RECEIVED: May 8, 2021

REVISED: June 14, 2021

ACCEPTED: August 12, 2021

PUBLISHED: September 2, 2021

Intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and BKP hierarchy

Alexander Alexandrov

*Center for Geometry and Physics, Institute for Basic Science (IBS),
Pohang 37673, Korea*

E-mail: alexandrovsash@gmail.com

ABSTRACT: In their recent inspiring paper, Mironov and Morozov claim a surprisingly simple expansion formula for the Kontsevich-Witten tau-function in terms of the Schur Q-functions. Here we provide a similar description for the Brézin-Gross-Witten tau-function. Moreover, we identify both tau-functions of the KdV hierarchy, which describe intersection numbers on the moduli spaces of punctured Riemann surfaces, with the hypergeometric solutions of the BKP hierarchy.

KEYWORDS: 2D Gravity, Integrable Hierarchies, Matrix Models

ARXIV EPRINT: [2012.07573](https://arxiv.org/abs/2012.07573)

Dedicated to the memory of Sergey Mironovich Natanzon

Contents

1	Introduction	1
2	KW and BGW tau-functions in intersection theory	2
2.1	Intersection numbers and their generating functions	2
2.2	Heisenberg-Virasoro constraints and cut-and-join description	4
2.3	Generalized Brézin-Gross-Witten model	6
3	Q-Schur expansion of the KW and BGW tau-functions	7
4	BKP hierarchy	8
4.1	Symmetries of BKP	8
4.2	Hypergeometric solutions of BKP hierarchy	10
4.3	Miller-Morita-Mumford classes	12

1 Introduction

The Kontsevich-Witten (KW) tau-function [21, 38] of the KdV hierarchy plays a special role in modern mathematical physics. It is the basic building block for several universal constructions, including Chekhov-Eynard-Orantin topological recursion and Givental decomposition. This makes the Kontsevich-Witten tau-function one of the most well-studied tau-functions of the integrable solitonic hierarchies. However, unexpected new properties of this tau-function continue to surprise.

In their new paper [24] Mironov and Morozov came with a new idea. According to their conjecture, expansion of the Kontsevich-Witten tau-function in the basis of the Schur Q-functions Q_λ is unexpectedly simple:

$$\tau_{KW} = \sum_{\lambda \in \text{DP}} \left(\frac{\hbar}{16} \right)^{|\lambda|/3} \frac{Q_\lambda(\mathbf{t}) Q_\lambda(\delta_{k,1}) Q_{2\lambda}(\delta_{k,3}/3)}{Q_{2\lambda}(\delta_{k,1})}. \quad (1.1)$$

Here the summation runs over strict partitions.

The Brézin-Gross-Witten (BGW) model was introduced in lattice gauge theory 40 years ago [8, 14]. It has a natural enumerative geometry interpretation given by the intersection theory of Norbury's Θ -classes, also related to super Riemann surfaces [31, 32]. Moreover, the BGW tau-function is another fundamental element of topological recursion/Givental decomposition, which corresponds to the hard edge case [3, 9]. Similarly to the KW case, it is a tau-function of the KdV integrable hierarchy and can be described by the generalized Kontsevich model [27]. All this makes the BGW model interesting and, in many respects, similar to the KW tau-function.

In this paper we consider¹ a Schur Q-function expansion of the BGW tau-function. Similarly to the Mironov-Morozov formula for the KW tau-function, this expansion is described by simple coefficients made of specifications of the Schur Q-functions

$$\tau_{\text{BGW}} = \sum_{\lambda \in \text{DP}} \left(\frac{\hbar}{16} \right)^{|\lambda|} \frac{Q_{\lambda}(\mathbf{t}) Q_{\lambda}(\delta_{k,1})^3}{Q_{2\lambda}(\delta_{k,1})^2}. \quad (1.2)$$

These expansions lead to the question about the relation between KW and BGW tau-functions on one side, and the BKP hierarchy on the other. Indeed, Schur Q-functions are known to provide a natural basis for the expansion of the BKP tau-functions. Surprisingly enough, we found that KW and generalized BGW tau-functions after a simple rescaling of the times satisfy the BKP integrable hierarchy. Moreover, they belong to the family of *hypergeometric* BKP tau-functions, related to *spin Hurwitz numbers*.

While a non-linear relation between tau-functions of the KP and BKP hierarchies is well-known [11], the identification of KdV and BKP tau-functions is new and unexpected (see, however, [6]). In this paper we investigate only a few examples of such a relation.

After the first version of this paper was posted on the arXiv, the author has proven [4] that any solution of the KdV hierarchy also solves the BKP hierarchy, or symbolically

$$\text{KdV} \cap \text{BKP} = \text{KdV}. \quad (1.3)$$

More precisely, for all functions τ such that $\tau(\mathbf{t})$ is a tau-function of the KdV hierarchy, $\tau(\mathbf{t}/2)$ is a tau-function of the BKP hierarchy. Interpretation of this relation in terms of usual and orthogonal Sato Grassmannians is rather intriguing. This relation between two integrable hierarchies makes natural the Schur Q-function expansion of KdV tau-functions.

The present paper is organized as follows. In section 2 we remind the reader some basic elements of the intersection theory on the moduli spaces and its relation to the Kontsevich-Witten and Brézin-Gross-Witten tau-functions. Section 3 is devoted to Schur Q-function expansion of these tau-functions. In section 4 we describe the interpretation of these tau-functions as hypergeometric solutions of the BKP hierarchy.

2 KW and BGW tau-functions in intersection theory

2.1 Intersection numbers and their generating functions

Denote by $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford compactification of the moduli space of all compact Riemann surfaces of genus g with n distinct marked points. It is a non-singular complex orbifold of dimension $3g - 3 + n$. It is empty unless the stability condition

$$2g - 2 + n > 0 \quad (2.1)$$

is satisfied.

New directions in the study of $\overline{\mathcal{M}}_{g,n}$ were initiated by Witten in his seminal paper [38]. For each marking index i consider the cotangent line bundle $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{g,n}$, whose fiber

¹After the first version of this paper was posted on the arXiv, the proof of this expansion, based on the cut-and-join description and the symmetries of the BKP hierarchy, was given by the author in [5].

over a point $[\Sigma, z_1, \dots, z_n] \in \overline{\mathcal{M}}_{g,n}$ is the complex cotangent space $T_{z_i}^* \Sigma$ of Σ at z_i . Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the first Chern class of $\mathbb{L}_i$. We consider the intersection numbers

$$\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}. \quad (2.2)$$

The integral on the right-hand side of (2.2) vanishes unless the stability condition (2.1) is satisfied, all a_i are non-negative integers, and the dimension constraint

$$3g - 3 + n = \sum_{i=1}^n a_i \quad (2.3)$$

holds true. Let $T_i, i \geq 0$, be formal variables and let

$$\tau_{KW} := \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n} \right), \quad (2.4)$$

where

$$F_{g,n} := \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g \frac{\prod T_{a_i}}{n!}. \quad (2.5)$$

Witten's conjecture [38], proved by Kontsevich [21], states that the partition function τ_{KW} becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n+1)!! t_{2n+1}$.

On the same moduli space one can consider other types of intersection numbers. An interesting family of such intersection numbers was recently considered by Norbury [31]. Namely, he introduced Θ -classes, $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$, see [32, section 2] for the definition. Norbury also described their intersections with the ψ -classes

$$\langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g^\Theta = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}. \quad (2.6)$$

Again, the integral on the right-hand side vanishes unless the stability condition (2.1) is satisfied, all a_i are non-negative integers, and the dimension constraint

$$g - 1 = \sum_{i=1}^n a_i \quad (2.7)$$

holds true. Consider the generating function of the intersection numbers of Θ -classes and ψ -classes

$$F_{g,n}^\Theta = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g^\Theta \frac{\prod T_{a_i}}{n!} \quad (2.8)$$

then, we have a direct analog of the Kontsevich-Witten tau-function [31]:

Theorem (Norbury). *The generating function*

$$\tau_\Theta = \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n}^\Theta \right) \quad (2.9)$$

becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n+1)!! t_{2n+1}$.

Norbury also proved that τ_Θ is nothing but the tau-function of the Brézin-Gross-Witten model [8, 14]

$$\tau_\Theta = \tau_{\text{BGW}}. \quad (2.10)$$

We refer the reader to [31, 32] for a detailed description.

Both KW and BGW tau-functions can be described by matrix models. Consider a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_M)$. For any function f , dependent on the infinite set of variables $\mathbf{t} = (t_1, t_2, t_3, \dots)$, let

$$f\left([\Lambda^{-1}]\right) := f(\mathbf{t}) \Big|_{t_k = \frac{1}{k} \text{Tr } \Lambda^{-k}} \quad (2.11)$$

be the *Miwa parametrization*. The KW tau-function can be described by the Kontsevich matrix integral [21]

$$\tau_{\text{KW}}([\Lambda^{-1}]) := \mathcal{C}^{-1} \int [d\Phi] \exp \left(-\frac{1}{\hbar} \text{Tr} \left(\frac{\Phi^3}{3!} + \frac{\Lambda \Phi^2}{2} \right) \right). \quad (2.12)$$

Here one takes the asymptotic expansion of the integral over Hermitian $M \times M$ matrices. The BGW tau-function in the Miwa parametrization is given by the unitary matrix integral

$$Z_{\text{BGW}}([\Lambda^{-1}]) = \int [dU] e^{\frac{1}{2\hbar} \text{Tr} (A^\dagger U + A U^\dagger)}. \quad (2.13)$$

It also has another integral description [27], similar to the Kontsevich integral (2.12), see section 2.3 below. KdV integrability of the BGW model easily follows from this description.

2.2 Heisenberg-Virasoro constraints and cut-and-join description

The KW and BGW tau-functions are solutions of the KdV hierarchy, which is a reduction of the KP hierarchy. In terms of tau-function $\tau(\mathbf{t})$ it is described by the Hirota bilinear identity

$$\oint_\infty e^{\xi(\mathbf{t}-\mathbf{t}', z)} \tau(\mathbf{t} - [z^{-1}]) \tau(\mathbf{t}' + [z^{-1}]) dz = 0. \quad (2.14)$$

This bilinear identity encodes all nonlinear equations of the KP hierarchy. Here we use the standard short-hand notations

$$\mathbf{t} \pm [z^{-1}] := \left\{ t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \dots \right\} \quad (2.15)$$

and

$$\xi(\mathbf{t}, z) = \sum_{k \geq 0} t_k z^k. \quad (2.16)$$

If a tau-function $\tau(\mathbf{t})$ of the KP hierarchy does not depend on even time variables,

$$\frac{\partial}{\partial t_{2k}} \tau(\mathbf{t}) = 0 \quad \forall k > 0, \quad (2.17)$$

then it is a tau-function of the KdV hierarchy.

Symmetries of the KP hierarchy can be described in terms of a central extension of the $GL(\infty)$ group. Let us consider the *Heisenberg-Virasoro subalgebra* of the corresponding $gl(\infty)$ algebra. It is generated by the operators

$$\hat{J}_k = \begin{cases} \frac{\partial}{\partial t_k} & \text{for } k > 0, \\ 0 & \text{for } k = 0, \\ -kt_{-k} & \text{for } k < 0, \end{cases} \quad (2.18)$$

unit, and

$$\hat{L}_m = \frac{1}{2} \sum_{a+b=-m} abt_a t_b + \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}. \quad (2.19)$$

The KW and BGW tau-functions can be naturally described in terms of this Heisenberg-Virasoro algebra. Let us introduce the notation

$$\tau_1 = \tau_{KW}, \quad \tau_0 = \tau_{BGW}. \quad (2.20)$$

Below we assume that $\alpha \in \{0, 1\}$. Both KW and BGW tau-functions are solutions of the KdV hierarchy, hence they satisfy the Heisenberg constraints

$$\frac{\partial}{\partial t_{2k}} \tau_\alpha = 0, \quad k > 0. \quad (2.21)$$

The dimensional constraints (2.3) and (2.7) can be represented as

$$\hat{L}_0 \cdot \tau_\alpha = (1 + 2\alpha)\hbar \frac{\partial}{\partial \hbar} \tau_\alpha. \quad (2.22)$$

Moreover, these tau-functions satisfy the Virasoro constraints

$$\hat{L}_k^\alpha \cdot \tau_\alpha = 0, \quad k \geq -\alpha, \quad (2.23)$$

where the Virasoro operators are given by

$$\hat{L}_k^\alpha = \frac{1}{2} \hat{L}_{2k} - \frac{1}{2\hbar} \frac{\partial}{\partial t_{2k+1+2\alpha}} + \frac{\delta_{k,0}}{16}. \quad (2.24)$$

Following the ideas of [28] and combining (2.22) with (2.21) and (2.23) one gets [1, 2]

$$\frac{\partial}{\partial \hbar} \tau_\alpha = \widehat{W}_\alpha \cdot \tau_\alpha, \quad (2.25)$$

where

$$\widehat{W}_\alpha = \frac{1}{2\alpha + 1} \sum_{k=0}^{\infty} (2k + 1) t_{2k+1} \left(\hat{L}_{2k-2\alpha} + \frac{\delta_{k,\alpha}}{8} \right). \quad (2.26)$$

Operators $\widehat{W}_\alpha$ are called *cut-and-join* operators because of their similarity to the cut-and-join operator description of simple Hurwitz numbers [13, 37]. Below we will work with the

space of odd times only, therefore we can represent these operators as follows

$$\begin{aligned}\widehat{W}_0 &= \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(kmt_k t_m \frac{\partial}{\partial t_{k+m-1}} + \frac{1}{2}(k+m+1)t_{k+m+1} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1}{8}, \\ \widehat{W}_1 &= \frac{1}{3} \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(kmt_k t_m \frac{\partial}{\partial t_{k+m-3}} + \frac{1}{2}(k+m+3)t_{k+m+3} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1^3}{3!} + \frac{t_3}{8},\end{aligned}\quad (2.27)$$

where we denote by $\mathbb{Z}_{\text{odd}}^+$ the set of all positive odd integers. Equation (2.25) leads to the cut-and-join description for the KW and BGW tau-functions [1, 2]

$$\tau_\alpha = e^{\hbar \widehat{W}_\alpha} \cdot 1. \quad (2.28)$$

This description is convenient for perturbative computations, for example

$$\begin{aligned}\log \tau_0 &= \frac{t_1}{8} \hbar + \frac{t_1^2}{16} \hbar^2 + \left(\frac{t_1^3}{24} + \frac{9t_3}{128} \right) \hbar^3 \\ &\quad + \left(\frac{t_1^4}{32} + \frac{27t_3 t_1}{128} \right) \hbar^4 + \left(\frac{t_1^5}{40} + \frac{27t_3 t_1^2}{64} + \frac{225t_5}{1024} \right) \hbar^5 \\ &\quad + \left(\frac{t_1^6}{48} + \frac{45t_3 t_1^3}{64} + \frac{1125t_5 t_1}{1024} + \frac{567t_3^2}{1024} \right) \hbar^6 + O(\hbar^7), \\ \log \tau_1 &= \left(\frac{t_1^3}{6} + \frac{t_3}{8} \right) \hbar + \left(\frac{t_3 t_1^3}{2} + \frac{5t_5 t_1}{8} + \frac{3t_3^2}{16} \right) \hbar^2 \\ &\quad + \left(\frac{15t_1 t_3 t_5}{4} + \frac{3t_3^2 t_1^3}{2} + \frac{5t_5 t_1^4}{8} + \frac{35t_7 t_1^2}{16} + \frac{3t_3^3}{8} + \frac{105t_9}{128} \right) \hbar^3 + O(\hbar^4),\end{aligned}\quad (2.29)$$

$$(2.30)$$

The relation of these cut-and-join operators to the KdV integrability is not known yet. However, operator $\widehat{W}_0$ has a natural interpretation in terms of another integrable system, namely the BKP hierarchy, see section 4.1 below.

2.3 Generalized Brézin-Gross-Witten model

The *Generalized Brézin-Gross-Witten model* was introduced in [27] and further investigated in [1]. In the Miwa parametrization it is given by the asymptotic expansion of the matrix integral

$$\tau_{\text{BGW}}([\Lambda^{-1}], N) := \tilde{C}^{-1} \int [d\Phi] \exp \left(\frac{1}{2\hbar} \text{Tr} \left(\Lambda^2 \Phi + \Phi^{-1} + 2\hbar(N-M) \log \Phi \right) \right). \quad (2.31)$$

For $N = 0$ it reduces to the original BGW model, $\tau_{\text{BGW}}(\mathbf{t}, 0) = \tau_{\text{BGW}}(\mathbf{t})$. For arbitrary N , this is a tau-function of the KdV hierarchy. Moreover, the tau-functions for different values of the parameter N (do not confuse with M , the size of the matrices) are related to each other by the MKP hierarchy. The Virasoro constraints for the generalized BGW model can be derived with the help of the Kac-Schwarz approach [1]. It leads to the cut-and-join description [1, Lemma 3.4]

$$\tau_{\text{BGW}}(\mathbf{t}, N) = e^{\hbar \widehat{W}_0(N)} \cdot 1, \quad (2.32)$$

with the cut-and-join operator given by a deformation of $\widehat{W}_0$ in (2.27),

$$\widehat{W}_0(N) = \widehat{W}_0 - \frac{N^2}{2} t_1. \quad (2.33)$$

We expect that for arbitrary N this generalized model has interesting enumerative geometry interpretation, related to Norbury's Θ -classes.

3 Q-Schur expansion of the KW and BGW tau-functions

Schur Q-functions were introduced by Schur [35] for the description of the projective representations of the symmetric groups. These functions are labeled by *strict partitions*. A partition λ is strict, if $\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = 0$. We denote the set of strict partitions, including the empty one, by DP.

For the Schur Q-functions we use the same normalization as in [24, 25]. It is related to the one considered by Macdonald in section 3.8 of his book [22], by

$$Q_\lambda = 2^{-\ell(\lambda)/2} Q_\lambda^{\text{Mac}}. \quad (3.1)$$

In many aspects the Schur Q-functions are similar to the usual Schur functions [22]. For example, let us mention the *Cauchy formula*

$$\sum_{\lambda \in \text{DP}} Q_\lambda(\mathbf{t}) Q_\lambda(\mathbf{t}') = \exp \left(2 \sum_{k \in \mathbb{Z}_{\text{odd}}^+} k t_k t'_k \right), \quad (3.2)$$

and an analog of the standard hook formula [35]

$$Q_\lambda(\delta_{k,1}) = 2^{|\lambda| - \ell(\lambda)/2} \frac{1}{\prod_{j=1}^{\ell(\lambda)} \lambda_j!} \prod_{k < m} \frac{\lambda_k - \lambda_m}{\lambda_k + \lambda_m}. \quad (3.3)$$

Recently Mironov and Morozov [24] conjectured a simple expansion formula for the KW tau-function

$$\tau_{KW} = \sum_{\lambda \in \text{DP}} \left(\frac{\hbar}{16} \right)^{|\lambda|/3} \frac{Q_\lambda(\mathbf{t}) Q_\lambda(\delta_{k,1}) Q_{2\lambda}(\delta_{k,3}/3)}{Q_{2\lambda}(\delta_{k,1})}. \quad (3.4)$$

Only partitions with the weights divisible by 3 contribute to this expansion.

Remark 3.1. *This formula immediately follows [15] from Proposition (K') of Di Francesco, Itzykson and Zuber [12]. Indeed, the main step in its derivation, that is, formula (55) of [24] is nothing but a reformulation of this proposition which uses (3.3) and the identification of the f -functions of [12] with the Schur Q-functions [19].*

Let us suggest an analog of the Mironov-Morozov formula for the BGW tau-function:

Result 1.²

$$\tau_{\text{BGW}} = \sum_{\lambda \in \text{DP}} \left(\frac{\hbar}{16} \right)^{|\lambda|} \frac{Q_\lambda(\mathbf{t}) Q_\lambda(\delta_{k,1})^3}{Q_{2\lambda}(\delta_{k,1})^2}. \quad (3.5)$$

²When the first version of this paper was posted on the arXiv, this result was stated as a conjecture. This result is a particular case of the more general one (see Result 2 below), which was recently proved by the author [5].

Remark 3.2. One can argue that it is much more natural to consider the expansion of these tau-functions in the basis of Schur functions. For the KW tau-function this expansion was investigated already by Itzykson and Zuber in the early 90's [17]. The coefficients of this expansion, which can be described by the determinants of the affine coordinates on the Sato Grassmannian, are rather complicated [7, 40], and their relation to geometrical interpretation of the Kontsevich-Witten tau-function is not known yet. This type of expansion for the BGW tau-function is discussed in [41].

Formulas (3.4) and (3.5) are rather surprising. Indeed, as discussed in [24], Schur Q-functions are natural elements of the theory of the BKP hierarchy, but not KdV.³ In the next section we will show that there is a natural explanation of the appearance of the Schur Q-functions here. Using the cut-and-join description we prove that the generalized BGW model is described by the BKP hierarchy, see Theorem 1.

Remark 3.3. Another way to relate BKP with KdV (so-called 1-constrained BKP) was described in [10, 33]. To the best of our understanding, this reduction does not coincide with the one considered in this paper.

4 BKP hierarchy

The BKP hierarchy was introduced by Date, Jimbo, Kashiwara and Miwa in [11, 18]. It can be represented in terms of a tau-function $\tau(\mathbf{t})$ satisfying the BKP Hirota bilinear identity, similar to (2.14),

$$\frac{1}{2\pi i} \oint_{\infty} e^{\sum_{k \in \mathbb{Z}_{\text{odd}}^+} (t_k - t'_k) z^k} \tau(\mathbf{t} - 2[z^{-1}]) \tau(\mathbf{t}' + 2[z^{-1}]) \frac{dz}{z} = \tau(\mathbf{t}) \tau(\mathbf{t}'). \quad (4.1)$$

Here

$$\mathbf{t} \pm 2[z^{-1}] := \left\{ t_1 \pm 2z^{-1}, t_3 \pm \frac{2}{3}z^{-3}, t_5 \pm \frac{2}{5}z^{-5}, \dots \right\}. \quad (4.2)$$

4.1 Symmetries of BKP

The vertex operator

$$X(z, w) = \exp \left(\sum_{k \in \mathbb{Z}_{\text{odd}}^+} t_k (z^k - w^k) \right) \exp \left(-2 \sum_{k \in \mathbb{Z}_{\text{odd}}^+} \left(\frac{1}{kz^k} - \frac{1}{kw^k} \right) \frac{\partial}{\partial t_k} \right) \quad (4.3)$$

generates the (additional) symmetries of the BKP hierarchy [11]. In particular, the *Heisenberg-Virasoro subalgebra* of BKP symmetry algebra is generated by the operators

$$\widehat{J}_k^B = \begin{cases} 2 \frac{\partial}{\partial t_k} & \text{for } k > 0, \\ -kt_{-k} & \text{for } k < 0, \end{cases} \quad (4.4)$$

³After the first version of this paper was posted on the arXiv, the author has proved that any tau-function of the KdV hierarchy after a simple rescaling of the variables solves the BKP hierarchy [4]. This observation justifies the Schur Q-function expansion of the KW and BGW tau-functions.

for odd k and Virasoro operators

$$\widehat{L}_k^B = \frac{1}{2} \sum_{i+j=k} : \widehat{J}_i^B \widehat{J}_j^B : \quad (4.5)$$

for even k . Here the bosonic normal ordering puts all $\widehat{J}_m^B$ with positive m to the right of all $\widehat{J}_m^B$ with negative m . Let us also consider the $W^{(3)}$ -algebra, which is generated by

$$\widehat{M}_k^B = \frac{1}{3} \sum_{i+j+l=k} : \widehat{J}_i^B \widehat{J}_j^B \widehat{J}_l^B : \quad k \in \mathbb{Z}_{\text{odd}}. \quad (4.6)$$

Below we will need two of these generators, namely

$$\begin{aligned} \widehat{M}_{-1}^B &= \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(2kmt_k t_m \frac{\partial}{\partial t_{k+m-1}} + 4(k+m+1)t_{k+m+1} \frac{\partial^2}{\partial t_k \partial t_m} \right), \\ \widehat{M}_{-3}^B &= \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(2kmt_k t_m \frac{\partial}{\partial t_{k+m-3}} + 4(k+m+3)t_{k+m+3} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1^3}{3}. \end{aligned} \quad (4.7)$$

These generators are similar to the cut-and-join operators (2.27) and (2.33). Namely, after a transformation $t_k \mapsto t_k/2$ we have

$$\begin{aligned} \widehat{W}_0(N) &= \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(\frac{1}{2}kmt_k t_m \frac{\partial}{\partial t_{k+m-1}} + (k+m+1)t_{k+m+1} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \left(\frac{1}{16} - \frac{N^2}{4} \right) t_1, \\ \widehat{W}_1 &= \frac{1}{3} \sum_{k,m \in \mathbb{Z}_{\text{odd}}^+}^{\infty} \left(\frac{1}{2}kmt_k t_m \frac{\partial}{\partial t_{k+m-3}} + (k+m+3)t_{k+m+3} \frac{\partial^2}{\partial t_k \partial t_m} \right) + \frac{t_1^3}{2^3 3!} + \frac{t_3}{16}. \end{aligned} \quad (4.8)$$

Therefore we can identify

$$\begin{aligned} \widehat{W}_0(N) &= \frac{1}{4} \widehat{M}_{-1}^B + \left(\frac{1}{16} - \frac{N^2}{4} \right) t_1, \\ \widehat{W}_1 &= \frac{1}{12} \widehat{M}_{-3}^B - \frac{t_1^3}{2^4 \cdot 3^2} + \frac{t_3}{16}. \end{aligned} \quad (4.9)$$

We see that $\widehat{W}_0(N)$ for any N belongs the algebra of BKP symmetries. Then the cut-and-join representation (2.32) implies

Theorem 1. *Tau-function of the generalized BGW model $\tau_{\text{BGW}}(\mathbf{t}/2, N)$ is a tau-function of the BKP hierarchy.*

Proof. Operator $\widehat{W}_0(N)$ for any N belongs the algebra of BKP symmetries, therefore corresponding group operator $e^{\hbar \widehat{W}_0(N)}$ acting on any BKP tau-function yields a tau-function. The unit is a trivial tau-function of the BKP hierarchy, then the statement of the theorem follows from the cut-and-join formula (2.32). $\square$

Remark 4.1. *Let us note that one can construct another two-parameter family of the solutions of the BKP hierarchy, namely*

$$\tau(\mathbf{t}) = \exp \left(\hbar \left(\widehat{W}_1 + \frac{t_1^3}{2^4 \cdot 3^2} + bt_3 \right) \right) \cdot 1, \quad (4.10)$$

where b and $\hbar$ are arbitrary parameters.

4.2 Hypergeometric solutions of BKP hierarchy

There are several interesting families of BKP tau-functions described by Schur Q-functions. For example, Schur Q-functions provide all possible polynomial solutions of the BKP hierarchy [11, 20, 30, 39].

Remark 4.2. *For $N \in \mathbb{Z} + \frac{1}{2}$ the tau-function $\tau_{\text{BGW}}(\mathbf{t}, N)$ of the generalized BGW model is polynomial, given by a shifted Schur function for a triangular Young diagram [1],*

$$\tau_{\text{BGW}}(\mathbf{t}, N) \in \mathbb{C}[\mathbf{t}] \quad \forall N \in \mathbb{Z} + \frac{1}{2}. \quad (4.11)$$

From Theorem 1 it follows that after rescaling of times, $\tau_{\text{BGW}}(\mathbf{t}/2, N)$, these functions are also solutions of the BKP hierarchy. Hence [20] they are shifted Schur Q-functions.

The class of hypergeometric solutions of the BGW hierarchy was introduced by Orlov [34]. It was shown by Mironov, Morozov, and Natanzon [25] that they can be interpreted as generating functions of the *spin Hurwitz numbers*. Hypergeometric solutions of the BKP hierarchy are given by the following sums over strict partitions

$$\begin{aligned} \tau &= \sum_{\lambda \in \text{DP}} 2^{-\ell(\lambda)} r_\lambda Q_\lambda^{\text{Mac}}(\mathbf{t}/2) Q_\lambda^{\text{Mac}}(\mathbf{t}^*/2) \\ &= \sum_{\lambda \in \text{DP}} r_\lambda Q_\lambda(\mathbf{t}/2) Q_\lambda(\mathbf{t}^*/2), \end{aligned} \quad (4.12)$$

where

$$r_\lambda = \prod_{j=1}^{\ell(\lambda)} r(1)r(2) \dots r(\lambda_j) \quad (4.13)$$

for some $r(z)$. It is also convenient to introduce alternative parametrization by $\xi(n)$ for $n \in \mathbb{Z}^+$ with

$$e^{\xi(n)} := \prod_{j=1}^n r(j). \quad (4.14)$$

In this parametrization $r_\lambda = e^{\sum_{j=1}^{\ell(\lambda)} \xi(\lambda_j)}$.

We claim that the KW and generalized BGW tau-functions belong to this family. These observations are implied by the expansions (3.4) and (3.5). Let us start from the generalized BGW model.

Result 2.⁴ *The partition function of the generalized BGW model in the properly normalized times, $\tau_{\text{BGW}}(\mathbf{t}/2, N)$ is a hypergeometric tau-function of the BKP hierarchy (4.12), described by*

$$r(z) = \hbar \frac{(2z-1)^2 - 4N^2}{16}, \quad (4.15)$$

and $t_k^* = 2\delta_{k,1}$.

Let us stress that for the BGW model ($N = 0$) this result is equivalent to Result 1. Namely, from (3.3) we have

$$\frac{Q_\lambda(\delta_{k,1})}{Q_{2\lambda}(\delta_{k,1})} = \prod_{j=1}^{\ell(\lambda)} (2\lambda_j - 1)!!. \quad (4.16)$$

To describe the KW tau-function let us introduce

$$A_k = \prod_{j=1}^k \frac{(6j-1)(6j-5)}{16}. \quad (4.17)$$

Then for arbitrary $\beta \neq 0$ we define

$$\begin{aligned} e^{\xi_{KW}(3k)} &= \hbar^k A_k, \\ e^{\xi_{KW}(3k-1)} &= -\hbar^{k-1/3} \frac{2}{(6k-1)\beta} A_k, \\ e^{\xi_{KW}(3k-2)} &= \hbar^{k-2/3} \frac{8\beta}{(6k-1)} A_k. \end{aligned} \quad (4.18)$$

Result 3.⁵ *The KW tau-function in the properly normalized times, $\tau_{KW}(\mathbf{t}/2)$ is a hypergeometric tau-function of the BKP hierarchy, described by (4.12) with $r_\lambda^{KW} = e^{\sum_{j=1}^{\ell(\lambda)} \xi_{KW}(\lambda_j)}$ given by (4.18) and $t_k^* = \frac{2}{3}\delta_{k,3}$.*

This result is equivalent to combination of the result of Mironov and Morozov [24] and the relation

$$r_\lambda^{KW} Q_\lambda(\delta_{k,3}/3) = \left(\frac{\hbar}{16}\right)^{|\lambda|/3} \frac{Q_\lambda(\delta_{k,1})}{Q_{2\lambda}(\delta_{k,1})} Q_{2\lambda}(\delta_{k,3}/3) \quad \forall \lambda \in \text{DP}. \quad (4.19)$$

Let us remind the reader that $Q_\lambda(\delta_{k,3}/3)$ vanishes for all partitions with weight not divisible by 3. Moreover, it also vanishes for some partitions with the weight divisible by 3.

In Results 2 and 3 the variables t_k^* are taken at the points, associated with the corresponding dilaton shifts.

⁴When the first version of this paper was posted on the arXiv, this result was stated as a conjecture. After the first version of this paper was posted on the arXiv, this conjecture was proven by the author [5]. The proof is based on the cut-and-join description (2.32). Namely, the cut-and-join group element can be factorized using the diagonal group element, and this diagonal group element leads to the weight (4.15).

⁵When the first version of this paper was posted on the arXiv, this result was stated as a conjecture. After the first version of this paper was posted on the arXiv, relation (4.19) was proven by Mironov-Morozov-Natanzon-Orlov [26]. This result, together with the early results [12, 19], implies Result 3.

Remark 4.3. With John Stembridge’s Maple packages *SF* and *QF* [36] we have checked the statements of Results 1–3 perturbatively for $|\lambda| \leq 39$.

Results 2 and 3 rise numerous questions. In particular they lead to the relatively simple interpretation of the intersection theory on the moduli spaces by spin Hurwitz numbers (see [25] and references therein). Moreover, restoring the second set of times t^* one can consider natural deformations to the 2-component BKP hierarchy, which should describe a family of double spin Hurwitz numbers. The geometric interpretation of these families is not known yet. Let us also note that using Results 2 and 3 one can find explicit description of the $\tau_{KW}(\mathbf{t})$ and $\tau_{BGW}(\mathbf{t}, N)$ in terms of neutral fermions [16, 34]. These topics will be considered elsewhere.

4.3 Miller-Morita-Mumford classes

With the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ we define the *Miller-Morita-Mumford tautological classes* [29], $\kappa_k := \pi_* \psi_{n+1}^{k+1} \in H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. According to Manin and Zograf [23], insertion of these classes can be described by the translation of the times t_k responsible for the insertion of the ψ -classes. Translations are symmetries of BKP hierarchy, generated by (4.4), hence from Theorem 1 we have

Corollary 4.1. *Let us consider the generating function of the higher Θ -Weil-Petersson volumes*

$$\tau_{GWP}^\Theta(\mathbf{t}, \mathbf{s}) := \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} \mathcal{F}_{g,n}(\mathbf{t}, \mathbf{s}) \right), \quad (4.20)$$

where

$$\mathcal{F}_{g,n}(\mathbf{t}, \mathbf{s}) = \sum_{a_1, \dots, a_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} e^{\sum_{k=1}^{\infty} s_k \kappa_k} \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n} \frac{\prod (2a_i + 1)!! t_{2a_i+1}}{n!}. \quad (4.21)$$

For arbitrary values of the parameters $\mathbf{s}$ the function $\tau_{GWP}^\Theta(\mathbf{t}/2, \mathbf{s})$ is a tau-function of the BKP hierarchy in variables $\mathbf{t}$.

From Results 3 a direct analog of this corollary follows for the generating function $\tau_{GWP}(\mathbf{t}/2, \mathbf{s})$ for the case without Norbury’s Θ -classes.

Acknowledgments

The author is grateful to A. Mironov, A. Morozov, and A. Orlov for useful discussions and the anonymous referee for the suggested improvements. This work was supported by IBS-R003-D1.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] A. Alexandrov, *Cut-and-join description of generalized Brezin-Gross-Witten model*, *Adv. Theor. Math. Phys.* **22** (2018) 1347 [[arXiv:1608.01627](#)] [[INSPIRE](#)].
- [2] A. Alexandrov, *Cut-and-join operator representation for Kontsevich-Witten τ -function*, *Mod. Phys. Lett. A* **26** (2011) 2193 [[arXiv:1009.4887](#)] [[INSPIRE](#)].
- [3] A. Alexandrov, A. Mironov and A. Morozov, *BGWM as second constituent of complex matrix model*, *JHEP* **12** (2009) 053 [[arXiv:0906.3305](#)] [[INSPIRE](#)].
- [4] A. Alexandrov, *KdV solves BKP*, *Proc. Nat. Acad. Sci.* **118** (2021) e2101917118 [[arXiv:2012.10448](#)] [[INSPIRE](#)].
- [5] A. Alexandrov, *Generalized Brézin-Gross-Witten tau-function as a hypergeometric solution of the BKP hierarchy*, [arXiv:2103.17117](#) [[INSPIRE](#)].
- [6] K. Aokage, E. Shinkawa and H.-F. Yamada, *Pfaffian identities and Virasoro operators*, *Lett. Math. Phys.* **110** (2020) 1381.
- [7] F. Balogh and D. Yang, *Geometric interpretation of Zhou's explicit formula for the Witten-Kontsevich tau function*, *Lett. Math. Phys.* **107** (2017) 1837 [[arXiv:1412.4419](#)].
- [8] E. Brézin and D.J. Gross, *The external field problem in the large N limit of QCD*, *Phys. Lett. B* **97** (1980) 120 [[INSPIRE](#)].
- [9] L. Chekhov and P. Norbury, *Topological recursion with hard edges*, *Int. J. Math.* **30** (2019) 1950014 [[arXiv:1702.08631](#)] [[INSPIRE](#)].
- [10] Y. Cheng, *Constraints of the Kadomtsev-Petviashvili hierarchy*, *J. Math. Phys.* **33** (1992) 3774.
- [11] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations. 4. A new hierarchy of soliton equations of KP type*, *Physica D* **4** (1982) 343.
- [12] P. Di Francesco, C. Itzykson and J.B. Zuber, *Polynomial averages in the Kontsevich model*, *Commun. Math. Phys.* **151** (1993) 193 [[hep-th/9206090](#)] [[INSPIRE](#)].
- [13] I.P. Goulden and D.M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, *Proc. A.M.S.* **125** (1997) 51.
- [14] D.J. Gross and E. Witten, *Possible third order phase transition in the large N lattice gauge theory*, *Phys. Rev. D* **21** (1980) 446 [[INSPIRE](#)].
- [15] J. Harnad, private communication.
- [16] J. Harnad, J.W. van de Leur and A.Y. Orlov, *Multiple sums and integrals as neutral BKP tau functions*, *Theor. Math. Phys.* **168** (2011) 951 [[arXiv:1101.4216](#)] [[INSPIRE](#)].
- [17] C. Itzykson and J.B. Zuber, *Combinatorics of the modular group. 2. The Kontsevich integrals*, *Int. J. Mod. Phys. A* **7** (1992) 5661 [[hep-th/9201001](#)] [[INSPIRE](#)].
- [18] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, *Publ. Res. Inst. Math. Sci. Kyoto* **19** (1983) 943.
- [19] T. Józefiak, *Symmetric functions in the Kontsevich-Witten intersection theory of the moduli space of curves*, *Lett. Math. Phys.* **33** (1995) 347 [[INSPIRE](#)].
- [20] V. Kac and J. van de Leur, *Polynomial τ -functions of BKP and DKP hierarchies*, *J. Math. Phys.* **60** (2019) 071702 [[arXiv:1811.08733](#)] [[INSPIRE](#)].

- [21] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* **147** (1992) 1 [INSPIRE].
- [22] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford U.K. (1995).
- [23] Y.I. Manin and P. Zograf, *Invertible cohomological field theories and Weil-Peterson volumes*, [math/9902051](#) [INSPIRE].
- [24] A. Mironov and A. Morozov, *Superintegrability of Kontsevich matrix model*, *Eur. Phys. J. C* **81** (2021) 270 [[arXiv:2011.12917](#)] [INSPIRE].
- [25] A. Mironov, A. Morozov and S. Natanzon, *Cut-and-join structure and integrability for spin Hurwitz numbers*, *Eur. Phys. J. C* **80** (2020) 97 [[arXiv:1904.11458](#)] [INSPIRE].
- [26] A.D. Mironov, A.Y. Morozov, S.M. Natanzon and A.Y. Orlov, *Around spin Hurwitz numbers*, [arXiv:2012.09847](#) [INSPIRE].
- [27] A. Mironov, A. Morozov and G.W. Semenoff, *Unitary matrix integrals in the framework of generalized Kontsevich model. 1. Brezin-Gross-Witten model*, *Int. J. Mod. Phys. A* **11** (1996) 5031 [[hep-th/9404005](#)] [INSPIRE].
- [28] A. Morozov and S. Shakirov, *Generation of matrix models by W-operators*, *JHEP* **04** (2009) 064 [[arXiv:0902.2627](#)] [INSPIRE].
- [29] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, *Prog. Math.* **36** (1983) 271 [INSPIRE].
- [30] J.J.C. Nimmo, *Hall-Littlewood symmetric functions and the BKP equation*, *J. Phys. A* **23** (1990) 751.
- [31] P. Norbury, *A new cohomology class on the moduli space of curves*, [arXiv:1712.03662](#).
- [32] P. Norbury, *Enumerative geometry via the moduli space of super Riemann surfaces*, [arXiv:2005.04378](#) [INSPIRE].
- [33] A.Y. Orlov, *Volterra operator algebra for zero curvature representation. Universality of KP*, in *Nonlinear processes in physics*, A.S. Fokas et al. eds., Springer, Berlin, Germany (1993), p. 126.
- [34] A.Y. Orlov, *Hypergeometric functions related to Schur Q-polynomials and the BKP equation*, *Theor. Math. Phys.* **137** (2003) 1574.
- [35] J. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, *J. Reine Angew. Math.* **139** (1911) 155.
- [36] J. Stembridge, <http://www.math.lsa.umich.edu/~jrs/maple.html>.
- [37] R. Vakil, *Enumerative geometry of curves via degeneration methods*, Ph.D. thesis, Harvard University, Cambridge U.S.A. (1997).
- [38] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in *Surveys in differential geometry*, H. Blaine Lawson Jr. and S.-T. Yau eds., International Press of Boston, Boston U.S.A. (1991).
- [39] Y. You, *Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups*, *Adv. Ser. Math. Phys.* **7** (1989) 449.
- [40] J. Zhou, *Explicit formula for Witten-Kontsevich tau-function*, [arXiv:1306.5429](#) [INSPIRE].
- [41] J. Zhou, *Grothendieck's dessins d'enfants in a web of dualities*, [arXiv:1905.10773](#) [INSPIRE].