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$T\bar{T}$ -flow effects on torus partition functions

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ABSTRACT: In this paper, we investigate the partition functions of conformal field theories (CFTs) with the $T\bar{T}$ deformation on a torus in terms of the perturbative QFT approach. In Lagrangian path integral formalism, the first- and second-order deformations to the partition functions of 2D free bosons, free Dirac fermions, and free Majorana fermions on a torus are obtained. The corresponding Lagrangian counterterms in these theories are also discussed. The first two orders of the deformed partition functions and the first-order vacuum expectation value (VEV) of the first quantum KdV charge obtained by the perturbative QFT approach are consistent with results obtained by the Hamiltonian formalism in literature.

KEYWORDS: Conformal Field Theory, Field Theories in Lower Dimensions, Integrable Field Theories, Renormalization Regularization and Renormalons

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1 Introduction

The $T\bar{T}$ deformation of field theory has attracted much research interest in recent years both from viewpoint of field theory and in the context of holographic duality. The $T\bar{T}$ deformation of 2D field theory is typically defined on the plane or cylinder by [1, 2]

$$\frac{d\mathcal{L}^\lambda}{d\lambda} = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}T_{\mu\rho}^\lambda T_{\nu\sigma}^\lambda, \quad (1.1)$$

where T^λ depending on λ is stress tensor of the theory $\mathcal{L}^\lambda$. Though the r.h.s. is a composite operator, it is well-defined quantum mechanically [3]. Remarkably, the $T\bar{T}$ deformation keeps the integrability of the un-deformed theory and the deformed theory is solvable in some sense [1, 2, 4–8]. Since the deformation is irrelevant, the density of states of the deformed theory in the UV exhibits Hagedorn growth behavior, which implies the $T\bar{T}$ deformation is non-local in the UV [2, 9, 10]. With many intriguing properties discovered, the $T\bar{T}$ deformation has subsequently been generalized to many directions, for instance, to other integrable deformations such as the $J\bar{T}$ deformation [11–13], to supersymmetric cases [14–17], to various dimensions [18–21] and spin chain models [22–26]. For some other developments of the $T\bar{T}$ deformation, please refer to [27–35].

Among these progresses, the partition functions as well as correlation functions in deformed CFTs are of particular interest in our present study. The partition functions of the $T\bar{T}$ deformed CFTs have been computed in [6] by using the known deformed spectrum. Since the results in [6] are nonperturbative, the modular properties can be discussed, and it was shown that the partition functions are modular covariant. From other perspective, the deformed partitions were discussed from random metric point of view [29], and also in the context of holographic duality [36]. As for correlation functions, the deformed one-point functions of KdV charges operators were considered non-perturbatively based on the deformed spectrum [37]. Also the general deformed correlation functions in the UV were considered by J. Cardy in [38].

On the other hand, one can study the $T\bar{T}$ deformation in a perturbative way. More concretely, suppose that one can expand $\mathcal{L}^\lambda$ around $\lambda = 0$,

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda \mathcal{L}^{(1)} + \frac{\lambda^2}{2!} \mathcal{L}^{(2)} + \dots, \quad (1.2)$$

where the first term $\mathcal{L}^{(0)}$ corresponds to the un-deformed theory, the second term is the $T\bar{T}$ operator of un-deformed theory as appeared in the r.h.s. of (2.2) with $\lambda = 0$, the third term and the terms omitted are presented since the stress tensor T^λ is not fixed but also flow under the deformation. In other words, the stress tensor depends on λ .

A number of works were done in the framework of perturbation method, for example, in [1] the renormalization of free theory under the $T\bar{T}$ deformation is investigated by matching the S-matrix. Meanwhile, other physical quantities were also computed perturbatively, such as entanglement entropies, wilson loop and correlation functions [39–41]. In this work, we will continue to study the partition functions (which can be treated as zero-point functions) of deformed CFTs in a perturbative manner. The correlation functions of deformed theories were considered earlier in [42–44], where two-point functions and three-point functions were calculated, as well as the correlation functions of stress tensors. Later, these results were generalized to higher-point function cases [45, 46], as well as including supersymmetry [47], torus CFTs [48], and especially the holographic dual of stress tensor correlation function in large c limit was considered in [49].

In these studies of correlation functions, it is worthwhile to note that computation is mainly performed in the first-order perturbation of CFT or in the case where the CFT is defined on the plane. Naturally, to make progress, the next step is that can we go

beyond the first-order perturbation. However, this is a nontrivial question as can be seen as follow. As discussed above, in the first-order perturbation, the $T\bar{T}$ operator is known which is just constructed from the stress tensor of the un-deformed CFT, while in higher-order perturbations, one must take the corrections of $T\bar{T}$ operator into consideration, namely, $T\bar{T}$ -flow effects. Unfortunately, in a general CFT, we do not have such an explicit expression on such kinds of corrections. Nevertheless, as the first step towards higher-order perturbations, we can start with free theory, where the corrections of stress tensor and Lagrangian under the $T\bar{T}$ deformation can be constructed explicitly order by order. Based on this setup, we will study the corrections of deformed partition functions up to second-order by employing perturbation method. This also generalize our previous work [48], where the first-order partition functions of deformed CFTs on torus were computed. Moreover, since we work in free theories, we will use Wick contraction rather than the Ward identity obtained in [48] to figure out the deformed correlation functions. Finally, the two methods will lead to the same results.

The organization of this paper is as follows. In section 2, we review the general method to obtain the deformed Lagrangian and stress tensor order by order, which can be used to expand the partition function up to the second-order that we are interested in. In section 3, section 4 and section 5, we computed the first- and second-order corrections to the partition functions of free bosons, Dirac fermions and Majorana fermions respectively. We use Wick contraction to computed the deformed partition functions, also some proper regularization methods are chosen. In section 6 we continue to calculate the VEV of the first KdV charge in the deformed free theories up to the first-order, by using the perturbative QFT approach. We end in section 7 with a conclusion and discussion. Our conventions, useful formulae, and some calculation details are presented in the appendices.

2 $T\bar{T}$ deformed partition function for generic 2d theory

In this section, we would like to compute the perturbation expansion of $T\bar{T}$ deformed partition function beyond the first-order. The procedure is based on the method first introduced in [2] (also see [50]), where deformed Lagrangian is obtained order by order. Let us first review this method below.

Consider a $T\bar{T}$ deformed QFT living in a two-dimensional Euclidean spacetime $(\mathcal{M}, g_{ab})$ whose dynamics is governed by the local action

$$S^\lambda = \int_{\mathcal{M}} \sqrt{g} d^2x \mathcal{L}^\lambda(\phi, \nabla_a \phi, g_{ab}). \quad (2.1)$$

Here $\mathcal{L}^\lambda$ denotes the deformed Lagrangian parameterized by λ . The $T\bar{T}$ deformation can then be defined by the following flow equation

$$\frac{d\mathcal{L}^\lambda}{d\lambda} = \frac{1}{2} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} T_{\mu\rho}^\lambda T_{\nu\sigma}^\lambda, \quad (2.2)$$

where $\epsilon_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} \epsilon^{\rho\sigma}$ is the volume element of the spacetime, and $T_{\mu\nu}^\lambda$ is the stress tensor of the deformed theory, which is defined as

$$T_{\mu\nu}^\lambda = \frac{2}{\sqrt{g}} \frac{\delta S^\lambda}{\delta g^{\mu\nu}} = 2 \frac{\partial \mathcal{L}^\lambda}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}^\lambda. \quad (2.3)$$

Now expand of deformed Lagrangian and stress tensor in the power of λ

$$\mathcal{L}^\lambda = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathcal{L}^{(n)}, \quad T_{\mu\nu}^\lambda = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} T_{\mu\nu}^{(n)}. \quad (2.4)$$

In order to figure out $\mathcal{L}^{(n)}$, one can plugging (2.4) into both (2.2) and (2.3). By comparing each order in the resulting expressions, eventually, we obtain the following recursion relations¹

$$\mathcal{L}^{(n+1)} = \frac{1}{2} \sum_{i=0}^n C_n^i \left(T_{\mu}^{\mu(i)} T_{\nu}^{\nu(n-i)} - T_{\nu}^{\mu(i)} T_{\mu}^{\nu(n-i)} \right), \quad (2.5)$$

$$T_{\mu\nu}^{(n)} = 2 \frac{\partial \mathcal{L}^{(n)}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}^{(n)}, \quad (2.6)$$

where $C_n^i \equiv \frac{n!}{i!(n-i)!}$. Note this recursion relations allow us to obtain $\mathcal{L}^{(n)}$ and $T_{\mu\nu}^{(n)}$ for arbitrary n , once $\mathcal{L}^{(0)}$, i.e. the un-deformed theory, is given.

With perturbations of $\mathcal{L}^\lambda$ acquired, we continue to derive the corrections of the partition function to higher-order in perturbation theory in path integral language, which is

$$\begin{aligned} \mathcal{Z}^\lambda &= \int \mathcal{D}\phi \, e^{-\int_{\mathcal{M}} \mathcal{L}^\lambda[\phi]} \\ &= \mathcal{Z}^{(0)} - \lambda \mathcal{Z}^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle + \frac{\lambda^2}{2} \mathcal{Z}^{(0)} \left(\int_{\mathcal{M}} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \right) + \mathcal{O}(\lambda^3) \\ &\equiv \mathcal{Z}^{(0)} + \lambda \mathcal{Z}^{(1)} + \frac{\lambda^2}{2} \mathcal{Z}^{(2)} + \mathcal{O}(\lambda^3), \end{aligned} \quad (2.7)$$

where

$$\mathcal{Z}^{(0)} = \int \mathcal{D}\phi \, e^{-\int_{\mathcal{M}} \mathcal{L}^{(0)}[\phi]}, \quad (2.8)$$

$$\mathcal{Z}^{(1)} = -\mathcal{Z}^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle, \quad (2.9)$$

$$\mathcal{Z}^{(2)} = \mathcal{Z}^{(0)} \left(\int_{\mathcal{M}} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \right). \quad (2.10)$$

In what follows, we will focus on the $T\bar{T}$ deformed free theories on a torus, including free bosons, Dirac fermions, and Majorana fermions, where the deformed partition functions up to the second-order (2.9)–(2.10) can be worked out analytically.

3 Free bosons

At first, what we would like to consider is the $T\bar{T}$ deformed free scalar on a torus T^2 . The corresponding action of the un-deformed theory reads

$$S = \frac{g}{2} \int_{T^2} d^2x \partial_\mu \phi \partial^\mu \phi, \quad (3.1)$$

¹The identity $g^{\mu\nu} g^{\rho\sigma} - g^{\rho\nu} g^{\mu\sigma} = \epsilon^{\mu\rho} \epsilon^{\nu\sigma}$ is used.

where g is a normalization constant. According to the recursion relations (2.5)–(2.6) mentioned above, one could obtain the deformed Lagrangian and stress tensor starting from $\mathcal{L}^{(0)}$,

$$\mathcal{L}^{(0)} = 2g\partial\phi\bar{\partial}\phi. \quad (3.2)$$

Then the un-deformed stress tensor is²

$$T^{(0)} = -2\pi g(\partial\phi)^2, \quad \bar{T}^{(0)} = -2\pi g(\bar{\partial}\phi)^2, \quad \Theta^{(0)} = 0, \quad (3.3)$$

from which the first-order Lagrangian is given by

$$\mathcal{L}^{(1)} = -\frac{1}{\pi^2}T^{(0)}\bar{T}^{(0)} = -4g^2(\partial\phi\bar{\partial}\phi)^2, \quad (3.4)$$

and the corresponding first-order stress tensor is

$$T^{(1)} = 8\pi g^2(\partial\phi)^3(\bar{\partial}\phi), \quad \bar{T}^{(1)} = 8\pi g^2(\bar{\partial}\phi)^3(\partial\phi), \quad \Theta^{(1)} = -4\pi g^2(\partial\phi\bar{\partial}\phi)^2. \quad (3.5)$$

Reusing eq. (2.5), we end up with the second-order Lagrangian

$$\mathcal{L}^{(2)} = -\frac{1}{\pi^2}(T^{(0)}\bar{T}^{(1)} + \bar{T}^{(0)}T^{(1)}) = 32g^3(\partial\phi\bar{\partial}\phi)^3, \quad (3.6)$$

We then could write out the corrections of the partition function (2.9) and (2.10) more concretely for bosonic fields

$$\mathcal{Z}^{(1)} = \frac{1}{\pi^2}\mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \langle T\bar{T}^{(0)}(z, \bar{z}) \rangle = 4g^2\mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \langle (\partial\phi\bar{\partial}\phi)^2 \rangle, \quad (3.7)$$

$$\begin{aligned} \mathcal{Z}^{(2)} &= \frac{1}{\pi^4}\mathcal{Z}^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle + \frac{1}{\pi^2}\mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \langle T^{(0)}\bar{T}^{(1)} + T^{(1)}\bar{T}^{(0)} \rangle \\ &= 16g^4\mathcal{Z}^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle (\partial_1\phi\bar{\partial}_1\phi)^2 (\partial_2\phi\bar{\partial}_2\phi)^2 \rangle - 32g^3\mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \langle (\partial\phi\bar{\partial}\phi)^3 \rangle. \end{aligned} \quad (3.8)$$

Note that the expectation values in (3.7)–(3.8) are defined in free theory, all of them could be evaluated directly by applying Wick contraction since the propagator is well-known for torus free scalar field [51],

$$\langle \phi(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \rangle = (4\pi g)^{-1} \left(-\log \left| \frac{\vartheta_1(z_{12})}{\eta(\tau)} \right|^2 + 2\pi \frac{(\text{Im}[z_{12}])^2}{\tau_2} \right). \quad (3.9)$$

Here $\vartheta_1(z)$ is one of Jacobi ϑ -functions and $\eta(\tau)$ is Dedekind η -function. Performing derivatives on (3.9) gives various two-point functions³

$$\langle \partial\phi(z_1, \bar{z}_1) \partial\phi(z_2, \bar{z}_2) \rangle = (4\pi g)^{-1} \left(\frac{\pi}{\tau_2} - 2\eta_1 - P(z_{12}) \right), \quad (3.10)$$

$$\langle \bar{\partial}\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_2, \bar{z}_2) \rangle = (4\pi g)^{-1} \left(\frac{\pi}{\tau_2} - 2\bar{\eta}_1 - \bar{P}(\bar{z}_{12}) \right), \quad (3.11)$$

$$\langle \partial\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_2, \bar{z}_2) \rangle = (4\pi g)^{-1} \left(\pi\delta^{(2)}(z_{12}) - \frac{\pi}{\tau_2} + \sum_{\{m,n\} \neq \{0,0\}} \pi\delta^{(2)}(z_{12} - (m+n\tau)) \right), \quad (3.12)$$

²In this paper, we use the conventional notation that $T \equiv -2\pi T_{zz}$, $\bar{T} \equiv -2\pi T_{\bar{z}\bar{z}}$, and $\Theta \equiv 2\pi T_{z\bar{z}}$. The complex coordinates $z := x + iy$, where y is Euclidean time. $\partial := (\partial_x - i\partial_y)/2$. The metric $g_{z\bar{z}} = \frac{1}{2}$.

³We neglect the last term of eq. (3.12) in the later part of this paper, since $\delta^{(2)}(z_{12} - (m+n\tau))$ is always zero in the integral region we're considering. We did the same thing later on when we deal with fermionic fields.

where $P(z)$ is Weierstrass elliptic function and we have applied the formula $\bar{\partial}(z^{-1}) = \partial(\bar{z}^{-1}) = \pi\delta^{(2)}(\vec{x}) \equiv \pi\delta^{(2)}(z)$. For more details on elliptic functions please refer to appendix A. The subsequent derivation of Wick contraction indicates that the expectation values of the composite operators $(\partial\phi(z_1, \bar{z}_1))^2$, $(\bar{\partial}\phi(z_1, \bar{z}_1))^2$, and $|\partial\phi(z_1, \bar{z}_1)|^2$ also make contributions. We regularize them by utilizing the point-splitting method

$$\langle \partial\phi(z_1, \bar{z}_1) \partial\phi(z_1, \bar{z}_1) \rangle = \lim_{z_2 \rightarrow z_1} \left(\langle \partial\phi(z_1, \bar{z}_1) \partial\phi(z_2, \bar{z}_2) \rangle + \frac{1}{4\pi g z_{12}^2} \right) = (4\pi g)^{-1} \left(\frac{\pi}{\tau_2} - 2\eta_1 \right), \quad (3.13)$$

$$\langle \bar{\partial}\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_1, \bar{z}_1) \rangle = \lim_{z_2 \rightarrow z_1} \left(\langle \bar{\partial}\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_2, \bar{z}_2) \rangle + \frac{1}{4\pi g \bar{z}_{12}^2} \right) = (4\pi g)^{-1} \left(\frac{\pi}{\tau_2} - 2\bar{\eta}_1 \right), \quad (3.14)$$

$$\langle \partial\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_1, \bar{z}_1) \rangle = \lim_{z_2 \rightarrow z_1} \left(\langle \partial\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_2, \bar{z}_2) \rangle - \frac{\delta(z_{12})}{4g} \right) = \frac{-1}{4g\tau_2}. \quad (3.15)$$

With all ingredients in place, we next go on to investigate the corrections to the partition function of free bosons.

3.1 First-order

First, we note that the partition function of the free scalar on a torus is

$$\mathcal{Z}^{(0)} = \frac{1}{\sqrt{\tau_2} |\eta(\tau)|^2}. \quad (3.16)$$

According to eq. (3.7), at the first-order we shall just compute the value of $\int_{T^2} d^2x \langle T\bar{T}^{(0)}(z, \bar{z}) \rangle$,⁴

$$\begin{aligned} \int_{T_1^2} d^2x_1 \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) \rangle &= 4\pi^2 g^2 \tau_2 \langle \partial\phi(z_1, \bar{z}_1) \partial\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_1, \bar{z}_1) \rangle \\ &= 4\pi^2 g^2 \tau_2 (2\langle 1\bar{1} \rangle^2 + \langle 11 \rangle \langle \bar{1}\bar{1} \rangle) \\ &= \frac{3\pi^2}{4\tau_2} + |\eta_1|^2 \tau_2 - \frac{\pi}{2} (\eta_1 + \bar{\eta}_1) \\ &= \frac{4\pi^2}{\mathcal{Z}^{(0)}} \tau_2 \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)}, \end{aligned} \quad (3.17)$$

which is consistent with [48]. Thus the first-order correction of the partition function is

$$\mathcal{Z}^{(1)} = \frac{\mathcal{Z}^{(0)}}{\pi^2} \int_{T_1^2} d^2x_1 \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) \rangle = 4\tau_2 \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)}. \quad (3.18)$$

⁴Here $i \equiv \partial\phi(z_i, \bar{z}_i)$, $\bar{i} \equiv \bar{\partial}\phi(z_i, \bar{z}_i)$, ($i = 1, 2, 3 \dots$).

3.2 Second-order

We next go on to consider the second-order correction to the partition function. We begin with calculating the first term of (3.8), whose integrand can be contracted as⁵

$$\begin{aligned}
 \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle &= (2\pi g)^4 \langle 11\bar{1}\bar{1}22\bar{2}\bar{2} \rangle \\
 &= (2\pi g)^4 \left[\langle 11 \rangle \langle \bar{1}\bar{1} \rangle \langle 22 \rangle \langle \bar{2}\bar{2} \rangle + 2 \times (\langle 11 \rangle \langle \bar{1}\bar{1} \rangle \langle 2\bar{2} \rangle^2 + \langle 11 \rangle \langle 22 \rangle \langle \bar{1}\bar{2} \rangle^2 + \langle 11 \rangle \langle \bar{2}\bar{2} \rangle \langle \bar{1}2 \rangle^2 + \langle \bar{1}\bar{1} \rangle \langle 22 \rangle \langle 1\bar{2} \rangle^2 \right. \\
 &\quad + \langle \bar{1}\bar{1} \rangle \langle \bar{2}\bar{2} \rangle \langle 12 \rangle^2 + \langle 22 \rangle \langle \bar{2}\bar{2} \rangle \langle 1\bar{1} \rangle^2) + 8 \times (\langle 11 \rangle \langle \bar{1}\bar{2} \rangle \langle 2\bar{2} \rangle \langle \bar{2}\bar{1} \rangle + \langle \bar{1}\bar{1} \rangle \langle 12 \rangle \langle 2\bar{2} \rangle \langle \bar{2}1 \rangle \\
 &\quad + \langle 22 \rangle \langle 1\bar{1} \rangle \langle \bar{1}\bar{2} \rangle \langle \bar{2}1 \rangle + \langle \bar{2}\bar{2} \rangle \langle 1\bar{1} \rangle \langle \bar{1}2 \rangle \langle 21 \rangle) + 4 \times (\langle 1\bar{1} \rangle^2 \langle 2\bar{2} \rangle^2 + \langle 12 \rangle^2 \langle \bar{1}\bar{2} \rangle^2 + \langle \bar{1}2 \rangle^2 \langle 1\bar{2} \rangle^2) \\
 &\quad \left. + 16 \times (\langle 1\bar{1} \rangle \langle \bar{1}\bar{2} \rangle \langle 2\bar{2} \rangle \langle \bar{2}1 \rangle + \langle 1\bar{1} \rangle \langle \bar{1}\bar{2} \rangle \langle \bar{2}\bar{2} \rangle \langle 21 \rangle + \langle 1\bar{2} \rangle \langle \bar{2}\bar{1} \rangle \langle \bar{1}2 \rangle \langle 21 \rangle) \right] \\
 &= \frac{1}{16} \left(24A^4 + 8A^2|B|^2 + |B|^4 + 4|B - P(z_{12})|^4 + 32A^2|B - P(z_{12})|^2 + 4\text{Re}[\bar{B}^2(B - P(z_{12}))^2] \right. \\
 &\quad \left. + 32A^2 \cdot \text{Re}[\bar{B}(B - P(z_{12}))] - 48\pi A^3 \delta(z_{12}) - 72\pi A|B|^2 \delta(z_{12}) \right), \tag{3.19}
 \end{aligned}$$

where $B \equiv (\frac{\pi}{\tau_2} - 2\eta_1)$, $\bar{B} \equiv (\frac{\pi}{\tau_2} - 2\bar{\eta}_1)$, and $A \equiv \frac{\pi}{\tau_2}$. Integrating the above expression amounts to compute the following integrals

$$\int_{T_1^2} \int_{T_2^2} (B - P(z_{12})) = 0, \tag{3.20}$$

$$\int_{T_1^2} \int_{T_2^2} (B - P(z_{12}))^2 = \frac{g_2 \tau_2^2}{12} - \tau_2^2 B^2, \tag{3.21}$$

$$\int_{T_1^2} \int_{T_2^2} |B - P(z_{12})|^2 = -\pi^2, \tag{3.22}$$

$$\int_{T_1^2} \int_{T_2^2} |B - P(z_{12})|^4 = \tau_2^2 |B|^4 + \frac{|g_2|^2 \tau_2^2}{12^2} - 4\tau_2^2 A^2 |B|^2 - B^2 \frac{\bar{g}_2 \tau_2^2}{12} - \bar{B}^2 \frac{g_2 \tau_2^2}{12}, \tag{3.23}$$

where g_2 is one of Weierstrass invariants whose definition can be found in appendix A. We collect the detailed computation of the above integrals in appendix B.2. Note some of the integrals are divergent, thus a proper regularization scheme is needed, which will be presented in appendix B.1.

With the help of (3.19)–(3.23) and the following identity relating the quantity g_2 with η_1

$$g_2 = 48(i\pi \partial_\tau \eta_1 + \eta_1^2), \tag{3.24}$$

⁵To obtain eq. (3.19), we have discarded terms purely divergent (i.e., they have no finite contribution to the final result under the minimal subtraction principle).

the double integral of eq. (3.19) is derived as

$$\begin{aligned}
 & \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle \\
 &= \tau_2^2 |\eta_1|^4 - \pi^2 |\eta_1|^2 + 4\pi^2 \tau_2^2 |\partial_\tau \eta_1|^2 - \frac{\pi^2}{4} (\eta_1^2 + \bar{\eta}_1^2) + \frac{3\pi^3}{4\tau_2} (\eta_1 + \bar{\eta}_1) + \tau_2 \pi |\eta_1|^2 (\eta_1 + \bar{\eta}_1) \\
 &+ \frac{i\pi^3}{2} (\partial_{\bar{\tau}} \bar{\eta}_1 - \partial_\tau \eta_1) + 2i\pi^2 \tau_2 (\bar{\eta}_1 \partial_\tau \eta_1 - \eta_1 \partial_{\bar{\tau}} \bar{\eta}_1) + 2i\pi \tau_2^2 (\bar{\eta}_1^2 \partial_\tau \eta_1 - \eta_1^2 \partial_{\bar{\tau}} \bar{\eta}_1) - \frac{15\pi^4}{2^4 \tau_2^2} \\
 &- 18\pi^2 |\eta_1|^2 + \frac{9\pi^3}{\tau_2} (\eta_1 + \bar{\eta}_1) - \frac{15\pi^4}{2\tau_2^2} \\
 &= \frac{16\pi^4}{\mathcal{Z}^{(0)}} (\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}^{(0)} - \frac{72\pi^4}{\mathcal{Z}^{(0)}} \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)} + \frac{6\pi^4}{\tau_2^2}. \tag{3.25}
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \mathcal{Z}^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle \mathcal{L}^{(1)}(z_1, \bar{z}_1) \mathcal{L}^{(1)}(z_2, \bar{z}_2) \rangle \\
 &= \frac{1}{\pi^4} \mathcal{Z}^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle \\
 &= 16(\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}^{(0)} - 72\partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)} + 6\tau_2^{-2} \mathcal{Z}^{(0)}. \tag{3.26}
 \end{aligned}$$

We next move to evaluate the second term in (3.8). Using Wick contraction, the integrand is

$$\begin{aligned}
 & \langle T^{(0)}(z_1, \bar{z}_1) \bar{T}^{(1)}(z_1, \bar{z}_1) \rangle + \langle T^{(1)}(z_1, \bar{z}_1) \bar{T}^{(0)}(z_1, \bar{z}_1) \rangle \\
 &= -32\pi^2 g^3 \langle 111\bar{1}\bar{1}\bar{1} \rangle \\
 &= -288\pi^2 g^3 \times \langle 1\bar{1} \rangle \langle 11 \rangle \langle \bar{1}\bar{1} \rangle - 192\pi^2 g^3 \times \langle 1\bar{1} \rangle^3 \\
 &= \frac{18}{\tau_2} |\eta_1|^2 - \frac{9\pi}{\tau_2^2} (\eta_1 + \bar{\eta}_1) + \frac{15\pi^2}{2\tau_2^3} \\
 &= \frac{72\pi^2}{\tau_2 \mathcal{Z}^{(0)}} \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)} - \frac{6\pi^2}{\tau_2^3}. \tag{3.27}
 \end{aligned}$$

After simple integration, one has

$$\mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \mathcal{L}^{(2)}(z, \bar{z}) = -\frac{1}{\pi^2} \mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \langle T^{(0)} \bar{T}^{(1)} + T^{(1)} \bar{T}^{(0)} \rangle = -72\partial_\tau \partial_{\bar{\tau}} \mathcal{Z}^{(0)} + 6\tau_2^{-2} \mathcal{Z}^{(0)}. \tag{3.28}$$

Putting together (3.26) and (3.28), we obtain the second-order correction of the partition function under the $T\bar{T}$ deformation

$$\begin{aligned}
 \mathcal{Z}^{(2)} &= \mathcal{Z}^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle \mathcal{L}^{(1)}(z_1, \bar{z}_1) \mathcal{L}^{(1)}(z_2, \bar{z}_2) \rangle - \mathcal{Z}^{(0)} \int_{\mathbb{T}^2} \mathcal{L}^{(2)}(z, \bar{z}) \\
 &= 16(\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}^{(0)}, \tag{3.29}
 \end{aligned}$$

which is consistent with [6]. Note that we have minimally subtracted the divergent terms⁶ when deriving the r.h.s. of (3.25), and thus (3.29). It is possible to implement this minimal

⁶The details of minimal subtraction ((B.14), (B.17)) are presented in appendix B.2.

subtraction by adding the following counterterm⁷

$$\mathcal{L}_{\text{FB,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi\epsilon^2} (\partial\phi\bar{\partial}\phi)^2 + \frac{1}{24\pi^3\epsilon^6} \right\}, \quad (3.30)$$

where ϵ stands for the radius of the infinitesimal disk regulator.

4 Free Dirac fermions

For the rest of the examples, we turn our attention to the fermionic fields defined on a torus. We first focus on a massless Dirac field whose action is

$$S = \frac{g}{2} \int_{\mathbb{T}^2} (\Psi^\dagger \gamma^0 \gamma^a \partial_a \Psi - \partial_a \Psi^\dagger \gamma^0 \gamma^a \Psi) \quad (4.1)$$

with

$$\Psi = [\psi \quad \bar{\psi}]^T, \quad \Psi^\dagger = [\psi^* \quad \bar{\psi}^*]. \quad (4.2)$$

Our convention for gamma matrices are $\{\gamma^0, \gamma^1\} = \{\sigma^1, \sigma^2\}$, where $\sigma^i, i = 1, 2$ are Pauli matrices.

As before we make the expansion

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda \mathcal{L}^{(1)} + \dots, \quad T_{\mu\nu}^\lambda = T_{\mu\nu}^{(0)} + \lambda T_{\mu\nu}^{(1)} + \dots, \quad (4.3)$$

remarkably, the case of fermions will simplify a lot comparing with bosons by the fact that the higher-order terms of $\mathcal{L}^{(n)}, n \geq 2$ are completely vanishing [50], due to the Grassmannian nature of fermionic fields. Following the derivation presented in [50], we obtain the full expression of $\mathcal{L}^\lambda$ and $T_{\mu\nu}^\lambda$ written in complex coordinates⁸

$$\mathcal{L}^{(0)} = g(\psi^* \overleftrightarrow{\partial} \psi + \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}), \quad (4.4)$$

$$\begin{aligned} \mathcal{L}^{(1)} &= \frac{1}{\pi^2} \left((\Theta^{(0)})^2 - T^{(0)} \bar{T}^{(0)} \right) \\ &= \frac{g^2}{2} \left((\psi^* \overleftrightarrow{\partial} \psi)(\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) + (\psi^* \bar{\psi} \partial \bar{\psi}^* \bar{\psi} + \bar{\psi}^* \bar{\psi} \partial \bar{\psi}^* \bar{\psi}) \right) - g^2 (\psi^* \overleftrightarrow{\partial} \psi)(\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}), \end{aligned} \quad (4.5)$$

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda \cdot \mathcal{L}^{(1)}, \quad (4.6)$$

and

$$T^{(0)} = -\pi g \cdot \psi^* \overleftrightarrow{\partial} \psi, \quad \bar{T}^{(0)} = -\pi g \cdot \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}, \quad \Theta^{(0)} = -\frac{\pi g}{2} (\psi^* \overleftrightarrow{\partial} \psi + \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}), \quad (4.7)$$

$$T^{(1)} = -\frac{\pi g^2}{2} \left(\psi^* \bar{\psi} (\partial \bar{\psi}^* \bar{\psi} + \partial \bar{\psi}^* \bar{\psi}) - (\psi^* \overleftrightarrow{\partial} \psi)(\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) \right), \quad (4.8)$$

$$\bar{T}^{(1)} = -\frac{\pi g^2}{2} \left(\bar{\psi}^* \bar{\psi} (\partial \bar{\psi}^* \bar{\psi} + \partial \bar{\psi}^* \bar{\psi}) - (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi})(\psi^* \overleftrightarrow{\partial} \psi) \right), \quad \Theta^{(1)} = 0, \quad (4.9)$$

$$T_{\mu\nu}^\lambda = T_{\mu\nu}^{(0)} + \lambda \cdot T_{\mu\nu}^{(1)}. \quad (4.10)$$

⁷Please refer to appendix C for a discussion of this Lagrangian counterterm.

⁸For the derivation, one can refer to appendix D.

It is well-known that the un-deformed partition function for Dirac fermions is given by

$$\mathcal{Z}_\nu^{(0)} = (d_\nu \bar{d}_\nu)^2, \quad d_\nu(\tau) = \left(\frac{\vartheta_\nu(\tau)}{\eta(\tau)} \right)^{1/2}. \quad (4.11)$$

where $\nu = 1, 2, 3, 4$ denotes the spin structures of fermions, corresponding to different boundary conditions,⁹ ϑ_ν are Jacobi ϑ -functions. The non-vanishing two-point functions for Dirac fermions with spin structure ν are

$$\langle \psi^*(z_1) \psi(z_2) \rangle_\nu = (2\pi g)^{-1} P_\nu(z_{12}), \quad (4.12)$$

$$\langle \bar{\psi}^*(\bar{z}_1) \bar{\psi}(\bar{z}_2) \rangle_\nu = (2\pi g)^{-1} \bar{P}_\nu(\bar{z}_{12}), \quad \nu = 2, 3, 4. \quad (4.13)$$

where

$$P_\nu(z) := \sqrt{P(z) - e_{\nu-1}} = \frac{\vartheta_\nu(z) \partial_z \vartheta_1(0)}{\vartheta_\nu(0) \vartheta_1(z)}. \quad (4.14)$$

Performing derivatives on the propagators leads to the following correlation functions

$$\langle \partial \psi^*(z_1) \psi(z_2) \rangle_\nu = (2\pi g)^{-1} \partial P_\nu(z_{12}), \quad (4.15)$$

$$\langle \partial \psi^*(z_1) \partial \psi(z_2) \rangle_\nu = -(2\pi g)^{-1} \partial^2 P_\nu(z_{12}), \quad (4.16)$$

$$\langle \bar{\partial} \psi^*(z_1) \psi(z_2) \rangle_\nu = (2g)^{-1} \delta^{(2)}(z_{12}), \quad (4.17)$$

$$\langle \psi^*(z_1) \bar{\partial} \psi(z_2) \rangle_\nu = -(2g)^{-1} \delta^{(2)}(z_{12}). \quad (4.18)$$

We need further to regularize these correlation functions when two points coincide with each other, in parallel with the bosonic case, we use the point-splitting method

$$\langle \psi^*(z_1) \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \psi^*(z_1) \psi(z_2) \rangle_\nu - (2\pi g z_{12})^{-1}) = 0, \quad (4.19)$$

$$\langle \partial \psi^*(z_1) \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \partial \psi^*(z_1) \psi(z_2) \rangle_\nu + (2\pi g z_{12}^2)^{-1}) = -(4\pi g)^{-1} e_{\nu-1}, \quad (4.20)$$

$$\langle \partial \psi^*(z_1) \partial \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \partial \psi^*(z_1) \partial \psi(z_2) \rangle_\nu + (\pi g z_{12}^3)^{-1}) = 0, \quad (4.21)$$

$$\langle \bar{\partial} \psi^*(z_1) \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial} \psi^*(z_1) \psi(z_2) \rangle_\nu - (2g)^{-1} \delta^{(2)}(z_{12})) = 0, \quad (4.22)$$

$$\langle \bar{\partial} \psi^*(z_1) \partial \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial} \psi^*(z_1) \partial \psi(z_2) \rangle_\nu + (2g)^{-1} \partial \delta^{(2)}(z_{12})) = 0, \quad (4.23)$$

$$\langle \bar{\partial} \psi^*(z_1) \bar{\partial} \psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial} \psi^*(z_1) \bar{\partial} \psi(z_2) \rangle_\nu + (2g)^{-1} \bar{\partial} \delta^{(2)}(z_{12})) = 0. \quad (4.24)$$

Now we have all the required ingredients to calculate the corrections to the partition function.

4.1 First-order

Using Wick contraction and the propagators and their derivatives listed above, we can compute the expectation value of $T^{(0)} \bar{T}^{(0)}$ and $(\Theta^{(0)})^2$

$$\langle T^{(0)} \bar{T}^{(0)} \rangle_\nu = \frac{1}{4} |e_{\nu-1}|^2 = \frac{4\pi^2}{\mathcal{Z}_\nu^{(0)}} \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}_\nu^{(0)}, \quad \langle (\Theta^{(0)})^2 \rangle_\nu = 0. \quad (4.25)$$

⁹ $\mathcal{Z}_1^{(0)}$ that corresponding to fermions with the double periodic boundary condition is zero, due to the property of Grassmann number [51].

Therefore the first-order correction of the partition function is

$$\mathcal{Z}_\nu^{(1)} = - \int_{\mathbb{T}^2} \mathcal{L}^{(1)}(z, \bar{z}) = \frac{1}{\pi^2} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}^2} \langle T^{(0)} \bar{T}^{(0)} \rangle_\nu = 4\tau_2 \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}_\nu^{(0)}. \quad (4.26)$$

Note that the first-order correction of free Dirac fermions shares the same structure with that of free bosons (3.18), which matches the conclusion in [6] obtained by the operator formalism. We're going to show that this is also true for the second-order correction.

4.2 Second-order

We now proceed to compute the second-order correction. Since there are no higher-order terms in Lagrangian ($\mathcal{L}^{(n)} = 0$ for $n \geq 2$) for free massless Dirac fermions, (2.10) reduces to

$$\begin{aligned} \mathcal{Z}_\nu^{(2)} &= \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle \mathcal{L}^{(1)}(z_1, \bar{z}_1) \mathcal{L}^{(1)}(z_2, \bar{z}_2) \rangle_\nu \\ &= \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle (\Theta^{(0)})^2(z_1, \bar{z}_1) (\Theta^{(0)})^2(z_2, \bar{z}_2) \rangle_\nu \\ &\quad - \frac{2}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T \bar{T}^{(0)}(z_1, \bar{z}_1) (\Theta^{(0)})^2(z_2, \bar{z}_2) \rangle_\nu + \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T \bar{T}^{(0)}(x_1) T \bar{T}^{(0)}(x_2) \rangle_\nu. \end{aligned} \quad (4.27)$$

After using Wick contraction and discarding the purely divergent terms,¹⁰ we obtain

$$\begin{aligned} \mathcal{Z}_\nu^{(2)} &= \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle \mathcal{L}^{(1)}(z_1, \bar{z}_1) \mathcal{L}^{(1)}(z_2, \bar{z}_2) \rangle_\nu \\ &= \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T \bar{T}^{(0)}(z_1, \bar{z}_1) T \bar{T}^{(0)}(z_2, \bar{z}_2) \rangle_\nu \\ &= \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \left\{ \frac{1}{4} |e_{\nu-1}|^4 + \frac{1}{4} |\partial P_\nu(z_{12})|^4 + \frac{1}{4} |P_\nu(z_{12}) \partial^2 P_\nu(z_{12})|^2 \right. \\ &\quad - \frac{1}{4} \left((\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) + (\partial P_\nu(z_{12}))^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) \right) \\ &\quad + \frac{1}{8} \left(e_{\nu-1}^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) + \bar{e}_{\nu-1}^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \right) \\ &\quad \left. - \frac{1}{8} \left(e_{\nu-1}^2 (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 + \bar{e}_{\nu-1}^2 (\partial P_\nu(z_{12}))^2 \right) \right\}. \end{aligned} \quad (4.28)$$

The integrals of the nontrivial integrands shown above are listed below

$$\int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} (\partial P_\nu(z_{12}))^2 = \tau_2 e_{\nu-1} (\pi - 2\tau_2 \eta_1) + \tau_2^2 \left(e_{\nu-1}^2 - \frac{g_2}{6} \right), \quad (4.29)$$

$$\int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) = - \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} (\partial P_\nu(z_{12}))^2, \quad (4.30)$$

$$\begin{aligned} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} |\partial P_\nu(z_{12})|^4 &= \tau_2^2 \left| e_{\nu-1}^2 - \frac{g_2}{6} \right|^2 \\ &\quad + |e_{\nu-1}|^2 (4\tau_2^2 |\eta_1|^2 - 2\pi\tau_2(\eta_1 + \bar{\eta}_1)) \end{aligned}$$

¹⁰This is similar to the case of the free bosons in the previous section.

$$\begin{aligned}
 & + \left(\tau_2 e_{\nu-1} \left(\bar{e}_{\nu-1}^2 - \frac{g_2}{6} \right) (\pi - 2\tau_2 \eta_1) \right. \\
 & \quad \left. + \tau_2 \bar{e}_{\nu-1} \left(e_{\nu-1}^2 - \frac{g_2}{6} \right) (\pi - 2\tau_2 \bar{\eta}_1) \right), \quad (4.31)
 \end{aligned}$$

$$\int_{T_1^2} \int_{T_2^2} |P_\nu(z_{12}) \partial^2 P_\nu(z_{12})|^2 = \int_{T_1^2} \int_{T_2^2} |\partial P_\nu(z_{12})|^4, \quad (4.32)$$

$$\int_{T_1^2} \int_{T_2^2} (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) = - \int_{T_1^2} \int_{T_2^2} |(\partial P_\nu(z_{12}))|^4. \quad (4.33)$$

For the detailed discussions of the above integrals please refer to appendix B.3.

With the help of the above nontrivial integrals and identity involving g_2 , $e_{\nu-1}$, and η_1

$$g_2 = 6(e_{\nu-1}^2 - i\pi \partial_\tau e_{\nu-1} - 2\eta_1 e_{\nu-1}), \quad (4.34)$$

one can find that (4.28) equals

$$\begin{aligned}
 & \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{T_1^2} \int_{T_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle_\nu \\
 & = \frac{\tau_2^2}{2^4 \pi^4} |e_{\nu-1}|^4 + \frac{\tau_2^2}{\pi^2} \tau_2^2 |\partial_\tau e_{\nu-1}|^2 + \frac{i\tau_2^2}{4\pi^3} (e_{\nu-1}^2 \partial_{\bar{\tau}} \bar{e}_{\nu-1} - \bar{e}_{\nu-1}^2 \partial_\tau e_{\nu-1}) \\
 & \quad + \frac{i\tau_2}{\pi^2} (\bar{e}_{\nu-1} \partial_\tau e_{\nu-1} - e_{\nu-1} \partial_{\bar{\tau}} \bar{e}_{\nu-1}) - \frac{\tau_2}{4\pi^3} (e_{\nu-1}^2 \bar{e}_{\nu-1} + \bar{e}_{\nu-1}^2 e_{\nu-1}) \\
 & = \frac{16}{\mathcal{Z}_\nu^{(0)}} (\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}_\nu^{(0)}. \quad (4.35)
 \end{aligned}$$

Therefore the second-order corrections of the partition function with spin structure ν are

$$\begin{aligned}
 \mathcal{Z}_\nu^{(2)} & = \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{T_1^2} \int_{T_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle_\nu \\
 & = 16(\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}_\nu^{(0)}, \quad (4.36)
 \end{aligned}$$

which has the same structure with the bosonic case (3.29), and agrees with the result in [6]. Similar to the case of free bosons, for the deformed free Dirac fermions we can find the counterterm corresponding to the minimum subtraction scheme as follows¹¹

$$\mathcal{L}_{\text{DF,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi\epsilon^2} \partial\psi^* \psi \bar{\partial}\bar{\psi}^* \bar{\psi} + \frac{1}{24\pi^3 \epsilon^6} \right\}. \quad (4.37)$$

5 Free Majorana fermions

As the last example, we investigate the deformation of free massless Majorana fermions, whose un-deformed action is given by

$$S = \frac{g}{2} \int_{T^2} (\Psi^T \gamma^0 \gamma^a \partial_a \Psi - \partial_a \Psi^T \gamma^0 \gamma^a \Psi), \quad (5.1)$$

where $\Psi = [\psi \quad \bar{\psi}]^T$, the gamma matrices are defined in the previous section.

¹¹The derivation is presented in appendix C.

Similar to the case of complex fermions, the $T\bar{T}$ flow of Lagrangian truncates at the first order, that is we have

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda \mathcal{L}^{(1)}, \quad T_{\mu\nu}^\lambda = T_{\mu\nu}^{(0)} + \lambda T_{\mu\nu}^{(1)}, \quad (5.2)$$

where

$$\mathcal{L}^{(0)} = 2g(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}), \quad \mathcal{L}^{(1)} = \frac{1}{\pi^2} \left((\Theta^{(0)})^2 - T^{(0)}\bar{T}^{(0)} \right) = g^2(2\psi\bar{\partial}\psi\bar{\psi}\partial\bar{\psi} - 4\psi\partial\psi\bar{\psi}\bar{\partial}\bar{\psi}), \quad (5.3)$$

and

$$T^{(0)} = -2\pi g \cdot \psi\partial\psi, \quad \Theta^{(0)} = -\pi g \cdot (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}), \quad \bar{T}^{(0)} = -2\pi g \cdot \bar{\psi}\bar{\partial}\bar{\psi}, \quad (5.4)$$

$$T^{(1)} = 2\pi g^2 \cdot \psi\partial\psi\bar{\psi}\partial\bar{\psi}, \quad \Theta^{(1)} = 0, \quad \bar{T}^{(1)} = 2\pi g^2 \cdot \bar{\psi}\bar{\partial}\bar{\psi}\psi\bar{\partial}\psi. \quad (5.5)$$

Note that one could obtain (5.3)–(5.5) by simply removing the “*” in (4.4)–(4.9).

The un-deformed partition function with spin structure ν is [51]

$$\mathcal{Z}_\nu^{(0)} = d_\nu \bar{d}_\nu, \quad d_\nu(\tau) = \left(\frac{\vartheta(\tau)_\nu}{\eta(\tau)} \right)^{1/2}, \quad (5.6)$$

The two-point functions for Majorana fermions with spin structure ν are [51]

$$\langle \psi(z_1)\psi(z_2) \rangle_\nu = (4\pi g)^{-1} P_\nu(z_{12}), \quad (5.7)$$

$$\langle \bar{\psi}(\bar{z}_1)\bar{\psi}(\bar{z}_2) \rangle = (4\pi g)^{-1} \bar{P}_\nu(\bar{z}_{12}), \quad (5.8)$$

$$\text{others} = 0, \quad \nu = 2, 3, 4. \quad (5.9)$$

Taking derivatives on above propagators gives

$$\langle \partial\psi(z_1)\psi(z_2) \rangle_\nu = (4\pi g)^{-1} \partial P_\nu(z_{12}), \quad \langle \bar{\partial}\psi(z_1)\psi(z_2) \rangle_\nu = (4g)^{-1} \delta^{(2)}(z_{12}). \quad (5.10)$$

The regularized expectation value of the propagators and their derivatives when two points coincide are

$$\langle \psi(z_1)\psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \psi(z_1)\psi(z_2) \rangle_\nu - (4\pi g z_{12})^{-1}) = 0, \quad (5.11)$$

$$\langle \partial\psi(z_1)\psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \partial\psi(z_1)\psi(z_2) \rangle_\nu + (4\pi g z_{12}^2)^{-1}) = -(8\pi g)^{-1} e_{\nu-1}. \quad (5.12)$$

$$\langle \bar{\partial}\psi(z_1)\psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial}\psi(z_1)\psi(z_2) \rangle_\nu - (4g)^{-1} \delta(z_{12})) = 0, \quad (5.13)$$

$$\langle \bar{\partial}\psi(z_1)\partial\psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial}\psi(z_1)\partial\psi(z_2) \rangle_\nu + (4g)^{-1} \partial\delta(z_{12})) = 0, \quad (5.14)$$

$$\langle \bar{\partial}\psi(z_1)\bar{\partial}\psi(z_1) \rangle_\nu \equiv \lim_{z_2 \rightarrow z_1} (\langle \bar{\partial}\psi(z_1)\bar{\partial}\psi(z_2) \rangle_\nu + (4g)^{-1} \bar{\partial}\delta(z_{12})) = 0. \quad (5.15)$$

In analogy to the Dirac fermion case we now go on to compute the corrections to the partition function.

5.1 First-order

According to (5.3), the first-order correction of the partition function is

$$\begin{aligned}\mathcal{Z}_\nu^{(1)} &= \frac{1}{\pi^2} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}^2} \left(T\bar{T}^{(0)}(z, \bar{z}) - (\Theta^{(0)})^2(z, \bar{z}) \right) = 4g^2\tau_2 \mathcal{Z}_\nu^{(0)} \langle \psi \partial \psi \bar{\psi} \bar{\partial} \bar{\psi} \rangle - g\tau_2 \mathcal{Z}_\nu^{(0)} \langle \psi \bar{\partial} \psi \bar{\psi} \partial \bar{\psi} \rangle \\ &= \frac{\tau_2}{(4\pi)^2} \mathcal{Z}_\nu^{(0)} |e_{\nu-1}|^2 \\ &= 4\tau_2 \partial_\tau \partial_{\bar{\tau}} \mathcal{Z}_\nu^{(0)},\end{aligned}\tag{5.16}$$

which takes the same form that of in free massless bosons and free massless Dirac fermions.

5.2 Second-order

For the second-order correction, in full analogy with the case of Dirac fermions, there is no contribution that comes from $\langle (\Theta^{(0)})^2(z_1, \bar{z}_1) (\Theta^{(0)})^2(z_2, \bar{z}_2) \rangle$ and $\langle T\bar{T}^{(0)}(z_1, \bar{z}_1) (\Theta^{(0)})^2(z_2, \bar{z}_2) \rangle$, hence we go on to compute the only nonzero contribution $\langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle$ and its integral.

$$\begin{aligned}&\langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle \\ &= (2\pi g)^4 \langle \psi(z_1) \partial \psi(z_1) \bar{\psi}(z_1) \bar{\partial} \bar{\psi}(z_1) \psi(z_2) \partial \psi(z_2) \bar{\psi}(z_2) \bar{\partial} \bar{\psi}(z_2) \rangle \\ &= \frac{1}{16} \left\{ \frac{1}{16} |e_{\nu-1}|^4 + |\partial P_\nu(z_{12})|^4 + \left| P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \right|^2 \right. \\ &\quad - \left((\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) + (\partial P_\nu(z_{12}))^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) \right) \\ &\quad + \frac{1}{4} \left(e_{\nu-1}^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) + \bar{e}_{\nu-1}^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \right) \\ &\quad \left. - \frac{1}{4} \left(e_{\nu-1}^2 (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 + \bar{e}_{\nu-1}^2 (\partial P_\nu(z_{12}))^2 \right) \right\}.\end{aligned}\tag{5.17}$$

Utilizing the nontrivial integrals and the identity (4.29)–(4.34) mentioned before, the double integral of (5.17) equals

$$\begin{aligned}&\int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle \\ &= \frac{\tau_2^2}{4^4} |e_{\nu-1}|^4 + \frac{\pi^2 \tau_2^2}{4} |\partial_\tau e_{\nu-1}|^2 + \frac{i\pi \tau_2^2}{32} (e_{\nu-1}^2 \partial_{\bar{\tau}} \bar{e}_{\nu-1} - \bar{e}_{\nu-1}^2 \partial_\tau e_{\nu-1}) \\ &\quad - \frac{\pi \tau_2}{32} (e_{\nu-1}^2 \bar{e}_{\nu-1} + \bar{e}_{\nu-1}^2 e_{\nu-1}) - \frac{i\pi^2 \tau_2}{4} (e_{\nu-1} \partial_{\bar{\tau}} \bar{e}_{\nu-1} - \bar{e}_{\nu-1} \partial_\tau e_{\nu-1}) \\ &= \frac{16\pi^4}{\mathcal{Z}_\nu^{(0)}} \left(\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau) \right) \mathcal{Z}_\nu^{(0)}.\end{aligned}\tag{5.18}$$

According to (5.18), we can obtain that the second-order correction of the partition function for deformed free Majorana fermions

$$\begin{aligned}\mathcal{Z}_\nu^{(2)} &= \frac{1}{\pi^4} \mathcal{Z}_\nu^{(0)} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} \langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle_\nu \\ &= 16(\tau_2^2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) \mathcal{Z}_\nu^{(0)},\end{aligned}\tag{5.19}$$

as was expected, the second-order corrections of Majorana fermions share the same structure as Dirac fermions (4.36) and free bosons (3.29), the conclusion of ref. [6] is confirmed again. Once again the counterterm can be found as¹²

$$\mathcal{L}_{\text{MF,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi\epsilon^2} \partial\psi\psi\bar{\partial}\bar{\psi}\bar{\psi} + \frac{1}{96\pi^3\epsilon^6} \right\}. \quad (5.20)$$

It's natural to ask whether the counterterms eq. (4.37)(5.20) introduced in the first two orders are enough or not to cancel the divergences of the higher-order partition function in free fermionic theories. From the perspective of the $T\bar{T}$ deformation as a kind of irrelevant deformation, one can expect new divergent terms to appear in the higher-order, and there is no a priori reason that new counterterms added to canceling these divergences should be vanishing, although the higher-order deformations of the Lagrangians (4.6)(5.3) are truncated due to the Grassmannian structure of the fermion. It is an interesting future problem to perform higher-order calculations to determine the exact higher-order counterterms.

6 The first KdV charge

In the previous sections, the corrections of various $T\bar{T}$ deformed partition functions evaluated by the conformal perturbation theory based on Lagrangian path integral are in good agreement with results obtained by the non-perturbative approach [6]. In this section, we proceed with the perturbation method to study the $T\bar{T}$ -flow effects of the first quantum KdV charge¹³ [52], for which there have been studies based on non-perturbation methods [4, 37].

Let's first consider a generic CFT, for the sake of convenience we call it a seed later, on a cylinder with coordinate $\{z, \bar{z}\}$ and circumference L .¹⁴ After the $T\bar{T}$ deformation, the deformed left-moving KdV charges P_s^λ in the resulting QFT take the form

$$P_s^\lambda = \frac{1}{2\pi} \int_0^L (dz T_{s+1}^\lambda + d\bar{z} \Theta_{s-1}^\lambda), \quad (6.1)$$

where the superscript λ represents the deformation parameter. For $s = 1$

$$P_1^\lambda = \frac{1}{2\pi} \int_0^L (dz T^\lambda + d\bar{z} \Theta^\lambda) \equiv -\frac{H^\lambda + P^\lambda}{2}, \quad (6.2)$$

where $H^\lambda = -\int_0^L dx T_{yy}^\lambda$ is deformed Hamiltonian and $P^\lambda = -i \int_0^L dx T_{xy}^\lambda$ is deformed momentum. The expectation value of P_1^λ in the deformed state $|n\rangle^\lambda$ thus reads

$${}^\lambda \langle n | P_1^\lambda | n \rangle^\lambda = -\frac{\mathcal{E}_n^\lambda + P_n^\lambda}{2}, \quad (6.3)$$

where $\mathcal{E}_n^\lambda$ and P_n^λ represent the energy and momentum of the state $|n\rangle^\lambda$ respectively. From the $T\bar{T}$ -flow equations of $\mathcal{E}_n^\lambda$, P_n^λ [1–3, 5]

$$\mathcal{E}_n^\lambda = \frac{L}{2\lambda} \left(\sqrt{1 + \frac{4\lambda E_n}{L} + \frac{4\lambda^2 (P_n)^2}{L^2}} - 1 \right), \quad P_n^\lambda = P_n, \quad (6.4)$$

¹²The derivation is presented in appendix C.

¹³We are grateful to the anonymous referee's suggestion to study the $T\bar{T}$ deformation of KdV charge.

¹⁴ $z = x + iy$, $\bar{z} = x - iy$, $x \sim x + L$.

where E_n and P_n are energy and momentum of the undeformed eigenstate $|n\rangle$ in the seed, we could get the closed form for ${}^\lambda\langle n|P_1^\lambda|n\rangle^\lambda$ depended only on E_n , P_n , λ , and L

$${}^\lambda\langle n|P_1^\lambda|n\rangle^\lambda = \frac{L}{4\lambda} \left(1 - \sqrt{1 + \frac{4\lambda E_n}{L} + \frac{4\lambda^2 (P_n)^2}{L^2}} \right) - \frac{P_n}{2}. \quad (6.5)$$

From now on, we're going to focus on the case where $n = 0$ (i.e., the ground state) and the seed theory is free bosons or free Dirac fermions or free Majorana fermions.

6.1 Non-perturbative approach

For the seed theory being free bosons with periodic boundary condition ($\phi(z+L) = \phi(z)$), or Dirac and Majorana fermions with anti-periodic boundary condition ($\psi(z+L) = -\psi(z)$), the vacuum energy and momentum are

$$E_0 = -\frac{\pi c}{6L}, \quad P_0 = 0, \quad (6.6)$$

respectively, which leads to ${}^\lambda\langle 0|P_1^\lambda|0\rangle^\lambda$ equals¹⁵

$$\begin{aligned} \langle P_1^\lambda \rangle_o^\lambda &\equiv {}^\lambda\langle 0|P_1^\lambda|0\rangle^\lambda = \frac{L}{4\lambda} \left(1 - \sqrt{1 - \frac{2\lambda\pi c}{3L^2}} \right) \\ &= \frac{c\pi}{12L} + \lambda \cdot \frac{c^2\pi^2}{72L^3} + O(\lambda^2) \\ &= \frac{c\pi}{12} + \lambda \cdot \frac{c^2\pi^2}{72} + O(\lambda^2), \quad (\text{for } L = 1). \end{aligned} \quad (6.7)$$

For the Dirac or Majorana fermions with periodic boundary condition ($\psi(z+L) = \psi(z)$), the vacuum energy and momentum are

$$E_0 = \frac{\pi c}{3L}, \quad P_0 = 0, \quad (6.8)$$

which leads to $\langle P_1^\lambda \rangle_o^\lambda$ equals

$$\begin{aligned} \langle P_1^\lambda \rangle_o^\lambda &= \frac{L}{4\lambda} \left(1 - \sqrt{1 + \frac{4\lambda\pi c}{3L^2}} \right) \\ &= -\frac{c\pi}{6L} + \lambda \cdot \frac{c^2\pi^2}{18L^3} + O(\lambda^2) \\ &= -\frac{c\pi}{6} + \lambda \cdot \frac{c^2\pi^2}{18} + O(\lambda^2), \quad (\text{for } L = 1). \end{aligned} \quad (6.9)$$

We next to reproduce the above results (6.7), (6.9) by utilizing conformal perturbative approach.

¹⁵We denote ${}^\lambda\langle 0|\mathcal{O}^\lambda|0\rangle^\lambda$ as $\langle \mathcal{O}^\lambda \rangle_o^\lambda$ for any flowing operator $\mathcal{O}^\lambda$ on a cylinder.

6.2 Perturbative approach

According to (6.2), computing $\langle P_1^\lambda \rangle_o^\lambda$ amounts to compute the deformed one-point functions $\langle T^\lambda \rangle_o^\lambda$ and $\langle \Theta^\lambda \rangle_o^\lambda$. Thanks to our previous setup, we may obtain the one-point functions on cylinder by taking the zero temperature limit of the corresponding one-point functions on torus, namely,

$$\lim_{\beta \rightarrow \infty} \langle \mathcal{O}^\lambda \rangle_{\text{tor.}}^\lambda \equiv \lim_{\beta \rightarrow \infty} \left\{ \text{Tr} \left[e^{-\beta H^\lambda} \right]^{-1} \cdot \text{Tr} \left[e^{-\beta H^\lambda} \mathcal{O}^\lambda \right] \right\} = \langle \mathcal{O}^\lambda \rangle_o^\lambda. \quad (6.10)$$

In Lagrangian path integral formalism, $\langle \mathcal{O}^\lambda \rangle_{\text{tor.}}^\lambda$ equals to

$$\begin{aligned} & \langle \mathcal{O}^\lambda \rangle_{\text{tor.}}^\lambda \\ &= \frac{1}{\mathcal{Z}^\lambda} \int \mathcal{D}\phi \mathcal{O}^\lambda \exp \left\{ - \int_{\text{T}^2} \mathcal{L}^\lambda \right\} \\ &= \langle \mathcal{O}^{(0)} \rangle_{\text{tor.}} + \lambda \cdot \left\{ \langle \mathcal{O}^{(1)} \rangle_{\text{tor.}} + \langle \mathcal{O}^{(0)} \rangle_{\text{tor.}} \int_{\text{T}^2} \langle \mathcal{L}^{(1)} \rangle_{\text{tor.}} - \int_{\text{T}_1^2} \langle \mathcal{O}^{(0)} \mathcal{L}^{(1)}(z_1, \bar{z}_1) \rangle_{\text{tor.}} \right\} + O(\lambda^2), \end{aligned} \quad (6.11)$$

where $\mathcal{O}^\lambda = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathcal{O}^{(n)}$, $\langle \mathcal{O} \rangle_{\text{tor.}} \equiv \text{Tr} \left[e^{-\beta H} \right]^{-1} \cdot \text{Tr} \left[e^{-\beta H} \mathcal{O} \right]$. We then make use of (6.11) to calculate the $\langle T^\lambda \rangle_{\text{tor.}}^\lambda$ and $\langle \Theta^\lambda \rangle_{\text{tor.}}^\lambda$ of free bosons and free fermions respectively.

With the help of free propagators given in previous sections, after doing Wick contraction and simple integral on a torus, the final results are listed as follows. For free bosons, we get¹⁶

$$\langle T^\lambda \rangle_{\text{tor.FB}}^\lambda = \left(\eta_1 - \frac{\pi}{2\tau_2} \right) + \lambda \cdot \left(\frac{2|\eta_1|^2}{\pi} - \frac{1}{2\tau_2} (\eta_1 + \bar{\eta}_1) + \left(\frac{2}{\pi} \tau_2 \bar{\eta}_1 - 1 \right) i \partial_\tau \eta_1 \right) + O(\lambda^2), \quad (6.12)$$

$$\langle \Theta^\lambda \rangle_{\text{tor.FB}}^\lambda = \lambda \cdot \left(-\frac{|\eta_1|^2}{\pi} + \frac{2}{\tau_2} (\eta_1 + \bar{\eta}_1) - \frac{3\pi}{4\tau_2^2} \right) + O(\lambda^2). \quad (6.13)$$

For free Dirac fermions,

$$\langle T^\lambda \rangle_{\nu;\text{tor.DF}}^\lambda = -\frac{e_{\nu-1}}{2} + \lambda \cdot \left(\frac{|e_{\nu-1}|^2}{2\pi} + \frac{i\tau_2 \bar{e}_{\nu-1} \partial_\tau e_{\nu-1}}{2\pi} \right) + O(\lambda^2), \quad (6.14)$$

$$\langle \Theta^\lambda \rangle_{\nu;\text{tor.DF}}^\lambda = \lambda \cdot \frac{-|e_{\nu-1}|^2}{4\pi} + O(\lambda^2), \quad \nu = 2, 3, 4. \quad (6.15)$$

And for free Majorana fermions,

$$\langle T^\lambda \rangle_{\nu;\text{tor.MF}}^\lambda = -\frac{e_{\nu-1}}{4} + \lambda \cdot \left(\frac{|e_{\nu-1}|^2}{8\pi} + \frac{i\tau_2 \bar{e}_{\nu-1} \partial_\tau e_{\nu-1}}{8\pi} \right) + O(\lambda^2), \quad (6.16)$$

$$\langle \Theta^\lambda \rangle_{\nu;\text{tor.MF}}^\lambda = \lambda \cdot \frac{-|e_{\nu-1}|^2}{16\pi} + O(\lambda^2), \quad \nu = 2, 3, 4. \quad (6.17)$$

¹⁶For the definition of η_1 and $e_{\nu-1}$, please refer to appendix A.

Take the zero temperature limits of (6.12)–(6.17) respectively, one obtains¹⁷

$$\langle T^\lambda \rangle_{\text{o,FB}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle T^\lambda \rangle_{\text{tor.FB}}^\lambda = \frac{\pi^2}{6} + \lambda \cdot \frac{\pi^3}{18} + O(\lambda^2), \quad (6.18)$$

$$\langle \Theta^\lambda \rangle_{\text{o,FB}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle \Theta^\lambda \rangle_{\text{tor.FB}}^\lambda = \lambda \cdot \frac{-\pi^3}{36} + O(\lambda^2), \quad (6.19)$$

$$\langle T^\lambda \rangle_{\text{o,DF}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle T^\lambda \rangle_{\nu;\text{tor.DF}}^\lambda = \begin{cases} \frac{-\pi^2}{3} + \lambda \cdot \frac{2\pi^3}{9} + O(\lambda^2), & \nu = 2, \\ \frac{\pi^2}{6} + \lambda \cdot \frac{\pi^3}{18} + O(\lambda^2), & \nu = 3, 4, \end{cases} \quad (6.20)$$

$$\langle \Theta^\lambda \rangle_{\text{o,DF}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle \Theta^\lambda \rangle_{\nu;\text{tor.DF}}^\lambda = \begin{cases} \lambda \cdot \frac{-\pi^3}{9} + O(\lambda^2), & \nu = 2, \\ \lambda \cdot \frac{-\pi^3}{36} + O(\lambda^2), & \nu = 3, 4, \end{cases} \quad (6.21)$$

$$\langle T^\lambda \rangle_{\text{o,MF}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle T^\lambda \rangle_{\nu;\text{tor.MF}}^\lambda = \begin{cases} \frac{-\pi^2}{6} + \lambda \cdot \frac{\pi^3}{18} + O(\lambda^2), & \nu = 2, \\ \frac{\pi^2}{12} + \lambda \cdot \frac{\pi^3}{72} + O(\lambda^2), & \nu = 3, 4, \end{cases} \quad (6.22)$$

$$\langle \Theta^\lambda \rangle_{\text{o,MF}}^\lambda = \lim_{\tau_2 \rightarrow \infty} \langle \Theta^\lambda \rangle_{\nu;\text{tor.MF}}^\lambda = \begin{cases} \lambda \cdot \frac{-\pi^3}{36} + O(\lambda^2), & \nu = 2, \\ \lambda \cdot \frac{-\pi^3}{144} + O(\lambda^2), & \nu = 3, 4. \end{cases} \quad (6.23)$$

Note that for the fermion cases, $\nu = 2, 3, 4$ correspond to the periodic(space)-antiperiodic(time), antiperiodic-periodic and antiperiodic-antiperiodic sectors respectively. It means that for fermions on a cylinder of circumference unity with periodic B. C.

$$\langle T \rangle_{\text{o,DF}}^\lambda = \frac{-\pi^2}{3} + \lambda \cdot \frac{2\pi^3}{9} + O(\lambda^2), \quad \langle \Theta \rangle_{\text{o,DF}}^\lambda = -\lambda \cdot \frac{\pi^3}{9} + O(\lambda^2), \quad (6.24)$$

$$\langle T \rangle_{\text{o,MF}}^\lambda = \frac{-\pi^2}{6} + \lambda \cdot \frac{\pi^3}{18} + O(\lambda^2), \quad \langle \Theta \rangle_{\text{o,MF}}^\lambda = -\lambda \cdot \frac{\pi^3}{36} + O(\lambda^2), \quad (6.25)$$

and

$$\langle T \rangle_{\text{o,DF}}^\lambda = \frac{\pi^2}{6} + \lambda \cdot \frac{\pi^3}{18} + O(\lambda^2), \quad \langle \Theta \rangle_{\text{o,DF}}^\lambda = -\lambda \cdot \frac{\pi^3}{36} + O(\lambda^2), \quad (6.26)$$

$$\langle T \rangle_{\text{o,MF}}^\lambda = \frac{\pi^2}{12} + \lambda \cdot \frac{\pi^3}{72} + O(\lambda^2), \quad \langle \Theta \rangle_{\text{o,MF}}^\lambda = -\lambda \cdot \frac{\pi^3}{144} + O(\lambda^2), \quad (6.27)$$

for antiperiodic B.C.

With the help of (6.2), (6.18)–(6.19), and (6.24)–(6.27), we obtain the $T\bar{T}$ -flow of the KdV charge P_1 up to the first-order for three free theories

$$\langle P_1^\lambda \rangle_{\text{o,FB}}^\lambda = \frac{\pi}{12} + \lambda \cdot \frac{\pi^2}{72} + O(\lambda^2), \quad (\text{periodic B.C.}) \quad (6.28)$$

$$\langle P_1^\lambda \rangle_{\text{o,DF}}^\lambda = \frac{\pi}{12} + \lambda \cdot \frac{\pi^2}{72} + O(\lambda^2), \quad (\text{antiperiodic B.C.}) \quad (6.29)$$

$$\langle P_1^\lambda \rangle_{\text{o,MF}}^\lambda = \frac{\pi}{24} + \lambda \cdot \frac{\pi^2}{288} + O(\lambda^2), \quad (\text{antiperiodic B.C.}) \quad (6.30)$$

$$\langle P_1^\lambda \rangle_{\text{o,DF}}^\lambda = -\frac{\pi}{6} + \lambda \cdot \frac{\pi^2}{18} + O(\lambda^2), \quad (\text{periodic B.C.}) \quad (6.31)$$

$$\langle P_1^\lambda \rangle_{\text{o,MF}}^\lambda = -\frac{\pi}{12} + \lambda \cdot \frac{\pi^2}{72} + O(\lambda^2). \quad (\text{periodic B.C.}) \quad (6.32)$$

¹⁷Here we set the modular parameter $\tau = i\tau_2 = i\beta$.

It's easy to check that the perturbative results eq. (6.28)–(6.30) (eq. (6.31)–(6.32)) match the results come from non-perturbative method eq. (6.7) (eq. (6.9)).¹⁸ The discussions on whether these two approaches match each other at the second-order, which is technically involved, will be served as our future work.

7 Conclusion and discussion

In this work, we perturbatively calculate the flow effects of $T\bar{T}$ deformation on the torus partition functions and the VEV of the first KdV charge P_1 under the Lagrangian path integral formalism. In previous cases [45, 47, 48], the authors have studied the correlation functions perturbatively up to the first-order deformation. Generally speaking, to evaluate the correlation functions and higher ordered partition functions perturbatively, the flow of stress tensor must be taken into consideration. As a preliminary study, we focus on the discussions of free theories, including free bosons, Dirac fermions and Majorana fermions, where the flow of stress tensor can be constructed explicitly.¹⁹ In terms of Wick contraction, we first compute the first- (3.18), (4.26), (5.16) and the second-order (3.29), (4.36), (5.19) deformations to the partition functions, then we calculate a certain kind of 1-pt function (6.28)–(6.32), i.e. VEV of the first KdV charge. It turns out that the first two orders of the deformed partition functions and the first-order VEV of the first quantum KdV charge are in good agreement with the results obtained in Hamiltonian formalism [6, 37] respectively, provided we make minimum subtraction in dealing with the divergence.

Although the results obtained from the Hamiltonian formalism are reproduced in the Lagrangian path integral formalism, in general, due to the emergence of higher derivative terms in the deformed Lagrangian (3.4), (3.6), the equivalence between the Lagrangian path integral formalism and the Hamiltonian path integral formalism remains as a mystery. For instance, though Legendre transformation, it can be found that the Minkowski Hamiltonian of the deformed free bosons takes the form [43, 54]

$$\begin{aligned}\mathcal{H}^\lambda &= \frac{1}{2\lambda} \left(-1 + \sqrt{1 + 2\lambda(\pi^2 + \phi'^2) + 4\lambda^2(\pi\phi')^2} \right) \\ &= \frac{1}{2}(\pi^2 + \phi'^2) + \frac{\lambda}{4}(-\pi^4 + 2\pi^2\phi'^2 - \phi'^4) + \frac{\lambda^2}{4}(\pi^6 - \pi^4\phi'^2 - \pi^2\phi'^4 + \phi'^6) + \mathcal{O}(\lambda^3),\end{aligned}\tag{7.1}$$

where ϕ' is the spatial derivative of ϕ and π the canonical momentum conjugate to ϕ . The higher power terms of π presented in (7.1) prevent us from getting the Lagrangian path integral directly from the corresponding Hamiltonian path integral, since how to deal with the generic integrals go beyond Gaussian integrals, for now, is still a major problem for mathematicians and physicists. It also leads to an open question of whether the Hamiltonian formalism is more fundamental than the Lagrangian formalism [55–58]. Fortunately, our results show, for the $T\bar{T}$ deformed theory, the use of disk regularization [59] together with minimum subtraction in Lagrangian formalism seems to be sufficient to match the Hamiltonian formalism. For the instances considered in this paper, the second-order Lagrangian counterterms corresponding to the minimum subtraction are presented

¹⁸The center charges $c_{FB} = c_{DF} = 2c_{MF} = 1$.

¹⁹For the discussions of T and $\bar{T}$ flow in generic CFTs, please refer to [38, 53].

in (3.30), (4.37), (5.20) respectively. To match the partition functions and correlation functions between the Lagrangian formalism and the Hamiltonian formalism up to the higher-order deformations will be interesting future work.

Further, it will be interesting to study the second-order deformation to the partition function in the interacting theories, e.g. massive fermions and bosons, Liouville field theory [60], and so on. The generic correlation functions with the $T\bar{T}$ -flow effects in SUSY extended CFTs will be also an interesting future direction with the following [47].

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A Details of Weierstrass functions

In this appendix, we give the definitions and properties of Weierstrass functions that appear in the calculations.

We first note that, in our convention, torus (T^2) is defined by the identification on complex plane $z \sim z + 2w$ and $z \sim z + 2w'$ with $2w = 1$, $2w' = \tau = \tau_1 + i\tau_2$.

The first Weierstrass function $P(z)$, called *Weierstrass P-function*, is defined as

$$P(z) = \frac{1}{z^2} + \sum_{\{m,n\} \neq \{0,0\}} \left(\frac{1}{(z - \tilde{w})^2} - \frac{1}{\tilde{w}^2} \right), \quad \tilde{w} = 2mw + 2nw'. \quad (\text{A.1})$$

The Laurent series expansion of $P(z)$ in the neighborhood of $z = 0$ is

$$P(z) \sim \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \mathcal{O}(z^6), \quad (\text{A.2})$$

hence we have

$$\partial P(z) \sim -\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + \mathcal{O}(z^5), \quad \partial^2 P(z) \sim \frac{6}{z^4} + \frac{g_2}{10} + \frac{3g_3}{7}z^2 + \mathcal{O}(z^4), \quad (\text{A.3})$$

where g_2 and g_3 are called *Weierstrass Invariants*

$$g_2 := \sum_{\{m,n\} \neq \{0,0\}} \frac{60}{\tilde{w}^4}, \quad g_3 := \sum_{\{m,n\} \neq \{0,0\}} \frac{140}{\tilde{w}^6}. \quad (\text{A.4})$$

The second Weierstrass function $\zeta(z)$, called *Weierstrass zeta-function*, is a primitive function of $-P(z)$

$$\zeta(z) = \frac{1}{z} + \sum_{\{m,n\} \neq \{0,0\}} \left(\frac{1}{z - \tilde{w}} + \frac{1}{\tilde{w}} + \frac{z}{\tilde{w}^2} \right), \quad \partial \zeta(z) = -P(z). \quad (\text{A.5})$$

Figure 1. The regularized cell for $f(z, \bar{z})$ contains three singularities (red points). The grey part bounded by the solid lines is the regularized integral region.

We then define

$$\eta_1 := \zeta(w), \quad \eta_2 := \zeta(w'), \quad (\text{A.6})$$

and

$$e_1 := P(w), \quad e_2 := P(-w - w'), \quad e_3 := P(w'), \quad (\text{A.7})$$

which are functions of the modular parameter τ . Note that there is an identity about $\eta_1(\tau)$ and Dedekind eta function $\eta(\tau)$,

$$\frac{\partial_\tau \eta}{\eta} = \frac{i}{2\pi} \eta_1. \quad (\text{A.8})$$

which has been used in the bosonic calculations (3.17).

B Details of some integrations

B.1 Prescription for regularization

Since the integrands over a torus we are interested in may contain singularities, in this appendix we will discuss how to deal with these singularities based on the prescription given in [59].

Let us consider an integrand $f(z, \bar{z})$ defined on a torus, which contains N number of singularities $(r_1, r_2 \dots r_N)$. Following the prescription in [59], when performing the integrals, we integrate over not the whole torus T^2 , but over the regularized parallelogram — the parallelogram with small disks around the singularities removed (see figure 1 for example). In the following, we denote the regularized torus by T'^2 . Suppose we find that

$$f(z, \bar{z}) = \partial_\mu F^\mu(z, \bar{z}), \quad (\text{B.1})$$

then with the Stoke's theorem in 2D space²⁰

$$\int_\Sigma f(z, \bar{z}) d^2x = \frac{i}{2} \oint_{\partial\Sigma} (F^z d\bar{z} - F^{\bar{z}} dz), \quad (\text{B.2})$$

²⁰Since $z = x + iy$, $\int_\Sigma d^2x \equiv \int_\Sigma dx \wedge dy = \frac{i}{2} \int_\Sigma dz \wedge d\bar{z} \equiv \frac{i}{2} \int_\Sigma d^2z$.

which can be applied to the regularized torus leading to

$$\int_{T'^2} f(z, \bar{z}) d^2x = \frac{i}{2} \left[\oint_{\partial T^2} - \oint_{\partial D(\text{poles})} \right] (F^z d\bar{z} - F^{\bar{z}} dz), \quad (\text{B.3})$$

where the contour integrals are anticlockwise. In this paper, we focus further on the case that $F^\mu(z, \bar{z})$ can be written as $F^\mu(z, \bar{z}) = F_1^\mu(z) F_2^\mu(\bar{z})$, where F_1^μ is holomorphic function and F_2^μ is anti-holomorphic. For the j -th pole $(r_j, \bar{r}_j)$ of $f(z, \bar{z})$ in T^2 , $F^\mu(z, \bar{z})$ could be expanded around it as follows

$$F^\mu(z, \bar{z}) = \sum_m \sum_n C_{j,m}^{1,\mu} C_{j,n}^{2,\mu} (z - r_j)^m (\bar{z} - \bar{r}_j)^n, \quad (\text{B.4})$$

then

$$\begin{aligned} \oint_{|z-r_j|=\epsilon} (F^z d\bar{z} - F^{\bar{z}} dz) &= \int_0^{2\pi} \sum_m \sum_n C_{j,m}^{1,z} C_{j,n}^{2,z} (\epsilon e^{i\theta})^m (\epsilon e^{-i\theta})^n (-i\epsilon) e^{-i\theta} d\theta \\ &\quad - \int_0^{2\pi} \sum_m \sum_n C_{j,m}^{1,\bar{z}} C_{j,n}^{2,\bar{z}} (\epsilon e^{i\theta})^m (\epsilon e^{-i\theta})^n (i\epsilon) e^{i\theta} d\theta \\ &= -2\pi i \sum_n \epsilon^{2(n+1)} \left(C_{j,n}^{1,\bar{z}} C_{j,n+1}^{2,\bar{z}} + C_{j,n+1}^{1,z} C_{j,n}^{2,z} \right). \end{aligned} \quad (\text{B.5})$$

Therefore, on the grounds of the prescription in [59], we have

$$\begin{aligned} \int_{T^2} f(z, \bar{z}) d^2x &:= \int_{T'^2} f(z, \bar{z}) d^2x \\ &= \lim_{\epsilon \rightarrow 0} G(\epsilon) + \frac{i}{2} \oint_{\partial T^2} (F^z d\bar{z} - F^{\bar{z}} dz), \end{aligned} \quad (\text{B.6})$$

where

$$G(\epsilon) := -\pi \sum_{j,n} \epsilon^{2(n+1)} \left(C_{j,n}^{1,\bar{z}} C_{j,n+1}^{2,\bar{z}} + C_{j,n+1}^{1,z} C_{j,n}^{2,z} \right). \quad (\text{B.7})$$

It is worth noting that for the case of F^z holomorphic, meanwhile, $F^{\bar{z}}$ anti-holomorphic, it must have $\lim_{\epsilon \rightarrow 0} G(\epsilon) = 0$.

B.2 Integrals for bosonic fields

In this appendix we record the details of integrals appearing in the calculations of free bosons part (3.20)–(3.23).

Since all the integrands are double periodic, we can shift the variable of the integration to make life easier without changing the value of the integrals, i.e., $\int_{T_1^2} \int_{T_2^2} f(z_{12}, \bar{z}_{12}) = \tau_2 \int_{T^2} f(z, \bar{z})$ for double periodic function f .

We start with the integration of the P -function in a cell. Since $P(z) = -\frac{\partial\zeta(z)}{\partial z}$, with the integral strategy shown in appendix B.1, we have²¹

$$\begin{aligned}
\int_{T^2} d^2x P(z) &= -\int_{T^2} d^2x \frac{\partial\zeta(z)}{\partial z} = \frac{-i}{2} \oint_{\partial T'^2} \zeta(z) d\bar{z} \\
&= -\frac{i}{2} \left(\int_{z_0}^{z_0+2w} + \int_{z_0+2w}^{z_0+2w+2w'} + \int_{z_0+2w+2w'}^{z_0+2w'} + \int_{z_0+2w'}^{z_0} \right) \zeta(z) d\bar{z} \\
&= -\frac{i}{2} \int_{z_0}^{z_0+2w} (\zeta(z) - \zeta(z+2w')) + \frac{i}{2} \int_{z_0}^{z_0+2w'} d\bar{z} (\zeta(z) - \zeta(z+2w)) \\
&= 2i\bar{w}\zeta(w') - 2i\bar{w}'\zeta(w) = \pi - 2\tau_2\eta_1,
\end{aligned} \tag{B.8}$$

where we have used the identity

$$w'\zeta(w) - w\zeta(w') = \frac{i\pi}{2} \tag{B.9}$$

to eliminate $\zeta(w')$.

Next Let us consider $\int_{T^2} d^2x P(z)^2$. Since $P(z)^2$ is still a double periodic meromorphic function, we can expand $P(z)^2$ in terms of $\zeta(z)$ and its derivatives [61],

$$P(z)^2 = \frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z), \tag{B.10}$$

where the constant $\frac{g_2}{12}$ is fixed by comparing the constant terms of Laurent expansion of two functions, $P(z)^2$ and $\zeta^{(3)}(z)$, at zero. Then

$$\begin{aligned}
\int_{T^2} d^2x P(z)^2 &= \frac{g_2}{12}\tau_2 - \frac{1}{6} \int_{T^2} d^2x \zeta^{(3)}(z) \\
&= \frac{g_2}{12}\tau_2 - \frac{1}{6} \int_{T^2} d^2x \frac{\partial\zeta^{(2)}(z)}{\partial z} = \frac{g_2\tau_2}{12}.
\end{aligned} \tag{B.11}$$

We next turn to the integrand $|P(z)|^2$. Since $|P(z)|^2$ is no longer analytic, we can not expand it in terms of $\zeta(z)$ as what we did for $P(z)^2$. Instead, we will adopt the following

²¹In this case $\lim_{\epsilon \rightarrow 0} G(\epsilon) = 0$.

approach²²

$$\begin{aligned}
 \int_{\mathbb{T}^2} d^2x P(z) \bar{P}(\bar{z}) &= - \int_{\mathbb{T}^2} d^2x \partial(\zeta(z) \bar{P}(\bar{z})) \\
 &= - \lim_{\epsilon \rightarrow 0} G(\epsilon) - \frac{i}{2} \oint_{\partial \mathbb{T}^2} d\bar{z} \zeta(z) \bar{P}(\bar{z}) \\
 &= \lim_{\epsilon \rightarrow 0} \pi \epsilon^{-2} - \frac{i}{2} \left(\int_{z_0}^{z_0+2w} + \int_{z_0+2w}^{z_0+2w+2w'} + \int_{z_0+2w+2w'}^{z_0+2w'} + \int_{z_0+2w'}^{z_0} \right) \zeta(z) \bar{P}(\bar{z}) d\bar{z} \\
 &= 2i(\eta_1 \bar{\eta}_2 - \eta_2 \bar{\eta}_1) + \lim_{\epsilon \rightarrow 0} \pi \epsilon^{-2} = 4\tau_2 |\eta_1|^2 - 2\pi(\eta_1 + \bar{\eta}_1) + \lim_{\epsilon \rightarrow 0} \pi \epsilon^{-2}. \quad (\text{B.13})
 \end{aligned}$$

Note that the integration is divergent, which is consistent with the intuitive expectation to the integral process, since $|P(z)|^2 \sim \frac{1}{|z|^4}$ when z close to zero. We regularize it by simply subtracting the divergent part, that is, we set²³

$$\int_{\mathbb{T}^2} d^2x |P(z)|^2 = 4\tau_2 |\eta_1|^2 - 2\pi(\eta_1 + \bar{\eta}_1). \quad (\text{B.14})$$

Next consider the integrand $P(z)^2 \bar{P}(\bar{z}) = \frac{g_2}{12} \bar{P}(\bar{z}) - \frac{1}{6} \zeta^{(3)}(z) \bar{P}(\bar{z})$, where we can use (B.10) to rewrite it as follows

$$\begin{aligned}
 \int_{\mathbb{T}^2} d^2x P(z)^2 \bar{P}(\bar{z}) &= \frac{g_2}{12} \int_{\mathbb{T}^2} d^2x \bar{P}(\bar{z}) - \frac{1}{6} \int_{\mathbb{T}^2} d^2x \zeta^{(3)}(z) \bar{P}(\bar{z}) \\
 &= \frac{g_2}{12} (\pi - 2\bar{\eta}_1 \tau_2) - \frac{1}{6} \int_{\mathbb{T}^2} d^2x \partial(\zeta^{(2)}(z) \bar{P}(\bar{z})) \\
 &= \frac{g_2}{12} (\pi - 2\bar{\eta}_1 \tau_2) - \frac{i}{12} \oint_{\partial \mathbb{T}^2} \zeta^{(2)}(z) \bar{P}(\bar{z}) d\bar{z} - \frac{1}{6} \lim_{\epsilon \rightarrow 0} G(\epsilon) \\
 &= \frac{g_2}{12} (\pi - 2\bar{\eta}_1 \tau_2). \quad (\text{B.15})
 \end{aligned}$$

²²Here we have omitted the term $\int_{\mathbb{T}^2} d^2x \zeta(z) \partial \bar{P}(\bar{z})$, since for any holomorphic function $f(z)$ with poles r_i of order n_i and anti-holomorphic function $\bar{g}(\bar{z})$ with poles $\bar{p}_k$ of order m_k

$$\begin{aligned}
 \int_{\mathbb{T}^2} d^2x f(z) \partial \bar{g}(\bar{z}) &\sim \int_{\mathbb{T}^2} d^2x f(z) \sum_k (\bar{\partial}^{m_k-1} \delta^{(2)}(z - p_k)) \\
 &\sim \sum_k \int_{\mathbb{T}^2} d^2x (\bar{\partial}^{m_k-1} f(z)) \delta^{(2)}(z - p_k) \\
 &\sim \sum_{i,k} \int_{\mathbb{T}^2} d^2x (\partial^{n_i-1} \bar{\partial}^{m_k-2} \delta^{(2)}(z - r_i)) \delta^{(2)}(z - p_k) \\
 &\sim \sum_{i,k} \partial^{n_i-1} \bar{\partial}^{m_k-2} \delta^{(2)}(p_k - r_i). \quad (\text{B.12})
 \end{aligned}$$

For $f(z) = \zeta(z)$ and $\bar{P}(\bar{z}) = \bar{g}(\bar{z})$, the result is purely divergent thus we have dropped it. We have discarded the similar terms in later integrals.

²³In plane case, there is a similar divergence, which is moved out by dimensional regularization [47].

Finally, let us consider integration of $|P(z)|^4 = (\frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z))(\frac{\bar{g}_2}{12} - \frac{1}{6}\bar{\zeta}^{(3)}(\bar{z}))$,

$$\begin{aligned}
 & \int_{T^2} d^2x P(z)^2 \bar{P}(\bar{z})^2 \\
 &= \int_{T^2} d^2x \left(\frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z) \right) \left(\frac{\bar{g}_2}{12} - \frac{1}{6}\bar{\zeta}^{(3)}(\bar{z}) \right) \\
 &= \left| \frac{g_2}{12} \right|^2 \tau_2 - \frac{g_2}{72} \int_{T^2} d^2x \zeta^{(3)}(\bar{z}) - \frac{\bar{g}_2}{72} \int_{T^2} d^2x \zeta^{(3)}(z) + \frac{1}{36} \int_{T^2} d^2x \zeta^{(3)}(z) \bar{\zeta}^{(3)}(\bar{z}) \\
 &= \frac{|g_2|^2 \tau_2}{12^2} + \frac{1}{36} \int_{T^2} d^2x \partial(\zeta^{(2)}(z) \bar{\zeta}^{(3)}(\bar{z})) \\
 &= \frac{|g_2|^2 \tau_2}{12^2} + \frac{i}{72} \oint_{\partial T^2} \zeta^{(2)}(z) \bar{\zeta}^{(3)}(\bar{z}) d\bar{z} + \frac{1}{36} \lim_{\epsilon \rightarrow 0} G(\epsilon) \\
 &= \frac{|g_2|^2 \tau_2}{12^2} + \lim_{\epsilon \rightarrow 0} \frac{\pi}{3\epsilon^6}.
 \end{aligned} \tag{B.16}$$

Similar to the case (B.14), we regularize the integral by simply discarding the divergent part, which gives

$$\int_{T^2} d^2x |P(z)|^4 = \frac{|g_2|^2 \tau_2}{12^2}. \tag{B.17}$$

According to the results of (B.8), (B.11), (B.14), (B.15) and (B.17), we have

$$\int_{T_1^2} \int_{T_2^2} (B - P(z_{12})) = \tau_2^2 \left(\frac{\pi}{\tau_2} - 2\eta_1 \right) - \tau_2 (\pi - 2\tau_2 \eta_1) = 0, \tag{B.18}$$

$$\int_{T_1^2} \int_{T_2^2} (B - P(z_{12}))^2 = \tau_2 \int_{T^2} d^2x (B^2 + P(z)^2 - 2BP(z)) = \frac{g_2 \tau_2^2}{12} - \tau_2^2 B^2, \tag{B.19}$$

$$\int_{T_1^2} \int_{T_2^2} |B - P(z_{12})|^2 = \tau_2 \int_{T^2} d^2x (|B|^2 - B\bar{P}(\bar{z}) - \bar{B}P(z) + |P(z)|^2) = -\pi^2, \tag{B.20}$$

and

$$\begin{aligned}
 & \int_{T_1^2} \int_{T_2^2} |B - P(z_{12})|^4 \\
 &= \tau_2 \int_{T^2} d^2x \left(|B|^4 + |P(z)|^4 + 4|B|^2 |P(z)|^2 + (B^2 \bar{P}(\bar{z})^2 + \bar{B}^2 P(z)^2) \right. \\
 &\quad \left. - 2|B|^2 (B\bar{P}(\bar{z}) + \bar{B}P(z)) - 2(BP(z)\bar{P}(\bar{z}) + \bar{B}\bar{P}(z)P(z)^2) \right) \\
 &= \tau_2^2 |B|^4 + \frac{|g_2|^2 \tau_2^2}{12^2} - 4\tau_2^2 A^2 |B|^2 - B^2 \frac{\bar{g}_2 \tau_2^2}{12} - \bar{B}^2 \frac{g_2 \tau_2^2}{12}.
 \end{aligned} \tag{B.21}$$

B.3 Integrals for fermionic fields

In this appendix we present the details of integrals appearing in the calculations of free fermions part (4.29)–(4.33).

We first note that both $(\partial P_\nu(z))^2$ and $P_\nu(z)\partial^2 P_\nu(z)$ are elliptic functions with the modular parameter τ ,

$$(\partial P_\nu(z))^2 = \frac{(\partial P(z))^2}{4(P(z) - e_{\nu-1})}, \quad P_\nu(z)\partial^2 P_\nu(z) = \frac{1}{2}\partial^2 P(z) - \frac{(\partial P(z))^2}{4(P(z) - e_{\nu-1})}, \tag{B.22}$$

where $e_1 := P(w)$, $e_2 := P(w + w')$, $e_3 := P(w')$. Hence we can expand $(\partial P_\nu(z))^2$ and $P_\nu(z)\partial^2 P_\nu(z)$ in terms of $\zeta(z)$ and its derivatives, the results are

$$(\partial P_\nu(z))^2 = \frac{1}{6}\partial^2 P(z) + e_{\nu-1}P(z) + e_{\nu-1}^2 - \frac{g_2}{6}, \quad (\text{B.23})$$

$$P_\nu(z)\partial^2 P_\nu(z) = \frac{1}{3}\partial^2 P(z) - e_{\nu-1}P(z) - e_{\nu-1}^2 + \frac{g_2}{6}. \quad (\text{B.24})$$

Consequently, with the integral strategy shown in appendix B.1, the first two integrals

$$\begin{aligned} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} (\partial P_\nu(z_{12}))^2 &= \tau_2 \int_{\mathbb{T}^2} d^2x \left(\frac{1}{6}\partial^2 P(z) + e_{\nu-1}P(z) + e_{\nu-1}^2 - \frac{g_2}{6} \right) \\ &= \tau_2 e_{\nu-1}(\pi - 2\tau_2\eta_1) + \tau_2^2 \left(e_{\nu-1}^2 - \frac{g_2}{6} \right), \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned} \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} P_\nu(z_{12})\partial^2 P_\nu(z_{12}) &= \frac{\tau_2}{2} \int_{\mathbb{T}^2} d^2x \partial^2 P(z) - \int_{\mathbb{T}_1^2} \int_{\mathbb{T}_2^2} (\partial P_\nu(z_{12}))^2 \\ &= -\tau_2 e_{\nu-1}(\pi - 2\tau_2\eta_1) - \tau_2^2 \left(e_{\nu-1}^2 - \frac{g_2}{6} \right), \end{aligned} \quad (\text{B.26})$$

where we have utilized the integral²⁴

$$\int_{\mathbb{T}^2} d^2x \partial^2 P(z) = \lim_{\epsilon \rightarrow 0} G(\epsilon) + \frac{i}{2} \oint_{\partial \mathbb{T}^2} \partial P(z) d\bar{z} = 0 \quad (\text{B.27})$$

To compute the remaining three integrations, we need to work out the following integrals first

$$\begin{aligned} \int_{\mathbb{T}^2} d^2x \left| \partial^2 P(z) \right|^2 &= \int_{\mathbb{T}^2} d^2x \partial(\partial P(z) \bar{\partial}^2 \bar{P}(\bar{z})) \\ &= \frac{i}{2} \oint_{\partial \mathbb{T}^2} \partial P(z) \bar{\partial}^2 \bar{P}(\bar{z}) d\bar{z} + \lim_{\epsilon \rightarrow 0} G(\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} G(\epsilon) = \lim_{\epsilon \rightarrow 0} 12\pi\epsilon^{-6}. \end{aligned} \quad (\text{B.28})$$

In analogy with the bosonic case, in our regularization scheme, we simply drop out the divergent part to obtain the finite answer, that is,

$$\int_{\mathbb{T}^2} d^2x \left| \partial^2 P(z) \right|^2 = 0. \quad (\text{B.29})$$

Next consider the integrand $\bar{P}(\bar{z})\partial^2 P(z)$

$$\begin{aligned} \int_{\mathbb{T}^2} d^2x \bar{P}(\bar{z})\partial^2 P(z) &= \int_{\mathbb{T}^2} d^2x \partial(\bar{P}(\bar{z})\partial P(z)) \\ &= \frac{i}{2} \oint_{\partial \mathbb{T}^2} \bar{P}(\bar{z})\partial P(z) d\bar{z} + \lim_{\epsilon \rightarrow 0} G(\epsilon) \\ &= \frac{i}{2} \left(\int_{z_0}^{z_0+2w} + \int_{z_0+2w}^{z_0+2w+2w'} + \int_{z_0+2w+2w'}^{z_0+2w'} + \int_{z_0+2w'}^{z_0} \right) \bar{P}(\bar{z})\partial P(z) d\bar{z} \\ &= 0. \end{aligned} \quad (\text{B.30})$$

²⁴For the definition of $G(\epsilon)$, please refer to appendix B.1.

According to the results of (B.14), (B.25), (B.26), (B.29) and (B.30), we can evaluate the last three integrals now, which are listed in the following

$$\begin{aligned}
 & \int_{T_1^2} \int_{T_2^2} |\partial P_\nu(z_{12})|^4 \\
 &= \tau_2 \int_{T^2} d^2x \left| \frac{1}{6} \partial^2 P(z) + e_{\nu-1} P(z) + e_{\nu-1}^2 - \frac{g_2}{6} \right|^2 \\
 &= \tau_2 \int_{T^2} d^2x \left(\frac{1}{36} |\partial^2 P(z)|^2 + |e_{\nu-1}|^2 |P(z)|^2 + \frac{1}{6} (\bar{e}_{\nu-1} \bar{P}(\bar{z}) \partial^2 P(z) + e_{\nu-1} P(z) \bar{\partial}^2 \bar{P}(\bar{z})) \right. \\
 &\quad \left. + \left(\bar{e}_{\nu-1}^2 - \frac{\bar{g}_2}{6} \right) (\partial P_\nu(z))^2 + (e_{\nu-1}^2 - \frac{g_2}{6}) (\bar{\partial} \bar{P}_\nu(\bar{z}))^2 - \left| e_{\nu-1}^2 - \frac{g_2}{6} \right|^2 \right) \\
 &= \tau_2^2 \left| e_{\nu-1}^2 - \frac{g_2}{6} \right|^2 + |e_{\nu-1}|^2 (4\tau_2^2 |\eta_1|^2 - 2\pi\tau_2(\eta_1 + \bar{\eta}_1)) \\
 &\quad + \left(\tau_2 e_{\nu-1} \left(\bar{e}_{\nu-1}^2 - \frac{\bar{g}_2}{6} \right) (\pi - 2\tau_2 \eta_1) + \tau_2 \bar{e}_{\nu-1} \left(e_{\nu-1}^2 - \frac{g_2}{6} \right) (\pi - 2\tau_2 \bar{\eta}_1) \right), \tag{B.31}
 \end{aligned}$$

$$\begin{aligned}
 & \int_{T_1^2} \int_{T_2^2} \left| P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \right|^2 \\
 &= \int_{T_1^2} \int_{T_2^2} \left(\frac{1}{2} \partial^2 P(z_{12}) - (\partial P_\nu(z_{12}))^2 \right) \left(\frac{1}{2} \bar{\partial}^2 \bar{P}(\bar{z}_{12}) - (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 \right) \tag{B.32}
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{T_1^2} \int_{T_2^2} \left(\frac{1}{4} |\partial^2 P(z_{12})|^2 - \frac{1}{2} \partial^2 P(z_{12}) (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 - \frac{1}{2} \bar{\partial}^2 \bar{P}(\bar{z}_{12}) (\partial P_\nu(z_{12}))^2 \right. \\
 &\quad \left. + |\partial P_\nu(z_{12})|^4 \right) = \int_{T_1^2} \int_{T_2^2} |\partial P_\nu(z_{12})|^4, \tag{B.33}
 \end{aligned}$$

$$\begin{aligned}
 & \int_{T_1^2} \int_{T_2^2} (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) = \int_{T_1^2} \int_{T_2^2} (\bar{\partial} \bar{P}_\nu(\bar{z}_{12}))^2 \left(\frac{1}{2} \partial^2 P(z) - (\partial P_\nu(z_{12}))^2 \right) \\
 &= - \int_{T_1^2} \int_{T_2^2} |(\partial P_\nu(z_{12}))|^4. \tag{B.34}
 \end{aligned}$$

C Derivation of the counterterms

In this appendix, we present the derivations of the counterterms that appear in free bosons (3.30), free Dirac fermions (4.37), and free Majorana fermions (5.20). It's clear that in our cases all the counterterms are proportional to λ^2 , thus from the expression of the second-order correction of the partition function,

$$\mathcal{Z}^{(2)} = \mathcal{Z}^{(0)} \left(\int_{T_1^2} \int_{T_2^2} \langle \mathcal{L}^{(1)}(x_1) \mathcal{L}^{(1)}(x_2) \rangle - \int_{T^2} \langle \mathcal{L}^{(2)} \rangle - \frac{2}{\lambda^2} \int_{T^2} \langle \mathcal{L}_{ct} \rangle \right), \tag{C.1}$$

we know that $\int_{T^2} \langle \mathcal{L}_{ct} \rangle$ only need to cancel the divergent parts in $\frac{\lambda^2}{2} \int_{T_1^2} \int_{T_2^2} \langle \mathcal{L}^{(1)}(x_1) \mathcal{L}^{(1)}(x_2) \rangle$ since $\int_{T^2} \langle \mathcal{L}^{(2)} \rangle$ converges.

For free boson, we first rewrite the integrand (3.19) as

$$\langle T\bar{T}^{(0)}(z_1, \bar{z}_1) T\bar{T}^{(0)}(z_2, \bar{z}_2) \rangle = (|B|^2 + 2A^2) |P(z_{12})|^2 + \frac{1}{4} |P(z_{12})|^4 + \dots, \tag{C.2}$$

where “...” stands for terms giving finite integral results. As shown in (B.2), under the disk regularization (B.1), we have²⁵

$$\left(|B|^2 + 2A^2\right) \int_{T_1^2} \int_{T_2^2} |P(z_{12})|^2 = \pi\tau_2 \left(|B|^2 + 2A^2\right) \epsilon^{-2} + \text{convergent part}, \quad (\text{C.3})$$

$$\frac{1}{4} \int_{T_1^2} \int_{T_2^2} |P(z_{12})|^4 = \frac{\pi\tau_2}{12} \epsilon^{-6} + \text{convergent part}, \quad (\text{C.4})$$

then the divergent part in $\frac{\lambda^2}{2} \int_{T_1^2} \int_{T_2^2} \langle \mathcal{L}_{\text{FB}}^{(1)}(x_1) \mathcal{L}_{\text{FB}}^{(1)}(x_2) \rangle$ is given by

$$\begin{aligned} \frac{\tau_2 \lambda^2}{2\pi^3} \left(|B|^2 + 2A^2\right) \epsilon^{-2} + \frac{\tau_2 \lambda^2}{24\pi^3} \epsilon^{-6} \\ = \frac{8\tau_2 \lambda^2 g^2}{\pi \epsilon^2} \langle (\partial\phi)^2 (\bar{\partial}\phi)^2 \rangle + \frac{\tau_2 \lambda^2}{24\pi^3} \epsilon^{-6} \\ = \int_{T^2} \langle \mathcal{L}_{\text{FB,ct}} \rangle = \tau_2 \langle \mathcal{L}_{\text{FB,ct}} \rangle. \end{aligned} \quad (\text{C.5})$$

Finally, to implement the minimal subtraction, from (C.5) the following choice of counterterm is the simplest one

$$\mathcal{L}_{\text{FB,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi \epsilon^2} (\partial\phi \bar{\partial}\phi)^2 + \frac{1}{24\pi^3 \epsilon^6} \right\}, \quad (\text{C.6})$$

In the following, we would like to determine the counterterms of the deformed free Dirac fermion. Though the integrals given in appendix (B.3), we can find the divergent terms which we have omitted in the previous text (4.28),

$$\begin{aligned} \int_{T_1^2} \int_{T_2^2} \langle \mathcal{L}_{\text{DF}}^{(1)}(z_1, \bar{z}_1) \mathcal{L}_{\text{DF}}^{(1)}(z_2, \bar{z}_2) \rangle &= \frac{\tau_2}{12\pi^3 \epsilon^6} + \frac{\tau_2 |e_{\nu-1}|^2}{\pi^3 \epsilon^2} + \text{convergent part} \\ &= \frac{\tau_2}{12\pi^3 \epsilon^6} + \frac{16\tau_2 g^2}{\pi \epsilon^2} \langle \partial\psi^* \psi \bar{\partial}\bar{\psi}^* \bar{\psi} \rangle + \text{convergent part}. \end{aligned} \quad (\text{C.7})$$

According to (C.1), similar to the bosonic case, the simplest choice of Lagrangian counterterm for free Dirac fermions is

$$\mathcal{L}_{\text{DF,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi \epsilon^2} \partial\psi^* \psi \bar{\partial}\bar{\psi}^* \bar{\psi} + \frac{1}{24\pi^3 \epsilon^6} \right\}. \quad (\text{C.8})$$

Finally, we look at the free Majorana fermion. Similar to the case of Dirac fermion,

$$\begin{aligned} \frac{\lambda}{2} \int_{T_1^2} \int_{T_2^2} \langle \mathcal{L}_{\text{MF}}^{(1)}(z_1, \bar{z}_1) \mathcal{L}_{\text{MF}}^{(1)}(z_2, \bar{z}_2) \rangle &= \frac{\lambda^2 \tau_2}{8\pi^3 \epsilon^2} |e_{\nu-1}|^2 + \frac{\lambda^2 \tau_2}{96\pi^3 \epsilon^6} + \text{convergent part} \\ &= \lambda^2 \frac{8\tau_2 g^2}{\pi \epsilon^2} \langle \partial\psi \psi \bar{\partial}\bar{\psi} \bar{\psi} \rangle + \frac{\lambda^2 \tau_2}{96\pi^3 \epsilon^6} + \text{convergent part}, \end{aligned} \quad (\text{C.9})$$

then the simplest form of counterterm leading to minimal subtraction is

$$\mathcal{L}_{\text{MF,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi \epsilon^2} \partial\psi \psi \bar{\partial}\bar{\psi} \bar{\psi} + \frac{1}{96\pi^3 \epsilon^6} \right\}. \quad (\text{C.10})$$

²⁵Here ϵ represents the radius of the infinitesimal disk regulator.

D Derivation of the $T\bar{T}$ -flow for 2d fermions

In this appendix, we reproduce the derivation of the $T\bar{T}$ -flow for 2d fermionic theories as shown in [50].

The action of the un-deformed fermionic theory living in a 2d Euclidean flat spacetime is given by

$$\mathcal{L}^{(0)} = \frac{g}{2}(\bar{\Psi}\gamma^a\partial_a\Psi - \partial_a\bar{\Psi}\gamma^a\Psi) + V[\Psi]. \quad (\text{D.1})$$

One can rewrite it in a more general form, i.e., the form in curved spacetime, which is

$$\mathcal{L}^{(0)} = \frac{g}{2}(\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - \nabla_\mu\bar{\Psi}\gamma^\mu\Psi) + V = e^\mu_a X^a_\mu + V, \quad (\text{D.2})$$

where

$$X^a_\mu := \frac{g}{2}(\bar{\Psi}\gamma^a\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^a\Psi). \quad (\text{D.3})$$

X^a_μ is independent of the metric. We then utilize the recursion relation (2.5)–(2.6) to derive the expansion of $\mathcal{L}^\lambda$. First of all, the stress tensor of the un-deformed theory is²⁶

$$T_{ab}^{(0)} = e^\mu_a e^\nu_b \left(2 \frac{\partial \mathcal{L}^{(0)}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}^{(0)} \right) = 2e^\mu_a e^\nu_b \frac{\partial e^\lambda_c}{\partial g^{\mu\nu}} X^c_\lambda - \delta_{ab} \mathcal{L}^{(0)} = X_{(ab)} - \delta_{ab} \mathcal{L}^{(0)}. \quad (\text{D.4})$$

It is useful to introduce a new notation to mark the symmetrized tensor $\hat{X}_{ab} := X_{(ab)}$. Then according to (2.5)

$$\mathcal{L}^{(1)} = \frac{1}{2}(T^{a(0)}_a)^2 - \frac{1}{2}T^{a(0)}_b T^{b(0)}_a = \frac{1}{2}(\text{Tr}[\hat{X}]^2 - \text{Tr}[\hat{X}^2] + 2V\text{Tr}[\hat{X}] + 2V^2), \quad (\text{D.5})$$

from which we can derive $T_{ab}^{(1)}$, the resulting expression is

$$T_{ab}^{(1)} = 2e^\mu_a e^\nu_b \frac{\partial \mathcal{L}^{(1)}}{\partial g^{\mu\nu}} - \delta_{ab} \mathcal{L}^{(1)} = e^\mu_a e^\nu_b \left(\frac{\partial \text{Tr}[\hat{X}]^2}{\partial g^{\mu\nu}} - \frac{\partial \text{Tr}[\hat{X}^2]}{\partial g^{\mu\nu}} + 2V \frac{\partial \text{Tr}[\hat{X}]}{\partial g^{\mu\nu}} \right) - \delta_{ab} \mathcal{L}^{(1)}, \quad (\text{D.6})$$

where

$$\frac{\partial \text{Tr}[\hat{X}]^2}{\partial g^{\mu\nu}} = 2\text{Tr}[\hat{X}] \frac{\partial e^\lambda_c}{\partial g^{\mu\nu}} X^c_\lambda = \text{Tr}[\hat{X}] \hat{X}_{\mu\nu}, \quad (\text{D.7})$$

$$\frac{\partial \text{Tr}[\hat{X}^2]}{\partial g^{\mu\nu}} = \hat{X}^a_b \frac{\partial (e^{\lambda b} X_{a\lambda} + e^\lambda_a X^b_\lambda)}{\partial g^{\mu\nu}} = (\hat{X} \cdot X)_{(\mu\nu)}. \quad (\text{D.8})$$

Hence

$$T_{ab}^{(1)} = (\text{Tr}[\hat{X}] + V) \hat{X}_{ab} - (\hat{X} \cdot X)_{(ab)} - \delta_{ab} \mathcal{L}^{(1)}. \quad (\text{D.9})$$

We continue to evaluate $\mathcal{L}^{(2)}$

$$\mathcal{L}^{(2)} = T^{a(0)}_a T^{b(1)}_b - T^{a(0)}_b T^{b(1)}_a = \text{Tr}[\hat{X}^3] - \frac{3}{2}\text{Tr}[\hat{X}]\text{Tr}[\hat{X}^2] + \frac{1}{2}\text{Tr}[\hat{X}]^3 + V(\text{Tr}[\hat{X}]^2 - \text{Tr}[\hat{X}^2]), \quad (\text{D.10})$$

²⁶The formula $\frac{\partial e^\lambda_c}{\partial g^{\mu\nu}} = \frac{1}{4}(e_{\mu c}\delta^\lambda_\nu + e_{\nu c}\delta^\lambda_\mu)$ is used.

from which we finally obtain $T_{ab}^{(2)}$ as follows

$$T_{ab}^{(2)} = 2e^\mu_a e^\nu_b \frac{\partial}{\partial g^{\mu\nu}} \left(\text{Tr}[\hat{X}^3] - \frac{3}{2} \text{Tr}[\hat{X}] \text{Tr}[\hat{X}^2] + \frac{1}{2} \text{Tr}[\hat{X}]^3 + V(\text{Tr}[\hat{X}]^2 - \text{Tr}[\hat{X}^2]) \right) - \delta_{ab} \mathcal{L}^{(2)}, \quad (\text{D.11})$$

where

$$\frac{\partial \text{Tr}[\hat{X}^3]}{\partial g^{\mu\nu}} = 3 \frac{\partial \hat{X}_{ab}}{\partial g^{\mu\nu}} \hat{X}_{bc} \hat{X}_{ca} = \frac{3}{2} (\hat{X}^2 \cdot X)_{(\mu\nu)}, \quad (\text{D.12})$$

$$\frac{\partial \text{Tr}[\hat{X}]^3}{\partial g^{\mu\nu}} = 3 \text{Tr}[\hat{X}]^2 \frac{\partial \text{Tr}[\hat{X}]}{\partial g^{\mu\nu}} = \frac{3}{2} \text{Tr}[\hat{X}]^2 \hat{X}_{\mu\nu}. \quad (\text{D.13})$$

Therefore $T_{ab}^{(2)}$ is

$$\begin{aligned} T_{ab}^{(2)} &= 3(\hat{X}^2 \cdot X)_{(ab)} - (3\text{Tr}[\hat{X}] + 2V)(\hat{X} \cdot X)_{(ab)} \\ &\quad + \left(\frac{3}{2} \text{Tr}[\hat{X}]^2 - \frac{3}{2} \text{Tr}[\hat{X}^2] + 2V \text{Tr}[\hat{X}] \right) \hat{X}_{ab} - \delta_{ab} \mathcal{L}^{(2)}. \end{aligned} \quad (\text{D.14})$$

According to the nature of Grassmann variables, one actually could find two identities to reduce (D.10) and (D.14), that is

$$\text{Tr}[\hat{X}^3] - \frac{3}{2} \text{Tr}[\hat{X}] \text{Tr}[\hat{X}^2] + \frac{1}{2} \text{Tr}[\hat{X}]^3 \equiv 0, \quad (\text{D.15})$$

$$3(\hat{X}^2 \cdot X)_{(ab)} - 3\text{Tr}[\hat{X}](\hat{X} \cdot X)_{(ab)} + \frac{3}{2} (\text{Tr}[\hat{X}]^2 - \text{Tr}[\hat{X}^2]) \hat{X}_{ab} \equiv \mathbf{0}_{ab}, \quad (\text{D.16})$$

where $\mathbf{0}$ is the 2×2 null matrix. We present all reduced results as follows

$$\mathcal{L}^{(0)} = \text{Tr}[\hat{X}] + V, \quad (\text{D.17})$$

$$\mathcal{L}^{(1)} = \frac{1}{2} \text{Tr}[\hat{X}]^2 - \frac{1}{2} \text{Tr}[\hat{X}^2] + V \text{Tr}[\hat{X}] + V^2, \quad (\text{D.18})$$

$$\mathcal{L}^{(2)} = V(\text{Tr}[\hat{X}]^2 - \text{Tr}[\hat{X}^2]), \quad (\text{D.19})$$

$$T_{ab}^{(0)} = \hat{X}_{ab} - \delta_{ab} \mathcal{L}^{(0)}, \quad (\text{D.20})$$

$$T_{ab}^{(1)} = (\text{Tr}[\hat{X}] + V) \hat{X}_{ab} - (\hat{X} \cdot X)_{(ab)} - \delta_{ab} \mathcal{L}^{(1)}, \quad (\text{D.21})$$

$$T_{ab}^{(2)} = 2V \text{Tr}[\hat{X}] \hat{X}_{ab} - 2V(\hat{X} \cdot X)_{(ab)} - \delta_{ab} \mathcal{L}^{(2)}, \quad (\text{D.22})$$

where $\hat{X}_{ab}$ is

$$\hat{X}_{ab} = \frac{g}{2} (\bar{\Psi} \gamma_{(a} \partial_{b)} \Psi - \partial_{(a} \bar{\Psi} \gamma_{b)} \Psi). \quad (\text{D.23})$$

Although we can continue to calculate the higher-order corrections, as mentioned in [50], for the free massive fermions (i.e., $V[\Psi] = m\bar{\Psi}\Psi$), the $T\bar{T}$ -flow of $\mathcal{L}^\lambda$ terminates at the second-order.

The explicit forms of (D.17)–(D.22), for massive Dirac fermions, in complex coordinates are

$$T_{zz}^{(0)} = \frac{g}{2} \psi^* \overleftrightarrow{\partial} \psi, \quad T_{z\bar{z}}^{(0)} = -\frac{g}{4} (\psi^* \overleftrightarrow{\partial} \psi + \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) - \frac{m}{2} (\psi^* \bar{\psi} + \bar{\psi}^* \psi), \quad T_{\bar{z}\bar{z}}^{(0)} = \frac{g}{2} \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}, \quad (\text{D.24})$$

$$T_{zz}^{(1)} = \frac{g^2}{4} (\psi^* \psi (\bar{\partial} \psi^* \partial \psi + \partial \psi^* \bar{\partial} \psi) - (\psi^* \overleftrightarrow{\partial} \psi) \cdot (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi})) - \frac{gm}{2} \psi^* \psi (\bar{\psi}^* \partial \psi - \partial \psi^* \bar{\psi}), \quad (\text{D.25})$$

$$T_{z\bar{z}}^{(1)} = \frac{gm}{4} (\psi \bar{\psi} (\psi^* \bar{\partial} \psi^* - \bar{\psi}^* \partial \bar{\psi}^*) - \psi^* \bar{\psi}^* (\psi \bar{\partial} \psi - \bar{\psi} \partial \bar{\psi})) + m^2 \psi^* \psi \bar{\psi}^* \bar{\psi}, \quad (\text{D.26})$$

$$T_{\bar{z}\bar{z}}^{(1)} = \frac{g^2}{4} (\bar{\psi}^* \bar{\psi} (\partial \bar{\psi}^* \bar{\partial} \bar{\psi} + \bar{\partial} \bar{\psi}^* \partial \bar{\psi}) - (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) \cdot (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi})) - \frac{gm}{2} \bar{\psi}^* \bar{\psi} (\psi^* \bar{\partial} \bar{\psi} - \bar{\partial} \bar{\psi}^* \psi), \quad (\text{D.27})$$

$$T_{zz}^{(2)} = \frac{g^2 m}{2} \psi^* \psi \bar{\psi}^* \bar{\psi} (\partial \psi^* \bar{\partial} \bar{\psi} + \bar{\partial} \bar{\psi}^* \partial \psi), \quad T_{z\bar{z}}^{(2)} = 0, \quad T_{\bar{z}\bar{z}}^{(2)} = \frac{g^2 m}{2} \bar{\psi}^* \bar{\psi} \psi^* \psi (\bar{\partial} \bar{\psi}^* \partial \bar{\psi} + \partial \bar{\psi}^* \bar{\partial} \bar{\psi}). \quad (\text{D.28})$$

$$\mathcal{L}^{(0)} = g (\psi^* \overleftrightarrow{\partial} \psi + \bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) + m (\psi^* \bar{\psi} + \bar{\psi}^* \psi), \quad (\text{D.29})$$

$$\begin{aligned} \mathcal{L}^{(1)} = & \frac{g^2}{2} ((\psi^* \overleftrightarrow{\partial} \psi) (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) + (\psi^* \psi \bar{\partial} \bar{\psi}^* \partial \psi + \bar{\psi}^* \bar{\psi} \partial \bar{\psi}^* \partial \bar{\psi})) - g^2 (\psi^* \overleftrightarrow{\partial} \psi) (\bar{\psi}^* \overleftrightarrow{\partial} \bar{\psi}) \\ & - gm (\psi \bar{\psi} (\psi^* \bar{\partial} \bar{\psi}^* - \bar{\psi}^* \partial \bar{\psi}^*) - \psi^* \bar{\psi}^* (\psi \bar{\partial} \psi - \bar{\psi} \partial \bar{\psi})) - 2m^2 \psi^* \psi \bar{\psi}^* \bar{\psi}, \end{aligned} \quad (\text{D.30})$$

$$\mathcal{L}^{(2)} = g^2 m \psi^* \psi \bar{\psi}^* \bar{\psi} (2 \partial \psi^* \bar{\partial} \bar{\psi} + 2 \bar{\partial} \bar{\psi}^* \partial \bar{\psi} - \partial \bar{\psi}^* \partial \bar{\psi} - \bar{\partial} \bar{\psi}^* \bar{\partial} \bar{\psi}). \quad (\text{D.31})$$

Let $m = 0$, the above results degenerate to the results in section 4.

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