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$\mathrm{AdS}_3/\mathrm{AdS}_2$ degression of massless particles

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Abstract: We study a 3d/2d dimensional degression which is a Kaluza-Klein type mechanism in AdS_3 space foliated into AdS_2 hypersurfaces. It is shown that an AdS_3 massless particle of spin $s = 1, 2, \ldots, \infty$ degresses into a couple of AdS_2 particles of equal energies $E = s$. Note that the Kaluza-Klein spectra in higher dimensions are always infinite. To formulate the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression we consider branching rules for AdS_3 isometry algebra o(2,2) representations decomposed with respect to AdS_2 isometry algebra o(1,2). We find that a given o(2,2) higher-spin representation lying on the unitary bound (i.e. massless) decomposes into two equal o(1,2) modules. In the field-theoretical terms, this phenomenon is demonstrated for spin-2 and spin-3 free massless fields. The truncation to a finite spectrum can be seen by using particular mode expansions, (partial) diagonalizations, and identities specific to two dimensions.

Keywords: Field Theories in Lower Dimensions, Topological Field Theories, Higher Spin Gravity

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1 Introduction

A distinctive feature of Kaluza-Klein compactification [1, 2] of a top theory are infinite towers of fields of increasing masses and decreasing spins in the resulting bottom theory.[1] The same property holds both in the AdS dimensional degression [3, 4] and the AdS waveguide compactification [5]. In this paper we show that the spectrum of a bottom theory can be finite if one starts with a *topological* theory in higher-dimensional AdS space. Specifically, we study the AdS_3/AdS_2 degression for a spin-s massless field theory.

[1] In this paper, we use the terms *top* and *bottom* theories to designate respective higher-dimensional and lower-dimensional theories in the context of the Kaluza-Klein type mechanism.

The resulting AdS_2 spectrum consists of finitely many *propagating* modes. To some extent this phenomenon is similar to the Brown-Henneaux relation for $3d$ Einstein gravity with the cosmological term without local degrees of freedom which under appropriate AdS_3 boundary conditions leads to $2d$ boundary $\mathbb{R}^2$ local degrees of freedom spanning Virasoro algebra representations [6].

In the $\mathrm{AdS}_{d+1}/\mathrm{AdS}_d$ degression the spectrum of a bottom AdS_d theory is defined by branching rules for a particular representation of the AdS_{d+1} isometry algebra $\mathrm{o}(d,2)$ with respect to the AdS_d isometry algebra $\mathrm{o}(d-1,2) \subset \mathrm{o}(d,2)$. The $\mathrm{o}(d,2)$ representation may be chosen to describe massive or (partially-)massless AdS_{d+1} particle of any spin. Then, in $d > 2$ the branching rules for a massless spin-s $\mathrm{o}(d,2)$ representation give the following general pattern [4]: an infinite upper spin-s sequence of $\mathrm{o}(d-1,2)$ representations with running energies and a finite collection of lower spin-q $\mathrm{o}(d-1,2)$ representations with fixed energies, where $q = 0, 1, \ldots, s-1$. In the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression the branching rules for relevant $\mathrm{o}(2,2)$ representations with respect to $\mathrm{o}(1,2)$ subalgebra are drastically simplified, involving only finitely many $\mathrm{o}(1,2)$ representations. Namely, a massless $\mathrm{o}(2,2)$ representation of spin $s = 1, 2, 3, \ldots$ turns out to be isomorphic to two $\mathrm{o}(1,2)$ representations which are Verma modules of equal weights s.

On the field-theoretical level, when considering respective field theory of higher rank tensor (gauge) fields, such a truncation is basically due to the so-called Schouten identities for two-dimensional kinetic operators. It follows that a massless spin-s AdS_3 top theory degresses into an AdS_2 bottom theory consisting of the Klein-Gordon and Proca fields of equal energies $E = s$ (but different masses $m^2_{KG} = E(E-1)$ and $m^2_P = E(E-1) - 1$, respectively). On the contrary, degressing spin-0 massive/massless theories in AdS_3 which do have local degrees of freedom we still obtain an infinite spectrum of the respective bottom theory in AdS_2.[2] In general, an infinite AdS_2 spectrum persists for all massive spin-s field AdS_3 theories.

The paper is organized as follows. In section 2 we discuss the general known facts about compactification/degression in AdS spaces. We summarize the basic features of the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression focusing on differences with the higher-dimensional case which result in truncating the spectrum from infinite to finite one. In section 3 we shortly recall a few basic facts about relevant $\mathrm{o}(d,2)$ representations paying particular attention to $d = 1$ and $d = 2$ cases.[3] Also, we formulate the branching rules specifying which $\mathrm{o}(1,2)$ representations occur in a given type of $\mathrm{o}(2,2)$ representations. In section 4 we explicitly consider how the spin-2 massless field theory degresses from three to two dimensions. To this end we first consider a spin-2 massless field theory in AdS_{d+1} spacetime (i.e. the linearized Einstein gravity with the cosmological constant) and then restrict to $d = 2$ dimensions. In section 5 we extend the results and techniques of the previous section to the case of spin-3 massless fields. The conclusions and outlooks are given in section 6. Appendix A summarizes our notation and conventions for AdS spaces. In appendix B we discuss the representation the-

[2] Discussion of AdS_3 field dynamics and review of the $\mathrm{SL}(2,\mathbb{R})$ group representations can be found e.g. in [3, 7]. AdS_2 field equations and their solutions attracted recently some attention mainly because of the SYK/JT duality problem and the conformal bootstrap, see e.g. recent [8–10].

[3] For more discussion in the present context see e.g. [11–14] and appendix B.

ory of AdS_2 and AdS_3 isometry algebras in the context of adding discrete transformations inherited from the respective Lie groups. Appendix C contains an extended discussion of the Jacobi polynomials and associated basis functions needed to perform the AdS_3/AdS_2 degression. Appendix D describes the Schouten identities in two and higher dimensions. All intermediate calculations from section 5 are collected in appendix E.

2 Summary of the AdS_3/AdS_2 degression

By analogy with the Poincare coordinates representing AdS_{d+1} space as a stack of Minkowski spaces $\mathbb{R}^{d-1,1}$ with the growing warp factor, one may introduce new coordinates to represent AdS_{d+1} as sliced by AdS_d spaces of the growing radius which is effectively the $(d+1)$-th dimension [3, 15]. Similar to the standard Kaluza-Klein analysis on a manifold being a direct product of two other manifolds, one considers a space $\mathcal{M}^{d+1}$ sliced into AdS_d hypersurfaces continuously parameterized by a finite interval variable with the full line element

$$ds(\mathcal{M}^{d+1})^2 = \frac{1}{\cos^2\theta}\left[ds(\mathrm{AdS}_d)^2 + \ell^2_{\mathrm{AdS}}d\theta^2\right] . \tag{2.1}$$

Here, ℓ_{AdS} is the AdS_d radius and the slicing variable θ belongs to the closed interval $\theta \in [-\alpha, \alpha]$ with α restricted as $\alpha < \pi/2$ [5]. Note that parameters ℓ_{AdS} and α define two independent scales. An asymptotic $\alpha \to \pi/2$ defines the decompactification limit analogous to the infinite radius limit of the Kaluza-Klein manifold. In this limit $\mathcal{M}^{d+1}$ goes back to AdS_{d+1}. The conformal boundary ∂AdS_{d+1} consists of two limiting $\alpha = \pm\pi/2$ hypersurfaces (see [15] for more details).

At $\alpha \neq \pi/2$ one may develop a sort of compactification of AdS_{d+1} called the AdS waveguide compactification [5]. This is strictly analogous to the original Kaluza-Klein compactification (see e.g. [16] for recent discussion). In the decompactifying limit $\alpha = \pi/2$ one deals instead with a different mechanism called the dimensional degression [3, 4] (see also [17]).

In practice, at $\alpha = \pi/2$ it is more convenient to work with another coordinate form of the line element (2.1) obtained by the change $\tan\theta = \sinh z$ with $z \in (-\infty, \infty)$ [4]

$$ds(\mathrm{AdS}_{d+1})^2 = \cosh^2 z\left[ds(\mathrm{AdS}_d)^2 + \frac{\ell^2_{\mathrm{AdS}}}{\cosh^2 z}dz^2\right] , \tag{2.2}$$

where we explicitly singled out the warp factor to identify the conformal boundary at $z = \pm\infty$. In the sequel we use the $d = 2$ version of (2.2).[4]

Spin-s massless fields. Below we outline the basic ingredients of the AdS_3/AdS_2 degression for spin-s massless fields. The general approach is then illustrated by the examples of spins $s = 2$ and $s = 3$ in sections 4 and 5. More extended discussion of the higher-dimensional AdS_{d+1}/AdS_d degression can be found in [3–5].

Let us consider the Fronsdal theory of double-traceless totally-symmetric tensor gauge fields $\Phi^{m_1\ldots m_s}(x, z)$ [19–21] which describe free massless spin-s particles propagating in

[4] A foliation of AdS_3 by AdS_2 slices was recently discussed in [18].

the AdS_{d+1} space with the metric (2.2) in the local AdS_d coordinates x and the slicing coordinate z. It is convenient to keep the dimension of space d arbitrary. Of course, in $d+1=3$ dimensions the Fronsdal theory becomes topological that is explicitly manifested within the Chern-Simons theory with the gauge algebra $sl(N)\oplus sl(N)$ [22–24].

A rank-s tensor field can be decomposed into $s+1$ lower rank component fields with convention that Latin indices run $d+1$ values and Greek indices run d values:

$$\Phi^{m_1\ldots m_s}(x,z) = \left\{\Phi^{\mu_1\cdots\mu_s}(x,z),\ \Phi^{\mu_1\cdots\mu_{s-1}}(x,z),\ \ldots,\Phi^{\mu_1}(x,z),\ \Phi(x,z)\right\}. \tag{2.3}$$

The component fields here satisfy appropriate trace conditions followed from the Fronsdal double-traceless condition.

Eigenfunction expansions. In order to obtain a degressed theory in one less dimension one integrates out the slicing coordinate z. This can be done by expanding the component functions (2.3) with respect to orthonormal sets of eigenfunctions (basis functions) $P_n^t(z)$ of auxiliary second-order differential operators in the z-coordinate. These operators naturally arise in the original AdS_{d+1} Fronsdal action written in terms of the metric (2.2). In this way, a rank-t component field

$$\Phi^{\mu_1\cdots\mu_t}(x,z) = \sum_{n=0}^{\infty}\phi_n^{\mu_1\cdots\mu_t}(x)\,P_n^t(z)\,, \qquad t=0,\ldots,s\,, \tag{2.4}$$

decomposes into an infinite collection of rank-t totally-symmetric tensor fields in AdS_d space. Note that the basis functions $P_n^t(z)$ are labelled by t which means that a rank-t component field $\Phi^{\mu_1\cdots\mu_t}(x,z)$ can have its own mode expansion. Remarkably, there is a unified choice of basis functions in terms of the Jacobi polynomials $J_n^{\alpha,\beta}$ of running degrees for all rank-t component fields, $P_n^t(z)\sim(\cosh z)^\gamma J_n^{\alpha,\beta}(-\tanh z)$ (for some α,β,γ in terms of d,n,t; see appendix C). To integrate out the slicing coordinate z in the original action one calculates the following overlaps of two basis functions

$$\int_{-\infty}^{+\infty} dz\cosh^k z\,\left(AP_n^t\right)(z)\left(BP_m^{t'}\right)(z)\,, \tag{2.5}$$

for some integers k,n,m,t,t', and some differential-algebraic operators A and B acting on the basis functions. All ingredients in the overlap integrals (2.5) are constituents of the original second-order AdS_{d+1} action written in the coordinates x,z (2.2) so that the operators A and B are at most linear in the z-derivatives and involve hyperbolic coefficient functions.

Triangular transformations. In general, taking overlap integrals (2.5) is a technically complicated problem. It can be largely overcome by making triangular field redefinitions of the components fields $\Phi^{\mu_1\cdots\mu_t}(x,z)$. E.g. a rank-2 component can be redefined as

$$\Phi^{\mu_1\mu_2}\to\widetilde{\Phi}^{\mu_1\mu_2} = \Phi^{\mu_1\mu_2} - \frac{1}{(d-2)}\mathrm{sech}^2 z\,g^{\mu_1\mu_2}\Phi\,, \tag{2.6}$$

where $g^{\mu_1\mu_2}$ is the AdS_d metric, see (A.4). Such a redefinition is typical in the Kaluza-Klein type compactifications (see e.g. a review [25]).

Simultaneously, the above triangular transformations diagonalize the bottom action that makes it possible to recognize the spectrum in a standard fashion as a sum of free field theories with fixed spins and masses. In particular, cancelling off-diagonal terms fixes the numerical coefficient in (2.6). Strictly speaking, a diagonalization may also require partial gauge fixing. On the other hand, the presence of a pole in the space dimension does not allow making such a substitution in the $d = 2$ case. A properly modified diagonalization procedure in two dimensions involves the use of particular basis functions different from those used in the higher-dimensional case as well as some extra identities to be discussed later in this section.

Stueckelberg shift symmetry. Since AdS_{d+1} covariant derivatives contain algebraic contributions, then the gauge transformations of the component fields inherit such algebraic terms which are generally treated as Stueckelberg-type transformations. Indeed, the original Fronsdal field transforms as $\delta\Phi^{m_1\ldots m_s}(x,z) = \overline{\nabla}^{(m_1}\Xi^{m_2\ldots m_s)}(x,z)$, where $\Xi^{m_1\ldots m_{s-1}}$ is a traceless totally-symmetric tensor gauge parameter and $\overline{\nabla}$ is the AdS_{d+1} covariant derivative (see appendix A). Similar to (2.3) the gauge parameter decomposes as

$$\Xi^{m_1\ldots m_{s-1}}(x,z) = \{\xi^{\mu_1\ldots\mu_{s-1}}(x,z),\ \xi^{\mu_1\ldots\mu_{s-2}}(x,z),\ \ldots,\xi^{\mu_1}(x,z),\ \xi(x,z)\}\,, \tag{2.7}$$

into s independent gauge parameters subjected to their own trace conditions. Then, the component fields schematically transform as

$$\delta\Phi^{\mu_1\cdots\mu_t}(x,z) = \nabla^{(\mu_1}\xi^{\mu_2\ldots\mu_s)}(x,z) + \mathrm{S}^{\mu_1\cdots\mu_t}\left[\xi(x,z), g(x), \cosh z, \partial_z\right], \tag{2.8}$$

where $t = 0,\ldots,s-1$ and S denotes a Stueckelberg-type gauge operator which generally depends on component gauge parameters, AdS_d metric, hyperbolic functions in z, and first derivatives in z. The basic rationale behind this formula is that the AdS_d covariant derivative ∇ encoded in the AdS_{d+1} derivative $\overline{\nabla}$ is separated from other contributions which, therefore, contain the AdS_d metric and are at most of first order in the z-variable.

The mode expansion for the component gauge parameters goes along the same lines as for the component fields (2.4) yielding infinite sets of gauge parameters $\xi_n^{\mu_1\cdots\mu_t}(x)$ with $n = 0,1,2,\ldots,\infty$. It turns out that most of the field expansion modes $\phi_n^{\mu_1\cdots\mu_t}$ are Stueckelberg fields which can be gauged away. The remaining fields define a bottom theory: these are generally massive if compared with the original Fronsdal massless AdS_{d+1} field. The group-theoretical analysis in higher dimensions $d > 2$ shows that there should arise infinite collections of massive AdS_d fields with running masses and spins $0,1,\ldots,s$ [4, 5].

Schouten identities and PDoF. We now move on to considering the $\text{AdS}_3/\text{AdS}_2$ degression. As discussed earlier, the bottom action in this case cannot be directly made diagonal so that there is a problem of identifying the spectrum. Nonetheless, only finitely many (off-)diagonal lower-rank terms remain non-vanishing on-shell while an infinite number of higher-rank contributions can be systematically neglected by using specific two-dimensional identities. As a by-product, an expected infinite collection of massive fields on AdS_2 space reduces to just two massive spinless modes described by the Klein-Gordon and Proca actions.[5]

Indeed, AdS_2 kinetic operators for rank-s massive fields of the vacuum energy E are of the form

$$\left(-\Box + \ldots + m_s^2\right)\phi^{\mu_1\cdots\mu_s}(x)\,, \quad \text{where [26]:} \qquad m_s^2 = E(E-1) - s\,, \tag{2.9}$$

where the ellipses denote second-order derivative and trace terms which can be set to zero by imposing the TT (transverse + traceless) gauge conditions. However, all these kinetic operators trivialize except for rank 0 and 1 fields described by the Klein-Gordon and Proca actions. In those cases we have the field equations

$$\begin{aligned} &\mathrm{AdS}_2 \text{ Klein-Gordon theory:} && [-\Box + E(E-1)]\phi = 0\,, \\ &\mathrm{AdS}_2 \text{ Maxwell-Proca theory:} && [-\Box + E(E-1) - 1]\phi^\mu = 0\,, \quad \nabla_\mu\phi^\mu = 0\,. \end{aligned} \tag{2.10}$$

These two theories describe the same physical degrees of freedom spanned by the $o(1,2)$ Verma module of weight (energy) E. This agrees with our understanding that the only local degrees of freedom in AdS_2 are massive spinless modes. Both the Klein-Gordon and Proca theories with masses as those in (2.10) have the same on-shell but differently realized off-shell dynamical content.

The above-mentioned vanishing of the higher-rank kinetic operators is due to the so-called Schouten identities which tell us that the derivative part of a given kinetic operator is equivalent to a combination of algebraic terms (see appendix D). So, roughly speaking, in the higher-spin sector of the AdS_2 theory almost all of infinite number of equations of motion become algebraic. Basically, this means that all fields $\phi_n^{\mu_1\cdots\mu_s}(x)$ either vanish on-shell except for the zeroth scalar and vector modes $\phi_0(x)$ and $\phi_0^\mu(x)$, or are expressed in terms of $\phi_0(x)$ and $\phi_0^\mu(x)$ (i.e. are auxiliary fields).

To summarize, the spectrum of a bottom theory in the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression can be read off by means of the following multistage procedure: **(1)** specific basis functions, **(2)** Stueckelberg-type gauge fixing, **(3)** Schouten identities, **(4)** eliminating auxiliary fields. The resulting sharp shortening of the physical spectra in two dimensions as compared to higher dimensions has a clear group-theoretical explanation given in the next section.

3 Branching rules from $o(2,2)$ to $o(1,2)$

The global isometry of AdS_{d+1} is $o(d,2)$ algebra and its lowest-weight (non-)unitary representations identified with elementary particles are generalized Verma modules $\mathcal{D}_{o(d,2)}(E,s)$ characterized by energy E and spin s (here, we consider only totally-symmetric representations, see e.g. [27, 28]). At critical values $E_0 = s+d-t-2$ defined by the depth parameter

[5]Note that e.g. for gravitons the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression should be contrasted with the standard Kaluza-Klein compactification on $\mathbb{R}^{1,1}\times S^1$. The latter possesses the lowest eigenfunction with zero eigenvalue so that after the gauge fixing the remaining fields are massless. It follows that the two-dimensional bottom theory has non-dynamical fields only and, therefore, there are no PDoF [16]. Contrary, in the AdS degression the lowest modes are massive and so that there are finitely many dynamical fields in the bottom theory that changes the count of PDoF. We are grateful to K. Hinterbichler for a useful discussion of this issue.

$t \in \{0, 1, \ldots, s-1\}$, there are singular submodules $\mathcal{S}_t \subset \mathcal{D}(E_0, s)$ generated from singular vectors on the $(t+1)$-th level. These are given by $\mathcal{S}_t = \mathcal{D}(E_0', s')$, where $E_0' = E_0 + t + 1$ and $s' = s - t - 1$. Factoring them out yields irreducible quotients

$$\mathcal{H}(E_0, s) = \mathcal{D}(E_0, s)/\mathcal{S}_t\,. \tag{3.1}$$

In AdS_{d+1} space the resulting representations $\mathcal{H}(E_0, s)$ describe either non-unitary partially-massless spin-s fields (depth $t > 0$) [29] or unitary massless spin-s fields $(t = 0)$ [19]. Representations $\mathcal{D}(E, s)$ with the energy above the unitary bound $E > E_0(t = 0)$ describe unitary massive spin-s fields.

In lower dimensions $d = 1$ and $d = 2$ the above-described construction remains basically the same with a few additional specifications. See appendix B for a detailed discussion.

$\mathrm{o}(1,2)$ representations. In $d = 1$ all representations of the AdS_2 isometry algebra $\mathrm{o}(1, 2)$ are characterized by a single parameter which can be interpreted as the energy $E \in \mathbb{R}$, while the spin number is absent. The respective lowest-weight representations are $\mathrm{o}(1, 2)$ Verma modules $\mathcal{D}_E$. At $E > 0$ the representations are unitary, irreducible, and infinite-dimensional. At negative values of energy $E \leqslant 0$ the representations are no longer unitary. Moreover, when $E_0 = -j \in \frac{1}{2}\mathbb{Z}_{\leqslant 0}$, there is a singular submodule $\mathcal{S}_{E_0} = \mathcal{D}_{-E_0+1}$ so that one may consider the quotient $\mathcal{H}_{E_0} = \mathcal{D}_{E_0}/\mathcal{S}_{E_0}$ which is a finite-dimensional ($\dim \mathcal{H}_{E_0} = 2j + 1$) non-unitary $\mathrm{o}(1, 2)$ representation.[6]

$\mathrm{o}(2,2)$ representations. In $d = 2$ the AdS_3 isometry algebra is not simple and decomposes as $\mathrm{o}(2, 2) \approx \mathrm{o}(1, 2) \oplus \mathrm{o}(1, 2)$. Parameterizing the energy and spin as $E = h_1 + h_2 \geqslant 0$ and $s = |h_1 - h_2|$ one can see that the lowest-weight $\mathrm{o}(2, 2)$ spin-s representations $\mathcal{D}(E, s)$ decompose into $\mathrm{o}(1, 2)$ representations as

$$\mathcal{D}(h_1 + h_2, h_1 - h_2) = \left[\mathcal{D}_{h_1} \otimes \mathcal{D}_{h_2}\right] \oplus \left[\mathcal{D}_{h_2} \otimes \mathcal{D}_{h_1}\right], \tag{3.2}$$

at $s \neq 0$ and

$$\mathcal{D}^*(2h, 0) = \mathcal{D}_h \otimes \mathcal{D}_h\,, \tag{3.3}$$

at $s = 0$. Each [factor] in (3.2) corresponds to modes $\pm s$ which form together a parity-invariant combination. In the spin $s = 0$ case the representation (3.3) is parity-invariant per se. In particular, it follows that $\mathcal{D}(2h, 0) = \mathcal{D}^*(2h, 0) \oplus \mathcal{D}^*(2h, 0)$. Such representations are realized in $\mathrm{O}(2, 2)$ invariant QFT, i.e. when the respective space of states is invariant under $\mathrm{O}(2, 2)$ discrete symmetries which are time reversal and space reflection, see appendix B.

Since $\mathrm{o}(2, 2)$ representations under considerations are generally given by tensor products of $\mathrm{o}(1, 2)$ Verma modules, then they can be explicitly evaluated by means of the Clebsch-Gordan decompositions (see appendix B)

$$\mathcal{D}_{h_1} \otimes \mathcal{D}_{h_2} = \bigoplus_{h=h_1+h_2}^{\infty} \mathcal{D}_h\,, \tag{3.4}$$

$$\mathcal{D}_{h_3} \otimes \mathcal{H}_{h_4} = \bigoplus_{h=h_3+h_4}^{h_3-h_4} \mathcal{D}_h\,, \tag{3.5}$$

where $\forall h_1, h_2$ and $h_3 > 0$, $h_3 > |h_4|$, $h_4 \in \frac{1}{2}\mathbb{Z}_{\leqslant 0}$, summation in n goes with a step 1.

[6] Except for $\mathcal{H}_0$ which is unitarizable since it is a one-dimensional representation.

Branching rules for massive representations. Substituting (3.4) in (3.2) we obtain

$$\mathcal{D}(E,s) = \bigoplus_{n=0}^{\infty} \left[\mathcal{D}_{E+n} \oplus \mathcal{D}_{E+n}\right] \equiv 2 \bigoplus_{n=0}^{\infty} \mathcal{D}_{E+n}\,, \tag{3.6}$$

where the factor of 2 indicates that each representation is duplicated. The scalar representation (3.3) decomposes as

$$\mathcal{D}(E,0) = \bigoplus_{n=0}^{\infty} \mathcal{D}_{E+n}\,, \tag{3.7}$$

that precisely matches the branching rule for a scalar $o(d,2)$ representation [4] evaluated at $d = 2$. The above branching rules show that a given $\mathrm{o}(2,2)$ generalized Verma module decomposes into an infinite collection of $\mathrm{o}(1,2)$ Verma modules with an equidistant spectrum of weights.

Branching rules for (partially-)massless representations. In order to formulate the branching rules for the quotient representations (3.1) in $d = 2$ dimensions we use the Clebsch-Gordan series (3.4) and obtain both the original representation and the singular submodule in the form

$$\mathcal{D}(s-t,s) = 2\bigoplus_{n=0}^{\infty} \mathcal{D}_{s-t+n} \equiv \left[2\bigoplus_{k=0}^{t} \mathcal{D}_{s-k}\right] \oplus \left[2\bigoplus_{m=1}^{\infty} \mathcal{D}_{s+m}\right] \tag{3.8}$$

$$\mathcal{S}_t = \mathcal{D}(s+1, s-t-1) = 2\bigoplus_{m=1}^{\infty} \mathcal{D}_{s+m}\,. \tag{3.9}$$

Here, a summation in (3.8) is reorganized to isolate the singular submodule (3.9). Taking the quotient (3.8)/(3.9) we are left with the first factor in (3.8) given by

$$\mathcal{H}(s-t,s) = \bigoplus_{n=0}^{t} \left[\mathcal{D}_{s-n} \oplus \mathcal{D}_{s-n}\right]. \tag{3.10}$$

This is the branching rule for the (partially-)massless $\mathrm{o}(2,2)$ representations. We find out that, contrary to infinite towers of representations in $d \geqslant 3$, we have just a finite number of representations in $d = 2$.

According to [12, 13] the maximal-depth case $t = s-1$ has a degenerate interpretation. The corresponding singular submodule (3.9) is given by the duplicated scalar representation $\mathcal{D}^*(s+1,0) \oplus \mathcal{D}^*(s+1,0)$, where each factor is parity-invariant on its own. In principle, one can consider a quotient representation with either one or two scalar submodules factored out. Factoring out two copies $\mathcal{D}^*(s+1,0) \oplus \mathcal{D}^*(s+1,0)$ we obtain (3.10), while factoring out only one copy $\mathcal{D}^*(s+1,0)$ yields

$$\widetilde{\mathcal{H}}(s-t,s) = \mathcal{H}(s-t,s) \oplus \mathcal{D}^*(s+1,0) \equiv \bigoplus_{n=0}^{t} \left[\mathcal{D}_{s-n} \oplus \mathcal{D}_{s-n}\right] \bigoplus_{n=0}^{\infty} \mathcal{D}_{s+n+1}\,. \tag{3.11}$$

On the other hand, the right-hand side of the branching rule (3.10) can be packed again into the tensor product by virtue of the Clebsch-Gordan series (3.5) to obtain [12]

$$\mathcal{H}(s-t,s) = \left[\mathcal{D}_{s-\frac{t}{2}} \otimes \mathcal{H}_{-\frac{t}{2}}\right] \oplus \left[\mathcal{H}_{-\frac{t}{2}} \otimes \mathcal{D}_{s-\frac{t}{2}}\right], \tag{3.12}$$

where $\mathcal{H}_{-\frac{t}{2}}$ is a $(t+1)$-dimensional $o(1,2)$ representation. In particular, (3.12) explicitly shows that all $t \neq 0$ representations are non-unitary due to $\mathcal{H}_{-\frac{t}{2}}$ factor which is necessarily non-unitary being finite-dimensional. The unitarity holds in the case $t = 0$ only.

Summary. Relations (3.6), (3.7), (3.10), and (3.11) are the branching rules for, respectively, massive spin-s, massive(massless) spin-0, and (partially-)massless spin-s depth-t $o(2,2)$ representations. Below is the summary list of these representations.

$$
\begin{aligned}
&\text{massive(massless) spin-0:} && \mathcal{D}(E,0) = \bigoplus_{n=0}^{\infty} \mathcal{D}_{E+n} \\
&\text{massive spin-}s\text{:} && \mathcal{D}(E,s) = \bigoplus_{n=0}^{\infty} \left[\mathcal{D}_{E+n} \oplus \mathcal{D}_{E+n}\right] \\
&\text{massless spin-}s\text{:} && \mathcal{H}(s,s) = \mathcal{D}_s \oplus \mathcal{D}_s \\
&\text{partially-massless spin-}s\text{ (I): any depth} && \mathcal{H}(s-t,s) = \left[\mathcal{D}_{s-t} \oplus \cdots \oplus \mathcal{D}_s\right] \oplus \left[\mathcal{D}_{s-t} \oplus \cdots \oplus \mathcal{D}_s\right] \\
&\text{partially-massless spin-}s\text{ (II): maximal depth} && \widetilde{\mathcal{H}}(1,s) = \mathcal{H}(1,s) \bigoplus_{n=0}^{\infty} \mathcal{D}_{s+n+1}
\end{aligned}
\tag{3.13}
$$

In the next sections we consider the field-theoretical realization of the massless spin-s representations $\mathcal{H}(s,s)$. In particular, from our discussion in section 2 we conclude that regardless of spin value $\mathcal{H}(s,s)$ are described by the Klein-Gordon and Proca equations in AdS_2 space which correspond to two factors on the right-hand side of the respective branching rule. Note that the maximal depth partially-massless systems can have or not have local degrees of freedom depending on a local field theory formulation. In particular, the Maxwell theory is simultaneously a massless spin-1 and a maximal depth partially-massless spin-1 system which is classified in the list (3.13) by the last line. It follows that the AdS_3/AdS_2 degression of the Maxwell theory yields an infinite spectrum. Thus, the phenomenon of finite spectra begins to fully manifest itself when $s = 2$.

4 AdS_3/AdS_2 degression of spin-2 massless fields

Consider first the linearized action of $(d+1)$-dimensional gravity with the cosmological term. The resulting theory of symmetric traceful rank-2 tensor field $\Phi^{mn} = \Phi^{mn}(x,z)$ describes a massless spin-2 particle propagating in AdS_{d+1} spacetime,

$$
\begin{aligned}
S = \int d\mu_{d+1}\Big\{ &-(\overline{\nabla}_a\Phi^{mn})^2 + 2(\overline{\nabla}_m\Phi^{mn})^2 - 2\overline{\nabla}_m\Phi\overline{\nabla}_n\Phi^{mn} + (\overline{\nabla}_a\Phi)^2 + \\
&+ b\left(2(\Phi^{mn})^2 + (d-2)\Phi^2\right)\Big\}, \qquad \Phi \equiv \Phi^{mn}G_{mn}\,,
\end{aligned}
\tag{4.1}
$$

where for future convenience we introduced the concise notation for the integration measure $d\mu_{d+1}$ (A.5) and $G_{mn}(x,z)$ is the AdS_{d+1} metric (A.4) (see appendix A for other conventions and notation). The action is invariant under the gauge transformations with the gauge parameter Ξ^m

$$\delta\Phi^{mn}(x,z) = \overline{\nabla}^m\Xi^n(x,z) + \overline{\nabla}^n\Xi^m(x,z)\,, \tag{4.2}$$

where $\overline{\nabla}$ stands for the AdS$_{d+1}$ covariant derivative. Throughout this section we keep the dimension of space d arbitrary and set $d = 2$ in the concluding subsection 4.3.

4.1 $d + 1$ split and component actions

According to the general strategy described in section 2 we decompose a rank-2 field Φ^{mn} into rank-$0, 1, 2$ component fields as

$$\Phi^{mn}(x,z) = \{\Phi^{\mu\nu}(x,z),\ \Phi^{\mu}(x,z),\ \Phi(x,z)\} \coloneqq \{h^{\mu\nu}(x,z),\ A^{\mu}(x,z),\ \phi(x,z)\}\,, \tag{4.3}$$

along with the gauge parameter

$$\Xi^m(x,z) = \{\Xi^\mu(x,z),\ \Xi(x,z)\} \coloneqq \{ab^{-1}\cosh^2 z\,\xi^\mu(x,z),\ ab^{-1}\cosh^2 z\,\xi(x,z)\}\,, \tag{4.4}$$

where for future convenience we renamed and redefined component tensors by means of the hyperbolic functions, and a, b are convenient numerical factors parameterizing AdS curvatures (see appendix A). Then, plugging (4.3) and (4.4) as well as the metric G_{mn} (A.4) into the action (4.1) we find the component representation

$$S = \sum_{m\geqslant n}\ \sum_{n=0,1,2} S_{mn}\,, \tag{4.5}$$

where

$$S_{22} = a\iint \cosh^2 z\Big\{-(\nabla_\alpha h^{\mu\nu})^2 + 2(\nabla_\mu h^{\mu\nu})^2 - 2\nabla_\mu h\nabla_\nu h^{\mu\nu} + (\nabla_\alpha h)^2 + \\ + a\big[2(h^{\mu\nu})^2 + (d-3)h^2 + h_{\mu\nu}\,Ł_d\,(\cosh^2 z\,Ł_2\,h^{\mu\nu}) - hŁ_d(\cosh^2 z\,Ł_2\,h)\big]\Big\}\,, \tag{4.6}$$

$$S_{21} = 4a\iint \cosh^2 z\Big\{\nabla_\mu h^{\mu\nu}\,Ł_d\,A_\nu - \nabla_\mu h\,Ł_d\,A^\mu\Big\}\,, \tag{4.7}$$

$$S_{20} = 2\iint \Big\{\phi[\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h] + a(d-1)\phi h - a(d-1)\phi\tanh z\cosh^2 z\,Ł_2\,h\Big\}\,, \tag{4.8}$$

$$S_{11} = 2\iint \Big\{-(\nabla_\mu A^\nu)^2 + (\nabla_\mu A^\mu)^2 - a(d-1)(A^\mu)^2\Big\}\,, \tag{4.9}$$

$$S_{10} = 4(d-1)\iint \tanh z\,\nabla_\mu A^\mu\phi\,, \tag{4.10}$$

$$S_{00} = d(d-1)\iint \tanh^2 z\,\phi^2\,. \tag{4.11}$$

Here, $h = g_{\mu\nu}h^{\mu\nu}$ and $\Box = \nabla_\mu\nabla^\mu$, also we introduced the double-integral notation (A.5)

$$\iint = b^{-1}\int d\mu_{d+1} = a^{d/2}b^{-(d+3)/2}\int d\mu_d\int dz\,\cosh^d z\,. \tag{4.12}$$

The component actions (4.6)–(4.11) are essentially those obtained in [4]. Here we slightly rearranged terms and denoted different action components as S_{mn} to indicate contributions of rank-m and rank-n components. Also, following [5] we introduced a convenient notation $Ł_m$ (C.15) for particular first-order differential operators in the z-coordinate.

The gauge transformations (4.2) are given by

$$\delta h^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu + 2\tanh z\, g^{\mu\nu}\xi\,, \tag{4.13}$$

$$\delta A^\mu = \nabla^\mu \xi + a\cosh^2\! z\, Ł_2 \xi^\mu\,, \tag{4.14}$$

$$\delta\phi = 2a\cosh^2\! z\, Ł_2 \xi\,, \tag{4.15}$$

according to the general form of the Stueckelberg-type transformations (2.8).

4.2 Integrating out the slicing coordinate

The mode expansions for fields and parameters read

$$h^{\mu\nu}(x,z) = \sum_{n=0}^{\infty} h_n^{\mu\nu}(x) P_n^2(z)\,, \quad A^\mu(x,z) = \sum_{n=0}^{\infty} A_n^\mu(x) P_n^1(z)\,, \quad \phi(x,z) = \sum_{n=0}^{\infty} \phi_n(x) P_n^1(z)\,, \tag{4.16}$$

$$\xi^\mu(x,z) = \sum_{n=0}^{\infty} \xi_n^\mu(x) P_n^2(z)\,, \qquad \xi(x,z) = \sum_{n=0}^{\infty} \xi_n(x) P_n^2(z)\,, \tag{4.17}$$

where the basis functions P_n^s are related to the Jacobi polynomials (see appendix C.1 for exact expressions). There is an essential difference between $d = 2$ and $d > 2$ cases shortly discussed in section 2. Namely, making the field redefinition (2.6) in $d > 2$ one can see that the most convenient mode expansion for the scalar component is given by [4]

$$\phi(x,z) = \sum_{n=0}^{\infty} \phi_n(x) P_n^0(z)\,, \tag{4.18}$$

where the basis functions P_n^0 are used instead of P_n^1. In this case the overlap integrals possess the properties (C.13) and (C.19) which are sufficient to integrate out the z-coordinate in a straightforward manner. However, it turns out that the basis functions P_n^0 do not exist in $d = 2$ as their inner products do not converge (see the beginning of appendix C.1). This fact along with the $d = 2$ pole in the substitution (2.6) lead us to fix in the mode expansions (4.16) the same basis functions both for scalar and vector components. It follows the resulting inner products between basis functions are much more complicated than those in $d \neq 2$ dimensions. However, using the calculation technique elaborated in appendix C.2 all relevant inner products can be explicitly evaluated in analytical terms. On the other hand, the choice of basis functions for the component gauge parameters as in (4.17) greatly simplifies the Stueckelberg-type operator (2.8) after the mode expansion, see (4.28) below. Of course, one can equally choose other basis functions, but practical computations show that the mode expansions (4.16) are optimal against other possible choices.

Now, inserting the mode expansions (4.16) into the component actions (4.6)–(4.11) and introducing the notation

$$\int \equiv a^{d/2} b^{-(d+3)/2} \int d\mu_d \,, \tag{4.19}$$

we obtain

$$\begin{aligned} S_{22} &= a \int \sum_{n=0}^{\infty} \Big\{ -(\nabla_\alpha h_n^{\mu\nu})^2 + 2(\nabla_\mu h_n^{\mu\nu})^2 - 2\nabla_\mu h_n \nabla_\nu h_n^{\mu\nu} + (\nabla_\alpha h_n)^2 + \\ &\qquad + a\Big[(-(\gamma_{2|n})^2 + 2)(h_n^{\mu\nu})^2 + ((\gamma_{2|n})^2 + d - 3)h_n^2\Big]\Big\} \,, \end{aligned} \tag{4.20}$$

$$S_{21} = 4a \int \sum_{n=1}^{\infty} \gamma_{2|n-1} \Big\{ -\nabla_\mu h_{n-1}^{\mu\nu} A_{n\nu} + \nabla_\mu h_{n-1} A_n^\mu \Big\} \,, \tag{4.21}$$

$$\begin{aligned} S_{20} &= 2 \int \sum_{n,m=0}^{\infty} \Big\{ \phi_n [\nabla_\mu \nabla_\nu h_m^{\mu\nu} - \Box h_m + a(d-1)h_m](P_n^1, P_m^2)_1 - \\ &\qquad - a(d-1)\gamma_{2|m}\phi_n h_m (P_n^1, \tanh z P_{m+1}^1)_1 \Big\} \,, \end{aligned} \tag{4.22}$$

$$S_{11} = 2 \int \sum_{n=0}^{\infty} \Big\{ -(\nabla_\mu A_n^\nu)^2 + (\nabla_\mu A_n^\mu)^2 - a(d-1)(A_n^\mu)^2 \Big\} \,, \tag{4.23}$$

$$S_{10} = 4(d-1) \int \sum_{n,m=0}^{\infty} \nabla_\mu A_n^\mu \, \phi_m \, (P_n^1, \tanh z P_m^1)_1 \,, \tag{4.24}$$

$$S_{00} = d(d-1) \int \sum_{n,m=0}^{\infty} \phi_n \phi_m \, (P_n^1, \tanh^2 z P_m^1)_1 \,, \tag{4.25}$$

where the constant $\gamma_{2|n}$ is given by

$$\gamma_{2|n} = \sqrt{(n+1)(n+d)} \,, \tag{4.26}$$

(a general definition is given in (C.18)). The inner products $(A,B)_l$ here are defined in (C.13).

The gauge transformations (4.13)–(4.15) are similarly expanded as

$$\delta h_n^{\mu\nu} = \nabla^\mu \xi_n^\nu + \nabla^\nu \xi_n^\mu + 2g^{\mu\nu} \sum_{m=0}^{\infty} (P_n^2, \tanh z P_m^1)_2 \, \xi_m \,, \tag{4.27}$$

$$\delta A_n^\mu = \nabla^\mu \xi_n + a\gamma_{2|n-1} \, \xi_{n-1}^\mu \,, \qquad \delta\phi_n = 2a\gamma_{2|n-1} \, \xi_{n-1} \,, \tag{4.28}$$

where $n = 0, 1, \ldots, \infty$. As a consistency check, one can directly verify that the total component action (4.20)–(4.25) is invariant under the mode transformations (4.27)–(4.28). Their Stueckelberg-type form suggests that the parameters ξ_m^μ and ξ_n with $m, n = 0, 1, \ldots, \infty$ can be used to gauge away the fields A_m^μ and ϕ_n with $m, n = 1, 2, \ldots, \infty$. It follows that no gauge parameters and residual gauge symmetry are left for the other fields A_0^μ, ϕ_0, and $h_n^{\mu\nu}$ with $n = 0, 1, \ldots, \infty$. The corresponding total action S is drastically simplified. It

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takes the form

$$S_{22} = a\sum_{n=0}^{\infty}\int\Big\{ -(\nabla_\alpha h_n^{\mu\nu})^2 + 2(\nabla_\mu h_n^{\mu\nu})^2 - 2\nabla_\mu h_n \nabla_\nu h_n^{\mu\nu} + (\nabla_\alpha h_n)^2 + \\ + a\Big[(-(\gamma_{2|n})^2+2)(h_n^{\mu\nu})^2 + ((\gamma_{2|n})^2+d-3)h_n^2\Big]\Big\}\,, \tag{4.29}$$

$$S_{21} = 0\,, \tag{4.30}$$

$$S_{20} = 2\int\Big\{\phi_0\big[\nabla_\mu\nabla_\nu h_0^{\mu\nu} - \Box h_0 + a(d-1)h_0\big](P_0^1,P_0^2)_1 - \\ - a(d-1)\gamma_{2|0}\,\phi_0 h_0\,(P_0^1,\tanh z P_1^1)_1\Big\}\,, \tag{4.31}$$

$$S_{11} = 2\int\Big\{-(\nabla_\alpha A_0^\mu)^2 + (\nabla_\mu A_0^\mu)^2 - a(d-1)(A_0^\mu)^2\Big\}\,, \tag{4.32}$$

$$S_{10} = 0\,, \tag{4.33}$$

$$S_{00} = d(d-1)\int\Big\{\phi_0^2\,(P_0^1,\tanh^2 z P_0^1)_1\Big\}\,, \tag{4.34}$$

where the inner products $(P_0^1,P_0^2)_1$, $(P_0^1,\tanh z P_1^1)_1$, $(P_0^1,\tanh^2 z P_0^1)_1$ are constants (depending on d) which can be calculated using the technique elaborated in appendix C.2.

At this stage, despite the use of different sets of the basis functions we observe that the resulting set of fields does coincide with that one found in [4]: an infinite tower of rank-2 fields, a single vector field, a single scalar field. However, the action above contains off-diagonal terms and a priori it is not clear if the spectrum of masses and spins is correct. In particular, from the component actions (4.31) and (4.34) involving a scalar field ϕ_0 it follows that ϕ_0 misses a standard kinetic term. It is a triangular algebraic field redefinition (2.6) that introduces a required kinetic term for ϕ_0. Being expanded in modes the triangular redefinition formula goes like $\widetilde{h}_n^{\mu\nu} = h_n^{\mu\nu} + g^{\mu\nu}\sum_m T_{mn}\phi_m$, where T_{mn} are some overlap coefficients which can be explicitly calculated. In order to diagonalize the action more field redefinitions are required which are not explicitly seen for the above choice of the basis functions.

4.3 Degressed equations of motion in two dimensions

Let us finally fix $d = 2$. Then, various constants arising in the higher dimensional action can be explicitly calculated as

$$\gamma_{2|0} = \sqrt{2}\,,\quad (P_0^1,P_0^2)_1 = \sqrt{\frac{2}{3}}\,,\quad (P_0^1,\tanh z P_1^1)_1 = -\frac{1}{\sqrt{3}}\,,\quad (P_0^1,\tanh^2 z P_0^1)_1 = \frac{1}{3}\,. \tag{4.35}$$

The equations of motion that follow from the total action (4.29)–(4.34) at $d = 2$ are given by

$$\frac{\delta S}{\delta h_n^{\mu\nu}} = 0: \quad n=0: \quad \Box h_0^{\mu\nu} - \nabla^{(\mu}\nabla_\rho h_0^{\nu)\rho} + \nabla^\mu\nabla^\nu h_0 + g^{\mu\nu}\nabla_\rho\nabla_\sigma h_0^{\rho\sigma} - g^{\mu\nu}\Box h_0 + \\ + a g^{\mu\nu} h_0 + \nabla^\mu\nabla^\nu\phi_0 - g^{\mu\nu}\Box\phi_0 + 2a g^{\mu\nu}\phi_0 = 0\,, \tag{4.36}$$

$$n\geqslant 1: \quad \Box h_n^{\mu\nu} - \nabla^{(\mu}\nabla_\rho h_n^{\nu)\rho} + \nabla^\mu\nabla^\nu h_n + g^{\mu\nu}\nabla_\rho\nabla_\sigma h_n^{\rho\sigma} - g^{\mu\nu}\Box h_n + \\ + a\big[(-(\gamma_{2|n})^2+2)h_n^{\mu\nu} + ((\gamma_{2|n})^2-1)g^{\mu\nu}h_n\big] = 0\,, \tag{4.37}$$

$$\frac{\delta S}{\delta A_0^\mu} = 0: \quad \Box A_0^\mu - \nabla^\mu\nabla_\nu A_0^\nu - a A_0^\mu = 0\,, \tag{4.38}$$

$$\frac{\delta S}{\delta \phi_0} = 0: \quad \nabla_\mu\nabla_\nu h_0^{\mu\nu} - \Box h_0 + 2a h_0 + a\phi_0 = 0\,. \tag{4.39}$$

To simplify the resulting expressions here we redefined the scalar field as $a^{-1}\sqrt{2/3}\phi_0 \to \phi_0$. The equation (4.38) describes a massive vector field A_0^μ which decouples from other fields. The other three equations (4.36), (4.37), (4.39) form a coupled system of relations on the tower of rank-2 fields $h_n^{\mu\nu}$ and a scalar field ϕ_0.

Now, the Schouten identity (D.2) for rank-2 tensor fields allows eliminating all derivative terms in (4.36) and (4.37). Indeed, all derivative terms in the first lines of (4.36) and (4.37) (note that the first lines are identical) constitute the derivative part of the Schouten identity (D.2) so that they are equal to particular combination of purely algebraic terms. Thus, these equations take much more elegant form

$$n=0: \qquad \nabla^\mu\nabla^\nu\phi_0 - g^{\mu\nu}\Box\phi_0 + 2ag^{\mu\nu}\phi_0 + 2a\big[-h_0^{\mu\nu} + g^{\mu\nu}h_0\big] = 0\,, \tag{4.40}$$

$$n\geqslant 1: \qquad (\gamma_{2|n})^2\big[-h_n^{\mu\nu} + g^{\mu\nu}h_n\big] = 0\,. \tag{4.41}$$

Since the parameter $(\gamma_{2|n})^2 \neq 0$ for any n (4.26), then the second equation (4.41) can be easily solved. Taking the trace results in $h_n = 0$ and, hence, $h_n^{\mu\nu} = 0$ at $n = 1, 2, \ldots, \infty$. Thus, these modes do not propagate. Taking the trace in the first equation (4.40) yields

$$h_0 = \frac{1}{2a}(\Box - 4a)\phi_0\,. \tag{4.42}$$

Substituting this expression back into (4.40) we can express $h_0^{\mu\nu}$ in terms of the scalar field as

$$h_0^{\mu\nu} = \frac{1}{2a}(\nabla^\mu\nabla^\nu - 2ag^{\mu\nu})\phi_0\,. \tag{4.43}$$

In other words, the rank-2 tensor field is auxiliary. Finally, substituting (4.42)–(4.43) into the last equation of motion (4.39) we obtain the following fourth-order equation

$$\frac{1}{2a}\big[\nabla_\mu\nabla_\nu\nabla^\mu\nabla^\nu - 2a\Box - (\Box - 2a)(\Box - 4a)\big]\phi_0 + a\phi_0 = 0\,. \tag{4.44}$$

However, one can see that the terms quartic in covariant derivatives cancel each other so that the resulting equation takes the standard second-order form

$$(\Box - 2a)\phi_0 = 0\,. \tag{4.45}$$

To summarize, we obtain the infinite sequences of modes vanishing on-shell

$$h_n^{\mu\nu} = 0\,, \qquad A_n^\mu = 0\,, \qquad \phi_n = 0\,, \qquad n = 1, 2, \ldots, \infty\,, \tag{4.46}$$

along with the zeroth rank-2 mode which is an auxiliary field

$$h_0^{\mu\nu} = \frac{1}{2a}\left[\nabla^\mu\nabla^\nu - 2ag^{\mu\nu}\right]\phi_0\,. \tag{4.47}$$

The remaining fields are the scalar and vector zeroth modes subjected to the second-order equations

$$\begin{aligned} \big[-\Box + 2a\big]\phi_0 &= 0\,, \\ \big[-\Box + a\big]A_0^\mu = 0\,, \quad \nabla_\mu A_0^\mu &= 0\,, \end{aligned} \tag{4.48}$$

which are the Klein-Gordon and Proca equations, respectively. Fixing $a=1$ we conclude from (2.10) that $E=2$ and PDoF are organized into two Verma modules $\mathcal{D}_2 \oplus \mathcal{D}_2$ according to the branching rules (3.13) for $s=2$.

To conclude this section, from (4.39) we notice that the scalar field ϕ_0 can equivalently be treated as auxiliary with respect to the rank-2 field, i.e. $\phi_0 = a^{-1}(\nabla_\mu \nabla_\nu h_0^{\mu\nu} - \Box h_0 + 2ah_0)$. Substituting this expression into (4.38) we arrive at the forth-order equation on $h_0^{\mu\nu}$. In this form the final system becomes diagonalized at the cost of having higher-order equations. However, our previous consideration explicitly shows that the resulting higher-order equation is essentially of second-order by means of introducing an additional scalar variable. This phenomenon of a higher-derivative theory describing unitary PDoF is similar to that of the NMG theory in three dimensions [30].

5 $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression of spin-3 massless fields

A massless spin-3 particle in AdS_{d+1} is described by a symmetric traceful tensor Φ^{mnk} with the following gauge transformation[7]

$$\delta\Phi^{mnk} = \overline{\nabla}^m \Xi^{nk} + \overline{\nabla}^n \Xi^{mk} + \overline{\nabla}^k \Xi^{mn} \,, \tag{5.1}$$

where the gauge parameter Ξ^{mn} is a symmetric traceless tensor. Again, we begin in arbitrary dimension, then specify to dimension two. The action is given by

$$\begin{aligned} S = \int d\mu_{d+1} \Big\{ &- (\overline{\nabla}_a \Phi^{mnk})^2 + 3(\overline{\nabla}_m \Phi^{mnk})^2 - 6\overline{\nabla}_m \Phi_n \overline{\nabla}_k \Phi^{mnk} + 3(\overline{\nabla}_a \Phi^m)^2 + \\ &+ \frac{3}{2}(\overline{\nabla}_m \Phi^m)^2 + b\big[-(d-2)(\Phi^{mnk})^2 + 6d\,(\Phi^m)^2\big]\Big\} \,, \qquad \Phi^m \equiv \Phi^{mnk} G_{nk} \,. \end{aligned} \tag{5.2}$$

Decomposing the original field as $\Phi^{mnk}(x,z) = \{\Phi^{\mu\nu\rho}(x,z),\, \Phi^{\mu\nu}(x,z),\, \Phi^\mu(x,z),\, \Phi(x,z)\}$ we introduce new notations for the component fields along with some convenient redefinitions

$$w^{\mu\nu\rho} := \Phi^{\mu\nu\rho} + (ad)^{-1}\operatorname{sech}^2 z\, g^{(\mu\nu}\Phi^{\rho)} \,, \quad h^{\mu\nu} := \Phi^{\mu\nu} \,, \quad A^\mu := \Phi^\mu \,, \quad \phi := \Phi \,. \tag{5.3}$$

For the component fields we choose the following mode expansions

$$\begin{aligned} w^{\mu\nu\rho}(x,z) &= \sum_{n=0}^\infty w_n^{\mu\nu\rho}(x) P_n^3(z) \,, \qquad & h^{\mu\nu}(x,z) &= \sum_{n=0}^\infty h_n^{\mu\nu}(x) P_n^2(z) \,, \\ A^\mu(x,z) &= \sum_{n=0}^\infty A_n^\mu(x) P_n^1(z) \,, & \phi(x,z) &= \sum_{n=0}^\infty \phi_n(x) P_n^1(z) \,. \end{aligned} \tag{5.4}$$

Similar to the spin-2 case the basis functions of the scalar and vector components are chosen to be the same since P_n^0 do not exist in $d=2$ (see appendix C.1) so that P_n^1 are used instead.

[7] The AdS_{d+1} waveguide compactification for massless spin-3 particles was considered in [5].

The $\mathrm{AdS}_{d+1}/\mathrm{AdS}_d$ degression for massless spin-3 fields goes along the same lines as for massless spin-2 fields in section 4. Therefore, all the details of calculating the degressed component actions are relegated to appendix E. Here, we will assume that the Stueckelberg-type gauge symmetry with parameters ξ_n^μ and ξ_n has been partially used to gauge away fields A_n^μ and ϕ_n with $n = 1, 2, \ldots, \infty$ and the resulting degressed equations of motion in AdS_d are explicitly known, see eqs. (E.37)–(E.40). Now, we fix $d = 2$.

As shown in section 4 the Schouten identities (D.2) for rank-2 tensor fields are sufficient to turn the spin-2 kinetic operators into purely algebraic terms that results in truncating the spectrum. However, the Schouten identities (D.4) for rank-3 tensor fields are generally different from the spin-3 kinetic operators. Nonetheless, the gauge parameters $\xi_n^{\mu\nu}$ (E.42) can be used to impose the TT gauge on the rank-3 tensor fields

$$\nabla_\mu w_n^{\mu\nu\rho} = 0\,, \qquad w_n^\mu = 0\,, \qquad n = 0, 1, 2, \ldots, \infty\,. \tag{5.5}$$

Combining the TT gauge conditions (5.5) and the Schouten identities (D.4) one finds that

$$\begin{aligned}&\Box w_n^{\mu\nu\rho} - \nabla^{(\mu}\nabla_\sigma w_n^{\nu\rho)\sigma} + \frac{1}{2}\nabla^{(\mu}\nabla^\nu w_n^{\rho)} - g^{(\mu\nu}\Box w_n^{\rho)} + g^{(\mu\nu}\nabla_\sigma\nabla_\varsigma w_n^{\rho)\sigma\varsigma} - \frac{1}{2}g^{(\mu\nu}\nabla^{\rho)}\nabla_\sigma w_n^\sigma\\ &= 3a w_n^{\mu\nu\rho}\,, \qquad n = 0, 1, 2, \ldots, \infty\,,\end{aligned} \tag{5.6}$$

where the left-hand side is the first three lines in the equation of motion (E.37) containing second derivatives of $w_n^{\mu\nu\rho}$. Reorganizing the equations of motion (E.37)–(E.40) and redefining the scalar field as $a^{-1}\sqrt{2/3}\,\phi_0 \to \phi_0$, we finally obtain a simplified system

$$\begin{cases} n \geqslant 0: & \gamma_{3|n}\big[-2g^{(\mu\nu}\nabla_\sigma h_{n+1}^{\rho)\sigma} + \nabla^\mu h_{n+1}^{\nu\rho)} + \tfrac{1}{2}g^{(\mu\nu}\nabla^{\rho)}h_{n+1}\big] - \\ & - a((\gamma_{3|n})^2 - 1)w_n^{\mu\nu\rho} = 0\,, \\ n \geqslant 1: & -6h_n^{\mu\nu} + \tfrac{3}{2}((\gamma_{2|n})^2 + 2)g^{\mu\nu}h_n = 0\,, \end{cases} \tag{5.7}$$

$$\Box A_0^\mu - \nabla^\mu\nabla_\nu A_0^\nu - 5aA_0^\mu = 0\,, \tag{5.8}$$

$$\begin{cases} [\nabla^\mu\nabla^\nu - g^{\mu\nu}(\Box - 7a)]\phi_0 + 6a\big[-h_0^{\mu\nu} + g^{\mu\nu}h_0\big] = 0\,, \\ [\Box - 11a]\phi_0 - \tfrac{3}{2}[\nabla_\mu\nabla_\nu - g_{\mu\nu}(\Box - 7a)]h_0^{\mu\nu} = 0\,. \end{cases} \tag{5.9}$$

From the first subsystem (5.7) it follows that the modes $h_n^{\mu\nu}$ with $n = 1, 2, \ldots, \infty$ and $w_n^{\mu\nu\rho}$ with $n = 0, 1, 2, \ldots, \infty$ vanish on-shell thereby carrying no degrees of freedom. The second subsystem (5.8) is the Proca equation which can be equivalently represented as

$$[\Box - 5a]A_0^\mu = 0\,, \qquad \nabla_\mu A_0^\mu = 0\,. \tag{5.10}$$

The third subsystem (5.9) is not diagonal. However, it can be solved similar to the spin-2 case. Taking the trace in the first equation of (5.9) we find

$$h_0 = \frac{1}{6a}[\Box - 14a]\phi_0\,. \tag{5.11}$$

Then, substituting this expression back into (5.9) one can express the rank-2 tensor field $h_0^{\mu\nu}$ in terms of the scalar field ϕ_0 as

$$h_0^{\mu\nu} = \frac{1}{6a}[\nabla^\mu\nabla^\nu - 7ag^{\mu\nu}]\phi_0\,. \tag{5.12}$$

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Finally, using this expression in the second equation of (5.9) one finds a forth-order expression

$$[\Box - 11a]\phi_0 - \frac{1}{4a}[\nabla_\mu\nabla_\nu\nabla^\mu\nabla^\nu - 7a\Box - (\Box - 7a)(\Box - 14a)]\phi_0 = 0\,, \tag{5.13}$$

where all fourth order derivatives cancel each other so that the resulting expression is the standard second-order equation

$$-\frac{9}{4}[\Box - 6a]\phi_0 = 0\,. \tag{5.14}$$

To summarize, we obtain the infinite sequences of modes vanishing on-shell

$$\begin{aligned} w_n^{\mu\nu\rho} &= 0\,, \qquad n = 0,1,2\ldots,\infty\,, \\ h_n^{\mu\nu} &= 0\,, \qquad A_n^\mu = 0\,, \quad \phi_n = 0\,, \quad n = 1,2,3\ldots,\infty\,, \end{aligned} \tag{5.15}$$

along with the zeroth rank-2 mode which is an auxiliary field

$$h_0^{\mu\nu} = \frac{1}{6a}[\nabla^\mu\nabla^\nu - 7ag^{\mu\nu}]\phi_0\,. \tag{5.16}$$

The remaining fields are again the scalar and vector zeroth modes subjected to the second-order equations

$$\begin{aligned} &[-\Box + 6a]\phi_0 = 0\,, \\ &[-\Box + 5a]A_0^\mu = 0\,, \quad \nabla_\mu A_0^\mu = 0\,, \end{aligned} \tag{5.17}$$

which are the Klein-Gordon and Proca equations, respectively. Fixing $a = 1$ we conclude from (2.10) that $E = 3$ and PDoF are organized into two $o(1,2)$ Verma modules $\mathcal{D}_3 \oplus \mathcal{D}_3$ according to the branching rules (3.13) for $s = 3$. Cf. with the equations (4.46)–(4.48).

6 Conclusion and outlooks

We have analyzed the AdS_3/AdS_2 dimensional degression and showed that a given spin-s massless theory in AdS_3 degresses into the sum of the Klein-Gordon and Proca theories in AdS_2. Thus, contrary to higher-dimensional Kaluza-Klein type theories the spectrum of a bottom theory in the AdS_3/AdS_2 degression is finite. Moreover, a top theory in this case is topological while a bottom theory propagates local degrees of freedom. This is consistent with the branching rules showing how a respective $o(2,2)$ representation decomposes into a direct sum of $o(1,2)$ Verma modules.

Note that our definition of a topological field theory in $d+1$ dimensions is that there are no propagating (local) degrees of freedom, i.e. the corresponding field equations are solved in terms of arbitrary functions of less than d continuous variables. A useful concept here is the Gelfand-Kirillov dimension $\#_{\mathbb{GK}}$ [31] that provides an interface between the representation theory and partial differential equations (in the present context see e.g. [32, 33]). For AdS_3 topological systems the Gelfand-Kirillov dimension of the spin-s (partially-)massless $o(2,2)$ representations equals $\#_{\mathbb{GK}} = 1$, while $\#_{\mathbb{GK}} = 2$ corresponds to massive representations, cf. the summary list (3.13). It is instructive to compare with the Jackiw-Teitelboim (JT) gravity in two dimensions where $\#_{\mathbb{GK}} = 0$ which means there are only finitely many constants parameterizing the space of solutions. We conclude that generally a topological top theory nonetheless encodes non-trivial degrees of freedom to be interpreted as true dynamical degrees of freedom in a bottom theory. The AdS_3/AdS_2 degression is merely a manifestation of this fact. And, conversely, local degrees of freedom of some dynamical theory can be uplifted to higher dimensions to be described in terms of a topological higher-dimensional theory which is presumably simpler and more tractable.

Note that the standard formulation in terms of (gauge) tensor fields discussed in this paper can be equivalently reformulated in terms of $\mathcal{G}$-connections with Chern-Simons or BF actions for some gauge algebra $\mathcal{G}$. Moreover, such a formulation is known to be extremely useful in building and analyzing interactions. It would be important to understand the AdS compactification/degression directly in terms of connections. Some discussion in this direction can be found in [34–36] (see also recent [37–39]). One of most interesting models here is the AdS_3 higher-spin gravity [22, 23, 40] and its $sl(3)\oplus sl(3)$ version [23] that can also be reformulated perturbatively in terms of spin-2 and spin-3 tensor fields [41, 42]. Since the interacting theory here also remains topological, it is expected that its dimensional degression yields a version of the AdS_2 higher-spin gravity [43–45] which describes finitely many massive spinless excitations with higher-spin-gravitational interactions.[8]

Another implication of the AdS_3/AdS_2 degression may be relevant in the context of AdS_2 higher-spin gravity with infinitely many massive spinless excitations governed by the higher-spin algebra $hs[\lambda]$, where a real parameter λ defines an equidistant mass spectrum [14, 47]. The AdS_3/AdS_2 spectrum suggests that AdS_2 modes may arise in the form of scalar/vector covariant fields which are indistinguishable on-shell but may participate in different types of interactions. Also, it becomes relevant in searching a bulk theory in the context of AdS_2/SYK correspondence, where the bulk degrees are given by an infinite tower of massive spinless modes [17, 48]. In particular, AdS_3/AdS_2 degressing ϕ^3 scalar theory yields an infinite tower of AdS_2 scalars with certain coupling constants parametrized by the triple overlap integrals which differ however from the SYK spectrum. It may imply that a bulk theory is more complex, presumably a single scalar field should be replaced with an infinite tower of higher-spin massless AdS_3 fields. Then, each of them yields a couple of spinless massive modes in AdS_2 thereby producing an infinite massive spectrum as well.

In this respect, let us mention the CFT bootstrap analogy used to control the high-energy behaviour of bottom theories with infinitely many fields arising through the Kaluza-Klein reduction of Yang-Mills and Einstein theories in higher dimensions [49, 50]. This is suggested by the observation that the coupling constants of a bottom theory are given by multiple overlap integrals subjected to the bootstrap-like constraints. Our results revealing finite spectra suggest that the bootstrap analogy introduces here the class of "Kaluza-Klein minimal models". Hopefully, it may provide an example of exact solutions to the unitary sum rules in bottom theories provided they are degressed from a topological top theory.

[8]In particular, the equivalence of $2d$ and $3d$ partition functions of higher-spin fields in the near-horizon region of the near-extremal BTZ black hole was recently shown in [46].

To conclude, it would be important to develop the AdS_3/AdS_2 degression for AdS_3 massless rank-s fields $\phi_{m_1 \ldots m_s}$ thereby extending the results of the present paper to $s \geqslant 4$. A non-trivial aspect here would be to take a proper account of the double-tracelessness condition activated at $s \geqslant 4$. Another important issue is to extend our analysis to partially-massless spin-s depth-t AdS_3 fields relevant in the context of respective interacting theories [51, 52]. Also, it would be interesting to study the degression for continuous spin AdS_3 fields [53].

Finally, let us remark that when considering the AdS_3/AdS_2 degression of the linearized Einstein gravity with the cosmological term Λ in section 4 we did not see any trace of the linearized JT theory which presumably should arise in $2d$ gravitational systems. We expect that the JT gravity will show up either when considering the AdS_3 waveguide compactification with an extra parameter α measuring the size of extra dimension (see (2.1); in this case partially-massless fields may arise [5]), or one should start with the (linearized) AdS_3 partially-massless gravity instead. Both options agree with our current understanding that higher-spin fields in AdS_2 are to be interpreted as partially-massless [43].

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A Notation and conventions for AdS spaces

In this appendix we describe our notation and conventions for AdS spaces. Consider first the AdS_{d+1} spacetime as a hyperboloid embedded in the ambient spacetime $\mathbb{R}^{2,d}$

$$- Y_{\bar{0}}^2 - Y_0^2 + Y_1^2 + \ldots + Y_d^2 = -\frac{1}{b}\,, \tag{A.1}$$

where the parameter $b > 0$ will correspond to the constant negative scalar curvature of AdS_{d+1}. A foliation into AdS_d slices can be parametrized as follows [4]

$$Y_{\bar{0}} = \sqrt{\frac{a}{b}} \cosh z \, y_{\bar{0}}(x)\,, \quad \ldots\,, \quad Y_{d-1} = \sqrt{\frac{a}{b}} \cosh z \, y_{d-1}(x)\,, \quad Y_d = \frac{1}{\sqrt{b}} \sinh z\,, \tag{A.2}$$

where $a > 0$ is a constant associated with the AdS_d scalar curvature, $x = \{x^\mu, \mu = 0, \ldots d-1\}$ are local coordinates in AdS_d spacetime and z is the slicing coordinate. AdS_d spacetime can also be represented as a hyperboloid in the ambient spacetime $\mathbb{R}^{2,d-1}$ of one less dimension

$$- y_{\bar{0}}^2 - y_0^2 + y_1^2 + \ldots + y_{d-1}^2 = -\frac{1}{a}\,. \tag{A.3}$$

This construction yields the following block-diagonal form of the metric G_{mn} in AdS_{d+1} (indices m, n run $0, \ldots, d$)

$$G_{mn}(x,z) = \frac{1}{b} \left(\begin{array}{c|c} a \cosh^2 z g_{\mu\nu}(x) & 0 \\ \hline 0 & 1 \end{array} \right)\,, \tag{A.4}$$

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where $g_{\mu\nu}(x)$ is the metric in AdS_d ($\mu \equiv m = 0, \ldots, d-1$). The respective integration measures read

$$d\mu_d = d^d x \sqrt{-g}\,, \qquad d\mu_{d+1} = d^d x dz \sqrt{-G} = d\mu_d dz\, a^{d/2} b^{-(d+1)/2} \cosh^d z\,, \tag{A.5}$$

where G and g are the determinants of the AdS_{d+1} and AdS_d metrics, respectively.

The $(d+1)$-th value of an index m is denoted by $\bullet \equiv m = d$. Note that the metric (A.4) becomes identical to (2.2) provided that $a = b = \ell_{\mathrm{AdS}}^{-2}$, i.e. both original AdS_{d+1} and its slices AdS_d have the same radius. The reason behind keeping parameters a and b arbitrary is to control mass-like terms in the respective actions of AdS_d and AdS_{d+1} theories.

The Riemann tensor and covariant derivatives are defined as

$$\overline{R}^{m}{}_{nkl} = \partial_{[k}\overline{\Gamma}^{m}_{l]n} + \overline{\Gamma}^{m}_{p[k}\overline{\Gamma}^{p}_{l]n}\,, \tag{A.6}$$

$$\left[\overline{\nabla}_m, \overline{\nabla}_n\right] T^{k\cdots}{}_{l\cdots} = \overline{R}_{mn}{}^{k}{}_{p} T^{p\cdots}{}_{l\cdots} + \ldots + \overline{R}_{mnl}{}^{p} T^{k\cdots}{}_{p\cdots} + \ldots\,, \tag{A.7}$$

where the bars over the Riemann tensor, Christoffel symbols and covariant derivatives refer to AdS_{d+1} geometry, otherwise this is AdS_d geometry. The brackets $(mn\ldots)$ and $[mn\ldots]$ denote (anti)symmetrization of indices, which are defined as a sum of essentially different terms with a unit weight. Then, the scalar curvatures of AdS_{d+1} and AdS_d along with the Christoffel symbols read

$$\bar{R} = -bd(d+1)\,, \qquad R = -ad(d-1)\,, \tag{A.8}$$

$$\overline{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}\,, \qquad \overline{\Gamma}^{\mu}_{\nu\bullet} = \tanh z\,\delta^{\mu}_{\nu}\,, \qquad \overline{\Gamma}^{\bullet}_{\mu\nu} = -a\cosh^2 z \tanh z\, g_{\mu\nu}\,. \tag{A.9}$$

Note that curvatures R and $\bar{R}$ of the original space and its slices are generally different.

By way of example, AdS_{d+1} covariant derivatives acting on a rank-3 symmetric tensor W^{mnk} with respect to the metric (A.4) is given by

$$\begin{aligned} \overline{\nabla}_\alpha W^{\mu\nu\bullet} &= \partial_\alpha W^{\mu\nu\bullet} + \overline{\Gamma}^{(\mu}_{\alpha p} W^{\nu)p\bullet} + \overline{\Gamma}^{\bullet}_{\alpha p} W^{\mu\nu p} = \\ &= \nabla_\alpha W^{\mu\nu\bullet} + \tanh z\,\delta^{(\mu}_{\alpha} W^{\nu)\bullet\bullet} - a\cosh^2 z \tanh z\, W_\alpha{}^{\mu\nu}\,, \\ \overline{\nabla}_\bullet W^{\mu\nu\rho} &= \partial W^{\mu\nu\rho} + \overline{\Gamma}^{(\mu}_{\bullet p} W^{\nu\rho)p} = (\partial + 3\tanh z) W^{\mu\nu\rho}\,, \end{aligned} \tag{A.10}$$

where we introduced the notation $\partial \equiv \partial/\partial z$. Also, the Laplace-Beltrami operator in AdS_d spacetime is denoted as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

In order to simplify our notations we denote quadratic combinations of symmetric tensors $X^{m\cdots}$ in AdS_{d+1} as

$$(\overline{\nabla}_a X^{m\ldots})^2 \equiv \overline{\nabla}_a X^{m\ldots} \overline{\nabla}^a X_{m\ldots}\,, \qquad (X^{m\ldots})^2 \equiv X^{m\ldots} X_{m\ldots}\,. \tag{A.11}$$

The same notation is used for AdS_d tensors and their derivatives.

B Discrete transformations and Lie algebra representations

B.1 AdS_3 isometry group

AdS_3 isometry group is given by the split indefinite orthogonal Lie group $O(2,2)$ which leaves invariant $\eta_{AB} = \mathrm{diag}(-1,-1,1,1)$, where $A, B = \bar{0}, 0, 1, 2$. There are four components; the identity component $O^+(2,2)$ yields the Lie algebra $o(2,2)$. The discrete transformations are generated by the elements

$$\mathrm{T} = \mathrm{diag}(1, -1, 1, 1)\,, \qquad \mathrm{P} = \mathrm{diag}(1, 1, -1, 1)\,, \tag{B.1}$$

which correspond to time reversal and space reflection (parity transformation). In addition to $\mathrm{O}^+(2,2)$ the three other components are generated from $\mathrm{O}^+(2,2)$ by acting on its elements with T, P, PT. The component $\mathrm{PT} \cdot \mathrm{O}^+(2,2) \subset \mathrm{SO}(2,2)$. The commutation relations of the Lie algebra $o(2,2)$ read

$$i\,[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC} - \eta_{BD}M_{AC}\,. \tag{B.2}$$

The generators are all Hermitian $M^\dagger_{AB} = M_{AB}$.

To build a highest-weight or lowest-weight (generalized) Verma module the six $o(2,2)$ generators are rearranged as

$$\widehat{E} = M_{\bar{0}0}\,, \qquad \widehat{S} = M_{12}\,, \qquad \widehat{L}_{\pm k} = M_{0k} \pm iM_{\bar{0}k}\,, \quad k = 1, 2\,, \tag{B.3}$$

which are, respectively, energy, spin, lowering/raising boosts. The generators are conjugated as $\widehat{E}^\dagger = \widehat{E}$, $\widehat{S}^\dagger = \widehat{S}$, $(\widehat{L}_{+k})^\dagger = \widehat{L}_{-k}$, while the discrete symmetries (B.1) act as

$$\mathrm{T}\,\widehat{E}\,\mathrm{T}^{-1} = \widehat{E}\,, \qquad \mathrm{T}\,\widehat{S}\,\mathrm{T}^{-1} = -\widehat{S}\,, \qquad \mathrm{T}\,\widehat{L}_{\pm k}\,\mathrm{T}^{-1} = \widehat{L}_{\pm k}\,, \tag{B.4}$$

$$\mathrm{P}\,\widehat{E}\,\mathrm{P}^{-1} = \widehat{E}\,, \qquad \mathrm{P}\,\widehat{S}\,\mathrm{P}^{-1} = -\widehat{S}\,, \qquad \mathrm{P}\,\widehat{L}_{\pm k}\,\mathrm{P}^{-1} = (-)^k\widehat{L}_{\pm k}\,, \tag{B.5}$$

where T and P are, respectively, antilinear and linear operators [54].[9]

A *lowest-weight* state $|E, s\rangle$ spans an irreducible representation of $o(2)\oplus o(2)$ subalgebra of $\widehat{E}$ and $\widehat{S}$ (two-dimensional space)

$$\widehat{E}|E, s\rangle = E|E, s\rangle\,, \qquad \widehat{S}|E, s\rangle = s|E, s\rangle\,, \qquad \widehat{L}_{-k}|E, s\rangle = 0\,, \tag{B.6}$$

where the energy $E \in \mathbb{R}$, the spin is integer $s \in \mathbb{Z}$ (we consider only bosonic modules). A lowest-weight module is then generated by acting with basis monomials of the raising boosts as

$$\mathcal{V}_+(E, s) = \{\, \widehat{L}_{+k_1}\widehat{L}_{+k_2}\cdots\widehat{L}_{+k_l}|E, s\rangle, \;\; l = 0, 1, 2, \ldots \}\,. \tag{B.7}$$

Here, l denotes the level and each level-l space is organized into $o(2)$ finite-dimensional irreps of spin numbers in the range $|s+l|, |s+l-2|, \ldots, |s-l|$. For lowest-weight representations the energy is bounded from below. At specific values $E = E_0(s,t)$ there are singular vectors arising on the $(t+1)$-th level. The resulting $o(2,2)$ quotient modules $\mathcal{H}(E_0, s)$ are described in section 3.

Let us now discuss how discrete symmetries (B.4)–(B.5) act on the modules $\mathcal{V}_+(E, s)$ (B.7). The energy operator is invariant under all discrete transformations P, T, PT. On the contrary, the spin operator changes the sign under P and T, and remains PT invariant. One can define the vacuum $|\widetilde{E, s}\rangle := |E, -s\rangle \oplus |E, s\rangle$ which stays

[9] For unitary representations these operators become (anti)-unitary. On the other hand, assuming that both T and P are linear we find that highest-weight and lowest-weight conditions (i.e. positive/negative discrete series) are interchanged.

invariant under all discrete symmetries. Recalling how the discrete symmetries act on the boosts (B.4)–(B.5) we conclude that other lowest-weight conditions in (B.6) remain intact. Thus, a module generated from a discrete-invariant vacuum is the following direct sum of (generalized) Verma modules

$$\mathcal{D}(E,s) = \mathcal{V}_{-}(E,-s) \oplus \mathcal{V}_{+}(E,s)\,, \tag{B.8}$$

with a lowest-weight space containing states of opposite spins $\pm s$. From now on we set the spin to be non-negative integer $s \in \mathbb{N}$. This definition can be equivalently replaced by imposing a weaker condition $\mathbb{C}_2[\mathrm{o}(2)]|\widetilde{E,s}\rangle = s^2|\widetilde{E,s}\rangle$, where $|\widetilde{E,s}\rangle$ is the lowest-weight state of $\mathcal{D}(E,s)$ and $\mathbb{C}_2$ denotes the quadratic Casimir operator. It is these modules (B.8) which are discussed in section 3.

One can consider two other bases in the Lie algebra $\mathrm{o}(2,2)$: the Lorentz basis and the factorized basis. The Lorentz basis can be built by the standard block decomposition of an antisymmetric matrix M_{AB} as

$$P_a = M_{\bar{0}a}\,, \qquad L_{ab} = M_{ab}\,, \qquad a,b = 0,1,2\,, \tag{B.9}$$

which are momentum and Lorentz rotation generators commuting in the standard fashion. On the other hand, the decomposition of $\mathrm{o}(2,2)$ in simple subalgebras can be achieved by introducing linear combinations

$$J_a^{(\varepsilon)} = \frac{1}{2}\left(P_a + \varepsilon M_a\right)\,, \quad \varepsilon = \pm\,; \qquad [J_a^{(-)}, J_b^{(+)}] = 0\,, \tag{B.10}$$

where $M_a = \frac{1}{2}\epsilon_{abc}M^{bc}$ and $\epsilon_{012} = 1$, cf. (B.9). Then, each set of elements $J_a^{(\varepsilon)}$ defines simple factors $\mathrm{o}(1,2)\oplus\mathrm{o}(2,1) = \mathrm{o}(2,2)$. The discrete symmetries act on them as $\mathrm{P}J_0^{(+)}\mathrm{P}^{-1} = J_0^{(-)}$ and $\mathrm{P}J_0^{(+)}\mathrm{P}^{-1} = J_0^{(-)}$ that means the two factors are interchanged. The energy and spin operators are given by

$$\widehat{E} = J_0^{(-)} + J_0^{(+)}\,, \qquad \widehat{S} = J_0^{(-)} - J_0^{(+)}\,. \tag{B.11}$$

Each copy of $\mathrm{o}(1,2)$ defines its own Verma module. To this end one introduces diagonal and raising/lowering operators as

$$J_0^{(\varepsilon)} := J_0^{(\varepsilon)}\,, \qquad J_\pm^{(\varepsilon)} := J_1^{(\varepsilon)} \pm iJ_2^{(\varepsilon)}\,, \tag{B.12}$$

which define lowest-weight conditions imposed in a lowest-weight state $|h_\varepsilon\rangle$ as

$$J_0^{(\varepsilon)}|h_\varepsilon\rangle = h_\varepsilon|h_\varepsilon\rangle\,, \qquad J_-^{(\varepsilon)}|h_\varepsilon\rangle = 0\,, \tag{B.13}$$

for some $h_\varepsilon \in \mathbb{R}$. A lowest-weight $\mathrm{o}(1,2)$ representation is built as

$$\mathcal{D}_{h_\varepsilon} = \left\{ \left(J_+^{(\varepsilon)}\right)^m |h_\varepsilon\rangle,\;\; m = 0,1,2,3,\ldots\,, \varepsilon = \pm \right\}\,. \tag{B.14}$$

Now, lowest-weight $\mathrm{o}(2,2)$ representations can be constructed by tensoring respective $\mathrm{o}(1,2)$ representations (B.14). Introducing a lowest-weight vector $|h_-\rangle\otimes|h_+\rangle$ one constructs $\mathrm{o}(2,2)$ representation $\mathcal{D}_{h_-}\otimes\mathcal{D}_{h_+}$. Here, a basis energy and spin are parametrized as $E = h_- + h_+$ and $s = |h_- - h_+|$, cf. (B.11). The discrete transformations interchange the weights as $h_- \leftrightarrow h_+$ and so $E \to E$, $s \to -s$. It follows that P and T invariant representations take a duplicated form $(\mathcal{D}_{h_-}\otimes\mathcal{D}_{h_+})\oplus(\mathcal{D}_{h_+}\otimes\mathcal{D}_{h_-})$ which is in fact (B.8).

B.2 AdS$_2$ isometry group

AdS$_2$ isometry group is given by the indefinite orthogonal Lie group $\mathrm{O}(2,1)$ which leaves invariant $\eta_{AB} = \mathrm{diag}(-1,-1,1)$, where $A,B = 0',0,1$ (see e.g. [55, 56]). It has four components; the identity component $\mathrm{O}^+(2,1)$ yields the Lie algebra $\mathrm{o}(2,1)$. The discrete transformations are generated by the elements

$$\mathrm{T} = \mathrm{diag}(1,-1,1)\,, \qquad \mathrm{P} = \mathrm{diag}(1,1,-1)\,, \tag{B.15}$$

which correspond to time reversal and space reflection (parity transformation). Here, we kept the same notation as in (B.1). The commutation relations of the Lie algebra $\mathrm{o}(2,1)$

$$i\,[M_A, M_B] = -\epsilon_{ABC}\eta^{CD}M_D\,, \tag{B.16}$$

can be obtained from (B.2) when A,B,C are three-dimensional by substitution $M_A = \frac{1}{2}\epsilon_{ABC}M^{BC}$, where $\epsilon_{0'01} = +1$. The generators are all Hermitian $M_A^\dagger = M_A$.

To build a highest-weight or lowest-weight Verma module the three $\mathrm{o}(2,1)$ generators are rearranged as

$$\widehat{E} = M_{0'0}\,, \qquad \widehat{L}_\pm = M_{01} \pm iM_{0'1}\,, \tag{B.17}$$

which are the energy and raising/lowering boosts (no spin operators in this case). The generators are conjugated as $\widehat{E}^\dagger = \widehat{E}$ and $(\widehat{L}_+)^\dagger = \widehat{L}_-$, while the discrete symmetries (B.15) act as

$$\begin{aligned} &\mathrm{T}\,\widehat{E}\,\mathrm{T}^{-1} = \widehat{E}\,, \qquad \mathrm{T}\,\widehat{L}_\pm\,\mathrm{T}^{-1} = -\widehat{L}_\pm\,,\\ &\mathrm{P}\,\widehat{E}\,\mathrm{P}^{-1} = \widehat{E}\,, \qquad \mathrm{P}\,\widehat{L}_\pm\,\mathrm{P}^{-1} = -\widehat{L}_\pm\,, \end{aligned} \tag{B.18}$$

where T and P are, respectively, antilinear and linear operators. One can introduce $\mathrm{o}(1,2)$ Verma lowest-weight modules $\mathcal{D}_E$ (B.14) with the lowest energy E. From (B.18) it follows that $\mathrm{O}(2,1)$ discrete symmetry leaves $\mathcal{D}_E$ invariant so that no duplication similar to (B.8) is required.

Consider now the Clebsch-Gordan problem for representations $\mathcal{D}_E$ and $\mathcal{H}_E$ (see section 3). For the coupling of such representations we have [13, 57–62]

$$\mathcal{D}_{h_1} \otimes \mathcal{D}_{h_2} = \bigoplus_{h=h_1+h_2}^{\infty} \mathcal{D}_h\,, \tag{B.19}$$

$$\mathcal{D}_{h_3} \otimes \mathcal{H}_{h_4} = \bigoplus_{h=h_3+h_4}^{h_3-h_4} (-)^{h_3+h_4-h}\,\mathcal{D}_h\,, \tag{B.20}$$

$$\mathcal{H}_{h_4} \otimes \mathcal{H}_{h_5} = \bigoplus_{h=h_4+h_5}^{-|h_4-h_5|} (-)^{h_4+h_5-h}\,\mathcal{H}_h\,, \tag{B.21}$$

for $\forall h_1, h_2$; $h_3 > 0$, $h_3 > |h_4|$, and $h_4, h_5 \in \frac{1}{2}\mathbb{Z}_{\leqslant 0}$; the summation goes with a step 1. An essential property here is that $\mathcal{D}_{h_3} \otimes \mathcal{H}_{h_4}$ has finitely many components similar to $\mathcal{H}_{h_4} \otimes \mathcal{H}_{h_5}$. In the last two products (B.20) and (B.21) we added the sign factors which mean that two modules are summed as $V_a \oplus (-V_b)$, where $-V_b$ has the opposite signature inner product [61]: the flipping sings in the tensor products with finite-dimensional modules indicate inevitable non-unitarity except for the identity module $\mathcal{H}_0$ of zeroth weight. In other cases, for simplicity, we ignore the sign-flipping, if any.

C Jacobi polynomials and basis functions

Here, we recap some basics about the Jacobi polynomials, see e.g. [63]. Also, we explicitly compute some of the integral overlaps arising in the dimensional degression.

The Jacobi polynomials $J_n^{\alpha,\beta}(x)$ ($n \in \mathbb{N}_0$) are polynomials in the domain $x \in (-1,1)$ with two real parameters $\alpha, \beta > -1$. They are orthogonal with respect to the Jacobi weight function $\omega^{\alpha,\beta} = (1-x)^\alpha(1+x)^\beta$, namely

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx = \left\| J_n^{\alpha,\beta} \right\|^2 \delta_{mn} , \tag{C.1}$$

where

$$\left\| J_n^{\alpha,\beta} \right\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} . \tag{C.2}$$

The Jacobi polynomials are the eigenfunctions of the following Sturm-Liouville operator

$$\begin{aligned} \mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta}(x) &\equiv -(1-x)^{-\alpha}(1+x)^{-\beta}\partial_x \left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x J_n^{\alpha,\beta}(x)\right] \\ &= \left(x^2-1\right)\partial_x^2 J_n^{\alpha,\beta}(x) + (\alpha-\beta+(\alpha+\beta+2)x)\partial_x J_n^{\alpha,\beta}(x) = \lambda_n^{\alpha,\beta} J_n^{\alpha,\beta}(x) , \end{aligned} \tag{C.3}$$

where the eigenvalues are $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$. The Jacobi polynomials are given by

$$\begin{aligned} J_n^{\alpha,\beta}(x) &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} \left(\frac{x-1}{2}\right)^k = \\ &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) . \end{aligned} \tag{C.4}$$

The Jacobi polynomials can be defined by the recurrence equations

$$\begin{aligned} &J_0^{\alpha,\beta}(x) = 1 , \quad J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta) , \\ &J_{n+1}^{\alpha,\beta}(x) = (a_n x - b_n) J_n^{\alpha,\beta}(x) - c_n J_{n-1}^{\alpha,\beta}(x) , \quad n \geqslant 1 , \end{aligned} \tag{C.5}$$

where

$$\begin{aligned} a_n &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} , \\ b_n &= \frac{\left(\beta^2-\alpha^2\right)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} , \\ c_n &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} . \end{aligned} \tag{C.6}$$

Note that in this paper we use the Jacobi polynomials with $\alpha = \beta$ so that $b_n = 0$ for any n. The following differential constraint satisfied by the Jacobi polynomials is useful in practice

$$\partial_x J_n^{\alpha,\beta}(x) = \mu_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta+1}(x) , \quad \mu_n^{\alpha,\beta} = \frac{1}{2}(n+\alpha+\beta+1) . \tag{C.7}$$

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Two families of the Jacobi polynomials with parameters α, β and a, b can be related to each other by the linear transformation

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \widehat{c}_k^n J_k^{a,b}(x)\,, \tag{C.8}$$

where the transition coefficients are given by

$$\begin{aligned} \widehat{c}_k^n = &\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} \frac{(2k+a+b+1)\Gamma(k+a+b+1)}{\Gamma(k+a+1)} \times \\ &\times \sum_{m=0}^{n-k} \frac{(-1)^m \Gamma(n+k+m+\alpha+\beta+1)\Gamma(m+k+a+1)}{m!(n-k-m)!\Gamma(k+m+\alpha+1)\Gamma(m+2k+a+b+2)}\,. \end{aligned} \tag{C.9}$$

C.1 Basis functions

To perform the $\mathrm{AdS}_3/\mathrm{AdS}_2$ degression we use the following basis functions built in terms of the Jacobi polynomials

$$\begin{aligned} P_n^s(z) &= \widetilde{N}_n^s (\cosh z)^{-d-2s+2} J_n^{\frac{d+2s-4}{2},\frac{d+2s-4}{2}}(-\tanh z) = \\ &= N_n^s (\cosh z)^{-d-2s+2} {}_2F_1\left(-n, n+d+2s-3; \frac{d+2s-2}{2}; \frac{1+\tanh z}{2}\right)\,, \end{aligned} \tag{C.10}$$

where the normalization constants read

$$N_n^s = \widetilde{N}_n^s \frac{\Gamma(n+\frac{d+2s-2}{2})}{n!\Gamma(\frac{d+2s-2}{2})} = \frac{\sqrt{d+2s-3+2n}}{2^{\frac{d+2s-3}{2}}\Gamma(\frac{d+2s-2}{2})} \sqrt{\frac{\Gamma(d+2s-3+n)}{n!}}\,. \tag{C.11}$$

Such a set of the basis functions turns out to be convenient when describing the higher-dimensional $\mathrm{AdS}_{d+1}/\mathrm{AdS}_d$ degression [4]. In the $d=2$ case the parameter s must be $s>1$ otherwise $\frac{d+2s-4}{2} \leqslant -1$. It follows that in two dimensions we cannot consider functions P_n^0 as their inner products built using (C.1) do not converge. This is another critical reason why the two-dimensional case is different from the higher-dimensional case.

For the later convenience, let us denote

$$J_n^s \equiv J_n^{\frac{d+2s-4}{2},\frac{d+2s-4}{2}}\,. \tag{C.12}$$

The functions $\{P_n^s\}_{n\in\mathbb{N}_0}$ defined above form an orthonormal basis

$$\begin{aligned} (P_n^s, P_m^s)_s &\equiv \int_{-\infty}^{+\infty} dz (\cosh z)^{d+2s-2} P_n^s(z) P_m^s(z) = \\ &= \widetilde{N}_n^s \widetilde{N}_m^s \int_{-\infty}^{+\infty} dz (\cosh z)^{-d-2s+2} J_n^s(-\tanh z) J_m^s(-\tanh z) = \\ &= \widetilde{N}_n^s \widetilde{N}_m^s \int_{-1}^{+1} dx (1-x)^{\frac{d+2s-4}{2}} (1+x)^{\frac{d+2s-4}{2}} J_n^s(x) J_m^s(x) = \delta_{nm}\,. \end{aligned} \tag{C.13}$$

The relation (C.13) introduces an inner product $(A,B)_s$ between two basis functions. It has the following obvious properties

$$(A,B)_s = (B,A)_s\,, \qquad (A, \tanh^n z B)_s = (\tanh^n z A, B)_s\,. \tag{C.14}$$

A convenient way to organize terms arising in a top action is to introduce a first-order differential operator $Ł_n$ originating from AdS$_{d+1}$ covariant derivatives (similar to the one used in [5]: $\cos^{-1}\theta$ changed to $\cosh z$)

$$Ł_n A \equiv \cosh^{-n} z\partial(\cosh^n zA) \equiv (\partial + n\tanh z)A\,. \tag{C.15}$$

The Ł-operator has the property that

$$(A, Ł_n B)_s = -(Ł_{d+2s-2-n}A, B)_s\,, \qquad (A, \cosh^2 zŁ_2 B)_s = -(Ł_{d+2s-2}A, B)_{s+1}\,. \tag{C.16}$$

The functions $\{P^s_n(z)\}_{n\in\mathbb{N}_0}$ can be defined as eigenfunctions of the corresponding Sturm-Liouville operator[10]

$$Ł_{d+2s-4}\left(\cosh^2 zŁ_2 P^s_n\right) = \left[\cosh^2 zŁ_2 Ł_{d+2s-2} - (d+2s-4)\right]P^s_n = -\left(\gamma_{s|n}\right)^2 P^s_n\,, \tag{C.17}$$

where

$$\gamma_{s|n} = \sqrt{(n+1)(n+d+2s-4)}\,. \tag{C.18}$$

Thus, these functions form a basis in the $(\cosh z)^{d+2s-2}$-weighted $L^2(\mathbb{R})$ space.

Using the equations (C.7) and (C.16) one can obtain differential relations between neighbouring families of the basis functions (parameterized by s)

$$\cosh^2 z\, Ł_2 P^s_n = \gamma_{s|n} P^{s-1}_{n+1}, \qquad Ł_{d+2s-2}P^s_n = -\gamma_{s+1|n-1}P^{s+1}_{n-1}\,. \tag{C.19}$$

It is worth noting that $\cosh^2 z\, Ł_2$ and $Ł_{d+2s-2}$ can be considered as lowering/raising operators in s or n, respectively.

Now, using the recurrence relations (C.5) for the Jacobi polynomials we obtain the similar recurrence formula (at $\alpha = \beta = (d+2s-4)/2$)

$$\tanh z\, P^s_n = -\frac{1}{a_n}\frac{\widetilde{N}^s_n}{\widetilde{N}^s_{n+1}}P^s_{n+1} - \frac{c_n}{a_n}\frac{\widetilde{N}^s_n}{\widetilde{N}^s_{n-1}}P^s_{n-1}\,, \tag{C.20}$$

where the second term is non-vanishing only if $n \geqslant 1$.

C.2 Inner products

The recurrence relation (C.20) allows computing non-trivial inner products with $\tanh z$ and $\tanh^2 z$ (in general, with $\tanh^m z$ for higher m)

$$\begin{aligned}(P^s_n, \tanh z P^s_m)_s &= \begin{cases} -\frac{1}{a_n}\frac{\widetilde{N}^s_{n-1}}{\widetilde{N}^s_n}\,, & n = m+1\,,\\ -\frac{c_{n+1}}{a_{n+1}}\frac{\widetilde{N}^s_{n+1}}{\widetilde{N}^s_n}\,, & n = m-1\,, m \geqslant 1\,,\end{cases}\\ &= \begin{cases} -\sqrt{\frac{n(n+d+2s-4)}{(2n+d+2s-3)(2n+d+2s-5)}}\,, & n = m+1\,,\\ -\sqrt{\frac{(n+1)(n+d+2s-3)}{(2n+d+2s-1)(2n+d+2s-3)}}\,, & n = m-1\,, m \geqslant 1\,.\end{cases}\end{aligned} \tag{C.21}$$

[10] In refs. [3, 4, 17] it was chosen to be the Pöschl-Teller Hamiltonian.

$$(P_n^s, \tanh^2 z P_m^s)_s = \begin{cases} \frac{1}{a_{n-2}a_{n-1}}\frac{\widetilde{N}^s_{n-2}}{\widetilde{N}^s_n}\,, & n=m+2\,, \\ \frac{c_{n+1}}{a_n a_{n+1}} + \frac{c_n}{a_{n-1}a_n}\,, & n=m\,, m\geqslant 1\,, \\ \frac{c_1}{a_0 a_1}\,, & n=m=0\,, \\ \frac{c_{n+1}c_{n+2}}{a_{n+1}a_{n+2}}\frac{\widetilde{N}^s_{n+2}}{\widetilde{N}^s_n}\,, & n=m-2\,, m\geqslant 2\,, \end{cases}$$
$$= \begin{cases} \frac{\sqrt{n(n-1)(n+d+2s-4)(n+d+2s-5)}}{(2n+d+2s-5)\sqrt{(2n+d+2s-3)(2n+d+2s-7)}}\,, & n=m+2\,, \\ \frac{2n(n+d+2s-3)+d+2s-5}{(2n+d+2s-1)(2n+d+2s-5)}\,, & n=m\,, \\ \frac{\sqrt{(n+2)(n+1)(n+d+2s-2)(n+d+2s-3)}}{(2n+d+2s-1)\sqrt{(2n+d+2s+1)(2n+d+2s-3)}}\,, & n=m-2\,, m\geqslant 2\,. \end{cases} \tag{C.22}$$

Using (C.8) one can derive ($\alpha=\beta=(d+2s-4)/2$ and $a=b=(d+2s-2)/2$)

$$\begin{aligned} (P_n^{s+1}, P_m^s)_s &= \left(P_n^{s+1}, \widetilde{N}^s_m \sum_{k=0}^m \widehat{c}^{\,m}_k \frac{1}{\widetilde{N}^{s+1}_k} P_k^{s+1}\right)_{s+1} = \\ &= \widehat{c}^{\,m}_n \frac{\widetilde{N}^s_m}{\widetilde{N}^{s+1}_n}\,, \quad n=\{m, m-2\} \\ &= \begin{cases} \sqrt{\frac{(n+d+2s-2)(n+d+2s-3)}{(2n+d+2s-1)(2n+d+2s-3)}}\,, & n=m\,, \\ -\sqrt{\frac{(n+2)(n+1)}{(2n+d+2s+1)(2n+d+2s-1)}}\,, & n=m-2\,, m\geqslant 2\,. \end{cases} \end{aligned} \tag{C.23}$$

It turns out that $\widehat{c}^{\,m}_k$ are non-vanishing only when $n=m$ and $n=m-2\,, m\geqslant 2$.

All other relevant inner products can be obtained in the similar way. Below we list those ones used in this paper. Recall that the coefficients $\widehat{c}^{\,m}_n$ here implicitly depend on α, β, a, b, cf. (C.9).

$$\begin{aligned} (P_n^s, Ł_i P_m^s)_s &= (P_n^s, [Ł_{d+2s-2} + (i-d-2s+2))\tanh z] P_m^s)_s = \\ &= -\gamma_{s+1|m-1}(P_n^s, P_{m-1}^{s+1})_s + (i-d-2s+2)(P_n^s, \tanh z P_m^s)_s\,. \end{aligned} \tag{C.24}$$

$$(P_n^{s+2}, P_m^s)_s = \left(P_n^{s+2}, \widetilde{N}^s_m \sum_{k=0}^m \widehat{c}^{\,m}_k \frac{1}{\widetilde{N}^{s+1}_k} P_k^{s+1}\right)_{s+1} = \widehat{c}^{\,m}_n \frac{\widetilde{N}^s_m}{\widetilde{N}^{s+2}_n}\,, \quad n=\{m, m-2, m-4\}\,. \tag{C.25}$$

$$(P_n^{s+1}, \tanh z P_m^s)_s = \left(P_n^{s+1}, \tanh z \widetilde{N}^s_m \sum_{k=0}^m \widehat{c}^{\,m}_k \frac{1}{\widetilde{N}^{s+1}_k} P_k^{s+1}\right)_{s+1}\,. \tag{C.26}$$

$$\begin{aligned} &(P_n^{s+1}, \cosh^2 z Ł_i Ł_j P_m^s)_s = \\ &\quad = \left(\widetilde{N}^{s+1}_n \sum_{k=0}^n \widehat{c}^{\,n}_k \frac{1}{\widetilde{N}^s_k} P_k^s, Ł_i Ł_j P_m^s\right)_s, k=\left\{n, n-2, n-4, \ldots, n-2\left\lfloor\frac{n}{2}\right\rfloor\right\}\,. \end{aligned} \tag{C.27}$$

$$
\begin{aligned}
&(P^s_n, \L_i \L_j P^s_m)_s = \\
&\quad = (P^s_n, [\L_{i+j-d-2s+2}\L_{d+2s-2} + (i-d-2s+1)(j-d-2s+2)\tanh^2 z + \\
&\qquad + (j-d-2s+2)]P^s_m)_s = \\
&\quad = -\gamma_{s+1|m-1}(P^s_n, [\L_{d+2s} + (i+j-2d-4s+2)\tanh z]P^{s+1}_{m-1})_s + \\
&\qquad + (i-d-2s+1)(j-d-2s+2)(P^s_n, \tanh^2 z P^s_m)_s + (j-d-2s+2)\delta_{nm} = \\
&\quad = \gamma_{s+1|m-1}\gamma_{s+2|m-2}(P^s_n, P^{s+2}_{m-2})_s - (i+j-2d-4s+2)\gamma_{s+1|m-1}(P^s_n, \tanh z P^{s+1}_{m-1})_s + \\
&\qquad + (i-d-2s+1)(j-d-2s+2)(P^s_n, \tanh^2 z P^s_m)_s + (j-d-2s+2)\delta_{nm} \,.
\end{aligned}
\tag{C.28}
$$

D Schouten identities

In two dimensions there are useful identities that follow from the fact that any tensor with three antisymmetrized indices is identically zero. In refs. [64–66] such identities with over-antisymmetrization were generally called the Schouten identities. In particular, antisymmetrizing a product of three Kronecker deltas $\delta^{\mu\nu\rho}_{\alpha\beta\gamma} = \delta^{[\mu}_{\alpha}\,\delta^{\nu}_{\beta}\,\delta^{\rho]}_{\gamma}$ and then contracting with a second derivative of a symmetric rank-2 tensor field $X^{\mu\nu}$ one obtains

$$
\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla_\nu\nabla^\beta X_\rho{}^\gamma \equiv 0\,, \tag{D.1}
$$

where the first factor vanishes identically because all indices take two values, $\alpha, \beta, \gamma, \mu, \nu, \rho = 0, 1$. Raising the index α yields the following relation

$$
\Box X^{\mu\nu} - \nabla^{(\mu}\nabla_\rho X^{\nu)\rho} + \nabla^\mu\nabla^\nu X - g^{\mu\nu}\Box X + g^{\mu\nu}\nabla_\rho\nabla_\sigma X^{\rho\sigma} + a\big[2X^{\mu\nu} - g^{\mu\nu}X\big] \equiv 0\,, \tag{D.2}
$$

where the algebraic terms arise from commutating covariant derivatives (see appendix A) and $X = g^{\mu\nu}X_{\mu\nu}$ is the trace. Remarkably, the left-hand side of (D.2) represents the linearized Einstein equation in AdS_2 with the curvature $R = -2a$. This is yet another manifestation that the Einstein-Hilbert action in two dimensions is topological.

The analogous Schouten identities exist for tensor fields of rank $s = 3, 4, \ldots$, while there are no any non-trivial (i.e. containing $\Box X^\mu$) identities for $s = 1$ tensor fields. For instance, the Schouten identity for a symmetric rank-3 tensor field $X^{\gamma\rho\sigma}$,

$$
\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla_\nu\nabla^\beta X_\rho{}^{\gamma\sigma} \equiv 0\,, \tag{D.3}
$$

is expanded as follows

$$
\begin{aligned}
6\Box X^{\mu\nu\rho} &- 4\nabla^{(\mu}\nabla_\sigma X^{\nu\rho)\sigma} + \nabla^{(\mu}\nabla^\nu X^{\rho)} - 2g^{(\mu\nu}\Box X^{\rho)} \\
&+ 2g^{(\mu\nu}\nabla_\sigma\nabla_\varsigma X^{\rho)\sigma\varsigma} + a\big[-18X^{\mu\nu\rho} + 4g^{(\mu\nu}X^{\rho)}\big] \equiv 0\,.
\end{aligned}
\tag{D.4}
$$

Formally, one can say that in two dimensions there are two series of second-order kinetic operators for rank-s tensor fields $X^{\mu_1\cdots\mu_s}$: the Schouten series and the Fronsdal series. They are different in general, but coincide at $s = 2$ giving rise to the Einstein tensor. Both of them are gauge invariant with respect to $X^{\mu_1\cdots\mu_s} \to X^{\mu_1\cdots\mu_s} + \nabla^{(\mu_s}Y^{\mu_1\cdots\mu_{s-1})}$. However, the Schouten identities are trivially invariant since the gauge variation is another

Schouten identity with three derivatives acting on $Y^{\mu_1\cdots\mu_{s-1}}$. On the other hand, the Schouten identities can be used to express the Laplace-Beltrami operator $\Box X^{\mu_1\cdots\mu_s}$ in terms of divergences $\nabla_\nu X^{\nu\mu_1\cdots\mu_{s-1}}$ and algebraic terms $X^{\mu_1\cdots\mu_s}$ and $g_{\nu\rho}X^{\nu\rho\mu_1\cdots\mu_{s-2}}$ that makes the Fronsdal operators to be proportional to $X^{\mu_1\cdots\mu_s}$ provided that the TT gauge is imposed.

Finally, the Schouten type identities also exist in higher dimensions $n\geqslant 3$ when one acts with an antisymmetrized product of $n+1$ Kronecker symbols $\delta^{\mu_1\dots\,\mu_{n+1}}_{\alpha_1\dots\,\alpha_{n+1}}$ on a higher derivative combination of a tensor field, e.g.

$$\delta^{\mu_1\dots\,\mu_{n+1}}_{\alpha_1\dots\,\alpha_{n+1}}\, X^{\nu_1\dots\nu_r}{}_{\mu_1}{}^{\alpha_1}\nabla_{\mu_2}\nabla^{\alpha_2}\dots\nabla_{\mu_n}\nabla^{\alpha_n}X_{\mu_{n+1}}{}^{\alpha_{n+1}}{}_{\nu_1\dots\nu_r}\equiv 0\,, \tag{D.5}$$

where $X^{\mu_1\cdots\mu_{r+2}}$ is a totally-symmetric rank-$(r{+}2)$ tensor (mixed-symmetry tensor could be also considered). Performing index contractions the identity (D.5) can be cast into the form $X^{\mu_1\cdots\mu_{r+2}}\Box^{n-1}X_{\mu_1\dots\mu_{r+2}}+\ldots\equiv 0$. Such identities could be useful in studying higher-order theories in higher-dimensional spaces with typical kinetic terms of the form $X\Box^k X+\ldots$ with some critical relation between n and k that makes such theories topological. In this paper these numbers are $n=2$ and $k=1$.

E Mode expansion in the spin-3 theory

Having the decomposition $\Phi^{mnk}(x,z)=\{\Phi^{\mu\nu\rho}(x,z),\,\Phi^{\mu\nu}(x,z),\,\Phi^{\mu}(x,z),\,\Phi(x,z)\}$ we introduce new notations for the component fields along with some convenient redefinitions

$$w^{\mu\nu\rho}:=\Phi^{\mu\nu\rho}+(ad)^{-1}\operatorname{sech}^2 z\, g^{(\mu\nu}\Phi^{\rho)}\,,\quad h^{\mu\nu}:=\Phi^{\mu\nu}\,,\quad A^{\mu}:=\Phi^{\mu}\,,\quad \phi:=\Phi\,. \tag{E.1}$$

The component fields are all traceful with respect to the AdS$_d$ metric. Similarly, one decomposes the gauge parameter $\Xi^{mn}(x,z)=\{\Xi^{\mu\nu}(x,z),\,\Xi^{\mu}(x,z),\,\Xi(x,z),\,\Xi(x,z)\}$ and redefines the components as

$$\xi^{\mu\nu}:=a^{-1}b\operatorname{sech}^2 z\left[\Xi^{\mu\nu}+(ad)^{-1}\operatorname{sech}^2 z\, g^{\mu\nu}\,\Xi\right]\,, \tag{E.2}$$

$$\xi^{\mu}:=a^{-1}b\operatorname{sech}^2 z\,\Xi^{\mu}\,,\quad \xi:=a^{-1}b\operatorname{sech}^2 z\,\Xi\,, \tag{E.3}$$

where the gauge parameter $\xi^{\mu\nu}$ is made traceless. Then, the total action decomposed according to (E.1)

$$S=\sum_{m\geqslant n}\ \sum_{n=0,1,2,3}S_{mn}\,, \tag{E.4}$$

is built from the following component actions

$$\begin{aligned}S_{33}=a^2\iint\cosh^4 z\Big[&-(\nabla_\alpha w^{\mu\nu\rho})^2+3(\nabla_\mu w^{\mu\nu\rho})^2-6\nabla_\mu w_\nu\nabla_\rho w^{\mu\nu\rho}+3(\nabla_\alpha w^{\mu})^2+\\&+\frac{3}{2}(\nabla_\mu w^\mu)^2+a\big[-(d-3)(w^{\mu\nu\rho})^2+6(d-1)(w^\mu)^2+\\&+w_{\mu\nu\rho}Ł_{d+2}(\cosh^2 zŁ_2 w^{\mu\nu\rho})-3w_\mu Ł_{d+2}(\cosh^2 zŁ_2 w^\mu)\big]\Big]\,,\end{aligned} \tag{E.5}$$

$$S_{32}=3a^2\iint\cosh^4 z\Big[-4\nabla_\mu w_\nu Ł_{d+2}h^{\mu\nu}+2\nabla_\mu w^{\mu\nu\rho}Ł_{d+2}h_{\nu\rho}+\nabla_\mu w^\mu Ł_{d+2}h\Big]\,, \tag{E.6}$$

$$S_{31}=6a^2(d+1)d^{-1}\iint\cosh^4 z w_\mu Ł_{d+2}Ł_d A^\mu\,, \tag{E.7}$$

$$S_{30} = 3a \int\int \cosh^2 z \nabla_\mu w^\mu \L_d \phi\,, \tag{E.8}$$

$$\begin{aligned} S_{22} = 3a \iint \cosh^2 z \Big[&- (\nabla_\alpha h^{\mu\nu})^2 + 2(\nabla_\mu h^{\mu\nu})^2 - 2\nabla_\mu h \nabla_\nu h^{\mu\nu} + (\nabla_\alpha h)^2 + \\ &+ a\Big[- 2d(h^{\mu\nu})^2 + \frac{1}{2}(3d-2)h^2 - \frac{3}{2} h \L_d(\cosh^2 z \L_2 h)\Big]\Big]\,, \end{aligned}$$

$$S_{21} = 12(d+1)ad^{-1} \int\int \cosh^2 z \Big[\nabla_\mu h^{\mu\nu} \L_d A_\nu - \nabla_\mu h \L_d A^\mu\Big]\,, \tag{E.9}$$

$$\begin{aligned} S_{20} = 3 \iint &2\phi[\nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h] + \\ &+ ah[- \cosh^2 z \L_{-2d-2} \L_d + 2(3d+1)]\phi\,, \end{aligned} \tag{E.10}$$

$$\begin{aligned} S_{11} = -3(d+1)d^{-2} \iint &d[(\nabla_\alpha A^\mu)^2 - (\nabla_\mu A^\mu)^2] + \\ &+ a\Big[(3d^2+d-4)(A^\mu)^2 + (d+2)A_\mu \L_{d-2}(\cosh^2 z \L_2 A^\mu)\Big]\,, \end{aligned} \tag{E.11}$$

$$S_{10} = -6(d+1)d^{-1} \iint \nabla_\mu A^\mu \L_{-d}\phi\,, \tag{E.12}$$

$$\begin{aligned} S_{00} = a^{-1} \iint &2\cosh^{-2} z (\nabla_\alpha \phi)^2 + \\ &+ \frac{a}{2}\phi[- \L_{-6d}\L_d + d(3d-1)\tanh^2 z + 11d + 4]\phi\,, \end{aligned} \tag{E.13}$$

where $w^\mu \equiv w^{\mu\nu\rho} g_{\nu\rho}$ and $h \equiv h^{\mu\nu} g_{\mu\nu}$. Here, the integration measure is defined as

$$\iint = b^{-2} \int d\mu_{d+1} = a^{d/2} b^{-(d+5)/2} \int d\mu_d \int dz \cosh^d z\,,$$

which differs from (4.12) by an additional factor of b^{-1}. The gauge transformations (5.1) given in terms of the component fields read

$$\begin{aligned} \delta w^{\mu\nu\rho} &= \nabla^{(\mu} \xi^{\nu\rho)} + 2d^{-1} \L_{d+2}\, g^{(\mu\nu} \xi^{\rho)}\,, \\ \delta h^{\mu\nu} &= \nabla^{(\mu} \xi^{\nu)} + a\cosh^2 z\, \L_2\, \xi^{\mu\nu} - d^{-1} g^{\mu\nu} \L_{-2d}\, \xi\,, \\ \delta A^\mu &= \nabla^\mu \xi + 2a\cosh^2 z\, \L_2\, \xi^\mu\,, \\ \delta\phi &= 3a\cosh^2 z\, \L_2\, \xi\,. \end{aligned} \tag{E.14}$$

The mode expansions for the component fields and parameters are chosen to be

$$\begin{aligned} w^{\mu\nu\rho}(x,z) &= \sum_{n=0}^{\infty} w_n^{\mu\nu\rho}(x) P_n^3(z)\,, \quad & h^{\mu\nu}(x,z) &= \sum_{n=0}^{\infty} h_n^{\mu\nu}(x) P_n^2(z)\,, \\ A^\mu(x,z) &= \sum_{n=0}^{\infty} A_n^\mu(x) P_n^1(z)\,, & \phi(x,z) &= \sum_{n=0}^{\infty} \phi_n(x) P_n^1(z)\,, \end{aligned} \tag{E.15}$$

$$\xi^{\mu\nu}(x,z) = \sum_{n=0}^{\infty} \xi_n^{\mu\nu}(x) P_n^3(z)\,, \qquad \xi^\mu(x,z) = \sum_{n=0}^{\infty} \xi_n^\mu(x) P_n^2(z)\,, \quad \xi(x,z) = \sum_{n=0}^{\infty} \xi_n(x) P_n^2(z)\,. \tag{E.16}$$

Note that the fields A^μ and ϕ as well as the gauge parameters ξ^μ and ξ are expanded with respect to the same basis functions. Plugging (E.15)–(E.16) into the gauge transformations (E.14) one finds the mode transformations

$$
\begin{aligned}
\delta w_n^{\mu\nu\rho} &= \nabla^{(\mu}\xi_n^{\nu\rho)} - 2d^{-1}\,\gamma_{3|n}\,g^{(\mu\nu}\xi_{n+1}^{\rho)}\,,\\
\delta h_n^{\mu\nu} &= \nabla^{(\mu}\xi_n^{\nu)} + a\,\gamma_{3|n-1}\,\xi_{n-1}^{\mu\nu} - d^{-1}g^{\mu\nu}\sum_{m=0}^{\infty}(P_n^2,\L_{-2d}P_m^2)_2\,\xi_m\,,\\
\delta A_n^{\mu} &= \nabla^{\mu}\sum_{m=0}^{\infty}(P_n^1,P_m^2)_1\,\xi_m + 2a\,\gamma_{2|n-1}\,\xi_{n-1}^{\mu}\,,\\
\delta\phi_n &= 3a\,\gamma_{2|n-1}\,\xi_{n-1}\,.
\end{aligned}
\tag{E.17}
$$

Then, integrating out the slicing coordinate z yields the total action (E.4) in the form

$$
\begin{aligned}
S_{33} &= a^2\int\sum_{n=0}^{\infty}\Big\{-(\nabla_\alpha w_n^{\mu\nu\rho})^2 + 3(\nabla_\mu w_n^{\mu\nu\rho})^2 - 6\nabla_\mu w_{n\nu}\nabla_\rho w_n^{\mu\nu\rho} + 3(\nabla_\alpha w_n^{\mu})^2 + \qquad (E.18)\\
&\quad + \frac{3}{2}(\nabla_\mu w_n^{\mu})^2 + a\big[-((\gamma_{3|n})^2 + d - 3)(w_n^{\mu\nu\rho})^2 + (3(\gamma_{3|n})^2 + 6(d-1))(w_n^{\mu})^2\big]\Big\}\,,\\
S_{32} &= 3a^2\int\sum_{n=1}^{\infty}\gamma_{3|n-1}\Big\{4\nabla_\mu w_{n-1\nu}h_n^{\mu\nu} - 2\nabla_\mu w_{n-1}^{\mu\nu\rho}h_{n\nu\rho} - \nabla_\mu w_{n-1}^{\mu}h_n\Big\}\,, \qquad (E.19)\\
S_{31} &= \frac{6(d+1)a^2}{d}\int\sum_{n=2}^{\infty}\gamma_{2|n-1}\gamma_{3|n-2}w_{n-2|\mu}A_n^{\mu}\,, \qquad (E.20)\\
S_{30} &= -3a\int\sum_{m=0,n=1}^{\infty}\gamma_{2|n-1}\nabla_\mu w_m^{\mu}\phi_n(P_m^3,P_{n-1}^2)_2\,,\\
S_{22} &= 3a\int\sum_{n=0}^{\infty}\Big\{-(\nabla_\alpha h_n^{\mu\nu})^2 + 2(\nabla_\mu h_n^{\mu\nu})^2 - 2\nabla_\mu h_n\nabla_\nu h_n^{\mu\nu} + (\nabla_\alpha h_n)^2 + \qquad (E.21)\\
&\quad + a\Big[-2d(h_n^{\mu\nu})^2 + \frac{1}{2}(3(\gamma_{2|n})^2 + 3d - 2)(h_n)^2\Big]\Big\}\,,\\
S_{21} &= \frac{12(d+1)a}{d}\int\sum_{n=1}^{\infty}\gamma_{2|n-1}\Big\{-\nabla_\mu h_{n-1}^{\mu\nu}A_{n\nu} + \nabla_\mu h_{n-1}A_n^{\mu}\Big\}\,, \qquad (E.22)\\
S_{20} &= 3\int\sum_{m,n=0}^{\infty}\Big\{2\phi_m[\nabla_\mu\nabla_\nu h_n^{\mu\nu} - \Box h_n](P_m^1,P_n^2)_1 + \qquad (E.23)\\
&\quad + a\phi_m h_n([-\cosh^2 z\L_{-2d-2}\L_d + 2(3d+1)]P_m^1,P_n^2)_1\Big\}\,,\\
S_{11} &= -\frac{3(d+1)}{d^2}\int\sum_{n=0}^{\infty}\Big\{d[(\nabla_\alpha A_n^{\mu})^2 - (\nabla_\mu A_n^{\mu})^2] + \qquad (E.24)\\
&\quad + a\Big[3d^2 + d - 4 - (d+2)(\gamma_{1|n})^2\Big](A_n^{\mu})^2\Big\}\,,\\
S_{10} &= -\frac{6(d+1)}{d}\int\sum_{m,n=0}^{\infty}\nabla_\mu A_m^{\mu}\phi_n(P_m^1,\L_{-d}P_n^1)_1\,, \qquad (E.25)\\
S_{00} &= \frac{1}{a}\int\sum_{m,n=0}^{\infty}\Big\{2\nabla_\alpha\phi_m\nabla^{\alpha}\phi_n(P_m^1,[1-\tanh^2 z]P_n^1)_1 + \qquad (E.26)\\
&\quad + \frac{a}{2}\phi_m\phi_n(P_m^1,[-\L_{-6d}\L_d + d(3d-1)\tanh^2 z + 11d + 4]P_n^1)_1\Big\}\,.
\end{aligned}
$$

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where the integration measure is introduced

$$\int = a^{d/2} b^{-(d+5)/2} \int d\mu_d \,. \tag{E.27}$$

The fields A_n^μ and ϕ_n with $n = 1, 2, \ldots, \infty$ can be gauged away by means of the Stueckelberg-type gauge transformations using all vector and scalar gauge parameters ξ_n^μ and ξ_n with $n = 0, 1, 2, \ldots, \infty$. The modes $w_n^{\mu\nu\rho}$ and $h_n^{\mu\nu}$ remain intact. Then, the resulting partially gauged fixed action (E.18)–(E.26) reads

$$\begin{aligned}
S_{33} &= a^2 \int \sum_{n=0}^{\infty} \Big\{ -(\nabla_\alpha w_n^{\mu\nu\rho})^2 + 3(\nabla_\mu w_n^{\mu\nu\rho})^2 - 6\nabla_\mu w_{n\nu} \nabla_\rho w_n^{\mu\nu\rho} + 3(\nabla_\alpha w_n^\mu)^2 + \\
&\quad + \frac{3}{2}(\nabla_\mu w_n^\mu)^2 + a\big[-((\gamma_{3|n})^2 + d - 3)(w_n^{\mu\nu\rho})^2 + (3(\gamma_{3|n})^2 + 6(d-1))(w_n^\mu)^2\big] \Big\} ,
\end{aligned} \tag{E.28}$$

$$S_{32} = 3a^2 \int \sum_{n=1}^{\infty} \gamma_{3|n-1} \Big\{ 4\nabla_\mu w_{n-1\nu} h_n^{\mu\nu} - 2\nabla_\mu w_{n-1}^{\mu\nu\rho} h_{n\nu\rho} - \nabla_\mu w_{n-1}^\mu h_n \Big\} , \tag{E.29}$$

$$S_{31} = 0 \,, \qquad S_{30} = 0 \,, \tag{E.30}$$

$$\begin{aligned}
S_{22} &= 3a \int \sum_{n=0}^{\infty} \Big\{ -(\nabla_\alpha h_n^{\mu\nu})^2 + 2(\nabla_\mu h_n^{\mu\nu})^2 - 2\nabla_\mu h_n \nabla_\nu h_n^{\mu\nu} + (\nabla_\alpha h_n)^2 + \\
&\quad + a\Big[-2d(h_n^{\mu\nu})^2 + \frac{1}{2}(3(\gamma_{2|n})^2 + 3d - 2)(h_n)^2 \Big] \Big\} , \qquad S_{21} = 0 \,,
\end{aligned} \tag{E.31}$$

$$\begin{aligned}
S_{20} &= 3 \int \sum_{n=0}^{\infty} \Big\{ 2\phi_0 [\nabla_\mu \nabla_\nu h_n^{\mu\nu} - \Box h_n] (P_0^1, P_n^2)_1 + \\
&\quad + a\phi_0 h_n \big([-\cosh^2 z Ł_{-2d-2} Ł_d + 2(3d+1)] P_0^1, P_n^2 \big)_1 \Big\} = \\
&= 3 \int \left\{ 2\phi_0 [\nabla_\mu \nabla_\nu h_0^{\mu\nu} - \Box h_0] \sqrt{\frac{d}{d+1}} + a\phi_0 h_0 \left(0 + 2(3d+1) \sqrt{\frac{d}{d+1}} \right) \right\} = \\
&= 6\sqrt{\frac{d}{d+1}} \int \phi_0 [\nabla_\mu \nabla_\nu h_0^{\mu\nu} - \Box h_0 + a(3d+1) h_0] \,,
\end{aligned} \tag{E.32}$$

$$S_{11} = -\frac{3(d+1)}{d^2} \int \Big\{ d[(\nabla_\alpha A_0^\mu)^2 - (\nabla_\mu A_0^\mu)^2] + a(2d^2 + d)(A_0^\mu)^2 \Big\} , \tag{E.33}$$

$$S_{10} = -\frac{6(d+1)}{d} \int \nabla_\mu A_0^\mu \phi_0 (P_0^1, Ł_{-d} P_0^1)_1 = \tag{E.34}$$

$$\begin{aligned}
&= -\frac{6(d+1)}{d} \int \nabla_\mu A_0^\mu \phi_0 (P_0^1, [Ł_d - 2d \tanh z] P_0^1)_1 = \\
&= -\frac{6(d+1)}{d} \int \nabla_\mu A_0^\mu \phi_0 (0 - 2d \cdot 0) = 0 \,,
\end{aligned} \tag{E.35}$$

$$\begin{aligned}
S_{00} &= \frac{1}{a} \int \Big\{ 2(\nabla_\alpha \phi_0)^2 (P_0^1, [1 - \tanh^2 z] P_0^1)_1 + \\
&\quad + \frac{a}{2}(\phi_0)^2 (P_0^1, [-Ł_{-6d} Ł_d + d(3d-1)\tanh^2 z + 11d + 4] P_0^1)_1 \Big\} = \\
&= \frac{1}{a} \int \left\{ 2(\nabla_\alpha \phi_0)^2 \left(1 - \frac{1}{d+1} \right) + \frac{a}{2}(\phi_0)^2 \left(0 + d(3d-1) \cdot \frac{1}{d+1} + 11d + 4 \right) \right\} = \\
&= \frac{1}{a(d+1)} \int \Big\{ 2d(\nabla_\alpha \phi_0)^2 + a(7d^2 + 7d + 2)(\phi_0)^2 \Big\} .
\end{aligned} \tag{E.36}$$

Evaluating the inner products and varying the action (E.28)–(E.36) one finds the equations of motion of the bottom theory:

$$\frac{\delta S}{\delta w_n^{\mu\nu\rho}}=0: \quad n\geqslant 0: \quad \Box w_n^{\mu\nu\rho}-\nabla^{(\mu}\nabla_\sigma w_n^{\nu\rho)\sigma}+\frac{1}{2}\nabla^{(\mu}\nabla^\nu w_n^{\rho)}-g^{(\mu\nu}\Box w_n^{\rho)}+ \tag{E.37}$$
$$+g^{(\mu\nu}\nabla_\sigma\nabla_\zeta w_n^{\rho)\sigma\zeta}-\frac{1}{2}g^{(\mu\nu}\nabla^{\rho)}\nabla_\sigma w_n^\sigma+$$
$$+a\big[-((\gamma_{3|n})^2+d-3)w_n^{\mu\nu\rho}+((\gamma_{3|n})^2+2(d-1))g^{(\mu\nu}w_n^{\rho)}\big]+$$
$$+\gamma_{3|n}\left[-2g^{(\mu\nu}\nabla_\sigma h_{n+1}^{\rho)\sigma}+\nabla^\mu h_{n+1}^{\nu\rho)}+\frac{1}{2}g^{(\mu\nu}\nabla^{\rho)}h_{n+1}\right]=0\,,$$

$$\frac{\delta S}{\delta h_n^{\mu\nu}}=0: \quad n=0: \quad \Box h_0^{\mu\nu}-\nabla^{(\mu}\nabla_\sigma h_0^{\nu)\sigma}+\nabla^\mu\nabla^\nu h_0-g^{\mu\nu}\Box h_0+\nabla_\sigma\nabla_\rho h_0^{\sigma\rho}+ \tag{E.38}$$
$$+a\left[-2dh_0^{\mu\nu}+\frac{1}{2}(3(\gamma_{2|0})^2+3d-2)g^{\mu\nu}h_0\right]+$$
$$+\frac{1}{a}\sqrt{\frac{d}{d+1}}[\nabla^\mu\nabla^\nu-g^{\mu\nu}\Box+a(3d+1)g^{\mu\nu}]\phi_0=0\,, \tag{E.39}$$
$$n\geqslant 1: \quad \Box h_n^{\mu\nu}-\nabla^{(\mu}\nabla_\sigma h_n^{\nu)\sigma}+\nabla^\mu\nabla^\nu h_n-g^{\mu\nu}\Box h_n+\nabla_\sigma\nabla_\rho h_n^{\sigma\rho}+$$
$$+a\left[-2dh_n^{\mu\nu}+\frac{1}{2}(3(\gamma_{2|n})^2+3d-2)g^{\mu\nu}h_n\right]+$$
$$+a\left[\nabla^{(\mu}w_{n-1}^{\nu)}-\nabla_\rho w_{n-1}^{\mu\nu\rho}-\frac{1}{2}g^{\mu\nu}\nabla_\rho w_{n-1}^\rho\right]=0\,,$$

$$\frac{\delta S}{\delta A_0^\mu}=0: \quad \Box A_0^\mu-\nabla^\mu\nabla_\nu A_0^\nu-a(2d+1)A_0^\mu=0\,, \tag{E.40}$$

$$\frac{\delta S}{\delta\phi_0}=0: \quad \left[\Box-\frac{a(7d^2+7d+2)}{2d}\right]\phi_0- \tag{E.41}$$
$$-\frac{3}{2}\sqrt{\frac{d+1}{d}}a[\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box+a(3d+1)g_{\mu\nu}]h_0^{\mu\nu}=0\,.$$

Similar to the spin-2 case the vector mode A_0^μ decouples from the other fields. Also, there is the residual gauge invariance

$$\delta w_n^{\mu\nu\rho}=\nabla^{(\mu}\xi_n^{\nu\rho)}\,, \qquad \delta h_n^{\mu\nu}=a\gamma_{3|n-1}\xi_{n-1}^{\mu\nu}\,, \tag{E.42}$$

with respect to the traceless parameters $\xi_n^{\mu\nu}$ with $n=0,1,2,\ldots,\infty$. The invariance can be used to impose the TT gauge for rank-3 fields (see section 5).

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