Universal renormalization procedure for higher curvature gravities in $D \leq 5$

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ABSTRACT: We implement a universal method for renormalizing AdS gravity actions applicable to arbitrary higher curvature theories in up to five dimensions. The renormalization procedure considers the extrinsic counterterm for Einstein-AdS gravity given by the *Kounterterms* scheme, but with a theory-dependent coupling constant that is fixed by the requirement of renormalization for the vacuum solution. This method is shown to work for a generic higher curvature gravity with arbitrary couplings except for a zero measure subset, which includes well-known examples where the asymptotic behavior is modified and the AdS vacua are degenerate, such as Chern-Simons gravity in 5D, Conformal Gravity in 4D and New Massive Gravity in 3D. In order to show the universality of the scheme, we perform a decomposition of the equations of motion into their normal and tangential components with respect to the Poincare coordinate and study the Fefferman-Graham expansion of the metric. We verify the cancellation of divergences of the on-shell action and the well-posedness of the variational principle.

KEYWORDS: Classical Theories of Gravity, AdS-CFT Correspondence

ARXIV EPRINT: [2108.01126](https://arxiv.org/abs/2108.01126)

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1 Introduction

Higher curvature gravity (HCG) has been extensively studied as a possible effective field theory extension of Einstein's general relativity (GR), motivated by the fact that curvature terms of arbitrarily high order are compatible with the diffeomorphism invariance symmetry, and that these terms can generically be considered as quantum corrections to

the Einstein-Hilbert action at high energies [1]. The presence of higher curvature terms on the effective gravitational action are a robust prediction of string theory [2] —there are indeed hints of a possible mutual implication [3], and the resulting effective theories have better UV properties than GR. However, generic higher curvature theories have dynamical instabilities and ghost-like degrees of freedom. The best studied HCG is Lovelock gravity, which was constructed in order to have second-order field equations and no dynamical (Ostrogradsky) instabilities [4]. Other HCGs that have been constructed to be of second order around specific backgrounds, such as spherically-symmetric or cosmological (FLRW) backgrounds, are quasi-topological gravities [5] and generalized quasi-topological gravities (GQGs) [6–8]. Stability conditions have been studied perturbatively for these theories and other concerns regarding their dynamics have been already addressed in the literature.

In the context of the gauge/gravity duality [9], the role of HCGs can be understood as a tool to accommodate the description of a broader class of conformal field theories (CFTs), having for example different central charges and anomaly coefficients than those dual to GR [10]. The holographic properties of HCGs such as Lovelock and GQGs have also been discussed in the literature, where anomaly coefficients and central charges have been holographically computed and restrictions have been placed on the couplings for the higher curvature terms based on CFT considerations, such as the unitarity requirement for the CFT dual to the particular HCG in the bulk (see [11–13] for Lanczos-Gauss-Bonnet, [14–16] for Lovelock and [17, 18] for GQGs).

As it is standard in the saddle-point approximation of anti-de Sitter/CFT (AdS/CFT) holography, the use of the on-shell classical gravity action as the generating functional for connected correlators of the dual CFT requires the finiteness of said action and also requires the corresponding variational principle to be well-posed. The latter is needed in order for arbitrary variations of the action to correspond entirely to variations with respect to the boundary field values, which are identified as the CFT sources. In the case of GR and Lovelock gravity, both requirements have been implemented following the holographic renormalization (HR) prescription ([19–23] for GR and [24] for Lovelock). In its original proposal, this prescription considers first fixing the variational principle at an arbitrary radial boundary in the bulk of an asymptotically locally AdS (AlAdS) manifold, using the standard Gibbons-Hawking-York (GHY) term or the Myers term [25] respectively. Then, the radius at which the boundary is placed is taken as a regulator of the theory and it is extended to the conformal boundary of the space. This produces divergent terms at different orders in the regulator, which can be thus isolated. Finally, boundary terms which are intrinsic in the induced metric at the boundary, and thus depend on its Riemannian curvature and covariant derivatives thereof, are added in order to remove said divergences without modifying the variational principle.

In the case of arbitrary HCGs, the boundary term required for fixing the variational principle at fixed radius and the HR counterterms required for the finiteness of the action on AlAdS spaces are not known. However, it has been suggested that for Einsteinian cubic gravity (a GQG of cubic order in $D = 4$), the action can be rendered finite with the same combination of the GHY and the HR counterterms used in GR, up to a coupling-dependent overall factor [26]. It is therefore suggestive to explore the possibility of defining a uni-

versal renormalization prescription, that would work for an arbitrary HCG, allowing not only to cancel the divergences in the gravitational action, but also to pose the variational principle properly.

In this work, we develop such a generic renormalization scheme for arbitrary HCGs, in spacetimes with dimension $D \leq 5$. In order to do so, we study the radial decomposition of the equations of motion (EOM) for the arbitrary HCG, expanded in the Poincare coordinate z considering the Fefferman-Graham (FG) expansion of the metric [27], which is standard for AlAdS manifolds. We show that the divergent terms which contribute to the on-shell action at the boundary (for bulk $D \leq 5$) depend only on the first four FG coefficients of the metric (associated to the powers in z up to z^3). We also show that for the generic combination of couplings, the second coefficient (of the order z term) and the fourth coefficient (order z^3) are zero, and that the third coefficient (order z^2) has a universal form. Using these facts, we are able to show that the extrinsic boundary counterterms discussed in [28, 29], with theory-dependent coupling constants, successfully implement the renormalization and fix the variational principle at the conformal boundary. In showing this, we consider the asymptotic equivalence between the *Kounterterm* renormalization scheme and the HR prescription for manifolds of up to $D = 5$, as discussed in [30].

The paper is organized as follows. In section 2, we consider the radially decomposed EOM of an arbitrary HCG of the form $\mathcal{L}$ (Riemann), in order to check that for the generic case, the second and fourth coefficients in the FG expansion of the metric ($g_{ij}^{(1)}$ and $g_{ij}^{(3)}$) are zero, which is needed in order to verify the renormalization of the theory. In section 3, we present the extrinsic boundary counterterms, with the correct theory-dependent couplings, which implement the renormalization in arbitrary HCGs (in $D \leq 5$ spacetime dimensions). In section 4, we verify the cancellation of divergences in HCG actions renormalized with the generic prescription. In section 5, we verify that the addition of the extrinsic counterterm used in the generic renormalization prescription achieves the well-posedness of the variational principle for a generic HCG. Finally, in section 6, we summarize and discuss our results.

2 Projected equations of motion in HCG

We consider a HCG in $D \leq 5$ whose action is given by

$$I = \int_M d^D X \sqrt{-G} \mathcal{L} \left(R_{\mu\nu}^{\rho\sigma} \right), \quad (2.1)$$

where the Lagrangian includes any possible term constructed from arbitrary contractions of the Riemann tensor and the metric. The equations of motion are given by [31]

$$\mathcal{E}_\mu^\nu = P_{\mu\alpha}^{\beta\gamma} R_{\beta\gamma}^{\nu\alpha} - \frac{1}{2} \delta_\mu^\nu \mathcal{L} - 2 \nabla^\alpha \nabla_\beta P_{\mu\alpha}^{\beta\nu}, \quad (2.2)$$

where the tensor $P_{\rho\sigma}^{\mu\nu}$ is defined as

$$P_{\rho\sigma}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}^{\rho\sigma}}. \quad (2.3)$$

Since we are interested in renormalizing theories of gravity on AlAdS backgrounds, we consider that the Riemann tensor near the AdS boundary behaves as

$$R_{\mu\nu}^{\rho\sigma} \rightarrow -\frac{1}{L^2}(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\nu^\rho \delta_\mu^\sigma), \quad (2.4)$$

where L is the effective AdS radius, which is related to that appearing in the cosmological term, $\Lambda = -(D-1)(D-2)/(2L_0^2)$, through a relation $L = L(L_0)$ that depends on the particular theory. Notice that $P_{\rho\sigma}^{\mu\nu}$ has the same symmetries as the Riemann tensor (2.3); thereby, close to the boundary it becomes

$$P_{\rho\sigma}^{\mu\nu} \rightarrow \frac{1}{2}C(L)(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu). \quad (2.5)$$

The constant $C(L)$ can be obtained by replacing these expressions in the field equations [32],

$$C(L) = -\frac{L^2}{2(D-1)}\mathcal{L}|_{\text{AdS}} = \frac{L^3}{2D(D-1)}\frac{\partial\mathcal{L}|_{\text{AdS}}}{\partial L}, \quad (2.6)$$

where $\mathcal{L}|_{\text{AdS}}$ is the Lagrangian evaluated in the AdS vacuum solution. In the case of Einstein-Hilbert gravity, this constant is nothing but $1/(16\pi G_N)$.

Now, we consider the FG expansion of the bulk metric by splitting the coordinates, $X^\mu = (x^i, z)$, into normal and tangent to the boundary,

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = \frac{L^2}{z^2} dz^2 + h_{ij}(x, z) dx^i dx^j, \quad (2.7)$$

where h_{ij} is given by¹

$$h_{ij}(x, z) = \frac{1}{z^2} g_{ij}(x, z) = \frac{1}{z^2} \left(g_{ij}^{(0)}(x) + z g_{ij}^{(1)}(x) + z^2 g_{ij}^{(2)}(x) + z^3 g_{ij}^{(3)}(x) + \dots \right), \quad (2.8)$$

z being the Poincare coordinate, defined such that the asymptotic conformal boundary is located at $z \rightarrow 0$. The coefficients $g_{ij}^{(n)}(x)$ with $n < D-1$ are determined completely in terms of $g_{ij}^{(0)}(x)$ by means of the projected equations of motion $\mathcal{E}_z^z = 0$, $\mathcal{E}_j^i = 0$ and $\mathcal{E}_i^z = 0$. Since we are considering $D \leq 5$, we want to scrutinize the coefficients in this expansion up to order z^3 , in particular to see whether they are the same as those in Einstein's gravity,

$$\begin{aligned} g_{ij}^{(1)} &= 0, \\ g_{ij}^{(2)} &= -\frac{L^2}{D-3} \left(\mathcal{R}_{ij}(g^{(0)}) - \frac{1}{2(D-2)} \mathcal{R}(g^{(0)}) g_{ij}^{(0)} \right), \\ g_{ij}^{(3)} &= 0, \end{aligned} \quad (2.9)$$

or not. Here, $\mathcal{R}_{ij}(g^{(0)})$ and $\mathcal{R}(g^{(0)})$ are respectively the Ricci tensor and curvature scalar computed from $g_{ij}^{(0)}$. In the form (2.7), the metric is naturally decomposed in the parts that are normal (zz component) and tangent to the boundary (ij components). The tangent

¹Logarithmic terms also appear in this expansion for odd-dimensions, even in Einstein's gravity.

indices are raised with the inverse tangent metric $h^{ij}(x, z) = z^2 g^{ij}(x, z)$, which has an expansion in z such that $h_{ik}h^{jk} = \delta_i^j$. The vector normal to the boundary, n , is defined as

$$n = n^\mu \partial_\mu = n^z \partial_z = -\frac{z}{L} \partial_z, \quad n^\mu n_\mu = 1, \quad (2.10)$$

where the minus sign appears due to the fact that the boundary is located at the lowest limit of the possible range of values of the radial coordinate z . We can also define a covariant derivative compatible with the tangent part of the metric, $\tilde{\nabla}_i$, such that

$$\tilde{\nabla}_i h_{jk} = \tilde{\nabla}_i g_{jk} = 0, \quad (2.11)$$

since $h_{jk} = g_{jk}/z^2$ and $\tilde{\nabla}_i z = 0$ by definition. Since h_{ij} and g_{ij} depend on z , the tangent covariant derivative will also admit an expansion in this normal coordinate. Apart from this, the usual covariant derivative is compatible with the global metric, $\nabla_\mu G_{\rho\sigma} = 0$. For completeness, we write the explicit form of the different non-vanishing components of the Christoffel symbols computed from the metric (2.7),

$$\Gamma_{zz}^z(\nabla) = -\frac{1}{z}, \quad \Gamma_{zj}^i(\nabla) = \frac{1}{2} h^{ik} \partial_z h_{jk}, \quad \Gamma_{ij}^z(\nabla) = -\frac{1}{2} \frac{z^2}{L^2} \partial_z h_{ij}, \quad (2.12)$$

whereas $\Gamma_{ij}^k(\nabla) = \Gamma_{ij}^k(\tilde{\nabla}) + \mathcal{O}(z)$. Again, since $h_{ij} = h_{ij}(x, z)$, most of these admit an expansion in powers of z , in terms of the different coefficients in (2.8). We further need to know the form of the extrinsic curvature of this metric, given by

$$K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = \frac{1}{2} n^z \partial_z h_{ij} = -\frac{1}{2} \frac{z}{L} \partial_z \left(\frac{1}{z^2} g_{ij} \right). \quad (2.13)$$

Now, we can use the Gauss-Codazzi equations, together with the expression of the extrinsic curvature, to obtain the different components of the Riemann tensor that will be needed to compute the projected equations of motion,

$$\begin{aligned} R_{izjz} &= \frac{L^2}{z^2} \left(-\mathcal{L}_n K_{ij} + K_{ik} K_j^k - a_i a_j + \tilde{\nabla}_{(i} a_{j)} \right), \\ R_{ijkz} &= -n_z \left(\tilde{\nabla}_i K_{jk} - \tilde{\nabla}_j K_{ik} \right), \\ R_{ijkl} &= \mathcal{R}_{ijkl}(h) + K_{il} K_{jk} - K_{ik} K_{jl}, \end{aligned} \quad (2.14)$$

where $a_\mu = n^\alpha \nabla_\alpha n_\mu$, and $\mathcal{R}_{ijkl}(h)$ is the usual Riemann tensor of the tangent metric $h_{ij}(x, z)$, which can be expanded also in powers of z ,

$$\mathcal{R}_{ijkl}(h) = \frac{1}{z^2} \mathcal{R}_{ijkl}(g) = \frac{1}{z^2} \left(\mathcal{R}_{ijkl}(g^{(0)}) + \mathcal{O}(z) \right). \quad (2.15)$$

Notice that the indices of $\mathcal{R}_{ijkl}(h)$ are raised with the tangent inverse metric h^{ij} , while those of $\mathcal{R}_{ijkl}(g)$ must be raised with g^{ij} .

2.1 Vanishing of $g_{ij}^{(1)}$ in a general HCG

If we compute the different components of the Riemann tensor explicitly to next-to-leading order in the holographic coordinate, we find that they match the expresion

$$R_{\rho\sigma}^{\mu\nu} = -\frac{2}{L^2} \delta_{[\rho}^{[\mu} \delta_{\sigma]}^{\nu]} + z \frac{2}{L^2} \delta_{[\rho}^{[\mu} g^{(1)\nu]}_{\sigma]} + \mathcal{O}(z^2), \quad (2.16)$$

where by definition only the components of $g^{(1)\nu}_{\sigma}$ with both indices in the directions tangent to the boundary can be non-zero. Regarding $P_{\rho\sigma}^{\mu\nu}$, we can use its definition (2.3) and the expression of the Riemann tensor (2.16) to constrain its tensorial form,

$$P_{\rho\sigma}^{\mu\nu} = C(L) \delta_{\rho}^{[\mu} \delta_{\sigma]}^{\nu]} + z \left(A^{(1)}(L) \delta_{[\rho}^{[\mu} g^{(1)\nu]}_{\sigma]} + B^{(1)}(L) \delta_{\rho}^{[\mu} \delta_{\sigma]}^{\nu]} \text{Tr } g^{(1)} \right) + \mathcal{O}(z^2), \quad (2.17)$$

where $C(L)$ was given in (2.6), whereas $A^{(1)}(L)$ and $B^{(1)}(L)$ are scalar functions depending on the couplings of the theory and the effective AdS radius, L ; also, $\text{Tr } g^{(1)} := g^{(1)i}_i$.

The Lagrangian also appears in the general expression of the equations of motion (2.2), so we expand it symbolically to first order in z ,

$$\mathcal{L} = \mathcal{L}^{(0)} + P_{\mu\nu}^{\rho\sigma} \delta R_{\rho\sigma}^{\mu\nu} + \dots = \mathcal{L}^{(0)} + z \mathcal{L}^{(1)} + \mathcal{O}(z^2), \quad (2.18)$$

where $\mathcal{L}^{(0)}$ is the Lagrangian evaluated in the background solution (2.4) and, given that $\delta R_{\rho\sigma}^{\mu\nu}$ is the deviation of the Riemann tensor with respect to this background, thereby already of order z ,

$$\mathcal{L}^{(1)} = \frac{D-1}{L^2} C(L) \text{Tr } g^{(1)}. \quad (2.19)$$

With all these ingredients, the equations of motion, decomposed into their radial and tangential components and expanded in powers of z , read:

$$\mathcal{E}_z^z = \left(\frac{(1-D)C(L)}{L^2} - \frac{\mathcal{L}^{(0)}}{2} \right) + \frac{z}{2L^2} \left(a^{(1)}(L) + (D-1)b^{(1)}(L) \right) \text{Tr } g^{(1)} + \mathcal{O}(z^2), \quad (2.20)$$

$$\mathcal{E}_j^i = \left(\frac{(1-D)C(L)}{L^2} - \frac{\mathcal{L}^{(0)}}{2} \right) \delta_j^i + \frac{z(D-2)}{2L^2} \left[a^{(1)}(L) g^{(1)i}_j + b^{(1)}(L) \delta_j^i \text{Tr } g^{(1)} \right] + \mathcal{O}(z^2), \quad (2.21)$$

$$\mathcal{E}_i^z = \frac{1}{2(D-2)L^2} \left(a^{(1)}(L) \tilde{\nabla}_j g^{(1)j}_i + b^{(1)}(L) \tilde{\nabla}_i \text{Tr } g^{(1)} \right) + \mathcal{O}(z^3), \quad (2.22)$$

where

$$a^{(1)}(L) := C(L) - A^{(1)}(L), \quad (2.23)$$

$$b^{(1)}(L) := -C(L) - A^{(1)}(L) - 4B^{(1)}(L). \quad (2.24)$$

The zeroth order gives the aforementioned result for $C(L)$ and all the information of the equations at the lowest orders is encoded in $a^{(1)}(L)$ and $b^{(1)}(L)$. The main feature of these equations is that, except in some particular cases, they imply

$$g^{(1)i}_j = 0. \quad (2.25)$$

More specifically, the equations of motion fix the $\mathcal{O}(z)$ FG coefficient to be zero, except in the following cases:

- If $a^{(1)}(L) \neq 0$ and $b^{(1)}(L) = -\frac{a^{(1)}(L)}{(D-1)}$, the off-diagonal components of $g^{(1)i}_j$ are fixed to zero and the elements in the diagonal are fixed to be equal to each other, but their value is free.
- If $a^{(1)}(L) = 0$ and $b^{(1)}(L) \neq 0$, $\text{Tr} g^{(1)}$ is fixed to zero, but otherwise it is free.
- And finally, if both $a^{(1)}(L) = 0$ and $b^{(1)}(L) = 0$, $g^{(1)i}_j$ is fully free and therefore it is not restricted by the equations of motion.

This is interesting as it means that, for the generic HCG theory, its equations of motion require the $\mathcal{O}(z)$ coefficient in the FG expansion of the metric to vanish, just like for Einstein-AdS gravity. This universality is analogous to the universality of the $\mathcal{O}(z^2)$ coefficient as implied by the PBH transformations [33], and both are central to the applicability of the generic renormalization procedure presented in this work.

An example: quadratic curvature gravity. Let us illustrate this further by working out explicitly the case of quadratic curvature gravity. Its action can be written as

$$I = \int_M d^D X \sqrt{-G} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha_1 R_{\mu\nu} R^{\mu\nu} + \alpha_2 R^2 + \alpha_3 \chi_{\text{GB}} \right], \quad (2.26)$$

where χ_{GB} is the Lanczos-Gauss-Bonnet combination. Once we compute and plug in the values of $C(L)$, $a^{(1)}(L)$ and $b^{(1)}(L)$, we have

$$C(L) = \frac{1}{\kappa} - \frac{2(D-1)}{L^2} \left(\alpha_1 + D\alpha_2 + \frac{(D-2)(D-3)}{D-1} \alpha_3 \right), \quad (2.27)$$

$$a^{(1)}(L) = \frac{1}{\kappa} - \frac{1}{L^2} \left[(3D-4)\alpha_1 + 2D(D-1)\alpha_2 + 2(D-3)(D-4)\alpha_3 \right], \quad (2.28)$$

$$b^{(1)}(L) = -\frac{1}{\kappa} + \frac{1}{L^2} \left[(D-4)\alpha_1 + 2(D-1)(D-4)\alpha_2 + 2(D-3)(D-4)\alpha_3 \right]. \quad (2.29)$$

These equations imply $g^{(1)}_{ij} = 0$, unless the conditions discussed after (2.25) are met. Some examples of quadratic curvature gravity theories where $a^{(1)}(L) = b^{(1)}(L) = 0$, such that $g^{(1)}_{ij}$ is not fixed by the equations of motion, include:

- Einstein-Lanczos-Gauss-Bonnet gravity at the (dimensionally continued) Chern-Simons point. In [34], the authors find that in this theory the coefficients of the FG expansion are not fixed by the equations of motion, as they vanish identically.
- Conformal Gravity in 4 dimensions [35–37], where it is found again that the equations of motion at the lowest orders vanish for any form of the coefficients in the expansion.
- New Massive Gravity [38, 39] at the special point (in the language of [38]).

A feature all these cases share is that their AdS vacua are degenerate. Indeed, we checked that this is true for any quadratic or cubic gravity theory fulfilling $a^{(1)}(L) = b^{(1)}(L) = 0$. While it is true that these particular cases allow for $g^{(1)}_{ij} \neq 0$, they do not enforce it. In

particular, $g_{ij}^{(1)}$ cannot be given as an expression in terms of $g_{ij}^{(0)}$, thereby its value must be fixed as a boundary condition [37]. Therefore, in general we could pick $g_{ij}^{(1)} = 0$ for any theory in vacuum, and build our discussion on top of this assumption.

We have also repeated this analysis for general theories of gravity with cubic contractions of the curvature tensors. The results are written in appendix A.

2.2 Universality of $g_{ij}^{(2)}$ from the PBH transformations

The aim of this section is to obtain the form of the coefficient $g_{ij}^{(2)}$ in terms of the conformal boundary metric $g_{ij}^{(0)}$ for general theories of gravity. While this could in principle be computed from the equations of motion $\mathcal{E}_\nu^\mu = 0$ at order z^2 , here we will review the approach of [33], which can be applied directly to any HCG with an asymptotically AdS solution.

The argument is based on the invariance of the bulk metric under PBH (Penrose-Brown-Henneaux) transformations, which are a subset of the bulk diffeomorphisms that reduce to Weyl transformations on the boundary metric, and leave the form of the bulk metric unchanged [33]. Imposing this on (2.7), one obtains transformation properties for the coefficients in the FG expansion of the tangent metric $g_{ij}(x, z)$.

The authors of [33] take the terms with odd powers of z in the expansion of the tangent metric (2.8) to be zero. This had already been shown to be true for Einstein gravity [27], and they assume that it holds for a general theory with an AdS solution. In our case we have proven that $g_{ij}^{(1)} = 0$ for general HCGs, except in some very particular cases.² Since we are restricting ourselves to this more general scenario, and we are currently interested in the coefficient $g_{ij}^{(2)}$, it is enough to take the results of [33]. In particular, it is found that invariance under PBH transformations constrains the form of $g_{ij}^{(2)}$ to be

$$g_{ij}^{(2)} = -\frac{L^2}{D-3} \left(\mathcal{R}_{ij}(g^{(0)}) - \frac{1}{2(D-2)} \mathcal{R}(g^{(0)}) g_{ij}^{(0)} \right), \quad (2.30)$$

where we took care of the fact that, due to scaling properties, the term $g_{ij}^{(n)}$ contains n derivatives with respect to the tangent coordinates x^i .

We should note that, since the argument builds upon invariance under boundary Weyl transformations, it fails to capture further contributions made of contractions of the Weyl tensor of the boundary that might appear.³ However, for 2- and 3-dimensional boundaries, which correspond respectively to $D = 3$ and $D = 4$ bulk dimensions, the Weyl tensor is identically zero, so these contributions do not occur. While for $D = 5$ this is not true, when treating the well-posedness of the variational problem we will need to assume asymptotic conformal flatness [40], which implies that the Weyl tensor of the boundary must be zero. Therefore, we see that under these assumptions the coefficient $g_{ij}^{(2)}$ takes the same form in a general HCG as in Einstein's theory (2.9).

²Strictly speaking, it has been known for a while that there are theories where $g_{ij}^{(1)}$ is nonzero [37, 38].

³Of course, $\mathcal{W}^{(0)ij}_{ik}$ vanishes identically, so this term is ruled out. However, at this order there could appear contractions, for example, with the schematic form of $\sqrt{\mathcal{W}^{(0)}\mathcal{W}^{(0)}}$ with two free indices, as seen for the case of Chern-Simons gravity in [34].

2.3 Vanishing of $g_{ij}^{(3)}$ in a general HCG

Until now, we have found that, for a general HCG, the coefficients $g_{ij}^{(1)}$ and $g_{ij}^{(2)}$ in the FG expansion (2.8) take the same form as in Einstein gravity, written in (2.9). We will show shortly that this is true also for the coefficient multiplied by z^3 , and in the next sections we will see that these are the only relevant ones in order to study the renormalization of theories with up to 5 bulk dimensions.

Following the same logic as in section 2.1, we expect the coefficient $g_{ij}^{(3)}$ to be fixed, in general, by the contributions of the equations of motion multiplied by z^3 . In order to obtain this we need to expand the different objects that appear in these equations as written in (2.2), and in particular we use symbolic expressions for both $\mathcal{L}$ and $P_{\rho\sigma}^{\mu\nu}$. Since we have already fixed $g_{ij}^{(1)} = 0$, and we know the form of the zeroth order coefficients in these two objects through equations (2.5) and (2.6), their expansions can be written as

$$\mathcal{L} = -\frac{2(D-1)}{L^2}C(L) + z^2\mathcal{L}^{(2)} + z^3\mathcal{L}^{(3)} + \dots, \quad (2.31)$$

$$P_{\alpha\beta}^{\mu\nu} = C(L)\delta_{\alpha}^{[\mu}\delta_{\beta}^{\nu]} + z^2P^{(2)\mu\nu}_{\alpha\beta} + z^3P^{(3)\mu\nu}_{\alpha\beta} + \dots. \quad (2.32)$$

We are interested only on the third order terms in the equations, so we should understand the form of the coefficient $P^{(3)\mu\nu}_{\alpha\beta}$. This will depend on the theory, of course, but we can follow the same reasoning as before and use its tensorial structure to write the components $P^{(3)ij}_{kl}$ and $P^{(3)iz}_{jz}$ as the combinations

$$\begin{aligned} P^{(3)ij}_{kl} &= A^{(3)}(L)\delta_{[k}^{[i}g^{(3)j]}_{l]} + B^{(3)}(L)\delta_{[k}^{[i}\delta_{l]}^{j]}\text{Tr } g^{(3)}, \\ P^{(3)iz}_{jz} &= D^{(3)}(L)g^{(3)i}_j + E^{(3)}(L)\delta_j^i\text{Tr } g^{(3)}, \end{aligned} \quad (2.33)$$

where the constants $A^{(3)}(L)$, $B^{(3)}(L)$, $D^{(3)}(L)$ and $E^{(3)}(L)$ depend on the AdS radius L and the higher-curvature couplings. Since $g_{ij}^{(1)} = 0$, terms of the form $g^{(1)i}_j g^{(2)k}_l$ will not appear in these general expressions. Also, the components with one index in the normal direction, such as $P^{(3)ij}_{zk}$, will have terms proportional to $\tilde{\nabla}_i g^{(2)j}_k$, but these appear in the equations of motion at higher orders in z , as they must be contracted with $n^z = z/L$. Regarding the expansion of the Lagrangian, $\mathcal{L}^{(3)}$ cannot have contributions of the form $\tilde{\nabla}_i g^{(2)j}_k$, as there are no objects with an odd number of tangent indices to contract it producing a term of order z^3 . Therefore, it can only contain terms that are proportional to $\text{Tr } g^{(3)}$, and if we expand it as in equation (2.18) we see that they can only be produced by $P^{(0)\mu\nu}_{\rho\sigma}\delta R^{(3)\rho\sigma}_{\mu\nu}$. Plugging in the third order components of the Riemann tensor,

$$R^{(3)ij}_{kl} = \frac{6}{L^2}\delta_{[k}^{[i}g^{(3)j]}_{l]}, \quad R^{(3)zi}_{zj} = -\frac{3}{2L^2}g^{(3)i}_j, \quad (2.34)$$

we find

$$\mathcal{L}^{(3)} = \frac{3(D-3)}{L^2}C(L)\text{Tr } g^{(3)}. \quad (2.35)$$

With all of this, we can compute the terms of third order in the projections of the equations of motion, which read

$$\mathcal{E}^{(3)z}_z = \frac{1}{2L^2} \left(a^{(3)}(L) + (D-1)b^{(3)}(L) \right) \text{Tr } g^{(3)}, \quad (2.36)$$

$$\mathcal{E}^{(3)i}_j = \frac{D-4}{2L^2} \left[a^{(3)}(L)g^{(3)i}_j + b^{(3)}(L)\delta^i_j \text{Tr } g^{(3)} \right], \quad (2.37)$$

where the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ are related to those introduced in (2.33) as

$$\begin{aligned} a^{(3)}(L) &= 3C(L) + 4(D-6)D^{(3)}(L) - (D-3)A^{(3)}(L), \\ b^{(3)}(L) &= -3C(L) - A^{(3)}(L) - 2(D-2)B^{(3)}(L) + 4(D-6)E^{(3)}(L). \end{aligned} \quad (2.38)$$

For completeness, we give the values of the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ for general quadratic and cubic theories of gravity in appendix B.

From the discussion above on the different contributions to $\mathcal{L}^{(3)}$ and $P^{(3)\mu\nu}_{\alpha\beta}$, it is clear that the equation $\mathcal{E}^{(3)z}_z = 0$ fixes —for general⁴ HCGs— $\text{Tr } g^{(3)} = 0$, and this in turn means, when we consider the equation $\mathcal{E}^{(3)i}_j = 0$, that

$$g^{(3)i}_j = 0. \quad (2.39)$$

As we commented when fixing $g^{(1)}_{ij} = 0$, there are families of theories for which the value of $g^{(3)}_{ij}$ is not determined by the equations of motion. In particular, the same analysis discussed after (2.25) applies to $g^{(3)}$ as well, but considering the $a^{(3)}(L)$ and $b^{(3)}(L)$ coefficients instead. All the quadratic and cubic theories considered in section 2.1 and appendix A, at the particular points mentioned, allow for $g^{(3)}_{ij} \neq 0$ even when choosing $g^{(1)}_{ij} = 0$ as a boundary condition.⁵ However, the conditions that leave $g^{(3)}_{ij}$ undetermined do not imply the degeneracy of the different AdS vacua.

3 Renormalization counterterms for generic HCG in $D \leq 5$

It was suggested in [26], for higher-curvature theories with second-order linearized EOM around maximally symmetric backgrounds, that one can renormalize their actions using the same boundary term that is used in the Holographic Renormalization of Einstein-AdS gravity, i.e., the GHY term plus the HR counterterm, but with a coupling-dependent overall coefficient. The renormalization was framed in terms of the cancellation of divergences of the gravity action, and not in terms of the well-posedness of the variational principle, as for higher-curvature gravities (with the exception of Lovelock) this is an open problem. As it is explained in what follows, this idea for generating counterterms based on the Einstein-AdS case can be generalized to arbitrary HCGs considering the asymptotic behaviour of AlAdS spaces.

⁴Notice that $\text{Tr } g^{(3)}$ must vanish for $D = 4$ while $g^{(3)i}_j$ is left undetermined. This is expected given that in three dimensions $g^{(3)i}_j$ is dual to the stress-energy tensor and there is no conformal anomaly.

⁵With the exception of New Massive Gravity, since it is a 3-dimensional theory, and therefore the coefficient $g^{(3)}_{ij}$ is sub-normalizable and should not be considered in the expansion (2.8).

When considering pure AdS vacua, a minimal requirement for the renormalization procedure is to render the Euclidean on-shell action equal to either zero or the vacuum energy of the maximally-symmetric configuration. As it is usual, the vacuum energy appears in odd-dimensional bulk manifolds, and in the context of AdS/CFT, it is related to the Casimir energy in the CFT side. One can then assume that the boundary term for HCGs is equal to the one for Einstein gravity but with a coupling-dependent overall factor. This overall factor can then be fixed by requiring the cancellation of divergences in the action for the maximally-symmetric solution. As in the case of $\mathcal{L}(\text{Riemann})$ theories, said action evaluated in the vacuum solution is proportional to the AdS volume, with an overall constant dependent of the couplings of the theory. One can then check if the same boundary term works for other AlAdS solutions besides the pure AdS configuration.

A similar approach was pursued in [41] and [42], where the authors considered some counterterms with a multiplicative constant—that matches the prescription in [26]—in order to compute the Noether-Wald charges for quadratic curvature gravity theories in even-dimensional asymptotically AdS spacetimes. In [43] the same terms are introduced to obtain renormalized entanglement entropies.

The counterterms considered in these last three references are, however, different from the usual HR proposal. The latter prescription produces a series of terms, whose complexity depends on the dimension and can not be expressed in any closed form. The new approach adds to the action some topological quantities dubbed *Kounterterms*—because they can be naturally written in terms of the extrinsic curvature of the boundary—and were originally proposed in [28, 29, 44–47] to renormalize the Einstein-Hilbert action and obtain a well-posed variational principle, which then allows to compute finite conserved charges in AdS gravity. Moreover, another interesting application of this method is the computation of renormalized entanglement entropies [40, 48, 49].

In the present work, we aim to expand this prescription to more general theories of gravity admitting AlAdS solutions in up to 5 dimensions, which can be expanded in terms of the radial coordinate as in (2.7), with the coefficients given in (2.9). First of all, we will simply write the form of the Kounterterms for general even and odd dimensional bulks, as given in the literature. The only modification that we make is the multiplicative constant $C(L)$ that accompanies these boundary terms, which was defined in (2.6) and is the only theory-dependent part of the entire expression. In sections 4 and 5 we will see that this constant appears naturally in the divergent terms that need to be cancelled, and hence it is motivated.

3.1 Kounterterms for even bulk dimensions

The Kounterterms for $D = 2n$ dimensions are given by [28]

$$I_{\text{Kt}} = c_{2n-1} \int_{\partial M} d^{2n-1}x B_{2n-1}[h, K, \mathcal{R}], \quad (3.1)$$

where B_{2n-1} is the n -th Chern form⁶

$$B_{2n-1} = -2n\sqrt{-h} \int_0^1 dt \delta_{j_1 \dots j_{2n-1}}^{i_1 \dots i_{2n-1}} K_{i_1}^{j_1} \left(\frac{1}{2} \mathcal{R}_{i_2 i_3}^{j_2 j_3} - t^2 K_{i_2}^{j_2} K_{i_3}^{j_3} \right) \\ \times \dots \times \left(\frac{1}{2} \mathcal{R}_{i_{2n-2} i_{2n-1}}^{j_{2n-2} j_{2n-1}} - t^2 K_{i_{2n-2}}^{j_{2n-2}} K_{i_{2n-1}}^{j_{2n-1}} \right), \quad (3.2)$$

and we write the constant c_{2n-1} as

$$c_{2n-1} = -\frac{(-L^2)^{n-1}}{n(2n-2)!} C(L). \quad (3.3)$$

This recovers the usual value of the constant for Einstein gravity, presented for example in [30], since in that case $C(L) = 1/\kappa$ with our conventions. However, we claim that this counterterm is suitable for more general theories of gravity whose Lagrangian is made of arbitrary contractions of the Riemann tensor, in particular, whose bulk is 4-dimensional.

As shown in [30], for Einstein gravity the Kounterterm (3.1) is exactly equivalent to the usual HR prescription in $D = 4$ — and, at least, in $D = 6$, as long as the boundary is conformally flat; i.e., the Weyl tensor of the boundary vanishes. We will see this explicitly in section 4, when we show that it cancels the divergences of the on-shell action in 4 dimensions. Besides, this even-dimensional Kounterterm can be written also as a bulk integral, by means of Euler's theorem. In particular,

$$\int_{M_{2n}} d^{2n}x \mathcal{E}_{2n} = (4\pi)^n n! \chi(M_{2n}) + \int_{\partial M_{2n}} d^{2n-1}x B_{2n-1}, \quad (3.4)$$

where $\chi(M_{2n})$ is the Euler characteristic of M_{2n} , and $\mathcal{E}_{2n}$ is the $2n$ -dimensional Euler density

$$\mathcal{E}_{2n} = \frac{\sqrt{-G}}{2^n} \delta_{\nu_1 \dots \nu_{2n}}^{\mu_1 \dots \mu_{2n}} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \dots R_{\mu_{2n-1} \mu_{2n}}^{\nu_{2n-1} \nu_{2n}}. \quad (3.5)$$

3.2 Kounterterms for odd bulk dimensions

For $D = 2n + 1$ bulk dimensions, the Kounterterm [29] reads:

$$I_{\text{Kt}} = c_{2n} \int_{\partial M} d^{2n}x B_{2n}[h, K, \mathcal{R}], \quad (3.6)$$

where B_{2n} is given by

$$B_{2n} = -2n\sqrt{-h} \int_0^1 dt \int_0^t ds \delta_{j_1 \dots j_{2n}}^{i_1 \dots i_{2n}} K_{i_1}^{j_1} \delta_{i_2}^{j_2} \left(\frac{1}{2} \mathcal{R}_{i_3 i_4}^{j_3 j_4} - t^2 K_{i_3}^{j_3} K_{i_4}^{j_4} + \frac{s^2}{L^2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4} \right) \\ \times \dots \times \left(\frac{1}{2} \mathcal{R}_{i_{2n-1} i_{2n}}^{j_{2n-1} j_{2n}} - t^2 K_{i_{2n-1}}^{j_{2n-1}} K_{i_{2n}}^{j_{2n}} + \frac{s^2}{L^2} \delta_{i_{2n-1}}^{j_{2n-1}} \delta_{i_{2n}}^{j_{2n}} \right), \quad (3.7)$$

⁶In these expressions, $\delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p}$ is the generalized Kronecker delta [50],

$$\delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p} = \det [\delta_{\nu_i}^{\mu_j}] = p! \delta_{[\nu_1}^{\mu_1} \delta_{\nu_2}^{\mu_2} \dots \delta_{\nu_p]}^{\mu_p]}.$$

and the coupling constant c_{2n} is

$$c_{2n} = -\frac{(-L^2)^{n-1}}{2^{2n-2}n(n-1)!^2}C(L). \quad (3.8)$$

As in the even-dimensional case, we recover the values of this constant presented in [30] if we set $C(L) = 1/\kappa$, as should be for Einstein gravity. Also, this is equivalent to the counterterms derived with the HR proposal, up to logarithmically divergent terms, in $D = 3, 5$ and 7 , as long as the boundary is conformally flat [30].

In this case, B_{2n} can not be written as the pullback of a topological quantity in the $D = 2n + 1$ manifold, contrasting with what is found for even dimensions. Also, the fact that B_{2n} depends on the AdS radius L , while B_{2n-1} does not, is a consequence of the topological origin of the latter.

4 Divergence cancellation for HCG actions up to $D = 5$

In this section, we shall address the problem of renormalizing the action of a general HCG when evaluated on an AlAdS background. For this matter, we will first find the form of the divergent terms at the boundary, with a general expression valid for up to $D = 5$. This latter restriction comes from the fact that we are interested on holographic applications in realistic situations; i.e., strongly coupled gauge theories in (at most) four dimensions. Then, we will analyze the divergent terms explicitly for $D = 3, 4$ and 5 , and show that they are indeed cancelled by the Kounterterms presented in section 3.

4.1 Divergent terms in the on-shell action

Let us consider the action of a general HCG in $D \leq 5$,

$$I = \int_M d^D X \sqrt{-G} \mathcal{L} \left(R_{\mu\nu}^{\rho\sigma} \right), \quad (4.1)$$

evaluated on an asymptotically locally AdS spacetime. In order to look at the structure of the divergences, we will write the metric as (2.7), with the coefficients of its FG expansion given by (2.9). Since in these coordinates the boundary is located at $z \rightarrow 0$, it is very natural to identify the divergences in this region by looking at the terms with negative powers of z in the expansion of the action (4.1). Given the form of the metric (2.7), to leading order near $z \rightarrow 0$ the square root of the determinant behaves as

$$\sqrt{-G} \sim \frac{1}{z^D}, \quad (4.2)$$

and it has more contributions of higher order in z , which thus decay faster near the boundary. However, this is enough to identify the terms in the Lagrangian that will produce divergences. As shown in section 2, the odd coefficients in the FG expansion up to the order that we are interested in vanish, so the on-shell Lagrangian can only have terms of the form z^{2i} . In the action, these produce

$$\int dz \sqrt{-G} z^{2i} \sim \int dz z^{2i-D} \sim z^{2i-(D-1)}. \quad (4.3)$$

So this can produce three different behaviors as $z \rightarrow 0$:

- If $i < \frac{D-1}{2}$ the term is divergent, and it needs to be subtracted.
- If $i > \frac{D-1}{2}$ the term vanishes at the boundary.
- For odd spacetime dimensions, there can be contributions with $i = (D-1)/2$, which produce a logarithm divergence in the boundary. This is universal and related to the conformal anomaly of the dual CFT [30, 40], and we will see that it is not cancelled by the topological Kounterterms.

Therefore, depending on the dimensionality of the spacetime the last term that produces divergences will be different. In our case, for up to 5 dimensions we will have to look at terms with $i \leq 2$.

In order to isolate the divergent terms, first of all we have to obtain an expansion of the Lagrangian $\mathcal{L}(R_{\rho\sigma}^{\mu\nu})$ close to the boundary, which can be written as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{(0)} + P_{\rho\sigma}^{\mu\nu} \delta R_{\mu\nu}^{\rho\sigma} + \dots \\ &= \mathcal{L}^{(0)} + P_{ij}^{kl} \delta R_{kl}^{ij} + 4P_{jk}^{zi} \delta R_{zi}^{jk} + 4P_{zi}^{zj} \delta R_{zj}^{zi} + \dots,\end{aligned}\tag{4.4}$$

where $\delta R_{\mu\nu}^{\rho\sigma}$ denotes the terms in the components of the Riemann tensor that are different from the background value (2.4), this is, those that depend on z . They can be computed using the Gauss-Codazzi equations (2.14), finding

$$\begin{aligned}\delta R_{kl}^{ij} &= \frac{z^2}{(D-2)(D-3)} \mathcal{R}^{(0)} \delta_{[k}^i \delta_{l]}^j - \frac{4z^2}{D-3} \mathcal{R}^{(0)[i} \delta_{[k} \delta_{l]}^{j]} + z^2 \mathcal{R}^{(0)ij}_{kl} + \mathcal{O}(z^4), \\ \delta R_{zi}^{jk} &= \frac{L^2 z^3}{D-3} \left(\tilde{\nabla}^{(0)j} \mathcal{R}_i^k - \tilde{\nabla}^{(0)k} \mathcal{R}^{(0)j}_i \right) \\ &\quad + \frac{L^2 z^3}{2(D-3)(D-2)} \left(\delta_i^j \tilde{\nabla}^{(0)k} \mathcal{R}^{(0)} - \delta_i^k \tilde{\nabla}^{(0)j} \mathcal{R}^{(0)} \right) + \mathcal{O}(z^5), \\ \delta R_{zi}^{zj} &= \mathcal{O}(z^4),\end{aligned}\tag{4.5}$$

where we introduced the notation $\tilde{\nabla}^{(0)} \equiv \tilde{\nabla}(g^{(0)})$ and $\mathcal{R}^{(0)} \equiv \mathcal{R}(g^{(0)})$ in order to write simpler expressions. Therefore, since $P^{(0)zi}_{jk} = 0$, in the expansion of the Lagrangian it is enough to take

$$\mathcal{L} = \mathcal{L}^{(0)} + P^{(0)kl}_{ij} \delta R_{kl}^{ij} + \mathcal{O}(z^4),\tag{4.6}$$

where $\mathcal{O}(z^4)$ includes terms coming from $P^{(0)zj}_{zi} \delta R_{zj}^{zi}$ and $P^{(0)zi}_{jk} \delta R_{zi}^{jk}$, among other terms with higher powers of z from the expansion of $P_{\rho\sigma}^{\mu\nu}$. Also, although we are not writing them explicitly, in this expression there should be terms with higher derivatives with respect to the Riemann tensor. However, since any derivative of the Lagrangian with respect to the Riemann tensor is, to the lowest possible order, constant in z , a term of the form $(\partial^n \mathcal{L} / (\partial R)^n) (\delta R)^n$ will be at least $\mathcal{O}(z^{2n})$. Therefore, second or higher derivatives are unimportant when looking for the divergences in low dimensions.

If we now compute $P^{(0)kl}_{ij} \delta R^{ij}_{kl}$ explicitly with the expressions above for $\delta R^{\mu\nu}_{\rho\sigma}$ and $P^{(0)\mu\nu}_{\rho\sigma}$ given by (2.5), we find that it vanishes to the lowest order. So we conclude that the only divergent part of $\mathcal{L}(R^{\mu\nu}_{\rho\sigma})$ is

$$\mathcal{L} = \mathcal{L}^{(0)} + z^4 \mathcal{L}^{(4)}, \quad (4.7)$$

where the term $\mathcal{L}^{(4)}$ will contribute only to the logarithmic divergence in $D = 5$ dimensions. It contains all the terms of order z^4 mentioned in the paragraph above, but its particular form is not relevant for our computations, since we assume that our method of renormalization will not cancel divergences of this type.

Now that we have expanded the Lagrangian in the coordinate z , we need to do the same with the determinant factor that appears in (4.1). Using again the FG expansion of the metric (2.7) and (2.8), this is

$$\sqrt{-G} = \frac{L}{z} \sqrt{-h} = \frac{L \sqrt{-g^{(0)}}}{z^D} \left(1 + \frac{z^2}{2} \text{Tr } g^{(2)} + \mathcal{O}(z^4) \right). \quad (4.8)$$

Plugging everything in, the divergent terms of the general action (4.1) for $D \leq 5$ are

$$\begin{aligned} I_{\text{diver}} &= \int_M d^D X \sqrt{-G} \left(\mathcal{L}^{(0)} + z^4 \mathcal{L}^{(4)} \right) \\ &= L \int d^{D-1} x \sqrt{-g^{(0)}} \int_{z=z_0} dz \left[\frac{1}{z^D} \mathcal{L}^{(0)} + \frac{1}{z^{D-2}} \frac{1}{2} \mathcal{L}^{(0)} \text{Tr } g^{(2)} + \mathcal{O}(z^{4-D}) \right], \end{aligned} \quad (4.9)$$

where we introduced the cutoff z_0 in the lower limit of integration, which must be taken to zero once the divergences are canceled. The actual form of these divergent contributions once integrated in z will depend on the dimension of the spacetime, and in particular the second and third terms will produce the aforementioned logarithmic divergences at $D = 3$ and $D = 5$, respectively. However, the terms that we want to cancel, the power-law divergences, at these low dimensions always appear multiplied by $\mathcal{L}^{(0)}$. This is proportional to the constant $C(L)$ appearing at the lowest order in $P^{\mu\nu}_{\rho\sigma}$ through the equation (2.6), or equivalently

$$\mathcal{L}^{(0)} = -\frac{2(D-1)}{L^2} C(L), \quad (4.10)$$

Therefore, this supports our claim that the Kounterterms which cancel these divergences are the same as those introduced for Einstein gravity, with the explicit prefactor $C(L)$.

4.2 Explicit analysis in different dimensions

We will now show how the divergences (4.9) are cancelled by the Kounterterms introduced in section 3, explicitly in 3-, 4- and 5-dimensional spacetimes. Notice that the computations carried out here were already done in section 3.4 of [30], and the only difference in our results is the generic constant $C(L)$ multiplying both the Kounterterms and the divergent terms in the on-shell action. In order to see that, we will have to write the objects in (3.1) and (3.6) in terms of the intrinsic curvature of the boundary metric $g^{(0)}_{ij}$.

In particular, provided the coefficients of the FG expansion are given by (2.9), we will need the extrinsic curvature (2.13)

$$\begin{aligned} K_j^i &= h^{ik} K_{kj} = \frac{1}{L} \delta_j^i - \frac{z^2}{L} g^{(2)i}{}_j + \frac{z^4}{L} \left(g^{(2)ik} g_{jk}^{(2)} - 2g^{(4)i}{}_j \right) \\ &= \frac{1}{L} \delta_j^i + z^2 \frac{L}{D-3} \left(\mathcal{R}^{(0)i}{}_j - \frac{1}{2(D-2)} \mathcal{R}^{(0)} \delta_j^i \right) + \mathcal{O}(z^4), \end{aligned} \quad (4.11)$$

and the determinant of the tangent metric in terms of the intrinsic curvature (4.8), which by means of (2.30) can be written as

$$\sqrt{-h} = \frac{\sqrt{-g^{(0)}}}{z^{D-1}} \left(1 - z^2 \frac{L^2}{4(D-2)} \mathcal{R}^{(0)} + \mathcal{O}(z^4) \right). \quad (4.12)$$

4.2.1 $D = 3$ dimensions

In this case, the general divergent terms in (4.9) become, after integrating in z ,

$$I_{\text{diver}} = -C(L) \int d^2x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^2} + L \log z_0 \mathcal{R}^{(0)} \right], \quad (4.13)$$

where we used (2.9) to rewrite $\text{Tr } g^{(2)}$ and (4.10) for $\mathcal{L}_0$. We need to find whether the Kounterterm (3.6), particularized for $D = 2n + 1 = 3$ dimensions, cancels the divergences found here. In this case, the constant c_2 and the function B_2 are equal to

$$c_2 = -C(L), \quad B_2 = -\sqrt{-h} K. \quad (4.14)$$

Therefore, replacing the determinant h and the extrinsic curvature K in terms of the intrinsic curvature of $g_{ij}^{(0)}$, we find that the total Kounterterm for $D = 3$ dimensions reads

$$I_{\text{Kt}} = c_2 \int d^2x B_2 = C(L) \int d^2x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^2} + \mathcal{O}(z_0^2) \right], \quad (4.15)$$

which cancels the power-law divergence found in (4.13), but not the logarithmic one, as we had anticipated in section 3.

4.2.2 $D = 4$ dimensions

The divergent terms in 4 dimensions (4.9) become

$$I_{\text{diver}} = -C(L) \int d^3x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^3} - \frac{3L}{4z_0} \mathcal{R}^{(0)} \right]. \quad (4.16)$$

The coupling of the Kounterterm that should cancel this and the second Chern form read

$$c_3 = \frac{L^2}{4} C(L), \quad B_3 = -4\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} K_{i_1}^{j_1} \left(\frac{1}{2} \mathcal{R}_{i_2 i_3}^{j_2 j_3} - \frac{1}{3} K_{i_2}^{j_2} K_{i_3}^{j_3} \right). \quad (4.17)$$

Writing it all together, the Kounterterm for a general theory in $D = 4$ dimensions is

$$I_{\text{Kt}} = C(L) \int d^3x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^3} - \frac{3L}{4z_0} \mathcal{R}^{(0)} + \mathcal{O}(z_0) \right], \quad (4.18)$$

which cancels exactly the divergent terms as written in (4.16). As was pointed out in [30] for the case of Einstein gravity, we find here that also for a general HCG the Kounterterm cancels exactly the divergences in AlAdS spaces in $D = 4$.

4.2.3 $D = 5$ dimensions

In this case, the divergent part of the action will have an additional logarithmic term, which depends on $\text{Tr } g^{(4)}$ and $\mathcal{L}^{(4)}$,

$$I_{\text{diver}} = -C(L) \int d^4x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^4} - \frac{L}{3z_0^2} \mathcal{R}^{(0)} + \mathcal{O}(1) \times \log z_0 \right]. \quad (4.19)$$

As before, this should be regularized by the Kounterterm (3.6) particularized for $D = 2n + 1 = 5$ dimensions. The value of the constant c_4 and the function B_4 are

$$c_4 = \frac{L^2}{8} C(L), \quad B_4 = -\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} K_{i_1}^{j_1} \left(\mathcal{R}_{i_2 i_3}^{j_2 j_3} - K_{i_2}^{j_2} K_{i_3}^{j_3} + \frac{1}{3L^2} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \right). \quad (4.20)$$

Therefore, the total Kounterterm in this case is

$$I_{\text{Kt}} = C(L) \int d^4x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^4} - \frac{L}{3z_0^2} \mathcal{R}^{(0)} + \mathcal{O}(1) \right], \quad (4.21)$$

which as in the case $D = 3$, cancels the divergences (4.19) except for the logarithmic one. As mentioned before, these divergences are universal terms, proportional to the conformal anomaly of the dual field theory, and therefore they were not expected to be cancelled out by this renormalization procedure.

5 Variational principle in HCG up to $D = 5$

We will now show that the same Kounterterms presented in section 3 also render finite the boundary terms that appear when varying the action, and they depend only on the variation of the metric of the conformal boundary $g_{ij}^{(0)}$, thus producing a well posed variational problem. First we will obtain the form of the boundary terms that we need to cancel for general dimensions up to 5, and then particularize the analysis to $D = 3, 4$ and 5 as done in the previous section to treat the divergences of the on-shell action.

5.1 Divergences in the boundary term of the variation of a general HCG

As in the preceding sections, we start by considering a general action that can contain any contraction of the Riemann tensor (4.1). Its variation produces two contributions [31, 32],

$$\delta I = \int_M d^Dx \sqrt{-G} \mathcal{E}_{\mu\nu} \delta G^{\mu\nu} + \epsilon \int_{\partial M} d^{D-1}x \sqrt{-h} n_\mu \delta v^\mu, \quad (5.1)$$

the first of which is proportional to the equations of motion (2.2), and thus vanishes on-shell. The second term is a contraction of the vector normal to the boundary ∂M (normalized such that $n \cdot n = \epsilon = \pm 1$) and the quantity

$$\delta v^\mu = -2P^{\mu\rho\sigma}{}_\nu \delta \Gamma_{\rho\sigma}^\nu - 2\nabla_\nu P^{\mu\rho\sigma\nu} \delta G_{\rho\sigma}. \quad (5.2)$$

We want to evaluate the boundary term in a solution of the equations of motion with AdS asymptotics, so we consider the metric to be given by the usual FG expansion (2.7). The vector normal to the boundary is given by (2.10), and therefore $\epsilon = 1$.

In order to evaluate the boundary term in δI we need expressions for the variation of the Christoffel symbols. In particular, the ones we need can be written in terms of variations of the extrinsic curvature (2.13) as

$$\begin{aligned}\delta\Gamma_{jk}^i &= \frac{1}{2}h^{il}(\nabla_k\delta h_{jl} + \nabla_j\delta h_{kl} - \nabla_l\delta h_{jk}) , \\ \delta\Gamma_{ij}^z &= \frac{z}{L}\delta K_{ij} , \\ \delta\Gamma_{zj}^i &= -\frac{L}{z}\delta K_j^i .\end{aligned}\tag{5.3}$$

Also, we expand the tensor $P_{\rho\sigma}^{\mu\nu}$ asymptotically as

$$P_{\rho\sigma}^{\mu\nu} \equiv \frac{C(L)}{2}(\delta_\rho^\mu\delta_\sigma^\nu - \delta_\sigma^\mu\delta_\rho^\nu) + \delta P_{\rho\sigma}^{\mu\nu} ,\tag{5.4}$$

where $\delta P_{\rho\sigma}^{\mu\nu}$ is not a variation, but a symbolic way of writing all the terms in $P_{\rho\sigma}^{\mu\nu}$ containing powers of z (see below, (5.6)). As mentioned before, when evaluated on-shell the variation of the action contains only the boundary term given by (5.2), which after replacing the expressions above for $\delta\Gamma_{\rho\sigma}^\mu$ and $P_{\rho\sigma}^{\mu\nu}$ becomes

$$\begin{aligned}\delta I = - \int_{\partial M} d^{D-1}x \sqrt{-h} \Bigg[& C(L) \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right) + 2\delta P_{zj}^{zi} \left(2\delta K_i^j + (h^{-1}\delta h)_k^j K_i^k \right) \\ & - 4n_z \nabla_l \delta P_{jk}^{zi} h^{jl} h^{km} \delta h_{im} - 2n^z h^{jk} \nabla_z \delta P_{zk}^{zi} \delta h_{ij} \Bigg] ,\end{aligned}\tag{5.5}$$

where $(h^{-1}\delta h)_j^i \equiv h^{ik}\delta h_{kj}$, and the variation $\delta h_{ij} = \delta g_{ij}/z^2$ can be written as a variation of the metric of the conformal boundary $\delta g_{ij}^{(0)}$ on-shell, due to the relations (2.8) and (2.9). Notice that if we restricted ourselves to Einstein gravity, we would have $C(L) = 1/\kappa$ and $\delta P_{\alpha\beta}^{\mu\nu} = 0$, independently of the background. Therefore, only the first term in this expression would contribute, recovering the results in appendix D of [40] for the boundary term of the variation on-shell.

Until now we have written this expression in terms of variations of the metric and the extrinsic curvature. However, on-shell they are related through (2.9) and (2.13), and therefore the variations of K_{ij} can be written as variations of the metric $g_{ij}^{(0)}$, thus leading to a well-posed Dirichlet problem once we get rid of the divergences. The next step is to expand $\delta P_{\rho\sigma}^{\mu\nu}$ in (5.5) in powers of z . Since we are not interested in more than $D = 5$ bulk dimensions, it is enough to keep only the terms with powers up to z^4 , as the determinant in the integrand behaves to leading order as $\sqrt{-h} \sim z^{-(D-1)}$. Knowing that $g_{ij}^{(1)} = 0$, we can expand $\delta P_{\rho\sigma}^{\mu\nu}$ as

$$\delta P_{\rho\sigma}^{\mu\nu} \equiv z^2 P_{\rho\sigma}^{(2)\mu\nu} + z^3 P_{\rho\sigma}^{(3)\mu\nu} + z^4 P_{\rho\sigma}^{(4)\mu\nu} + \dots .\tag{5.6}$$

Plugging this into (5.5) and evaluating the covariant derivatives explicitly yields

$$\begin{aligned} \delta I = - \int_{\partial M} d^{D-1}x \sqrt{-h} & \left[C(L) \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right) \right. \\ & + \frac{z_0^2}{L} \left(-2(2D-7)P_{zi}^{(2)zk} + 4P_{li}^{(2)lk} \right) (h^{-1}\delta h)_k^i \\ & \left. + \frac{z_0^3}{L} \left(-4(D-4)P_{zi}^{(3)zk} + 4P_{li}^{(3)lk} \right) (h^{-1}\delta h)_k^i + z_0^4 \mathcal{O}(\delta g_{ij}^{(0)}) + \dots \right]. \end{aligned} \quad (5.7)$$

The term of order z_0^3 inside the brackets contains the contractions $P_{zi}^{(3)zk}$ and $P_{li}^{(3)lk}$, which can be seen to vanish⁷ provided $g_{ij}^{(3)} = 0$ following the reasoning of section 2.3. So these divergences do not appear in the general theories that we are interested in.⁸ The contributions at order z_0^2 in (5.7) also vanish, as can be seen by showing that, on-shell,

$$P_{zi}^{(2)zk} = P_{li}^{(2)lk} = 0. \quad (5.8)$$

The arguments leading to this result are explained in appendix C. Finally, the term proportional to z_0^4 in (5.7) vanishes as $z_0 \rightarrow 0$ in $D = 3$ and $D = 4$, even when multiplied by $\sqrt{-h}$. In $D = 5$, however, it produces a constant term which corresponds to a well posed variational problem with Dirichlet boundary conditions, as all the variations that it contains can be written in terms of $\delta g_{ij}^{(0)}$.

Gathering everything up, we see that the boundary term of the variation reads

$$\delta I = - \int_{\partial M} d^{D-1}x \sqrt{-h} \left[C(L) \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right) + z_0^4 \mathcal{O}(\delta g_{ij}^{(0)}) + \dots \right]. \quad (5.9)$$

By simple inspection, we see that the only divergent part up to $D = 5$ is the same as that of Einstein gravity, presented for example in [40], multiplied by the constant $C(L)$ which depends on the couplings of the theory at hand. Therefore, we will assume that the divergences in this object can be regularized by adding to the original Lagrangian the usual boundary Kounterterms that are known to work for Einstein gravity, multiplied by the constant $C(L)$ as given in section 3, and which we already know that are enough to cancel the divergences of the on-shell action in these dimensions.

5.2 Explicit analysis in different dimensions

We will now show how the Kounterterms presented in section 3 can be used to cancel the divergences in the boundary term that appears when varying the action on-shell (5.9). Although the following computations are carried out more generally in appendix D of [40], here we will show the results more explicitly in up to 5 bulk dimensions.

⁷Notice that in $D = 4$ only $\text{Tr } g^{(3)} = 0$, the other components (the off-trace part) being free, corresponding to the holographic stress-tensor in the dual CFT. In this case, $P_{li}^{(3)lk} \neq 0$, thus inducing a constant term at the boundary which is standard.

⁸The expansion of $P_{\alpha\beta}^{(3)\mu\nu}$ also contains terms of the form $\nabla g^{(2)}$. However, notice that these are only present in the components $P_{jk}^{(3)zi}$ or $P_{zk}^{(3)ij}$, which are absent at this order —albeit they might appear at higher orders in the expansion of the boundary term of the variation (5.7).

5.2.1 $D = 3$ dimensions

Since $\sqrt{-h} \sim z^{-2}$ in this case, the non-vanishing terms of the variation (5.9) are simply

$$\delta I = -C(L) \int_{\partial M} d^2x \sqrt{-h} \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right). \quad (5.10)$$

In order to regularize this, we add the Kounterterm (3.6) particularized to $D = 3$, which reads

$$I_{\text{Kt}} = C(L) \int_{\partial M} d^2x \sqrt{-h} K. \quad (5.11)$$

It is straightforward to compute its variation, finding

$$\delta I_{\text{Kt}} = C(L) \int_{\partial M} d^2x \sqrt{-h} \left(\delta K_i^i + \frac{1}{2} (h^{-1}\delta h)_i^i K \right). \quad (5.12)$$

Then, adding this to δI above we get the variation of the renormalized action in $D = 3$,

$$\delta I_{\text{reg}} = \delta I + \delta I_{\text{Kt}} = C(L) \int_{\partial M} d^2x \sqrt{-h} \left(\frac{1}{2} (h^{-1}\delta h)_i^i K - (h^{-1}\delta h)_j^i K_i^j - \delta K_i^i \right). \quad (5.13)$$

To the lowest orders in z , the extrinsic curvature behaves as

$$K_j^i = \frac{1}{L} \delta_j^i + z^2 L S_j^i(g^{(0)}) + \dots, \quad (5.14)$$

where $S_j^i(g^{(0)})$ is the Schouten tensor of the metric $g_{ij}^{(0)}$. So we see that the terms in the parenthesis start at order z^2 , which is finite when multiplied by the determinant factor. Therefore, the Kounterterm cancels the divergences in the variation for this dimension of the spacetime, and also it allows the variation to be written only in terms of variations with respect to $g_{ij}^{(0)}$, thus leading to a well-posed variational problem.

5.2.2 $D = 4$ dimensions

The non-vanishing terms of the boundary variation of the action in 4 dimensions are

$$\delta I = -C(L) \int_{\partial M} d^3x \sqrt{-h} \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right). \quad (5.15)$$

These should be cancelled by the Kounterterm (3.1) with $n = 2$, which is

$$I_{\text{Kt}} = -L^2 C(L) \int_{\partial M} d^3x \sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} K_{i_1}^{j_1} \left(\frac{1}{2} \mathcal{R}_{i_2 i_3}^{j_2 j_3} - \frac{1}{3} K_{i_2}^{j_2} K_{i_3}^{j_3} \right). \quad (5.16)$$

Its variation δI_{Kt} can be evaluated explicitly term by term, and the final result reads [40]

$$\begin{aligned} \delta I_{\text{Kt}} = & C(L) \int_{\partial M} d^3x \sqrt{-h} \left(2\delta K_i^i + (h^{-1}\delta h)_j^i K_i^j \right) \\ & + L^2 C(L) \int_{\partial M} d^3x \sqrt{-h} \left[W_{jl}^{il} \left((h^{-1}\delta h)_k^j K_i^k + 2\delta K_i^j \right) \right. \\ & \quad \left. - \delta_{lmn}^{ijk} K_i^l \tilde{\nabla}^n \tilde{\nabla}_j (h^{-1}\delta h)_k^m \right]. \end{aligned} \quad (5.17)$$

The first term in this expression cancels exactly the divergent terms in the variation (5.15). The second integral in δI_{Kt} is finite if we assume asymptotic conformal flatness (which is

natural, since any metric in three dimensions is Weyl-flat). Indeed, in this case the relevant components of the bulk Weyl tensor behave as $W_{jl}^{il} \sim z^{D-1} = z^3$, and since $\delta K_j^i \sim z^2$ due to (5.14), only the first term in that parenthesis contributes. The last term of (5.17) can also be shown to vanish for this number of dimensions. Assuming that the boundary submanifold is infinite, we can integrate by parts without adding a boundary term,

$$\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} K_{i_1}^{j_1} \tilde{\nabla}^{j_3} \tilde{\nabla}_{i_2} (h^{-1} \delta h)_{i_3}^{j_2} \longrightarrow \sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (h^{-1} \delta h)_{i_3}^{j_2} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_2}^{j_1}. \quad (5.18)$$

But since $\tilde{\nabla}_l K_{i_2}^{j_1} \sim z^2$ at least (the zeroth order of K_j^i is proportional to δ_j^i) and the indices of the covariant derivative are raised with the metric $h^{ij} = z^2 g^{ij}$, we have

$$\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (h^{-1} \delta h)_{i_3}^{j_2} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_2}^{j_1} \sim z, \quad (5.19)$$

so this term vanishes as $z \rightarrow 0$. Therefore, the boundary term of the variation of the regularized action in $D = 4$ reads

$$\delta I_{\text{reg}} = \delta I + \delta I_{\text{Kt}} = L^2 C(L) \int_{\partial M} d^3 x \sqrt{-h} W_{jl}^{il} (h^{-1} \delta h)_k^j K_i^k, \quad (5.20)$$

which is finite and can be written as depending only on variations of $g_{ij}^{(0)}$, thus leading to a well-posed variational problem with Dirichlet boundary conditions.

5.2.3 $D = 5$ dimensions

In 5 dimensions the form of the divergent contribution is the same as before, with the difference that now the terms of order z_0^4 also contribute,

$$\delta I = - \int_{\partial M} d^4 x \sqrt{-h} \left[C(L) \left(2\delta K_i^i + (h^{-1} \delta h)_j^i K_i^j \right) + z_0^4 \mathcal{O}(\delta g_{ij}^{(0)}) \right]. \quad (5.21)$$

Notice that in this case only the first term inside the brackets produces divergences, and the second one results in a constant in the integrand, so it does not need to be cancelled. The boundary Kounterterm that should cancel these divergences is (3.6) particularized to $n = 2$,

$$I_{\text{Kt}} = -\frac{L^2}{8} C(L) \int_{\partial M} d^4 x \sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} K_{i_1}^{j_1} \left(\mathcal{R}_{i_2 i_3}^{j_2 j_3} - K_{i_2}^{j_2} K_{i_3}^{j_3} + \frac{1}{3L^2} \delta_{i_2}^{j_2} \delta_{i_3}^{j_3} \right). \quad (5.22)$$

Obtaining the variation of this Kounterterm requires a rather involved computation, which again can be carried out following appendix D of [40]. The final result reads

$$\delta I_{\text{Kt}} = C(L) \int_{\partial M} d^4 x \sqrt{-h} \left(2\delta K_i^i + (h^{-1} \delta h)_j^i K_i^j \right) + \delta I^{(W)} + \delta I^{(0)} + \delta I^{(\tilde{\nabla})}, \quad (5.23)$$

where the first term cancels exactly the divergent part of (5.21), and we have defined

$$\delta I^{(W)} = -\frac{L^2}{8} C(L) \int_{\partial M} d^4 x \sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} W_{i_2 i_3}^{j_2 j_3} \left(2\delta K_{i_1}^{j_1} + (h^{-1} \delta h)_k^{j_1} K_{i_1}^k \right), \quad (5.24)$$

$$\begin{aligned} \delta I^{(0)} = \frac{L^2}{16} C(L) \int_{\partial M} d^4 x \sqrt{-h} \delta_{j_1 j_2 j_3 j_4}^{i_1 i_2 i_3 i_4} & \left(\mathcal{R}_{i_3 i_4}^{j_3 j_4} - K_{i_3}^{j_3} K_{i_4}^{j_4} + \frac{1}{L^2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4} \right) \\ & \times \left((h^{-1} \delta h)_k^{j_1} (K_{i_1}^k \delta_{i_2}^{j_2} - \delta_{i_1}^k K_{i_2}^{j_2}) + 2\delta_{i_1}^{j_1} \delta K_{i_2}^{j_2} \right), \end{aligned} \quad (5.25)$$

$$\delta I^{(\tilde{\nabla})} = \frac{L^2}{4} C(L) \int_{\partial M} d^4 x \sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (h^{-1} \delta h)_{i_3}^{j_2} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1}. \quad (5.26)$$

Then, the variation of the total regularized action is

$$\delta I_{\text{reg}} = \delta I^{(W)} + \delta I^{(0)} + \delta I^{(\tilde{V})} + \delta I^{(z_0^4)}, \quad (5.27)$$

where $\delta I^{(z_0^4)}$ corresponds to the terms of order z_0^4 in δI that produce a constant in the integrand, and whose particular form depends on the theory. In order to show that the variation (5.27) of the regularized action is finite, we should count the powers of z appearing in each of the terms.

- $\delta I^{(W)}$ can be rewritten by expanding the sum in the indices of the antisymmetric δ , and using $W_{ij}^{ij} = 0$, which follows from $W_{\mu\nu}^{\mu\nu} = W_{\mu i}^{\mu i} = 0$. We find:

$$\delta I^{(W)} = \frac{L^2}{2} C(L) \int_{\partial M} d^4x \sqrt{-h} W_{jl}^{il} \left(2\delta K_i^j + (h^{-1}\delta h)_k^j K_i^k \right). \quad (5.28)$$

But now recall that $\delta K_j^i \sim z^2$, and under the assumption of asymptotic conformal flatness, $W_{jl}^{il} \sim z^{D-1} = z^4$. Then, since $\sqrt{-h} \sim z^{-4}$, the term with δK_i^j in the parenthesis does not contribute, and we can write simply

$$\delta I^{(W)} = \frac{L^2}{2} C(L) \int_{\partial M} d^4x \sqrt{-h} W_{jl}^{il} (h^{-1}\delta h)_k^j K_i^k. \quad (5.29)$$

- The first parenthesis in $\delta I^{(0)}$ can be rewritten in terms of the Weyl tensor of the bulk metric, using the Gauss-Codazzi equation

$$R_{kl}^{ij} = \mathcal{R}_{kl}^{ij} - 2K_{[k}^i K_{l]}^j, \quad (5.30)$$

and the definition of the Weyl tensor, which to the lowest order in z yields:

$$W_{kl}^{ij} = R_{kl}^{ij} + \frac{2}{L^2} \delta_{[k}^i \delta_{l]}^j. \quad (5.31)$$

These two expressions can be combined to find

$$\mathcal{R}_{kl}^{ij} = W_{kl}^{ij} + 2K_{[k}^i K_{l]}^j - \frac{2}{L^2} \delta_{[k}^i \delta_{l]}^j. \quad (5.32)$$

The first parenthesis in $\delta I^{(0)}$ (taking into account the prefactor $\delta_{j_1 j_2 j_3 j_4}^{i_1 i_2 i_3 i_4}$) can now be rewritten as

$$\mathcal{R}_{i_3 i_4}^{j_3 j_4} - K_{i_3}^{j_3} K_{i_4}^{j_4} + \frac{1}{L^2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4} = W_{i_3 i_4}^{j_3 j_4} + K_{i_3}^{j_3} K_{i_4}^{j_4} - \frac{1}{L^2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4}, \quad (5.33)$$

and since $W_{kl}^{ij} \sim z^4$ and $K_j^i = \delta_j^i/L + \mathcal{O}(z^2)$, we see that, to the lowest order,

$$\mathcal{R}_{i_3 i_4}^{j_3 j_4} - K_{i_3}^{j_3} K_{i_4}^{j_4} + \frac{1}{L^2} \delta_{i_3}^{j_3} \delta_{i_4}^{j_4} \sim z^2. \quad (5.34)$$

In the second parenthesis in $\delta I^{(0)}$, we have

$$K_{i_1}^k \delta_{i_2}^{j_2} - \delta_{i_1}^k K_{i_2}^{j_2} \sim z^2, \quad \delta K_{i_2}^{j_2} \sim z^2. \quad (5.35)$$

Therefore, the whole integrand starts at order z^4 , and when integrated with $d^4x \sqrt{-h}$ it produces a term that is constant and thus non-divergent when $z \rightarrow 0$.

- If we do a naive power counting in the term $\delta I^{(\tilde{\nabla})}$, we could find that it produces a constant at the boundary, $z \rightarrow 0$. Indeed, as we have seen in $D = 4$, $\tilde{\nabla}_l K_{i_2}^{j_1} \sim z^2$. Therefore, we might be tempted to think that

$$\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (h^{-1} \delta h)_{i_3}^{j_2} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1} \sim 1. \quad (5.36)$$

However, we will now show that this expression vanishes if we impose the boundary to be conformally flat. Indeed, since we can expand K_j^i in terms of the Schouten tensor as in (5.14), to leading order in z we can write

$$\begin{aligned} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1} &= z^2 L \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} S_{i_1}^{(0)j_1} + \dots = z^4 L \tilde{\nabla}_{i_2}^{(0)} \tilde{\nabla}^{(0)j_3} S_{i_1}^{(0)j_1} + \dots \\ &= z^4 \frac{L}{2} \tilde{\nabla}_{i_2}^{(0)} \left(\tilde{\nabla}^{(0)j_3} S_{i_1}^{(0)j_1} - \tilde{\nabla}^{(0)j_1} S_{i_1}^{(0)j_3} \right) + \dots, \end{aligned} \quad (5.37)$$

where $\tilde{\nabla}_i^{(0)}$ is the covariant derivative compatible with the boundary metric $g_{ij}^{(0)}$, and the indices of the objects with superscript (0) are raised using the inverse metric $g_{ij}^{(0)}$ (thus the extra explicit z^2 factor in the second step). In the last step, we used the fact that this object is contracted with a generalized Kronecker δ , so we antisymmetrized it explicitly in the indices j_1 and j_3 . This allows us to use the definition of the Cotton tensor,

$$C_{ijk}^{(0)} = \tilde{\nabla}_k^{(0)} S_{ij}^{(0)} - \tilde{\nabla}_j^{(0)} S_{ik}^{(0)}, \quad (5.38)$$

in order to rewrite the above expression as

$$\tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1} = z^4 \frac{L}{2} \tilde{\nabla}_{i_2}^{(0)} C_{i_1}^{(0)j_3 j_1} + \dots. \quad (5.39)$$

But the Cotton tensor of $g_{ij}^{(0)}$ is related to its Weyl tensor [30]

$$C_{j}^{(0)kl} = \frac{1}{D-4} \tilde{\nabla}^{(0)i} W_{ij}^{(0)kl}, \quad (5.40)$$

which is zero if we impose asymptotic conformal flatness, $W_{kl}^{(0)ij} = 0$. Therefore, the term of order z^4 in $\tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1}$ vanishes, and $\tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1} \sim z^6$, which means that the total integrand in $\delta I^{(\tilde{\nabla})}$ is zero in $D = 5$ dimensions, since

$$\sqrt{-h} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (h^{-1} \delta h)_{i_3}^{j_2} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K_{i_1}^{j_1} \sim z^2 \quad (5.41)$$

vanishes at the boundary.

Gathering everything up, we find that the boundary term of the variation of the regularized action in $D = 5$ is

$$\delta I_{\text{reg}} = \int_{\partial M} d^4 x \sqrt{-h} \left[\frac{L^2}{2} C(L) W_{jl}^{il} (h^{-1} \delta h)_k^j K_i^k + z_0^4 \mathcal{O}(\delta g_{ij}^{(0)}) \right], \quad (5.42)$$

which yields a well-posed Dirichlet variational problem. The last term between brackets contains the terms of order z_0^4 appearing in the original variation (5.21), as well as those coming from $\delta I^{(0)}$.

6 Discussion

In this work, we have implemented a universal renormalization procedure applicable to arbitrary higher curvature gravity theories evaluated on AlAdS manifolds of $D \leq 5$ dimensions.⁹ This method uses the extrinsic boundary counterterms of [28, 29], but with a theory-dependent coupling constant, as given in equations (3.3) and (3.8). In order to show the universality of the method, we decompose the equations of motion of an arbitrary HCG into their radial and tangential components (with respect to the holographic Poincare coordinate), and we are able to argue, on general grounds —by symmetry arguments— that the odd coefficients of the FG expansion of the bulk metric, $g_{ij}^{(1)}$ and $g_{ij}^{(3)}$, are zero. Furthermore, by virtue of the PBH transformation relations [33], it can be argued that $g_{ij}^{(2)}$ is constrained to have the universal form (2.30). Then, considering these general features of the FG expansion, we verify the cancellation of divergences (section 4) and the well-posedness of the variational principle (section 5), keeping the expansion terms up to the normalizable order.

The argument fails for particular theories (discussed in section 2.1 and appendix A), for which the equations of motion do not constrain the form of the $g_{ij}^{(1)}$ and/or the $g_{ij}^{(3)}$ coefficients. Albeit these theories correspond to zero measure submanifolds in theory space (parameterized by the couplings of the higher curvature terms), they are interesting on their own, as they include theories displaying degenerate AdS vacua —those that leave $g^{(1)}$ undetermined— and modified AdS asymptotics (as discussed in section 2.1).

It is natural to think about applying this procedure to obtain the finite asymptotic charges for black hole solutions in generic HCGs. Also, in the context of the AdS/CFT correspondence, one could use the method for renormalizing holographic entanglement entropy, as in [40, 43, 48, 49, 51]. Finally, one could use the conditions discussed after (2.25), which are required in order for $g_{ij}^{(1)}$ not to be fixed by the equations of motion, to seek for new theories with degenerate AdS vacua and/or modified asymptotic behaviour. Certainly, many interesting avenues of exploration are possible using this universal renormalization approach.

Acknowledgments

IJA thanks Giorgos Anastasiou and Rodrigo Olea for interesting discussions. The work of IJA is funded by Agencia Nacional de Investigación y Desarrollo (ANID), REC Convocatoria Nacional Subvención a Instalación en la Academia Convocatoria Año 2020, Folio PAI77200097. The work of JDE, AR, DVR and AVL is supported by MINECO FPA2017-84436-P, Xunta de Galicia ED431C 2017/07, Xunta de Galicia (Centro singular de investigación de Galicia accreditation 2019-2022), the European Union (European Regional Development Fund — ERDF), the “María de Maeztu” Units of Excellence MDM-2016-0692, and

⁹In the case of $D = 5$, the extra condition of Asymptotically Conformal Flatness [40] is required, which assumes the manifold to have a conformally flat boundary. This condition is needed to guarantee that $g_{(2)}$ has the universal form of (2.30), and for the variational principle to be well-posed, as discussed in section 5.2.3.

the Spanish Research State Agency. ARS is supported by the Spanish MECD fellowship FPU18/03719. DVR is supported by Xunta de Galicia under the grant ED481A-2019/115. AVL is supported by the Spanish MECD fellowship FPU16/06675. AVL is pleased to thank the Instituto de Física Teórica (IFT, Madrid) where part of this work was done, for their warm hospitality.

A Conditions to leave $g_{ij}^{(1)}$ undetermined in cubic theories

In the end of section 2.1 we mentioned some quadratic theories of gravity that do not impose $g_{ij}^{(1)} = 0$, but leave it free to be determined as a boundary condition. Here, we will do the same for theories constructed with cubic contractions of the curvature tensors. Let us consider the most general theory constructed from the Einstein-Hilbert action supplemented by all possible terms that are cubic in the curvature,

$$\begin{aligned}
 I = \int_M d^D x \sqrt{-g} \Bigg(& \frac{R - 2\Lambda_0}{\kappa} + \lambda_1 R^{\mu\nu\rho\sigma} R_{\mu}{}^{\gamma}{}_{\rho}{}^{\delta} R_{\nu\gamma\sigma\delta} + \lambda_2 R_{\mu\nu}{}^{\gamma\delta} R^{\mu\nu\rho\sigma} R_{\rho\sigma\gamma\delta} \\
 & + \lambda_3 R^{\mu\nu} R_{\mu}{}^{\rho\sigma\gamma} R_{\nu\rho\sigma\gamma} + \lambda_4 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \lambda_5 R^{\mu\nu} R^{\rho\sigma} R_{\mu\nu\rho\sigma} \\
 & + \lambda_6 R^{\mu\nu} R_{\nu\rho} R_{\mu}{}^{\rho} + \lambda_7 R_{\mu\nu} R^{\mu\nu} R + \lambda_8 R^3 \Bigg), \tag{A.1}
 \end{aligned}$$

where the coupling constants λ_i are arbitrary for the moment. The value of the constant $C(L)$ that multiplies the Kounterterms can be computed using equation (2.6),

$$\begin{aligned}
 C(L) = \frac{1}{\kappa} + \frac{3(D-1)}{L^4} \Bigg(& \frac{D-2}{D-1} \lambda_1 + \frac{4}{D-1} \lambda_2 + 2\lambda_3 + 2D\lambda_4 + (D-1)\lambda_5 \\
 & + (D-1)\lambda_6 + D(D-1)\lambda_7 + D^2(D-1)\lambda_8 \Bigg). \tag{A.2}
 \end{aligned}$$

The equations of motion at order z (2.20), (2.21) and (2.22) are determined by the constants $a^{(1)}(L)$ and $b^{(1)}(L)$, which in this case are given by

$$\begin{aligned}
 a^{(1)}(L) = \frac{1}{\kappa} + \frac{1}{L^4} \Bigg[& 6(D-3)\lambda_1 + 36\lambda_2 + 2(7D-9)\lambda_3 + 10D(D-1)\lambda_4 + (5D^2 - 13D + 9)\lambda_5 \\
 & + (6D^2 - 15D + 9)\lambda_6 + D(4D^2 - 9D + 5)\lambda_7 + 3D^2(D-1)^2\lambda_8 \Bigg], \\
 b^{(1)}(L) = -\frac{1}{\kappa} + \frac{1}{L^4} \Bigg[& 6\lambda_1 + 12\lambda_2 + 2(D+5)\lambda_3 - 2(D^2 - 17D + 16)\lambda_4 - (D^2 - 15D + 17)\lambda_5 \\
 & + 9(D-1)\lambda_6 - (2D^3 - 23D^2 + 37D - 16)\lambda_7 - 3D(D-1)^2(D-8)\lambda_8 \Bigg].
 \end{aligned}$$

As explained in section 2.1, the equations of motion imply $g_{ij}^{(1)} = 0$, unless the conditions discussed after (2.25) are met. Thus, only for both $a^{(1)}(L) = b^{(1)}(L) = 0$, $g_{ij}^{(1)}$ is fully unconstrained by the equations of motion, which happens only in a zero measure region of the space of parameters λ_i . Besides, as what was found for quadratic theories of gravity,

the conditions $a^{(1)}(L) = b^{(1)}(L) = 0$ end up implying that the corresponding cubic theory has degenerate AdS vacua.

In the case of cubic curvature gravity, let us quote two examples of usually well-behaved theories which have the above properties:

- Cubic Lovelock theory in general dimensions [4], $D \geq 7$. This corresponds to setting in (A.1) the values of the couplings to be

$$\begin{aligned} \lambda_1 = -8\mu, \quad \lambda_2 = 4\mu, \quad \lambda_3 = -24\mu, \quad \lambda_4 = 3\mu, \quad \lambda_5 = 24\mu, \\ \lambda_6 = 16\mu, \quad \lambda_7 = -12\mu, \quad \lambda_8 = \mu. \end{aligned} \quad (\text{A.3})$$

The particular value of the coupling μ at which $a^{(1)}(L) = b^{(1)}(L) = 0$ corresponds to

$$\mu = -\frac{L^4}{3(D-3)(D-4)(D-5)(D-6)\kappa}, \quad (\text{A.4})$$

which corresponds to the critical value [15].

- Einsteinian Cubic Gravity [52]. We could consider the Lagrangian $(R-2\Lambda_0)/\kappa + \mu_{\mathcal{P}}\mathcal{P}$, which amounts to

$$\lambda_1 = 12\mu_{\mathcal{P}}, \quad \lambda_2 = \mu_{\mathcal{P}}, \quad \lambda_5 = -12\mu_{\mathcal{P}}, \quad \lambda_6 = 8\mu_{\mathcal{P}}. \quad (\text{A.5})$$

while the remaining couplings vanish. The coefficient $g_{ij}^{(1)}$ becomes undetermined at the critical value of the coupling

$$\mu_{\mathcal{P}} = \frac{L^4}{12(D-3)(D-6)\kappa}, \quad (\text{A.6})$$

which corresponds to the critical value found in [26] when studying the AdS vacua of the theory in 4 dimensions.

One could also consider the Lagrangian density $\mathcal{C}$, defined in [53]. In particular, the combination $\mathcal{P} - 8\mathcal{C}$ in $D = 4$, introduced in [8, 54] due to its interesting cosmological properties, also leaves the coefficient $g_{ij}^{(1)}$ undetermined for the value of the coupling (A.6). This makes sense, as it is known that $\mathcal{C}$ does not modify the AdS vacuum in four dimensions.

It would be interesting to explore whether the $a^{(1)}(L) = b^{(1)}(L) = 0$ condition can be used as a tool to look for new theories in higher dimensions and of higher order in the curvature, with analogous behaviour to the examples discussed here.

B Conditions to leave $g_{ij}^{(3)}$ undetermined in quadratic and cubic theories

In this appendix we give the values of the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ introduced in section 2.3 for different families of theories. These determine the projected equations of motion at third order in z , (2.36) and (2.37). First let us consider the general quadratic

gravity action (2.26). These constants are expressions of the AdS radius L and the coupling constants in the Lagrangian, and read:

$$a^{(3)}(L) = \frac{3}{\kappa} + \frac{1}{L^2} \left[-3(5D-14)\alpha_1 - 6D(D-1)\alpha_2 - 6(D-3)(D-4)\alpha_3 \right],$$

$$b^{(3)}(L) = -\frac{3}{\kappa} + \frac{1}{L^2} \left[-3(D-6)\alpha_1 + 6(D^2-9D+24)\alpha_2 + 6(D-3)(D-4)\alpha_3 \right].$$

For the general theory containing cubic contractions of the curvature tensors (A.1), they take the values:

$$a^{(3)}(L) = \frac{3}{\kappa} + \frac{1}{L^4} \left[36\lambda_1 + 36(4D-17)\lambda_2 + 6(4D^2-13D-9)\lambda_3 + 6D(D-1)(4D-15)\lambda_4 \right. \\ \left. + 9(3D^2-13D+13)\lambda_5 + 9(D-1)(4D-13)\lambda_6 \right. \\ \left. + 9D(D-1)(2D-5)\lambda_7 + 9D^2(D-1)^2\lambda_8 \right], \quad (\text{B.1})$$

$$b^{(3)}(L) = -\frac{3}{\kappa} + \frac{1}{L^4} \left[18(D-4)\lambda_1 + 36\lambda_2 + 30(D-3)\lambda_3 - 6(D^2-33D+96)\lambda_4 \right. \\ \left. + 9(D^2-D-9)\lambda_5 + 9(D-1)(2D-7)\lambda_6 \right. \\ \left. + 9(D-1)(9D-32)\lambda_7 - 9D(D-1)(D^2-17D+48)\lambda_8 \right]. \quad (\text{B.2})$$

Studying the particular points where they vanish might lead us to theories of gravity whose dynamics differs from that of Einstein gravity.

C Proof that the boundary term of δI vanishes on-shell at $\mathcal{O}(z^2)$

In this appendix we will show that the terms of order z_0^2 in (5.7) vanish for a general HCG. This contribution contains the quantity

$$-2(2D-7)P^{(2)zk}_{zi} + 4P^{(2)lk}_{li}, \quad (\text{C.1})$$

and thus, in order to prove that it vanishes it is enough to show that the two contractions of $P^{(2)\mu\nu}_{\rho\sigma}$ appearing here vanish when evaluated on-shell. The tensor $P^{(2)\mu\nu}_{\rho\sigma}$ is defined as the derivative of the Lagrangian $\mathcal{L}(R^{\alpha\beta}_{\gamma\delta})$ with respect to the Riemann tensor. Then, its components will be given by contractions of the Riemann with four free indices that fulfill the symmetries of the Riemann tensor itself. Since we are interested in the form of the terms at order z^2 in this tensor, $P^{(2)\mu\nu}_{\rho\sigma}$, we need to study the components of the Riemann up to this order, which are given once the equations of motion are fulfilled by

$$R^{ij}_{kl} = -\frac{2}{L^2}\delta^{[i}_k\delta^{j]}_l + z^2 \left(\frac{4}{L^2}\delta^{[i}_k g^{(2)j]}_l + \mathcal{R}^{ij}_{kl} \right) + \mathcal{O}(z^4),$$

$$R^{zi}_{zj} = -\frac{1}{L^2}\delta^i_j + \mathcal{O}(z^4), \quad (\text{C.2})$$

$$R^{zi}_{jk} = \mathcal{O}(z^3).$$

As said, also contractions of the curvature can contribute to $P_{\rho\sigma}^{\mu\nu}$ in a general theory. In particular, it is enough to consider R_{jk}^{ik} and R_{zi}^{zi} . If we impose that the equations of motion are fulfilled, $g_{ij}^{(2)}$ is given by (2.9) and thus the form of these contractions is found to be

$$\begin{aligned} R_{jk}^{ik} &= -\frac{D-2}{L^2}\delta_j^i + \mathcal{O}(z^4), \\ R_{zi}^{zi} &= -\frac{D-1}{L^2} + \mathcal{O}(z^4). \end{aligned} \tag{C.3}$$

Thereby we see that the uncontracted components R_{kl}^{ij} are the only ones that can contribute to $P_{\rho\sigma}^{\mu\nu}$ at order z^2 on-shell. Therefore, we can write

$$\begin{aligned} P_{kl}^{(2)ij} &= C^{(2)} \left(\frac{4}{L^2} \delta_{[k}^{[i} g_{l]}^{(2)j]} + \mathcal{R}_{kl}^{ij} \right), \\ P_{zj}^{(2)zi} &= P_{jk}^{(2)zi} = 0, \end{aligned} \tag{C.4}$$

where $C^{(2)}$ is a constant depending upon the parameters of the particular theory that we consider. However, these expressions are enough to see that

$$P_{zi}^{(2)zk} = P_{li}^{(2)lk} = 0, \tag{C.5}$$

and thus the terms at order z^2 in the boundary term of the variation of the action (5.7) are zero on-shell for any theory of gravity.

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