_T factorization of correlators $\langle \mathcal{O}_H | T_{\mu\nu} \dots T_{\alpha\beta} | \mathcal{O}_H \rangle$ in heavy states.

this universality in all large-C_T theories. Also note that (1.2) is written for multi-trace operators $T_{\tau,s}^k$ which do not contain derivatives, but the presence of derivatives does not affect the statement of universality.

In section 3, we check the universality by computing a number of the multi stress tensor OPE coefficients in a free $SU(N)$ adjoint scalar theory in $d = 4$ dimensions. We compare the leading Δ_H behavior in the free theory with results from holography/bootstrap and find perfect agreement in all cases listed below. After fixing the coefficients for the stress tensor case in section 3.1, we look at the first nontrivial case, $T_{4,4}^2$ in section 3.2. Section 3.3 is devoted to the double stress tensor with two derivatives, $T_{4,6}^2$. This is an operator whose finite temperature expectation value vanishes in the large volume limit (on the plane), but is finite on the sphere. In section 3.4 we consider minimal twist multi stress tensors of the type $T_{2k,2k}^k$. Section 3.5 is devoted to multi stress tensors with non-minimal twist, $T_{6,2}^2$ and $T_{8,0}^2$.

In section 4, we verify that (1.1) holds in the free adjoint scalar theory for a variety of operators. In this section we again consider $d = 4$, but in addition, take the infinite volume limit. This is for technical reasons — it is easier to compute a finite temperature expectation value on the plane than on the sphere. We spell out the index structure in (1.1) in detail and go over all the examples discussed in the previous section. In addition, we discuss some triple stress tensor operators.

We continue in section 5 by studying thermal two-point functions in the free adjoint scalar model in $d = 4$. By decomposing the correlator into thermal blocks we read off the product of thermal one-point functions and the OPE coefficients for several operators of low dimension and observe agreement with the results of sections 3 and 4. Due to the presence of multiple operators with the same dimension and spin, we have to solve a mixing problem to find which operators contribute to the thermal two-point function.

In section 6 we explain the relation between our results and the Eigenstate Thermalization Hypothesis. We observe that unlike multi stress tensors, other light operators explicitly violate the Eigenstate Thermalization Hypothesis and do not thermalize. We end with a discussion in section 7.

Appendices A, B, and C contain explicit calculations of OPE coefficients while in appendices D and E thermal one-point functions are calculated. In appendix F we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$. In appendix G we study a free scalar in two dimensions and calculate thermal two-point functions of certain quasi-primary operators. In appendix H we consider a free scalar vector model in four dimensions. Appendix I discusses the factorization of multi-trace operators in the large volume limit.

2 Thermalization and universality

In the following we consider large-C_T CFTs on a $(d-1)$-dimensional sphere of radius R, which we set to unity for most of this section. As reviewed in [71], the stress tensor sector of conformal four-point functions consists of the contributions of the stress tensor and all its composites (multi stress tensors). The HHLL correlators we consider involve two heavy operators inserted at $x_E^0 = \pm\infty$ and two light operators inserted on the Euclidean cylinder,

with angular separation φ and time separation x_E^0. The correlator in a heavy state (the HHLL correlator on the cylinder) is related to the correlator on the plane by a conformal transformation

$$\langle \mathcal{O}_H | \mathcal{O}(x_E^0, \varphi) \mathcal{O}(0) | \mathcal{O}_H \rangle = \lim_{x_4 \to \infty} x_4^{2\Delta_H} (z\bar{z})^{-\Delta/2} \langle \mathcal{O}_H(x_4) \mathcal{O}(1) \mathcal{O}(z, \bar{z}) \mathcal{O}_H(0) \rangle, \qquad (2.1)$$

where the cross-ratios $(z, \bar{z})$ on the plane are related to the coordinates (x_E^0, φ) via

$$z = e^{-x_E^0 - i\varphi}, \qquad \bar{z} = e^{-x_E^0 + i\varphi}. \qquad (2.2)$$

The stress tensor sector of the HHLL correlator is given by

$$\mathcal{G}(z, \bar{z}) = \lim_{x_4 \to \infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4) \mathcal{O}(1) \mathcal{O}(z, \bar{z}) \mathcal{O}_H(0) \rangle \Big|_{\text{multi stress tensors}} \qquad (2.3)$$

and can be expanded in conformal blocks

$$\mathcal{G}(z, \bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^\Delta} \sum_{T_{\tau,s}^k} P_{T_{\tau,s}^k}^{(HH,LL)} g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z}), \qquad (2.4)$$

where τ, s, k label the twist, spin, and multiplicity of multi stress tensors. We are interested in the double scaling limit where the central charge and the dimension of $\mathcal{O}_H$ are large, $C_T, \Delta_H \to \infty$ with their ratio $\mu \propto \Delta_H / C_T$ fixed. In this limit the products of the OPE coefficients which appear in (2.4) are given by

$$P_{T_{\tau,s}^k}^{(HH,LL)} = \left(-\frac{1}{2} \right)^s \lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k} \lambda_{\mathcal{O}_H\mathcal{O}_H T_{\tau,s}^k} \Bigg|_{\left(\frac{\Delta_H}{C_T} \right)^k}, \qquad (2.5)$$

where we only keep the leading, $\left(\frac{\Delta_H}{\sqrt{C_T}} \right)^k$ term in the OPE coefficients $\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\tau,s}^k}$, but retain all terms in the OPE coefficients of the light operators $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k}$. The contribution of the conformal family of a multi stress operator $T_{\tau,s}^k$ to the HHLL correlator is therefore

$$\langle \mathcal{O}_H | \mathcal{O}(x_E^0, \varphi) \mathcal{O}(0) | \mathcal{O}_H \rangle |_{T_{\tau,s}^k} = \frac{P_{T_{\tau,s}^k}^{(HH,LL)} g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z})}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^\Delta}. \qquad (2.6)$$

We now consider these CFTs at finite temperature β^{-1}. To isolate the contribution of the conformal family associated with $T_{\tau,s}^k$, we can write the thermal correlator as

$$\langle \mathcal{O}(x_E^0, \varphi) \mathcal{O}(0) \rangle_\beta = \frac{1}{Z(\beta)} \sum_i e^{-\beta \Delta_i} \langle \mathcal{O}_i | \mathcal{O}(x_E^0, \varphi) \mathcal{O}(0) | \mathcal{O}_i \rangle$$

$$= \frac{1}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^\Delta} \sum_{T_{\tau,s}^k} \left(-\frac{1}{2} \right)^s \lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k} g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z}) \langle T_{\tau,s}^k \rangle_\beta$$

$$+ \dots, \qquad (2.7)$$

where

$$\langle T_{\tau,s}^k \rangle_\beta = \frac{1}{Z(\beta)} \sum_i e^{-\beta \Delta_i} \lambda_{\mathcal{O}_i \mathcal{O}_i T_{\tau,s}^k} \qquad (2.8)$$

– 4 –

is the finite temperature one-point function on the sphere of the $T_{\tau,s}^k$ operator and the dots denote contributions from other operators. In (2.8) $Z(\beta)$ is the partition function and the sum runs over all operators, including descendants.[6] Note that

$$\langle T_{\tau,s}^k \rangle_\beta = \beta^{-(\tau+s)} f_{\tau,s}^k(\beta). \tag{2.9}$$

Here and below the indices are suppressed (see e.g. [17] for the explicit form) and $f_{\tau,s}^k(\beta) \sim C_T^{k/2}$ is a theory-dependent nontrivial function of β which approaches a constant $f_{\tau,s}^k(0)$ in the large volume ($\beta \to 0$) limit.

Consider the thermalization of the stress tensor sector:

$$\langle \mathcal{O}_H | T_{\tau,s}^k | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T_{\tau,s}^k} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T_{\tau,s}^k \rangle_\beta. \tag{2.10}$$

Note that $T_{\tau,s}^k$ is unit-normalized, so all terms in (2.10) scale like $C_T^{k/2}$. Eq. (2.10) implies the equality between (2.6) and the corresponding term in (2.7). Note that the left-hand side of (2.10) is a function of the energy density while the right-hand side is a function of temperature. The relationship is fixed by considering the stress tensor case: the corresponding function $f_{d-2,2}^1(\beta)$ is determined by the free energy on the sphere (see section 6).

In the following, we will first discuss the case where the multi stress operators $T_{\tau,s}^k$ do not have any derivatives inserted, and then show that the derivatives do not change the conclusions. Assuming large-C_T factorization, the leading C_T behavior of $\langle T_{\tau,s}^k \rangle_\beta$ on the sphere is determined by that of the stress tensor. Schematically,

$$\langle T_{\tau,s}^k \rangle_\beta = c_{\tau,s}^k \, (\langle T_{d-2,2}^1 \rangle_\beta)^k + \dots, \tag{2.11}$$

where $c_{\tau,s}^k$ are numerical coefficients, which depend on k, τ, s, but are independent of the details of the theory and the dots stand for terms subleading in C_T^{-1}. By combining (2.11) and (2.10), one can formulate a universality condition

$$\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\tau,s}^k} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = c_{\tau,s}^k (\lambda_{\mathcal{O}_H \mathcal{O}_H T_{d-2,2}^1})^k = c_{\tau,s}^k \left(\frac{d}{1-d} \right)^k \frac{\Delta_H^k}{C_T^{\frac{k}{2}}}, \tag{2.12}$$

where the last equality follows from the stress tensor Ward identity for the three-point function which fixes $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{d-2,2}^1}$ ($T_{d-2,2}^1$ here is unit-normalized). In other words, thermalization and large-C_T factorization imply that the leading $\Delta^k/C_T^{k/2}$ behavior of the multi stress tensor OPE coefficients is completely fixed and given by (2.12) in all large-C_T CFTs.

In the paragraph above we considered multi stress tensor operators that did not contain any derivatives in them. However, the story largely remains the same when the derivatives are included, as long as their number does not scale with C_T. Indeed, the three-point function involving the stress-tensor with added derivatives, $\partial_\alpha \dots \partial_\beta T_{\mu\nu}$ still behaves like $\lambda_{\mathcal{O}_H \mathcal{O}_H \partial_\alpha \dots \partial_\beta T_{\mu\nu}} \simeq \Delta_H/\sqrt{C_T}$ up to a theory-independent coefficient. Hence, (2.12) still holds, provided thermalization and large-C_T factorization hold on the sphere.

---

[6]The corresponding conformal blocks can be obtained in the usual way by applying the quadratic conformal Casimir and solving the resulting differential equation [77].

Note that due to conformal invariance, correlators on the sphere depend on R only through the ratio β/R. Moreover, in the large volume limit, factors of R need to drop out of (2.6) and (2.7) to have a well defined limit. To see this we use that $(1 - z) \to 0$ and $(1 - \bar{z}) \to 0$ when $R \to \infty$ and the conformal blocks behave as (see e.g. [6])

$$
\begin{aligned}
g_{\tau,s}^{(0,0)}(1 - z, 1 - \bar{z}) &\sim \mathcal{N}_{d,s}[(1 - z)(1 - \bar{z})]^{\frac{\tau+s}{2}} C_s^{(d/2-1)} \left(\frac{(1 - z) + (1 - \bar{z})}{2\sqrt{(1 - z)(1 - \bar{z})}} \right) \\
&\sim \mathcal{N}_{d,s} \frac{|x|^{\tau+s}}{R^{\tau+s}} C_s^{(d/2-1)} \left(\frac{x_E^0}{|x|} \right),
\end{aligned}
\tag{2.13}
$$

where $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2}$, $C_s^{(d/2-1)}(\frac{x_E^0}{|x|})$ is a Gegenbauer polynomial and $\mathcal{N}_{d,s} = \frac{s!}{(d/2-1)_s}$. Including the factor $[(1 - z)(1 - \bar{z})]^{-\Delta}$ from (2.6) in (2.13) this agrees with the thermal block on $S^1 \times \mathbf{R}^{d-1}$ in [13]. Now from the thermalization of the stress tensor we will find in the large volume limit that

$$
\frac{\Delta_H}{C_T} \propto \left(\frac{R}{\beta} \right)^d,
\tag{2.14}
$$

and from (2.12) and (2.13) it follows that

$$
g_{\tau,s}^{(0,0)}(1 - z, 1 - \bar{z})\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k}\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\tau,s}^k}\Big|_{\frac{\Delta_H^k}{C_T^k}} \propto R^{dk-(\tau+s)}\beta^{-dk}.
\tag{2.15}
$$

The dimension of multi stress tensors $T_{\tau,s}^k$ is given by $\tau + s = dk + n$ where $n = 0, 2, \ldots$. Therefore, the only multi stress tensors that contribute in the large volume limit have dimensions dk. Restoring R in (2.6)–(2.7) and inserting (2.15) one finds that R drops out in the large volume limit. The correct dependence $\beta^{-(\tau+s)}$ from (2.9) in the $R \to \infty$ limit is also recovered in (2.6) using (2.15). The multi stress tensor operators that contribute in the large volume limit are therefore of the schematic form $T_{\mu_1\nu_1}T_{\mu_2\nu_2}\cdots T_{\mu_k\nu_k}$ with arbitrarily many contractions and no derivatives.

In holographic theories thermalization and the Wilson line prescription for the correlator allows one to compute the universal part of the OPE coefficients (see [66, 78] for explicit computations in the $d = 4$ case). It is also easy to check explicitly that the universality (2.12) holds for holographic theories with a Gauss-Bonnet gravitational coupling added. While the statement was shown to be true for the leading twist OPE coefficients in [63], it was not immediately obvious for multi stress tensors of non-minimal twist. Some such OPE coefficients were computed in [63, 71]. (See e.g. eqs. (5.48), (5.51), (5.52), (5.57) and (D.1)-(D.5) in [71]). Indeed, the leading $\Delta^k/C_T^{k/2}$ behavior of these OPE coefficients is independent of the Gauss-Bonnet coupling.

What about a general large-C_T theory? We first consider the OPE coefficients of double-stress tensors. To this end, consider the four point function[7] $\langle \mathcal{O}T_{\mu\nu}T_{\rho\sigma}\mathcal{O} \rangle$ where $\mathcal{O}$ is a scalar operator with scaling dimension Δ. In the direct channel $\mathcal{O}\times\mathcal{O} \to \mathcal{O}' \to T_{\mu\nu}\times T_{\rho\sigma}$ for finite Δ and large C_T, the leading contribution in the large-C_T limit comes from the

---

[7]This correlator for finite Δ was recently considered in holographic CFTs with $\Delta_{\text{gap}} \gg 1$ and $\Delta \ll \Delta_{\text{gap}}$ in [73].

identity operator $\mathcal{O} \times \mathcal{O} \to \mathbf{1} \to T_{\mu\nu} \times T_{\rho\sigma}$. The subleading contributions in the direct-channel are due to single trace operators as well as double trace operators made out of the external operators of the schematic form $T_{\tau,s}^2$ and $[\mathcal{O}\mathcal{O}]_{n,l} =: \mathcal{O}\partial^{2n}\partial_1 \ldots \partial_l \mathcal{O} :$. The exchange of the identity operator is reproduced in the cross-channel $\mathcal{O} \times T_{\mu\nu} \to [\mathcal{O}T_{\alpha\beta}]_{n,l} \to \mathcal{O} \times T_{\rho\sigma}$ by mixed double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ with OPE coefficients fixed by the MFT [79–81]. The subleading contributions in $1/C_T$ are then due to corrections to the anomalous dimension and OPE coefficients of $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ and single trace operators in the $\mathcal{O} \times T_{\mu\nu}$ OPE. An important example of the latter is the exchange of the single trace operator $\mathcal{O}$, whose contribution is universally fixed by the stress tensor Ward identity to be $(\lambda_{\mathcal{O}T_{d-2,2}^1\mathcal{O}})^2 \propto \Delta^2/C_T$ times the conformal block. This gives a universal contribution to $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}$ as was also noted in [73].

We now want to consider the case where $\Delta \sim C_T$ and study the OPE coefficients of the double-stress tensor operators in the $\mathcal{O} \times \mathcal{O}$ OPE. Firstly, note that the contribution from $T_{\tau,s}^2$ to the four-point function expanded in the direct channel is proportional to $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}\lambda_{TTT_{\tau,s}^2}$. The OPE coefficients $\lambda_{TTT_{\tau,s}^2}$ are fixed by the MFT and are independent of Δ and therefore the dependence on the scaling dimension comes solely from the OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}$. In the cross-channel, we analyze two kinds of contributions: from the exchanged operator $\mathcal{O}$ and from all other operators $\mathcal{O}' \neq \mathcal{O}$. From the operator $\mathcal{O}$ we get a universal contribution to the OPE coefficients in the direct channel $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}$, that we denote by $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}^{(1)}$. This contribution is universal since it only depends on $(\lambda_{\mathcal{O}T_{d-2,2}^1\mathcal{O}})^2 \propto \Delta^2/C_T$ in the cross-channel, which is fixed by the Ward identity. The contributions from other operators $\mathcal{O}'$ to the same OPE coefficient will be denoted by $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}^{(2)}$, such that $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2} = \lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}^{(1)} + \lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}^{(2)}$. Note that it also follows from the stress tensor Ward identity that the only scalar primary that appears in the cross-channel is $\mathcal{O}$. The operator $\mathcal{O}'$ therefore necessarily has spin $s \neq 0$.

To prove universality we need to show that $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^2}^{(2)} \ll \Delta^2/C_T$ in limit $1 \ll \Delta \propto C_T$ by studying the Δ dependence of the OPE coefficients $\lambda_{\mathcal{O}T_{d-2,2}^1\mathcal{O}'}$ in the cross-channel. For operators $\mathcal{O}'$, such that $\Delta_{\mathcal{O}'} \ll \Delta$, we expect that these OPE coefficients are heavily suppressed. It would be interesting to understand if one could put a general bound on the contribution of these operators in the cross-channel in any large-C_T theory. On the other hand, assuming thermalization, the OPE coefficients due to operators $\mathcal{O}'$ such that $\Delta_{\mathcal{O}'} \sim \Delta$ have been calculated in [23]. The obtained results are in agreement with our expectation, namely, these OPE coefficients are suppressed in $1 \ll \Delta \propto C_T$ limit. Additionally, in the cross-channel we have double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$, whose OPE is fixed by the MFT and it does not get Δ-enhanced. Note that in holographic theories with a large gap, $1 \ll \Delta_{\text{gap}} \ll C_T$, in the regime $\Delta \ll \Delta_{\text{gap}}$ there is a coupling $\lambda_{\mathcal{O}T_{d-2,2}^1 T_{d-2,2}^1}$ which scales like $\frac{1}{\Delta_{\text{gap}}^2\sqrt{C_T}}$ and its contribution to multi-stress tensor OPE coefficients was studied in [73]. This is different from the regime considered in this paper where $\Delta \gg \Delta_{\text{gap}}$.

One can iteratively extend the argument given here to multi stress tensors operators (with $k > 2$) by considering multi stress tensors as external operators. For example, to argue the universality of $\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^3}$ one may consider $\langle \mathcal{O}T_{d-2,2}^1 T_{\tau,s}^2 \mathcal{O}\rangle$. The bootstrap

argument above can be applied again by using the fact that OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ are universal, and the OPE coefficients $\lambda_{\mathcal{O}T^2_{\tau,s}\mathcal{O}'}$ are again expected to be subleading.

3 OPE coefficients in the free adjoint scalar model

In this section we consider a four-dimensional theory of a free scalar in the adjoint representation of SU(N), see [82–87] for related work. The relation between N and the central charge C_T in this theory is [88]

$$C_T = \frac{4}{3}(N^2 - 1), \tag{3.1}$$

and we consider the large-N (large-C_T) limit. The propagator for the scalar field $\phi^i{}_j$ is given by

$$\langle \phi^i{}_j(x)\phi^k{}_l(y)\rangle = \left(\delta^i{}_l\delta^k{}_j - \frac{1}{N}\delta^i{}_j\delta^k{}_l\right)\frac{1}{|x-y|^2}. \tag{3.2}$$

A single trace scalar operator with dimension Δ is given by

$$\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta}N^{\frac{\Delta}{2}}} : Tr(\phi^\Delta) : (x), \tag{3.3}$$

where $:\dots:$ denotes the oscillator normal ordering and the normalization is fixed by

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{3.4}$$

The CFT data that we compute in this section are the OPE coefficients of multi stress tensors in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. Assuming we can take $\Delta \to \Delta_H \sim C_T$, the large-$\Delta$ limit of these OPE coefficients is shown to be universal. One may worry that for $\Delta_H \sim C_T$ we can no longer trust the planar expansion, but, as we show in appendix C, the large-Δ limit of the planar result yields the correct expression even for $\Delta_H \sim C_T$.

3.1 Stress tensor

The stress tensor operator is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x), \tag{3.5}$$

where the normalization

$$\langle T^{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{|x|^8}\left(I^{(\mu}{}_\rho(x)I^{\nu)}{}_\sigma(x) - (\text{traces})\right), \tag{3.6}$$

with $I^\mu{}_\nu(x) := \delta^\mu{}_\nu - \frac{2x^\mu x_\nu}{|x|^2}$. The OPE coefficient is fixed by the stress tensor Ward identity to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^1_{2,2}} = -\frac{4\Delta}{3\sqrt{C_T}}. \tag{3.7}$$

It is also useful to find (3.7) using Wick contractions since an analogous calculation will be necessary for multi stress tensors. We do this explicitly in appendix A.

3.2 Double-stress tensor with minimal twist

In this section we study the minimal-twist composite operator made out of two stress tensors

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - \text{(traces)}, \tag{3.8}$$

with the normalization

$$\langle (T^2)^{\mu\nu\rho\sigma}(x)(T^2)_{\kappa\lambda\delta\omega}(0) \rangle = \frac{1}{|x|^{16}} \left(I^{(\mu}{}_\kappa I^\nu{}_\lambda I^\rho{}_\delta I^{\sigma)}{}_\omega - \text{(traces)} \right). \tag{3.9}$$

Consider the following three-point function

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4} \left(Z_\mu Z_\nu Z_\rho Z_\sigma - \text{(traces)} \right), \tag{3.10}$$

where $Z^\mu = \frac{x_{13}^\mu}{|x_{13}|^2} - \frac{x_{12}^\mu}{|x_{12}|^2}$. It is shown in appendix A that the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{3.11}$$

Evaluating $P^{(HH,LL)}_{T^2_{4,4}}$ defined by (2.5) in the large-Δ limit,[8] we obtain

$$
\begin{aligned}
P^{(HH,LL)}_{T^2_{4,4}} &= \left(-\frac{1}{2} \right)^4 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} \Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\
&= \frac{8}{81}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta \right) \right) = \mu^2 \left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta) \right),
\end{aligned}
\tag{3.12}
$$

where we use the following relation

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{3.13}$$

The result (3.12) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [66, 68].

3.3 Double-stress tensor with minimal twist and spin $s = 6$

We consider double-stess tensor operator with two (uncontracted) derivatives inserted

$$(T^2)_{\mu\nu\rho\sigma\eta\kappa}(x) = \frac{1}{2\sqrt{182}} : \left(T_{(\mu\nu}\partial_\rho\partial_\sigma T_{\eta\kappa)}(x) - \frac{7}{6}\left(\partial_{(\rho}T_{\mu\nu} \right)\left(\partial_\sigma T_{\eta\kappa)} \right)(x) - \text{(traces)} \right) : . \tag{3.14}$$

Using the conformal algebra eq. (C.2), it is straightforward to check that this operator is primary. It is unit-normalized such that

$$\langle (T^2)^{\mu\nu\rho\sigma\eta\kappa}(x)(T^2)_{\alpha\beta\gamma\delta\xi\epsilon}(0) \rangle = \frac{1}{|x|^{20}} \left(I^{(\mu}{}_\alpha I^\nu{}_\beta I^\rho{}_\gamma I^\sigma{}_\delta I^\eta{}_\xi I^{\kappa)}{}_\epsilon - \text{(traces)} \right). \tag{3.15}$$

---

[8]By the large-Δ limit, we strictly speaking mean $1 \ll \Delta \ll C_T$. However in this paper we often extrapolate this to the $\Delta \sim C_T$ regime.

By a calculation similar to those summarized in appendix A, we observe that the OPE coefficient of $(T^2)_{\mu\nu\rho\sigma\eta\kappa}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{4,6}^2} = \frac{8}{3}\sqrt{\frac{2}{91}}\frac{\Delta(\Delta - 1)}{C_T}. \tag{3.16}$$

Evaluating $P_{T_{4,6}^2}^{(HH,LL)}$, defined by (2.5), in the large-Δ limit, we obtain

$$\begin{aligned}
P_{T_{4,6}^2}^{(HH,LL)} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H \mathcal{O}_H T_{4,6}^2} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{4,6}^2}\bigg|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\
&= \frac{2}{819}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{1164800} + \mathcal{O}(\Delta)\right).
\end{aligned} \tag{3.17}$$

The result (3.17) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [66, 68].

3.4 Minimal-twist multi stress tensors

We now consider multi stress tensors $T_{2k,2k}^k$. Just like the double stress tensor ($k = 2$), we show that these have universal OPE coefficients in the large-Δ limit for any k.

Consider the unit-normalized minimal-twist multi stress tensor operator given by

$$(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}(x) = \frac{1}{\sqrt{k!}} : T_{(\mu_1\mu_2}T_{\mu_3\mu_4}\cdots T_{\mu_{2k-1}\mu_{2k})} : (x) - (\text{traces}). \tag{3.18}$$

The OPE coefficient of $(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, in the large-C_T limit is given by[9]

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{2k,2k}^k} = \left(-\frac{4}{3}\right)^k \frac{1}{\sqrt{k!}C_T^{k/2}}\frac{\Gamma(\Delta + 1)}{\Gamma(\Delta - k + 1)}. \tag{3.19}$$

First, we write $P_{T_{6,6}^3}^{(HH,LL)}$, defined by (2.5), in the large-Δ limit. We obtain this OPE coefficient from (3.19) for $k = 3$,

$$\begin{aligned}
P_{T_{6,6}^3}^{(HH,LL)} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H \mathcal{O}_H T_{6,6}^3} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{6,6}^3}\bigg|_{\left(\frac{\Delta_H}{C_T}\right)^3} \\
&= \frac{32}{2187}\frac{\Delta_H^3}{C_T^3}\left(\Delta^3 + \mathcal{O}(\Delta^2)\right) = \mu^3\left(\frac{\Delta^3}{10368000} + \mathcal{O}(\Delta^2)\right).
\end{aligned} \tag{3.20}$$

The result (3.20) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [68].

---

[9]See appendix A for detailed computations of similar OPE coefficients.

Additionally, we consider the OPE coefficient $P^{(HH,LL)}_{T^k_{2k,2k}}$ in the large-Δ limit for general k,

$$
\begin{aligned}
P^{(HH,LL)}_{T^k_{2k,2k}} &= \left(-\frac{1}{2}\right)^{2k} \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{2k,2k}} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{2k,2k}} \Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^k} \\
&= \frac{1}{k!} \left(\frac{2}{3}\right)^{2k} \frac{\Delta^k_H}{C^k_T} \left(\Delta^k + \mathcal{O}\left(\Delta^{k-1}\right)\right) = \mu^k \left(\frac{\Delta^k}{120^k k!} + \mathcal{O}(\Delta^{k-1})\right).
\end{aligned}
\tag{3.21}
$$

If we consider the limit $1 - \bar{z} \ll 1 - z \ll 1$, such that $\mu(1 - \bar{z})(1 - z)^3$ is held fixed, only operators $T^k_{2k,2k}$ contribute to the heavy-heavy-light-light four-point function given by eq. (2.3). The conformal blocks of $T^k_{2k,2k}$ in this limit are given by

$$
g^{(0,0)}_{2k,2k}(1 - z, 1 - \bar{z}) \approx (1 - \bar{z})^k (1 - z)^{3k},
\tag{3.22}
$$

and we can sum all contributions in eq. (2.4) explicitly to obtain

$$
\mathcal{G}(z, \bar{z}) \approx \frac{1}{((1 - z)(1 - \bar{z}))^\Delta} e^{\frac{\mu\Delta}{120}(1-\bar{z})(1-z)^3}.
\tag{3.23}
$$

Notice that the term in the exponential is precisely the stress-tensor conformal block in the limit $1 - \bar{z} \ll 1 - z \ll 1$ times its OPE coefficient. Therefore, the OPE coefficients (3.19) imply the exponentiation of stress-tensor conformal block. We conclude that these OPE coefficients are the same as the ones computed using holography and bootstrap in the limit of large Δ.

3.5 Double-stress tensors with non-minimal twist

So far we have shown that the minimal-twist multi stress tensor OPE coefficients are universal in the limit of large Δ. In this subsection, we extend this to show that the simplest non-minimal twist double-stress tensors also have universal OPE coefficients at large Δ.

The subleading twist double-stress tensor with twist $\tau = 6$ is of the schematic form $: T^\mu{}_\alpha T^{\alpha\nu} :$ and has dimension $\Delta = 8$ and spin $s = 2$. It is given by

$$
(T^2)_{\mu\nu}(x) = \frac{1}{\sqrt{2}} : T_{\mu\alpha} T^\alpha{}_\nu : (x) - (\text{trace}).
\tag{3.24}
$$

The normalization in (3.24) is again chosen such that $(T^2)_{\mu\nu}$ is unit-normalized, see appendix B for details.

The OPE coefficient of $(T^2)_{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is found from the three-point function in the large-C_T limit, for details see appendix B,

$$
\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) (T^2)^{\mu\nu}(x_3) \rangle = \frac{4\sqrt{2}\Delta(\Delta - 1)}{9C_T} \frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6},
\tag{3.25}
$$

from which we read off the OPE coefficient

$$
\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{6,2}} = \frac{4\sqrt{2}\Delta(\Delta - 1)}{9C_T}.
\tag{3.26}
$$

Evaluating $P_{T_{6,2}^2}^{(HH,LL)}$, defined by (2.5), in the large-Δ limit, we obtain

$$
\begin{aligned}
P_{T_{6,2}^2}^{(HH,LL)} &= \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_H \mathcal{O}_H T_{6,2}^2} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{6,2}^2}\Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\
&= \frac{8}{81}\frac{\Delta_H{}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2 \left(\frac{\Delta^2}{28800} + \mathcal{O}\left(\Delta\right)\right).
\end{aligned}
\tag{3.27}
$$

The result (3.27) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [71].

We further consider the scalar double-stress tensor with $\Delta = 8$ and spin $s = 0$ which is given by

$$
(T^2)(x) = \frac{1}{3\sqrt{2}} : T_{\mu\nu}T^{\mu\nu} : (x).
\tag{3.28}
$$

The three point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ is found in appendix B to be

$$
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8},
\tag{3.29}
$$

from which we read off the OPE coefficient

$$
\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{8,0}^2} = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}.
\tag{3.30}
$$

We write $P_{T_{8,0}^2}^{(HH,LL)}$ in the large-Δ limit

$$
\begin{aligned}
P_{T_{8,0}^2}^{(HH,LL)} &= \lambda_{\mathcal{O}_H \mathcal{O}_H T_{8,0}^2} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{8,0}^2}\Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\
&= \frac{8}{81}\frac{\Delta_H{}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2 \left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right).
\end{aligned}
\tag{3.31}
$$

The result (3.31) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [71].

4 Thermal one-point functions in the free adjoint scalar model

In this section we explicitly show that multi stress tensor operators thermalize in the free theory by calculating the thermal one-point function of some of these operators on $S^1 \times \mathbf{R}^3$. One-point functions of primary symmetric traceless operators at finite temperature are fixed by symmetry up to a dimensionless coefficient $b_\mathcal{O}$ (see e.g. [8, 13])

$$
\langle \mathcal{O}_{\mu_1 \cdots \mu_{s_\mathcal{O}}}\rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}}\left(e_{\mu_1}\cdots e_{\mu_{s_\mathcal{O}}} - (\text{traces})\right).
\tag{4.1}
$$

Here e_μ is a unit vector along the thermal circle.

To compare the thermal one-point functions and OPE coefficients from the previous section, we need to derive a relation between $\frac{\Delta_H}{C_T}$ and the temperature[10] β^{-1}. Here $\Delta_H \sim N^2$ refers to the scaling dimension of a heavy operator $\mathcal{O}_H$ with OPE coefficients given by the large-Δ limit of those obtained in section 3. One can relate the inverse temperature β to the parameter $\mu = \frac{160}{3} \frac{\Delta_H}{C_T}$ using the Stefan-Boltzmann's law $E/\text{vol}(S^3) = N^2 \pi^2/30\beta^4$. The energy of the state E is related to its conformal dimension Δ via $E = \Delta/R$. One can then use $\text{vol}(S^3) = 2\pi^2 R^3$ and the relation between N and C_T given by (3.1), to find

$$\mu = \frac{160}{3} \frac{\Delta_H}{C_T} = \frac{160}{3} E \frac{R}{C_T} = \frac{8}{3} \left(\frac{\pi R}{\beta} \right)^4. \tag{4.2}$$

4.1 Stress tensor

The thermal one-point function for the stress tensor $T_{2,2}^1 = T_{\mu\nu}$ is calculated in appendix D where we find that $b_{T_{2,2}^1}$ is given by

$$b_{T_{2,2}^1} = -\frac{2\pi^4 N}{15\sqrt{3}}. \tag{4.3}$$

Using (4.2) and (D.6) one arrives at

$$b_{T_{2,2}^1} \beta^{-4} = \lambda_{\mathcal{O}_H \mathcal{O}_H T_{2,2}^1}. \tag{4.4}$$

4.2 Double-stress tensor with minimal twist

In this section we calculate the thermal one-point function of the double-stress tensor operator with $\tau = 4$ and spin $s = 4$. The operator is written explicitly in (3.8). The leading contribution to the thermal one-point function of $(T^2)_{\mu\nu\rho\sigma}$ follows from the large-N factorization and is given by

$$\begin{aligned}
\langle (T^2)_{\mu\nu\rho\sigma} \rangle_\beta &= \frac{1}{\sqrt{2}} \langle T_{(\mu\nu} \rangle_\beta \langle T_{\rho\sigma)} \rangle_\beta - (\text{traces}) \\
&= \frac{2\sqrt{2}\pi^8 N^2}{675\beta^8} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right).
\end{aligned} \tag{4.5}$$

Using the relation (4.2) and the OPE coefficient (3.11), we observe the thermalization of this operator,

$$b_{T_{4,4}^2} \beta^{-8} = \lambda_{\mathcal{O}_H \mathcal{O}_H T_{4,4}^2} \Big|_{\frac{\Delta_H^2}{C_T}}. \tag{4.6}$$

4.3 Minimal-twist multi stress tensors

Consider now multi stress tensors $T_{2k,2k}^k$ with twist $\tau = 2k$ and spin $s = 2k$. We show that these operators thermalize for any k by calculating their thermal one-point functions:

$$\langle (T^k)_{\mu_1\mu_2...\mu_{2k}} \rangle_\beta = \frac{b_{T_{2k,2k}^k}}{\beta^{4k}} (e_{\mu_1} e_{\mu_2} \cdots e_{\mu_{2k}} - (\text{traces})), \tag{4.7}$$

---

[10]See also section 6 and appendix D for alternative derivations.

where the leading behavior of $b_{T^k_{2k,2k}}$ follows from the large-N factorization:

$$b_{T^k_{2k,2k}} = \frac{1}{\sqrt{k!}}(b_{T^1_{2,2}})^k = \frac{(-\frac{2}{5})^k N^k \pi^{4k}}{3^{\frac{3k}{2}}\sqrt{k!}}. \tag{4.8}$$

Eqs. (4.2) and (4.8) may be combined to yield

$$b_{T^k_{2k,2k}}\beta^{-4k} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{2k,2k}}\Big|_{\frac{\Delta^k_H}{C_T^{k/2}}}. \tag{4.9}$$

4.4 Double-stress tensors with non-minimal twist

The subleading twist double-stress tensor is of the schematic form $: T^\mu_{\ \alpha} T^{\alpha\nu} :$ and has twist $\tau = 6$ and spin $s = 2$. The explicit form can be found in (3.24). The leading term in the thermal one-point function is given by

$$\begin{aligned}
\langle (T^2)^{\mu\nu}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}\rangle_\beta \langle T^\nu_{\ \alpha}\rangle_\beta - \text{(trace)} \\
&= \frac{b^2_{T^1_{2,2}}}{2\sqrt{2}\beta^8}\left(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}\right) \\
&= \frac{\sqrt{2}N^2\pi^8}{675\beta^8}\left(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}\right),
\end{aligned} \tag{4.10}$$

therefore,

$$b_{T^2_{6,2}} = \frac{\sqrt{2}N^2\pi^8}{675}. \tag{4.11}$$

Taking the large-Δ limit of the OPE coefficient in (3.26) and substituting (4.2), we observe thermalization,

$$b_{T^2_{6,2}}\beta^{-8} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^2_{6,2}}\Big|_{\frac{\Delta^2_H}{C_T}}. \tag{4.12}$$

We further consider the scalar double-stress tensor with $\tau = 8$ and $s = 0$ which is given by (3.28). The thermal one-point function for this operator is

$$\begin{aligned}
\langle (T^2)\rangle_\beta &= \frac{1}{3\sqrt{2}}\langle T_{\mu\nu}\rangle_\beta \langle T^{\mu\nu}\rangle_\beta \\
&= \frac{1}{3\sqrt{2}}\frac{3}{4}b^2_{T^1_{2,2}}\beta^{-8} = \frac{\pi^8 N^2}{675\sqrt{2}\beta^8},
\end{aligned} \tag{4.13}$$

where the factor of $\frac{3}{4}$ in the first line comes from the index contractions. Hence,

$$b_{T^2_{8,0}} = \frac{\pi^8 N^2}{675\sqrt{2}}. \tag{4.14}$$

Using (4.14), (3.30) and (4.2), we again observe thermalization,

$$b_{T^2_{8,0}}\beta^{-8} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^2_{8,0}}\Big|_{\frac{\Delta^2_H}{C_T}}. \tag{4.15}$$

4.5 Triple-stress tensors with non-minimal twist

We consider the triple stress tensors with $\tau = 8, s = 4$ and $\tau = 10, s = 2$. The unit-normalized triple stress tensor with $\tau = 8$ can be written as

$$(T^3)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{3}} \left(: T_{(\mu\nu}T_{\rho|\alpha|}T^{\alpha}{}_{\sigma)} : (x) - (\text{traces}) \right), \tag{4.16}$$

where $|\alpha|$ denotes that index α is excluded from the symmetrization. The thermal one-point function follows from large-N factorization

$$\begin{aligned}
\langle (T^3)_{\mu\nu\rho\sigma} \rangle_\beta &= \frac{1}{\sqrt{3}} \left(\langle T_{(\mu\nu} \rangle_\beta \langle T_{\rho|\alpha|} \rangle_\beta \langle T^{\alpha}{}_{\sigma)} \rangle_\beta - (\text{traces}) \right) \\
&= \frac{1}{2\sqrt{3}} \frac{b^3_{T^1_{2,2}}}{\beta^{12}} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right) \\
&= -\frac{4\pi^{12}N^3}{30375\beta^{12}} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right),
\end{aligned} \tag{4.17}$$

therefore,

$$b_{T^3_{8,4}} = -\frac{4\pi^{12}N^3}{30375}. \tag{4.18}$$

The OPE coefficient of the operator with same quantum numbers ($\Delta = 12$, $s = 4$) is calculated holographically and is given by (D.1) in [71]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{8,4}} = \left(-\frac{1}{2} \right)^4 \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{8,4}} \lambda_{\mathcal{O}_H \mathcal{O}_H T^3_{8,4}} \Bigg|_{\left(\frac{\Delta_H}{C_T} \right)^3} = \frac{64}{2187} \frac{\Delta_H^3 \Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{4.19}$$

Now, one can easily read-off $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{8,4}}$ in the large-Δ limit

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{8,4}} = -\frac{32\Delta^3}{27\sqrt{3}C_T^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{4\Delta^3}{9N^3} + \mathcal{O}(\Delta^2), \tag{4.20}$$

where we use the relation between central charge C_T and N given by (3.1). Using (4.2) one can obtain

$$b_{T^3_{8,4}} \beta^{-12} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^3_{8,4}} \Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{4.21}$$

We also consider the triple stress tensors with quantum numbers $\Delta = 12$ and $s = 2$. There are two linearly independent such operators that schematically can be written as $: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} :$ and $: T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} :$. We write the following linear combinations of these operators

$$(T^3)_{\mu\nu}(x) = \frac{1}{10\sqrt{2}} \left(: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} : (x) + 4 : T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} : (x) - (\text{trace}) \right), \tag{4.22}$$

$$(\tilde{T}^3)_{\mu\nu}(x) = \frac{7}{20} \left(: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} : (x) - \frac{12}{7} : T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} : (x) - (\text{trace}) \right). \tag{4.23}$$

Both $(T^3)_{\mu\nu}$ and $(\tilde{T}^3)_{\mu\nu}$ are unit-normalized and their overlap vanishes in the large-N limit

$$\langle (T^3)_{\mu\nu}(x)(\tilde{T}^3)^{\rho\sigma}(y)\rangle = \mathcal{O}(1/N^2). \tag{4.24}$$

The thermal one-point functions of these operators, obtained by large-N factorization, in the large-N limit are given by

$$\langle (T^3)_{\mu\nu}\rangle_\beta = -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125\beta^{12}}\left(e_\mu e_\nu - (\text{trace})\right),$$
$$\langle (\tilde{T}^3)_{\mu\nu}\rangle_\beta = \mathcal{O}(N), \tag{4.25}$$

therefore,

$$b_{T^3_{10,2}} = -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125},$$
$$b_{\tilde{T}^3_{10,2}} = 0. \tag{4.26}$$

The holographic OPE coefficient of the operator with the same quantum numbers ($\Delta = 12$, $s = 2$), with external scalar operators is given by (5.57) in [71]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{10,2}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{10,2}}\lambda_{\mathcal{O}_H \mathcal{O}_H T^3_{10,2}}\Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{32}{729}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{4.27}$$

We can read-off $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{10,2}}$:

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{10,2}} = -\frac{8\sqrt{2}}{27}\frac{\Delta^3}{C_T^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{\sqrt{2}}{3\sqrt{3}}\frac{\Delta^3}{N^3} + \mathcal{O}(\Delta^2). \tag{4.28}$$

Again, using (4.2), one can confirm that this operator thermalizes

$$b_{T^3_{10,2}}\beta^{-12} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^3_{10,2}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{4.29}$$

5 Thermal two-point function and block decomposition

In this section we study the thermal two-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle_\beta$ and decompose it in thermal blocks. We determine the contributions of a few low-lying operators, including the stress tensor $T^1_{2,2}$ and the double stress tensor $T^2_{4,4}$. They exactly match the corresponding OPE coefficients and thermal expectation values computed in previous sections. Due to the presence of multiple operators with equal scaling dimension and spin, there is a mixing problem which we solve explicitly in a few cases. Related appendices include appendix F, where we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$ and appendix G, where we consider two-dimensional thermal two-point functions. In appendix H we do a similar analysis for the vector model in four dimensions.

5.1 Thermal two-point function of a single trace scalar operator

The correlator at finite temperature β^{-1} in the free theory can be calculated by Wick contractions using the propagators on $S^1 \times \mathbf{R}^3$. Explicitly, the two-point function at finite temperature is given by[11]

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \frac{\pi^4 \Delta(\Delta - 2)}{9\beta^4} \tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \dots, \tag{5.1}$$

where

$$\begin{aligned}
\tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^\infty \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\
&= \frac{\pi}{2\beta|\mathbf{x}|} \left[\mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\right) + \mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\right) \right].
\end{aligned} \tag{5.2}$$

The dots in (5.1) contain contributions due to further self-contractions which will not be important below.[12] Taking the $\beta \to \infty$ limit of (5.1) we can read off the decomposition of the two-point function in terms of thermal conformal blocks on $S^1 \times \mathbf{R}^3$ with coordinates $x = (x_E^0, \mathbf{x})$.

Following [13], if $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2} \le \beta$ the two-point function can be evaluated using the OPE:

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle_\beta = \sum_\mathcal{O} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}} |x|^{\tau - 2\Delta} x_{\mu_1} \cdots x_{\mu_{s_\mathcal{O}}} \langle \mathcal{O}^{\mu_1 \cdots \mu_{s_\mathcal{O}}} \rangle_\beta, \tag{5.3}$$

where $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}}$ is the OPE coefficient, τ and $s_\mathcal{O}$ is the twist and spin of $\mathcal{O}$, respectively. Using (4.1) together with (5.3), the two-point function on $S^1 \times \mathbf{R}^3$ can be organized in the following way [13]:

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle_\beta = \sum_{\mathcal{O}_{\tau,s} \in \mathcal{O}_\Delta \times \mathcal{O}_\Delta} \frac{a_{\mathcal{O}_{\tau,s}}}{\beta^{\Delta_\mathcal{O}}} \frac{1}{|x|^{2\Delta - \tau + s}} C_s^{(1)}\left(\frac{x_E^0}{|x|}\right), \tag{5.4}$$

where we sum over primary operators $\mathcal{O}_{\tau,s}$, with twist τ and spin s, appearing in the OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta \sim \mathcal{O}_{\tau,s} + \dots$. In (5.4) $C_s^{(1)}(x_E^0/|x|)$ is a Gegenbauer polynomial which, together with a factor of $|x|^{-2\Delta + \tau - s}$, forms a thermal conformal block in $d = 4$ dimensions and the coefficients $a_{\mathcal{O}_{\tau,s}}$ are given by

$$a_{\mathcal{O}_{\tau,s}} = \left(\frac{1}{2}\right)^s \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{\tau,s}} b_{\mathcal{O}_{\tau,s}}. \tag{5.5}$$

Expanding (5.1) for $\beta \to \infty$ one finds:

$$\begin{aligned}
\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle_\beta = \frac{1}{|x|^{2\Delta}} \Bigg[&1 + \frac{\pi^2 \Delta}{3\beta^2} |x|^2 \\
&+ \frac{\pi^4 \Delta}{90\beta^4} |x|^2 (3\mathbf{x}^2(5\Delta - 9) + (15\Delta - 19)(x_E^0)^2) + \mathcal{O}(\beta^{-6}) \Bigg].
\end{aligned} \tag{5.6}$$

---

[11]Here and below we assume that $\Delta > 4$. We further drop the disconnected term $\langle \mathcal{O}_\Delta \rangle_\beta^2 \sim N^2$.

[12]These terms will be proportional to $\beta^{-2a} \tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-a}$, with $a \ge 4$. When decomposed into thermal blocks, these will not affect the operators with dimension $\Delta < 8$ or $\Delta = 8$ with non-zero spin $s \ne 0$.

From the expansion (5.6), we can read off the coefficients $a_{\tau',s'} := \sum_{\mathcal{O}_{\tau',s'}} a_{\mathcal{O}_{\tau',s'}}$ where we sum over all operators with twist τ' and spin s':

$$a_{2,0} = \frac{\pi^2 \Delta}{3},$$
$$a_{4,0} = \frac{\pi^4 \Delta(3\Delta - 5)}{18}, \tag{5.7}$$
$$a_{2,2} = \frac{\pi^4 \Delta}{45}.$$

For future reference, expanding (5.1) to $\mathcal{O}(\frac{1}{\beta^8})$ one finds

$$a_{2,4} = \frac{2\pi^6 \Delta}{945},$$
$$a_{4,4} = \frac{\pi^8 \Delta(\Delta - 1)}{1050}. \tag{5.8}$$

Note that due to the mixing of operators with the same twist and spin, $a_{\tau,s}$ generically contains the contribution from multiple operators. In the following section we calculate the OPE coefficients and thermal one-point functions of operators which are not multi stress tensors but contribute to (5.7) and (5.8).

5.2 CFT data of scalar operators with dimensions two and four

We explicitly calculate the thermal one-point functions $\langle \mathcal{O} \rangle_\beta = b_\mathcal{O} \beta^{-\Delta_\mathcal{O}}$ and OPE coefficients $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}}$ for scalar operators $\mathcal{O}$ with twist $\tau' = 2$ and $\tau' = 4$ using Wick contractions. This is done to find which operators contribute to the thermal two-point function and to resolve a mixing problem.

For $\tau' = 2$ there is only one such operator, the single trace operator $\mathcal{O}_2(x) = \frac{1}{\sqrt{2N}} : Tr(\phi^2) : (x)$ given in (3.3). The OPE coefficient is found by considering the three-point correlator

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \mathcal{O}_2(x_3) \rangle = \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2}}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}. \tag{5.9}$$

The three-point function is calculated in appendix A, in the large-N limit, and it is given by

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \mathcal{O}_2(x_3) \rangle = \frac{\sqrt{2}\Delta}{N} \frac{1}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{5.10}$$

and therefore $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2} = \frac{\sqrt{2}\Delta}{N}$ to leading order in $1/N$. To calculate the thermal one-point function $\propto \langle Tr(\phi^2) \rangle_\beta$, we include self-contractions, i.e. contractions of fundamental fields within the same composite operator separated by a distance $m\beta$ along the thermal circle for $m \neq 0$ and integer. Explicitly, the one-point function of $\mathcal{O}_2$ is given by

$$\langle \mathcal{O}_2(x) \rangle_\beta = \frac{1}{\sqrt{2N}} \sum_{m \neq 0} \frac{N^2}{(m\beta)^2} = \frac{\pi^2 N}{3\sqrt{2}\beta^2}, \tag{5.11}$$

therefore,

$$b_{\mathcal{O}_2} = \frac{\pi^2 N}{3\sqrt{2}}. \tag{5.12}$$

The contribution to the thermal two-point function $a_{\mathcal{O}_2}$ is found using (5.10) and (5.12)

$$a_{2,0} = b_{\mathcal{O}_2} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2} = \frac{\pi^2 \Delta}{3}. \tag{5.13}$$

This agrees with (5.7) which was obtained from the thermal two-point function.

We now continue with scalar operators of twist four. There are two such linearly independent operators appearing in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. In order to construct an orthonormal basis, consider the following single and double trace operators:

$$\mathcal{O}_4(x) = \frac{1}{2N^2} : Tr(\phi^4) : (x),$$
$$\mathcal{O}_{4,\mathrm{DT}}(x) = \frac{1}{2\sqrt{2}N^2} : Tr(\phi^2) Tr(\phi^2) : (x). \tag{5.14}$$

We further construct the operator $\tilde{\mathcal{O}}_4$ that has vanishing overlap with $\mathcal{O}_{4,\mathrm{DT}}(x)$ as follows:

$$\tilde{\mathcal{O}}_4 = \mathcal{N}\left[\mathcal{O}_4 - c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}} \mathcal{O}_{4,\mathrm{DT}}\right], \tag{5.15}$$

with $\mathcal{N}$ a normalization constant and $c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}}$ is the overlap defined by

$$\langle \mathcal{O}_4(x) \mathcal{O}_{4,\mathrm{DT}}(y) \rangle = \frac{c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}}}{|x-y|^8}. \tag{5.16}$$

Explicit calculation gives $c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}} = \frac{2\sqrt{2}}{N}$ and $\mathcal{N} = \frac{1}{\sqrt{2}}$ in the large-N limit, and the scalar dimension four operator orthogonal to the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$ is therefore

$$\tilde{\mathcal{O}}_4 = \frac{1}{\sqrt{2}} \left[\mathcal{O}_4 - \frac{2\sqrt{2}}{N} \mathcal{O}_{4,\mathrm{DT}} \right]. \tag{5.17}$$

Note that even though the second term in (5.15) is suppressed by $1/N$, it can still contribute to the thermal two-point function due to the scaling of OPE coefficients and one-point function of a k-trace operator $\mathcal{O}^{(k)}$:

$$b_{\mathcal{O}^{(k)}} \sim N^k,$$
$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{(k)}} \sim \frac{1}{N^k}, \tag{5.18}$$

in the limit $N \to \infty$.

The one-point function and the OPE coefficient for $\mathcal{O}_4$ is found analogously to that of $\mathcal{O}_2$ in the large-N limit

$$b_{\mathcal{O}_4} = \frac{\pi^4 N}{9},$$
$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_4} = \frac{2\Delta}{N}. \tag{5.19}$$

Consider now the double trace operator given in (5.14). The one-point function factorizes in the large-N limit:

$$\langle \mathcal{O}_{4,\mathrm{DT}}(x) \rangle_\beta = \frac{1}{\sqrt{2}} (\langle \mathcal{O}_2(x) \rangle_\beta)^2$$
$$= \frac{\pi^4 N^2}{18\sqrt{2}\beta^4}. \tag{5.20}$$

Likewise, the OPE coefficient can be computed in the large-N limit (see appendix A)

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{4,\mathrm{DT}}} = \frac{\sqrt{2}\Delta(3\Delta - 5)}{N^2}. \qquad (5.21)$$

Consider now the thermal one-point function of $\tilde{\mathcal{O}}_4$ in (5.17)

$$\langle \tilde{\mathcal{O}}_4 \rangle_\beta = \frac{1}{\sqrt{2}\beta^4} \left[b_{\mathcal{O}_4} - \frac{2\sqrt{2}}{N} b_{\mathcal{O}_{4,\mathrm{DT}}} \right] \qquad (5.22)$$
$$= \mathcal{O}(N^{-1}),$$

where we have used (5.19) and (5.20). Since the corresponding OPE coefficient is suppressed by N^{-1}, it follows that the only scalar operator with dimension four contributing to the thermal two-point function is the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$. From the OPE coefficient and thermal one-point function of this double trace operator, using (5.20) and (5.21), we find the following contribution to the thermal two-point function

$$a_{4,0} = \frac{\pi^4 \Delta(3\Delta - 5)}{18}, \qquad (5.23)$$

which agrees with (5.7).

5.3 CFT data of single-trace operator with twist two and spin four

The primary single trace operator $\Xi = \mathcal{O}_{2,4}$ with twist $\tau = 2$ and spin $s = 4$ is given by

$$\Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} : Tr(\phi(\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi) - 16(\partial_{(\mu}\phi)(\partial_\nu \partial_\rho \partial_{\sigma)}\phi) \qquad (5.24)$$
$$+ 18(\partial_{(\mu}\partial_\nu \phi)(\partial_\rho \partial_{\sigma)}\phi) - (\text{traces})) : (x).$$

The relative coefficients follow from requiring that the operator is a primary, see appendix E for details.

The thermal one-point function of this operator is found from Wick contractions in the large-N limit to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8\pi^6 N}{27\sqrt{35}\beta^6} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right). \qquad (5.25)$$

Moreover, the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE can again be calculated using Wick contractions similarly to how it was done for $T_{4,4}^2$ in appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \qquad (5.26)$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \qquad (5.27)$$

Now, it is easy to check that

$$\frac{1}{2^4} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6 \Delta}{945}, \qquad (5.28)$$

which agrees with $a_{2,4}$ in (5.8).

5.4 CFT data of double-trace operators with twist and spin equal to four

To find the full contribution to the thermal two-point function from the operators with $\tau = 4$ and $s = 4$ we need to take into account the contribution of all operators with these quantum numbers and solve a mixing problem. In addition to the double-stress tensor operator with these quantum numbers, the other double trace primary operator which contributes is given by

$$
\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} : Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\
+18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big) : (x),
\tag{5.29}
$$

where the operator is unit-normalized. Notice that this is the double trace operator obtained by taking the normal ordered product of two single trace operators, the scalar operator with dimension 2 and the single trace spin-4 operator with dimension 6. There are more double trace operators with these quantum numbers which are, however, not simply products of single trace operators. These do not contribute to the thermal two-point function to leading order in $\frac{1}{N^2}$ (see appendix F).

Note that it follows from large-N factorization that the overlap of this operator with $(T^2)_{\mu\nu\rho\sigma}$ is suppressed by powers of $\frac{1}{N}$; since both of these are double trace operators and obey the scaling (5.18), to study the thermal two-point function to leading order in N^2, one can therefore neglect this overlap.

The thermal one-point function of $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ follows from the large-N factorization and we find that

$$
b_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\pi^8 N^2}{81},
\tag{5.30}
$$

where we used the thermal one-point functions for each single trace operator given by (5.11) and (5.25). The OPE coefficient is calculated in appendix A,

$$
\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta - 1)}{N^2}.
\tag{5.31}
$$

Using the thermal one point function and the OPE coefficient in (5.30) and (5.31) respectively, it is found that it the operator $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ gives the following contribution to the thermal two point function:

$$
a_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \left(\frac{1}{2}\right)^4 b_{\mathcal{O}^{\mathrm{DT}}_{4,4}}\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \frac{2\pi^8\Delta(\Delta - 1)}{2835}.
\tag{5.32}
$$

The total contribution from $T^2_{4,4}$ together with that of $\mathcal{O}^{\mathrm{DT}}_{4,4}$, using (3.11), (4.5) and (5.32), is

$$
a_{4,4} = (a_{T^2_{4,4}} + a_{\mathcal{O}^{\mathrm{DT}}_{4,4}}) = \frac{\pi^8\Delta(\Delta - 1)}{1050}.
\tag{5.33}
$$

This agrees with $a_{4,4}$ in (5.8).

6 Comparison with the eigenstate thermalization hypothesis

In this section we discuss the relation of our results to the eigenstate thermalization hypothesis (ETH). We argue that the stress tensor sector of the free $SU(N)$ adjoint scalar theory in $d = 4$ satisfies the ETH to leading order in $C_T \sim N^2 \gg 1$. We explain the equivalence of the micro-canonical and canonical ensemble when $\Delta_H \sim C_T$ in large-C_T theories. In this regime, the diagonal part of the ETH is (up to exponentially suppressed terms which we do not consider), equivalent to thermalization. Note that in two dimensions the Virasoro descendants of the identity satisfy the ETH (see e.g. [42] for a recent discussion).

We begin by showing the equivalence between the micro-canonical and the canonical ensemble on $S^1_\beta \times S^{d-1}$ when $\Delta_H \sim C_T \gg 1$. See [9, 11, 12, 14, 89] for a similar discussion at infinite volume as well as [47] in the two-dimensional case. The expectation value in the micro-canonical ensemble of an operator $\mathcal{O}$, which we take to be a scalar for simplicity, at energy $E = \Delta_H/R$ is given by

$$\langle \mathcal{O} \rangle_E^{(\text{micro})} = \frac{1}{N(E)} \sum_{\tilde{\mathcal{O}}} \langle \tilde{\mathcal{O}} | \mathcal{O} | \tilde{\mathcal{O}} \rangle, \tag{6.1}$$

where we sum over states $|\tilde{\mathcal{O}}\rangle$ with energy $(E, E + \delta E)$ and $N(E)$ is the number of states in this interval. On the other hand, consider the partition function at inverse temperature β given by

$$Z(\beta) = \sum_{\tilde{\mathcal{O}}} e^{-\frac{\beta\tilde{\Delta}}{R}} = \int d\tilde{\Delta}\rho(\tilde{\Delta}) e^{-\frac{\beta\tilde{\Delta}}{R}}, \tag{6.2}$$

where we sum over all states in the theory. In the second line in (6.2) we have approximated the sum of delta-functions by a continuous function $\rho(\tilde{\Delta})$. Expectation values in the canonical ensemble is then computed by[13]

$$\langle \mathcal{O} \rangle_\beta = Z(\beta)^{-1} \int d\tilde{\Delta}\rho(\tilde{\Delta})\langle \mathcal{O} \rangle_E^{(\text{micro})} e^{-\frac{\beta\tilde{\Delta}}{R}}. \tag{6.3}$$

Consider the partition function in (6.2) with a free energy $F = -\beta^{-1} \log Z(\beta)$. By an inverse Laplace transform of (6.2) we find the density of states

$$\rho(\Delta_H) = \frac{1}{2\pi i R} \int d\beta' e^{\beta'\left(\frac{\Delta_H}{R} - F(\beta')\right)}. \tag{6.4}$$

For $\Delta_H \sim C_T$ and a large free energy[14] $F \sim C_T$, we can evaluate (6.4) using a saddlepoint approximation with the saddle at β given by

$$\frac{\Delta_H}{R} = \partial_{\beta'}(\beta' F)|_\beta. \tag{6.5}$$

Consider now the thermal expectation value in (6.3), multiplying both sides by $Z(\beta)$ and doing an inverse Laplace transform evaluated at $\Delta_H \sim C_T$ we find

$$\rho(\Delta_H)\langle \mathcal{O} \rangle_{\Delta_H/R}^{(\text{micro})} = \frac{1}{2\pi i R} \int d\beta' \langle \mathcal{O} \rangle_{\beta'} e^{\beta'\left(\frac{\Delta_H}{R} - F(\beta')\right)}. \tag{6.6}$$

---

[13]It was argued in [14] that the existence of the thermodynamic limit implies that we only need to sum over operators with low spin.

[14]We consider a CFT in a high temperature phase.

For $F \sim C_T \gg 1$ we again use a saddlepoint approximation to evaluate (6.6) with the saddle at β determined by (6.5), assuming $\langle \mathcal{O} \rangle_{\beta'}$ does not grow exponentially with C_T. The r.h.s. of (6.6) is therefore the thermal expectation value $\langle \mathcal{O} \rangle_\beta$ multiplied by the saddlepoint approximation of the density of states in (6.4). It then follows that

$$\langle \mathcal{O} \rangle^{(\text{micro})}_{\Delta_H / R} \approx \langle \mathcal{O} \rangle_\beta, \tag{6.7}$$

with β determined by (6.5). In particular, in the infinite volume limit $R \to \infty$, the free energy is given by[15]

$$F = \frac{b_{T^{(\text{can})}_{\mu\nu}} S_d R^{d-1}}{d\beta^d}, \tag{6.8}$$

where $S_d = Vol(S^{d-1}) = 2\pi^{\frac{d}{2}} / \Gamma(\frac{d}{2})$. Inserting (6.8) in (6.5) we find [9]

$$\frac{\beta}{R} = \left(\frac{-(d-1) b_{T^{(\text{can})}_{\mu\nu}} S_d}{d\Delta_H} \right)^{\frac{1}{d}}. \tag{6.9}$$

We can use (6.7) to see the thermalization of the stress tensor. The free energy is related to the expectation value of the stress tensor $T^{(\text{can})}_{\mu\nu}$ [5]

$$\langle T^{(\text{can})}_{00} \rangle_\beta = \frac{1}{S_d R^{d-1}} \partial_\beta (-\beta F(\beta)). \tag{6.10}$$

On the other hand, the expectation value of $T^{(\text{can})}_{00}$ in a heavy state $|\mathcal{O}_H\rangle$ is fixed by the Ward identity to be

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H \rangle = -\frac{\Delta_H}{S_d R^d}. \tag{6.11}$$

Multiplying (6.5) with $(S_d R^{d-1})^{-1}$ and comparing with (6.10)–(6.11) we find that

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H \rangle = \langle T^{(\text{can})}_{00} \rangle_\beta. \tag{6.12}$$

This shows the thermalization of the stress tensor in heavy states where $F \sim \Delta_H \sim C_T$ in large-C_T theories. Note that this follows from (6.7) since we can replace the micro-canonical expectation value at $E = \Delta_H / R$, on the l.h.s., with the expectation value in any single heavy state with dimension Δ_H due to the Ward identity, independent of the heavy state. Put differently, the stress tensor satisfies the ETH as we will review below.

We now consider the eigenstate thermalization hypothesis for CFTs at finite temperature on the sphere S^{d-1} of radius R. The diagonal part of the ETH is given by

$$\langle \mathcal{O}_H | \mathcal{O}_{\tau, s} | \mathcal{O}_H \rangle = \langle \mathcal{O}_{\tau, s} \rangle^{(\text{micro})}_E + \mathcal{O} \left(e^{-S(E)} \right), \tag{6.13}$$

where $\mathcal{O}_H$ and $\mathcal{O}_{\tau, s}$ are local primary operators and $\langle \mathcal{O}_{\tau, s} \rangle^{(\text{micro})}_E$ is the expectation value of $\mathcal{O}_{\tau, s}$ in the micro-canonical ensemble at energy $E = \frac{\Delta_H}{R}$. Here we assume that the

---

[15]Here we denote the canonically normalized stress tensor by $T^{(\text{can})}_{\mu\nu}$, whose two-point function is given by $\langle T^{(\text{can})\mu\nu}(x) T^{(\text{can})}_{\rho\sigma}(y) \rangle = \frac{C_T}{S_d^2} (I^\mu_{(\rho} I^\nu_{\sigma)} - (\text{trace}))$.

operator $\mathcal{O}_H$ is a heavy scalar operator with large conformal dimension $\Delta_H \propto C_T \gg 1$. The operator $\mathcal{O}_{\tau,s}$ on the other hand can have non-zero spin.[16] In (6.13), $e^{S(E)}$ is the density of states at energy $E = \Delta_H/R$. As shown in (6.7), in the limit $\Delta_H \sim C_T \gg 1$, the micro-canonical ensemble is equivalent to the canonical ensemble at inverse temperature β determined by (6.5). It then follows from (6.13) that the diagonal part of the ETH can written in terms of OPE coefficients and thermal one-point functions:

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H \mathcal{O}_{\tau,s}}}{R^{\tau+s}} = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\beta/R)}{\beta^{\tau+s}} + \mathcal{O}\left(e^{-S(E)}\right), \tag{6.14}$$

where $f_{\mathcal{O}_{\tau,s}}$ also appears in (2.9). This is equivalent to the statement of thermalization discussed in the rest of the paper.

In this paper we observed that the multi stress tensor operators satisfy (6.14). One can also ask if (6.14) holds for any operator in the specific heavy state we considered. By comparing eqs. (5.10) and (5.11) using (4.2), one can check that operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy (6.14). Since this is a free theory, it is not a surprise that the ETH is not satisfied by all operators in the spectrum which is seen explicitly in this case.

7 Discussion

In this paper we argued that multi stress tensor operators $T_{\tau,s}^k$ in CFTs with a large central charge C_T thermalize: their expectation values in heavy states are the same as their thermal expectation values. This is equivalent to the statement that multi stress tensor operators in higher-dimensional CFTs satisfy the diagonal part of the ETH in the thermodynamic limit. The analogous statement in the $d = 2$ case is that the Virasoro descendants of the identity satisfy the ETH condition in the large-C_T limit.

We observed that the operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy the ETH. This is seen by comparing eqs. (5.10) and (5.11) using (4.2). While this operator does not thermalize in the heavy states we considered, the OPE coefficient averaged over all operators with $\Delta_H \sim C_T$ is expected to be proportional to the thermal one-point function. The averaged OPE coefficients should therefore scale like $\sim \sqrt{\Delta_H}$ compared to $\lambda_{\mathcal{O}_H \mathcal{O}_H \mathcal{O}_2} \sim \Delta_H/\sqrt{C_T}$ for the heavy states we considered. It would be interesting to exhibit heavy operators that produce the former scaling.

We provided a bootstrap argument in favor of the thermalization of multi stress tensor operators. One should be able to refine it to give an explicit form for leading behavior of the multi stress tensor OPE coefficients — we leave it for future work. The holographic/bootstrap OPE coefficients for the leading twist double stress tensor operators can be found in e.g. [66] — they are nontrivial functions of the spin. As explained in [66, 68], the leading Δ behavior of the minimal-twist double- and triple-stress tensor OPE coefficients is consistent with the exponentiation of the near lightcone stress tensor conformal block. One can go beyond the leading twist multi stress tensors. In holographic HHLL correlators each term of the type $(\Delta\mu)^k \sim (\Delta\Delta_H/C_T)^k$ comes from the exponentiation of

---

[16]The tensor structure in (6.13) is suppressed.

the stress-tensor block — this follows from the Wilson line calculation of the correlator in the AdS-Schwarzschild background [66, 78, 90].

In this paper we argue that this behavior is universal, and is not just confined to holographic theories. Hence, one can formulate another statement equivalent to the thermalization of multi stress tensor operators. Namely, scalar correlators of pairwise identical operators of dimensions $\Delta_{1,2}$ in large-C_T theories in the limit $\Delta_{1,2} \gg 1$, $\Delta_1\Delta_2/C_T$ fixed are given by the exponentials of the stress tensor conformal block.[17] This is similar to what happens in two-dimensional CFTs.

Note that the universality of the OPE coefficients is naively in tension with the results of [73], where finite gap (Δ_{gap}) corrections to the multi stress tensor OPE coefficients were considered. In particular, for double stress tensors, such corrections behave like $\Delta^3/\Delta_{\text{gap}}$ which is clearly at odds with the universality statement. Of course, the results of [73] are obtained in the limit $\Delta \ll \Delta_{\text{gap}}$, while in this paper we consider the opposite regime $\Delta \gg \Delta_{\text{gap}}$.

One may also wonder what happens with the universality of the OPE coefficients beyond leading order in Δ. In particular, in [78], it was shown that the bootstrap result for the HHLL correlator exactly matches the holographic Wilson line calculation (in the double scaling limit where only the minimal twist multi stress tensor operators contribute). This corresponds to including terms beyond the exponential of the stress tensor block — one needs to compute the HHLL correlator, take a logarithm of the result, divide by Δ, and then take the large-Δ limit. The result is sensitive to terms subleading in the large-Δ limit of the multi stress tensor OPE coefficients. In four spacetime dimensions the result in [78] is given by an elliptic integral — is it applicable beyond holography?

In [66] terms subleading in Δ were shown to be important for the computation of the phase shift. The simplest nontrivial case in two spacetime dimensions is the operator Λ_4 which is a level four Virasoro descendant of the identity (see e.g. [91]). One could also get it by using the CFT normal ordering and imposing the quasi-primary condition [92]. Consider now the case of minimal twist (twist four) operators in four dimensions. How do we determine the analog of Λ_4? There is no Virasoro algebra now.

Presumably, one can reconstruct the analog of Λ_4 in four spacetime dimensions by considering a CFT normal ordered product of stress tensors, and adding a single trace term to ensure that the resulting operator is a primary and is orthogonal to the stress tensor itself. Note that the CFT normal ordering differs from the oscillator normal ordering in a free theory by the addition of a single trace operator, as reviewed in appendix G. This procedure can then be generalized to other multi-trace operators. We leave it for future work.

It is also helpful to imagine what happens in a theory like $\mathcal{N} = 4$ Super Yang-Mills, where there is a marginal line connecting the weak and the strong coupling (the latter admits a holographic description). Presumably, as the coupling is turned on, only one operator remains light (with dimension eight and spin four), while others get anomalous dimensions. It would be interesting to see this explicitly even to the leading nontrivial order in the 't Hooft coupling. It would also be interesting to see how the corresponding OPE coefficient interpolates between its free and strong coupling values.

---

[17]See [33] for previous work on the eikonalization of the multi stress tensor OPE coefficients at large spin.

Using crossing symmetry, we argued that the universality of multi stress tensor OPE coefficients is related to the OPE coefficients $\lambda_{\mathcal{O}_H T_{\mu\nu} \mathcal{O}'}$, with $\mathcal{O}' \neq \mathcal{O}_H$ being either heavy or light, present in the cross-channel expansion. Such OPE coefficients with at least one operator being heavy were recently studied in [23, 93]. It would be interesting to further study the connection of our results to this work.

Another interesting question concerns the fate of the double trace operators of the schematic form $[\mathcal{O}_\Delta \mathcal{O}_\Delta]_{n,l}$. Consider the $d = 4$ case in the large volume limit and $n, l = 0$, for simplicity. We expect that the corresponding OPE coefficients in the free theory behave like $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}} \propto \Delta_H^2 / C_T \propto C_T \mu^2,$[18] while their thermal one-point functions behave like $\langle [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0} \rangle_\beta \propto C_T \beta^{-2\Delta}$. Comparing the two results with the help of (4.2) one observes that such operators do not thermalize in the free theory for generic Δ. The situation is more nontrivial in holography where we do not know the large μ behavior of the OPE coefficients.[19] As pointed out in [63], the contribution of double-trace operators to thermal two-point functions is different from that of multi stress tensors. The latter is only sensitive to the behavior of the metric near the boundary, but the former knows about the full black hole metric. This seems to indicate that the thermalization of the double trace operators in holographic theories is also unlikely.[20]

It is a natural question how generic are the heavy states for which the stress tensor sector thermalizes. The results of our paper seem to suggest that such thermalization is more generic than the thermalization of other light operators.[21] Other interesting questions include generalizations to the case of finite but large central charge and to non-conformal quantum field theories.

Acknowledgments

A OPE coefficients from Wick contractions

In this appendix we go through the calculations needed for finding the OPE coefficients of various operators using Wick contractions. This mainly amounts to counting the number of contractions leading to a planar diagram. For simplicity, the figures are shown for external

---

[18]This scaling is obtained by computing the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}}$ for $1 \ll \Delta_H \ll C_T$ and extrapolating it to the $\Delta_H \sim C_T$ regime.

[19]Note that the large-N scaling in holography is different. Both the OPE coefficients and the thermal expectation values behave like C_T^0 as opposed to $C_T \sim N^2$.

[20]A simple way to decouple such operators is to take the large-Δ limit.

[21]A closely related question of finding "typical" states where the stress tensor sector thermalizes in the large volume limit in $d = 2$ was recently discussed in [58]. There it was observed that such states are Virasoro descendants when the central charge is finite.
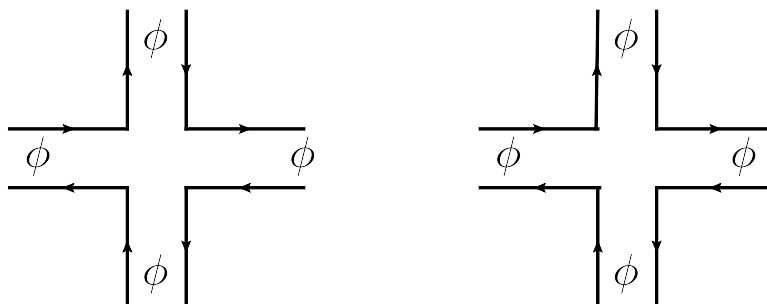
Figure 1. The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ before any contractions.
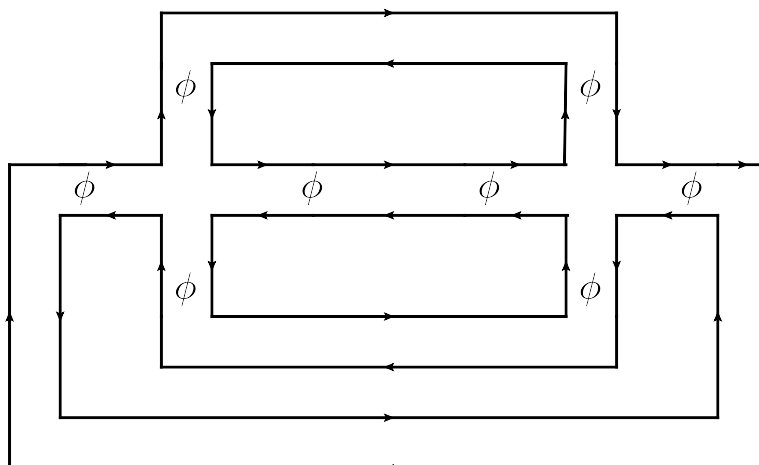

Figure 2. The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

operators with $\Delta = 4$ while we write down the result for general Δ as this is needed for the main body of the paper.

To begin with, since we consider a large-N matrix theory, it is convenient to use the double-line notation for fundamental field propagators. In figure 1 the two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ is visualised. In figure 2, the planar diagram is shown for $\Delta = 4$ and there are Δ number of such contractions giving a planar diagram

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta): \rangle} = \Delta, \tag{A.1}$$

where the $P_{\langle \ldots \rangle}$ denotes the number of planar diagrams for $\langle \ldots \rangle$.

We further need the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2}$. This is shown in figure 3 for $\Delta = 4$ and there are 2Δ possibilities for step (1), Δ number of possibilites for step (2) after which everything is fixed assuming that the diagram is planar. This gives

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^2): \rangle} = 2\Delta^2. \tag{A.2}$$

In figure 4 the three-point function $\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) :: Tr(\phi^4) : \rangle$ for $\Delta = 4$ is shown. For the first contraction (1) there are 2Δ possibilites, for the second contraction there are Δ and for step (3) there are two possibilites. This gives overall

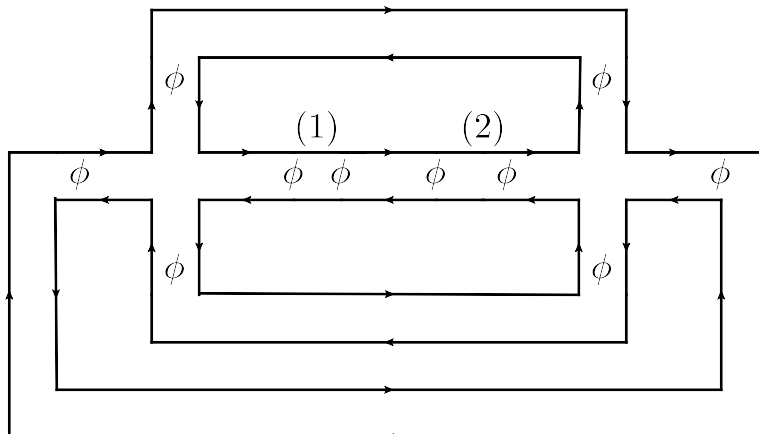$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^4): \rangle} = 4\Delta^2. \tag{A.3}$$

Figure 3. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2) : \rangle$ completely contracted.
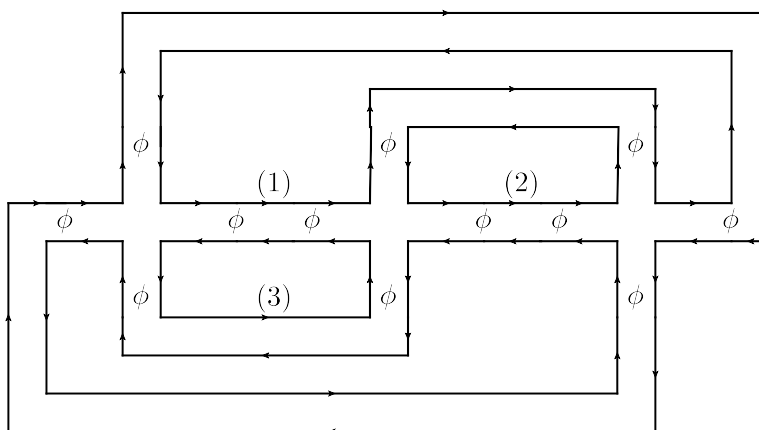

Figure 4. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

In figure 5 and figure 6, the three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : Tr(\phi^2)Tr(\phi^2) : \rangle$ is shown. The reason for there being two different types of diagrams is because each trace term in the double trace operator $: Tr(\phi^2)Tr(\phi^2) :$ can either be contracted with the same $: Tr(\phi^4) :$ (figure 5, type B), or to both (figure 6, type A).

Consider first the type of diagrams in figure 5. For the first contraction there are 2Δ such terms and the second contraction gives another factor of 2. Contraction (3) and (4) contributes factors of Δ and 2 respectively. What remains is equivalent to the two-point function $\langle : Tr(\phi^{\Delta-2}) :: Tr(\phi^{\Delta-2}) : \rangle$ which further give a factor of $(\Delta - 2)$ and therefore there are $8\Delta^2(\Delta - 2)$ contractions of type B in figure 5.

Continuing with figure 6, the first contraction gives a factor of 2Δ, the second contraction Δ and the third one a factor of $2(\Delta - 1)$. What remains is then fixed by imposing that the diagram is planar. The type A diagrams in figure 6 therefore further contributes $4\Delta^2(\Delta - 1)$ planar diagrams to $\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) : Tr(\phi^2)Tr(\phi^2) : \rangle$. It is therefore found that

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta):Tr(\phi^2)Tr(\phi^2):\rangle} = 4\Delta^2(3\Delta - 5). \tag{A.4}$$
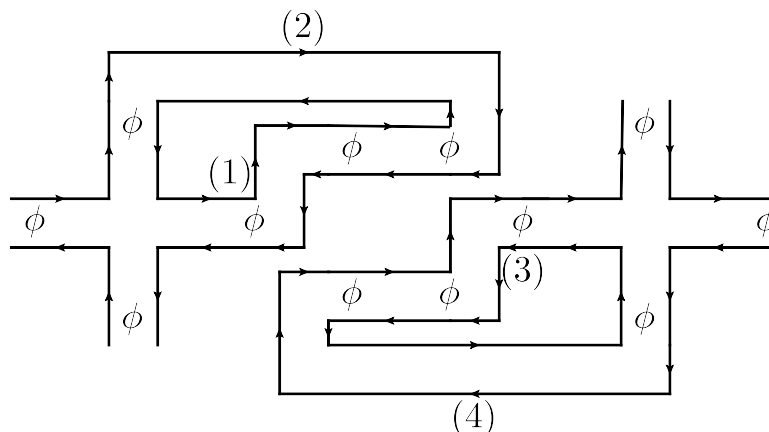
Figure 5. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) :\rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to a separate $: Tr(\phi^4) :$.
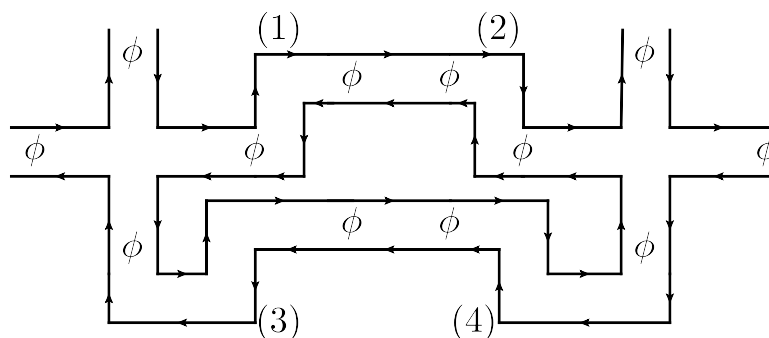

Figure 6. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) :\rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to both $: Tr(\phi^4) :$ operators.

Consider now the stress tensor OPE coffiecient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}}$ where

$$T_{\mu\nu}(x) = \frac{1}{2\sqrt{3}N} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x) \tag{A.5}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu} \rangle$:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)T_{\mu\nu}(x_3) \rangle = \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}} \frac{Z_\mu Z_\nu - \text{traces}}{|x_{12}|^{2\Delta-2}|x_{23}|^2|x_{13}|^2}, \tag{A.6}$$

where $Z_\mu = \frac{x_{13\mu}}{|x_{13}|^2} - \frac{x_{12\mu}}{|x_{12}|^2}$. From the definition of $T_{\mu\nu}$ in (A.5) it is clear that the only term that contributes to term $x_{13\mu}x_{13\nu}$ comes from the second term in (A.5) that is of the form $\propto Tr(\phi\partial_\mu\partial_\nu\phi)$. Up to the derivatives, the diagram will look like those visualised in figure 3. The number of diagrams is half of that given in (A.2) since we restrict to terms proportional to $x_{13\mu}x_{13\nu}$:

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu} \rangle |_{x_{13\mu}x_{13\nu}}} = \Delta^2, \tag{A.7}$$

from which we reproduce (3.7).

Now we want to find the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}}$ for the double-stress tensor $T^2_{4,4}$. This is done similarly to the way the stress tensor OPE coefficient was found. First, the operator $(T^2)_{\mu\nu\rho\sigma}$ was given in (3.8) to be

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}) \tag{A.8}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta (T^2)_{\mu\nu\rho\sigma}\rangle$ is fixed by conformal symmetry to be

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{A.9}$$

Consider the term in (A.9) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$. This will be due to the term in $(T^2)_{\mu\nu\rho\sigma}$ of the form $Tr(\phi\partial_{(\mu}\partial_{\nu}\phi)Tr(\phi\partial_\rho\partial_{\sigma)}\phi)$. Using this we find that

$$
\begin{aligned}
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} &= \frac{1}{\Delta N^\Delta}\frac{1}{\sqrt{2}}\left(\frac{-1}{4\sqrt{3}N}\right)^2 8^2 N^\Delta \\
&\times \frac{P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}.
\end{aligned}
\tag{A.10}
$$

The number of contractions giving a planar diagram, $P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}$, come from diagrams of the form given in figure 6. Since we are considering the term proportional $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$, the number of such diagrams are reduced compared to scalar double trace operator. Instead the first contraction, (1) in figure 6, give a factor of Δ, the second contraction, (2), a factor of $(\Delta-1)$, the third contraction (3) gives a further factor Δ after which everything is fixed by imposing that the diagram is planar. We therefore find that

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}} = \Delta^2(\Delta-1), \tag{A.11}$$

and inserting this in (A.10) gives

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{3N^2}, \tag{A.12}$$

and therefore reproduces (3.11).

Similar to the double-stress tensor, consider the dimension-eight spin-four double trace operator

$$
\begin{aligned}
\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} : Tr(\phi^2)\Big(&Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\
&+ 18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big) : (x).
\end{aligned}
\tag{A.13}
$$

The three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x_3)\rangle$ is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{A.14}$$

By again considering terms in (A.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ we find that each term in (A.13) will contribute planar diagram of the type in figure 5, while only the

term $\sim Tr(\phi\partial^4\phi)$ also give a contribution of the type in figure 6. Considering first the terms coming from the diagram in figure 5, one finds that this contribution vanishes. The remaining contribution to the term (A.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ comes from the first term in (A.13) and the planar diagram pictured in figure 6; there are $2\Delta^2(\Delta-1)$ contractions giving such a planar diagram leading to

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_{\mu\nu\rho\sigma}^{\mathrm{DT}}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} = \frac{1}{\Delta N^\Delta}\frac{384}{96\sqrt{70}N^2}N^\Delta \\ \times \frac{2\Delta^2(\Delta-1)}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}, \tag{A.15}$$

where the 384 in the numerator come from the derivatives. This gives the OPE coefficient:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{\mu\nu\rho\sigma}^{\mathrm{DT}}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2} + \mathcal{O}(N^{-4}). \tag{A.16}$$

B Subleading twist double-stress tensors

In this appendix we study the subleading twist double-stress tensors, both with dimension 8 and spin $s = 0, 2$ denoted (T^2) and $(T^2)^{\mu\nu}$ respectively. The calculations needed to find the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE are reviewed as well as the normalization of $(T^2)^{\mu\nu}$.

The $(T^2)^{\mu\nu}$ was defined in (3.24) which we repeat here:

$$(T^2)^{\mu\nu}(x) = \frac{1}{\sqrt{2}} : T^\mu{}_\alpha T^{\alpha\nu} : (x) - \frac{\delta^{\mu\nu}}{4\sqrt{2}} : T^\beta{}_\alpha T^\alpha{}_\beta : (x). \tag{B.1}$$

The operator $(T^2)^{\mu\nu}$ can be seen to be unit-normalized to leading order in N:

$$\langle(T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle = \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}(x_1)T_{\rho\beta}(x_2)\rangle\langle T^\nu{}_\alpha(x_1)T^\beta{}_\sigma\rangle \\ + (\rho \longleftrightarrow \sigma) - (\text{traces}) + \mathcal{O}(N^{-2}). \tag{B.2}$$

Using the two-point function of the stress tensor in (3.6) and $I^\mu{}_\alpha I^\alpha{}_\rho = \delta^\mu{}_\rho$ one finds

$$\langle(T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle = \frac{1}{|x|^{16}}\left(I^{(\mu}{}_\rho I^{\nu)}{}_\sigma - (\text{traces})\right), \tag{B.3}$$

from which it is seen that $(T^2)^{\mu\nu}$ is unit-normalised.

We now want to find the OPE coefficient of $(T^2)^{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. It can be found from the basic objects $I^{(1)}_{\mu\nu\rho\sigma}$, $I^{(2)}_{\mu\nu\rho\sigma}$ and $I^{(3)}_{\mu\nu\rho\sigma}$ which we calculate below.

We first consider a similar quantity $J^{(1)\mu\nu\rho\sigma}$:

$$J^{(1)\mu\nu\rho\sigma} = \langle: Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\partial_\mu\phi\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ = \frac{2^4 N^\Delta}{|x_{13}|^8|x_{23}|^8|x_{12}|^{2\Delta-4}} \times \Big[(2\Delta)^2(\Delta-2)(x_{13}^\mu x_{13}^\nu x_{23}^\rho x_{23}^\sigma + x_{23}^\mu x_{23}^\nu x_{13}^\rho x_{13}^\sigma)+ \\ \Delta^2(\Delta-1)(x_{13}^\mu x_{23}^\nu(x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma) + x_{23}^\mu x_{13}^\nu(x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma))\Big]. \tag{B.4}$$

Defining $X_{13}^{\mu\nu} = \frac{1}{|x_{13}|^4}(-\delta^{\mu\nu} + 4\frac{x_{13}^\mu x_{13}^\nu}{|x_{13}|^2})$ we then study $J^{(2)\mu\nu\rho\sigma}$:

$$
\begin{aligned}
J^{(2)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi) Tr(\phi\partial_\rho\partial_\sigma\phi) : (x_3)\rangle \\
&= \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Bigg[\Delta^2(\Delta-1)2^2\left(X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}X_{13}^{\rho\sigma}\frac{1}{|x_{23}|^2} + X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}X_{23}^{\rho\sigma}\frac{1}{|x_{13}|^2}\right) \\
&\quad + ((2\Delta)^2(\Delta-2))2^2 X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}X_{23}^{\rho\sigma}\frac{1}{|x_{23}|^2} \\
&\quad + (13)\longleftrightarrow(23)\Bigg].
\end{aligned}
\tag{B.5}
$$

And lastly $J^{(3)\mu\nu\rho\sigma}$:

$$
\begin{aligned}
J^{(3)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi) Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\
&= \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Bigg[((2\Delta)^2(\Delta-2))2^3 X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}\frac{x_{23}^\rho x_{23}^\sigma}{|x_{23}|^8} + \\
&\quad + \Delta^2(\Delta-1)2^3 X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}\frac{x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma}{|x_{13}|^4|x_{23}|^4} \\
&\quad + (13)\longleftrightarrow(23)\Bigg].
\end{aligned}
\tag{B.6}
$$

We further need to make (B.4)–(B.6) traceless in the pairs (μ,ν) and (ρ,σ) and therefore define $I^{(i)\mu\nu\rho\sigma}$ as

$$
I^{(i)\mu\nu\rho\sigma} = J^{(i)\mu\nu\rho\sigma} - \frac{\delta^{\mu\nu}}{4}J^{(i)\alpha}{}_\alpha{}^{\rho\sigma} - \frac{\delta^{\rho\sigma}}{4}J^{(i)\mu\nu\alpha}{}_\alpha + \frac{\delta^{\mu\nu}\delta^{\rho\sigma}}{16}J^{(i)\alpha}{}_\alpha{}^\gamma{}_\gamma.
\tag{B.7}
$$

From (B.4)–(B.6), the three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle$ is given by

$$
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}\rangle = \frac{1}{12\sqrt{2}\Delta N^{\Delta+2}}\left(I^{(1)(\mu|\alpha}{}_\alpha{}^{|\nu)} - I^{(3)(\mu|\alpha}{}_\alpha{}^{|\nu)} + \frac{1}{4}I^{(2)(\mu|\alpha}{}_\alpha{}^{|\nu)} - (\text{trace})\right).
\tag{B.8}
$$

Explicitly we find that

$$
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{\sqrt{2}\Delta(\Delta-1)}{3N^2}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6} + \mathcal{O}(N^{-4}).
\tag{B.9}
$$

Consider now the scalar operator (T^2) defined by

$$
(T^2)(x) = \frac{1}{36\sqrt{2}N^2} : T_{\mu\nu}T^{\mu\nu} : (x).
\tag{B.10}
$$

The three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ can be found using $I^{(i)}$ defined in (B.7) as follows

$$
\begin{aligned}
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle &= \frac{1}{36\sqrt{2}\Delta N^{2+\Delta}}\left(I^{(1)\mu\nu}{}_{\mu\nu} - I^{(3)\mu\nu}{}_{\mu\nu} + \frac{1}{4}I^{(2)\mu\nu}{}_{\mu\nu}\right) + \mathcal{O}(N^{-4}) \\
&= \frac{\Delta(\Delta-1)}{3\sqrt{2}N^2}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8} + \mathcal{O}(N^{-4}).
\end{aligned}
\tag{B.11}
$$

C Single trace operator with dimension $\Delta \sim C_T$

In this appendix we study the single trace scalar operator $\mathcal{O}_{\Delta_H}$ given by

$$\mathcal{O}_H(x) = \frac{1}{\sqrt{\mathcal{N}_{\Delta_H}}} : Tr(\phi^{\Delta_H}) : (x), \tag{C.1}$$

with $\Delta_H \sim C_T$ and $\mathcal{N}_{\Delta_H}$ a normalization constant.[22] When calculating the normalization constant $\mathcal{N}_{\Delta_H}$ as well as the three-point functions $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}(x_3)\rangle$, non-planar diagrams generically gets enhanced by powers of Δ_H and therefore invalidates the naive planar expansion. The goal of this appendix is to show that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\hat{\mathcal{O}}(x_3)\rangle = \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\hat{\mathcal{O}}(x_3)\rangle|_{\Delta=\Delta_H}, \tag{C.2}$$

where $\hat{\mathcal{O}}$ is either $: Tr(\phi^2) :$ or, more importantly, minimal-twist multi stress tensors with any spin. Moreover, note that the l.h.s. in (C.2) is in principle exact in $C_T \sim N^2$ while the r.h.s. is obtained by keeping only planar diagrams with $\Delta \ll C_T$ and then setting $\Delta = \Delta_H$ in the end.

The propagator for the field ϕ was given in (3.2) by

$$\langle \phi^i{}_j(x)\phi^k{}_l(y)\rangle = \left(\delta^i{}_l\delta^k{}_j - \frac{1}{N}\delta^i{}_j\delta^k{}_l\right)\frac{1}{|x-y|^2}. \tag{C.3}$$

Consider now the three-point function $\langle : Tr(\phi^{\Delta_H}) : (x_1) : Tr(\phi^{\Delta_H}) : (x_2) : Tr(\phi^2) : (x_3)\rangle$. Due to the normal ordering, one ϕ field in $: Tr(\phi^2) : (x_3)$ need to be contracted with $: Tr(\phi^{\Delta_H}) : (x_1) :$ and the other one with $: Tr(\phi^{\Delta_H}) : (x_2) :$. Note that for this contraction the second term in (C.3) give a contribution proportional to $Tr(\phi(x_3)) = 0$. It is therefore seen that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3^2) :\rangle = 2\Delta_H \langle : Tr(\phi_3\phi_1^{\Delta_H-1}) :: Tr(\phi_2^{\Delta_H}) :\rangle, \tag{C.4}$$

where we introduced the notation $\phi_i = \phi(x_i)$ and dropped the $|x_{ij}|^{-2}$ coming from (C.3). The position dependence is easily restored in the end. Now it is seen that the r.h.s. of (C.4) is proportional to the two-point function[23] of $\mathcal{O}_H$ and we therefore find that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3^2) :\rangle = 2\Delta_H\mathcal{N}_{\Delta_H}, \tag{C.5}$$

which is exact to all orders in C_T. Including the normalization factor of $\mathcal{O}_H$ in (C.1) and $\mathcal{O}_2$ from (3.3) we find that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta_H}{N}\frac{1}{|x_{12}|^{2\Delta_H-2}|x_{13}|^2|x_{23}|^2} + \mathcal{O}(N^{-3}). \tag{C.6}$$

By comparing (C.6) with (5.10) we find that

$$\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}|_{\Delta=\Delta_H}. \tag{C.7}$$

---

[22]Mixing with other operators with $\Delta \sim C_T$ is not important for this discussion.

[23]Up to the position dependence.

Note that in (C.6) the normalization of $\mathcal{O}_H$ cancels the contribution from non-planar diagrams in limit $\Delta_H \sim C_T$. For $\Delta = 2$ in (3.3), it is trivial to compute the normalization exact in N to get the correction to $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2}$ in (C.6).

Consider now the stress tensor operator defined in (3.5) and the three-point function $\langle \mathcal{O}_H(x_1) \mathcal{O}_H(x_2) T_{\mu\nu}(x_3) \rangle$. This is fixed by the Ward identity but is an instructive example before considering more general multi stress tensors. In the same way as the OPE coefficient was found in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, due to the tensor structure being fixed by conformal symmetry, we consider the term proportional to $x_{13}^\mu x_{13}^\nu$ in the three-point function. This comes from the $-\frac{1}{6\sqrt{C_T}} Tr(\phi \partial_\mu \partial_\nu \phi)$ term in the stress tensor when $\partial_\mu \partial_\nu \phi$ is contracted with one of the Δ_H number of $\phi(x_1)$ fields. Doing this contraction we therefore see that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3 \partial_\mu \partial_\nu \phi_3) : \rangle |_{x_{13}^\mu x_{13}^\nu} = 8\Delta_H \langle : Tr(\phi_3 \phi_1^{\Delta_H - 1}) :: Tr(\phi_2^{\Delta_H}) : \rangle, \tag{C.8}$$

where the factor 8 comes from the derivatives and we again suppress the spacetime dependence. The r.h.s. of (C.8) is also proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization factor of the stress tensor in (3.5) and that of $\mathcal{O}_H$ in (C.1), the three-point function $\langle \mathcal{O}_H \mathcal{O}_H T_{\mu\nu} \rangle$ can be obtained from (C.8) from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}. \tag{C.9}$$

This agrees with (3.7).

We now want to show that is true for minimal-twist multi stress tensors with any spin. For simplicity, consider the double-stress tensor with spin 4 defined in (3.8)

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu} T_{\rho\sigma)} : (x) - (\text{traces}). \tag{C.10}$$

Similarly to the calculation of the three-point function with the stress tensor, we can obtain the three-point function $\langle \mathcal{O}_H(x_1) \mathcal{O}_H(x_2) (T^2)_{\mu\nu\rho\sigma}(x_3) \rangle$ by considering the term proportional to $x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma$. This will be due to the term $\frac{1}{\sqrt{2} 6^2 C_T} Tr(\phi \partial_\mu \partial_\nu \phi) Tr(\phi \partial_\rho \partial_\sigma \phi)$ when contracting $\partial_\mu \partial_\nu \phi$ with some $\phi(x_1)$ and likewise contracting $\partial_\rho \partial_\sigma \phi$ with some other $\phi(x_1)$. The number of such contractions is given by $\Delta_H(\Delta_H - 1)$ and we find that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3 \partial_\mu \partial_\nu \phi_3) Tr(\phi_3 \partial_\rho \partial_\sigma \phi_3) : \rangle |_{x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma}$$
$$= 8^2 \Delta_H(\Delta_H - 1) \langle : Tr(\phi_3^2 \phi_1^{\Delta_H - 2}) :: Tr(\phi_2^{\Delta_H}) : \rangle, \tag{C.11}$$

where the factor of 8^2 again is due to acting with the derivatives and note that the position of the ϕ_3 fields in the last line is not important. It is again seen that the r.h.s. of (C.11) is proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization in (3.8) and (C.1) we find the three-point function $\langle \mathcal{O}_H \mathcal{O}_H (T^2)_{\mu\nu\rho\sigma} \rangle$ and read off the OPE coefficient:

$$\lambda_{\mathcal{O}_H \mathcal{O}_H T_{4,4}^2} = \frac{8\sqrt{2}\Delta_H(\Delta_H - 1)}{9 C_T} + \mathcal{O}(C_T^{-3/2}), \tag{C.12}$$

which is seen to agree with (3.11) when setting $\Delta_H = \Delta$. Note that the corrections in (C.12) are solely due to corrections in the normalization of $T_{4,4}^2$ and therefore $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{4,4}^2} =$

$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{4,4}^2}$ to all orders in C_T. These arguments generalize straightforwardly to minimal-twist multi stress tensor with any spin such that the results are the same as those obtained in the planar limit for $\Delta \ll C_T^2$ in section 3 by setting $\Delta_H = \Delta$. The only correction in C_T is then due to the normalization of the multi stress tensor.

The same argument applies to any scalar primary multi-trace operator $\mathcal{O}_\Delta$, without any derivatives, with OPE coefficients given by (C.6), (C.9) and (C.12).

D   Stress tensor thermal one-point function

In order to calculate thermal one-point functions in the free adjoint scalar model we use the fact that the thermal correlation function is related to the zero-temperature case by summing over images. Consider now the thermal one-point function of the stress tensor. Generally, the one-point function of a spin-s symmetric traceless operator with dimension $\Delta_\mathcal{O}$ on $S^1 \times \mathbf{R}^{d-1}$ is given by [13]

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_s}(x) \rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}}(e^{\mu_1} \dots e^{\mu_s} - (\text{traces})), \tag{D.1}$$

where e^{μ_1} is a unit-vector along the thermal circle. Consider first the canonically normalized stress tensor given by $T_{\mu\nu}^{(\text{can})} = \frac{1}{3S_d}(Tr(\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2}Tr(\phi \partial_\mu \partial_\nu \phi) - (\text{traces}))$. In order to find the one-point function, use the following:

$$\langle Tr(\partial_\mu^{(x)}\phi(x)\partial_\nu^{(y)}\phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4}\left(\delta_{\mu\nu} - 4(y-x)_\mu(y-x)_\nu \frac{1}{|x-y|^2} \right) \tag{D.2}$$

and

$$\langle Tr(\partial_\mu^{(x)}\partial_\nu^{(x)}\phi(x)\phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4}\left(-\delta_{\mu\nu} + 4(y-x)_\mu(y-x)_\nu \frac{1}{|x-y|^2} \right). \tag{D.3}$$

To get the thermal correlator, we use (D.2) and (D.3) with x, y along the thermal circle separated by a distance $m\beta$, with m integer, and sum over $m \neq 0$. The relevant terms for calculating the one-point functions in terms of fundamental fields are therefore

$$\begin{aligned} \langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_{\beta,m} &= -\frac{8(N^2-1)}{(m\beta)^4}e^\mu e^\nu + \frac{2(N^2-1)}{(m\beta)^4}\delta_{\mu\nu}, \\ \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_{\beta,m} &= \frac{8(N^2-1)}{(m\beta)^4}e^\mu e^\nu - \frac{2(N^2-1)}{(m\beta)^4}\delta_{\mu\nu}, \end{aligned} \tag{D.4}$$

where we note that only the first term in each equation in (D.4) contribute to the non-trace term in (D.1).

We therefore find for the stress tensor one-point function:

$$\begin{aligned} \langle T_{\mu\nu}^{(\text{can})} \rangle_\beta &= \frac{1}{3S_d}(\langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_\beta - \frac{1}{2}\langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_\beta - \text{trace}) \\ &= \frac{-12(N^2-1)}{3S_d}\frac{2\zeta(4)}{\beta^4}(e_\mu e_\nu - (\text{trace})), \end{aligned} \tag{D.5}$$

where the $2\zeta(4)$ comes from summing over images and we therefore have

$$b_{T_{\mu\nu}^{(\text{can})}} = -\frac{4(N^2-1)}{S_d}2\zeta(4) = -\frac{4\pi^4}{45S_d}(N^2-1). \tag{D.6}$$

This agrees with $f = \frac{b_{T_{\mu\nu}^{(\text{can})}}}{d}$ in eq. (2.17) in [13] for (N^2-1) free scalar fields. This also agrees with $a_{2,2} = \frac{\pi^4\Delta}{45}$ found from the two-point thermal correlator using:

$$a_{2,2} = \frac{\pi^4\Delta}{45} = \left(\frac{1}{2}\right)^2 \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^{(\text{can})}}b_{T_{\mu\nu}^{(\text{can})}}}{\frac{C_T}{S_d^2}}, \tag{D.7}$$

using $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^{(\text{can})}} = -\frac{4\Delta}{3S_d}$ in this normalization and $C_T = \frac{4}{3}(N^2-1)$. This is simply related to the one-point function for the unit-normalized stress tensor by (to leading order in N)

$$\begin{aligned} b_{T_{\mu\nu}} &= \frac{b_{T_{\mu\nu}^{(\text{can})}}}{\frac{\sqrt{C_T}}{S_d}} \\ &\approx -\frac{2\pi^4 N}{15\sqrt{3}}. \end{aligned} \tag{D.8}$$

Let us now consider the thermalization of the stress tensor, keeping all the index structures. To compare the thermal two-point function with the heavy-heavy-light-light correlator, we want to relate the dimension of the heavy operator, Δ_H, to the inverse temperature β. Consider the expectation value of the stress tensor in a heavy state created by $\mathcal{O}_H$ on the cylinder $\mathbf{R}\times S^3$

$$\langle\mathcal{O}_H|T^{\mu\nu}(x_{E,2}^0,\hat{n})|\mathcal{O}_H\rangle_{\text{cyl}} = \lim_{x_3\to\infty}|x_3|^{2\Delta_H}|x_2|^4\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\frac{Z^\mu Z^\nu - \frac{1}{4}\delta^{\mu\nu}Z^\rho Z_\rho}{|x_{13}|^{2\Delta_H-2}|x_{23}|^2|x_{12}|^2}, \tag{D.9}$$

where the r.h.s. is found by a conformal transformation to the plane with $Z^\mu = \left(\frac{x_{12}^\mu}{|x_{12}|^2} + \frac{x_{23}^\mu}{|x_{23}|^2}\right)$. When $x_1 = 0$ and $x_3\to\infty$, it is seen that $Z^\mu = -\frac{x_2^\mu}{|x_2|^2}$ and (D.9) only depends on $\hat{x}^\mu = \frac{x_{21}^\mu}{|x_{21}|} = \hat{r}$, where $\hat{r}$ is a radial unit vector. In radial quantization it follows that

$$\langle\mathcal{O}_H|T^{\mu\nu}(x_{E,2}^0,\hat{n})|\mathcal{O}_H\rangle_{\text{cyl}} = \frac{\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}}{R^4}\left(\hat{e}_\mu\hat{e}_\nu - \frac{1}{4}\delta_{\mu\nu}\right) \tag{D.10}$$

where we reintroduced the radius of the sphere R, $\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}$ is the OPE coefficient of $T_{\mu\nu}$ in the $\mathcal{O}_H\times\mathcal{O}_H$ OPE and $\hat{e}_\mu = (1,0,0,0)$.

The thermal one-point function of an operator $\mathcal{O}_{\tau,s}$, with twist τ and spin s, on $S^1\times S^3$ is fixed by conformal symmetry [13]

$$\langle\mathcal{O}_{\tau,s}(x)\rangle_\beta = \frac{b_{\mathcal{O}_{\tau,s}}f_{\mathcal{O}_{\tau,s}}(\frac{\beta}{R})}{\beta^{\tau+s}}(e^{\mu_1}\cdots e^{\mu_s} - (\text{traces})), \tag{D.11}$$

where $f_{\mathcal{O}_{\tau,s}}(0) = 1$ and $e^\mu = (1,0,0,0)$.

We assume thermalization of the stress tensor in the heavy state:

$$\langle \mathcal{O}_H | T_{\mu\nu}(x) | \mathcal{O}_H \rangle = \langle T_{\mu\nu}(x) \rangle_\beta \tag{D.12}$$

where $\langle T^{\mu\nu}(x) \rangle_\beta$ is the thermal one-point function at inverse temperature β evaluated on $S^1 \times S^3$, with R being the radius of S^3. Using (D.10)–(D.12) we find

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} = \frac{b_{T_{\mu\nu}} f_{T_{\mu\nu}}(\frac{\beta}{R})}{\beta^4}. \tag{D.13}$$

Using (D.13) for $R \to \infty$ in the free adjoint scalar theory, together with the one-point function $b_{T_{\mu\nu}} = -\frac{2\pi^4 N}{15\sqrt{3}}$ and the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}$, one finds the following relation between $\mu = \frac{160\Delta_H}{3C_T}$ and the inverse temperature β:

$$\mu = \frac{8}{3}\left(\frac{\pi R}{\beta}\right)^4. \tag{D.14}$$

This agrees with (4.2).

E   Dimension-six spin-four single trace operator

We want to calculate the contribution of the single trace operator with $\tau = 2$ and $s = 4$. The unit-normalised $\mathcal{O}_{2,4}$ operator is given by[24]

$$\Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} : Tr(\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \tag{E.1}$$
$$+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})) : (x).$$

The relative coefficients are fixed by demanding that it is a primary operator $[K_\alpha, \Xi_{\mu\nu\rho\sigma}] = 0$. Explicitly, this is done using the conformal algebra

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \tag{E.2}$$
$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu),$$

and the action on the fundamental field ϕ

$$P_\mu\phi(0) = -i\partial_\mu\phi(0), \tag{E.3}$$
$$D\phi(0) = i\phi(0).$$

The relevant commutators in order to fix $\Xi_{\mu\nu\rho\sigma}$ are

$$[K_\alpha, P_\mu\phi] = -2\eta_{\alpha\mu}\phi,$$
$$[K_\alpha, P_\mu P_\nu\phi] = -4\eta_{\alpha\mu}P_\nu\phi - 4\eta_{\alpha\nu}P_\mu\phi + 2\eta_{\mu\nu}P_\alpha\phi,$$
$$[K_\alpha, P_\mu P_\nu P_\rho\phi] = -6\eta_{\alpha\mu}P_\nu P_\rho\phi - 6\eta_{\alpha\nu}P_\mu P_\rho\phi - 6\eta_{\alpha\rho}P_\nu P_\mu\phi$$
$$+ 2\eta_{\mu\nu}P_\rho P_\alpha\phi + 2\eta_{\rho\nu}P_\mu P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\alpha\phi,$$
$$[K_\alpha, P_\mu P_\nu P_\rho P_\sigma\phi] = -8\eta_{\alpha\mu}P_\nu P_\rho P_\sigma\phi - 8\eta_{\alpha\nu}P_\mu P_\rho P_\sigma\phi - 8\eta_{\alpha\rho}P_\nu P_\mu P_\sigma\phi - 8\eta_{\alpha\sigma}P_\nu P_\rho P_\mu\phi$$
$$+ 2\eta_{\mu\nu}P_\rho P_\sigma P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\sigma P_\alpha\phi + 2\eta_{\mu\sigma}P_\rho P_\nu P_\alpha\phi + 2\eta_{\nu\rho}P_\mu P_\sigma P_\alpha\phi$$
$$+ 2\eta_{\nu\sigma}P_\mu P_\rho P_\alpha\phi + 2\eta_{\rho\sigma}P_\mu P_\nu P_\alpha\phi, \tag{E.4}$$

which can also be found in e.g. appendix F in [94].

---

[24]We denote this operator either as $\mathcal{O}_{2,4}$ or $\Xi_{\mu\nu\rho\sigma}$ depending whether we want to explicitly list the indices or not.

The thermal one-point function of this operator is found from Wick contractions to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8(\pi T)^6 N}{27\sqrt{35}} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right). \tag{E.5}$$

Moreover, the three-point function with operators $\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta N^\Delta}} : Tr\left(\phi^\Delta\right) : (x)$ can again be calculated using Wick contractions similarly to how it was done for $T^2_{\mu\nu\rho\sigma}$ in appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{E.6}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{E.7}$$

Now, it is easy to check that

$$\frac{1}{2^4}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}}b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{E.8}$$

which agrees with $a_{2,4}$ in (5.8).

F Thermal one-point functions of multi-trace operators in the large-N limit

In (5.33), it was shown that $a_{4,4}$ was due to double trace operators which were normal ordered products of single trace operators without any derivatives. There are, however, other double trace operators that have the same quantum numbers and are schematically represented as $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$. Concretely, the double trace operators with twist and spin four besides $(T^2)_{\mu\nu\rho\sigma}$ and $(\mathcal{O}^{\mathrm{DT}})_{\mu\nu\rho\sigma}$ are $[\mathcal{O}_2\mathcal{O}_2]_{0,4}$ and $[\mathcal{O}_2 T_{\mu\nu}]_{0,2}$. We argue that the thermal one-point functions of these operators are subleading in the large-N limit when evaluated on the plane.

Consider the thermal one-point function of a double trace operator $[\mathcal{O}_a\mathcal{O}_b]_{n,l} = \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b + \ldots$, where $\mathcal{O}_a$ and $\mathcal{O}_b$ are single trace primary operators and dots represent terms where derivatives acts on $\mathcal{O}_a$ as well, in order to make $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$ a primary operator. The term in the thermal one-point function that behaves as N^k (N^2 for double trace operators) comes from contracting the fundamental field within each trace separately. Therefore we have

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b \rangle_\beta \approx \langle \mathcal{O}_a \rangle_\beta \langle \partial^{2n}\partial^l\mathcal{O}_b \rangle_\beta + \mathcal{O}(1), \tag{F.1}$$

which is simply due to large-N factorization. As $\partial^{2n}\partial^l\mathcal{O}_b$ is a descendant of $\mathcal{O}_b$, it is easy to explicitly show that $\langle \partial^{2n}\partial^l\mathcal{O}_b \rangle_\beta = 0$ for $n \neq 0$ or $l \neq 0$, from which it follows that

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b \rangle_\beta = \mathcal{O}(1). \tag{F.2}$$

Similar reasoning holds for all terms in $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$, so we conclude for $n \neq 0$ or $l \neq 0$ that

$$\langle [\mathcal{O}_a\mathcal{O}_b]_{n,l} \rangle_\beta = \mathcal{O}(1). \tag{F.3}$$

It is easy to generalise (n and/or l non-zero)

$$\langle [\mathcal{O}_{a_1} \ldots \mathcal{O}_{a_k}]_{n,l} \rangle_\beta = \mathcal{O}(N^{k-2}). \tag{F.4}$$

Using the canonical scaling for the OPE coefficients (5.18) it is found that these multi-trace operators give a suppressed contribution to the thermal two point function in the large-N limit:

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1} \ldots \mathcal{O}_{a_k}]_{n,l}} \langle [\mathcal{O}_{a_1} \ldots \mathcal{O}_{a_k}]_{n,l} \rangle_\beta = \mathcal{O}\left(\frac{1}{N^2}\right). \tag{F.5}$$

The conclusion is that these operators with $n \neq 0$ or $l \neq 0$ do not contribute to the thermal two-point functions to leading order in N. Note that for $n = l = 0$, the operator is just $: \mathcal{O}_{a_1} \mathcal{O}_{a_2} \ldots \mathcal{O}_{a_k} :$ and it does contribute to the thermal 2pt function since

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1} \ldots \mathcal{O}_{a_k}]_{n=0,l=0}} \langle [\mathcal{O}_{a_1} \ldots \mathcal{O}_{a_k}]_{n=0,l=0} \rangle_\beta = \mathcal{O}(1). \tag{F.6}$$

From (F.5) it is seen that multi stress tensor operators of the schematic form $[T^k]_{n,l}$ with either n or l, or both, being non-zero will not contribute to the thermal correlator to leading order in N on the plane.

G Free boson in two dimensions

In this appendix we discuss free scalars in two dimensions. We first consider a single scalar and then the case of the SU(N) adjoint scalar. We compute two-point functions of a particular class of quasi-primary operators at finite temperature $1/\beta$. These two-point functions are not determined by the conformal symmetry, because the quasi-primary operators do not transform covariantly from the plane to the cylinder. They transform covariantly only with respect to the global conformal transformations. The only operators that have the non-zero thermal one-point functions are the Virasoro descendants of the vacuum and therefore, only these operators contribute to the thermal two-point function of the quasi-primary operators.[25] Virasoro descendants of the vacuum have different OPE coefficients with external quasi-primary operators compared with the case when primary external operators are considered.[26]

G.1 Review free boson in two dimensions

We consider single free boson $\phi(z)$ in two dimensions. The stress tensor can be written in terms of Virasoro modes as

$$T(z) = \sqrt{2} \sum_n z^{-n-2} L_n. \tag{G.1}$$

This stress tensor is unit-normalized

$$\langle T(z) T(w) \rangle = \frac{1}{(z-w)^4}. \tag{G.2}$$

---

[25]We check this explicitly up to the $\mathcal{O}(1/\beta^4)$.

[26]Deviation from the Virasoro vacuum block in the Regge limit of four-point HHLL correlator is observed in [95] as well.

The fundamental field can be expressed as Laurent series

$$\partial\phi(z) = \sum_{n=-\infty}^{+\infty} z^{-n-1}\alpha_n, \tag{G.3}$$

where oscillators α_n obey the following algebra

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{G.4}$$

They act on the vacuum as

$$\alpha_n|0\rangle = 0, \qquad n \geq 0. \tag{G.5}$$

The two-point function of the fundamental fields is given by

$$\langle\partial\phi(z)\partial\phi(w)\rangle = \frac{1}{(z-w)^2}. \tag{G.6}$$

The unit-normalized stress tensor can be expressed in terms of the fundamental field as

$$T(z) = \frac{1}{\sqrt{2}} : \partial\phi\partial\phi : (z) = \frac{1}{\sqrt{2}}\sum_{m,n} z^{-m-n-2} : \alpha_m\alpha_n :, \tag{G.7}$$

where $: ab :$ denotes product of operators a and b with the corresponding free theory oscillators being normally ordered such that the operators annihilating the vacuum are put at the rightmost position. Then, it follows

$$L_n = \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m := \frac{1}{2}\left(\sum_{m\geq 0}\alpha_{n-m}\alpha_m + \sum_{m<0}\alpha_m\alpha_{n-m}\right). \tag{G.8}$$

G.2 Thermal two-point function of quasi-primary operator

We are interested in computing the thermal two-point function of quasi-primary operators at temperature $1/\beta$. Quasi-primary operators $\mathcal{O}(z)$ are defined as $[L_1, \mathcal{O}(z)] = 0$, or equivalently, in therms of their asymptotic in-states $\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$, as $L_1|\mathcal{O}\rangle = 0$. We denote the quantum numbers of quasi-primary operators that correspond to eigenvalues of L_0 and $\bar{L}_0$ by $(h, \bar{h})$. We consider the following unit-normalized quasi-primary operator with quantum numbers $(h, 0)$

$$\mathcal{O}_h(z) = \frac{1}{\sqrt{h!}} : (\partial\phi)^h : (z) = \frac{1}{\sqrt{h!}}\sum_{m_1, m_2, \ldots, m_h} z^{-\sum_{i=1}^h m_i - h} : \alpha_{m_1}\ldots\alpha_{m_h} :, \tag{G.9}$$

which is properly defined when h is a positive integer. Its asymptotic in-state is given by

$$|\mathcal{O}_h\rangle = \mathcal{O}_h(0)|0\rangle = \frac{1}{\sqrt{h!}}(\alpha_{-1})^h|0\rangle. \tag{G.10}$$

One can check that this operator is a quasi-primary but not a Virasoro primary.

The thermal two-point function of this operator for even h is given by

$$\begin{aligned}
\langle\mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = \sum_{n=0}^{\frac{1}{2}(h-2)} &\frac{h!}{4^n(h-2n)!}\left(\frac{2\zeta(2)}{\beta^2}\right)^{2n}\left(\sum_{m=-\infty}^{\infty}\frac{1}{(z+m\beta)^2}\right)^{h-2n} \\
&+ \frac{2^h\pi}{\Gamma\left(\frac{1}{2} - \frac{h}{2}\right)^2\Gamma(h+1)}\left(\frac{2\zeta(2)}{\beta^2}\right)^h.
\end{aligned} \tag{G.11}$$

This expression is obtained by writing all possible Wick contractions between fundamental fields $\partial\phi$, including those that belong to same operator $\mathcal{O}_h$, that we call self-contractions. Fundamental fields are separated along the thermal circle in all Wick contractions. Factors $\left(\frac{2\zeta(2)}{\beta^2}\right)$ are due to the self-contractions,

$$\sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{\beta^2 m^2} = \left(\frac{2\zeta(2)}{\beta^2}\right). \tag{G.12}$$

The sum over n comes from doing n self-contractions within each of the external operators. Term $\frac{h!}{4^n(h-2n)!}$ counts the number of Wick contractions with n self-contractions for each external operator, including $1/\sqrt{h!}$ normalization factors. The term in the second line of (G.11) is due to the case when we take $n = h/2$ self-contractions in both external operators, i.e. it represents the disconnected contribution.

Since the state $\mathcal{O}_h$ is quasi-primary, it transforms properly only with respect to the global conformal transformation. These are just the Möbius transformations in two-dimensional spacetime $z \to \frac{az+b}{cz+d}$, with $ad - bc = 1$. On the other hand, the usual way to calculate the thermal two-point function of primary operators in two dimensions is to do a conformal transformation from the plane to the cylinder with radius β, $z \to \frac{\beta}{2\pi}\log(z)$. This transformation is clearly not one of the Möbius transformations and that is why we can not use this method to compute the thermal two-point functions of quasi-primary operators.

Expanding (G.11) for $T = \frac{1}{\beta} \to 0$ one finds

$$z^{2h}\langle\mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = 1 + \frac{h}{3}\frac{(\pi z)^2}{\beta^2} + \frac{h(h-\frac{1}{5})}{12}\frac{(\pi z)^4}{\beta^4} + \mathcal{O}\left(\frac{1}{\beta^6}\right). \tag{G.13}$$

G.3 Quasi-primaries, OPE coefficients, and thermal one-point functions

In expansion (G.13), terms $\mathcal{O}(z^{h_1})$ are due to the quasi-primary operator with quantum numbers $(h_1, 0)$ in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$. Identity in the expansion is due to the identity operator. We show that the second term on the r.h.s. is due to the stress tensor. The quantum numbers of stress tensor $T(z)$ are $(2, 0)$. First, we evaluate the thermal one-point function of the stress tensor

$$\langle T\rangle_\beta = \frac{1}{\sqrt{2}}\sum_{m=-\infty, m\neq 0}^{\infty}\frac{1}{\beta^2 m^2} = \frac{\pi^2}{3\sqrt{2}\beta^2}. \tag{G.14}$$

This is obtained by the Wick contractions of fundamental fields in the stress tensor, that are separated along the thermal circle. The same result can be obtained by the transform of the stress tensor from the plane to the cylinder using the Schwarzian derivative.

We define the OPE coefficient of unit-normalized operator $\mathcal{O}$, with quantum numbers $(h_\mathcal{O}, 0)$, with two $\mathcal{O}_h$ operators as

$$\langle\mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}(z_3)\rangle = \frac{\lambda_{\mathcal{O}_h\mathcal{O}_h\mathcal{O}}}{(z_1-z_3)^{h_\mathcal{O}}(z_2-z_3)^{h_\mathcal{O}}(z_1-z_2)^{2h-h_\mathcal{O}}}. \tag{G.15}$$

Next, we evaluate its OPE coefficient of the stress tensor with $\mathcal{O}_h$ by doing the Wick contractions between fundamental fields

$$\langle\mathcal{O}_h(z_1)\mathcal{O}_h(z_2)T(z_3)\rangle = \sqrt{2}h\frac{1}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^{2(h-1)}}, \tag{G.16}$$

therefore $\lambda_{\mathcal{O}_h \mathcal{O}_h T} = \sqrt{2}h$. This OPE coefficient is fixed by the Ward identity. Now, it follows

$$z^2 \lambda_{\mathcal{O}_h \mathcal{O}_h T} \langle T \rangle_\beta = \frac{h}{3} \frac{(\pi z)^2}{\beta^2}, \tag{G.17}$$

which reproduces the second term on the r.h.s. of (G.13).

We are now interested in the contributions of quasi-primary operators with quantum numbers $(4, 0)$. There are only two linearly independent operators with these quantum numbers given by[27]

$$: TT : (z) = \frac{1}{\sqrt{24}} : (\partial\phi)^4 : (z) = \frac{1}{\sqrt{24}} \sum_{a,b,c,d} z^{-a-b-c-d-4} : \alpha_a \alpha_b \alpha_c \alpha_d :, \tag{G.18}$$

$$\Lambda_4(z) = \sqrt{\frac{10}{27}} \left(\sum_{m,n=-\infty}^{\infty} z^{-m-n-4} * L_m L_n * - \frac{3}{10} \sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m \right), \tag{G.19}$$

where $*ab*$ denotes the product where the relevant Virasoro generators are normally ordered. It should be noted that the operator $\Lambda_4(z)$ is Virasoro descendant of unity, while $: TT : (z)$ is not. The relevant asymptotic in-states are given by

$$| : TT : \rangle =: TT : (0)|0\rangle = \frac{1}{\sqrt{24}}(\alpha_{-1})^4|0\rangle,$$

$$|\Lambda_4\rangle = \Lambda_4(0)|0\rangle = \sqrt{\frac{10}{27}} \left(L_{-2}^2 - \frac{3}{5}L_{-4} \right) |0\rangle. \tag{G.20}$$

In terms of oscillators, $|\Lambda_4\rangle$ state can be represented as

$$|\Lambda_4\rangle = \sqrt{\frac{10}{27}} \left(\frac{1}{4}(\alpha_{-1})^4 + \frac{2}{5}\alpha_{-1}\alpha_{-3} - \frac{3}{10}(\alpha_{-2})^2 \right) |0\rangle. \tag{G.21}$$

From eqs. (G.20) and (G.21) one can see that $| : TT : \rangle$ and $|\Lambda_4\rangle$ are the only quasi-primary states with quantum numbers $(4, 0)$. Namely, all such states have to be linear combinations of the following states

$$\alpha_{-4}|0\rangle, \qquad \alpha_{-3}\alpha_{-1}|0\rangle, \qquad \alpha_{-2}^2|0\rangle, \qquad \alpha_{-2}\alpha_{-1}^2|0\rangle, \qquad \alpha_{-1}^4|0\rangle, \tag{G.22}$$

because

$$L_0 \left(\prod_{i=1}^{N} \alpha_{-k_i} \right) |0\rangle = \left(\sum_{i=1}^{N} k_i \right) \left(\prod_{i=1}^{N} \alpha_{-k_i} \right) |0\rangle, \tag{G.23}$$

where $k_i > 0$. It is straightforward to check

$$\begin{aligned}
L_1 \alpha_{-4}|0\rangle &= 4\alpha_{-3}|0\rangle, \\
L_1 \alpha_{-3}\alpha_{-1}|0\rangle &= 3\alpha_{-2}\alpha_{-1}|0\rangle, \\
L_1 \alpha_{-2}^2|0\rangle &= 4\alpha_{-2}\alpha_{-1}|0\rangle, \\
L_1 \alpha_{-2}\alpha_{-1}^2|0\rangle &= 2\alpha_{-1}^3|0\rangle, \\
L_1 \alpha_{-1}^4|0\rangle &= 0.
\end{aligned} \tag{G.24}$$

---

[27]Both of them are unit-normalized.

It follows that $\alpha_{-1}^4|0\rangle$ is already quasi-primary and one can make only one more as $\alpha_{-3}\alpha_{-1}|0\rangle - \frac{3}{4}\alpha_{-2}\alpha_{-2}|0\rangle$.[28] $|:TT:\rangle$ and $|\Lambda_4\rangle$ are just the linear combination of these two states with overall normalization.

Now, one can calculate the overlap of $|:TT:\rangle$ and $|\Lambda_4\rangle$ states as

$$\langle 0|\Lambda_4(0):TT:(0)|0\rangle = \frac{\sqrt{5}}{3}. \tag{G.25}$$

The state orthogonal to $|\Lambda_4\rangle$ can be written as

$$|\tilde{\Lambda}_4\rangle = \frac{3}{2}\left(:TT:(0) - \frac{\sqrt{5}}{3}\Lambda_4(0)\right)|0\rangle. \tag{G.26}$$

Using (G.20) and (G.21), it can be written in terms of free theory oscillators.

We compute the OPE coefficients of $:TT:$ and Λ_4 with two $\mathcal{O}_h$ operators. We express all states in terms of free theory oscillators and use algebra (G.4) to find

$$\lambda_{\mathcal{O}_h\mathcal{O}_h:TT:} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|:TT:\rangle = \frac{\sqrt{6}}{2}h(h-1), \tag{G.27}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\Lambda_4\rangle = \sqrt{\frac{5}{6}}h\left(h-\frac{1}{5}\right), \tag{G.28}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\tilde{\Lambda}_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\tilde{\Lambda}_4\rangle = \frac{2}{\sqrt{6}}h(h-2). \tag{G.29}$$

Now, we evaluate the thermal one-point functions of Λ_4 and $\tilde{\Lambda}_4$. From (3.4) in [58] we have

$$\langle *T^2* \rangle_\beta = \frac{3\pi^4}{20\beta^4}, \tag{G.30}$$

which is the thermal one-point function of the first term on the r.h.s. of (G.19). The second term can be written as $-\frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m = -\frac{3}{10\sqrt{2}}\partial^2 T(z)$. It is clear that it will not affect the thermal one-point function of $\Lambda_4(z)$, as $\langle\partial^2 T\rangle_\beta = 0$.

Therefore, from (G.19), we have

$$\langle\Lambda_4\rangle_\beta = \sqrt{\frac{10}{27}}\langle *T^2* \rangle_\beta = \frac{\pi^4}{2\sqrt{30}\beta^4}. \tag{G.31}$$

Now, it follows

$$z^4\langle\Lambda_4\rangle_\beta\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4 z^4}{12\beta^4}h\left(h-\frac{1}{5}\right), \tag{G.32}$$

which is the third therm at the r.h.s. of (G.13). On the other hand, we can evaluate the thermal one-point function of $:TT:(z)$ operator by Wick contractions of fundamental fields separated along the thermal circle

$$\langle:TT:\rangle_\beta = \frac{\pi^4}{6\sqrt{6}\beta^4}. \tag{G.33}$$

---

[28]These states are not unit-normalized.

Using (G.26), it is straightforward to confirm that $\langle \tilde{\Lambda}_4 \rangle_\beta = 0$. Therefore, as we expected, operator $\tilde{\Lambda}_4$ does not contribute to the thermal two-point function of $\mathcal{O}_h$ operators, even thought it is present in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$.

This is a general property of two-dimensional CFTs, that only the operators in the Virasoro vacuum module have non-zero expectation value on the cylinder.

G.4 Free adjoint scalar model in two dimensions

In this subsection we study a large-c theory. Consider the free adjoint $SU(N)$ scalar in 2d with

$$\partial \phi(z)^a{}_b = \sum_m z^{-m-1} (\alpha_m)^a{}_b \tag{G.34}$$

with

$$[(\alpha_m)^a{}_b, (\alpha_n)^c{}_d] = m \delta_{m+n} \left(\delta^a{}_d \delta^c{}_b - \frac{1}{N} \delta^a{}_b \delta^c{}_d \right). \tag{G.35}$$

The thermal two point of the quasi-primary operator $\mathcal{O}_h = \frac{1}{\sqrt{hN^h}} : Tr((\partial \phi)^h) :$ follows immediately from the result in four dimensions upon replacing the propagator of fundamental fields. We find that

$$\langle \mathcal{O}_h(z) \mathcal{O}_h(0) \rangle_\beta = g_{2d}(z)^h + \frac{\pi^4 h(h-2)}{9\beta^4} g_{2d}(z)^{h-2} + \ldots, \tag{G.36}$$

where

$$\begin{aligned} g_{2d}(z) &= \sum_{m=-\infty}^\infty \frac{1}{(z+m\beta)^2} \\ &= \left(\frac{\pi}{\beta \sin(\pi z/\beta)} \right)^2. \end{aligned} \tag{G.37}$$

Expanding (G.36) for $\beta \to \infty$ we find

$$\langle \mathcal{O}_h(z) \mathcal{O}_h(0) \rangle_\beta = z^{-2h} \left[1 + \frac{\pi^2 h}{3\beta^2} z^2 + \frac{\pi^4 h(15h-19)}{90\beta^4} z^4 + \mathcal{O}(\beta^{-6}) \right]. \tag{G.38}$$

Consider first the normalized stress tensor which is given by

$$T = \frac{1}{\sqrt{2}N} : Tr(\partial \phi \partial \phi) :, \tag{G.39}$$

with $c = N^2$ so that $\langle T(z) T(0) \rangle = \frac{1}{z^4}$. By calculating the OPE coefficient with $\mathcal{O}_h$ and the thermal one-point function of T, one finds that these are the same as those for the scalar $Tr(\phi^2)$ operator in four dimensions so that $\langle T \rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}$ and $\lambda_{\mathcal{O}_h \mathcal{O}_h T} = \frac{\sqrt{2}h}{N}$, and the product reproduces the weight two term in (G.38):

$$\langle T \rangle_\beta \lambda_{\mathcal{O}_h \mathcal{O}_h T} = \frac{\pi^2 h}{3\beta^2}. \tag{G.40}$$

Consider now $*TT*$ defined by

$$*TT*(0) = \lim_{z \to 0} T(z) T(0) - (\text{sing. terms}). \tag{G.41}$$

– 44 –

The OPE of the stress tensor in (G.39) can be found in the free theory by first performing Wick contractions

$$
\begin{aligned}
T(z)T(0) &= \frac{1}{2N^2} : Tr(\partial\phi(z)\partial\phi(z)) :: Tr(\partial\phi(0)\partial\phi(0)) : \\
&= : TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(z)\partial\phi(0)) : + \frac{1}{z^4},
\end{aligned}
\tag{G.42}
$$

and expanding the second term in (G.42) for $z \to 0$ we find

$$
\begin{aligned}
T(z)T(0) &= : TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(0)\partial\phi(0)) : \\
&+ \frac{2}{N^2 z} : Tr(\partial^2\phi(0)\partial\phi(0)) : + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : + \ldots \\
&+ \frac{1}{z^4},
\end{aligned}
\tag{G.43}
$$

where the dots refer to higher order terms in z. Inserting the OPE (G.43) in (G.41) we find that

$$
* TT * (0) = : TT : (0) + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : .
\tag{G.44}
$$

Consider the state $* TT * (0)|0\rangle$, which is given in terms of oscillator modes by

$$
* TT * (0)|0\rangle = \frac{1}{2N^2} Tr(\alpha_{-1}^2) Tr(\alpha_{-1}^2)|0\rangle + 2\frac{1}{N^2} Tr(\alpha_{-3}\alpha_{-1})|0\rangle.
\tag{G.45}
$$

Now $Tr(\alpha_{-1}^m)|0\rangle$ is a quasi-primary while $Tr(\alpha_{-3}\alpha_{-1})|0\rangle$ is not. One way to make it a quasi-primary is to simply remove the second term in (G.45) and then we get a quasi-primary state which is just $: TT : |0\rangle$. Another option is to remove a descendant of the stress tensor to construct $|\Lambda_4\rangle$. To do the latter we need to remove the descendant of the stress tensor with weight 4 given by $\partial^2 T$

$$
\partial^2 T = \frac{\sqrt{2}}{N} : Tr(\partial^3\phi\partial\phi) : + \frac{\sqrt{2}}{N} : Tr(\partial^2\phi\partial^2\phi) : .
\tag{G.46}
$$

Acting on the vacuum we find

$$
\partial^2 T(0)|0\rangle = \frac{2\sqrt{2}}{N} Tr(\alpha_{-3}\alpha_{-1})|0\rangle + \frac{\sqrt{2}}{N} Tr(\alpha_{-2}^2)|0\rangle.
\tag{G.47}
$$

Consider now $L_1 = \frac{\sqrt{2}}{N}(Tr(\alpha_{-1}\alpha_2) + Tr(\alpha_{-2}\alpha_3 + \ldots))$ which acts as $L_1 Tr(\alpha_{-2}^2)|0\rangle = \frac{4\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$ and as $L_1 Tr(\alpha_{-3}\alpha_{-1})|0\rangle = \frac{3\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$. We can therefore construct a quasi-primary state annihilated by L_1: $Tr(\alpha_{-3}\alpha_{-1})|0\rangle - \frac{3}{4} Tr(\alpha_{-2}^2)|0\rangle$. The quasi-primary $|\Lambda_4\rangle$ is then given by:

$$
\begin{aligned}
|\Lambda_4\rangle &= \frac{1}{\sqrt{2}} \left[* TT * (0)|0\rangle - \frac{3}{5\sqrt{2}N} \partial^2 T(0)|0\rangle \right] \\
&= \frac{1}{2\sqrt{2}N^2} \left[Tr(\alpha_{-1}^2) Tr(\alpha_{-1}^2)|0\rangle - \frac{6}{5} Tr(\alpha_{-2}^2)|0\rangle + \frac{8}{5} Tr(\alpha_{-1}\alpha_{-3})|0\rangle \right]
\end{aligned}
\tag{G.48}
$$

There are two more weight 4 single trace quasi-primary operators given by

$$
\begin{aligned}
\mathcal{O}^{(1)} &= \frac{1}{2N^2} Tr((\partial\phi)^4) \\
\mathcal{O}^{(2)} &= \frac{n_{\mathcal{O}^{(2)}}}{N} \left(Tr(\partial^3\phi\partial\phi) - \frac{3}{2} Tr(\partial^2\phi\partial^2\phi) \right), \\
&= \frac{n_{\mathcal{O}^{(2)}}}{N} \left(\frac{1}{2}\partial^2 Tr(\partial\phi\partial\phi) - \frac{5}{2} Tr(\partial^2\phi\partial^2\phi) \right),
\end{aligned}
\tag{G.49}
$$

where $n_{\mathcal{O}^{(2)}}$ is some N-independent normalization constant. The state $|\Lambda_4\rangle$ can be written in terms of $:TT:(0)|0\rangle + a\mathcal{O}_2(0)|0\rangle$ in the following way

$$
|\Lambda_4\rangle = \frac{1}{\sqrt{2}} \left[:TT:(0)|0\rangle + \frac{2}{5Nn_{\mathcal{O}^{(2)}}} \mathcal{O}^{(2)}|0\rangle \right].
\tag{G.50}
$$

The OPE coefficient for $:TT:$ is up to a normalization the same as the scalar dimension 4 double trace operator in 4d and is given by

$$
\begin{aligned}
\langle \mathcal{O}_h \mathcal{O}_h :TT: \rangle &= \frac{1}{hN^h} \frac{1}{2N^2} 4h^2(3h-5)N^h \frac{1}{z_{13}^4 z_{23}^4 z_{12}^{2h-4}} \\
&= \frac{1}{N^2} 2h(3h-5) \frac{1}{z_{13}^4 z_{23}^4 z_{12}^{2h-4}},
\end{aligned}
\tag{G.51}
$$

where $4h^2(3h-5)$ come from the number of contractions giving planar diagrams. Consider now the OPE coefficient for $\mathcal{O}^{(2)}$. One finds

$$
\begin{aligned}
\langle \mathcal{O}_h \mathcal{O}_h \mathcal{O}^{(2)} \rangle &= \frac{n_{\mathcal{O}^{(2)}}N^h}{hN^{h+1} z_{13}^4 z_{23}^4 z_{12}^{2h-2}} \left[(-2)(-3)h^2(z_{13}^2 + z_{23}^2) - \frac{3}{2} 2h^2(-2)^2 z_{13}z_{23} \right] \\
&= \frac{6hn_{\mathcal{O}^{(2)}}}{N z_{13}^4 z_{23}^4 z_{12}^{2h-4}}.
\end{aligned}
\tag{G.52}
$$

Using (G.51), (G.52) and (G.50) we find the OPE coefficient for $|\Lambda_4\rangle$

$$
\langle \mathcal{O}_h \mathcal{O}_h \Lambda_4 \rangle = \frac{\sqrt{2}h(15h-19)}{5N^2}.
\tag{G.53}
$$

Note that the h dependence matches that of the weight 4 term in the two-point function (G.38). Additionally, the OPE coefficient given by (G.53) can not be extrapolated to the limit when $h \sim C_T$, as in this limit the planar expansion used for calculating (G.53) breaks down. For this reason, we can not test the thermalization of Λ_4 in heavy state $\mathcal{O}_{h_H}$. Let us consider the thermal one-point function which is given by

$$
\langle \Lambda_4 \rangle_\beta = \left[\frac{1}{\sqrt{2}} b_T^2 + \mathcal{O}(1) \right] = \frac{\pi^4 N^2}{18\sqrt{2}\beta^4},
\tag{G.54}
$$

where the term $\propto \frac{1}{N}\langle \mathcal{O}^{(2)} \rangle_\beta$ is subleading since it is single trace. We find that

$$
\langle \Lambda_4 \rangle_\beta \lambda_{\mathcal{O}_h \mathcal{O}_h \Lambda_4} = \frac{\pi^4 h(15h-19)}{90\beta^4},
\tag{G.55}
$$

which agrees with the weight 4 term in (G.38).

– 46 –

Note that it is explicitly seen that one can write Λ_4 either as $*TT*+(\text{desc. of T})$ or as $:TT: + \frac{1}{N}\mathcal{O}_{ST}$ with $\mathcal{O}_{ST}$ a quasi-primary single trace operator. In this case the single trace operator which one needs to add to $:TT:$ to get Λ_4 can be written as a sum of descendants $\mathcal{O}^{(2)} \propto \partial^2 T - \frac{5}{\sqrt{2}}Tr(\partial^2\phi\partial^2\phi)$. Explicitly, we have

$$
\begin{aligned}
|\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\left[*TT*(0) - \frac{3}{5\sqrt{2}N}\partial^2 T(0)\right]|0\rangle \\
&= \frac{1}{\sqrt{2}}\left[:TT:(0) + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}\right]|0\rangle.
\end{aligned}
\tag{G.56}
$$

As we saw above, using the second line in (G.56) it is straightforward to calculate correlation functions using Wick contractions to see that Λ_4 gives the full weight four contributions to the thermal two-point function for large-N theories.

Now, we consider the following quasi-primary operator

$$
\mathcal{O}_\Delta(z,\bar{z}) = \frac{\sqrt{2}}{\sqrt{\Delta}N^{\Delta/2}}:Tr\left((\partial\phi\bar{\partial}\bar{\phi})^{\frac{\Delta}{2}}\right):(z,\bar{z}),
\tag{G.57}
$$

where we denote the anti-holomorphic part of the free field by $\bar{\phi} = \bar{\phi}(\bar{z})$. The thermal two-point function of this operator, up to the terms subleading in large-N expansion, is given by

$$
\begin{aligned}
\langle\mathcal{O}_\Delta(z,\bar{z})\mathcal{O}_\Delta(0,0)\rangle_\beta &= \frac{\pi^{2\Delta}}{\beta^{2\Delta}\sin^\Delta\left(\frac{\pi z}{\beta}\right)\sin^\Delta\left(\frac{\pi\bar{z}}{\beta}\right)} \\
&= \frac{1}{(z\bar{z})^\Delta}\left(1 + \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2} + \frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) + \frac{\pi^4\Delta^2}{36\beta^4}z^2\bar{z}^2 + \mathcal{O}\left(\frac{1}{\beta^6}\right)\right).
\end{aligned}
\tag{G.58}
$$

One can easily check that the OPE coefficients of stress tensor T and its anti-holomorphic partner $\bar{T}$ with $\mathcal{O}_\Delta$ are given by

$$
\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}} = \frac{\Delta}{\sqrt{2}N},
\tag{G.59}
$$

while their thermal one-point function are given by

$$
\langle T\rangle_\beta = \langle\bar{T}\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}.
\tag{G.60}
$$

It is easy to check that terms proportional to β^{-2} in (G.58) are contributions of T and $\bar{T}$ operators

$$
\langle T\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T}z^2 + \langle\bar{T}\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}}\bar{z}^2 = \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2}.
\tag{G.61}
$$

We compute the OPE coefficient of operators Λ_4, defined by (G.48), and its anti-holomorphic partner $\bar{\Lambda}_4$ with $\mathcal{O}_\Delta$ and obtain

$$
\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4} = \frac{\Delta(5\Delta+2)}{10\sqrt{2}N^2},
\tag{G.62}
$$

which agrees with (C.26) in [66]. Its thermal one-point function (which is the same as $\langle \bar{\Lambda}_4 \rangle_\beta$) is given by (G.54). Another operator that contributes to thermal two-point function (G.58) is $:T\bar{T}:$. Its OPE coefficient with $\mathcal{O}_\Delta$ and thermal one-point function are given by

$$
\begin{aligned}
\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:} &= \frac{\Delta^2}{2N^2} \\
\langle :T\bar{T}: \rangle_\beta &= \frac{\pi^4 N^2}{18\beta^4}.
\end{aligned}
\tag{G.63}
$$

Again, it is easy to check

$$
\begin{aligned}
\langle \Lambda_4 \rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \Lambda_4} z^4 + \langle \bar{\Lambda}_4 \rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{\Lambda}_4} \bar{z}^4 + \langle :T\bar{T}: \rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:} z^2 \bar{z}^2 &= \\
= \frac{\pi^4 \Delta(5\Delta + 2)}{360\beta^4}(z^4 + \bar{z}^4) + \frac{\pi^4 \Delta^2}{36\beta^4} z^2 \bar{z}^2,
\end{aligned}
\tag{G.64}
$$

which matches with the corresponding terms in (G.58).

The OPE coefficients $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \Lambda_4}$, $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{\Lambda}_4}$, and $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:}$ can be extrapolated to the limit $\Delta \sim N^2$, by the same logic as in appendix C. Then, we can explicitly check the thermalization property of Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$. To establish a relation between the inverse temperature β and the conformal dimension Δ_H of heavy state $\mathcal{O}_H = \mathcal{O}_{\Delta \sim N^2}$, we assume the thermalization of stress tensor

$$
\langle T \rangle_\beta = \lambda_{\mathcal{O}_H \mathcal{O}_H T},
\tag{G.65}
$$

which implies

$$
\frac{\Delta_H}{N^2} = \frac{\pi^2}{3\beta^2}.
\tag{G.66}
$$

Using this relation, it is easy to show

$$
\begin{aligned}
\langle \Lambda_4 \rangle_\beta &= \lambda_{\mathcal{O}_H \mathcal{O}_H \Lambda_4} \Big|_{\frac{\Delta_H^2}{N^2}}, \\
\langle \bar{\Lambda}_4 \rangle_\beta &= \lambda_{\mathcal{O}_H \mathcal{O}_H \bar{\Lambda}_4} \Big|_{\frac{\Delta_H^2}{N^2}}, \\
\langle :T\bar{T}: \rangle_\beta &= \lambda_{\mathcal{O}_H \mathcal{O}_H :T\bar{T}:} \Big|_{\frac{\Delta_H^2}{N^2}}.
\end{aligned}
\tag{G.67}
$$

This means that operators Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$ thermalize in the quasi-primary state $\mathcal{O}_H$ similarly to the thermalization in a Virasoro primary states in large-c theory, that was analyzed in [42].

H Vector model

In this section we study the free scalar vector model at large-N. Consider the scalar operator

$$
\mathcal{O}_\Delta = \frac{1}{\sqrt{\mathcal{N}(\Delta)}} : (\varphi^i \varphi^i)^{\frac{\Delta}{2}} : (x),
\tag{H.1}
$$

where $\mathcal{N}(\Delta)$ is a normalization constant which to leading order in N is given by

$$\mathcal{N}(\Delta) \approx (\Delta)!! N^{\frac{\Delta}{2}}. \tag{H.2}$$

The thermal two-point function is given by

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \left(\frac{\Delta}{2}\right)^2 \frac{1}{\Delta} \tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \dots, \tag{H.3}$$

where

$$
\begin{aligned}
\tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^{\infty} \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\
&= \frac{\pi}{2\beta|\mathbf{x}|} \left[\mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\right) + \mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\right) \right].
\end{aligned} \tag{H.4}
$$

The thermal $a_{\tau,J}$ coefficients $a_{2,2}$ and $a_{4,4}$ are the same as in the adjoint model (this is so since the second term in (H.3) does not affect these):

$$
\begin{aligned}
a_{2,2} &= \frac{\pi^4 \Delta}{45}, \\
a_{4,4} &= \frac{\pi^8 \Delta(\Delta-1)}{1050}.
\end{aligned} \tag{H.5}
$$

The unit-normalized stress tensor is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : \left(\partial_\mu \varphi^i \partial_\nu \varphi^i - \frac{1}{2}\varphi^i \partial_\mu \partial_\nu \varphi^i - (\text{trace})\right) : (x), \tag{H.6}$$

where $C_T = \frac{4}{3}N$. The OPE coefficient of the stress tensor is again found by Wick contractions to be

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}} = -\frac{4\Delta}{3\sqrt{C_T}}, \tag{H.7}$$

in agreement with the stress tensor Ward identity. The double-stress tensor is given by

$$T_{\mu\nu\rho\sigma}^2 = \frac{1}{\sqrt{2}} : T_{(\mu\nu} T_{\rho\sigma)} : -(\text{traces}), \tag{H.8}$$

and the OPE coefficient is calculated precisely as for the adjoint model and we find

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{4,4}^2} = \frac{8\sqrt{2}}{9 C_T} \Delta(\Delta-1). \tag{H.9}$$

There is another double-trace operator with twist 4 and spin 4 and takes the same form : $\mathcal{O}_2 \mathcal{O}_{2,4}$: as for the adjoint model

$$
\begin{aligned}
\mathcal{O}_{\mu\nu\rho\sigma}^{\mathrm{DT}}(x) = \frac{1}{96\sqrt{70}N} : \varphi^i \varphi^i \Big(& \varphi^j \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \varphi^j - 16\partial_{(\mu} \varphi^j \partial_\nu \partial_\rho \partial_{\sigma)} \varphi^j \\
& + 18\partial_{(\mu} \partial_\nu \varphi^j \partial_\rho \partial_{\sigma)} \varphi^j - (\text{traces}) \Big) : (x).
\end{aligned} \tag{H.10}
$$

The OPE coefficient and the thermal one-point function yields the same result as for the corresponding operator in the adjoint model.[29] It then follows that the $a_{4,4}$ extracted from (H.3) is reproduced by the sum of the double stress tensor and (H.10).

---

[29]Note that this is not true for all operators but is in line with the fact that $a_{4,4}$ is unaffected by the second term in (H.3).

I Factorization of thermal correlators

In this appendix we argue for the factorization of thermal expectation values of multi-trace operators in large-C_T theories on $S^1 \times \mathbf{R}^{d-1}$. Consider the thermal two-point function of a scalar operator $\mathcal{O}$ with dimension Δ:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\beta = \langle \mathcal{O}\rangle_\beta \langle \mathcal{O}\rangle_\beta + \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\beta,c}, \tag{I.1}$$

where the second term consist of the connected part of the correlator. Note that the disconnected term in (I.1) is independent of the position x. On the other hand we can evaluate (I.1) using the OPE on the plane which takes the form

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{1}{|x|^{2\Delta}} + \sum_{n,l} \lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}} x^{2n+l}[\mathcal{O}\mathcal{O}]_{n,l} + \dots, \tag{I.2}$$

when written in terms of primaries and the dots refer to terms surpressed in the large-C_T limit. Note that $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}}$ are the MFT OPE coefficient which are of order 1. The term in (I.2) that is independent of x is due to the $n = l = 0$ term in (I.2) and inserting the OPE on the l.h.s. of (I.2), we find that

$$\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} \langle [\mathcal{O}\mathcal{O}]_{0,0}\rangle_\beta = \langle \mathcal{O}\rangle_\beta^2. \tag{I.3}$$

When $[\mathcal{O}\mathcal{O}]_{0,0}$ is unit-normalized the OPE coefficient is given by $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} = \sqrt{2}$ and it follows that

$$\langle [\mathcal{O}\mathcal{O}]_{0,0}\rangle_\beta = \frac{1}{\sqrt{2}}\langle \mathcal{O}\rangle_\beta^2. \tag{I.4}$$

We therefore see the that the thermal one-point function of the double-trace operator factorizes on the plane. We expect a similar argument to hold for multi stress tensors.

References

[1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [`hep-th/9711200`] [INSPIRE].

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`] [INSPIRE].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [`hep-th/9802109`] [INSPIRE].

[4] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*, SpringerBriefs in Physics (2016) [DOI] [`arXiv:1601.05000`] [INSPIRE].

[5] D. Simmons-Duffin, *The Conformal Bootstrap*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 1–74 (2017) [DOI] [`arXiv:1602.07982`] [INSPIRE].

[6] D. Poland, S. Rychkov and A. Vichi, *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications*, *Rev. Mod. Phys.* **91** (2019) 015002 [arXiv:1805.04405] [ɪɴSPIRE].

[7] E. Hijano, P. Kraus, E. Perlmutter and R. Snively, *Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks*, *JHEP* **01** (2016) 146 [arXiv:1508.00501] [ɪɴSPIRE].

[8] S. El-Showk and K. Papadodimas, *Emergent Spacetime and Holographic CFTs*, *JHEP* **10** (2012) 106 [arXiv:1101.4163] [ɪɴSPIRE].

[9] D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, *OPE Convergence in Conformal Field Theory*, *Phys. Rev. D* **86** (2012) 105043 [arXiv:1208.6449] [ɪɴSPIRE].

[10] E. Katz, S. Sachdev, E.S. Sørensen and W. Witczak-Krempa, *Conformal field theories at nonzero temperature: Operator product expansions, Monte Carlo, and holography*, *Phys. Rev. B* **90** (2014) 245109 [arXiv:1409.3841] [ɪɴSPIRE].

[11] N. Lashkari, A. Dymarsky and H. Liu, *Eigenstate Thermalization Hypothesis in Conformal Field Theory*, *J. Stat. Mech.* **1803** (2018) 033101 [arXiv:1610.00302] [ɪɴSPIRE].

[12] N. Lashkari, A. Dymarsky and H. Liu, *Universality of Quantum Information in Chaotic CFTs*, *JHEP* **03** (2018) 070 [arXiv:1710.10458] [ɪɴSPIRE].

[13] L. Iliesiu, M. Koloğlu, R. Mahajan, E. Perlmutter and D. Simmons-Duffin, *The Conformal Bootstrap at Finite Temperature*, *JHEP* **10** (2018) 070 [arXiv:1802.10266] [ɪɴSPIRE].

[14] Y. Gobeil, A. Maloney, G.S. Ng and J.-q. Wu, *Thermal Conformal Blocks*, *SciPost Phys.* **7** (2019) 015 [arXiv:1802.10537] [ɪɴSPIRE].

[15] L.V. Delacrétaz, T. Hartman, S.A. Hartnoll and A. Lewkowycz, *Thermalization, Viscosity and the Averaged Null Energy Condition*, *JHEP* **10** (2018) 028 [arXiv:1805.04194] [ɪɴSPIRE].

[16] A.C. Petkou and A. Stergiou, *Dynamics of Finite-Temperature Conformal Field Theories from Operator Product Expansion Inversion Formulas*, *Phys. Rev. Lett.* **121** (2018) 071602 [arXiv:1806.02340] [ɪɴSPIRE].

[17] L. Iliesiu, M. Koloğlu and D. Simmons-Duffin, *Bootstrapping the 3d Ising model at finite temperature*, *JHEP* **12** (2019) 072 [arXiv:1811.05451] [ɪɴSPIRE].

[18] S. Banerjee, K. Papadodimas, S. Raju, P. Samantray and P. Shrivastava, *A Bound on Thermal Relativistic Correlators at Large Spacelike Momenta*, *SciPost Phys.* **8** (2020) 064 [arXiv:1902.07203] [ɪɴSPIRE].

[19] A. Manenti, *Thermal CFTs in momentum space*, *JHEP* **01** (2020) 009 [arXiv:1905.01355] [ɪɴSPIRE].

[20] P. Sabella-Garnier, K. Schalm, T. Vakhtel and J. Zaanen, *Thermalization/Relaxation in integrable and free field theories: an Operator Thermalization Hypothesis*, arXiv:1906.02597 [ɪɴSPIRE].

[21] A. Bukva, P. Sabella-Garnier and K. Schalm, *Operator thermalization vs eigenstate thermalization*, arXiv:1911.06292 [ɪɴSPIRE].

[22] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici and M. Smolkin, *Thermal Order in Conformal Theories*, *Phys. Rev. D* **102** (2020) 065014 [arXiv:2005.03676] [ɪɴSPIRE].

[23] L.V. Delacretaz, *Heavy Operators and Hydrodynamic Tails*, *SciPost Phys.* **9** (2020) 034 [arXiv:2006.01139] [INSPIRE].

[24] J. Engelsöy, J. Larana-Aragon, B. Sundborg and N. Wintergerst, *Operator thermalisation in $d > 2$: Huygens or resurgence*, *JHEP* **09** (2020) 103 [arXiv:2007.00589] [INSPIRE].

[25] L.F. Alday, M. Kologlu and A. Zhiboedov, *Holographic correlators at finite temperature*, *JHEP* **06** (2021) 082 [arXiv:2009.10062] [INSPIRE].

[26] J.M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43** (1991) 2046.

[27] M. Srednicki, *Chaos and quantum thermalization*, *Phys. Rev. E* **50** (1994) 888.

[28] M. Rigol, V. Dunjko and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, *Nature* **452** (2008) 854.

[29] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, *Adv. Phys.* **65** (2016) 239 [arXiv:1509.06411] [INSPIRE].

[30] J.M. Deutsch, *Eigenstate Thermalization Hypothesis*, *Rept. Prog. Phys.* **81** (2018) 082001.

[31] C.T. Asplund, A. Bernamonti, F. Galli and T. Hartman, *Holographic Entanglement Entropy from 2d CFT: Heavy States and Local Quenches*, *JHEP* **02** (2015) 171 [arXiv:1410.1392] [INSPIRE].

[32] P. Caputa, J. Simón, A. Štikonas and T. Takayanagi, *Quantum Entanglement of Localized Excited States at Finite Temperature*, *JHEP* **01** (2015) 102 [arXiv:1410.2287] [INSPIRE].

[33] A.L. Fitzpatrick, J. Kaplan and M.T. Walters, *Virasoro Conformal Blocks and Thermality from Classical Background Fields*, *JHEP* **11** (2015) 200 [arXiv:1501.05315] [INSPIRE].

[34] A.L. Fitzpatrick, J. Kaplan, M.T. Walters and J. Wang, *Hawking from Catalan*, *JHEP* **05** (2016) 069 [arXiv:1510.00014] [INSPIRE].

[35] A.L. Fitzpatrick and J. Kaplan, *Conformal Blocks Beyond the Semi-Classical Limit*, *JHEP* **05** (2016) 075 [arXiv:1512.03052] [INSPIRE].

[36] A.L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, *On information loss in AdS_3/CFT_2*, *JHEP* **05** (2016) 109 [arXiv:1603.08925] [INSPIRE].

[37] T. Anous, T. Hartman, A. Rovai and J. Sonner, *Black Hole Collapse in the $1/c$ Expansion*, *JHEP* **07** (2016) 123 [arXiv:1603.04856] [INSPIRE].

[38] J. de Boer and D. Engelhardt, *Remarks on thermalization in 2D CFT*, *Phys. Rev. D* **94** (2016) 126019 [arXiv:1604.05327] [INSPIRE].

[39] P. Kraus and A. Maloney, *A cardy formula for three-point coefficients or how the black hole got its spots*, *JHEP* **05** (2017) 160 [arXiv:1608.03284] [INSPIRE].

[40] A.L. Fitzpatrick and J. Kaplan, *On the Late-Time Behavior of Virasoro Blocks and a Classification of Semiclassical Saddles*, *JHEP* **04** (2017) 072 [arXiv:1609.07153] [INSPIRE].

[41] H.H. Lai and K. Yang, *Entanglement entropy scaling laws and eigenstate typicality in free fermion systems*, *Phys. Rev. B* **91** (2015) 081110 [arXiv:1409.1224].

[42] P. Basu, D. Das, S. Datta and S. Pal, *Thermality of eigenstates in conformal field theories*, *Phys. Rev. E* **96** (2017) 022149 [arXiv:1705.03001] [INSPIRE].

[43] S. He, F.-L. Lin and J.-j. Zhang, *Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT*, *JHEP* **08** (2017) 126 [arXiv:1703.08724] [INSPIRE].

[44] H. Chen, C. Hussong, J. Kaplan and D. Li, *A Numerical Approach to Virasoro Blocks and the Information Paradox*, *JHEP* **09** (2017) 102 [arXiv:1703.09727] [INSPIRE].

[45] S. He, F.-L. Lin and J.-j. Zhang, *Dissimilarities of reduced density matrices and eigenstate thermalization hypothesis*, *JHEP* **12** (2017) 073 [arXiv:1708.05090] [INSPIRE].

[46] E.M. Brehm, D. Das and S. Datta, *Probing thermality beyond the diagonal*, *Phys. Rev. D* **98** (2018) 126015 [arXiv:1804.07924] [INSPIRE].

[47] T. Faulkner and H. Wang, *Probing beyond ETH at large c*, *JHEP* **06** (2018) 123 [arXiv:1712.03464] [INSPIRE].

[48] A. Romero-Bermúdez, P. Sabella-Garnier and K. Schalm, *A Cardy formula for off-diagonal three-point coefficients; or, how the geometry behind the horizon gets disentangled*, *JHEP* **09** (2018) 005 [arXiv:1804.08899] [INSPIRE].

[49] Y. Hikida, Y. Kusuki and T. Takayanagi, *Eigenstate thermalization hypothesis and modular invariance of two-dimensional conformal field theories*, *Phys. Rev. D* **98** (2018) 026003 [arXiv:1804.09658] [INSPIRE].

[50] W.-Z. Guo, F.-L. Lin and J. Zhang, *Note on ETH of descendant states in 2D CFT*, *JHEP* **01** (2019) 152 [arXiv:1810.01258] [INSPIRE].

[51] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, *Thermal Correlation Functions of KdV Charges in 2D CFT*, *JHEP* **02** (2019) 044 [arXiv:1810.11053] [INSPIRE].

[52] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, *Generalized Gibbs Ensemble and the Statistics of KdV Charges in 2D CFT*, *JHEP* **03** (2019) 075 [arXiv:1810.11054] [INSPIRE].

[53] A. Dymarsky and K. Pavlenko, *Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit*, *JHEP* **01** (2019) 098 [arXiv:1810.11025] [INSPIRE].

[54] A. Dymarsky and K. Pavlenko, *Exact generalized partition function of 2D CFTs at large central charge*, *JHEP* **05** (2019) 077 [arXiv:1812.05108] [INSPIRE].

[55] E.M. Brehm and D. Das, *Korteweg-de Vries characters in large central charge CFTs*, *Phys. Rev. D* **101** (2020) 086025 [arXiv:1901.10354] [INSPIRE].

[56] A. Dymarsky and K. Pavlenko, *Generalized Eigenstate Thermalization Hypothesis in 2D Conformal Field Theories*, *Phys. Rev. Lett.* **123** (2019) 111602 [arXiv:1903.03559] [INSPIRE].

[57] T. Anous and J. Sonner, *Phases of scrambling in eigenstates*, *SciPost Phys.* **7** (2019) 003 [arXiv:1903.03143] [INSPIRE].

[58] S. Datta, P. Kraus and B. Michel, *Typicality and thermality in 2d CFT*, *JHEP* **07** (2019) 143 [arXiv:1904.00668] [INSPIRE].

[59] M. Beşken, S. Datta and P. Kraus, *Quantum thermalization and Virasoro symmetry*, *J. Stat. Mech.* **2006** (2020) 063104 [arXiv:1907.06661] [INSPIRE].

[60] S. Collier, A. Maloney, H. Maxfield and I. Tsiares, *Universal dynamics of heavy operators in CFT_2*, *JHEP* **07** (2020) 074 [arXiv:1912.00222] [INSPIRE].

[61] D. Das, Y. Kusuki and S. Pal, *Universality in asymptotic bounds and its saturation in 2D CFT*, *JHEP* **04** (2021) 288 [arXiv:2011.02482] [INSPIRE].

[62] M. Kulaxizi, G.S. Ng and A. Parnachev, *Black Holes, Heavy States, Phase Shift and Anomalous Dimensions*, *SciPost Phys.* **6** (2019) 065 [`arXiv:1812.03120`] [ɪɴSPIRE].

[63] A.L. Fitzpatrick and K.-W. Huang, *Universal Lowest-Twist in CFTs from Holography*, *JHEP* **08** (2019) 138 [`arXiv:1903.05306`] [ɪɴSPIRE].

[64] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, *Black Holes and Conformal Regge Bootstrap*, *JHEP* **10** (2019) 046 [`arXiv:1904.00060`] [ɪɴSPIRE].

[65] Y.-Z. Li, Z.-F. Mai and H. Lü, *Holographic OPE Coefficients from AdS Black Holes with Matters*, *JHEP* **09** (2019) 001 [`arXiv:1905.09302`] [ɪɴSPIRE].

[66] M. Kulaxizi, G.S. Ng and A. Parnachev, *Subleading Eikonal, AdS/CFT and Double Stress Tensors*, *JHEP* **10** (2019) 107 [`arXiv:1907.00867`] [ɪɴSPIRE].

[67] A.L. Fitzpatrick, K.-W. Huang and D. Li, *Probing universalities in $d > 2$ CFTs: from black holes to shockwaves*, *JHEP* **11** (2019) 139 [`arXiv:1907.10810`] [ɪɴSPIRE].

[68] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, *Leading Multi-Stress Tensors and Conformal Bootstrap*, *JHEP* **01** (2020) 076 [`arXiv:1909.05775`] [ɪɴSPIRE].

[69] Y.-Z. Li, *Heavy-light Bootstrap from Lorentzian Inversion Formula*, *JHEP* **07** (2020) 046 [`arXiv:1910.06357`] [ɪɴSPIRE].

[70] R. Karlsson, *Multi-stress tensors and next-to-leading singularities in the Regge limit*, *JHEP* **08** (2020) 037 [`arXiv:1912.01577`] [ɪɴSPIRE].

[71] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, *Stress tensor sector of conformal correlators operators in the Regge limit*, *JHEP* **07** (2020) 019 [`arXiv:2002.12254`] [ɪɴSPIRE].

[72] Y.-Z. Li and H.-Y. Zhang, *More on heavy-light bootstrap up to double-stress-tensor*, *JHEP* **10** (2020) 055 [`arXiv:2004.04758`] [ɪɴSPIRE].

[73] A.L. Fitzpatrick, K.-W. Huang, D. Meltzer, E. Perlmutter and D. Simmons-Duffin, *Model-dependence of minimal-twist OPEs in $d > 2$ holographic CFTs*, *JHEP* **11** (2020) 060 [`arXiv:2007.07382`] [ɪɴSPIRE].

[74] L.G. Yaffe, *Large N Limits as Classical Mechanics*, *Rev. Mod. Phys.* **54** (1982) 407 [ɪɴSPIRE].

[75] F.R. Brown and L.G. Yaffe, *The Coherent State Variational Algorithm: A Numerical Method for Solving Large N Gauge Theories*, *Nucl. Phys. B* **271** (1986) 267 [ɪɴSPIRE].

[76] T.A. Dickens, U.J. Lindqwister, W.R. Somsky and L.G. Yaffe, *The Coherent State Variational Algorithm. 2. Implementation and Testing*, *Nucl. Phys. B* **309** (1988) 1 [ɪɴSPIRE].

[77] F.A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, *Nucl. Phys. B* **678** (2004) 491 [`hep-th/0309180`] [ɪɴSPIRE].

[78] A. Parnachev, *Near Lightcone Thermal Conformal Correlators and Holography*, *J. Phys. A* **54** (2021) 155401 [`arXiv:2005.06877`] [ɪɴSPIRE].

[79] A.L. Fitzpatrick and J. Kaplan, *Unitarity and the Holographic S-matrix*, *JHEP* **10** (2012) 032 [`arXiv:1112.4845`] [ɪɴSPIRE].

[80] A.L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, *The Analytic Bootstrap and AdS Superhorizon Locality*, *JHEP* **12** (2013) 004 [`arXiv:1212.3616`] [ɪɴSPIRE].

[81] Z. Komargodski and A. Zhiboedov, *Convexity and Liberation at Large Spin*, *JHEP* **11** (2013) 140 [`arXiv:1212.4103`] [ɪɴSPIRE].

[82] B. Sundborg, *The Hagedorn transition, deconfinement and N = 4 SYM theory*, *Nucl. Phys. B* **573** (2000) 349 [hep-th/9908001] [INSPIRE].

[83] P. Haggi-Mani and B. Sundborg, *Free large N supersymmetric Yang-Mills theory as a string theory*, *JHEP* **04** (2000) 031 [hep-th/0002189] [INSPIRE].

[84] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn — deconfinement phase transition in weakly coupled large N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [hep-th/0310285] [INSPIRE].

[85] M. Brigante, G. Festuccia and H. Liu, *Inheritance principle and non-renormalization theorems at finite temperature*, *Phys. Lett. B* **638** (2006) 538 [hep-th/0509117] [INSPIRE].

[86] I. Amado, B. Sundborg, L. Thorlacius and N. Wintergerst, *Probing emergent geometry through phase transitions in free vector and matrix models*, *JHEP* **02** (2017) 005 [arXiv:1612.03009] [INSPIRE].

[87] I. Amado, B. Sundborg, L. Thorlacius and N. Wintergerst, *Black holes from large N singlet models*, *JHEP* **03** (2018) 075 [arXiv:1712.06963] [INSPIRE].

[88] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [hep-th/9307010] [INSPIRE].

[89] A. Dymarsky, N. Lashkari and H. Liu, *Subsystem ETH*, *Phys. Rev. E* **97** (2018) 012140 [arXiv:1611.08764] [INSPIRE].

[90] H. Maxfield, *A view of the bulk from the worldline*, arXiv:1712.00885 [INSPIRE].

[91] E. Perlmutter, *Virasoro conformal blocks in closed form*, *JHEP* **08** (2015) 088 [arXiv:1502.07742] [INSPIRE].

[92] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Springer-Verlag (1997) [DOI] [INSPIRE].

[93] A. Belin and J. de Boer, *Random statistics of OPE coefficients and Euclidean wormholes*, *Class. Quant. Grav.* **38** (2021) 164001 [arXiv:2006.05499] [INSPIRE].

[94] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, *JHEP* **03** (2011) 025 [arXiv:1011.1485] [INSPIRE].

[95] S. Giusto, M.R.R. Hughes and R. Russo, *The Regge limit of AdS_3 holographic correlators*, *JHEP* **11** (2020) 018 [arXiv:2007.12118] [INSPIRE].