

EFT anomalous dimensions from the S-matrix

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ABSTRACT: We use the on-shell S-matrix and form factors to compute anomalous dimensions of higher dimension operators in the Standard Model Effective Field Theory. We find that in many instances, these computations are made simple by using the on-shell method. We first compute contributions to anomalous dimensions of operators at dimension-six that arise at one-loop. Then we calculate two-loop anomalous dimensions for which the corresponding one-loop contribution is absent, using this powerful method.

KEYWORDS: Beyond Standard Model, Effective Field Theories, Renormalization Group, Scattering Amplitudes

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1 Introduction

The discovery of the Higgs particle [1, 2], Higgs coupling measurements, and a plethora of beyond the Standard Model (SM) searches suggest that there is an energy gap between the electroweak scale and any new physics scale. Physics within this energy gap is appropriately described by the SM, together with a tower of operators encoding the deformations generated by the new dynamics. Given such separation of scales, the Renormalisation Group (RG) mixing of higher-dimension SM Effective Field Theory (EFT) operators can lead to important physical effects on precision observables. The RG running of SM EFT operators down from a hypothetical new physics scale to the EW scale is governed by the operator anomalous dimensions.

The anomalous dimension of an operator can be computed using a number of different methods. For instance, the S-matrix elements, the effective action, or form factors, are

subject to a Callan-Symanzik equation, a.k.a. renormalisation group equation (RGE), that depends on the anomalous dimensions. In this work we will compute the anomalous dimensions of SM dimension-six operators through the RGE's satisfied by the form factors.

The form factors (FFs) are defined by

$$F_O(\vec{n}) \equiv {}_{\text{out}} \langle \vec{n} | O(0) | 0 \rangle, \tag{1.1}$$

where $O(x)$ is a local and gauge invariant operator and ${}_{\text{out}} \langle \vec{n} |$ is a multi-particle asymptotic state. These are the same states that are used to define the scattering S -matrix elements

$$S_{nm} \equiv \langle \vec{n} | \hat{S} | \vec{m} \rangle = {}_{\text{out}} \langle \vec{n} | \vec{m} \rangle_{\text{in}}. \tag{1.2}$$

In perturbation theory, the calculation of the FFs involves regularising IR and UV divergences. For definiteness we will be using dimensional regularisation and the $\overline{\text{MS}}$ scheme so that the FF computed at a fixed order in perturbation theory will depend on the 't Hooft scale μ . However, it satisfies the Callan-Symanzik equation

$$(\mu \partial_\mu + \gamma - \gamma_{\text{IR}} + \beta_g \partial_g) F_O(\vec{n}; \mu) = 0, \tag{1.3}$$

where γ is the anomalous dimension matrix, γ_{IR} is the IR anomalous dimension, and β_g denotes collectively the beta function of the couplings in the theory. γ_{IR} is diagonal in the space of kinematically independent operators and it is due to soft and/or collinear emission of particles.

Next we derive a relation between the form-factors and the S-matrix. We need three ingredients:

- 1) Unitarity $F_O(\vec{n}) \equiv {}_{\text{out}} \langle \vec{n} | O(0) | 0 \rangle = \sum_{\vec{m}} {}_{\text{out}} \langle \vec{n} | \vec{m} \rangle_{\text{in}} \langle \vec{m} | O(0) | 0 \rangle$.
- 2) The CPT theorem implies ${}_{\text{out}} \langle \vec{n} | O(0) | 0 \rangle = \langle 0 | O^\dagger(0) | \vec{n} \rangle_{\text{in}}$.
- 3) Analyticity $F_O^*(s_{ij} - i\epsilon) = F_O(s_{ij} + i\epsilon)$.

Regarding 2), recall that CPT is a symmetry transformation that is represented on the Hilbert space by an anti-linear and anti-unitary operator. Thus it relates in/out states when inserted in the inner product, while acts on local operators as $\text{CPT } O(x) [\text{CPT}]^{-1} = O^\dagger(-x)$. Thus, point 2) follows from the statement ${}_{\text{out}} \langle \vec{n} | O(0) | 0 \rangle = \langle 0 | [\text{CPT}]^{-1} \text{CPT } O^\dagger(0) [\text{CPT}]^{-1} \text{CPT} | \vec{n} \rangle_{\text{in}}$.

As for 3), note that due to Lorentz invariance $F_O(\vec{n}(s_{ij} + i\epsilon)) \equiv F_O(s_{ij} + i\epsilon)$ must depend on the Mandelstam invariants $s_{ij} = 2p_i \cdot p_j$, for $i, j = 1, \dots, n$, which we collectively denote by $F_O(s_{ij} + i\epsilon)$, where $\epsilon > 0$ is a small positive parameter arising from the Feynman $i\epsilon$ -prescription. The real analyticity relation in 3) is definitely true in perturbation theory because complex conjugation of F_O amounts to complex conjugating the time-ordered propagators into anti-time-ordered propagators and thus $s_{ij} + i\epsilon \rightarrow s_{ij} - i\epsilon$. This full counter-clockwise rotation in the s_{ij} complex plane can be generated by a complex dilatation of the momenta. Indeed, defining the momentum space dilatation generator

$$D \equiv \sum_{\text{all particles}} p_i \frac{\partial}{\partial p_i}, \tag{1.4}$$

the relation in 3) can be written as $e^{-i\pi D} F_O^*(s_{ij} + i\epsilon) = F_O(s_{ij} + i\epsilon)$, which together with 1) and 2) implies

$$e^{-i\pi D} F_O^*(\vec{n}) = \sum_{\vec{m}} S_{nm} F_O^*(\vec{m}). \tag{1.5}$$

This equation was derived in [3] using a version of the optical theorem. The derivation of (1.5) that we present here though is similar to the proof of the two-dimensional generalised Watson theorem [4].¹ In two-dimensions complex conjugation in the rapidity plane θ acts as $F_O^*(\theta) = F_O(-\theta)$, where $\theta = |\theta_1 - \theta_2|$ is the rapidity difference of two incoming particles of momenta $\{p_1, p_2\}$, and $(p_1 + p_2)^2 = 4m^2 \cosh^2(\theta_1 - \theta_2)/2$. Then, by requiring only the extra assumption of a factorised two-dimensional S-matrix, equation (1.5) for $n = 2$ reduces to the generalised Watson theorem $F_O(\theta) = F_O(-\theta) S(\theta)$ [4], where $S(\theta)$ denotes the $2 \rightarrow 2$ scattering element and we are omitting flavour indices; this equation is exactly satisfied by many solvable two-dimensional QFTs.

For the purposes of computing UV anomalous dimensions, one can consider all particles as massless. Next, following the observation of [3], we relate the Callan-Symanzik equation (1.3) to the action of the momentum dilatation operator

$$DF_O \approx -\mu \partial_\mu F_O^{(1)} = (\gamma - \gamma_{\text{IR}} + \beta_g \partial_g)^{(1)} F_O^{(0)}. \tag{1.6}$$

Indeed, in dimensional regularisation there is a single renormalisation scale μ and we can trade derivatives of $\sum_i p_i \partial_{p_i}$ by $-\mu \partial_\mu$. Thus, using (1.5), (1.6) and expanding in powers of D at first non-trivial order one gets [3]

$$\langle \vec{n} | O_j | 0 \rangle^{(0)} (\gamma_{ji} - \gamma_{\text{IR}}^i \delta_{ij})^{(1)} = -\frac{1}{\pi} \langle \vec{n} | \mathcal{M} \otimes O_i | 0 \rangle^{(0)}, \tag{1.7}$$

where as usual we have separated the S-matrix as $S = \mathbb{1} + i\mathcal{M}$. The first factor on the left $\langle \vec{n} | O_j | 0 \rangle^{(0)}$ is a minimal form factor, namely the lowest order FF such that does not vanish in the free theory limit. The minimal FF is a polynomial in the kinematic variables: the momenta and polarisations of the particles. The action of $\beta_g \partial_g$ can be neglected because $\langle \vec{n} | O_j | 0 \rangle^{(0)}$ does not depend on the couplings of the theory. On the right hand side, ‘ \otimes ’ denotes an insertion of $\mathbb{1} = \sum_{\vec{m}} |\vec{m}\rangle_{\text{in}} \langle \vec{m}|$ as in 2). The \mathcal{M} -operator and the form factor on the right are consistently computed in perturbation theory so that it leads to a polynomial function like the l.h.s. minimal FF. For instance, note that if the leading single log μ in the FF of the l.h.s. in (1.5) arises at ℓ -loops, then applying $-\mu \partial_\mu$ as in (1.6) extracts the anomalous dimension at ℓ -loop order. Thus ‘(1)’ in (1.7) indicates the coefficient of the leading single log μ , which typically (but not necessarily) arises at one-loop. From here on we will neglect the super-indices $(\dots)^{(n)}$, $n = 0, 1$.

In section 2 we will discuss the method and compute representative one-loop anomalous dimensions of the SM dimension-six operators, and compare the results with previous literature. In section 3 we begin by summarising the pattern of vanishing entries in the matrix of one-loop anomalous dimensions. This allows us to identify those two-loop anomalous dimensions that are most interesting, namely those two-loop RG mixings that vanish at

¹Watson’s theorem [5, 6] is e.g. reviewed in [7], formulated as a relation between components of the S-matrix.

one-loop. These can be efficiently obtained with the on-shell methodology and we compute two-loop anomalous dimensions of this type in section 4. Finally we summarise our results and conclude in section 5.

2 One-loop RGEs of dimension-six operators

In this section we apply (1.7) in a series of examples of one-loop RGE mixing of the SM dimension-six operators. The one loop contributions to the right hand side of (1.7) involve a tree-level form factor and a tree-level S-matrix element, that are contracted with a two-particle phase space integral. Note that the scattering matrix in the right hand side of (1.5) is not the fully connected S-matrix. Thus we should also consider contributions that involve a scattering matrix with disconnected pieces

$$\langle 1, \dots, n | O_j | 0 \rangle (\gamma_{ji} - \gamma_{\text{IR}}^i \delta_{ij}) = \sum_{k=2}^n \langle 1, \dots, k | i\mathcal{T} | 0 \rangle \langle k+1, \dots, n | O_i | 0 \rangle, \quad (2.1)$$

where the lines to the right of the last FF are the disconnected particles of the S-matrix.² The dots in (2.1) denote the sum of any possible permutation of the external particles, and higher loop contributions.

For the purposes of computing the UV anomalous dimensions, one can consider all SM particles as massless, or equivalently consider the SM in the symmetric phase of $SU(2)_L$. Then, since all the four-momenta involved in the problem (2.1) is null $p^2 = 0$, it is convenient to use spinor helicity formalism [8–10]. An on-shell massless spin s particle is characterised by its helicity $h = \pm|s|$ and a pair of spinors $\{\lambda_a, \tilde{\lambda}_{\dot{a}}\}$. The spinors $\{\lambda_a, \tilde{\lambda}_{\dot{a}}\}$ transform in the representations $(1/2, 0)$ and $(0, 1/2)$ of the universal cover of the Lorentz group $SL(2, \mathbb{C})$, respectively. The basic invariant tensors are the antisymmetric Levi-Civita ϵ^{ab} , $\epsilon^{\dot{a}\dot{b}}$, and $\sigma^\mu = (1, \sigma^i)$ with σ^i the Pauli matrices. Given two particles with spinors $\{(\lambda_i)_a, (\tilde{\lambda}_i)_{\dot{a}}\}$ where $i = 1, 2$, two simple Lorentz invariants are

$$(\lambda_1)_a (\lambda_2)_b \epsilon^{ab} \equiv \langle 12 \rangle \quad \text{and} \quad (\tilde{\lambda}_1)_{\dot{a}} (\tilde{\lambda}_2)_{\dot{b}} \epsilon^{\dot{a}\dot{b}} \equiv [12], \quad (2.2)$$

where we have introduced the commonly used angle and square bracket notation. Another basic invariant is $(p_i + p_j)^2 = \langle ij \rangle [ji] \equiv s_{ij}$, where we are using mostly minus $\eta = (+---)$ metric signature. Our convention is $p^\mu = \lambda^a (\sigma^\mu)_{a\dot{a}} \tilde{\lambda}^{\dot{a}}$, and as usual we consider λ and $\tilde{\lambda}$ as independent complex spinors (i.e. two copies of $SL(2, \mathbb{C})$), however for real physical momenta (with positive energy) we should identify $\tilde{\lambda} = +\lambda^*$. See for instance [11, 12] for further details on the spinor helicity formalism and on-shell scattering amplitudes techniques.

As usual, we factor out a Dirac delta of total momenta conservation from the interacting and connected $i\mathcal{M}$ matrix

$$\langle 12 \dots k | i\mathcal{M} | 1' 2' \dots k' \rangle = (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^k p_i - \sum_{i=1}^{k'} p'_i \right) iM(1 \dots k; 1' \dots k'), \quad (2.3)$$

²Indeed, from $\int \frac{dL_{i'}}{2\pi} \langle 1, \dots, n | i\mathcal{M} | 1', 2', 3', \dots, k' \rangle \langle 1', 2', 3', \dots, k' | O_i | 0 \rangle$ pick a connected $2 \rightarrow k$ scattering element to reduce it to $\int \frac{dL_{i'}}{2\pi} \langle 1, \dots, k | i\mathcal{M} | 1', 2' \rangle \langle 1', 2', k+1, \dots, n | O_i | 0 \rangle$. The particles to the right of the last FF in (2.1) may involve tree-level interactions of the disconnected S-matrix, but we have absorbed those into the FF, such that both the S-matrix and the FF are connected.

$O_{\parallel} = |H|^2 |D^{\mu} H|^2$ and $O_{\perp} = |H^{\dagger} D_{\mu} H|^2$. Their minimal FFs with a four particle state are

$$F_{\parallel}(1_i 2_j 3_k^* 4_l^*) = \delta_i^l \delta_j^k \langle 14 \rangle [14] + \delta_j^l \delta_i^k \langle 13 \rangle [13] = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array}, \quad (2.8)$$

$$F_{\perp}(1_i 2_j 3_k^* 4_l^*) = \delta_i^l \delta_j^k \langle 13 \rangle [13] + \delta_j^l \delta_i^k \langle 14 \rangle [14] = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array}, \quad (2.9)$$

where we have used momentum conservation $\sum_{i=1}^4 p_i = 0$.³ The two operators are in correspondence with the two possible ways to contract the momenta (solid line in (2.8), (2.9)) of two particles with two antiparticles that carry a flavour index (dotted line).

Next we need to connect the FFs in (2.8), (2.9) with the matrix element (2.7) in the various possible ways or channels

$$2 \begin{array}{c} 1_i \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 2_j \end{array} \begin{array}{c} 1'_{\alpha} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 2'_{\beta} \end{array} \begin{array}{c} -3_k \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ -4_l \end{array} + 4 \begin{array}{c} 1_i \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 3_k^* \end{array} \begin{array}{c} 3'_{\beta} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 1'_{\alpha} \end{array} \begin{array}{c} -4_l^* \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ -2_j \end{array} + 4 \begin{array}{c} 1_i \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 4_l^* \end{array} \begin{array}{c} 4'_{\beta} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 1'_{\alpha} \end{array} \begin{array}{c} -3_k^* \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ -2_j \end{array} \quad (2.10)$$

where from here on we represent on-shell scalar particle lines with a dashed line, a crossed dot denotes an insertion of the operator and a grey blob the \mathcal{M} -matrix. Regarding the multiplicative factor in each channel, note that a factor of two is due to crossing,⁴ while the last two channels get an extra factor of two because $|H_a, H_b^*\rangle$ and $|H_b, H_a^*\rangle$ are distinct states.

With this it is straightforward to compute the various channels. Starting with the F_{\perp} form factor, the first channel gives

$$-\frac{1}{16\pi^2} \int d\Omega_2 [-2\lambda(\delta_i^{\alpha} \delta_j^{\beta} + \delta_j^{\alpha} \delta_i^{\beta})] [\delta_{\alpha}^l \delta_{\beta}^k \langle 1'3 \rangle [1'3] + \delta_{\beta}^l \delta_{\alpha}^k \langle 1'4 \rangle [1'4]], \quad (2.11)$$

where $d\Omega_2$ is the angular integration in (2.6). The spinor $\langle 1'|$ is rotated according to (2.5), which gives $\langle 1'i \rangle = \cos\theta \langle 1i \rangle - \sin\theta e^{i\phi} \langle 2i \rangle$. The ϕ integral sets to zero the terms with a phase $e^{i\phi}$, thus $\int d\Omega_2 \langle 1'i \rangle [1'i] = \int d\Omega_2 (\cos^2\theta \langle 1i \rangle [1i] + \sin^2\theta \langle 2i \rangle [2i])$ for $i = 3, 4$. After doing the θ integral we get

$$(2.11) = \frac{2\lambda}{16\pi^2} [\delta_i^l \delta_j^k + \delta_j^l \delta_i^k] [\langle 13 \rangle [13] + \langle 14 \rangle [14]]. \quad (2.12)$$

The second and third channels in (2.10) are done in the same way. The second one gives

$$-\frac{1}{16\pi^2} \int d\Omega_2 M(1_i 3_k^*; 1'_{\alpha} 3'_{\beta}) F_{\perp}(1'_{\alpha} 2_j 3'_{\beta} 4_l^*) = -\frac{\lambda \langle 13 \rangle [13]}{16\pi^2} [\delta_i^k \delta_j^l (N-1) - 2\delta_i^l \delta_j^k], \quad (2.13)$$

where $N = 2$ is the dimension of the fundamental representation of the Higgs doublet. In this case we rotate $(|1'\rangle, |3'\rangle) = R(\theta, \phi) \cdot (|1\rangle, |3\rangle)$. Finally the third channel

$$-\frac{1}{16\pi^2} \int d\Omega_2 M(1_i 4_l^*; 1'_{\alpha} 4'_{\beta}) F_{\perp}(1'_{\alpha} 2_j 3_k 4'_{\beta}) = -\frac{\lambda \langle 14 \rangle [14]}{16\pi^2} [\delta_i^l \delta_j^k (N-1) - 2\delta_i^k \delta_j^l], \quad (2.14)$$

³FFs in momentum space, $F_O(p_1 \dots p_n; q) \equiv \int d^4x e^{iqx} F_O(p_1, \dots, p_n)$, have the support of the four-momentum conservation delta function $\delta^{(4)}(q + \Sigma_i p_i)$. We will indistinguishably refer to $F_O(p_1 \dots p_n; 0)$ by $F_O(p_1 \dots p_n)$.

⁴I.e. one gets the same result when exchanging $\{1, 2\} \leftrightarrow \{3, 4\}$, $\{1, 3\} \leftrightarrow \{2, 4\}$ and $\{1, 4\} \leftrightarrow \{2, 3\}$ in the first second and third channel respectively.

can be obtained from the second channel upon exchanging $3 \leftrightarrow 4$ and $k \leftrightarrow l$ on (2.13). After adding all the contributions, $2 \cdot$ (2.12) + $4 \cdot$ (2.13) + $4 \cdot$ (2.14), we get

$$\frac{4\lambda}{16\pi^2} [3 F_{\perp}(1_i 2_j 3_k^* 4_l^*) + (2 - N) F_{\parallel}(1_i 2_j 3_k^* 4_l^*)]. \quad (2.15)$$

Thus for a SM Higgs doublet, we have that $N = 2$ and the renormalisation is diagonal.

Proceeding in a similar fashion for the F_{\parallel} form factor we get the following result for the various channels: $t_1 = 2\kappa[\delta_i^l \delta_j^k + \delta_i^k \delta_j^l] \langle 12 \rangle [12]$, $t_2 = -\kappa[\delta_i^k \delta_j^l (2N + 1) + \delta_i^l \delta_j^k] \langle 13 \rangle [13]$ and $t_3 = -\kappa[\delta_j^k \delta_i^l (2N + 1) + \delta_j^l \delta_i^k] \langle 14 \rangle [14]$, where $\kappa \equiv -2\lambda(16\pi^2)^{-1}$. Adding the different channels, $2t_1 + 4t_2 + 4t_3$, gives

$$\frac{8\lambda}{16\pi^2} (2N - 1) F_{\parallel}(1_i 2_j 3_k^* 4_l^*). \quad (2.16)$$

From (2.15) and (2.16) we read the order λ anomalous dimension matrix of the $\partial^2 H^4$ -type SM operators $\gamma_{c_{\perp}} = 12\lambda\gamma_{c_{\perp}}/16\pi^2$ and $\gamma_{c_{\parallel}} = 24\lambda\gamma_{c_{\parallel}}/16\pi^2$, in agreement with [14–16].⁵

2.1.2 The rest of the order λ corrections

After having set the notation and basic steps, it is straightforward to compute the rest of the order λ corrections in the SM. We need to specify the minimal FFs that overlap with two Higgses

$$F_{FF}(1_i 2_j^* 3_a^- 4_b^-) = -2\delta_i^j \delta_{AB} \langle 34 \rangle^2, \quad (2.17)$$

$$F_6(1_a 2_b 3_c 4_d^* 5_e^* 6_f^*) = 6 \delta_a^d \delta_b^e \delta_c^f + \mathcal{P}_{d,e,f}, \quad (2.18)$$

$$F_y(1_i 2_j^* 3_k 4_l^- 5^-) = (\delta_i^j \delta_k^l + \delta_k^j \delta_i^l) \langle 45 \rangle, \quad (2.19)$$

$$F_{QH}(1_i 2_j^* 3_k^- 4_l^+) = -T_{ji}^A T_{kl}^A (\langle 31 \rangle [14] - \langle 32 \rangle [24]), \quad (2.20)$$

where \mathcal{P}_X means all the permutations of the previous term over the elements in X . The lower case and capital case indices in (2.17)–(2.20) are fundamental and adjoint indices of $SU(2)$, respectively. For operators containing fermions or vectors, one can generate FF with different helicity configurations. Throughout the text we will chose to do the calculations with minimal FF that have mostly negative helicity particles. Due to CPT, if the calculation is done with the mostly positive helicity FF the results are the same.

In table 1 we provide the normalisation of the operators involved in these FFs. There we also provide the rest of minimal FFs that we will use throughout out the paper. Note that when using on-shell methods, a natural base of dimensions-six operators to use is the so called Warsaw basis [17], because in such basis all operators that vanish on-shell have been traded by higher point contact interactions [18–22].

Next we contract each FF in (2.17)–(2.20) with (2.7) and perform the basic tensor multiplications and $d\Omega_2$ integrals as in (2.6). Note that the calculation of the anomalous dimension of the FFs F_{FF} , F_6 and F_y is rather simple because the integration over the

⁵The renormalization mixing pattern in (2.15), (2.16) for $N = 2$ can be understood using custodial symmetry and spurion analysis [15]. While O_{\perp} transforms non-trivially, the quartic interactions and O_{\parallel} are singlets under custodial transformation.

Operator	MFF		
O_{3F}	$\frac{f^{ABC}}{2 \cdot 3!} F_{A\nu}^\mu F_{B\rho}^\nu \bar{F}_C^\rho$	$F_3(1^-_A 2^-_B 3^-_C)$	$\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \quad if^{ABC} / \sqrt{2}$
O_{FF}	$\frac{1}{2} H^\dagger H F_{\mu\nu}^A \bar{F}^{A\mu\nu}$	$F_{FF}(1_i 2_j^* 3_A^- 4_B^-)$	$\langle 34 \rangle \langle 34 \rangle \quad -2\delta_i^j \delta_{AB}$
O_{qF}	$\bar{Q} \sigma^{\mu\nu} T^A q H F_{\mu\nu}^A$	$F_{qF}(1_i^- 2^- 3_k 4_A^-)$	$\langle 14 \rangle \langle 42 \rangle \quad 2\sqrt{2} T_{ik}^A$
O_{4F_1}	$(\bar{Q}_i u) \epsilon_{ij} (\bar{Q}_j d)$	$F_{4F_1}(1_i^- 2^- 3_j^- 4^-)$	$\langle 12 \rangle \langle 34 \rangle \quad \epsilon_{ij}$
O_y	$ H ^2 \bar{Q} q H$	$F_y(1_i 2_j^* 3_k 4_l^- 5^-)$	$\langle 45 \rangle \quad \delta_i^j \delta_k^l + \delta_k^j \delta_i^l$
O_6	$ H ^6$	$F_6(1_a 2_b 3_c 4_d^* 5_e^* 6_f^*)$	$1 \quad 6\delta_a^d \delta_b^e \delta_c^f + \mathcal{P}_{d,e,f}$
O_{4F_2}	$(\bar{Q} T^A \gamma^\mu Q)(\bar{Q} T^A \gamma_\mu Q)$	$F_{4F_2}(1_i^- 2_j^- 3_k^+ 4_l^+)$	$\langle 12 \rangle [34] \quad 2T_{il}^A T_{jk}^A - 2T_{jl}^A T_{ik}^A$
O_{QH}	$(\bar{Q} T^A \gamma^\mu Q)(i H^\dagger T^A \overleftrightarrow{D}_\mu H)$	$F_{QH}(1_i 2_j^* 3_k^- 4_l^+)$	$\langle 31 \rangle [14] \quad -2T_{ji}^A T_{kl}^A$
O_\perp	$(H^\dagger D_\mu H)(D^\mu H)^\dagger H$	$F_\perp(1_i 2_j 3_k^* 4_l^*)$	$\langle 13 \rangle [13] \quad \delta_i^l \delta_j^k + \mathcal{P}_{1_i, 2_j}$
O_\parallel	$ H ^2 (D^\mu H)^\dagger (D_\mu H)$	$F_\parallel(1_i 2_j 3_k^* 4_l^*)$	$\langle 14 \rangle [14] \quad \delta_i^k \delta_j^l + \mathcal{P}_{1_i, 2_j}$

Table 1. The Standard Model dimension-six operators and its minimal form factors (MFF) — with the least number of particles. The MFFs are given by the product of the two rightmost columns. In the right most column we give the $SU(2)_L$ and $SU(3)_c$ tensors that compound the MFF — omitting the identity in colour space and flavour mixing matrices. $\bar{F}_{\alpha\beta} = F_{\alpha\beta} - i\epsilon_{\alpha\beta\mu\nu} F^{\mu\nu}/2$, and T^A is either of the generators of the SM gauge groups. We are not displaying operators whose MFF involves a straightforward modification of the tensor structure — in the rightmost column — of the operator shown. e.g. like $H^\dagger \sigma^i H B_{\mu\nu} W^{i\mu\nu}$, $i\tilde{H}^\dagger D_\mu H (u\gamma^\mu d)$ and $|H|^2 \bar{Q} u \tilde{H}$, where $\tilde{H} = i\sigma_2 H^*$.

rotation angles is trivial and all what is left is to perform the contraction of the $SU(2)_L$ tensors. Indeed, for F_{FF} we get

$$-\frac{2\lambda}{16\pi^2} \left(\delta_\alpha^j \delta_i^\beta + \delta_\alpha^\beta \delta_i^j \right) \delta_\beta^\alpha \delta_{ab} 2 \langle 12 \rangle^2 = \frac{2\lambda}{16\pi^2} (N+1) F_{FF}(1_i 2_j^* 3_a^- 4_b^-) \quad (2.21)$$

The factor of 2 arises because there are two particle-antiparticle channels. Regarding the O_6 self-renormalisation, there are two types of channels the particle-particle channel (or equivalently antiparticle-antiparticle) and the particle-antiparticle channel. There are 6 particle-particle type channels, each of them gives $-M(1_a 2_b; 1'_a 2'_b) F_6(1'_a 2_b 3_c 4_d^* 5_e^* 6_f^*) = 4\lambda F_6(1_a 2_b 3_c 4_d^* 5_e^* 6_f^*)$. The second type, the particle-antiparticle channel, gives $-M(1_a 4_d^*; 1'_a 4'_d) F_6(1'_a 2_b 3_c 4_d^* 5_e^* 6_f^*) + \mathcal{P}_{d,e,f} = 14\lambda F_6(1_a 2_b 3_c 4_d^* 5_e^* 6_f^*)$ for a Higgs doublet. There are 6 types of contributions like this. Thus, all in all we get $6(4+14)\lambda/(16\pi^2)$, at leading order in λ . Regarding F_y , there are also two channels. The particle-particle gives $-M(1_i 3_k; 1'_a 3'_b) F_y(1'_a 2_j^* 3'_b 4_l^- 5^-) = 4\lambda F_y(1_i 2_j^* 3_k 4_l^- 5^-)$. The second type, the particle-antiparticle channel, gives $-M(1_i 2_j^*; 1'_a 2'_b) F_y(1'_a 2_b^* 3_b 4_l^- 5^-) + \mathcal{P}_{ik} = 2(N+3)\lambda F_y(1_i 2_j^* 3_k 4_l^- 5^-)$; there are two such terms because $|1'_a 2'_b\rangle$ and $|1'_b 2'_a\rangle$ are distinct states. Adding all the channels we get $24\lambda/(16\pi^2)$ for the anomalous dimension of F_y .

As for the calculation of F_{QH} , there is only a particle-antiparticle channel

$$\propto \int d\Omega_2 \langle 31' \rangle [1'4] = \langle 31 \rangle [14] + \langle 32 \rangle [24], \quad (2.22)$$

where the solid lines represent fermions. The former polynomial vanishes by momentum conservation $\sum_{i=1}^4 |i\rangle\langle i| = 0$.⁶ This is of course unsurprising since the mixing is only sensitive to the current $J_{\mu}^A = iH^{\dagger}T^A\partial_{\mu}H - i\partial_{\mu}H^{\dagger}T^AH$ in the operator $O_{QH} = J_{\mu}^A(\bar{Q}\gamma^{\mu}T^AQ)$. The current J_{μ}^A is conserved (up to terms proportional to the $SU(2)_L$ coupling g), hence it obeys a Ward identity and thus does not acquire an anomalous dimension proportional to $\lambda[1 + O(g)]$.

All in all, we recover the order λ anomalous dimension matrix of the dimensions-six operators

$$\begin{pmatrix} \gamma_{FF} \\ \gamma_{\perp} \\ \gamma_{\parallel} \\ \gamma_6 \\ \gamma_y \\ \gamma_{\psi H} \end{pmatrix} = \frac{4\lambda}{16\pi^2} \begin{pmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 6 & & & \\ & & & 27 & & \\ & & & & 6 & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} c_{FF} \\ c_{\perp} \\ c_{\parallel} \\ c_6 \\ c_y \\ c_{\psi H} \end{pmatrix}, \quad (2.23)$$

where we only show the non-vanishing matrix elements. Eq. (2.23) is in agreement with [14–16].

2.2 Contributions from gauge interactions

Now we consider contributions to anomalous dimensions that are proportional to a gauge coupling squared. These contributions arise when an $O(g^2)$ \mathcal{M} -matrix element is used in (2.6). In this case however, \mathcal{M} -matrix elements can have collinear and soft divergences, which generate a non-vanishing contribution to γ_{IR} . We will therefore describe how the formalism must be used when IR divergences are present, first calculating the anomalous dimension of a conserved Noether current as a simple example. Then, we will calculate gauge contributions to the anomalous dimensions of O_{\parallel} and O_{\perp} .

2.2.1 IR divergences

In a gauge theory, the IR anomalous dimension has the following expression at the one-loop level:

$$\gamma_{\text{IR}}(s_{ij}; \mu) = \sum_{i < j} T_i^A \cdot T_j^A \gamma_{\text{cusp}} \log \frac{\mu^2}{-s_{ij}} + \sum_i \gamma_i^{\text{coll}}, \quad (2.24)$$

see e.g. refs. [23, 24]. In (2.24) the sum runs over ordered pairs of external legs, T_i^A is a generator of the gauge group, and μ is the t'Hooft scale of dimensional regularisation. This form for γ_{IR} has also been confirmed to hold in a number of examples at two and three

⁶The corresponding diagram in an Effective Action calculation leads to $\partial|H|^2\bar{Q}\gamma Q \xrightarrow{e.o.m.} yO_y + yO_y^*$, where y is the Yukawa coupling and we have used the equations of motion. In the on-shell calculation, such order λy contribution comes from using $\langle HH^{\dagger}H\bar{Q}Q \rangle$ as an outgoing state in (2.6).

loops in QCD and $\mathcal{N} = 4$ SYM (see [24] and references therein), and has been conjectured to hold at all subsequent loop orders. For our current purposes it suffices that (2.24) holds at one loop for either of the SM gauge groups with collinear dimensions given by

$$\gamma_g^{\text{coll}} = -b_0 \frac{g^2}{16\pi^2}, \quad \gamma_q^{\text{coll}} = -3C_2 \frac{g^2}{16\pi^2}, \quad \gamma_H^{\text{coll}} = -4C_2 \frac{g^2}{16\pi^2}, \quad (2.25)$$

where b_0 is the one-loop beta function coefficient, $C_2 \mathbb{1} = \sum_A T^A T^A$, and at one loop $\gamma_{\text{cusp}} = g^2/(4\pi^2)$. The collinear dimensions $\gamma_{g/q}^{\text{coll}}$ are shown in [23] and γ_H^{coll} can be extracted from [25]. Using (2.24) in (2.4) it can be shown that [3]

$$\begin{aligned} \gamma_{ji} F_{O_j} = & -\frac{1}{16\pi^2} \int d\Omega_2 \left(M(12; 1'2') F_{O_i}(1'2'3 \dots n) + \frac{2g^2 T_1^A T_2^A F_{O_i}(123 \dots n)}{\sin^2 \theta \cos^2 \theta} \right) \\ & + F_{O_i} \sum_k \gamma_k^{\text{coll}}, \end{aligned} \quad (2.26)$$

where as in (2.4) we leave implicit the sum over permutations of ordered pairs of the n external particles.

The FF of a Noether current that corresponds to an ungauged global symmetry has a vanishing UV anomalous dimension. Next, we will calculate the anomalous dimension of such a FF and confirm that it is zero to provide a good consistency check on (2.26). We have an ungauged global symmetry in the SM Higgs sector, associated with SU(2) rotations of the Higgs doublet once the SU(2)_L gauge coupling has been set to zero. The corresponding Noether current is $J_\mu^A = i(H^\dagger T^A \partial_\mu H - \partial_\mu H^\dagger T^A H)$, and its associated MFF is

$$F_{JA}(1_i 2_j^*) = -T_{ij}^A (\lambda_1^\alpha \tilde{\lambda}_1^{\dot{\alpha}} - \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}}), \quad (2.27)$$

where $T^A = \sigma^A/2$. We will consider the contribution to the UV anomalous dimension of this FF coming from the SM Hypercharge interaction. To compute this contribution to $O(g'^2)$, we need the tree-level $2 \rightarrow 2$ scattering matrix element for Higgs scalars due to the Hypercharge gauge interaction:

$$M(1_i 2_j 3_k^* 4_l^*) = g'^2 Y_H^2 \delta_i^k \delta_j^l \left(1 - 2 \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right) + g'^2 Y_H^2 \delta_i^l \delta_j^k \left(1 - 2 \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle} \right). \quad (2.28)$$

Evaluating (2.26) for the non-abelian current in (2.27), we have

$$\begin{aligned} \gamma_{JJ} F_{JA}(1_i 2_j^*) = & -\frac{1}{16\pi^2} \int d\Omega_2 \left(M(1_i 2_j^*; 1'_\alpha 2'_\beta) F_{JA}(1'_\alpha 2'_\beta) + M(1_i 2_j^*; 1'^*_\alpha 2'_\beta) F_{JA}(1'^*_\alpha 2'_\beta) \right. \\ & \left. - \frac{g'^2 F_{JA}(1_i 2_j^*)}{2 \sin^2 \theta \cos^2 \theta} \right) + F_{JA}(1_i 2_j^*) \sum_k \gamma_k^{\text{coll}}, \end{aligned} \quad (2.29)$$

where we have used $T_1^A T_2^A = -1/4$ for the hypercharge generators $Y_H = 1/2$ acting on a Higgs scalar and its antiparticle. The integral in (2.29) is evaluated as usual. After performing the angular rotations and basic integrations, we find that it equals $2g'^2 F_{JA}(1_i 2_j^*)/16\pi^2$. Next, using the result for the collinear dimension of scalars from (2.25), and that $C_2 = 1/4$,

we find that the final term in eq. (2.29) equals $-2g'^2 F_{JA}(1_i 2_j^*)/16\pi^2$. We have therefore demonstrated that the required result $\gamma_{JJ} = 0$, can be recovered using this formalism.

In passing, we note that if the same exercise is instead carried out for the Noether current associated with the gauged $U(1)_Y$ symmetry, we get a non-vanishing anomalous dimension. This is because renormalisation of the naive Noether current occurs when the corresponding abelian symmetry is gauged. However, an improved current can be defined in this case that is both conserved and non-renormalised [26].

Having done some simple checks of (2.26), we now compute instances of gauge contributions to the SM dimension-six anomalous dimensions.

2.2.2 Gauge mixing

Coming back to the operators O_{\parallel} and O_{\perp} , let us discuss the mixing due to the $U(1)_Y$ -Hypercharge. We will use (2.26) to calculate the matrix of anomalous dimensions, which is needed as there is both operator mixing and a non-vanishing IR anomalous dimension for this interaction and choice of FFs. In this formula, we will plug in the matrix element for scalar scattering due to the Hypercharge interaction, given by (2.28).

Begin by first taking F_{\perp} in the integral. Just like the corresponding $O(\lambda)$ calculation in section 2.1, there will be three independent channels that contribute to the integral, corresponding to different pairs of scalars crossing the cut. The integral for the first channel gives

$$\begin{aligned}
 & -\frac{1}{16\pi^2} \int d\Omega_2 \left(M(1_i 2_j; 1'_\alpha 2'_\beta) F_{\perp}(1'_\alpha 2'_\beta 3_k^* 4_l^*) + \frac{g'^2 F_{\perp}(1_i 2_j 3_k^* 4_l^*)}{2 \sin^2 \theta \cos^2 \theta} \right) \\
 & = -\frac{g'^2}{64\pi^2} \delta_i^l \delta_j^k \{5 \langle 13 \rangle [31] - 3 \langle 14 \rangle [41]\} - \frac{g'^2}{64\pi^2} \delta_j^l \delta_i^k \{5 \langle 14 \rangle [41] - 3 \langle 13 \rangle [31]\}, \quad (2.30)
 \end{aligned}$$

where the Higgs Hypercharge value $Y_H = 1/2$ has been inputted. Individually, the two terms appearing in the integral in eq. (2.30) are divergent, but their sum is finite. The integral for the second channel is

$$\begin{aligned}
 & -\frac{1}{16\pi^2} \int d\Omega_2 \left(M(1_i 3_k^*; 1'_\alpha 3'_\beta) F_{\perp}(1'_\alpha 2_j 3'_\beta 4_l^*) + M(1_i 3_k^*; 1'_\alpha 3'_\beta) F_{\perp}(1'_\alpha 2_j 3'_\beta 4_l^*) \right. \\
 & \quad \left. - \frac{g'^2 F_{\perp}(1_i 2_j 3_k^* 4_l^*)}{2 \sin^2 \theta \cos^2 \theta} \right) \\
 & = \frac{2g'^2}{64\pi^2} \delta_i^l \delta_j^k \langle 13 \rangle [31] + \frac{g'^2}{64\pi^2} \delta_j^l \delta_i^k \left\{ \left(\frac{N}{3} + 3 \right) \langle 13 \rangle [31] + \left(\frac{2N}{3} + 8 \right) \langle 14 \rangle [41] \right\}. \quad (2.31)
 \end{aligned}$$

The integral for the third channel can be found from the second by making the substitutions $3 \leftrightarrow 4$ and $k \leftrightarrow l$. Finally, multiplying each of these three channels by two (to include the crossing symmetric contributions) and adding them gives the total value for the integral in (2.26). Using the equations for the FFs in (2.8) and (2.9), we can express this total as

$$\frac{g'^2}{16\pi^2} \left(\frac{N}{3} + \frac{5}{2} \right) F_{\perp}(1_i 2_j 3_k^* 4_l^*) + \frac{g'^2}{16\pi^2} \left(\frac{N}{6} + 3 \right) F_{\parallel}(1_i 2_j 3_k^* 4_l^*). \quad (2.32)$$

Using (2.25), we find that the collinear contribution is $\sum \gamma^{\text{coll}} = -4g'^2/16\pi^2$. Then by matching coefficients of F_{\perp} and F_{\parallel} in equation (2.26), we can extract order g' values for

two elements of the anomalous dimension matrix — $\gamma_{\perp} = g'^2(N/3 - 3/2)c_{\perp}/16\pi^2$ and $\gamma_{\parallel} = g'^2(N/6 + 3)c_{\parallel}/16\pi^2$.

By repeating this procedure except inputting F_{\parallel} to the integral in (2.26), the remaining two elements of the anomalous dimension matrix can be found. After combining the three channels in the same way as before, the integral term is

$$\frac{g'^2}{16\pi^2} \left(\frac{10}{3} F_{\perp}(1_i 2_j 3_k^* 4_l^*) + \frac{8}{3} F_{\parallel}(1_i 2_j 3_k^* 4_l^*) \right), \quad (2.33)$$

independently of N . After adding the collinear contribution and matching coefficients of F_{\perp} and F_{\parallel} in eq. (2.26), we can find the remaining two elements of the O_{\parallel} and O_{\perp} anomalous dimension matrix at order g'^2 . After setting $N = 2$ for the Higgs doublet, the final result is

$$\begin{pmatrix} \gamma_{\perp} \\ \gamma_{\parallel} \end{pmatrix} = \frac{g'^2}{16\pi^2} \begin{pmatrix} -\frac{5}{6} & \frac{10}{3} \\ \frac{10}{3} & -\frac{4}{3} \end{pmatrix} \begin{pmatrix} c_{\perp} \\ c_{\parallel} \end{pmatrix}, \quad (2.34)$$

in agreement with previous literature [15, 27].⁷ By similar means one can compute the mixing due to the $SU(2)_L$ gauge interactions. However, such order g^2 contributions are diagonal because, due to custodial symmetry, the RG mixing between O_{\parallel} and O_{\perp} must be proportional to the Hypercharge coupling g'^2 [15].

2.3 Non-minimal form factors

In the previous sections we have only encountered minimal form factors, i.e. FF that do not vanish at zero order in the SM couplings. This is because we focused on transitions between operators that have the same number of external fields. Instead, if the transition between operators increases the number of particles, we will potentially have contributions from a $2 \rightarrow 2$ S-matrix element acting on a non-minimal form factor.

Next we compute an illustrative example consisting on the dipole renormalisation by the F^3 operator. For concreteness we focus on the F^3 and dipole involving the gluon field strength. The MFFs that we need are⁸

$$F_{3G}(1_A^- 2_B^- 3_C^-) = \frac{i}{\sqrt{2}} f^{ABC} \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle, \quad F_{qG}(1_i^- 2_j^- 3 4_A^-) = 2\sqrt{2} T_{ij}^A \langle 14 \rangle \langle 42 \rangle, \quad (2.35)$$

where in F_{qG} we only show the color index, because the $SU(2)$ indices are diagonal throughout the calculation. There are three different contributions,

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \quad (2.36)$$

The first diagram in (2.36) involves a 5-point scattering amplitude. The amplitude between two fermions, a scalar and two gluons with all-plus helicity vanishes:

⁷The calculation of [15] was done in a basis where $O_B \equiv (H^\dagger D_\nu H) \partial_\mu B^{\mu\nu}$ is kept as an independent operator. After rotating away such operator and rotating the operators $\{O_H, O_T\}$ (there defined) into $\{O_{\parallel}, O_{\perp}\}$ full agreement is found. An early calculation of the RG mixing from O_H to O_T can be found in [28].

⁸Where we used the convention $\epsilon_{a\dot{a}}^- = -\sqrt{2}\lambda_a \tilde{\mu}_{\dot{a}} / [\tilde{\lambda}, \tilde{\mu}]$.

$M(\psi^+\psi^+\phi g^+g^+) = 0$. Therefore the only contribution is from the amplitude with negative helicity fermions $M(\psi^-\psi^-\phi g^+g^+)$, which is a potential contribution to the dipole $F_{qG}(1_i^-2_j^-3_4^+)$ in (2.35). The 5-point amplitude is

$$M(1_i^-2_A^+3_B^+4_j^-5) = -2yg_s^2(T^AT^B)_{ij} \frac{\langle 14 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} - 2yg_s^2(T^BT^A)_{ij} \frac{\langle 14 \rangle^2}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle}. \quad (2.37)$$

We need to perform the convolution of this amplitude with F_{3G} in (2.35),

$$-\frac{1}{16\pi^2} \int d\Omega_2 M(1_i^-3_j^-4; x_{\bar{X}}y_{\bar{Y}}) F_{3G}(x_{\bar{X}}y_{\bar{Y}}2_A^-) = -\frac{yg_s^2}{16\pi^2} \frac{N_c T_{ij}^A}{\sqrt{2}} 2\langle 13 \rangle^2 \int d\Omega_2 \frac{\langle y2 \rangle \langle 2x \rangle}{\langle 1x \rangle \langle y3 \rangle} \quad (2.38)$$

where we used $-if^{ABC}T^AT^B = \frac{1}{2}N_cT^C$, and a factor of two arises because both color orderings in (2.37) give the same integral since it is invariant under $x \leftrightarrow y$. Now we need to rotate the internal spinors in terms of the three external ones. We rotate $|x\rangle$ and $|y\rangle$ in terms of spinors $|a\rangle$ and $|b\rangle$ such that they are on-shell and $p_a + p_b = p_1 + p_3 + p_4$, a simple choice is [3]

$$|a\rangle = |1\rangle \sqrt{\frac{s_{134}}{s_{13} + s_{14}}}, \quad |b\rangle = (|3\rangle[13] + |4\rangle[14]) \frac{1}{\sqrt{s_{13} + s_{14}}} \quad (2.39)$$

Next we rotate $(|x\rangle, |y\rangle) = R(\theta, \phi) \cdot (|a\rangle, |b\rangle)$ like in (2.5) and we are lead to

$$\int_0^{\pi/2} 2c_\theta s_\theta d\theta \int \frac{d\phi}{2\pi} \frac{(\langle b2 \rangle c_\theta + s_\theta e^{-i\phi} \langle a2 \rangle)(\langle 2a \rangle c_\theta - s_\theta e^{i\phi} \langle 2b \rangle)}{-s_\theta e^{i\phi} \langle 1b \rangle (\langle b3 \rangle c_\theta + s_\theta e^{-i\phi} \langle a3 \rangle)}. \quad (2.40)$$

A convenient way to perform the ϕ and θ integrals is to change variables to $z \equiv e^{i\phi}$ and $t \equiv \tan \theta$. In this way, the denominator of the integrand is a product of terms like $(z - \langle ij \rangle / \langle ik \rangle t)$ so the z integral picks the residues inside the unit circle and the integral on t scans the position of those residues. The integral in (2.40) is

$$-2 \frac{\langle 2b \rangle \langle 2a \rangle}{\langle 1b \rangle \langle 3b \rangle} \int_0^\infty \frac{dt}{(1+t^2)^2} \oint_{|z|=1} \frac{dz}{2\pi i} \frac{(z + tr_2)(1 - zt/r_2)}{z^2(z + tr_3)}, \quad (2.41)$$

where we have defined $r_i = \langle ia \rangle / \langle ib \rangle$. After performing the integrals, we use the expressions in (2.39), and after a lengthy manipulation of the result using Schouten identities $[\xi i][jk] + [\xi j][ki] + [\xi k][ij] = 0$ we get

$$(2.41) = \frac{\langle 12 \rangle \langle 23 \rangle}{\langle 13 \rangle^2} \left(2 - \log \frac{(s_{13} + s_{14})(s_{13} + s_{34})}{s_{14}s_{34}} \right) + \frac{\langle 12 \rangle^2 \langle 34 \rangle}{\langle 14 \rangle \langle 13 \rangle^2} \frac{s_{14}}{s_{13} + s_{14}} + \frac{\langle 14 \rangle \langle 23 \rangle^2}{\langle 34 \rangle \langle 13 \rangle^2} \frac{s_{34}}{s_{13} + s_{34}}. \quad (2.42)$$

Next we compute the right diagram in (2.36), which involves the following four-particle amplitude and non-minimal FF

$$M(1_i^-2_A^+3_j^-4) = yg_s \sqrt{2} T_{ij}^A \frac{\langle 13 \rangle^2}{\langle 12 \rangle \langle 23 \rangle} \quad (2.43)$$

$$F_{3G}(1_i^-2_A^-3_B^+4_j^+) = -ig_s f^{ABC} T_{ij}^C \frac{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}{\langle 34 \rangle}. \quad (2.44)$$

Convoluting this scattering matrix element with the FF we get

$$\begin{aligned}
 & -\frac{2}{16\pi^2} \int d\Omega_2 M(1_i^- 4; x_{\bar{X}}^- y_k^+) F_{3G}(2_A^- x_{\bar{X}}^- 3_j^- y_k^+) \\
 & = -\frac{yg_s^2}{16\pi^2} \frac{N_c T_{ij}^A}{\sqrt{2}} 2 \frac{\langle 23 \rangle}{\langle 14 \rangle} \int d\Omega_2 \frac{\langle 1y \rangle^2 \langle 2x \rangle \langle 3x \rangle}{\langle 1x \rangle \langle 3y \rangle}.
 \end{aligned} \tag{2.45}$$

The factor of two comes because there are two channels corresponding to propagating either $|x_{\bar{X}}^- y_k^+\rangle$ or $|y_k^+ x_{\bar{X}}^-\rangle$, that give the same value. Next we rotate $(|x\rangle, |y\rangle) = R(\theta, \phi) \cdot (|1\rangle, |4\rangle)$ and perform again the change of variables into $z \equiv e^{i\phi}$ and $t \equiv \tan \theta$. We are left with the following integral

$$\begin{aligned}
 & 2 \int_0^\infty \frac{dt}{(1+t^2)^3} \oint_{|z|=1} \frac{dz}{2\pi i} \frac{\left(1 + z t \frac{\langle 24 \rangle}{\langle 12 \rangle}\right) \left(\frac{\langle 13 \rangle}{\langle 34 \rangle} + t z\right)}{z \left(t \frac{\langle 13 \rangle}{\langle 34 \rangle} - z\right)} \\
 & = \left(-\frac{3}{2} + \log \frac{s_{13} + s_{34}}{s_{34}} - \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} \frac{s_{34}}{s_{13} + s_{34}} \right).
 \end{aligned} \tag{2.46}$$

The final result for this contribution is $-\frac{yg_s^2}{16\pi^2} \frac{N_c T_{ij}^A}{\sqrt{2}} 2 \langle 12 \rangle \langle 23 \rangle \times (2.46)$.

The third contribution in (2.36) comes from exchanging 1 and 3 in the second contribution. Finally adding the three contributions, we see that the log's and rational functions cancel and we are left with the contact interaction

$$\frac{yg_s^2}{16\pi^2} \frac{N_c}{2} T_{ij}^A 2\sqrt{2} \langle 12 \rangle \langle 23 \rangle, \tag{2.47}$$

which we easily recognise as the dipole MFF in (2.35). Thus the anomalous dimension is

$$\gamma_{qG \leftarrow 3G} = \frac{g_s^2 y}{16\pi^2} \frac{N_c}{2}, \tag{2.48}$$

in agreement with [29].

In passing we note that if we do the calculation with the positive helicity gluons $F_{3G}(1_A^+ 2_B^+ 3_C^+)$, i.e. the CPT related FF, we get the complex conjugate dipole. The Wilson coefficient in $c_{3G} O_{3G}$ is taken as complex and thus we generate the operator O_{qG} with a complex coefficient, namely the CP even and CP odd dipole.

3 Structure of the anomalous dimension matrix

The on-shell formalism that we have been using suggests classifying the operators — or minimal FF — in five classes,

$$\langle \cdot \rangle^3, \quad \langle \cdot \rangle^2, \quad \langle \cdot \rangle, \quad 1 \quad \text{and} \quad \langle \cdot \rangle [\cdot] \tag{3.1}$$

depending on the number of λ and $\tilde{\lambda}$'s. For instance $F_{3G}(1_A^- 2_B^- 3_C^-) \propto \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \sim \langle \cdot \rangle^3$ and therefore is of the first class, see table 1 for the rest of FFs. The total number of spinors and dimensional analysis determines the number of particles involved in each FF.⁹

⁹This coincides with the classifications in [30, 31].

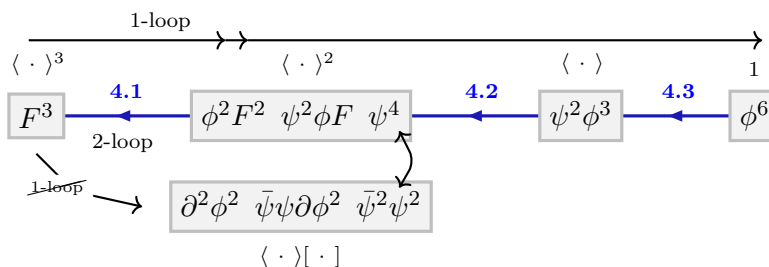


Figure 1. One-loop RG mixing can only occur from operators renormalising operators in boxes to the right or in the same box, with the only exception of the four-fermion of type $\langle \cdot \rangle^2$ and type $\langle \cdot \rangle[\cdot]$ that do mix. One-loop transitions from $\langle \cdot \rangle^3$ -type operators to $\langle \cdot \rangle[\cdot]$ is forbidden by helicity selection rules. These patterns are summarised by the black arrows in the figure. With blue arrows we indicate the two-loop anomalous dimensions that are computed in the sections indicated on top.

The on-shell formalism makes very transparent the pattern of the anomalous dimensions matrix, namely the non-vanishing entries of the matrix. When mixing from O_i to O_j , the phase-space integral will remove two legs from the form factor F_{O_i} , while the $S_{m \leftarrow 2}$ scattering matrix element generates $m \geq 2$ legs. Thus O_j has, the same number of external fields or more, but never less than O_i . Therefore operators can only renormalise, at one-loop, operators with equal or more fields. This is an instance of the more general theorem in [32].

The RG mixing between the two four-particle classes — $\langle \cdot \rangle^2$ and $\langle \cdot \rangle[\cdot]$ — requires the 2 to 2 S-matrix element to have a total net helicity $\sum_i h_i \neq 0$ with all particles outgoing. In the SM there is a unique four-point tree-level amplitude that has net helicity, namely the four-fermion amplitude proportional to the up and down Yukawas $y_u y_d$ [30, 33]. Therefore the only allowed mixing between operators from class $\langle \cdot \rangle^2$ and $\langle \cdot \rangle[\cdot]$ is between the four-fermion operators ψ^4 and $\psi^2 \bar{\psi}^2$. This effect was previously explained using SUSY as a spurious symmetry, since the up and down Yukawa interactions are not simultaneously holomorphic [34, 35]. Finally, helicity selection rules [30], or accidental supersymmetry [35], also exclude one-loop mixing from operators in $\langle \cdot \rangle^3$, into $\langle \cdot \rangle[\cdot]$.¹⁰

While chunks of the SM dimensions-six operators’ anomalous dimensions have been computed in many papers in the literature, see e.g. [14, 37–42] and [15, 43] for the anomalous dimensions relevant for Higgs and Electroweak physics, the full matrix has been computed in [16, 27, 44, 45]. With this we conclude the summary of the one-loop pattern, see figure 1 for an outline.

A particularly interesting class of two-loop anomalous dimensions consists of those RG mixings that are forbidden at one-loop. We distinguish two types of such two-loop RG mixings. The first type are those two loop RG mixings that decrease the number of particles by a unit, namely an operator with ℓ legs can mix, at two loops, with an operator with $\ell - 1$ legs. These transitions occur in the direction opposite to the long 1-loop arrow in figure 1. The second type are those two loop RG mixings that are forbidden at one loop

¹⁰Helicity selection rules also imply that the corresponding one-loop diagram vanishes up to a rational function of the kinematical variables; in many cases the rational factor vanishes as well [36], see also below.

where the spinors in the integrand are written in terms the two external spinor by the following rotation

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = \begin{pmatrix} \cos \theta_1 & -e^{i\phi} \cos \theta_3 \sin \theta_1 \\ \cos \theta_2 \sin \theta_1 & e^{i\phi} (\cos \theta_1 \cos \theta_2 \cos \theta_3 - e^{i\delta} \sin \theta_2 \sin \theta_3) \\ \sin \theta_1 \sin \theta_2 & e^{i\phi} (\cos \theta_1 \cos \theta_3 \sin \theta_2 + e^{i\delta} \cos \theta_2 \sin \theta_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad (4.2)$$

and the measure is $d\Omega_3 = 4 \cos \theta_1 \sin^3 \theta_1 d\theta_1 2 \cos \theta_2 \sin \theta_2 d\theta_2 2 \cos \theta_3 \sin \theta_3 d\theta_3 \frac{d\delta}{2\pi} \frac{d\phi}{2\pi}$ [3, 13]. The $1/3!$ in (4.1) arises only when considering the phase space for three identical particles. Further details on the phase-space integral can be found in appendix A.

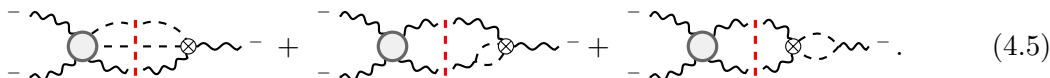
4.1 $\langle \cdot \rangle^2 \rightarrow \langle \cdot \rangle^3$: 4-point to 3-point

We start by computing the transitions from four-field operators into three-field operators. There are only two structurally different contributions at two loops, namely the $\phi^2 F^2 \rightarrow F^3$ and the $\psi^2 \phi F \rightarrow F^3$ mixings. We start with the former contribution. In the SM this transition is only possible when the gauge field belongs to the $SU(2)_L$, thus we compute the RG mixing from the operator O_{WW} to O_{3W} , see table 1 with $F = W$. The corresponding MFFs are

$$F_{3W}(1_A^- 2_B^- 3_C^-) = \frac{i}{\sqrt{2}} f^{ABC} \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \quad (4.3)$$

$$F_{WW}(1_i 2_A^- 3_B^- 4_j^*) = -2 \delta_i^j \delta^{AB} \langle 23 \rangle^2. \quad (4.4)$$

We find the following potential contributions

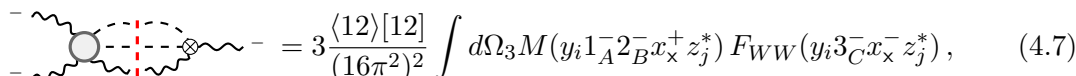


$$- + - + - \quad (4.5)$$

We have excluded one loop FFs that are contracted with the amplitude $H_i H_j \rightarrow W^- W^-$, because this process is zero at tree-level in the SM. Next, note that the one-loop FFs in (4.5) vanish. Indeed, the loop is given by $\int d^d \ell \frac{\ell^\mu}{[(\ell+p/2)^2 - i\epsilon][(\ell-p/2)^2 - i\epsilon]} = 0$.¹² We are left with the first term in (4.5), involving a three-particle phase-space integration between a tree-level matrix element and a minimal FF. The five-point amplitude is given by

$$M(1_i 2_{A_2}^- 3_{A_3}^- 4_{A_4}^+ 5_j^*) = 2\sqrt{2} g^3 [14]^2 [45]^2 \left(\frac{(T^{A_2} T^{A_3} T^{A_4})_{ij}}{[12][23][34][45][51]} + \text{perms. } \{2,3,4\} \right), \quad (4.6)$$

which has to be contracted with the MFF in (4.4),



$$- = 3 \frac{\langle 12 \rangle [12]}{(16\pi^2)^2} \int d\Omega_3 M(y_i 1_A^- 2_B^- x_x^+ z_j^*) F_{WW}(y_i 3_C^- x_x^- z_j^*), \quad (4.7)$$

where we denoted by x , y and z the momenta of the spinors we are integrating over. In (4.7) we included a factor of 3 that accounts for all possible ways to contract the external state

¹²The same integral appeared recently in [32] when computing the two-loop mixing of O_6 into O_{3W} .

$\langle 1_A^- 2_B^- 3_C^- |$ with the $M \otimes F$. The integral is done by rotating $\lambda_x, \lambda_y, \lambda_z$ in terms of λ_1, λ_2 as in (4.2). The integrand of (4.7) involves permutations of $[ijk] \equiv \frac{[xy]^2 [xz]^2 \langle x3 \rangle^3}{[yi][ij][jk][kz][zy]}$, with $i, j, k = 1, 2, x$, multiplied by the corresponding trace of T^A 's matrices, generated from the contraction of the color structure in (4.6) with the δ_{ij} of the F_{WW} form factor. The integral measure is invariant under $y \leftrightarrow z$, which is implemented with the change of variables $(\theta_3, \delta) \rightarrow (\pi/2 - \theta_3, \delta + \pi)$. Therefore, we can reverse the ordering of the arguments to get $[ijk] = -[kji]$, and write the integrands proportional to $\text{tr}(T^A T^B T^C)$ in terms of the ones proportional to $\text{tr}(T^C T^B T^A)$. Thus we are lead to

$$(4.7) = -\frac{\langle 12 \rangle [12]}{(16\pi^2)^2} 6\sqrt{2} g^3 i f^{ABC} \int d\Omega_3 ([12x] + [x12] + [2x1]), \quad (4.8)$$

where we have used $\text{tr}(T^A [T^B, T^C]) = i f^{ABC}/2$ and a factor of 3 comes from the permutations in (4.7). The angular integrals in (4.8) are given by

$$\int d\Omega_3 [12x] = \int d\Omega_3 [x12] = \frac{1}{2} \frac{\langle 13 \rangle \langle 23 \rangle}{[12]}, \quad \int d\Omega_3 [2x1] = \frac{3}{2} \frac{\langle 13 \rangle \langle 23 \rangle}{[12]}. \quad (4.9)$$

From this we read the two loop anomalous dimension

$$\gamma_{3W \leftarrow WW} = 30 \frac{g^3}{(16\pi^2)^2}. \quad (4.10)$$

Next we compute a transition of the type $\psi^2 \phi F \rightarrow F^3$. In particular we compute the renormalisation of the pure glue operator O_{3G} by the gluon dipole O_{qG} . This is the opposite mixing direction that we have computed in section 2.3. The MFFs are already given in (2.35). There are the following possible contributions



$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}. \quad (4.11)$$

However, the one loop FF vanishes because, due to the presence of $\sigma_{\mu\nu}$ in the dipole vertex, the loop of fermions involves the trace of an odd number of γ^μ -matrices. We have excluded one-loop FFs that are contracted with the amplitudes $\psi^+ \psi^+ \rightarrow g^- g^-$ or $\phi \phi \rightarrow g^- g^-$ which do not exist, at tree-level, in the SM Lagrangian. Again, the contribution reduces to compute the first term in (4.11) which involves a tree-level matrix element and a minimal FF. The five-point scattering amplitude has already appeared in (2.37). Since in the current case the $SU(2)$ indices are relevant, they are denoted by i, j , while the color ones by α, β . Both the amplitude and FF proportional to the trivial factor δ_{ij} . The convolution of the amplitude and the MFF is

$$\int d\Omega_3 M(y_{\alpha i}^+ 1_A^- 2_B^- z_{\beta}^+ x_j) F_{qG}(y_{\alpha i}^- 3_C^- z_{\beta}^- x_j^*) = -4\sqrt{2} y g_s^2 i f^{ABC} \int d\Omega_3 \frac{[yz]^2 \langle y3 \rangle \langle 3z \rangle}{[y1][12][2z]} \quad (4.12)$$

where we used the fact that the integral measure is invariant under $y \leftrightarrow z$ to write the two color orderings in a similar way, and that $i f^{ABC}/2 = \text{tr}(T^A [T^B, T^C])$ and $\delta_{ij} \delta_{ij} = 2$.

The integral over the phase space angles is straightforward to do and gives $\frac{1}{2}\langle 13 \rangle \langle 23 \rangle / [12]$. Finally, taking into account a factor of 3 for the permutations of the three different gauge bosons, we get

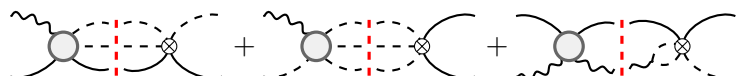
$$\gamma_{3G \leftarrow qG} = 12 \frac{y g_s^2}{(16\pi^2)^2}. \quad (4.13)$$

This computation, compared with the one loop reverse transition in 2.3, shows explicitly that the complexity of the computations in this method does not scale strongly with the number of loops, and two-loop computations can be easier than one-loop ones.

These two Lorentz structures, $\phi^2 F^2$ and $\psi \phi F$, are the only four-particle operators that mix with the three-particle structure F^3 at two loops with a single log.

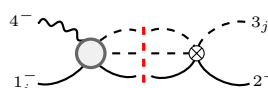
4.2 $\langle \cdot \rangle \rightarrow \langle \cdot \rangle^2$: 5-point to 4-point

We start by computing a transition of the type $\psi^2 \phi^3 \rightarrow \psi^2 \phi F$. For concreteness we compute the renormalisation of the hypercharge dipole of the leptons $\mathcal{O}_{eB} = \bar{\ell}_L \sigma_{\mu\nu} e_R H B_{\mu\nu} + h.c.$ due to the leptonic Yukawa $\mathcal{O}_{ye} = |H|^2 \bar{\ell}_L e_R H + h.c.$, with other pairings being similar since the non-abelian part of the amplitude plays no role in the computation. The potential contributions to the $\psi^2 \phi^3 \rightarrow \psi^2 \phi F$ mixing are given by



$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}. \quad (4.14)$$

The second contribution with the three scalar in the cut is zero. This can be seen by explicit computation, but also by noticing that the fermion structure $\langle 12 \rangle$ of the Yukawa is unaltered by the integration and there is no dimension-six invariant with the $F \psi^2 \phi$ particle content and proportional to the fermion spinors $\langle 12 \rangle$. The third contribution also vanishes, because the FF loop vanishes as we showed in the previous section. We are left with the tree-level amplitude contribution



$$= \frac{\langle 14 \rangle [14]}{(16\pi^2)^2} \int d\Omega_3 M(1_i^- x_a^+ y_b^* z_c 4^-) F_y(x_a^- 2^- 3_j y_b z_c^*). \quad (4.15)$$

The amplitude in (4.15) is given by

$$M(1_i^- 2_j^+ 3_k 4_l^* 5^-) = -2\sqrt{2} g' (g'^2 Y_L Y_H \delta_{ij} \delta_{kl} + g^2 T_{ij}^A T_{kl}^A) \left(Y_H \frac{[23][24]}{[12][35][45]} - Y_L \frac{[23][24]}{[15][25][34]} \right), \quad (4.16)$$

where we included both $U(1)_Y$ and $SU(2)_L$ contributions to the amplitude, Y_L refers to the lepton doublet and Y_H the Higgs doublet Hypercharge. In first term in the parenthesis of (4.16) the $U(1)_Y$ boson 5^- is attached to the scalars, while in the second term the gauge boson is attached to the fermions. The FF in (4.15) is

$$F_{ye}(1_i^- 2^- 3_j 4_k^* 5_l) = \langle 12 \rangle (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj}) \quad (4.17)$$

Next, the spinors associated to the particles x, y, z are rotated as in (4.2) leaving a simple trigonometric integral that gives

$$\int d\Omega_3 \left(Y_H \frac{[xy][xz]\langle x2 \rangle}{[1x][y5][z5]} - Y_L \frac{[xy][xz]\langle x2 \rangle}{[15][x5][yz]} \right) = -Y_H \frac{\langle 42 \rangle}{[14]} \quad (4.18)$$

with the term proportional to the fermion hypercharge vanishing since the integrand is odd under $y \leftrightarrow z$ while the integral measure is invariant. Therefore all in all we get

$$(4.15) = \frac{1}{(16\pi^2)^2} g' Y_H \left(g'^2 Y_L Y_H (N+1) + g^2 \frac{N^2-1}{2N} \right) 2\sqrt{2} \delta_{ij} \langle 14 \rangle \langle 42 \rangle, \quad (4.19)$$

where $N = 2$ for the doublet. Exchanging $1^- \leftrightarrow 2^-$ in (4.15) amounts to $Y_L \leftrightarrow Y_R$ in the order g'^3 contribution. Thus all in all, we get the anomalous dimension

$$\gamma_{eB \leftarrow y_e} = \frac{3}{(16\pi^2)^2} \left(g'^3 (Y_L + Y_R) Y_H^2 + g' g^2 Y_H \frac{1}{4} \right), \quad (4.20)$$

in agreement with [46].

The second case we consider is the renormalisation of the $\phi^2 F^2$ operators due to the dimension-six Yukawa O_y . For instance, take the operator to be O_{GG} . As before, denote by α, β the color indices and by i, j, \dots the $SU(2)_L$ ones. There are two types of contributions, given by the diagrams



$$(4.21)$$

The one-loop FF vanishes as before, and we have excluded those one-loop FFs that are contracted with $M(\phi\phi \rightarrow g^-g^-) = 0$ at tree-level. Thus we are left with the first contribution in (4.21). Both the MFF and the 5-point amplitude have already appeared and they are given in (2.19) and (2.37), respectively. The phase-space integral in (4.31) gives

$$\int d\Omega_3 M(y_{\alpha k}^+ 1_A^- 2_B^- z_{\beta}^+ x_l^*) F(y_{\alpha k}^- z_{\beta j}^- x_l 3_i 4_j^*) = -y g_s^2 \delta^{AB} \delta_{ij} (N+1) 2 \int d\Omega_3 \frac{[yz]^2 \langle yz \rangle}{[y1][12][2z]}, \quad (4.22)$$

where the factor 2 comes from the two color orderings giving the same integrand, $\delta^{AB} = 2\text{tr}(T^A T^B)$ and the $(N+1)$ factor, with $N = 2$, comes from the $SU(2)_L$ contraction. The integral over the trigonometric angles gives $-\langle 12 \rangle / [12]$. The result is

$$\gamma_{FF \leftarrow y} = -3 \frac{y g_s^2}{(16\pi^2)^2}. \quad (4.23)$$

The last contribution of the type $\langle \cdot \rangle \rightarrow \langle \cdot \rangle^2$ is the $\psi^2 \phi^3 \rightarrow \psi^4$ transition,



$$(4.24)$$

There is not a contribution from a one-loop FF. Note that the one-loop form factor where one scalar line is closed into a fermion leg vanishes on-shell because it is proportional to $\not{\partial}q$. Indeed such loops are proportional to $(\bar{q}\not{\partial}q)|H|^2 + h.c.$ in an effective action calculation, see e.g. [15].

For definiteness, we compute the renormalisation of the 4-fermion operator $O_{lequ} = \epsilon^{ij}(\bar{\ell}_L^i e_R)(\bar{q}^j u_R)$ by the up-quark Yukawa $O_{y_u} = |H|^2 \bar{q}_L^i u_R \epsilon_{ij} H^{j*}$ with MFF

$$F_{lequ}(1_i^- 2^- 3_j^- 4^-) = \epsilon^{ij} \langle 12 \rangle \langle 34 \rangle, \quad (4.25)$$

$$F_{y_u}(1_i^- 2^- 3_j^* 4_k 5_l^*) = (\epsilon_{ij} \delta_{kl} + \epsilon_{il} \delta_{kj}) \langle 12 \rangle. \quad (4.26)$$

Any other choice of Yukawa and 4-fermion operators will be computed in a similar fashion. The 5-point amplitude involved has three components, proportional to y_e^3 , $y_e \lambda$ and $y_e g^2$ or $y_e g'^2$. The contributions dependent on the gauge coupling are proportional to the difference of two scalar momenta, say $(p_5 - p_4)_\mu$, since there is always a scalar current coupled to a gauge boson. Since the FF is independent of the scalar momenta, the phase space integration sets these contributions to zero. The only relevant part of the amplitude is then

$$M(1_i^- 2^- 3_j 4_k^* 5_l) = -y_e^3 \left(\delta_i^j \delta_k^l \frac{[35]}{[13][25]} + \delta_i^l \delta_k^j \frac{[53]}{[15][23]} \right) + 2y_e \lambda (\delta_i^j \delta_k^l + \delta_i^l \delta_k^j) \frac{1}{[12]}. \quad (4.27)$$

The convolution is

$$\begin{aligned} & \int d\Omega_3 M(\ell_i^- e^- x_a y_b^* z_c) F_{y_u}(q_j^- u^- x_a^* y_b z_c^*) \\ &= 2(N+1) \epsilon_{ji} \int d\Omega_3 \left[-y_e^3 \frac{[xz]}{[lx][ez]} \langle qu \rangle + 2y_e \lambda \frac{\langle qu \rangle}{[le]} \right], \end{aligned} \quad (4.28)$$

where we labeled the external momenta as ℓ, e, q, u to keep track of the fermions' flavour, and $N = 2$ is the dimension of the doublet. The second trigonometric integral is trivial and gives 1, while the first one gives $2\langle qu \rangle/[le]$. After multiplying this by the phase space factor $\langle \ell e \rangle [le]$ and a $1/2!$ for the identical particles x and z , we identify the resulting factor $\epsilon^{ij} \langle \ell e \rangle \langle qu \rangle$ as the one associated with the operator $O_{lequ} = \epsilon^{ij}(\bar{\ell}_L^i e_R)(\bar{q}_L^j u_R)$, with an anomalous dimension

$$\gamma_{lequ \leftarrow y_u} = -\frac{6}{(16\pi^2)^2} (y_e \lambda - y_e^3). \quad (4.29)$$

4.3 $1 \rightarrow \langle \cdot \rangle$: 6-point to 5-point

Lastly, consider the transition between the operator $O_6 = |H|^6$ with MFF

$$F_6(1_i 2_j^* 3_k 4_l^* 5_m 6_n^*) = 6 \delta_i^j \delta_k^l \delta_m^n + 5 \text{ perm. of } \{j, l, n\} \quad (4.30)$$

and the dimension-six Yukawa O_y . The only non-vanishing contribution comes from the diagram



$$(4.31)$$

The integral over the phase space is the same we encountered in the preceding section for the $\psi^2\phi^3 \rightarrow \psi^4$ transition. The flavour contraction gives $2(N+2)(\delta_i^j\delta_k^l + \delta_i^l\delta_k^j)$. Thus we are led to

$$\gamma_{y_e\leftarrow 6} = \frac{48}{(16\pi^2)^2}(y_e\lambda - y_e^3). \tag{4.32}$$

5 Conclusions

In this work we have demonstrated how to retrieve the anomalous dimensions of the SM EFT operators using the on-shell S-matrix and form factors. In particular we showed how only tree-level S-matrix elements and FFs can be used to determine all of the one-loop and many two-loop anomalous dimensions of the SM EFT. Another nice aspect of this method is that it re-uses the same amplitudes to determine many anomalous dimensions. For instance, the amplitude $M(FF\phi\psi\psi)$ appears in the calculation of the one loop $F^3 \rightarrow \psi^2\phi F$, $\phi^2 F^2 \rightarrow \phi^3\psi^2$, and the two loop $\psi^2\phi^3 \rightarrow \phi^2 F^2$ and $\psi^2\phi F \rightarrow F^3$ mixings. This aspect of the method permits many self-consistency checks. Our calculations could be fully automated and extended to the entire one-loop anomalous dimension matrix of the leading higher dimensional operators in the SM EFT, or a more general field theory.

In section 2 we computed a variety of representative one-loop anomalous dimensions including the order λ corrections, examples involving IR anomalous dimensions and non-minimal form factors. In section 3 we have discussed the structure of the one-loop anomalous dimension matrix of dimension-six operators. This has allowed us to identify the most interesting two-loop anomalous dimensions, namely those that are forbidden at one-loop either by leg counting or by helicity selection rules only. In section 4 we have computed the two-loop anomalous dimensions that mix operators as indicated by the blue arrows in figure 1 as an illustration of the power of the on-shell method.

Current and future precision experiments probe and will probe the RG structure at two loops. The paradigmatic example is the measurement of the electron EDM [46, 47], whose current bound tests multi-TeV dynamics at two loops. Other future experiments, like the measurements of the $\text{Br}(\mu \rightarrow e\gamma)$ at the MEG-II experiment will reach a precision of $6 \cdot 10^{-14}$ [48]. This would translate into a bound of $\Lambda \gtrsim 10^3$ TeV for the operator $\mathcal{L} \supset \frac{\sqrt{y_e y_\mu}}{\Lambda^2} \bar{\mu} \sigma_{\mu\nu} e H F_{\mu\nu}$. If the UV dynamics is such that, instead of the dipole, an operator $\frac{\sqrt{y_e y_\mu}}{\Lambda_i^2} O_i$, which only affects the dipole at two loop order, is generated at the high scale, then the log-enhanced contribution due to the two-loop RG mixing would set a constraint of $\Lambda \gtrsim 10$ TeV. Having a complete map of the two-loop anomalous dimensions of the operators probed by this and other future precision experiments, will provide the complete characterisation of the dynamics affecting such processes through a log-enhanced contribution.

While we have shown how to compute the transitions $\langle \cdot \rangle^2 \rightarrow \langle \cdot \rangle^3$, $\langle \cdot \rangle \rightarrow \langle \cdot \rangle^2$ and $1 \rightarrow \langle \cdot \rangle$, it will be interesting to compute the rest of the two-loop anomalous dimensions for which the corresponding one-loop contributions are absent, as well as those for which the one-loop contribution are suppressed by a small coupling, like a Yukawa of a light particle. In addition, we look forward to investigating other aspects of this method. Whether we

can relate the branch-cut discontinuity of other physical quantities to the Callan-Symanzik equation using the dilatation operator (1.4), or compute certain anomalous dimensions at large orders, are some of the questions that we are eager to investigate further.

Note added. Simultaneously with the submission of this preprint ref. [49] appeared where the renormalisation of higher-dimensional operators from on-shell amplitudes is presented from a slightly different perspective. In ref. [49] all one-loop anomalous dimensions of the $SU(2)_L$ dipole operator of the electron (up to self-renormalisation) are reproduced.

Shortly after, ref. [50] appeared where the computation of SM EFT operator anomalous dimensions using unitarity cuts is also discussed and the self-renormalisation of $|H|^2 B_{\mu\nu}^2$ and $G_{\mu\nu}^3$ is reproduced. There instances of applying these techniques to operators of dimension larger than six are also discussed.

Before we submitted to the journal, ref. [51] appeared where the use of on-shell methods to compute one and two-loop anomalous dimensions in the context of EFTs containing higher-dimension operators is described. The pattern of vanishing entries in the two-loop anomalous dimension matrix is also analysed.

Acknowledgments

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A Phase space integrals

We are interested in evaluating “ $n \rightarrow 2$ ” integrals of the type

$$I_{n \rightarrow 2} = \int \frac{1}{n!} \prod_{i=1}^n \frac{d^3 p'_i}{(2\pi)^3 2p'_i} (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=1}^n p'_i \right) \langle 12 | M_{n \rightarrow 2} | 1' \dots n' \rangle \langle 1' \dots n' | \mathcal{O} | 0 \rangle \tag{A.1}$$

These integrals feature an S-matrix element with n particles in the initial state and 2 in the final state. We include a symmetry factor $1/n!$ for identical particles.

For massless particles, the integrand is expressed more conveniently in terms of spinor helicity variables λ'_i , rather than the momenta. Assuming that $s_{12} = \langle 12 \rangle [21]$ is nonzero, the two spinors corresponding to the external particles λ_1 and λ_2 are linearly independent, and form a complete basis. Therefore the spinors for the n exchanged particles (whose momenta are integrated over) may be expressed as

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \vdots \\ \lambda'_n \end{pmatrix} = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix}, \tag{A.2}$$

where the a_{ib} are arbitrary complex numbers. Next we recast the measure and momentum conserving delta function according to the change of variables in (A.2).

The momentum conserving delta function is then given by

$$\delta^4(P) = 4\delta(\lambda^1\bar{\lambda}^1)\delta(\lambda^1\bar{\lambda}^2)\delta(\lambda^2\bar{\lambda}^1)\delta(\lambda^2\bar{\lambda}^2) = \frac{4}{\langle 12 \rangle^2 [12]^2} \delta(u_{11})\delta(u_{12})\delta(u_{21})\delta(u_{22}), \quad (\text{A.3})$$

where $P^{\alpha\dot{\alpha}} \equiv \lambda^\alpha \bar{\lambda}^{\dot{\alpha}} \equiv \lambda_1^\alpha \bar{\lambda}_1^{\dot{\alpha}} + \lambda_2^\alpha \bar{\lambda}_2^{\dot{\alpha}} - \sum_{i=1}^n \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}$. And we have defined the 2×2 matrix $u_{nm} = \delta_{nm} - (A^\dagger A)_{nm}$. Thus, the momentum conserving delta function imposes the constraint $A^\dagger A = \mathbb{1}_2$.

Now we consider the measure for one of the n exchanged particles. We would like to express it in terms of the a_{ib} variables, perhaps like:

$$\frac{d^3 p'_i}{(2\pi)^3 2p_i^0} \stackrel{?}{=} f(a_{i1}, a_{i2}) da_{i1} da_{i1}^* da_{i2} da_{i2}^*. \quad (\text{A.4})$$

This measure only depends on a_{i1} and a_{i2} because the measure depends only upon p_i^μ and $p_i^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}$. However, there are three real variables to be integrated over on the l.h.s. but four on the r.h.s., so the relationship above cannot be true. The problem arises due to a redundant phase that emerges when the kinematic information is expressed in the a_{ib} variables rather than in the momenta. To be specific, we can rotate complex a_{i1} and a_{i2} by a common phase so that

$$a_{i1} \rightarrow e^{i\alpha_i} a_{i1}, \quad a_{i2} \rightarrow e^{i\alpha_i} a_{i2}, \quad \lambda'_i \rightarrow e^{i\alpha_i} \lambda'_i. \quad (\text{A.5})$$

This rotation leaves $p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}$ unchanged, so clearly the measure, and the momentum conserving delta function are invariant under this rotation. Equally, the integrand is also independent of this global phase

$$\langle 12 | \mathcal{M}_{n \rightarrow 2} | 1' \dots n' \rangle \langle 1' \dots n' | \mathcal{O} | 0 \rangle \rightarrow e^{2ih_i\alpha_i} \langle 12 | \mathcal{M}_{n \rightarrow 2} | 1' \dots n' \rangle \cdot e^{-2ih_i\alpha_i} \langle 1' \dots n' | \mathcal{O} | 0 \rangle, \quad (\text{A.6})$$

where h_i is the helicity of particle i . All quantities in the integral are therefore invariant under the global phase transformation, so that adjusting it is analogous to performing a gauge transformation in a gauge theory. Just as in gauge theory, it is convenient to fix a value for this ‘‘gauge parameter’’. This can be done in a way that still leaves the equivalence of different gauge fixing conditions manifest using the Fadeev-Popov trick: we multiply the measure by the following factor

$$\int d\alpha_i \delta(f_i(\alpha_i)) f'_i(\alpha_i) = 1. \quad (\text{A.7})$$

Defining the overall phase to be $\alpha_i \equiv \arg(a_{i1}) + \arg(a_{i2})$, the measure becomes

$$\frac{d^3 p'_i d\theta_i}{(2\pi)^3 2p_i^0} = \frac{\langle 12 \rangle [21]}{2(2\pi)^3} da_{i1} da_{i1}^* da_{i2} da_{i2}^*. \quad (\text{A.8})$$

Putting everything back together, the original integral can be written as

$$I_{n \rightarrow 2} = \frac{(\langle 12 \rangle [21])^{n-2}}{2^{n-2} n! (2\pi)^{3n-4}} \int \prod_{i=1}^n [d^2 a_{i1} d^2 a_{i2} \delta(f_i(\alpha_i)) f'_i(\alpha_i)] \delta^4(A^\dagger A - \mathbb{1}_2) \times \langle 12 | \mathcal{M}_{n \rightarrow 2} | 1' \dots n' \rangle \langle 1' \dots n' | \mathcal{O} | 0 \rangle. \quad (\text{A.9})$$

A.1 The $2 \rightarrow 2$ case

This is the simplest case. We would like to parametrize the matrix A so that the delta function constraints take a particularly simple form. This can be achieved with a polar decomposition

$$A = UP, \quad (\text{A.10})$$

where U is a unitary 2×2 matrix and P is a positive-semidefinite hermitian matrix. Without loss of generality we parametrise P and U as

$$P = \begin{pmatrix} r & u + iv \\ u - iv & t \end{pmatrix}, \quad U = \begin{pmatrix} e^{i\rho} \cos \theta & -e^{i\rho} \sin \theta e^{i\phi} \\ e^{i\eta} \sin \theta e^{-i\phi} & e^{i\eta} \cos \theta \end{pmatrix}, \quad (\text{A.11})$$

where r, u, v, t are real valued and the angles have the ranges $0 \leq \rho, \eta, \phi < 2\pi$ and $0 \leq \theta < \pi/2$. The delta function constraint acts purely on the hermitian matrix (A.10)

$$\delta^4(A^\dagger A - \mathbb{1}_2) = \frac{1}{32} \delta(r-1) \delta(u) \delta(v) \delta(t-1). \quad (\text{A.12})$$

Next, using (A.11) and fixing $P = \mathbb{1}_2$, the delta function constraints that eliminate gauge redundancy take a simpler form $\theta_1 = 2\rho + \phi$, $\theta_2 = 2\eta - \phi$. We can choose any topologically equivalent functions for the $f_i(\alpha_i)$, but it is convenient to pick the gauge where $f_1(\alpha_1) = \frac{1}{2}(\alpha_1 - \phi)$ and $f_2(\alpha_2) = \frac{1}{2}(\alpha_2 + \phi)$. The gauge fixing delta functions become $\delta(f_1(\alpha_1)) f_1'(\alpha_1) \delta(f_2(\alpha_2)) f_2'(\alpha_2) = \frac{1}{4} \delta(\rho) \delta(\eta)$. All in all, in terms of the new parameters, the integral measure becomes

$$d^2 a_{11} d^2 a_{12} d^2 a_{21} d^2 a_{22} = 32 \sin 2\theta dr du dv dt d\rho d\eta d\theta d\phi, \quad (\text{A.13})$$

once we have set $P = \mathbb{1}_2$. Putting everything together, the $2 \rightarrow 2$ integral becomes

$$I_{2 \rightarrow 2} = \frac{1}{16\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\pi/2} 2 \sin \theta \cos \theta d\theta \langle 12 | \mathcal{M}_{2 \rightarrow 2} | 1'2' \rangle \langle 1'2' | \mathcal{O} | 0 \rangle. \quad (\text{A.14})$$

A.2 The $3 \rightarrow 2$ case

Here we consider the phase space integral when 3 particles are exchanged across the cut. The calculation of the integral measure and transformation law for the spinors representing the exchanged particles proceeds in a similar way to the 2 particle case; the delta function constraints take a simpler form when the complex 3×2 matrix A is first expressed using a singular value decomposition, analogous to a polar decomposition for matrices that are not square

$$A = U \Sigma V^\dagger. \quad (\text{A.15})$$

U is a unitary 3×3 matrix, Σ is a 3×2 rectangular diagonal matrix with real, non-negative entries, and V is a unitary 2×2 matrix. This decomposition is not unique however. It can be seen that applying the following transformation leaves A unchanged:

$$U \rightarrow U \text{diag}\{e^{i\phi_a}, e^{i\phi_b}, e^{i\phi_c}\}, \quad V \rightarrow V \text{diag}\{e^{i\phi_a}, e^{i\phi_b}\}. \quad (\text{A.16})$$

To eliminate this redundancy, we choose to fix two of the undetermined phases in V and one in U . With these restrictions, U can be parametrized as

$$U = \text{diag}\{e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4}\} \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix} \text{diag}\{1, e^{i\phi}, 1\}, \quad (\text{A.17})$$

where $c_i \equiv \cos \theta_i$ and $s_i \equiv \sin \theta_i$. Note that alternative parametrizations of an arbitrary 3×3 unitary matrix can be found by taking any parametrization of the CKM matrix and multiplying on either side by diagonal matrices of phases. We parametrize Σ and V (unitary matrix minus two redundant phases) as

$$\Sigma = \begin{pmatrix} r & 0 \\ 0 & t \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} c_4 & -s_4e^{i\phi_5} \\ s_4e^{-i\phi_5} & c_4 \end{pmatrix}. \quad (\text{A.18})$$

In each case, the allowed ranges for the parameters is $0 \leq \theta_i < \pi/2$, $0 \leq r, t < \infty$ and $0 \leq \phi_i < 2\pi$. The delta function constraints depend only upon the parameters in Σ and V :

$$\delta(A^\dagger A - \mathbb{1}_2) = \frac{1}{8rt(r^2 - t^2)^2 s_4 c_4} \delta(r - 1) \delta(t - 1) \delta(\theta_4) \delta(\phi_5). \quad (\text{A.19})$$

The denominator is zero at the point in parameter space selected by the delta functions. This is just a harmless coordinate singularity though, which gets removed when combined with the integral measure in these new variables. This measure is

$$\prod_{i=1}^3 [d^2 a_{i1} d^2 a_{i2}] = 64r^3 t^3 (r^2 - t^2)^2 s_1^3 c_1 s_2 c_2 s_3 c_3 s_4 c_4 \prod_{j=1}^4 d\theta_j \prod_{k=1}^5 d\phi_k dr dt d\delta. \quad (\text{A.20})$$

Similarly to the $2 \rightarrow 2$ case, the gauge fixing delta functions become

$$\delta(f_1(\alpha_1)) f'_1(\alpha_1) \delta(f_2(\alpha_2)) f'_2(\alpha_2) \delta(f_3(\alpha_3)) f'_3(\alpha_3) = \frac{1}{8} \delta(\phi_2) \delta(\phi_3) \delta(\phi_4). \quad (\text{A.21})$$

Putting everything together into (A.9), yields the following result for the $I_{3 \rightarrow 2}$ integral

$$I_{3 \rightarrow 2} = \frac{\langle 12 \rangle [21]}{4^4 \pi^3 3!} \int d\Omega_3 \langle 12 | \mathcal{M}_{3 \rightarrow 2} | 1' 2' 3' \rangle \langle 1' 2' 3' | \mathcal{O} | 0 \rangle, \quad (\text{A.22})$$

with the measure given by

$$d\Omega_3 = 4 \cos \theta_1 \sin^3 \theta_1 d\theta_1 2 \cos \theta_2 \sin \theta_2 d\theta_2 2 \cos \theta_3 \sin \theta_3 d\theta_3 \frac{d\delta}{2\pi} \frac{d\phi}{2\pi}. \quad (\text{A.23})$$

After the all of the constraints have been applied, the transformation law for the spinors is

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = \begin{pmatrix} c_1 & -e^{i\phi} c_3 s_1 \\ c_2 s_1 & e^{i\phi} (c_1 c_2 c_3 - e^{i\delta} s_2 s_3) \\ s_1 s_2 & e^{i\phi} (c_1 c_3 s_2 + e^{i\delta} c_2 s_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \quad (\text{A.24})$$

B Amplitudes

We computed some high point amplitudes of the text using the BCFW technique [52].¹³ In this appendix we illustrate how to compute the five point amplitude in (2.37). An advantage of using BCFW w.r.t. Feynman diagrams is that it automatically produces nice compact expressions that simplify our phase-space integrations.

To apply BCFW recursion relations we need the lower order seeds, namely the three point amplitudes. The three point amplitude between two fermions and a gluon is¹⁴

$$M(1_i^- 2_j^+ 3_A^-) = -g\sqrt{2} T_{ij}^A \frac{\langle 13 \rangle^2}{\langle 12 \rangle} \quad \text{and} \quad M(1_i^- 2_j^+ 3_A^+) = -g\sqrt{2} T_{ij}^A \frac{[13]^2}{[12]}, \quad (\text{B.1})$$

and for a Yukawa interaction $\mathcal{L} \supset -yH\bar{q}_L q_R$,

$$M(1^- 2^- 3) = -y\langle 12 \rangle \quad \text{and} \quad M(1^+ 2^+ 3) = -y[12]. \quad (\text{B.2})$$

In this case we do not need the three point amplitude between two scalars and a gauge boson because the Higgs does not carry color.

To compute the five point amplitude (2.37) we need the 4-point amplitude between two same-helicity fermions, a scalar and a gluon. It is easy to do it with the usual techniques, but we will do it using the BCFW recursion relation. See [12] for a review.

BCFW proceeds by shifting the spinors by a complex parameter z so that the shifted momenta \hat{p} is on-shell and total momentum is preserved, $\hat{p}^2 = 0$ and $\sum_i p_i^\mu = 0$. For instance, in a n -particle amplitude a $[1, 2]$ -shift consists on shifting the spinors of p_1 and p_2 as $|\hat{1}\rangle = |1\rangle + z|2\rangle, |\hat{2}\rangle = |2\rangle, |\hat{1}\rangle = |1\rangle, |\hat{2}\rangle = |2\rangle - z|1\rangle$, and one can check that this preserves momentum conservation of on-shell momenta. For instance, applying this shift on the $M(1_i^- 2_A^\pm 3_j^- 4)$ amplitude, BCFW splits it into two on-shell 3-point amplitudes

$$= M_L(\hat{1}_i^-, \hat{p}_k^-, 4) \frac{1}{\langle 14 \rangle [14]} M_R(-\hat{p}_k^-, 3_j^-, \hat{2}_A^\pm). \quad (\text{B.3})$$

There are two different cases depending on the helicity of the gluon. If the gluon has negative helicity, the numerator of the right amplitude is $\langle 3\hat{2} \rangle^2$. However, this is evaluated at a $\hat{p}^2 = 0 = \langle \hat{2}3 \rangle [\hat{2}3] = \langle \hat{2}3 \rangle [23]$. This means that z is such that $\langle \hat{2}3 \rangle = 0$, and therefore we recover that the all-minus amplitude vanishes $M(1_i^- 2_A^- 3_j^- 4) = 0$. Instead, if the gluon has positive helicity we get,

$$M(1_i^- 2_A^+ 3_j^- 4) = yg\sqrt{2} T_{ij}^A \langle \hat{1}\hat{p} \rangle \frac{1}{\langle 14 \rangle [14]} \frac{[\hat{p}\hat{2}]^2}{[\hat{p}3]}, \quad (\text{B.4})$$

where we used $\hat{p} = -\hat{p}_1 - p_4, \langle 1\hat{p} \rangle [\hat{p}2] = -\langle 14 \rangle [42]$. A basic trick to get rid of \hat{p} is to multiply the amplitude by $\langle 2\hat{p} \rangle / \langle 2\hat{p} \rangle$, and then write $\langle 2\hat{p} \rangle [\hat{p}2] = \langle 23 \rangle [32]$ and $\langle 2\hat{p} \rangle [\hat{p}3] = \langle 2\hat{2} \rangle [23]$. As we showed, this last factor has to be evaluated at $\langle \hat{2}3 \rangle = 0 = \langle 23 \rangle - z\langle 13 \rangle$ so that

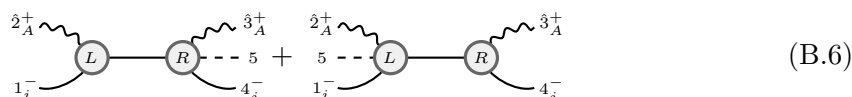
¹³Note that some of the SM amplitudes have a pole at infinity in the BCFW complex plane. In those cases we used standard Feynman diagram calculation.

¹⁴The normalisation is such that the covariant derivative is $D_\mu = \partial_\mu + iT_{ij}^A G_\mu^A$.

$z = -\langle 23 \rangle / \langle 13 \rangle$. At this value, $\langle 2\hat{2} \rangle = -z\langle 12 \rangle = \langle 12 \rangle \langle 23 \rangle / \langle 13 \rangle$. Then putting everything together, we get

$$M(1_i^- 2_A^+ 3_j^- 4) = -yg\sqrt{2}T_{ij}^A \frac{[24]^2}{[14][34]} = yg\sqrt{2}T_{ij}^A \frac{\langle 13 \rangle^2}{\langle 12 \rangle \langle 23 \rangle}. \quad (\text{B.5})$$

Next we use the 4-point amplitude in (B.5) to construct the 5-point amplitude $M(1_i^- 2_A^+ 3_B^+ 4_j^- 5)$. We do a $[2, 3]$ -shift. There are two contributions,



the two diagrams are given by

$$M_L(1_i^- \hat{p}_k^+ \hat{2}_A^+) \frac{1}{p_{12}^2} M_R(-\hat{p}_k^- \hat{3}_B^+ 4_j^- 5) + M_L(1_i^- \hat{2}_A^+ \hat{p}_k^- 5) \frac{1}{p_{34}^2} M_R(-\hat{p}_k^+ \hat{3}_B^+ 4_j^-), \quad (\text{B.7})$$

where we have fixed a particular color ordering, namely $(T^A T^B)_{ij}$. Exchanging $2 \leftrightarrow 3$ in the sub-amplitudes generates the other color ordering $(T^B T^A)_{ij}$. The first diagram vanishes because it is evaluated at $[1\hat{2}] = 0$, and $[\hat{p}][\hat{p}\hat{2}] = (p_1 + \hat{p}_2)[\hat{2}] = -[1][\hat{2}] = 0$. Hence $[\hat{p}\hat{2}] = 0$ as well, and therefore the left amplitude vanishes. The second diagram gives

$$M(1_i^- 2_A^+ 3_B^+ 4_j^- 5) = -2yg^2(T^A T^B)_{ij} \frac{\langle 1\hat{p} \rangle^2}{\langle 1\hat{2} \rangle \langle 2\hat{p} \rangle} \frac{1}{\langle 34 \rangle [34]} \frac{[\hat{p}\hat{3}]^2}{[\hat{p}4]}. \quad (\text{B.8})$$

The same argument does not work for this diagram, since the right amplitude is proportional to $[\hat{p}\hat{3}]$ but instead it is evaluated at $\langle \hat{3}4 \rangle = 0$. Now, it is straightforward to simplify the expression using $\langle 1\hat{p} \rangle [\hat{p}\hat{3}] = \langle 14 \rangle [43]$ and $\langle 2\hat{p} \rangle [\hat{p}4] = \langle 2\hat{3} \rangle [34]$; also, note that $\langle 2\hat{3} \rangle = \langle 23 \rangle$. Therefore the 5-point amplitude is

$$M(1_i^- 2_A^+ 3_B^+ 4_j^- 5) = -2yg^2(T^A T^B)_{ij} \frac{\langle 14 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} - 2yg^2(T^B T^A)_{ij} \frac{\langle 14 \rangle^2}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle}. \quad (\text{B.9})$$

The amplitude can be crosschecked versus the $N = 4$ SYM Parke-Taylor, up to color factors.

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