Semiclassical quantization of the mixed-flux AdS_3 giant magnon

Adam Varga

*Department of Mathematics, School of Mathematics,
Computer Science & Engineering, University of London,
City, EC1V 0HB London, U.K.*

E-mail: Adam.Varga@city.ac.uk

ABSTRACT: We obtain explicit formulas for the eight bosonic and eight fermionic fluctuations around the mixed-flux generalization of the Hofman-Maldacena giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. As a check of our results, we confirm that the semiclassical quantization of these fluctuations leads to a vanishing one-loop correction to the magnon energy, as expected from symmetry based arguments.

KEYWORDS: AdS-CFT Correspondence, Superstrings and Heterotic Strings

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1 Introduction

An important aspect of the $\text{AdS}_5/\text{CFT}_4$ correspondence [1] is integrability, a hidden symmetry present both on the $\mathcal{N} = 4$ super Yang-Mills gauge theory side [2–4] and $\text{AdS}_5 \times S^5$ type IIB superstring theory side [5–9] of the duality. Interactions in a quantum integrable theory reduce to a series of diffractionless two-body scattering processes, and in the decompactified worldsheet limit the spectrum is solvable using a Bethe Ansatz [10, 11]. Therefore,

the main object of interest in (the planar limit of) AdS/CFT is the S-matrix, encoding these two-body scatterings of elementary excitations, or magnons [12]. $\text{AdS}_5/\text{CFT}_4$ has a $\mathfrak{psu}(2,2|4)$ symmetry, and the subalgebra leaving the vacuum invariant is $\mathfrak{su}(2|2)^2$. The off-shell, centrally extended version of this residual algebra, $\mathfrak{su}(2|2)_{\text{c.e.}}^2$, fixes the S-matrix up to an overall phase [13, 14], which then can be calculated from the so-called crossing symmetry [15–19]. These algebraic arguments also determine the magnon dispersion relation to be

$$\epsilon = \sqrt{1 + 4h^2 \sin^2 \frac{p}{2}}, \quad (1.1)$$

where p is the magnon momentum, and the effective string tension h is related to the string tension α' and AdS radius R on the string theory side, and the 't Hooft coupling λ on the gauge theory side by

$$h = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}. \quad (1.2)$$

Solitons are particle-like solutions of integrable field theories, whose dynamics can be captured by a small number of collective degrees of freedom. Quantization of these collective coordinates [20–23] provides a window into regimes of the quantum theory not directly accessible to perturbation methods. The giant magnon of Hofman and Maldacena [24] is a soliton of the integrable $\text{AdS}_5 \times S^5$ worldsheet sigma-model [25], whose spacetime image is an open string uniformly rotating around an axis of an $S^2 \subset S^5$, stretched between two points on the equator. In fact, the worldsheet momentum p of the giant magnon is the angular distance between these two points, and its dispersion relation $\epsilon = 2h \sin \frac{p}{2}$ agrees with the large h limit of (1.1). An $\mathbb{R} \times S^3$ generalization of this solution, the dyonic giant magnon [26], has the dispersion relation

$$\epsilon = \sqrt{J_2 + 4h^2 \sin^2 \frac{p}{2}}, \quad (1.3)$$

where J_2 is the second angular momentum on S^3 . Upon semiclassical quantization J_2 takes integer values, and we recover the exact dispersion relation (1.1).

There are a number of calculations one can perform to check that the giant magnon is indeed the large coupling limit of the elementary excitation of the quantum theory. A semiclassical analysis of the worldsheet scattering of dyonic giant magnons [27] shows that their 1-loop S-matrix agrees with the Hernandez-Lopez phase [16], and also that the 1-loop correction to the giant magnon energy vanishes. From an algebraic perspective the magnon is a BPS state of the $\mathfrak{su}(2|2)_{\text{c.e.}}^2$ superalgebra, and accordingly, must be part of a 16 dimensional short multiplet [14]. As a consequence the giant magnon should have eight fermionic zero modes, as Hofman and Maldacena argued in [24]. These zero modes were explicitly constructed by Minahan [28], starting from the quadratic fermionic part of the Green-Schwarz action expanded around the giant magnon. Quantizing these modes he was also able to reproduce the odd generators of the residual algebra. Subsequently, building on Minahan's work, an explicit basis of the magnon's fluctuation spectrum was found by Papathanasiou and Spradlin [29], once again confirming that the dispersion relation receives no corrections, by showing that the 1-loop functional determinant vanishes.

With 32 supercharges $\text{AdS}_5/\text{CFT}_4$ has the maximal amount of supersymmetry possible for any 10 dimensional string theory, but integrability has proven to be a powerful tool in understanding other, less supersymmetric AdS/CFT dualities too. One example is $\text{AdS}_4/\text{CFT}_3$, the duality between ABJM Super Chern-Simons and type IIA string theory on $\text{AdS}_4 \times CP^3$ with 24 supersymmetries [30], however, for the rest of this paper we focus on $\text{AdS}_3/\text{CFT}_2$, and in particular two¹ backgrounds with maximal supersymmetry allowed for such geometries (16 supercharges). One of them is $\text{AdS}_3 \times S^3 \times T^4$, where the radii of AdS_3 and S^3 are equal, and the other one is $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, where the AdS radius R and the radii of the two 3-spheres $R_{\pm}$ satisfy [31]

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{R^2}. \quad (1.4)$$

Setting the AdS_3 radius to one, this geometry can be parametrized by an angle φ

$$R_+^2 = \frac{1}{\cos^2 \varphi}, \quad R_-^2 = \frac{1}{\sin^2 \varphi}, \quad (1.5)$$

and in fact the $\varphi \rightarrow 0$ limit covers the $\text{AdS}_3 \times S^3 \times T^4$ geometry too, once the blown up sphere is compactified on a torus. The type IIB supergravity equations allow these backgrounds to be supported by pure Ramond-Ramond (R-R) flux, pure Neveu-Schwarz-Neveu-Schwarz (NS-NS) flux, or mixed R-R and NS-NS fluxes

$$\begin{aligned} F &= 2\tilde{q} \left(\text{Vol}(\text{AdS}_3) + \cos \varphi \text{Vol}(S_+^3) + \sin \varphi \text{Vol}(S_-^3) \right), \\ H &= 2q \left(\text{Vol}(\text{AdS}_3) + \cos \varphi \text{Vol}(S_+^3) + \sin \varphi \text{Vol}(S_-^3) \right), \end{aligned} \quad (1.6)$$

where $q \in [0, 1]$ and $\tilde{q} = \sqrt{1 - q^2}$. While the pure NS-NS theory can be solved using a chiral decomposition [32–34], no such method exists when the R-R flux is turned on, instead, it is believed that the solution will be given in terms of integrable methods,² as both the pure R-R [39–41] and mixed-flux [42] theories were shown to be classically integrable.

The $\text{AdS}_3 \times S^3 \times T^4$ backgrounds with pure R-R and pure NS-NS fluxes arise as near horizon limits of the D1/D5 and F1/NS5 brane systems, respectively. Historically, the mixed-flux background has been thought of as the near-horizon limit of bound states of D1/D5- and F1/NS5-branes, but it was recently shown that the same worldsheet action arises in a pure NS-NS theory with an R-R modulus turned on [43]. Integrable structures have been identified in the CFT_2 dual to $\text{AdS}_3 \times S^3 \times T^4$ strings [44], and there has also been promising progress in understanding the CFT_2 dual of string theory on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ [45–50].

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¹There is a third maximally supersymmetric AdS_3 background, $\text{AdS}_3 \times S^3 \times K3$. It should be possible to apply integrable methods to this background, at least in the orbifold limit of $K3$, and then it would be interesting to see what the effect of turning on the blow-up modes is.

²Although it is worth noting that there have been attempts to understand the mixed-flux theory using the hybrid formalism of Berkovits, Vafa and Witten [35–38].

Similarly to the $\text{AdS}_5/\text{CFT}_4$ duality, the symmetry algebra can be used to determine both the S-matrix and the all-loop magnon dispersion relation [51–53]

$$\epsilon_{\pm} = \sqrt{\left(m \pm q\sqrt{\lambda}\frac{p}{2\pi}\right)^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}, \quad (1.7)$$

where $h = \frac{\sqrt{\lambda}}{2\pi}$ only in the classical string limit, and in general h will receive quantum corrections. The excitations are of mass $m = 1, 0$ for $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $m = 0, \sin^2 \varphi, \cos^2 \varphi, 1$ for $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. The mixed-flux $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ dyonic giant magnon was found by Hoare, Stepanchuk and Tseytlin [54], with the dispersion relation

$$E - J_1 = \sqrt{(J_2 \pm qhp)^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}, \quad (1.8)$$

where E is the spacetime energy and J_1, J_2 are two angular momenta on the S^3 . They also noted that upon semiclassical quantization J_2 takes integer values, and the lowest $J_2 = 1$ gives an exact match to the quantum dispersion relation (1.7). Just like in the AdS_5 case, there are a number of semiclassical checks on these string solutions. The 1-loop worldsheet S-matrix has been determined from multi-soliton scattering states in [55] in agreement with the finite-gap calculations of [56]. The 1-loop correction to the magnon energy can also be calculated from the algebraic curve [57], or directly from the GS action [41, 58]. The off-shell residual symmetry algebras of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ are the centrally extended $\mathfrak{psu}(1|1)^4$ [51, 53, 59, 60] and the centrally extended $\mathfrak{su}(1|1)^2$ [61, 62], and the BPS magnon must transform in 4 and 2 dimensional short multiplets of these superalgebras, respectively. Therefore, the mixed-flux magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ should have 4 and 2 fermion zero modes. We found these zero modes in [63], and showed how they can be used to construct the odd generators of the residual algebras. Our objective here is to find the complete spectrum of fluctuations around the AdS_3 giant magnon. Throughout, we will only consider the *stationary magnon*, a subclass of solutions we identified as the mixed-flux generalisation of the HM giant magnon in [63]. The rest of this paper is structured as follows.

In section 2 we first review the mixed-flux stationary giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, then write down the spectrum of small bosonic fluctuations around the classical solution. Although the perturbation equations are rather complicated, one can construct explicit solutions algebraically using the dressing method, which we adapt to be more suited to the fluctuation analysis. In section 3 we find the fermionic fluctuations, closely following the methods developed in [28, 63] extended to non-zero angular frequencies. Using the symmetries of the system and an explicit kappa-fixed ansatz, the full 2×32 component spinor equations are reduced to a 4 dimensional system, which we can solve explicitly.

Finally in section 4 we read off the stability angles of the fluctuations, and use them to evaluate the 1-loop functional determinant around the soliton background, following the method of Dashen, Hasslacher and Neveu [64]. We find that, in agreement with our expectations based on the superalgebra, the leading order quantum correction vanishes. We conclude in section 5 and present some of the lengthier or more technical details in the appendices.

2 Bosonic sector

In this section we review the mixed-flux AdS_3 stationary magnon, and solve for its bosonic fluctuations using a similar approach employed to study the AdS_5 magnon in [29]. We consider the case of the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ background in our calculations, and comment on how the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ modes can be obtained at the end of the section.

The conformal gauge bosonic action can be written in the form

$$S = \tilde{S}[Y] + \frac{1}{\cos^2 \varphi} S_+[X^+] + \frac{1}{\sin^2 \varphi} S_-[X^-], \quad (2.1)$$

with AdS_3 and $\text{S}^3_\pm$ components

$$\begin{aligned} \tilde{S}[Y] &= -\frac{\hbar}{2} \int_{\mathcal{M}} d^2x \left[\eta^{ab} \partial_a Y^i \partial_b Y_i + \tilde{\Lambda} (Y^2 + 1) \right] \\ &\quad - \frac{\hbar q}{3} \int_{\mathcal{B}} d^3x \epsilon^{abc} \epsilon_{\mu\nu\rho\sigma} Y^\mu \partial_a Y^\nu \partial_b Y^\rho \partial_c Y^\sigma \\ S_\pm[X] &= -\frac{\hbar}{2} \int_{\mathcal{M}} d^2x \left[\eta^{ab} \partial_a X^i \partial_b X_i + \Lambda_\pm (X^2 - 1) \right] \\ &\quad - \frac{\hbar q}{3} \int_{\mathcal{B}} d^3x \epsilon^{abc} \epsilon_{ijkl} X^i \partial_a X^j \partial_b X^k \partial_c X^l \end{aligned} \quad (2.2)$$

where $\eta^{ab} = \text{diag}(-1, +1)$, the embedding coordinates $Y \in \mathbb{R}^{2,2}$, $X^\pm \in \mathbb{R}^4$ are enforced to lie on the unit-radius surfaces

$$Y^2 = -1, \quad (X^\pm)^2 = 1 \quad (2.3)$$

by the Lagrange multipliers $\tilde{\Lambda}, \Lambda_\pm$, and the Wess-Zumino term is defined on a 3d manifold $\mathcal{B}$ such that its boundary is the worldsheet $\partial\mathcal{B} = \mathcal{M}$. The equations of motion

$$\begin{aligned} (\partial^2 - \tilde{\Lambda}) Y_\mu - q \tilde{K}_\mu &= 0, & \tilde{K}_\mu &= \epsilon^{ab} \epsilon_{\mu\nu\rho\sigma} Y^\nu \partial_a Y^\rho \partial_b Y^\sigma, \\ (\partial^2 - \Lambda_\pm) X_i^\pm - q K_i^\pm &= 0, & K_i^\pm &= \epsilon^{ab} \epsilon_{ijkl} X_j^\pm \partial_b X_k^\pm \partial_c X_l^\pm, \end{aligned} \quad (2.4)$$

need to be supplemented by the conformal gauge Virasoro constraints

$$\begin{aligned} (\partial_0 Y)^2 + (\partial_1 Y)^2 + \frac{1}{\cos^2 \varphi} ((\partial_0 X^+)^2 + (\partial_1 X^+)^2) \\ + \frac{1}{\sin^2 \varphi} ((\partial_0 X^-)^2 + (\partial_1 X^-)^2) &= 0, \\ \partial_0 Y \cdot \partial_1 Y + \frac{1}{\cos^2 \varphi} \partial_0 X^+ \cdot \partial_1 X^+ + \frac{1}{\sin^2 \varphi} \partial_0 X^- \cdot \partial_1 X^- &= 0. \end{aligned} \quad (2.5)$$

Taking scalar products of (2.4) with $Y, X^\pm$, it follows from (2.3) and

$$Y^\mu \tilde{K}_\mu = 0, \quad X^{\pm i} K_i^\pm = 0, \quad (2.6)$$

that the Lagrange multipliers take the classical values

$$\tilde{\Lambda} = -Y \cdot \partial^2 Y, \quad \Lambda_\pm = X^\pm \cdot \partial^2 X^\pm. \quad (2.7)$$

2.1 The stationary giant magnon

The classical solution we consider for the rest of this paper is the stationary mixed-flux giant magnon

$$\begin{aligned}
 Y^0 + iY^1 &= e^{it} \\
 X_1^- + iX_2^- &= e^{i\sin^2\varphi t} \\
 Z_1 \equiv X_1^+ + iX_2^+ &= e^{i\cos^2\varphi t} \left[\cos \frac{p}{2} + i \sin \frac{p}{2} \tanh \mathcal{Y} \right] \\
 Z_2 \equiv X_3^+ + iX_4^+ &= e^{-\frac{iq}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} \sin \frac{p}{2} \operatorname{sech} \mathcal{Y}
 \end{aligned} \tag{2.8}$$

where the scaled and boosted worldsheet coordinate is

$$\mathcal{Y} = \cos^2\varphi \, \gamma \sqrt{\tilde{q}^2 - u^2} \mathcal{X}, \quad \mathcal{X} = \gamma(x - ut) \tag{2.9}$$

and

$$\tilde{q} = \sqrt{1 - q^2}, \quad \gamma^2 = \frac{1}{1 - u^2}. \tag{2.10}$$

The parameter u , restricted to $u \in (-\tilde{q}, \tilde{q})$, can be regarded as the velocity of the magnon. The worldsheet momentum $p \in [0, 2\pi)$ is not a Noether charge of the action, rather a topological charge of the soliton, corresponding to the longitudinal distance between the two endpoints of the magnon on the equator of S_+^3 ($Z_2 = 0$). The parameters further satisfy

$$u = \tilde{q} \cos \frac{p}{2}. \tag{2.11}$$

This is a special case of the dyonic mixed-flux magnon, which was first constructed in [54] for the $\text{AdS}_3 \times S^3 \times T^4$ background. The stationary magnon was identified in [63] as the mixed-flux equivalent of the Hofman-Maldacena magnon [24], as compared to the more general AdS_5 dyonic magnon of [26]. The dispersion relation³

$$E - J_1 = 2h\tilde{q} \sin \frac{p}{2}, \tag{2.12}$$

bears witness to this analogy, to be compared to the similarly simple $E - J_1 = 2h \sin \frac{p}{2}$ for the $q = 0$ HM magnon. The Lagrange multipliers (2.7) evaluate to the classical values

$$\tilde{\Lambda} = 1, \quad \Lambda_- = -\sin^4\varphi, \quad \Lambda_+ = \cos^4\varphi (1 - 2\tilde{q}^{-2}\gamma^2(\tilde{q}^2 - u^2) \operatorname{sech}^2\mathcal{Y}). \tag{2.13}$$

2.2 AdS_3 fluctuation spectrum

Let us now determine the spectrum of fluctuations around the mixed-flux magnon (2.8), starting with the AdS_3 bosons. We denote the perturbed solution by

$$Y + \delta \tilde{y} \tag{2.14}$$

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³ E is the spacetime energy, J_1 is the angular momentum corresponding to the maximally supersymmetric geodesic along the equators of $S_\pm^3$.

where Y is the classical solution, $\delta \ll 1$ and the perturbation $\tilde{y} \in \mathbb{R}^{2,2}$ is bounded. Substituting into the equation (2.4) and expanding to first order in δ (note that $\tilde{\Lambda}$ also receives corrections) we get the perturbation equation

$$(\partial^2 - 1) \tilde{y}_\mu + (Y \cdot \partial^2 \tilde{y} + q \tilde{K} \cdot \tilde{y}) Y_\mu - q \tilde{k}_\mu = 0 \quad (2.15)$$

where $\tilde{K}_\mu$ is as in (2.4) and

$$\tilde{k}_\mu = \epsilon^{ab} \epsilon_{\mu\nu\rho\sigma} (\tilde{y}^\nu \partial_a Y^\rho \partial_b Y^\sigma + 2 Y^\nu \partial_a \tilde{y}^\rho \partial_b Y^\sigma). \quad (2.16)$$

Furthermore, to preserve the norm (2.3), the perturbation must be orthogonal to the classical solution

$$Y_\mu \tilde{y}^\mu = 0. \quad (2.17)$$

These equations have one massless and two massive solutions. To get the massless perturbation we make the ansatz

$$\tilde{y}^0 = -f \sin t, \quad \tilde{y}^1 = f \cos t, \quad (2.18)$$

for which (2.15) reduces to the free wave equation

$$\partial^2 f = 0 \quad \Rightarrow \quad f = e^{ikx - i\omega t} \quad (2.19)$$

satisfying the massless dispersion relation $\omega^2 = k^2$. The remaining two massive solutions lie in the transverse directions ($\tilde{y}^0 = \tilde{y}^1 = 0$) of AdS_3 , automatically satisfying (2.17). A simple plane-wave ansatz gives

$$\tilde{y}^2 = e^{ikx - i\omega t}, \quad \tilde{y}^3 = \mp i e^{ikx - i\omega t}, \quad \omega^2 = (1 \pm qk)^2 + \tilde{q}^2 k^2. \quad (2.20)$$

Note that this is the small p , fixed $k = \text{hp}$ limit of the mixed-flux $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ dispersion relation [53]

$$\epsilon_\pm = \sqrt{(m \pm q\text{hp})^2 + 4 \tilde{q}^2 h^2 \sin^2 \frac{p}{2}}. \quad (2.21)$$

with mass $m = 1$.

2.3 S^3_- fluctuation spectrum

The S^3_- fluctuations are very similar to the ones on AdS_3 . Substituting the perturbed solution

$$X^- + \delta \tilde{x}^- \quad (2.22)$$

into (2.4), we get the first order equations

$$(\partial^2 + \sin^4 \varphi) \tilde{x}_i^- + (X^- \cdot \partial^2 \tilde{x}^- - q K^- \cdot \tilde{x}^-) X_i - q k_i^- = 0 \quad (2.23)$$

where K_i^- is as in (2.4) and

$$k_i^- = \epsilon^{ab} \epsilon_{ijkl} \left(\tilde{x}_j^- \partial_b X_k^- \partial_c X_l^- + 2 X_j^- \partial_b \tilde{x}_k^- \partial_c X_l^- \right), \quad (2.24)$$

which needs to be supplemented by $X_i^- \tilde{x}^{-i} = 0$ to preserve the norm. Just like on AdS_3 , these equations admit a massless solution

$$\begin{aligned}\tilde{x}_1^- &= -e^{ikx-i\omega t} \sin(\sin^2\varphi t), \\ \tilde{x}_2^- &= e^{ikx-i\omega t} \cos(\sin^2\varphi t), \quad \omega^2 = k^2,\end{aligned}\tag{2.25}$$

and two perturbations of mass $m = \sin^2\varphi$

$$\begin{aligned}\tilde{x}_3^- &= e^{ikx-i\omega t}, \\ \tilde{x}_4^- &= \mp e^{ikx-i\omega t}, \quad \omega^2 = (\sin^2\varphi \pm qk)^2 + \tilde{q}^2 k^2.\end{aligned}\tag{2.26}$$

2.4 S_+^3 fluctuation spectrum

For the S_+^3 perturbed solution we write

$$X^+ + \delta \tilde{x}^+, \tag{2.27}$$

and also introduce the complex coordinates

$$z_1 = \tilde{x}_1^+ + i\tilde{x}_2^+, \quad z_2 = \tilde{x}_3^+ + i\tilde{x}_4^+, \tag{2.28}$$

so that the perturbed S_+^3 component of (2.8) can be written as

$$Z_1 + \delta z_1, \quad Z_2 + \delta z_2. \tag{2.29}$$

The equations of motion for the S_+^3 fluctuations read

$$\begin{aligned}(\partial^2 - \cos^4\varphi (1 - 2\tilde{q}^{-2}\gamma^2(\tilde{q}^2 - u^2)\text{sech}^2\mathcal{Y})) \tilde{x}_i^+ \\ + (X^+ \cdot \partial^2 \tilde{x}^+ - qK^+ \cdot \tilde{x}^+) X_i - qk_i^+ = 0\end{aligned}\tag{2.30}$$

where K_i^+ is as in (2.4),

$$k_i^+ = \epsilon^{ab} \epsilon_{ijkl} \left(\tilde{x}_j^+ \partial_b X_k^+ \partial_c X_l^+ + 2X_j^+ \partial_b \tilde{x}_k^+ \partial_c X_l^+ \right), \tag{2.31}$$

and to preserve the embedding norm

$$X_i^+ \tilde{x}^{+i} = 0. \tag{2.32}$$

These equations have two different classes of solutions.

Firstly, there are the zero modes, representing collective coordinates of the magnon. The BMN limit fixes the orientation of the magnon in the (X_1^+, X_2^+) plane, but there is a rotational freedom in traverse coordinates (X_3^+, X_4^+) leading to the zero mode

$$\begin{aligned}z_1 &= 0, \\ z_2 &= ie^{-\frac{iq}{\sqrt{\tilde{q}^2 - u^2}}\mathcal{Y}} \text{sech}\mathcal{Y}.\end{aligned}\tag{2.33}$$

Furthermore, the magnon breaks the x -translation symmetry of the BMN vacuum, leading to the zero mode

$$\begin{aligned} z_1 &= i e^{i \cos^2 \varphi t} \operatorname{sech}^2 \mathcal{Y}, \\ z_2 &= -e^{-\frac{i q}{\sqrt{q^2 - u^2}} \mathcal{Y}} \operatorname{sech} \mathcal{Y} \tanh \mathcal{Y}. \end{aligned} \quad (2.34)$$

These two normalizable zero modes are presented for completeness, but will not play any further role in our analysis.

The solutions we are interested in are plane-wave fluctuations of the form

$$e^{i k x - i \omega t} f(\mathcal{Y}), \quad (2.35)$$

where $f(\mathcal{Y})$ is a bounded profile that is stationary in the magnon's frame. The equations are too complicated for us to find solutions by substituting the plane-wave ansatz into (2.30), we need to look for another strategy. The authors of [29] suggest using the dressing method [65–67] to construct the scattering state of a magnon and a breather, only then to expand this solution in the breather momentum to find the fluctuation as the subleading term. We find, instead, that it is simpler to apply the dressing method to the perturbed BMN vacuum, i.e. the point-like string moving along the equator together with fluctuations like (2.25)–(2.26), which results in the perturbed magnon. The details of this calculation can be found in appendix A, here we just present the solutions. As further confirmation of the validity of our approach, we show in appendix B that applying our method in the $\varphi = q = 0$ limit we recover the expected subset of the $\text{AdS}_5 \times \text{S}^5$ fluctuations found in [29].

The massless plane-wave solution is given by

$$\begin{aligned} z_1 &= -i e^{i k x - i \omega t} e^{+i \cos^2 \varphi t} \left(\tilde{q} k - \omega \cos \frac{\mathcal{P}}{2} \right. \\ &\quad \left. - i \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \left(\omega - \tilde{q} k \cosh \left(\mathcal{Y} + i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right) \right), \\ \bar{z}_1 &= i e^{i k x - i \omega t} e^{-i \cos^2 \varphi t} \left(\tilde{q} k - \omega \cos \frac{\mathcal{P}}{2} \right. \\ &\quad \left. + i \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \left(\omega - \tilde{q} k \cosh \left(\mathcal{Y} - i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right) \right), \\ z_2 &= i e^{i k x - i \omega t} \sin \frac{\mathcal{P}}{2} e^{-\frac{i q}{\sqrt{q^2 - u^2}} \mathcal{Y}} \operatorname{sech} \mathcal{Y} \left(q k - i \tilde{q} k \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \right), \\ \bar{z}_2 &= -i e^{i k x - i \omega t} \sin \frac{\mathcal{P}}{2} e^{+\frac{i q}{\sqrt{q^2 - u^2}} \mathcal{Y}} \operatorname{sech} \mathcal{Y} \left(q k + i \tilde{q} k \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \right), \end{aligned} \quad (2.36)$$

with

$$\omega^2 = k^2. \quad (2.37)$$

Here $\bar{z}_i$ are not the complex conjugates of z_i , rather⁴

$$\bar{z}_1 = \tilde{x}_1^+ - i \tilde{x}_2^+, \quad \bar{z}_2 = \tilde{x}_3^+ - i \tilde{x}_4^+, \quad (2.38)$$

⁴To preserve the (relative) simplicity of the formulas we consider $\tilde{x}_i^+$ to be complex themselves. Real solutions to (2.30) can be readily obtained by taking the real parts of these fluctuations.

The two massive modes both have $m = \cos^2 \varphi$. One of them is

$$\begin{aligned}
 z_1 &= -ie^{ikx-i\omega t} e^{\frac{iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} e^{+i \cos^2 \varphi t} \sin \frac{\mathcal{P}}{2} \operatorname{sech} \mathcal{Y} \left(\omega + \cos^2 \varphi + qk - \tilde{q}k \cosh \left(\mathcal{Y} + i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right), \\
 \bar{z}_1 &= -ie^{ikx-i\omega t} e^{\frac{iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} e^{-i \cos^2 \varphi t} \sin \frac{\mathcal{P}}{2} \operatorname{sech} \mathcal{Y} \left(\omega - \cos^2 \varphi - qk - \tilde{q}k \cosh \left(\mathcal{Y} - i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right), \\
 z_2 &= ie^{ikx-i\omega t} \left(\tilde{q}k \sin^2 \frac{\mathcal{P}}{2} \operatorname{sech}^2 \mathcal{Y} - 2 \left(\tilde{q}k - \omega \cos \frac{\mathcal{P}}{2} \right) - 2i(\cos^2 \varphi + qk) \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \right) \\
 \bar{z}_2 &= ie^{ikx-i\omega t} e^{\frac{2iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} \tilde{q}k \sin^2 \frac{\mathcal{P}}{2} \operatorname{sech}^2 \mathcal{Y},
 \end{aligned} \tag{2.39}$$

with

$$\omega^2 = (\cos^2 \varphi + qk)^2 + \tilde{q}^2 k^2, \tag{2.40}$$

while the other one is

$$\begin{aligned}
 z_1 &= -ie^{ikx-i\omega t} e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} e^{+i \cos^2 \varphi t} \sin \frac{\mathcal{P}}{2} \operatorname{sech} \mathcal{Y} \left(\omega + \cos^2 \varphi - qk - \tilde{q}k \cosh \left(\mathcal{Y} + i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right), \\
 \bar{z}_1 &= -ie^{ikx-i\omega t} e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} e^{-i \cos^2 \varphi t} \sin \frac{\mathcal{P}}{2} \operatorname{sech} \mathcal{Y} \left(\omega - \cos^2 \varphi + qk - \tilde{q}k \cosh \left(\mathcal{Y} - i \frac{\mathcal{P}}{2} \right) \operatorname{sech} \mathcal{Y} \right), \\
 z_2 &= ie^{ikx-i\omega t} e^{-\frac{2iq}{\sqrt{\tilde{q}^2-u^2}} \mathcal{Y}} \tilde{q}k \sin^2 \frac{\mathcal{P}}{2} \operatorname{sech}^2 \mathcal{Y}, \\
 \bar{z}_2 &= ie^{ikx-i\omega t} \left(\tilde{q}k \sin^2 \frac{\mathcal{P}}{2} \operatorname{sech}^2 \mathcal{Y} - 2 \left(\tilde{q}k - \omega \cos \frac{\mathcal{P}}{2} \right) - 2i(\cos^2 \varphi - qk) \sin \frac{\mathcal{P}}{2} \tanh \mathcal{Y} \right)
 \end{aligned} \tag{2.41}$$

with

$$\omega^2 = (\cos^2 \varphi - qk)^2 + \tilde{q}^2 k^2. \tag{2.42}$$

2.5 Bosonic modes in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ string theory

In addition to the fluctuations we found above, there is of course the massless S^1 mode

$$e^{ikx-i\omega t} \quad \omega^2 = k^2. \tag{2.43}$$

However, in a proper quantization of $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ string theory the sigma-model action (2.1) would need to be supplemented by ghosts, cancelling the massless AdS_3 mode (2.19), and also a combination of the massless $\text{S}^3_{\pm}$ modes (2.25), (2.36), corresponding to the $\text{S}^3_+ \times \text{S}^3_-$ leg of the BMN geodesic. These are analogous to the longitudinal modes in light-cone gauge, and in our semiclassical analysis we will simply omit them [68, 69].

In summary, the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ magnon has two massless modes (one on the flat S^1 and another one perpendicular to the BMN angle on $\text{S}^3_+ \times \text{S}^3_-$), two $m = 1$ fluctuations on AdS_3 , two $m = \cos^2 \varphi$ modes on S^3_+ , and two $m = \sin^2 \varphi$ modes on S^3_- , all with the dispersion relations

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2. \tag{2.44}$$

2.6 Bosonic modes in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ string theory

Taking the $\varphi \rightarrow 0$ limit of $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ blows up the S^3_- factor, which we can recompactify on a T^3 to get the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ geometry. In this limit the AdS_3 and S^1 fluctuations are unchanged, the S^3_+ modes take the same form but become $m = 1$, while on S^3_- the massless mode becomes the one unaffected by the ghosts, and the two $m = \sin^2 \varphi$ modes become massless T^4 modes. In summary, the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ magnon has four massless, and four mass 1 bosonic fluctuations.

3 Fermionic sector

In this section we solve for the complete fermion fluctuation spectrum around the mixed-flux stationary magnon (2.8). Our approach will be mostly based on [63], but rather than normalizable zero modes, we will be looking for solutions with plane-wave asymptotes. The leading order (quadratic) action for fermion fluctuations around a general bosonic string solution $X^\mu(t, x)$ is given by [70]

$$S_F = \hbar \int d^2x \mathcal{L}_F, \quad \mathcal{L}_F = -i \left(\eta^{ab} \delta^{IJ} + \epsilon^{ab} \sigma_3^{IJ} \right) \bar{\vartheta}^I \rho_a \mathcal{D}_b \vartheta^J. \quad (3.1)$$

The ϑ^I are two ten-dimensional Majorana-Weyl spinors, σ_3^{IJ} is the Pauli matrix $\text{diag}(+1, -1)$, and ρ_a are projections of the ten-dimensional Dirac matrices

$$\rho_a \equiv e_a^A \Gamma_A, \quad e_a^A \equiv \partial_a X^\mu E_\mu^A(X). \quad (3.2)$$

Note the difference in notation compared to the previous section, X^μ are now the curved space coordinates of $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, and not coordinates of a flat embedding space. For the remainder of this section we use Hopf coordinates, where the only non-constant components of the stationary magnon are along $\mu = t, \theta^+, \phi_1^+, \phi_2^+, \phi_1^-$ corresponding to the tangent space components $A = 0, 3, 4, 5, 7$, respectively. The covariant derivative is

$$\mathcal{D}_a \vartheta^I = \left(\delta^{IJ} \left(\partial_a + \frac{1}{4} \omega_\mu^{AB} \partial_a X^\mu \Gamma_{AB} \right) + \frac{1}{48} \sigma_1^{IJ} \not{F} \rho_a + \frac{1}{8} \sigma_3^{IJ} \not{H}_a \right) \vartheta^J, \quad (3.3)$$

where ω_μ^{AB} is the usual spin-connection,

$$\not{H}_a \equiv e_a^A H_{ABC} \Gamma^{BC} = \frac{1}{6} (\rho_a \not{H} + \not{H} \rho_a), \quad (3.4)$$

and the contracted 3-form fluxes are

$$\begin{aligned} \not{F} &= 12\tilde{q} \left(\Gamma^{012} + \cos \varphi \Gamma^{345} + \sin \varphi \Gamma^{678} \right), \\ \not{H} &= 12q \left(\Gamma^{012} + \cos \varphi \Gamma^{345} + \sin \varphi \Gamma^{678} \right). \end{aligned} \quad (3.5)$$

3.1 The fluctuation equations

The equations of motion for (3.1) are

$$\begin{aligned} (\rho_0 + \rho_1)(\mathcal{D}_0 - \mathcal{D}_1) \vartheta^1 &= 0, \\ (\rho_0 - \rho_1)(\mathcal{D}_0 + \mathcal{D}_1) \vartheta^2 &= 0, \end{aligned} \quad (3.6)$$

We proceed by changing variables to the more natural scaled and boosted worldsheet coordinates (2.9) of the magnon

$$\mathcal{Y} = \cos^2 \varphi \zeta \mathcal{X}, \quad \mathcal{S} = \cos^2 \varphi \zeta \mathcal{T}, \quad \zeta = \gamma \sqrt{\tilde{q}^2 - u^2}, \quad (3.7)$$

yielding

$$\begin{aligned} (\rho_0 + \rho_1) [\zeta(1+u)\gamma(D - \partial_S) \vartheta^1 + \mathcal{O}\vartheta^2] &= 0, \\ (\rho_0 - \rho_1) [\zeta(1-u)\gamma(\tilde{D} + \partial_S) \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1] &= 0. \end{aligned} \quad (3.8)$$

Here we defined the mixing operators

$$\mathcal{O} = -\frac{1}{48 \cos^2 \varphi} \not{F}(\rho_0 - \rho_1), \quad \tilde{\mathcal{O}} = \frac{1}{48 \cos^2 \varphi} \not{F}(\rho_0 + \rho_1), \quad (3.9)$$

and fermion derivatives

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2} G \Gamma_{34} + \frac{1}{2} Q \Gamma_{35} - \frac{(1-u)\gamma}{48 \cos^2 \varphi \zeta} (\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1) \not{H}), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G} \Gamma_{34} + \frac{1}{2} Q \Gamma_{35} - \frac{(1+u)\gamma}{48 \cos^2 \varphi \zeta} (\not{H}(\rho_0 + \rho_1) + (\rho_0 + \rho_1) \not{H}), \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} G &= \frac{\tilde{q}^2(1-u) \cosh^2 \mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}(\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2)} \operatorname{sech} \mathcal{Y}, \\ \tilde{G} &= -\frac{\tilde{q}^2(1+u) \cosh^2 \mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}(\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2)} \operatorname{sech} \mathcal{Y}, \\ Q &= -\frac{q}{\tilde{q} \sqrt{\tilde{q}^2 - u^2}} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2} \operatorname{sech} \mathcal{Y}. \end{aligned} \quad (3.11)$$

The full Green-Schwarz superstring has a local fermionic symmetry (κ -symmetry), that we need to fix for physical solutions. Noting that the operators $(\rho_0 \pm \rho_1)$ are half-rank, nilpotent and commute with the fermion derivatives D and $\tilde{D}$, it is clear that the projectors

$$K_1 = \frac{1}{2} \sec \varphi \hat{\Gamma}^0(\rho_0 + \rho_1), \quad K_2 = \frac{1}{2} \sec \varphi \hat{\Gamma}^0(\rho_0 - \rho_1), \quad (3.12)$$

can be used to fix κ -gauge. Here we introduced a set of “boosted” gamma matrices

$$\hat{\Gamma}^0 = \sec \varphi (\Gamma^0 - \sin \varphi \Gamma^7), \quad \hat{\Gamma}^7 = \sec \varphi (\Gamma^7 - \sin \varphi \Gamma^0), \quad \hat{\Gamma}^A = \Gamma^A \quad (A \neq 0, 7), \quad (3.13)$$

that simplify the notation in what follows. The kappa-fixed spinors $\Psi^J = K_J \vartheta^J$ then satisfy

$$\begin{aligned} \zeta(1+u)\gamma(D - \partial_S) \Psi^1 + K_1 \mathcal{O} \Psi^2 &= 0, \\ \zeta(1-u)\gamma(\tilde{D} + \partial_S) \Psi^2 + K_2 \tilde{\mathcal{O}} \Psi^1 &= 0. \end{aligned} \quad (3.14)$$

Introducing the 6d chirality projector

$$P_{\pm} = \frac{1}{2} \left(\mathbb{1} \pm \hat{\Gamma}^{012345} \right), \quad [P_{\pm}, K_J] = 0, \quad (3.15)$$

and, with $\bar{\rho}_0 = -\hat{\Gamma}_0 \rho_0 \hat{\Gamma}_0$, the invertible matrix

$$R = \frac{1}{2} \sec \varphi \hat{\Gamma}^{012} (\bar{\rho}_0 - \rho_0), \quad (3.16)$$

we can rewrite the equations, using the boosted gamma matrix basis

$$\begin{aligned} \zeta(1+u)\gamma(D - \partial_S)\Psi^1 + \tilde{q} \left(RP_- - K_1 \Delta \hat{\Gamma}^{012} \right) \Psi^2 &= 0, \\ \zeta(1-u)\gamma(\tilde{D} + \partial_S)\Psi^2 - \tilde{q} \left(RP_- - K_2 \Delta \hat{\Gamma}^{012} \right) \Psi^1 &= 0. \end{aligned} \quad (3.17)$$

The fermion differential operators are

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2} G \hat{\Gamma}_{34} + \frac{1}{2} Q \hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12}) P_+ + \Delta_0 \hat{\Gamma}_{12} \right), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G} \hat{\Gamma}_{34} + \frac{1}{2} Q \hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12}) P_+ + \Delta_0 \hat{\Gamma}_{12} \right), \end{aligned} \quad (3.18)$$

and we define

$$\Delta = -\frac{1}{2} \tan \varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right) \Gamma^7 \equiv \Delta_0 \hat{\Gamma}^0 + \Delta_7 \hat{\Gamma}^7, \quad (3.19)$$

with

$$\Delta_0 = -\frac{1}{2} \tan^2 \varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right), \quad \Delta_7 = \csc \varphi \Delta_0. \quad (3.20)$$

Note that the only source of structural difference between the equations for $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ is a non-zero Δ , and in fact this was our main reason to introduce the boosted gamma matrix basis. A much more detailed derivation of these equations, together with a thorough explanation of κ -gauge fixing, can be found in [63].

3.2 Ansatz and reduced equations

To reduce the seemingly complicated (3.17) to a more manageable set of equations we will make an ansatz that reflects the symmetries of the system. Firstly, all of $\hat{\Gamma}^{012345}$, $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$ commute with the kappa projectors (3.12), so the kappa-fixed spinors can be written as

$$\Psi^J = \sum_{\lambda_P, \lambda_{12}, \lambda_{68} \in \{\pm\}} \mathcal{K}_J(\lambda_P \lambda_{12}) V_{\lambda_P, \lambda_{12}, \lambda_{68}}^J(\mathcal{S}, \mathcal{Y}), \quad (3.21)$$

where the eigenvalues of $V_{\lambda_P, \lambda_{12}, \lambda_{68}}^J$ under $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $i\lambda_{12}$, $i\lambda_{68}$ and λ_P , respectively. Note that λ_{12} , λ_{68} , λ_P all take values in ± 1 . There are multiple ways to make

the above ansatz satisfy $K_J \Psi^J = \Psi^J$, in [63] we chose to impose the additional constraint⁵ $\hat{\Gamma}^{34} V^J = +i V^J$ and found

$$\begin{aligned}\mathcal{K}_1(\lambda) &= e^{+i\chi} \sqrt{1 + \lambda Q_+ \operatorname{sech} \mathcal{Y}} - \lambda e^{-i\chi} \sqrt{1 - \lambda Q_+ \operatorname{sech} \mathcal{Y}} \hat{\Gamma}_{45}, \\ \mathcal{K}_2(\lambda) &= e^{+i\tilde{\chi}} \sqrt{1 - \lambda Q_- \operatorname{sech} \mathcal{Y}} + \lambda e^{-i\tilde{\chi}} \sqrt{1 + \lambda Q_- \operatorname{sech} \mathcal{Y}} \hat{\Gamma}_{45},\end{aligned}\quad (3.22)$$

where

$$Q_{\pm} = \frac{q \sqrt{\tilde{q}^2 - u^2}}{\tilde{q}(1 \pm u)}, \quad (3.23)$$

and

$$\begin{aligned}\chi(\mathcal{Y}) &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) - \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1 - Q_+^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right), \\ \tilde{\chi}(\mathcal{Y}) &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) + \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1 - Q_-^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right).\end{aligned}\quad (3.24)$$

While the zero modes are time-independent in the magnon's frame $\partial_S \Psi^J = 0$, for the $\mathcal{S}$ -dependence of the non-zero modes we make a Fourier ansatz

$$V^J(\mathcal{S}, \mathcal{Y}) = e^{-i\tilde{\omega} \mathcal{S}} V^J(\mathcal{Y}). \quad (3.25)$$

As opposed to the kappa-projectors, the equations of motion (3.17) only commute with $\hat{\Gamma}^{12}$ and $\hat{\Gamma}^{68}$, and the solutions will not have definite chirality under $\hat{\Gamma}^{012345}$, unless $\Delta = 0$, i.e. for the $\operatorname{AdS}_3 \times \operatorname{S}^3 \times \operatorname{T}^4$ background, or on the $\hat{\Gamma}^{1268} = -1$ spinor subspace for the $\operatorname{AdS}_3 \times \operatorname{S}^3 \times \operatorname{S}^3 \times \operatorname{S}^1$ background. With this in mind, we take the general ansatz

$$\Psi^J = e^{-i\tilde{\omega} \mathcal{S}} \left(f_J(\mathcal{Y}) \mathcal{K}_J(-\lambda_{12}) + g_J(\mathcal{Y}) \mathcal{K}_J(\lambda_{12}) \hat{\Gamma}_{07} \right) U, \quad (3.26)$$

where the constant Weyl⁶ spinor U , that is shared between Ψ^1 and Ψ^2 , has eigenvalues $i\lambda_{12}, i\lambda_{68}, +i, -1$ under $\hat{\Gamma}^{12}, \hat{\Gamma}^{68}, \hat{\Gamma}^{34}, \hat{\Gamma}^{012345}$, respectively. The P_- part of the solution is represented by the scalar functions f_1, f_2 , while g_1, g_2 correspond to the P_+ components. The validity of such an ansatz is further justified by a quick counting of the degrees of freedom. A general Weyl's spinor has 16 complex components, and after 4 mutually commuting projections, there is a single free component left, hence we can capture the $\mathcal{Y}$ -dependence with a single function f_J multiplying U . Substituting (3.26) into (3.17), after a considerable amount of simplification we get

$$\begin{aligned}e^{-i\tilde{\omega} \mathcal{S}} \left[\left((\partial_{\mathcal{Y}} + C_{f_1 f_1}) f_1 + C_{f_1 f_2} f_2 + C_{f_1 g_2} g_2 \right) \mathcal{K}_1(-\lambda_{12}) \right. \\ \left. \left((\partial_{\mathcal{Y}} + C_{g_1 g_1}) g_1 + C_{g_1 g_2} g_2 + C_{g_1 f_2} f_2 \right) \mathcal{K}_1(\lambda_{12}) \hat{\Gamma}_{07} \right] U = 0, \\ e^{-i\tilde{\omega} \mathcal{S}} \left[\left((\partial_{\mathcal{Y}} + C_{f_2 f_2}) f_2 + C_{f_2 f_1} f_1 + C_{f_2 g_1} g_1 \right) \mathcal{K}_2(-\lambda_{12}) \right. \\ \left. \left((\partial_{\mathcal{Y}} + C_{g_2 g_2}) g_2 + C_{g_2 g_1} g_1 + C_{g_2 f_1} f_1 \right) \mathcal{K}_2(\lambda_{12}) \hat{\Gamma}_{07} \right] U = 0,\end{aligned}\quad (3.27)$$

⁵Note that kappa-fixing reduces the degrees of freedom by half, and in our ansatz this is done at the level of the projections $\hat{\Gamma}^{34} V^J = +i V^J$, since $\mathcal{K}_J(\lambda)$ are invertible.

⁶We postpone the analysis of the Majorana condition until later, see the discussion around (3.54).

with coefficients $C_{..}$ listed in appendix C. The matrix structure matches that of the general kappa-fixed spinors, confirming that the kappa-projectors commute with the fermion derivatives $D, \tilde{D}$. Further substituting

$$\begin{aligned}
 f_1 &= \frac{1}{\sqrt{1+u}} e^{i \frac{\lambda_{12} q}{\sqrt{q^2 - u^2}} \left(\frac{1}{2} + p_{1268} \tan^2 \varphi \right) \mathcal{Y}} e^{-\frac{i}{2} \lambda_{12} \arctan \left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1 - Q_+^2}} \right)} \tilde{f}_1, \\
 g_1 &= \frac{i \lambda_{12}}{\sqrt{1+u}} e^{i \frac{\lambda_{12} q}{\sqrt{q^2 - u^2}} \left(\frac{1}{2} + p_{1268} \tan^2 \varphi \right) \mathcal{Y}} e^{+\frac{i}{2} \lambda_{12} \arctan \left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1 - Q_+^2}} \right)} \tilde{g}_1, \\
 f_2 &= \frac{\lambda_{12}}{\sqrt{1-u}} e^{i \frac{\lambda_{12} q}{\sqrt{q^2 - u^2}} \left(\frac{1}{2} + p_{1268} \tan^2 \varphi \right) \mathcal{Y}} e^{-\frac{i}{2} \lambda_{12} \arctan \left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1 - Q_-^2}} \right)} \tilde{f}_2, \\
 g_2 &= \frac{i}{\sqrt{1-u}} e^{i \frac{\lambda_{12} q}{\sqrt{q^2 - u^2}} \left(\frac{1}{2} + p_{1268} \tan^2 \varphi \right) \mathcal{Y}} e^{+\frac{i}{2} \lambda_{12} \arctan \left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1 - Q_-^2}} \right)} \tilde{g}_2,
 \end{aligned} \tag{3.28}$$

where p_{1268} is the eigenvalue of the projector $\frac{1}{2}(\mathbb{1} + \hat{\Gamma}^{1268})$

$$p_{1268} = \frac{1}{2}(1 - \lambda_{12} \lambda_{68}), \tag{3.29}$$

and defining

$$\xi = \frac{qu}{\sqrt{q^2 - u^2}}, \tag{3.30}$$

we arrive at the reduced equations

$$\begin{aligned}
 \partial_{\mathcal{Y}} \tilde{f}_1 + i(\tilde{\omega} + (1 + p_{1268} \tan^2 \varphi) \lambda_{12} \xi) \tilde{f}_1 + (1 + p_{1268} \tan^2 \varphi)(\tanh \mathcal{Y} - i \lambda_{12} \xi) \tilde{f}_2 \\
 - \lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} \tilde{g}_2 = 0,
 \end{aligned} \tag{3.31}$$

$$\begin{aligned}
 \partial_{\mathcal{Y}} \tilde{f}_2 - i(\tilde{\omega} + (1 + p_{1268} \tan^2 \varphi) \lambda_{12} \xi) \tilde{f}_2 + (1 + p_{1268} \tan^2 \varphi)(\tanh \mathcal{Y} + i \lambda_{12} \xi) \tilde{f}_1 \\
 + \lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} \tilde{g}_1 = 0,
 \end{aligned}$$

$$\begin{aligned}
 \partial_{\mathcal{Y}} \tilde{g}_1 + i(\tilde{\omega} + p_{1268} \tan^2 \varphi \lambda_{12} \xi) \tilde{g}_1 + \lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} \tilde{f}_2 \\
 + p_{1268} \tan^2 \varphi (\tanh \mathcal{Y} + i \lambda_{12} \xi) \tilde{g}_2 = 0,
 \end{aligned} \tag{3.32}$$

$$\begin{aligned}
 \partial_{\mathcal{Y}} \tilde{g}_2 - i(\tilde{\omega} + p_{1268} \tan^2 \varphi \lambda_{12} \xi) \tilde{g}_2 - \lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} \tilde{f}_1 \\
 + p_{1268} \tan^2 \varphi (\tanh \mathcal{Y} - i \lambda_{12} \xi) \tilde{g}_1 = 0.
 \end{aligned}$$

3.3 Solutions

Let us first find the solutions for $\varphi > 0$, i.e. for the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^1$ geometry. For $p_{1268} = 0$ the P_+ components $\tilde{g}_1$ and $\tilde{g}_2$ decouple, and we have the two solutions

$$\tilde{g}_1 = e^{i\tilde{k}\mathcal{Y}}, \quad \tilde{g}_2 = 0, \quad \tilde{k} = -\tilde{\omega}, \tag{3.33}$$

$$\tilde{g}_2 = e^{i\tilde{k}\mathcal{Y}}, \quad \tilde{g}_1 = 0, \quad \tilde{k} = +\tilde{\omega}, \tag{3.34}$$

while on the P_- subspace we have the equations

$$\begin{aligned}
 \partial_{\mathcal{Y}} \tilde{f}_1 + i(\tilde{\omega} + \lambda_{12} \xi) \tilde{f}_1 + (\tanh \mathcal{Y} - i \lambda_{12} \xi) \tilde{f}_2 &= 0, \\
 \partial_{\mathcal{Y}} \tilde{f}_2 - i(\tilde{\omega} + \lambda_{12} \xi) \tilde{f}_2 + (\tanh \mathcal{Y} + i \lambda_{12} \xi) \tilde{f}_1 &= 0,
 \end{aligned} \tag{3.35}$$

with the two solutions

$$\begin{aligned}\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}} \left(\tanh \mathcal{Y} - i(\tilde{k} - \tilde{\omega}) \right), \\ \tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}} \left(\tanh \mathcal{Y} - i(\tilde{k} + \tilde{\omega}) \right), \\ \tilde{k} &= \pm \sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi\tilde{\omega} - 1}.\end{aligned}\tag{3.36}$$

The case of $p_{1268} = 1$ is a bit more complicated, but solving the first two equations of (3.31) for $\tilde{g}_J$ and substituting into the second two, we get two second order differential equations for $\tilde{f}_1, \tilde{f}_2$. The difference of those two equations is

$$\partial_{\mathcal{Y}}^2 \left(\tilde{f}_1 - \tilde{f}_2 \right) + \left(\tilde{\omega}^2 + 2\lambda_{12}\xi \sec^2 \varphi \tilde{\omega} - \sec^4 \varphi \right) \left(\tilde{f}_1 - \tilde{f}_2 \right) = 0,\tag{3.37}$$

which is easily solved, and inserting the solution into the $(\tilde{f}_1 + \tilde{f}_2)$ equation we find

$$\begin{aligned}\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}} \lambda_{12} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}, & \tilde{g}_1 &= -e^{i\tilde{k}\mathcal{Y}} \left(\tan^2 \varphi \tanh \mathcal{Y} + i(\tilde{k} - \tilde{\omega}) \right), \\ \tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}} \lambda_{12} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}, & \tilde{g}_2 &= e^{i\tilde{k}\mathcal{Y}} \left(\tan^2 \varphi \tanh \mathcal{Y} + i(\tilde{k} + \tilde{\omega}) \right), \\ \tilde{k} &= \pm \sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi \tan^2 \varphi \tilde{\omega} - \tan^4 \varphi},\end{aligned}\tag{3.38}$$

and

$$\begin{aligned}\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}} \left(\sec^2 \varphi \tanh \mathcal{Y} - i(\tilde{k} - \tilde{\omega}) \right), & \tilde{g}_1 &= e^{i\tilde{k}\mathcal{Y}} \lambda_{12} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}, \\ \tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}} \left(\sec^2 \varphi \tanh \mathcal{Y} - i(\tilde{k} + \tilde{\omega}) \right), & \tilde{g}_2 &= -e^{i\tilde{k}\mathcal{Y}} \lambda_{12} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}, \\ \tilde{k} &= \pm \sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi \sec^2 \varphi \tilde{\omega} - \sec^4 \varphi}.\end{aligned}\tag{3.39}$$

Dispersion relation. The observant reader might have already noted that all of the these solutions come with a plane-wave factor $e^{i\tilde{k}\mathcal{Y} - i\tilde{\omega}\mathcal{S}}$, satisfying

$$\tilde{k}^2 = \tilde{\omega}^2 \pm 2\xi(\sec^2 \varphi m) \tilde{\omega} - (\sec^2 \varphi m)^2,\tag{3.40}$$

with masses $m = 0, \cos^2 \varphi, \sin^2 \varphi$, and 1. This is not quite the expected dispersion relation, and there are two reasons why. Firstly, $(\mathcal{S}, \mathcal{Y})$ are scaled versions of the boosted worldsheet coordinates $(\mathcal{T}, \mathcal{X})$, but more importantly, the dispersion relation (2.44) is not relativistically invariant. We therefore need to rewrite the fermion fluctuations in the form

$$e^{i\tilde{k}\mathcal{Y} - i\tilde{\omega}\mathcal{S}} \vartheta(\mathcal{Y}) = e^{i(\tilde{k} + \alpha)\mathcal{Y} - i\tilde{\omega}\mathcal{S}} e^{-i\alpha\mathcal{Y}} \vartheta(\mathcal{Y}) = e^{ikx - i\omega t} e^{-i\alpha\mathcal{Y}} \vartheta(\mathcal{Y}),\tag{3.41}$$

where α will be necessary to match (2.44). From (3.7) it follows that

$$\tilde{k} = \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}}(k - u\omega) - \alpha, \quad \tilde{\omega} = \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}}(\omega - uk),\tag{3.42}$$

and substituting these into (3.40) we get the expected relation

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2,\tag{3.43}$$

provided that

$$\alpha = \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} \lambda_{12} q m. \quad (3.44)$$

Using this transformation we can parametrize the fluctuations by their wavenumber k , and we find that for a given wavenumber there are two positive frequency, and two negative frequency solutions of each mass, $m = 0, \cos^2 \varphi, \sin^2 \varphi$, and 1. Further defining

$$\hat{w}_{\pm} = \frac{1}{2} \arctan \left(\frac{Q_{\pm} \tanh \mathcal{Y}}{\sqrt{1 - Q_{\pm}^2}} \right), \quad (3.45)$$

we collect these solutions below.

Fermion fluctuations with $m = 0$. The massless perturbations are somewhat special, with the positive and negative frequency solutions exciting only one of the two spinors Ψ^J . Writing the solutions as

$$\Psi^J = e^{ikx - i\omega t} \hat{g}_J(\mathcal{Y}) \mathcal{K}_J(\lambda) V_{\lambda}, \quad (3.46)$$

the positive and negative frequency fluctuations are

$$\begin{aligned} \hat{g}_2 &= e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_-}, & \hat{g}_1 &= 0, & \omega &= +k, \\ \hat{g}_1 &= e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_+}, & \hat{g}_2 &= 0, & \omega &= -k, \end{aligned} \quad (3.47)$$

and the eigenvalues of the (k -dependent) constant Weyl spinor V_{λ} under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $+i, i\lambda, i\lambda$ and $+1$, respectively.

Fermion fluctuations with $m = \cos^2 \varphi$. These solutions live on the same subspace as the normalizable zero modes ($\lambda_P = -1, \lambda_{12} \lambda_{68} = 1$) and are given by

$$\Psi^J = e^{ikx - i\omega t} \hat{f}_J(\mathcal{Y}) \mathcal{K}_J(-\lambda) U_{\lambda}, \quad (3.48)$$

where

$$\begin{aligned} \hat{f}_1 &= \frac{1}{\sqrt{1+u}} \left(\tanh \mathcal{Y} - i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1+u)(k-\omega) - \lambda q \cos^2 \varphi) \right) e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_+}, \\ \hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}} \left(\tanh \mathcal{Y} - i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1-u)(k+\omega) - \lambda q \cos^2 \varphi) \right) e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_-}, \\ w &= \pm \sqrt{(\cos^2 \varphi - \lambda q k)^2 + \tilde{q}^2 k^2}, \end{aligned} \quad (3.49)$$

and the (k -dependent) constant Weyl spinor U_{λ} has eigenvalues $+i, i\lambda, i\lambda$ and -1 under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$, respectively.

Fermion fluctuations with $m = \sin^2 \varphi$. These fluctuations live on the $\hat{\Gamma}^{1268} = 1$ subspace, and do not have a definite chirality under $P_{\pm}$

$$\Psi^J = e^{ikx - i\omega t} \left(\hat{f}_J(\mathcal{Y}) \mathcal{K}_J(-\lambda) + \hat{g}_J(\mathcal{Y}) \mathcal{K}_J(\lambda) \hat{\Gamma}_{07} \right) W_{\lambda}, \quad (3.50)$$

$$\begin{aligned} \hat{f}_1 &= \frac{1}{\sqrt{1+u}} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_+}, \\ \hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_-}, \\ \hat{g}_1 &= \frac{i}{\sqrt{1+u}} \left(\tan^2 \varphi \tanh \mathcal{Y} + i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1+u)(k-\omega) - \lambda q \sin^2 \varphi) \right) e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_+}, \\ \hat{g}_2 &= \frac{-i \lambda}{\sqrt{1-u}} \left(\tan^2 \varphi \tanh \mathcal{Y} + i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1-u)(k+\omega) - \lambda q \sin^2 \varphi) \right) e^{\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_-}, \\ \omega &= \pm \sqrt{(\sin^2 \varphi - \lambda q k)^2 + \tilde{q}^2 k^2}, \end{aligned} \quad (3.51)$$

and the eigenvalues of the (k -dependent) constant Weyl spinor W_{λ} under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $+i, i\lambda, -i\lambda$ and -1 , respectively.

Fermion fluctuations with $m = 1$. Finally, the heaviest fermions are

$$\Psi^J = e^{ikx - i\omega t} \left(\hat{f}_J(\mathcal{Y}) \mathcal{K}_J(-\lambda) + \hat{g}_J(\mathcal{Y}) \mathcal{K}_J(\lambda) \hat{\Gamma}_{07} \right) W_{\lambda}, \quad (3.52)$$

$$\begin{aligned} \hat{f}_1 &= \frac{1}{\sqrt{1+u}} \left(\sec^2 \varphi \tanh \mathcal{Y} - i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1+u)(k-\omega) - \lambda q) \right) e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_+}, \\ \hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}} \left(\sec^2 \varphi \tanh \mathcal{Y} - i \frac{\sec^2 \varphi}{\sqrt{\tilde{q}^2 - u^2}} ((1-u)(k+\omega) - \lambda q) \right) e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{-i \lambda \hat{w}_-}, \\ \hat{g}_1 &= \frac{-i}{\sqrt{1+u}} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_+}, \\ \hat{g}_2 &= \frac{i \lambda}{\sqrt{1-u}} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y} e^{-\frac{1}{2} \frac{i \lambda q}{\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y}} e^{i \lambda \hat{w}_-}, \\ \omega &= \pm \sqrt{(1 - \lambda q k)^2 + \tilde{q}^2 k^2}, \end{aligned} \quad (3.53)$$

and the constant spinor W_{λ} satisfies the same conditions as for $m = \sin^2 \varphi$.

Majorana condition. In a Majorana basis $(\hat{\Gamma}^A)^* = -\hat{\Gamma}^A$ and the Majorana condition is $(\Psi^J)^* = \Psi^J$. To impose this condition we need to consider linear combinations of two solutions (from the same mass group) such that the wavenumbers are k and $-k$, the frequencies are of opposite sign (apart from the massless case), and so are the λ eigenvalues. Noting that the dispersion relation is invariant under $(k \rightarrow -k, \lambda \rightarrow -\lambda)$, and

$$\mathcal{K}_1(\lambda)^* = -\lambda \mathcal{K}_1(-\lambda) \hat{\Gamma}_{45}, \quad \mathcal{K}_2(\lambda)^* = \lambda \mathcal{K}_2(-\lambda) \hat{\Gamma}_{45}, \quad (3.54)$$

it follows that $(\Psi^J)^* = \Psi^J$ will simply relate the constant spinor multipliers of the two components. We show explicitly how to construct solutions satisfying the Majorana condition in the massless case. Analogous expressions for the massive modes can also be found,

but these are quite lengthy. Since they do not play any role in the subsequent analysis we do not write them explicitly here. We start with the linear combination

$$\begin{aligned}\Psi^1 &= e^{+ik(x+t)} e^{+\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\hat{w}+} \mathcal{K}_1(+1)V_+^1 \\ &+ e^{-ik(x+t)} e^{-\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\hat{w}+} \mathcal{K}_1(-1)V_-^2,\end{aligned}\quad (3.55)$$

where the two components have opposite k , ω , and λ . Its complex conjugate is

$$\begin{aligned}(\Psi^1)^* &= -e^{-ik(x+t)} e^{-\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\hat{w}+} \mathcal{K}_1(-1)\hat{\Gamma}_{45}(V_+^1)^* \\ &+ e^{+ik(x+t)} e^{+\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\hat{w}+} \mathcal{K}_1(+1)\hat{\Gamma}_{45}(V_-^2)^*,\end{aligned}\quad (3.56)$$

and $(\Psi^J)^* = \Psi^J$ as long as

$$\hat{\Gamma}_{45}(V_+^1)^* = -V_-^2 \quad \text{and} \quad \hat{\Gamma}_{45}(V_-^2)^* = V_+^1. \quad (3.57)$$

These two conditions are equivalent, and consistent with the $\hat{\Gamma}^{34}$, $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ eigenvalues of V_+^1 and V_-^2 . We have found an explicit Majorana solution.

Solutions for $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$. Again, this geometry corresponds to the $\varphi \rightarrow 0$ limit, the reduced equations (3.31) decouple for the $P_\pm$ subspaces, and all of the solutions are the same form as the $p_{1268} = 0$ fluctuations above. In particular, we have four massless fermions

$$\begin{aligned}\Psi^J &= e^{ikx-i\omega t} \hat{g}_J(\mathcal{Y}) \mathcal{K}_J(\lambda) V_\lambda, \\ \hat{g}_2 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}-}, \quad \hat{g}_1 = 0, \quad \omega = +k, \\ \hat{g}_1 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}+}, \quad \hat{g}_2 = 0, \quad \omega = -k,\end{aligned}\quad (3.58)$$

and four massive fermions

$$\begin{aligned}\Psi^J &= e^{ikx-i\omega t} \hat{f}_J(\mathcal{Y}) \mathcal{K}_J(-\lambda) U_\lambda, \quad \omega = \pm\sqrt{(1-\lambda qk)^2 + \tilde{q}^2 k^2}, \\ \hat{f}_1 &= \frac{1}{\sqrt{1+u}} \left(\tanh \mathcal{Y} - i \frac{1}{\sqrt{\tilde{q}^2-u^2}} ((1+u)(k-\omega) - \lambda q) \right) e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}+}, \\ \hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}} \left(\tanh \mathcal{Y} - i \frac{1}{\sqrt{\tilde{q}^2-u^2}} ((1-u)(k+\omega) - \lambda q) \right) e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}-},\end{aligned}\quad (3.59)$$

where under the operators $\hat{\Gamma}^{34}$, $\hat{\Gamma}^{12}$ and $\hat{\Gamma}^{012345}$ the constant spinor V_λ has eigenvalues $+i, i\lambda$ and $+1$, while U_λ has eigenvalues $+i, i\lambda$ and -1 , respectively. The difference compared to (3.47), (3.49) is that the $\hat{\Gamma}^{68}$ eigenvalues of U_λ, V_λ are no longer constrained.

4 The 1-loop functional determinant

Using the fluctuations found in the previous two sections we now calculate the leading order quantum corrections to the energy of the stationary giant magnon. We follow a similar argument in [29], which is based on well-established quantization techniques for

solitons [22, 23, 64, 71]. By energy we mean the Noether charge combination $E - J_1$, where E is the conserved charge associated with translations in global AdS_3 time, while J_1 is the $U(1)$ charge associated with rotations along the BMN geodesic. In light-cone gauge, the quantity $E - J_1$ can be identified with the (transverse) Hamiltonian of physical string excitations [68]. In conformal gauge the sigma-model action has to be supplemented by ghosts to cancel two unphysical bosons, however, for the purposes of our semiclassical analysis it is sufficient to simply omit two of the massless bosonic modes, as discussed in section 2.

A detailed calculation (building on the $\text{AdS}_3 \times S^3 \times T^4$ case discussed in [53]) can be found in [63], here we just note that the mixed-flux dyonic giant magnon has classical charges

$$E - J_1 = \sqrt{(\cos^2 \varphi J_2 - hqp)^2 + 4h^2 \tilde{q}^2 \sin^2 \frac{p}{2}}, \quad (4.1)$$

where $\cos^2 \varphi$ is the mass of the magnon and J_2 is its second angular momentum. Remarkably, this classical expression is in agreement with the exact dispersion relation of elementary excitations

$$\epsilon = \sqrt{(m \pm qhp)^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}, \quad (4.2)$$

determined from supersymmetry [51–53], hence we expect no quantum corrections. The one-loop correction to the energy can be calculated as the functional determinant $\ln \det |\delta^2 S|$ around the classical background, and is given by

$$\frac{1}{2} \sum_{i,k} (-1)^{F} \nu_i, \quad (4.3)$$

where F is the fermion number operator, ν_i are the so-called stability angles, frequencies of small oscillations around the classical solution, and the sum is over excitations i and wavenumbers k . For a non-static soliton, like the giant magnon, we can apply the method of Dashen, Hasslacher and Neveu [64] to calculate these stability angles. We put the system in a box of length $L \gg 1$, with periodic boundary conditions $x \cong x + L$. It is clear from the form of the solution (2.8) that the system is also periodic in worldsheet time, with period $T = L/u$. Then, the stability angle ν of a generic fluctuation $\delta\phi$ can be read off from

$$\delta\phi(t + T, x) = e^{-i\nu} \delta\phi(t, x). \quad (4.4)$$

Although we had to write the oscillations in the original worldsheet coordinates (x, t) to get the correct dispersion relations, the magnon's stationary frame $(\mathcal{X}, \mathcal{T})$ is better suited to the analysis of stability angles. In sections 2 and 3 we found fluctuations with oscillatory terms

$$e^{ikx - i\omega t} \quad (4.5)$$

parametrized by mass m and an additional eigenvalue $\lambda = \pm 1$, and satisfying dispersion relations

$$\omega = \sqrt{(m - \lambda qk)^2 + \tilde{q}^2 k^2}. \quad (4.6)$$

Rewriting the plane-wave terms as⁷

$$e^{ikx-i\omega t} = e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{i\lambda q m \gamma \mathcal{X}}, \quad (4.7)$$

the new frequency and wavenumber satisfy

$$\hat{\omega} = -\lambda q u \gamma m + \sqrt{\tilde{q}^2 m^2 + \hat{k}^2}, \quad (4.8)$$

while $e^{i\lambda q m \gamma \mathcal{X}}$ can be absorbed into the rest of the $\mathcal{Y}$ -dependent solution.

4.1 1-loop correction in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ string theory

For each excitation, the stability angle can be further decomposed as

$$\nu_i(m, \lambda) = \nu_i^{(0)}(m, \lambda) + \nu_i^{(1)}(m, \lambda) + \lambda \nu_i^{(2)}(m), \quad (4.9)$$

where $\nu_i^{(0)}$ comes from the pure plane-wave $e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}$, $\nu_i^{(2)}$ from terms like $e^{i\lambda f(\mathcal{Y})}$ and $\nu_i^{(1)}$ corresponds to the rest. Since we have exactly one boson and one fermion for each of the 8 combinations of (m, λ) , and the first terms are the same

$$\nu_{\text{bos}}^{(0)}(m, \lambda) = \nu_{\text{ferm}}^{(0)}(m, \lambda) = \frac{L}{u} \gamma \left(\hat{\omega} + u \hat{k} \right), \quad (4.10)$$

the total contribution from these terms vanishes even before integrating over $\hat{k}$

$$\sum_{m, \lambda} \nu_{\text{bos}}^{(0)}(m, \lambda) - \nu_{\text{ferm}}^{(0)}(m, \lambda) = 0. \quad (4.11)$$

Furthermore, summing over $\lambda = \pm 1$ pairs of the same excitation the $\nu_i^{(2)}$ terms cancel, leaving us with the total correction

$$\sum_{i, k} (-1)^F \nu_i = \int d\hat{k} \sum_{m, \lambda} \left(\nu_{\text{bos}}^{(1)}(m, \lambda) - \nu_{\text{ferm}}^{(1)}(m, \lambda) \right). \quad (4.12)$$

Under the transformation (4.7) we have

$$k = \gamma(\hat{k} + u\hat{\omega}) + \lambda q \gamma^2 m, \quad \omega = \gamma(\hat{\omega} + u\hat{k}) + \lambda q u \gamma^2 m, \quad (4.13)$$

and it is then straightforward to read off the $\nu_i^{(1)}$ stability angles for the fluctuations in sections 2 and 3. The excitations with non-zero $\nu_i^{(1)}$ are the two $m = \cos^2 \varphi$ bosons (2.39), (2.41) with

$$e^{\nu_{\text{bos}}^{(1)}(\cos^2 \varphi, \lambda)} = E_{\text{bos}}(\cos^2 \varphi, \lambda), \quad (4.14)$$

and six massive fermions (3.49), (3.51), (3.53) with

$$\begin{aligned} e^{\nu_{\text{ferm}}^{(1)}(\cos^2 \varphi, \lambda)} &= E_{\text{ferm}}(\cos^2 \varphi, \lambda), \\ e^{\nu_{\text{ferm}}^{(1)}(\sin^2 \varphi, \lambda)} &= 1/E_{\text{ferm}}(\sin^2 \varphi, \lambda), \\ e^{\nu_{\text{ferm}}^{(1)}(1, \lambda)} &= E_{\text{ferm}}(1, \lambda), \end{aligned} \quad (4.15)$$

---

⁷Note that this is the inverse of the transformation (3.41) that we applied to the fermion fluctuations.

where we have defined

$$E_{\text{bos}}(m, \lambda) = \frac{\hat{k} - \frac{q^2 u}{\tilde{q}^2 - u^2} \hat{\omega} + \lambda q \gamma m + i \left(\gamma \sqrt{\tilde{q}^2 - u^2} m - \frac{\lambda q}{\sqrt{\tilde{q}^2 - u^2}} (\hat{k} + u \hat{\omega}) \right)}{\hat{k} - \frac{q^2 u}{\tilde{q}^2 - u^2} \hat{\omega} + \lambda q \gamma m - i \left(\gamma \sqrt{\tilde{q}^2 - u^2} m - \frac{\lambda q}{\sqrt{\tilde{q}^2 - u^2}} (\hat{k} + u \hat{\omega}) \right)}, \quad (4.16)$$

$$E_{\text{ferm}}(m, \lambda) = \frac{\hat{k} - \hat{\omega} + i \gamma \sqrt{\tilde{q}^2 - u^2} m}{\hat{k} - \hat{\omega} - i \gamma \sqrt{\tilde{q}^2 - u^2} m}.$$

With these, the integrand of (4.12) becomes

$$\sum_{m, \lambda} \left(\nu_{\text{bos}}^{(1)}(m, \lambda) - \nu_{\text{ferm}}^{(1)}(m, \lambda) \right) = -i \log \left(\prod_{\lambda=\pm 1} \frac{E_{\text{bos}}(\cos^2 \varphi, \lambda) E_{\text{ferm}}(\sin^2 \varphi, \lambda)}{E_{\text{ferm}}(\cos^2 \varphi, \lambda) E_{\text{ferm}}(1, \lambda)} \right). \quad (4.17)$$

Since

$$\frac{E_{\text{bos}}(m, +1) E_{\text{bos}}(m, -1)}{(E_{\text{ferm}}(m, +1) E_{\text{ferm}}(m, -1))^2} = 1 \quad (4.18)$$

holds for general m , (4.12) simplifies to

$$\sum_{i, k} (-1)^F \nu_i = -i \int d\hat{k} \log \left(\prod_{\lambda=\pm 1} \frac{E_{\text{ferm}}(\cos^2 \varphi, \lambda) E_{\text{ferm}}(\sin^2 \varphi, \lambda)}{E_{\text{ferm}}(1, \lambda)} \right). \quad (4.19)$$

Further noting that

$$E_{\text{ferm}}(m, +1) E_{\text{ferm}}(m, -1) = \frac{\hat{k} + i \gamma \sqrt{\tilde{q}^2 - u^2} m}{\hat{k} - i \gamma \sqrt{\tilde{q}^2 - u^2} m} \quad (4.20)$$

it is clear that the integrand is antisymmetric in $\hat{k}$. Moreover, we have the asymptotic expansion around $\hat{k} = \pm \infty$

$$\prod_{\lambda=\pm 1} \frac{E_{\text{ferm}}(\cos^2 \varphi, \lambda) E_{\text{ferm}}(\sin^2 \varphi, \lambda)}{E_{\text{ferm}}(1, \lambda)} = 1 + \frac{i}{2} \gamma^3 (\tilde{q}^2 - u^2)^{3/2} \sin^2 2\varphi \frac{1}{\hat{k}^3} + O\left(\frac{1}{\hat{k}^5}\right), \quad (4.21)$$

and taking logarithm, the integrand of (4.19) is $O(\hat{k}^{-3})$, hence the integral itself is bounded and well-defined. We conclude that the integral is zero, and, in agreement with our expectations, the giant magnon energy receives no corrections at one loop, providing another check on our results.

4.2 1-loop correction in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ string theory

On $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ the situation is even simpler. We have two bosons and two fermions for each of the 4 combinations of $m = 0, 1$, $\lambda = \pm 1$. Paring these up, the $\nu_i^{(0)}$ contribution vanishes, while the $\nu_i^{(2)}$ terms cancel between $\lambda = \pm 1$ pairs. With two of the massive bosons and four of the massive fermions contributing, the integrand of (4.12) becomes

$$\sum_{\lambda} \left(\nu_{\text{bos}}^{(1)}(1, \lambda) - 2\nu_{\text{ferm}}^{(1)}(1, \lambda) \right) = -i \log \left(\frac{E_{\text{bos}}(1, +1) E_{\text{bos}}(1, -1)}{(E_{\text{ferm}}(1, +1) E_{\text{ferm}}(1, -1))^2} \right), \quad (4.22)$$

which is the same as the $\varphi \rightarrow 0$ limit of (4.17). Using (4.18) we arrive at the expected zero one-loop correction result even before integrating over $\hat{k}$.

5 Conclusions

In this paper we found the full spectrum of fluctuations around the mixed-flux AdS_3 stationary giant magnon, the $q > 0$ generalisation of the Hofman-Maldacena giant magnon. To obtain the non-trivial bosonic fluctuations, we adapted the method used in [29]. Rather than dressing the vacuum twice to get a complicated breather-soliton superposition (only then to expand in small breather momentum), we dress the perturbed BMN vacuum once, keeping terms up to subleading order throughout the calculation. The leading order term in the dressed solution is the giant magnon, so the subleading term must be its perturbation. The fermionic fluctuations are obtained as solutions of the equations derived from the quadratic fermionic action, using the formalism developed in [63], which builds on the original developments of [28] for AdS_5 .

We find that all of the fluctuations can be written in the form

$$e^{ikx-i\omega t} f(x-ut) \quad (5.1)$$

where u is the magnon's speed on the worldsheet (2.11), and with k, ω satisfying

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2, \quad (5.2)$$

which is the small-momentum limit of the exact dispersion relation (1.7). Furthermore, the fluctuations can be arranged into short multiplets of the residual symmetry algebras, according to mass and chirality ($\pm$ sign in the dispersion relation). On $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ there are four 4 dimensional multiplets of $\mathfrak{psu}(1|1)_{\text{c.e.}}^4$ with two bosons and two fermions, while $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ has eight 2 dimensional multiplets of $\mathfrak{su}(1|1)_{\text{c.e.}}^2$, with a boson and a fermion each.

Finally, from the explicit form of each fluctuation we read off the so called stability angles, which sum to the one-loop functional determinant. In both of the geometries we were able to show that this one-loop determinant is zero, or in other words, the one-loop correction to the magnon energy vanishes. It is interesting to compare this result with other calculations of the one-loop correction to energies of AdS_3 string states. The expansion of the coupling h around the classical string limit

$$h(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \quad (5.3)$$

is equivalent to the expansion of the energy (1.7)

$$\epsilon = \epsilon_0 + \frac{4\tilde{q}^2 h_0^2 \sin^2 \frac{\text{p}}{2}}{h_0 \epsilon_0} c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \quad (5.4)$$

where the subscript 0 refers to the classical (string) values, and we see that our results translate to $c = 0$ for both geometries. The one-loop correction to the giant magnon energy on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ with pure R-R flux was derived in [41] directly from the GS action, and in [57] from the algebraic curve. They both found that the correction is dependent on the chosen regularisation scheme, with two naturally emerging prescriptions:

in the *physical* regularisation the cutoff is at the same mode number for all excitations, while in the *new* prescription the cutoff is proportional to the mass of the polarisation. The two prescriptions both give zero correction $c_{phys} = c_{new} = 0$ on the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ background, but differ for the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ theory

$$c_{phys} = \frac{\alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)}{2\pi}, \quad c_{new} = 0. \quad (5.5)$$

For the mixed-flux $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ background the direct GS action calculation [58] shows that there is no one-loop correction, $c = 0$, and the same conclusion can be drawn by considering the worldsheet scattering of giant magnons [55]. Our results are in agreement with the *new* prescription, although it is not clear that we work in either of the regularisation schemes, as in (4.19) we have an implicit cutoff⁸ on the mode numbers $\hat{k}$ in the magnon's frame. There have been recent advances in our understanding of the protected spectrum of $\text{AdS}_3/\text{CFT}_2$ using integrable methods [72, 73]. The protected spectrum for $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ agrees with the older results of [74], while the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ case was independently derived using supergravity and WZW methods in [75].

As we have discussed in [63], there is no stationary magnon for $q = 1$, hence the results of present paper do not apply in this limit. The pure NS-NS string theory has been long known to be solvable using a chiral decomposition [32–34], and there is now a good understanding of integrability for the microscopic excitations [76–79], but it would be interesting to see a soliton analysis on these backgrounds. In more recent developments, the CFT dual of the $k = 1$ WZW model, i.e. $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with minimal quantized NS-NS flux, has been identified as a symmetric product orbifold [49, 80–82].

Semiclassical methods continue to provide valuable insight into the $\text{AdS}_3/\text{CFT}_2$ duality, as one can see in this paper, the analysis of fermion zero modes [63], or the calculation of 1-loop corrections to the rigid spinning string dispersion relations [83]. Where they seem to fail is the description of massless modes. In the $\alpha \rightarrow 0$ limit our fluctuations simply reduce to the plane-wave perturbations of the BMN vacuum, shedding no further light on the nature of the massless soliton of the theory, in agreement with our previous findings [63], and the fact that the $\alpha \rightarrow 0$ limit of the spin-chain fails to capture these inherently non-perturbative modes on the other side of the duality [84]. Furthermore, massless modes render a perturbative computation of wrapping corrections impossible, once the theory is put on a compactified worldsheet⁹ [87]. Instead, wrapping corrections may be computed from a non-perturbative TBA using an alternative low-momentum expansion [88–90].

There are two interesting directions for future research. Firstly, we would like to get a better understanding of the solitons of the $q = 1$ theory, their zero modes and fluctuation spectrum. Secondly, and this is a more speculative direction, one could try and describe the elusive massless modes by finding solitons of the fermionic part of the action, motivated by the fact that the massless representation's highest weight state is a fermion [59].

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⁸The integrals should be computed separately for each mass before summing, instead we first sum, then compute the integral, which is equivalent to having the same cutoff on $\hat{k}$ for each mass.

⁹This is to be compared with the finite-size AdS_5 giant magnon, where wrapping interactions give the right correction to the dispersion relation [85, 86].

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A Dressing the perturbed BMN string

In this section we briefly review the dressing method for the $SU(2)$ principal chiral model [65–67], and describe how it can be used when working with perturbations. We then apply this recipe to the three different perturbed BMN strings to obtain the three S^3 fluctuations of the $AdS_3 \times S^3 \times T^4$ giant magnon. The S^3_+ perturbation of the $AdS_3 \times S^3 \times S^3 \times S^1$ magnon can be obtained from these, simply by scaling the worldsheet coordinates by $\cos^2\varphi$.

A.1 Review of the $SU(2)$ dressing method

The S^3 part of the sigma-model action (2.1) is equivalent to the $SU(2)$ principal chiral model with Wess-Zumino term

$$S = -\frac{\hbar}{2} \left[\int_{\mathcal{M}} d^2x \frac{1}{2} \text{tr}(\tilde{\mathfrak{J}} \mathfrak{J}) - q \int_{\mathcal{B}} d^3x \frac{1}{3} \varepsilon^{abc} \text{tr}(\mathfrak{J}_a \mathfrak{J}_b \mathfrak{J}_c) \right]. \quad (\text{A.1})$$

where the left currents are $\mathfrak{J}_a = g^{-1} \partial_a g$, the partial derivatives on $\mathfrak{J} = g^{-1} \partial g$, $\tilde{\mathfrak{J}} = g^{-1} \bar{\partial} g$ are with respect to $z = \frac{1}{2}(t - x)$ and $\bar{z} = \frac{1}{2}(t + x)$, and the embedding

$$g = \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2) \quad (\text{A.2})$$

connects to the $\mathbb{R}^4$ coordinates of (2.1)

$$\begin{aligned} Z_1 &= X_1 + iX_2, & \bar{Z}_1 &= X_1 - iX_2, \\ Z_2 &= X_3 + iX_4, & \bar{Z}_2 &= X_3 - iX_4. \end{aligned} \quad (\text{A.3})$$

Note that $\bar{Z}_i$ are the complex conjugates of Z_i for the real classical solution, but not necessarily for the perturbation that we will write as complex functions.

The equations of motion for the action (A.1) are

$$(1 + q) \bar{\partial} (\partial g g^{-1}) + (1 - q) \partial (\bar{\partial} g g^{-1}) = 0, \quad (\text{A.4})$$

and starting with a solution g , the dressing method aims to find the appropriate dressing factor $\chi(z, \bar{z})$ such that

$$g \rightarrow g' = \chi g \quad (\text{A.5})$$

is a new solution. The construction exploits the equivalence between the compatibility condition of the overdetermined *auxiliary system*

$$\bar{\partial} \Psi = \frac{A \Psi}{1 + (1 + q) \lambda}, \quad \partial \Psi = \frac{B \Psi}{1 - (1 - q) \lambda}, \quad (\text{A.6})$$

and (A.4) via

$$A = \bar{\partial}g g^{-1}, \quad B = \partial g g^{-1}. \quad (\text{A.7})$$

One then solves the auxiliary problem for general complex spectral parameter λ such that $\Psi(\lambda)$ satisfies

$$\Psi(0) = g, \quad (\text{A.8})$$

and the unitarity condition

$$\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = 1. \quad (\text{A.9})$$

The simplest non-trivial dressing factor is then given by

$$\chi(\lambda) = \mathbf{1} + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P, \quad (\text{A.10})$$

with the projector

$$P = \frac{v_1 v_1^\dagger}{v_1^\dagger v_1}, \quad v_1 = \Psi(\bar{\lambda}_1)e, \quad e = (1, 1). \quad (\text{A.11})$$

We will also refer to the matrix X and scalar y

$$X = v_1 v_1^\dagger, \quad y = v_1^\dagger v_1 : \quad P = \frac{X}{y}. \quad (\text{A.12})$$

In order for the dressed solution $\chi(0)\Psi(0)$ to have unit determinant, we need to introduce an additional constant phase $(\lambda_1/\bar{\lambda}_1)^{1/2}$, and with this, the dressed solution becomes

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left(\mathbb{1} - \left(1 - \frac{\bar{\lambda}_1}{\lambda_1} \right) P \right) g. \quad (\text{A.13})$$

A.2 Dressing the unperturbed BMN string

To set the scene and some notation, let us quickly run through the application of the dressing method to the BMN string $Z_1 = e^{it}, Z_2 = 0$. We solve the auxiliary problem

$$g_{\text{BMN}} = \begin{pmatrix} e^{-i(z-\bar{z})} & 0 \\ 0 & e^{i(z-\bar{z})} \end{pmatrix}, \quad A_{\text{BMN}} = -B_{\text{BMN}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A.14})$$

to find

$$\Psi_{\text{BMN}}(\lambda) = \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & e^{iZ(\lambda)} \end{pmatrix}, \quad Z(\lambda) = \frac{z}{1 - (1-q)\lambda} - \frac{\bar{z}}{1 + (1+q)\lambda}. \quad (\text{A.15})$$

Introducing the real variables

$$U = i(Z(\bar{\lambda}_1) - Z(\lambda_1)), \quad V = -Z(\bar{\lambda}_1) - Z(\lambda_1) - t, \quad (\text{A.16})$$

the projector (A.11) becomes

$$P_{\text{BMN}} = \frac{X_{\text{BMN}}}{y_{\text{BMN}}} : \quad y_{\text{BMN}} = 2 \cosh U, \quad X_{\text{BMN}} = \begin{pmatrix} e^{-U} & e^{i(t+V)} \\ e^{-i(t+V)} & e^U \end{pmatrix}. \quad (\text{A.17})$$

Parametrizing the pole as $\lambda_1 = r e^{i\frac{p}{2}}$, the dressing (A.13) yields the giant magnon

$$g_{\text{GM}} = \begin{pmatrix} e^{it} \left[\cos \frac{p}{2} + i \sin \frac{p}{2} \tanh U \right] & -i e^{iV} \sin \frac{p}{2} \operatorname{sech} U \\ -i e^{-iV} \sin \frac{p}{2} \operatorname{sech} U & e^{-it} \left[\cos \frac{p}{2} - i \sin \frac{p}{2} \tanh U \right] \end{pmatrix}. \quad (\text{A.18})$$

Furthermore, setting $r = \tilde{q}^{-1}$, we get the stationary magnon

$$U = \gamma \sqrt{\tilde{q}^2 - u^2} \mathcal{X}, \quad V = -q\gamma \mathcal{X}, \quad \mathcal{X} = \gamma(x - ut), \quad (\text{A.19})$$

where

$$\gamma^2 = \frac{1}{1 - u^2}, \quad \cot \frac{p}{2} = \frac{u}{\sqrt{\tilde{q}^2 - u^2}}. \quad (\text{A.20})$$

A.3 Dressing the perturbed BMN string

To apply the dressing method to the perturbed BMN string

$$g_0 = g_{\text{BMN}} + \delta g_{\text{pert}}, \quad (\text{A.21})$$

in each step we keep terms up to first order in δ . For example

$$g_0^{-1} = g_{\text{BMN}}^{-1} - \delta g_{\text{BMN}}^{-1} g_{\text{pert}} g_{\text{BMN}}^{-1}. \quad (\text{A.22})$$

The auxiliary problem can be written as

$$A_0 = A_{\text{BMN}} + \delta A_{\text{pert}}, \quad B_0 = B_{\text{BMN}} + \delta B_{\text{pert}}, \quad (\text{A.23})$$

and its solution

$$\Psi_0(\lambda) = \Psi_{\text{BMN}}(\lambda) + \delta \Psi_{\text{pert}}(\lambda). \quad (\text{A.24})$$

Then we expand the projector

$$P_0 = \frac{X_0}{y_0} = \frac{X_{\text{BMN}} + \delta X_{\text{pert}}}{y_{\text{BMN}} + \delta y_{\text{pert}}} \equiv P_{\text{BMN}} + \delta P_{\text{pert}}, \quad (\text{A.25})$$

i.e.

$$P_{\text{pert}} = \frac{X_{\text{pert}}}{y_{\text{BMN}}} - \frac{y_{\text{pert}}}{y_{\text{BMN}}} P_{\text{BMN}}, \quad (\text{A.26})$$

and the dressing factor (A.13)

$$\chi_0 = \chi_{\text{BMN}} + \delta \chi_{\text{pert}} : \quad \chi_{\text{pert}} = \frac{\bar{\lambda}_1 - \lambda_1}{|\lambda_1|} P_{\text{pert}}. \quad (\text{A.27})$$

Finally, the dressed solution is

$$g_1 = \chi_0 g_0 \approx \chi_{\text{BMN}} g_{\text{BMN}} + \delta (\chi_{\text{pert}} g_{\text{BMN}} + \chi_{\text{BMN}} g_{\text{pert}}) \quad (\text{A.28})$$

from which we can read off the perturbation as the first order term. Let us now apply these steps to the three perturbations we found in¹⁰ (2.25)–(2.26).

¹⁰Setting $\sin \varphi = 1$ for the S^3 perturbations of the $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ magnon gives the S^3 fluctuations of the $\text{AdS}_3 \times S^3 \times T^4$ BMN string.

Massless fluctuation. The massless BMN perturbation is

$$g_{\text{pert}} = e^{ikx-i\omega t} \begin{pmatrix} ie^{it} & 0 \\ 0 & -ie^{-it} \end{pmatrix}, \quad \omega^2 = k^2, \quad (\text{A.29})$$

for which the auxiliary problem has perturbations

$$A_{\text{pert}} = i(\omega - k)e^{ikx-i\omega t} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_{\text{pert}} = i(\omega + k)e^{ikx-i\omega t} \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \quad (\text{A.30})$$

and

$$\Psi_{\text{pert}}(\lambda) = \frac{i(k(1+q\lambda) + \omega\lambda)}{k(1 - (1-q)\lambda)(1 + (1+q)\lambda)} e^{ikx-i\omega t} \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & -e^{iZ(\lambda)} \end{pmatrix}. \quad (\text{A.31})$$

Then P_{pert} can be calculated in terms of

$$\begin{aligned} y_{\text{pert}} &= -\frac{2\tilde{q}k \sin \frac{p}{2}}{\omega - \tilde{q}k \cos \frac{p}{2}} e^{ikx-i\omega t} \sinh U \\ X_{\text{pert}} &= -\frac{i}{\omega - \tilde{q}k \cos \frac{p}{2}} e^{ikx-i\omega t} \begin{pmatrix} i\tilde{q}k \sin \frac{p}{2} e^{-U} & -(\omega + qk - \tilde{q}k \cos \frac{p}{2})e^{i(t+V)} \\ (\omega + qk - \tilde{q}k \cos \frac{p}{2})e^{-i(t+V)} & -i\tilde{q}k \sin \frac{p}{2} e^U \end{pmatrix} \end{aligned} \quad (\text{A.32})$$

Finally, from

$$\chi_{\text{pert}} g_{\text{BMN}} + \chi_{\text{BMN}} g_{\text{pert}} = \begin{pmatrix} z_1 & -iz_2 \\ -i\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \quad (\text{A.33})$$

we can read off the fluctuation components (after a constant rescaling)

$$\begin{aligned} z_1 &= -ie^{ikx-i\omega t} e^{+it} \left(\tilde{q}k - \omega \cos \frac{p}{2} \right. \\ &\quad \left. - i \sin \frac{p}{2} \tanh U \left(\omega - \tilde{q}k \cosh \left(U + i\frac{p}{2} \right) \text{sech} U \right) \right), \\ \bar{z}_1 &= ie^{ikx-i\omega t} e^{-it} \left(\tilde{q}k - \omega \cos \frac{p}{2} \right. \\ &\quad \left. + i \sin \frac{p}{2} \tanh U \left(\omega - \tilde{q}k \cosh \left(U - i\frac{p}{2} \right) \text{sech} U \right) \right), \\ z_2 &= ie^{ikx-i\omega t} \sin \frac{p}{2} e^{+iV} \text{sech} U \left(qk - i\tilde{q}k \sin \frac{p}{2} \tanh U \right), \\ \bar{z}_2 &= -ie^{ikx-i\omega t} \sin \frac{p}{2} e^{-iV} \text{sech} U \left(qk + i\tilde{q}k \sin \frac{p}{2} \tanh U \right), \end{aligned} \quad (\text{A.34})$$

Massive fluctuation (1). The first massive BMN fluctuation is

$$g_{\text{pert}} = e^{ikx-i\omega t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \omega^2 = (1 + qk)^2 + \tilde{q}^2 k^2, \quad (\text{A.35})$$

for which the auxiliary problem has perturbations

$$\begin{aligned} A_{\text{pert}} &= +(\omega + 1 - k)e^{it}e^{ikx-i\omega t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ B_{\text{pert}} &= -(\omega + 1 + k)e^{it}e^{ikx-i\omega t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A.36})$$

and

$$\Psi_{\text{pert}}(\lambda) = \frac{e^{it}e^{ikx-i\omega t}}{1 + (1+q)\frac{\omega-1-k}{\omega+1-k}\lambda} \begin{pmatrix} 0 & e^{iZ(\lambda)} \\ 0 & 0 \end{pmatrix}. \quad (\text{A.37})$$

Further substituting and using the identity

$$\frac{1}{1 + \frac{\omega-1-k}{\omega+1-k}\sqrt{\frac{1+q}{1-q}}e^{i\frac{p}{2}}} = \frac{1}{2} \frac{\omega + 1 + qk - \tilde{q}ke^{-i\frac{p}{2}}}{\omega - \tilde{q}k \cos \frac{p}{2}} \quad (\text{A.38})$$

one gets the magnon fluctuation (rescaled by a constant)

$$\begin{aligned} z_1 &= -ie^{ikx-i\omega t}e^{-iV}e^{+it}\sin\frac{p}{2}\operatorname{sech}U\left(\omega + 1 + qk - \tilde{q}k \cosh\left(U + i\frac{p}{2}\right)\operatorname{sech}U\right), \\ \bar{z}_1 &= -ie^{ikx-i\omega t}e^{-iV}e^{-it}\sin\frac{p}{2}\operatorname{sech}U\left(\omega - 1 - qk - \tilde{q}k \cosh\left(U - i\frac{p}{2}\right)\operatorname{sech}U\right), \\ z_2 &= ie^{ikx-i\omega t}\left(\tilde{q}k \sin^2\frac{p}{2}\operatorname{sech}^2U - 2\left(\tilde{q}k - \omega \cos\frac{p}{2}\right) - 2i(1+qk)\sin\frac{p}{2}\tanh U\right) \\ \bar{z}_2 &= ie^{ikx-i\omega t}e^{-2iV}\tilde{q}k \sin^2\frac{p}{2}\operatorname{sech}^2U. \end{aligned} \quad (\text{A.39})$$

Massive fluctuation (2). The other massive BMN fluctuation is

$$g_{\text{pert}} = e^{ikx-i\omega t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \omega^2 = (1 - qk)^2 + \tilde{q}^2k^2, \quad (\text{A.40})$$

for which the auxiliary problem has perturbations

$$\begin{aligned} A_{\text{pert}} &= +(\omega - 1 - k)e^{-it}e^{ikx-i\omega t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ B_{\text{pert}} &= -(\omega - 1 + k)e^{-it}e^{ikx-i\omega t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A.41})$$

and

$$\Psi_{\text{pert}}(\lambda) = \frac{e^{-it}e^{ikx-i\omega t}}{1 + (1+q)\frac{\omega+1-k}{\omega-1-k}\lambda} \begin{pmatrix} 0 & 0 \\ e^{-iZ(\lambda)} & 0 \end{pmatrix}. \quad (\text{A.42})$$

Further substituting, using the identity

$$\frac{1}{1 + \frac{\omega+1-k}{\omega-1-k}\sqrt{\frac{1+q}{1-q}}e^{i\frac{p}{2}}} = \frac{1}{2} \frac{\omega - 1 + qk - \tilde{q}ke^{-i\frac{p}{2}}}{\omega - \tilde{q}k \cos \frac{p}{2}} \quad (\text{A.43})$$

and after constant rescaling, one can read off the magnon fluctuation

$$\begin{aligned}
 z_1 &= -ie^{ikx-i\omega t} e^{iV} e^{+it} \sin \frac{P}{2} \operatorname{sech} U \left(\omega + 1 - qk - \tilde{q}k \cosh \left(U + i\frac{P}{2} \right) \operatorname{sech} U \right), \\
 \bar{z}_1 &= -ie^{ikx-i\omega t} e^{iV} e^{-it} \sin \frac{P}{2} \operatorname{sech} U \left(\omega - 1 + qk - \tilde{q}k \cosh \left(U - i\frac{P}{2} \right) \operatorname{sech} U \right), \\
 z_2 &= ie^{ikx-i\omega t} e^{2iV} \tilde{q}k \sin^2 \frac{P}{2} \operatorname{sech}^2 U, \\
 \bar{z}_2 &= ie^{ikx-i\omega t} \left(\tilde{q}k \sin^2 \frac{P}{2} \operatorname{sech}^2 U - 2 \left(\tilde{q}k - \omega \cos \frac{P}{2} \right) - 2i(1 - qk) \sin \frac{P}{2} \tanh U \right).
 \end{aligned} \tag{A.44}$$

B Comparison to $\text{AdS}_5 \times \text{S}^5$ fluctuations

In this appendix we compare our solutions, in the $\varphi = q = 0$ limit, to the fluctuations of the $\text{AdS}_5 \times \text{S}^5$ giant magnon found in [29]. To harmonize notation, we need to write the frequency and wavenumber in the boosted worldsheet basis

$$\begin{aligned}
 e^{ikx-i\omega t} &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}, \\
 k &= \gamma(\hat{k} + u\hat{\omega}) = \csc \frac{P}{2} \left(\hat{k} + \cos \frac{P}{2} \hat{\omega} \right), \\
 \omega &= \gamma(\hat{\omega} + u\hat{k}) = \csc \frac{P}{2} \left(\hat{\omega} + \cos \frac{P}{2} \hat{k} \right),
 \end{aligned} \tag{B.1}$$

where we also used the $q = 0$ version of (2.11).

B.1 Bosonic fluctuations

Although in the $q = 0$ limit the stationary magnon reduces to the HM giant magnon, due to obvious differences in the geometry we will only match a subset of our fluctuations to a subset of the ones found in [29]. The magnon on $\text{AdS}_5 \times \text{S}^5$ has four massive and one (unphysical) massless fluctuations on AdS_5 , and, four massive and one (unphysical) massless fluctuations on S^5 . Out of these, we will match both unphysical and four of the massive modes (two each on AdS_3 and S^3), while our massless modes on the T^4 have no counterparts on $\text{AdS}_5 \times \text{S}^5$. The pure plane-wave AdS_3 bosons (2.19), (2.20) are trivially the same as the AdS_5 bosons (2.11) of [29] (restricted to the $\text{AdS}_3 \subset \text{AdS}_5$ subspace), so let us focus on the S^3 fluctuations. Substituting (B.1), the massless solution (2.36) becomes

$$\begin{aligned}
 z_1 &= -ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{P}{2} \left(\hat{k} - \hat{\omega} \sinh \mathcal{X} \sinh \left(\mathcal{X} + i\frac{P}{2} \right) \right), \\
 \bar{z}_1 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{-it} \sin \frac{P}{2} \left(\hat{k} - \hat{\omega} \sinh \mathcal{X} \sinh \left(\mathcal{X} - i\frac{P}{2} \right) \right), \\
 z_2 &= \bar{z}_2 = e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{P}{2} \left(\hat{k} + \cos \frac{P}{2} \hat{\omega} \right) \operatorname{sech} \mathcal{Y} \tanh \mathcal{Y},
 \end{aligned} \tag{B.2}$$

which, up to a factor of $\sin \frac{p}{2}$, matches¹¹ equation (2.19) of [29]. In this limit the massive boson (2.39) reduces to

$$\begin{aligned}
 z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} + i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 \bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} - i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} \left(\left(\hat{k} + \cos \frac{p}{2} \hat{\omega} \right) \operatorname{sech}^2 \mathcal{X} - 2(\hat{k} + i \tanh \mathcal{X}) \right), \\
 \bar{z}_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} \left(\hat{k} + \cos \frac{p}{2} \hat{\omega} \right) \operatorname{sech}^2 \mathcal{X},
 \end{aligned} \tag{B.3}$$

while (2.41) becomes

$$\begin{aligned}
 z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} + i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 \bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} - i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} \left(\hat{k} + \cos \frac{p}{2} \hat{\omega} \right) \operatorname{sech}^2 \mathcal{X}, \\
 \bar{z}_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} \left(\left(\hat{k} + \cos \frac{p}{2} \hat{\omega} \right) \operatorname{sech}^2 \mathcal{X} - 2(\hat{k} + i \tanh \mathcal{X}) \right).
 \end{aligned} \tag{B.4}$$

Although the two $m = 1$ bosons do not mix for $q > 0$, as can be seen from their dispersion relations $\omega^2 = (1 \pm qk)^2 + \tilde{q}^2 k^2$, in the pure R-R limit they become degenerate, and one can take linear combinations to match the specific solutions of [29]. The difference $\frac{1}{2}((\text{B.4}) - (\text{B.3}))$

$$\begin{aligned}
 z_1 &= \bar{z}_1 = 0 \\
 z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} (\hat{k} + i \tanh \mathcal{X}), \\
 \bar{z}_2 &= -ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} (\hat{k} + i \tanh \mathcal{X}),
 \end{aligned} \tag{B.5}$$

reproduces the solution (2.22) of [29], with $\vec{m}$ pointing along the X_4 direction, while the sum $\frac{1}{2}((\text{B.3}) + (\text{B.4}))$

$$\begin{aligned}
 z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} + i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 \bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin \frac{p}{2} \operatorname{sech}^2 \mathcal{X} \left(\hat{k} \sinh \mathcal{X} + \hat{\omega} \sinh \left(\mathcal{X} - i\frac{p}{2} \right) + i \cosh \mathcal{X} \right), \\
 z_2 &= \bar{z}_2 = ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin \frac{p}{2} \left(\left(\hat{k} + \cos \frac{p}{2} \hat{\omega} \right) \operatorname{sech}^2 \mathcal{X} - (\hat{k} + i \tanh \mathcal{X}) \right),
 \end{aligned} \tag{B.6}$$

matches the solution (2.20) of [29], with $\vec{m} = \vec{n}$ pointing along the X_3 direction.

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¹¹Note that $\delta Z, \delta \vec{X}$ of [29] are related to our notation by $z_1 = \delta Z, z_2 = \delta X_3 + i\delta X_4$, and we have chosen the magnon-polarization vector $\vec{n}$ to point in the X_3 direction.

B.2 Fermionic fluctuations

Since $\text{AdS}_5 \times \text{S}^5$ is supported by 5-form fluxes, while $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ is supported by 3-form fluxes, the spinor structure of fermion fluctuations on the two backgrounds will be quite different, however, it is reasonable to expect similar functional forms. The kappa-fixed solutions (3.35), (3.37) in [29] are of the form

$$\begin{aligned}\Psi^1 &\sim \csc \frac{\text{p}}{4} \sqrt{\hat{\omega} + \hat{k}} \text{sech} \mathcal{X} \sqrt{\hat{\omega} \cosh 2\mathcal{X} + \hat{k}} e^{i\alpha} e^{\pm i\chi} U, \\ \Psi^2 &\sim \sec \frac{\text{p}}{4} \sqrt{\hat{\omega} - \hat{k}} \text{sech} \mathcal{X} \sqrt{\hat{\omega} \cosh 2\mathcal{X} - \hat{k}} e^{i\beta} e^{\pm i\tilde{\chi}} U,\end{aligned}\tag{B.7}$$

where $\chi, \tilde{\chi}$ are the same as our (3.24) and

$$\begin{aligned}e^{i\alpha} &= e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\frac{1 + i\hat{\omega} \sinh 2\mathcal{X}}{1 - i\hat{\omega} \sinh 2\mathcal{X}} \frac{1 - i\hat{k} \tanh 2\mathcal{X}}{1 + i\hat{k} \tanh 2\mathcal{X}} \right)^{1/4}, \\ e^{i\beta} &= e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\frac{1 - i\hat{\omega} \sinh 2\mathcal{X}}{1 + i\hat{\omega} \sinh 2\mathcal{X}} \frac{1 - i\hat{k} \tanh 2\mathcal{X}}{1 + i\hat{k} \tanh 2\mathcal{X}} \right)^{1/4}.\end{aligned}\tag{B.8}$$

At first glance these solutions seem rather different from (3.59), but for $\hat{\omega} = \sqrt{\hat{k}^2 + 1}$

$$\begin{aligned}\sqrt{\hat{\omega} + \hat{k}} \text{sech} \mathcal{X} \sqrt{\hat{\omega} \cosh 2\mathcal{X} + \hat{k}} e^{i\alpha} &= i e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\tanh \mathcal{X} - i(\hat{k} + \hat{\omega}) \right), \\ \sqrt{\hat{\omega} - \hat{k}} \text{sech} \mathcal{X} \sqrt{\hat{\omega} \cosh 2\mathcal{X} - \hat{k}} e^{i\beta} &= -i e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\tanh \mathcal{X} - i(\hat{k} - \hat{\omega}) \right).\end{aligned}\tag{B.9}$$

$$\csc \frac{\text{p}}{4} = \sqrt{\frac{2}{1-u}}, \quad \sec \frac{\text{p}}{4} = \sqrt{\frac{2}{1+u}},\tag{B.10}$$

and we can rewrite (B.7) as

$$\begin{aligned}\Psi^1 &\sim \frac{1}{\sqrt{1-u}} e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\tanh \mathcal{X} - i(\hat{k} + \hat{\omega}) \right) e^{\pm i\chi} U, \\ \Psi^2 &\sim \frac{1}{\sqrt{1+u}} e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} \left(\tanh \mathcal{X} - i(\hat{k} - \hat{\omega}) \right) e^{\pm i\tilde{\chi}} U,\end{aligned}\tag{B.11}$$

in agreement with the $q = 0$ limit of (3.59), with the caveat that in [29] the spinors are swapped $\Psi^1 \leftrightarrow \Psi^2$ compared to our notation.

C Coefficients in the reduced equations of motion

Here we present the coefficients of the reduced equations (3.27), and to do so in a relatively compact form we need to introduce the shorthands

$$p_{1268} = \frac{1}{2}(1 - \lambda_{12}\lambda_{68}),\tag{C.1}$$

$$\xi = \frac{qu}{\sqrt{\tilde{q}^2 - u^2}},\tag{C.2}$$

and define

$$N_{ab} = \frac{i}{2} \lambda_{12} \left(\frac{aq}{\sqrt{\tilde{q}^2 - u^2}} + \frac{Q_b \sqrt{1 - Q_b^2 \operatorname{sech}^2 \mathcal{Y}}}{1 - Q_b^2 \operatorname{sech}^2 \mathcal{Y}} \right), \quad a, b \in \{\pm\}. \quad (\text{C.3})$$

With these, we have

$$\begin{aligned} C_{f_1 f_1} &= +N_{-+} + i(\tilde{\omega} + \lambda_{12}(1 + p_{1268} \tan^2 \varphi) \xi) - i\lambda_{12} q \gamma \zeta^{-1} p_{1268} \tan^2 \varphi, \\ C_{g_1 g_1} &= -N_{++} + i(\tilde{\omega} + \lambda_{12} p_{1268} \tan^2 \varphi \xi) - i\lambda_{12} q \gamma \zeta^{-1} p_{1268} \tan^2 \varphi, \\ C_{f_2 f_2} &= +N_{--} - i(\tilde{\omega} + \lambda_{12}(1 + p_{1268} \tan^2 \varphi) \xi) - i\lambda_{12} q \gamma \zeta^{-1} p_{1268} \tan^2 \varphi, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} C_{g_2 g_2} &= -N_{+-} - i(\tilde{\omega} + \lambda_{12} p_{1268} \tan^2 \varphi \xi) - i\lambda_{12} q \gamma \zeta^{-1} p_{1268} \tan^2 \varphi, \\ C_{f_1 f_2} &= (1 - u) \gamma e^{\int (-N_{-+} + N_{--}) d\mathcal{Y}} (1 + p_{1268} \tan^2 \varphi) (\lambda_{12} \tanh \mathcal{Y} - i\xi), \\ C_{g_1 g_2} &= (1 - u) \gamma e^{\int (+N_{++} - N_{+-}) d\mathcal{Y}} p_{1268} \tan^2 \varphi (\lambda_{12} \tanh \mathcal{Y} + i\xi), \end{aligned} \quad (\text{C.5})$$

$$\begin{aligned} C_{f_2 f_1} &= (1 + u) \gamma e^{\int (-N_{--} + N_{-+}) d\mathcal{Y}} (1 + p_{1268} \tan^2 \varphi) (\lambda_{12} \tanh \mathcal{Y} + i\xi), \\ C_{g_2 g_1} &= (1 + u) \gamma e^{\int (+N_{+-} - N_{++}) d\mathcal{Y}} p_{1268} \tan^2 \varphi (\lambda_{12} \tanh \mathcal{Y} - i\xi), \\ C_{f_1 g_2} &= (1 - u) \gamma e^{\int (-N_{-+} - N_{+-}) d\mathcal{Y}} (i\lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}), \\ C_{g_1 f_2} &= (1 - u) \gamma e^{\int (+N_{++} + N_{--}) d\mathcal{Y}} (i\lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}), \\ C_{f_2 g_1} &= (1 + u) \gamma e^{\int (-N_{--} - N_{++}) d\mathcal{Y}} (-i\lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}), \\ C_{g_2 f_1} &= (1 + u) \gamma e^{\int (+N_{+-} + N_{-+}) d\mathcal{Y}} (-i\lambda_{12} p_{1268} \tan \varphi \sec \varphi \operatorname{sech} \mathcal{Y}). \end{aligned} \quad (\text{C.6})$$

Note that p_{1268} is the eigenvalue of the ansatz with respect to the projector $\frac{1}{2}(\mathbb{1} + \hat{\Gamma}^{1268})$, and $\Delta = 0$ exactly when $p_{1268} \tan \varphi = 0$. In this case we see that the last block of coefficients are zero, the $P_{\pm}$ parts of the equations decouple and we have solutions with definite $\hat{\Gamma}^{012345}$ chirality.

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References

- [1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ superYang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [3] N. Beisert, C. Kristjansen and M. Staudacher, *The Dilatation operator of conformal $N = 4$ superYang-Mills theory*, *Nucl. Phys. B* **664** (2003) 131 [[hep-th/0303060](#)] [[INSPIRE](#)].

- [4] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [[hep-th/0307042](#)] [[INSPIRE](#)].
- [5] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[INSPIRE](#)].
- [6] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT* , *JHEP* **05** (2004) 024 [[hep-th/0402207](#)] [[INSPIRE](#)].
- [7] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)] [[INSPIRE](#)].
- [8] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The Algebraic curve of classical superstrings on $AdS_5 \times S^5$* , *Commun. Math. Phys.* **263** (2006) 659 [[hep-th/0502226](#)] [[INSPIRE](#)].
- [9] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$* , *JHEP* **02** (2005) 059 [[hep-th/0411089](#)] [[INSPIRE](#)].
- [10] H. Bethe, *On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain*, *Z. Phys.* **71** (1931) 205 [[INSPIRE](#)].
- [11] L.D. Faddeev, *How algebraic Bethe ansatz works for integrable model*, in *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*, Les Houches France (1996), pg. 149 [[hep-th/9605187](#)] [[INSPIRE](#)].
- [12] M. Staudacher, *The Factorized S -matrix of CFT/AdS* , *JHEP* **05** (2005) 054 [[hep-th/0412188](#)] [[INSPIRE](#)].
- [13] N. Beisert, *The $SU(2|2)$ dynamic S -matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[INSPIRE](#)].
- [14] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry*, *J. Stat. Mech.* **01** (2007) P01017 [[nlin/0610017](#)].
- [15] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S -matrix and crossing symmetry*, *Phys. Rev. D* **73** (2006) 086006 [[hep-th/0603038](#)] [[INSPIRE](#)].
- [16] N. Beisert, R. Hernandez and E. Lopez, *A Crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [[hep-th/0609044](#)] [[INSPIRE](#)].
- [17] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [[hep-th/0610251](#)] [[INSPIRE](#)].
- [18] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the Singularities of the Magnon S -matrix*, *Phys. Rev. D* **76** (2007) 025011 [[hep-th/0703104](#)] [[INSPIRE](#)].
- [19] D. Volin, *Minimal solution of the AdS/CFT crossing equation*, *J. Phys. A* **42** (2009) 372001 [[arXiv:0904.4929](#)] [[INSPIRE](#)].
- [20] J.-L. Gervais and B. Sakita, *Extended Particles in Quantum Field Theories*, *Phys. Rev. D* **11** (1975) 2943 [[INSPIRE](#)].
- [21] J.-L. Gervais, A. Jevicki and B. Sakita, *Perturbation Expansion Around Extended Particle States in Quantum Field Theory. 1.*, *Phys. Rev. D* **12** (1975) 1038 [[INSPIRE](#)].
- [22] J.-L. Gervais, A. Jevicki and B. Sakita, *Collective Coordinate Method for Quantization of Extended Systems*, *Phys. Rept.* **23** (1976) 281 [[INSPIRE](#)].

- [23] J.-L. Gervais and A. Jevicki, *Quantum Scattering of Solitons*, *Nucl. Phys. B* **110** (1976) 113 [[INSPIRE](#)].
- [24] D.M. Hofman and J.M. Maldacena, *Giant Magnons*, *J. Phys. A* **39** (2006) 13095 [[hep-th/0604135](#)] [[INSPIRE](#)].
- [25] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys. B* **533** (1998) 109 [[hep-th/9805028](#)] [[INSPIRE](#)].
- [26] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [[hep-th/0605155](#)] [[INSPIRE](#)].
- [27] H.-Y. Chen, N. Dorey and R.F. Lima Matos, *Quantum scattering of giant magnons*, *JHEP* **09** (2007) 106 [[arXiv:0707.0668](#)] [[INSPIRE](#)].
- [28] J.A. Minahan, *Zero modes for the giant magnon*, *JHEP* **02** (2007) 048 [[hep-th/0701005](#)] [[INSPIRE](#)].
- [29] G. Papathanasiou and M. Spradlin, *Semiclassical quantization of the giant magnon*, *JHEP* **06** (2007) 032 [[arXiv:0704.2389](#)] [[INSPIRE](#)].
- [30] T. Klose, *Review of AdS/CFT Integrability, Chapter IV.3: $N = 6$ Chern-Simons and Strings on $AdS_4 \times CP^3$* , *Lett. Math. Phys.* **99** (2012) 401 [[arXiv:1012.3999](#)] [[INSPIRE](#)].
- [31] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Supersymmetry of rotating branes*, *Phys. Rev. D* **59** (1998) 025001 [[hep-th/9809065](#)] [[INSPIRE](#)].
- [32] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and $SL(2, \mathbb{R})$ WZW model 1.: The Spectrum*, *J. Math. Phys.* **42** (2001) 2929 [[hep-th/0001053](#)] [[INSPIRE](#)].
- [33] J.M. Maldacena, H. Ooguri and J. Son, *Strings in AdS_3 and the $SL(2, \mathbb{R})$ WZW model. Part 2. Euclidean black hole*, *J. Math. Phys.* **42** (2001) 2961 [[hep-th/0005183](#)] [[INSPIRE](#)].
- [34] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and the $SL(2, \mathbb{R})$ WZW model. Part 3. Correlation functions*, *Phys. Rev. D* **65** (2002) 106006 [[hep-th/0111180](#)] [[INSPIRE](#)].
- [35] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [[hep-th/9902098](#)] [[INSPIRE](#)].
- [36] S.K. Ashok, R. Benichou and J. Troost, *Asymptotic Symmetries of String Theory on $AdS_3 \times S^3$ with Ramond-Ramond Fluxes*, *JHEP* **10** (2009) 051 [[arXiv:0907.1242](#)] [[INSPIRE](#)].
- [37] L. Eberhardt and K. Ferreira, *The plane-wave spectrum from the worldsheet*, *JHEP* **10** (2018) 109 [[arXiv:1805.12155](#)] [[INSPIRE](#)].
- [38] L. Eberhardt and K. Ferreira, *Long strings and chiral primaries in the hybrid formalism*, *JHEP* **02** (2019) 098 [[arXiv:1810.08621](#)] [[INSPIRE](#)].
- [39] A. Babichenko, B. Stefański Jr. and K. Zarembo, *Integrability and the AdS_3/CFT_2 correspondence*, *JHEP* **03** (2010) 058 [[arXiv:0912.1723](#)] [[INSPIRE](#)].
- [40] O. Ohlsson Sax and B. Stefański Jr., *Integrability, spin-chains and the AdS_3/CFT_2 correspondence*, *JHEP* **08** (2011) 029 [[arXiv:1106.2558](#)] [[INSPIRE](#)].
- [41] P. Sundin and L. Wulff, *Classical integrability and quantum aspects of the $AdS_3 \times S^3 \times S_3 \times S_1$ superstring*, *JHEP* **10** (2012) 109 [[arXiv:1207.5531](#)] [[INSPIRE](#)].
- [42] A. Cagnazzo and K. Zarembo, *B-field in AdS_3/CFT_2 Correspondence and Integrability*, *JHEP* **11** (2012) 133 [*Erratum ibid.* **04** (2013) 003] [[arXiv:1209.4049](#)] [[INSPIRE](#)].

- [43] O. Ohlsson Sax and B. Stefański, *Closed strings and moduli in AdS_3/CFT_2* , *JHEP* **05** (2018) 101 [[arXiv:1804.02023](#)] [[INSPIRE](#)].
- [44] O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *Integrability and the Conformal Field Theory of the Higgs branch*, *JHEP* **06** (2015) 103 [[arXiv:1411.3676](#)] [[INSPIRE](#)].
- [45] H.J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter space-times and dual superconformal theories*, *Nucl. Phys. B* **533** (1998) 127 [[hep-th/9803231](#)] [[INSPIRE](#)].
- [46] S. Gukov, E. Martinec, G.W. Moore and A. Strominger, *The Search for a holographic dual to $AdS_3 \times S^3 \times S^3 \times S^1$* , *Adv. Theor. Math. Phys.* **9** (2005) 435 [[hep-th/0403090](#)] [[INSPIRE](#)].
- [47] D. Tong, *The holographic dual of $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **04** (2014) 193 [[arXiv:1402.5135](#)] [[INSPIRE](#)].
- [48] L. Eberhardt, M.R. Gaberdiel and W. Li, *A holographic dual for string theory on $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **08** (2017) 111 [[arXiv:1707.02705](#)] [[INSPIRE](#)].
- [49] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The Worldsheet Dual of the Symmetric Product CFT*, *JHEP* **04** (2019) 103 [[arXiv:1812.01007](#)] [[INSPIRE](#)].
- [50] L. Eberhardt and M.R. Gaberdiel, *Strings on $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **06** (2019) 035 [[arXiv:1904.01585](#)] [[INSPIRE](#)].
- [51] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański and A. Torrielli, *The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector*, *JHEP* **08** (2013) 043 [[arXiv:1303.5995](#)] [[INSPIRE](#)].
- [52] B. Hoare and A.A. Tseytlin, *Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux*, *Nucl. Phys. B* **873** (2013) 395 [[arXiv:1304.4099](#)] [[INSPIRE](#)].
- [53] T. Lloyd, O. Ohlsson Sax, A. Sfondrini and J. Stefański, Bogdan, *The complete worldsheet S matrix of superstrings on $AdS_3 \times S^3 \times T^4$ with mixed three-form flux*, *Nucl. Phys. B* **891** (2015) 570 [[arXiv:1410.0866](#)] [[INSPIRE](#)].
- [54] B. Hoare, A. Stepanchuk and A.A. Tseytlin, *Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux*, *Nucl. Phys. B* **879** (2014) 318 [[arXiv:1311.1794](#)] [[INSPIRE](#)].
- [55] A. Stepanchuk, *String theory in $AdS_3 \times S^3 \times T^4$ with mixed flux: semiclassical and 1-loop phase in the S-matrix*, *J. Phys. A* **48** (2015) 195401 [[arXiv:1412.4764](#)] [[INSPIRE](#)].
- [56] A. Babichenko, A. Dekel and O. Ohlsson Sax, *Finite-gap equations for strings on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux*, *JHEP* **11** (2014) 122 [[arXiv:1405.6087](#)] [[INSPIRE](#)].
- [57] M.C. Abbott, *Comment on Strings in $AdS_3 \times S^3 \times S^3 \times S^1$ at One Loop*, *JHEP* **02** (2013) 102 [[arXiv:1211.5587](#)] [[INSPIRE](#)].
- [58] P. Sundin and L. Wulff, *One- and two-loop checks for the $AdS_3 \times S^3 \times T^4$ superstring with mixed flux*, *J. Phys. A* **48** (2015) 105402 [[arXiv:1411.4662](#)] [[INSPIRE](#)].
- [59] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *The complete $AdS_3 \times S^3 \times T^4$ worldsheet S matrix*, *JHEP* **10** (2014) 066 [[arXiv:1406.0453](#)] [[INSPIRE](#)].
- [60] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *Towards the All-Loop Worldsheet S Matrix for $AdS_3 \times S^3 \times T^4$* , *Phys. Rev. Lett.* **113** (2014) 131601 [[arXiv:1403.4543](#)] [[INSPIRE](#)].

- [61] R. Borsato, O. Ohlsson Sax and A. Sfondrini, *A dynamic $SU(1|1)^2$ S -matrix for AdS_3/CFT_2* , *JHEP* **04** (2013) 113 [[arXiv:1211.5119](#)] [[INSPIRE](#)].
- [62] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, *The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S matrix*, *J. Phys. A* **48** (2015) 415401 [[arXiv:1506.00218](#)] [[INSPIRE](#)].
- [63] A. Varga, *Fermion zero modes for the mixed-flux AdS_3 giant magnon*, *JHEP* **02** (2019) 135 [[arXiv:1901.00530](#)] [[INSPIRE](#)].
- [64] R.F. Dashen, B. Hasslacher and A. Neveu, *The Particle Spectrum in Model Field Theories from Semiclassical Functional Integral Techniques*, *Phys. Rev. D* **11** (1975) 3424 [[INSPIRE](#)].
- [65] V.E. Zakharov and A.V. Mikhailov, *Relativistically Invariant Two-Dimensional Models in Field Theory Integrable by the Inverse Problem Technique* (in Russian), *Sov. Phys. JETP* **47** (1978) 1017 [[INSPIRE](#)].
- [66] J.P. Harnad, Y. Saint Aubin and S. Shnider, *Backlund Transformations for Nonlinear σ Models With Values in Riemannian Symmetric Spaces*, *Commun. Math. Phys.* **92** (1984) 329 [[INSPIRE](#)].
- [67] M. Spradlin and A. Volovich, *Dressing the Giant Magnon*, *JHEP* **10** (2006) 012 [[hep-th/0607009](#)] [[INSPIRE](#)].
- [68] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)] [[INSPIRE](#)].
- [69] I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: One-loop correction to energy in $SL(2)$ sector*, *JHEP* **03** (2005) 013 [[hep-th/0501203](#)] [[INSPIRE](#)].
- [70] M. Cvetič, H. Lü, C.N. Pope and K.S. Stelle, *T duality in the Green-Schwarz formalism, and the massless/massive IIA duality map*, *Nucl. Phys. B* **573** (2000) 149 [[hep-th/9907202](#)] [[INSPIRE](#)].
- [71] A. Jevicki, *Classical And Quantum Dynamics Of Two-dimensional Nonlinear Field Theories: A Review*, in *Proceedings of Nonlinear Partial Differential Equations In Engineering and Applied Science*, Kingston U.S.A. (1979), pg. 311.
- [72] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, *On the spectrum of $AdS_3 \times S^3 \times T^4$ strings with Ramond-Ramond flux*, *J. Phys. A* **49** (2016) 41LT03 [[arXiv:1605.00518](#)] [[INSPIRE](#)].
- [73] M. Baggio, O. Ohlsson Sax, A. Sfondrini, B. Stefański and A. Torrielli, *Protected string spectrum in AdS_3/CFT_2 from worldsheet integrability*, *JHEP* **04** (2017) 091 [[arXiv:1701.03501](#)] [[INSPIRE](#)].
- [74] J. de Boer, *Six-dimensional supergravity on $S^3 \times AdS_3$ and 2 – D conformal field theory*, *Nucl. Phys. B* **548** (1999) 139 [[hep-th/9806104](#)] [[INSPIRE](#)].
- [75] L. Eberhardt, M.R. Gaberdiel, R. Gopakumar and W. Li, *BPS spectrum on $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **03** (2017) 124 [[arXiv:1701.03552](#)] [[INSPIRE](#)].
- [76] M. Baggio and A. Sfondrini, *Strings on NS-NS Backgrounds as Integrable Deformations*, *Phys. Rev. D* **98** (2018) 021902 [[arXiv:1804.01998](#)] [[INSPIRE](#)].
- [77] A. Dei and A. Sfondrini, *Integrable spin chain for stringy Wess-Zumino-Witten models*, *JHEP* **07** (2018) 109 [[arXiv:1806.00422](#)] [[INSPIRE](#)].
- [78] A. Dei, M.R. Gaberdiel and A. Sfondrini, *The plane-wave limit of $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **08** (2018) 097 [[arXiv:1805.09154](#)] [[INSPIRE](#)].

- [79] A. Dei and A. Sfondrini, *Integrable S matrix, mirror TBA and spectrum for the stringy $AdS_3 \times S^3 \times S^3 \times S^1$ WZW model*, *JHEP* **02** (2019) 072 [[arXiv:1812.08195](#)] [[INSPIRE](#)].
- [80] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, *Superstrings on AdS_3 at $\parallel = 1$* , *JHEP* **08** (2018) 204 [[arXiv:1803.04420](#)] [[INSPIRE](#)].
- [81] M.R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on AdS_3* , *JHEP* **05** (2018) 085 [[arXiv:1803.04423](#)] [[INSPIRE](#)].
- [82] L. Eberhardt, *AdS_3/CFT_2 at higher genus*, *JHEP* **05** (2020) 150 [[arXiv:2002.11729](#)] [[INSPIRE](#)].
- [83] J.M. Nieto and R. Ruiz, *One-loop quantization of rigid spinning strings in $AdS_3 \times S^3 \times T^4$ with mixed flux*, *JHEP* **07** (2018) 141 [[arXiv:1804.10477](#)] [[INSPIRE](#)].
- [84] O. Ohlsson Sax, B. Stefański Jr. and A. Torrielli, *On the massless modes of the AdS_3/CFT_2 integrable systems*, *JHEP* **03** (2013) 109 [[arXiv:1211.1952](#)] [[INSPIRE](#)].
- [85] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size Effects from Giant Magnons*, *Nucl. Phys. B* **778** (2007) 1 [[hep-th/0606126](#)] [[INSPIRE](#)].
- [86] R.A. Janik and T. Lukowski, *Wrapping interactions at strong coupling: The Giant magnon*, *Phys. Rev. D* **76** (2007) 126008 [[arXiv:0708.2208](#)] [[INSPIRE](#)].
- [87] M.C. Abbott and I. Aniceto, *Massless Lüscher terms and the limitations of the AdS_3 asymptotic Bethe ansatz*, *Phys. Rev. D* **93** (2016) 106006 [[arXiv:1512.08761](#)] [[INSPIRE](#)].
- [88] D. Bombardelli, B. Stefański and A. Torrielli, *The low-energy limit of AdS_3/CFT_2 and its TBA*, *JHEP* **10** (2018) 177 [[arXiv:1807.07775](#)] [[INSPIRE](#)].
- [89] A. Fontanella, O. Ohlsson Sax, B. Stefański, Jr. and A. Torrielli, *The effectiveness of relativistic invariance in AdS_3* , *JHEP* **07** (2019) 105 [[arXiv:1905.00757](#)] [[INSPIRE](#)].
- [90] M.C. Abbott and I. Aniceto, *Integrable Field Theories with an Interacting Massless Sector*, [arXiv:2002.12060](#) [[INSPIRE](#)].