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Yang-Baxter deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model from 4D Chern-Simons theory

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ABSTRACT: We present homogeneous Yang-Baxter deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model as boundary conditions of a 4D Chern-Simons theory. We first generalize the procedure for the 2D principal chiral model developed by Delduc et al. [5] so as to reproduce the 2D symmetric coset sigma model, and specify boundary conditions governing homogeneous Yang-Baxter deformations. Then the conditions are applicable for the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model case as well. In addition, homogeneous bi-Yang-Baxter deformation is also discussed.

KEYWORDS: AdS-CFT Correspondence, Integrable Field Theories

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1 Introduction

A significant subject in mathematical physics is to establish a unified picture to describe integrable systems [1, 2]. By focusing upon 2D classical integrable systems including non-linear sigma models (NLSMs), such a nice way was originally proposed by Costello and Yamazaki [3] based on a 4D Chern-Simons (CS) theory with a meromorphic 1-form ω . Notably, this 1-form ω is identified with a twist function characterizing the Poisson structure of the integrable system by Vicedo [4]. Recently, this procedure has been elaborated by Delduc, Laxcroix, Magro and Vicedo [5] so as to describe systematic ways to perform integrable deformations of 2D principal chiral model (PCM) including the Yang-Baxter (YB) deformation [6–12] and the λ -deformation [13, 14]. For other recent works on this subject, see [15, 16].

Our aim here is to generalize the preceding result on the PCM [5] to symmetric coset sigma models. By starting from a twist function in the rational description (with a slightly different parametrization of the spectral parameter), we specify a boundary condition associated with a symmetric coset. Then, the boundary condition is generalized so as to describe homogeneous YB deformations. It is straightforward to carry out the same analysis for the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model. As a result, the homogeneous YB deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model have been derived as specific boundary conditions of the 4D CS theory.

This paper is organized as follows. Section 2 explains how to derive 2D NLSMs from 4D CS theory. In section 3, we derive 2D symmetric coset sigma models as boundary conditions of the 4D CS theory and then specify boundary conditions which describes homogeneous

Yang-Baxter deformation. In section 4, the results obtained in section 3 are generalized to the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model case. Section 5 is devoted to conclusion and discussion. Appendix A explains the computation concerned with a dressed R -operator in detail. In appendix B, we present homogeneous bi-Yang-Baxter deformed sigma models as boundary conditions of the 4D CS theory.

Note. Just before submitting this manuscript to the arXiv, we have found an interesting work [17]. The content of [17] has some overlap with us on the integrability of the $\text{AdS}_5 \times \text{S}^5$ superstring.

2 2D NLSM from 4D CS theory

This section explains how to derive 2D NLSMs from a 4D CS theory by following [3, 5].

Let us begin with a 4D CS action [3],¹

$$S[A] = -\frac{i}{4\pi} \int_{\mathcal{M} \times \mathbb{C}P^1} \omega \wedge CS(A), \tag{2.1}$$

where A is a $\mathfrak{g}^{\mathbb{C}}$ -valued 1-form and $CS(A)$ is the CS 3-form defined as

$$CS(A) \equiv \left\langle A, dA + \frac{2}{3} A \wedge A \right\rangle. \tag{2.2}$$

Then ω is a meromorphic 1-form defined as

$$\omega \equiv \varphi(z) dz \tag{2.3}$$

and φ is a meromorphic function on $\mathbb{C}P^1$. This function is identified with a twist function characterizing the Poisson structure of the underlying integrable field theory [4].

Note that the z -component of A can always be gauged away like

$$A = A_\sigma d\sigma + A_\tau d\tau + A_{\bar{z}} d\bar{z}, \tag{2.4}$$

because $\varphi(z)$ depends only on z and hence the action (2.1) has an extra gauge symmetry

$$A \mapsto A + \chi dz. \tag{2.5}$$

The pole and zero structure of φ will be important in the following discussion. The set of poles is denoted as \mathfrak{p} and that of zeros is \mathfrak{z} . At each point of \mathfrak{z} , the 1-form A cannot be regular because otherwise the action (2.1) is degenerate and hence the equations of motion at \mathfrak{z} cannot be determined.

By taking a variation of the classical action (2.1), we obtain the bulk equation of motion

$$\omega \wedge F(A) = 0, \quad F(A) \equiv dA + A \wedge A \tag{2.6}$$

and the boundary equation of motion

$$d\omega \wedge \langle A, \delta A \rangle = 0. \tag{2.7}$$

¹For the notation and convention here, see [18].

Note that the boundary equation of motion (2.7) has the support only on $\mathcal{M} \times \mathfrak{p} \subset \mathcal{M} \times \mathbb{C}P^1$, because

$$d\omega = \partial_{\bar{z}}\varphi(z) d\bar{z} \wedge dz$$

and only the pole of φ can contribute as a distribution. The boundary conditions satisfying (2.7) are crucial to describe integrable deformations [3, 5].

The bulk equation of motion (2.6) can be expressed in terms of the component fields:

$$\partial_{\sigma}A_{\tau} - \partial_{\tau}A_{\sigma} + [A_{\sigma}, A_{\tau}] = 0, \tag{2.8}$$

$$\omega (\partial_{\bar{z}}A_{\sigma} - \partial_{\sigma}A_{\bar{z}} + [A_{\bar{z}}, A_{\sigma}]) = 0, \tag{2.9}$$

$$\omega (\partial_{\bar{z}}A_{\tau} - \partial_{\tau}A_{\bar{z}} + [A_{\bar{z}}, A_{\tau}]) = 0. \tag{2.10}$$

The factor ω is kept in order to cover the case $\partial_{\bar{z}}A_{\sigma}$ and $\partial_{\bar{z}}A_{\tau}$ are distributions on $\mathbb{C}P^1$ supported by \mathfrak{z} .

It is also helpful to rewrite the boundary equation of motion (2.7) into the form

$$\sum_{x \in \mathfrak{p}} \sum_{p \geq 0} (\text{res}_x \xi_x^p \omega) \epsilon^{ij} \frac{1}{p!} \partial_{\xi_x}^p \langle A_i, \delta A_j \rangle |_{\mathcal{M} \times \{x\}} = 0, \tag{2.11}$$

where ϵ^{ij} is the antisymmetric tensor. Here the local holomorphic coordinates ξ_x is defined as $\xi_x \equiv z - x$ for $x \in \mathfrak{p} \setminus \{\infty\}$ and $\xi_{\infty} \equiv 1/z$ if \mathfrak{p} includes the point at infinity. The relation (2.11) manifestly shows that the boundary equation of motion does not vanish only on $\mathcal{M} \times \mathfrak{p}$.

Lax form. By taking a formal gauge transformation

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\mathcal{L}\hat{g}^{-1} \tag{2.12}$$

with a smooth function $\hat{g} : \mathcal{M} \times \mathbb{C}P^1 \rightarrow G^{\mathbb{C}}$, the following gauge is realized

$$\mathcal{L}_{\bar{z}} = 0. \tag{2.13}$$

Hence the 1-form \mathcal{L} takes the form

$$\mathcal{L} \equiv \mathcal{L}_{\sigma}d\sigma + \mathcal{L}_{\tau}d\tau, \tag{2.14}$$

and we call \mathcal{L} the Lax form. This will be specified as a Lax pair for 2D theory later.

In terms of the Lax form \mathcal{L} , the bulk equations of motion are expressed as

$$\partial_{\tau}\mathcal{L}_{\sigma} - \partial_{\sigma}\mathcal{L}_{\tau} + [\mathcal{L}_{\tau}, \mathcal{L}_{\sigma}] = 0, \tag{2.15}$$

$$\omega \wedge \partial_{\bar{z}}\mathcal{L} = 0. \tag{2.16}$$

It follows that \mathcal{L} is a meromorphic 1-form with poles at the zeros of ω , namely \mathfrak{z} is regarded as the set of poles of \mathcal{L} .

Reality condition. It is natural to suppose some condition for the form of ω and its boundary condition on A so as to ensure the reality of the 4D action (2.1) and the resulting action (2.21) [5].

For a complex coordinate z , complex conjugation $z \mapsto \bar{z}$ defines an involution $\mu_t : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. Let $\tau : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ be an anti-linear involution. Then the set of the fixed point under τ is a real Lie subalgebra \mathfrak{g} of $\mathfrak{g}^{\mathbb{C}}$. The anti-linear involution τ satisfies

$$\overline{\langle B, C \rangle} = \langle \tau B, \tau C \rangle, \quad \forall B, C \in \mathfrak{g}^{\mathbb{C}}. \quad (2.17)$$

The associated operation to the Lie group G is denoted by $\tilde{\tau} : G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$.

Introducing the involutions, one can see that the reality of the action (2.21) is ensured by the conditions

$$\bar{\omega} = \mu_t^* \omega, \quad (2.18)$$

$$\tau A = \mu_t^* A. \quad (2.19)$$

Recalling the relation (2.12), we suppose that

$$\tilde{\tau} \hat{g} = \mu_t^* \hat{g}, \quad \tau \mathcal{L} = \mu_t^* \mathcal{L}, \quad (2.20)$$

so as to satisfy (2.19).

From 4D to 2D via the archipelago conditions. When \hat{g} satisfies the archipelago conditions [5], the 4D action (2.1) is reduced to a 2D action with the WZ term by performing an integral over $\mathbb{C}P^1$ as follows:

$$S \left[\{g_x\}_{x \in \mathfrak{p}} \right] = \frac{1}{2} \sum_{x \in \mathfrak{p}} \int_{\mathcal{M}} \langle \text{res}_x(\varphi \mathcal{L}), g_x^{-1} dg_x \rangle + \frac{1}{2} \sum_{x \in \mathfrak{p}} (\text{res}_x \omega) \int_{\mathcal{M} \times [0, R_x]} I_{\text{WZ}} [g_x]. \quad (2.21)$$

Here R_x is the radius of the open disk U_x .

The action (2.21) is invariant under a gauge transformation

$$g_x \mapsto g_x h, \quad \mathcal{L} \mapsto h^{-1} \mathcal{L} h + h^{-1} dh, \quad (2.22)$$

with a local function $h : \mathcal{M} \rightarrow G^{\mathbb{C}}$. This gauge symmetry can be seen as the remnant after taking the gauge (2.13). Note here that we have not imposed the reality condition by following [18], in comparison to [5]. The reality condition will be introduced later when fixing a boundary condition of \hat{g} .

3 YB deformations of the symmetric coset sigma model

In this section, we will reproduce the action of a symmetric coset sigma model and homogeneous Yang-Baxter deformations of it from the 4D CS theory (2.1) by generalizing the work [5, 18]. The symmetric coset case has been discussed in [3] in a slightly different way.

Symmetric coset. Let G and H be a Lie group and its subgroup, and the Lie algebras for G and H are denoted as \mathfrak{g} and \mathfrak{h} , respectively. We assume that the Lie algebra \mathfrak{g} enjoys a \mathbb{Z}_2 -grading, namely, \mathfrak{g} is decomposed like $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as the vector space and the following relations are satisfied

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (3.1)$$

Twist function. The twist function for a symmetric coset sigma model is given by²

$$\omega = \varphi_c(z) dz = \frac{16Kz}{(z-1)^2(z+1)^2} dz, \quad (3.3)$$

where we have followed the notation in [19]. The meromorphic 1-form ω indeed satisfies the reality condition (2.18). The poles and zeros of $\varphi_c(z)$ are listed as

$$\mathfrak{p} = \{\pm 1\}, \quad \mathfrak{z} = \{0, \infty\}, \quad (3.4)$$

where these poles are double poles, and each zero is a single zero. As we will see later, the twist function (3.3) is applicable not only to symmetric cosets, but also to homogeneous YB deformed sigma models.

Boundary condition. In order to specify a 2D integrable model, we need to choose a solution to the boundary equations of motion,

$$\epsilon^{ij} \langle\langle (A_i, \partial_{\xi_p} A_i), \delta(A_j, \partial_{\xi_p} A_j) \rangle\rangle_p = 0, \quad p \in \mathfrak{p}. \quad (3.5)$$

Here the double bracket is defined as

$$\begin{aligned} \langle\langle (x, y), (x', y') \rangle\rangle_p &\equiv (\text{res}_p \omega) \langle x, x' \rangle + (\text{res}_p \xi_p \omega) (\langle x, y' \rangle + \langle x', y \rangle) \\ &= 4pK (\langle x, y' \rangle + \langle x', y \rangle). \end{aligned} \quad (3.6)$$

The boundary equations of motion (3.5) take the same form as in the PCM case.

In the following, we will consider two classes of solutions. Note that $A|_{z=\pm 1}$ and $\partial_z A|_{z=\pm 1}$ take values in the real Lie algebra \mathfrak{g} , supposing the reality conditions (2.18) and (2.19) and the points $z = \pm 1$ are fixed points of the involution μ_t .

The first class is

$$\text{i) } (A|_{z=1}, \partial_z A|_{z=1}) \in \{0\} \times \mathfrak{g}_{\text{ab}}, \quad (A|_{z=-1}, \partial_z A|_{z=-1}) \in \{0\} \times \mathfrak{g}_{\text{ab}}, \quad (3.7)$$

where $\{0\} \times \mathfrak{g}_{\text{ab}}$ is an abelian copy of \mathfrak{g} defined as

$$\{0\} \times \mathfrak{g}_{\text{ab}} \equiv \{(0, x) \mid x \in \mathfrak{g}\}. \quad (3.8)$$

²The twist function (3.3) is the same as the one for PCM, and they are related by a transformation

$$z = \frac{1+z'}{1-z'}, \quad (3.2)$$

where z' is the spectral parameter for PCM.

This configuration obviously solves the boundary equations of motion and lead to a symmetric coset sigma model as we will see later.

The second class is

$$\text{ii) } (A|_{z=1}, \partial_z A|_{z=1}) \in \mathfrak{g}_R, \quad (A|_{z=-1}, \partial_z A|_{z=-1}) \in \mathfrak{g}_{\tilde{R}}, \quad (3.9)$$

where \mathfrak{g}_R and $\mathfrak{g}_{\tilde{R}}$ are defined as

$$\mathfrak{g}_R \equiv \{(2\eta R(x), x) \mid x \in \mathfrak{g}\}, \quad \mathfrak{g}_{\tilde{R}} \equiv \{(-2\eta \tilde{R}(x), x) \mid x \in \mathfrak{g}\}. \quad (3.10)$$

Here the linear R -operator $R : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfies the homogeneous classical Yang-Baxter equation (hCYBE),

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = 0, \quad x, y \in \mathfrak{g}. \quad (3.11)$$

The other R -operator $\tilde{R} : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined as

$$\tilde{R} \equiv f \circ R \circ f, \quad (3.12)$$

where $f : \mathfrak{g} \rightarrow \mathfrak{g}$ is a \mathbb{Z}_2 -grading automorphism of \mathfrak{g} . An explicit representation will be given in (3.24). For f in (3.24), we can show that \tilde{R} also solves the hCYBE (3.11) if the R -operator R is a solution to the equation (3.11). Furthermore, thanks to the hCYBE (3.11), we can check that the second configuration in (3.9) solves the boundary equations of motion (3.5). The choice of the boundary conditions (3.9) is motivated by the one of homogeneous bi-YB deformations (For the details, see appendix B). Note that the first and the second solutions are related by a β -transformation at the Lie algebra level (For the details, see appendix A of [18]).

Lax form. Before deriving sigma model actions, we shall summarize our notation used in the following. We will take \hat{g} at each pole of the twist function (3.3) as

$$\hat{g}(\tau, \sigma, z)|_{z=1} = g(\tau, \sigma), \quad \hat{g}(\tau, \sigma, z)|_{z=-1} = \tilde{g}(\tau, \sigma), \quad (3.13)$$

where $g, \tilde{g} \in G$. Here g and \tilde{g} take values in G (not $G^{\mathbb{C}}$) due to the reality condition (2.20). The reality condition has been implicitly imposed at this moment. The associated left-invariant currents are defined as

$$j \equiv g^{-1} dg, \quad \tilde{j} \equiv \tilde{g}^{-1} d\tilde{g}. \quad (3.14)$$

Then, the relation between the gauge field and the Lax pair at each pole becomes

$$A|_{z=1} = -dgg^{-1} + \text{Ad}_g \mathcal{L}|_{z=1}, \quad A|_{z=-1} = -d\tilde{g}\tilde{g}^{-1} + \text{Ad}_{\tilde{g}} \mathcal{L}|_{z=-1}. \quad (3.15)$$

From the zeros of the twist function (3.3), we suppose an ansatz for the Lax pair as

$$\mathcal{L} = (U_+ + zV_+)d\sigma^+ + (U_- + z^{-1}V_-)d\sigma^-, \quad (3.16)$$

where $U_{\pm}, V_{\pm} \in \mathfrak{g}$ are undetermined functions of σ, τ , and the light-cone coordinates are defined as

$$\sigma^{\pm} \equiv \frac{1}{2}(\tau \pm \sigma). \quad (3.17)$$

As we will see, the ansatz (3.16) of the Lax pair works well for the two classes of boundary conditions.

i) symmetric coset sigma model. Let us first see the class i) that describes a symmetric coset sigma model.

Under the boundary condition (3.7), the relations in (3.15) are rewritten as

$$j_{\pm} = U_{\pm} + V_{\pm}, \quad \tilde{j}_{\pm} = U_{\pm} - V_{\pm}. \quad (3.18)$$

By solving these equations with respect to U_{\pm} and V_{\pm} , we obtain

$$U_{\pm} = \frac{j_{\pm} + \tilde{j}_{\pm}}{2}, \quad V_{\pm} = \frac{j_{\pm} - \tilde{j}_{\pm}}{2}. \quad (3.19)$$

As a result, the Lax pair is expressed as

$$\mathcal{L}_{\pm} = \frac{j_{\pm} + \tilde{j}_{\pm}}{2} + z^{\pm 1} \frac{j_{\pm} - \tilde{j}_{\pm}}{2}. \quad (3.20)$$

Then, the residues of $\varphi_c \mathcal{L}$ at $z = \pm 1$ are evaluated as

$$\begin{aligned} \text{res}_{z=1}(\varphi_c \mathcal{L}) &= 4K(V_+ d\sigma^+ - V_- d\sigma^-), \\ \text{res}_{z=-1}(\varphi_c \mathcal{L}) &= -4K(V_+ d\sigma^+ - V_- d\sigma^-). \end{aligned} \quad (3.21)$$

By substituting (3.21) into (2.21), the 2D action is given by

$$S[g, \tilde{g}] = K \int_{\mathcal{M}} \langle j_+ - \tilde{j}_+, j_- - \tilde{j}_- \rangle d\sigma \wedge d\tau. \quad (3.22)$$

If \tilde{g} is independent of g , then by using the gauge symmetry of the 4D CS theory, we can rewrite the 2D action (3.22) to that of PCM with Lie group G [3, 5].

Here we would like to impose a relation between j and \tilde{j} . Note that the resulting action (3.22) is invariant under the exchange of j and \tilde{j} . This invariance should be respected in a relation $\tilde{j} = f(j)$ and hence the automorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}$ should satisfy the following conditions:

$$f([x, y]) = [f(x), f(y)], \quad f \circ f(x) = x, \quad x, y \in \mathfrak{g}. \quad (3.23)$$

In order to obtain the known result, we will take f satisfying the following relations:

$$f(\mathbf{P}_{\hat{a}}) = -\mathbf{P}_{\hat{a}}, \quad f(\mathbf{J}_{\hat{a}}) = \mathbf{J}_{\hat{a}}. \quad (3.24)$$

Here we have introduced the generators of the decomposed vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as

$$\mathfrak{h} = \langle \mathbf{J}_{\hat{a}} \rangle, \quad \mathfrak{m} = \langle \mathbf{P}_{\hat{a}} \rangle, \quad (3.25)$$

where $\hat{a} = 1, \dots, \dim \mathfrak{h}$ and $\check{a} = 1, \dots, \dim \mathfrak{m}$.

By employing the automorphism (3.24), \tilde{j} is evaluated as

$$\tilde{j} = f(j) = f(P_{(0)}(j) + P_{(2)}(j)) = P_{(0)}(j) - P_{(2)}(j), \quad (3.26)$$

where the projection operators $P_{(0)}$ and $P_{(2)}$ are defined as, respectively,

$$P_{(0)} : \mathfrak{g} \rightarrow \mathfrak{h}, \quad P_{(2)} : \mathfrak{g} \rightarrow \mathfrak{m}. \quad (3.27)$$

Then, by using the expression of \tilde{j} in (3.26), the 2D action can be further rewritten as

$$S[g] = 4K \int_{\mathcal{M}} \langle j_+, P_{(2)}(j_-) \rangle d\sigma \wedge d\tau, \quad (3.28)$$

and the Lax pair (3.20) becomes

$$\mathcal{L}_{\pm} = P_{(0)}(j_{\pm}) + z^{\pm 1} P_{(2)}(j_{\pm}). \quad (3.29)$$

These are the standard expressions of the classical action and the associated Lax pair for a symmetric coset sigma model.

ii) homogeneous YB deformations. The next one we will discuss is the class ii) in (3.9) that describes homogeneous YB deformations of a symmetric coset sigma model.

The condition gives a constraint on the gauge field A at each pole of the twist function,

$$A|_{z=1} = 2\eta R(\partial_z A|_{z=1}), \quad A|_{z=-1} = -2\eta \tilde{R}(\partial_z A|_{z=-1}). \quad (3.30)$$

We again suppose the same ansatz (3.16) for the Lax pair. Then, the constraints in (3.30) lead to

$$j_{\pm} = U_{\pm} + (1 \mp 2\eta R_g)(V_{\pm}), \quad \tilde{j}_{\pm} = U_{\pm} - (1 \mp 2\eta \tilde{R}_{\tilde{g}})(V_{\pm}), \quad (3.31)$$

where we defined $R_g \equiv \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_g$. By solving these equations with respect to U_{\pm} and V_{\pm} , we obtain

$$U_{\pm} = \frac{j_{\pm} + \tilde{j}_{\pm}}{2} \pm \eta(R_g - \tilde{R}_{\tilde{g}})(V_{\pm}), \quad V_{\pm} = \frac{1}{1 \mp \eta R_g \mp \eta \tilde{R}_{\tilde{g}}} \left(\frac{j_{\pm} - \tilde{j}_{\pm}}{2} \right). \quad (3.32)$$

The residues of $\varphi_c \mathcal{L}$ at $z = \pm 1$ take the same forms as (3.21), but V_{\pm} are given by (3.32). Thus the 2D action is given by

$$S[g, \tilde{g}] = 4K \int_{\mathcal{M}} \left\langle \frac{j_+ - \tilde{j}_+}{2}, \frac{1}{1 + \eta R_g + \eta \tilde{R}_{\tilde{g}}} \left(\frac{j_- - \tilde{j}_-}{2} \right) \right\rangle d\sigma \wedge d\tau. \quad (3.33)$$

Note that in the present case, the resulting action is invariant under the exchange of g and \tilde{g} , not j and \tilde{j} .

The exchange symmetry of the action (3.33) at the level of group element leads to a slight change in the previous case: for group elements, we impose an additional condition

$$\tilde{g} = F(g), \quad (3.34)$$

where an automorphism $F : G \rightarrow G$ has the \mathbb{Z}_2 -grading property $F \circ F(g) = g$. To specify an explicit representation of F , let us take a parameterization of an element $g \in G$ as

$$g = \exp(X^{\hat{a}} \mathbf{P}_{\hat{a}} + X^{\hat{a}} \mathbf{J}_{\hat{a}}), \quad (3.35)$$

where $X^{\hat{a}}$ and $X^{\hat{a}}$ are functions of τ and σ . Then, in a neighborhood of the identity, $F(g)$ can be written by using the automorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}$ as follows,

$$F(g) \equiv \exp(X^{\hat{a}} f(\mathbf{P}_{\hat{a}}) + X^{\hat{a}} f(\mathbf{J}_{\hat{a}})). \quad (3.36)$$

or equivalently,

$$F(g) = \exp(-X^{\hat{a}}\mathbf{P}_{\hat{a}} + X^{\hat{a}}\mathbf{J}_{\hat{a}}). \quad (3.37)$$

Now let us rewrite the 2D action (3.33) by requiring (3.34). As shown in appendix A.1, we can show that the dressed R -operators R_g and $\tilde{R}_{\tilde{g}}$ satisfy the following relation:

$$(R_g + \tilde{R}_{\tilde{g}}) \circ P_{(2)}(x) = 2P_{(2)} \circ R_g \circ P_{(2)}(x). \quad (3.38)$$

The relation (3.38) indicates

$$V_{\pm} = P_{(2)} \left(\frac{1}{1 \mp 2\eta R_g \circ P_{(2)}} j_{\pm} \right). \quad (3.39)$$

Furthermore, by using (3.39), U_{\pm} can be rewritten as

$$U_{\pm} = j_{\pm} - (1 \mp \eta R_g)(V_{\pm}) = P_{(0)} \left(\frac{1}{1 \mp 2\eta R_g \circ P_{(2)}} j_{\pm} \right). \quad (3.40)$$

As a result, we obtain the 2D action

$$S[g] = 4K \int_{\mathcal{M}} \left\langle j_{-}, P_{(2)} \left(\frac{1}{1 - 2\eta R_g \circ P_{(2)}} j_{+} \right) \right\rangle d\sigma \wedge d\tau, \quad (3.41)$$

and the Lax pair

$$\mathcal{L}_{\pm} = P_{(0)} \left(\frac{1}{1 \mp 2\eta R_g \circ P_{(2)}} j_{\pm} \right) + z^{\pm 1} P_{(2)} \left(\frac{1}{1 \mp 2\eta R_g \circ P_{(2)}} j_{\pm} \right). \quad (3.42)$$

These are the standard expressions of the classical action and the Lax pair for a homogeneous YB deformed symmetric coset sigma model [12].

4 YB deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model

In this section, we shall reproduce the Green-Schwarz (GS) action of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model [20] and homogeneous YB deformations of it [11] from the 4D CS theory.

Supercoset. The action of the $\text{AdS}_5 \times \text{S}^5$ superstring in the GS formalism [20] is based on the following supercoset

$$\frac{\text{PSU}(2, 2|4)}{\text{SO}(1, 4) \times \text{SO}(5)}. \quad (4.1)$$

The gauge field A in the 4D CS action (2.1) takes a value in $\mathfrak{g} = \mathfrak{su}(2, 2|4)$. Usually, $\mathfrak{su}(2, 2|4)$ is represented by using 8×8 supermatrices satisfying the supertraceless and the reality conditions. Then the bracket $\langle \cdot, \cdot \rangle$ in the 4D action (2.1) is replaced by the supertrace Str .

Twist function. The Poisson structure of the $\text{AdS}_5 \times \text{S}^5$ superstring has been considered in [21, 22], and the twist function of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model is given by³

$$\omega = \varphi_{\text{str}}(z) dz = \frac{4z^3}{(z^4 - 1)^2} dz. \quad (4.2)$$

Here ω is invariant under the involution μ_t since φ_{str} satisfies

$$\overline{\varphi_{\text{str}}(z)} = \varphi_{\text{str}}(\bar{z}). \quad (4.3)$$

The poles and zeros of the twist function (4.2) are listed as

$$\mathfrak{p} = \{+1, -1, +i, -i\}, \quad \mathfrak{z} = \{0, \infty\}, \quad (4.4)$$

where the poles are double poles and the zeros are triple zeros.

Boundary condition. The associated boundary equations of motion are

$$\epsilon^{ij} \langle\langle (A_i, \partial_{\xi_p} A_i), \delta(A_j, \partial_{\xi_p} A_j) \rangle\rangle_p = 0, \quad p \in \mathfrak{p}, \quad (4.5)$$

where the double bracket is defined as

$$\begin{aligned} \langle\langle (x, y), (x', y') \rangle\rangle_p &\equiv (\text{res}_p \omega) \text{Str}(x \cdot x') + (\text{res}_p \xi_p \omega) (\text{Str}(x \cdot y') + \text{Str}(x' \cdot y)) \\ &= \frac{p}{4} (\text{Str}(x \cdot y') + \text{Str}(x' \cdot y)). \end{aligned} \quad (4.6)$$

As in the symmetric coset case, one may consider two classes of solutions to the boundary equations of motion (4.5). Now by considering

$$\begin{aligned} \tau((\partial_z A)|_{z=p}) &= (\tau(\partial_z A))|_{z=p} = (\partial_{\bar{z}}(\tau A))|_{z=p} = (\mu_t^*(\partial_z A))|_{z=p} \\ &= (\partial_z A)|_{z=\bar{p}}, \end{aligned} \quad (4.7)$$

the reality condition (2.19) leads to

$$(\tau(A)|_{z=p}, \tau(\partial_z A)|_{z=p}) = (A|_{z=\bar{p}}, (\partial_z A)|_{z=\bar{p}}). \quad (4.8)$$

Due to the reality condition (4.8), the relation

$$(\partial_z A)|_{z=+i} = (\partial_z A)|_{z=-i} \quad \left(\in \mathfrak{su}(2, 2|4)^{\mathbb{C}} \right) \quad (4.9)$$

holds for the following two boundary conditions.

For the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model, we take the following solution:

$$\text{i) } (A|_{z=p}, \partial_z A|_{z=p}) \in \{0\} \times \mathfrak{su}(2, 2|4)_{p,\text{ab}} \quad (p \in \mathfrak{p}), \quad (4.10)$$

where $\mathfrak{su}(2, 2|4)_{p,\text{ab}}$ is an abelian copy of $\mathfrak{su}(2, 2|4)$ and $\mathfrak{su}(2, 2|4)^{\mathbb{C}}$:

$$\mathfrak{su}(2, 2|4)_{p,\text{ab}} \equiv \begin{cases} \mathfrak{su}(2, 2, |4) & \text{for } p = \pm 1 \\ \mathfrak{su}(2, 2, |4)^{\mathbb{C}} & \text{for } p = \pm i. \end{cases} \quad (4.11)$$

³ $\varphi_{\text{str}}(z)$ is slightly different from $\phi_{\text{string}}(z)$ in (2.10) of [10]. These are related via $\varphi_{\text{str}}(z) = \frac{1}{z} \phi_{\text{string}}(z)$.

The second choice for a homogeneous YB deformed $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model is given by

$$\text{ii) } \quad (A|_{z=p}, \partial_z A|_{z=p}) \in \mathfrak{su}(2, 2|4)_{p, R_{n_p}} \quad (p \in \mathfrak{p}). \quad (4.12)$$

The subscript n_p of R denotes the label of the poles as $\{n_1, n_i, n_{-1}, n_{-i}\} \equiv \{1, 2, 3, 4\}$, and $\mathfrak{su}(2, 2|4)_{p, R_{n_p}}$ is defined as

$$\mathfrak{su}(2, 2|4)_{p, R_{n_p}} \equiv \begin{cases} \{(p \eta R_{n_p}(x), x) \mid x \in \mathfrak{su}(2, 2|4)\} & \text{for } p = \pm 1 \\ \{(p \eta R_{n_p}(x), x) \mid x \in \mathfrak{su}(2, 2|4)^{\mathbb{C}}\} & \text{for } p = \pm i. \end{cases} \quad (4.13)$$

Here the linear operators $R_k : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ ($k = 1, 2, 3, 4$) are

$$R_k \equiv f_s^{k-1} \circ R \circ f_s^{-(k-1)}, \quad (4.14)$$

where the linear R -operator $R : \mathfrak{su}(2, 2|4)^{\mathbb{C}} \rightarrow \mathfrak{su}(2, 2|4)^{\mathbb{C}}$ is a solution to the hCYBE for $\mathfrak{su}(2, 2|4)^{\mathbb{C}}$, and $f_s : \mathfrak{su}(2, 2|4)^{\mathbb{C}} \rightarrow \mathfrak{su}(2, 2|4)^{\mathbb{C}}$ is a \mathbb{Z}_4 -grading automorphism of $\mathfrak{g}^{\mathbb{C}}$. An explicit representation is given in (4.27). For this representation, one can show that the R -operator R_k also satisfies the hCYBE (3.11) for $\mathfrak{su}(2, 2|4)^{\mathbb{C}}$ if R is a solution to the equation (3.11). Therefore, the boundary conditions (4.12) can be taken as solutions to the boundary equations of motion (4.5).

Lax form. Similarly to the symmetric coset sigma model case, let us take \hat{g} at each pole of the twist function (4.2) as

$$\begin{aligned} \hat{g}(\tau, \sigma, z)|_{z=1} &= g_1(\tau, \sigma), & \hat{g}(\tau, \sigma, z)|_{z=i} &= g_2(\tau, \sigma), \\ \hat{g}(\tau, \sigma, z)|_{z=-1} &= g_3(\tau, \sigma), & \hat{g}(\tau, \sigma, z)|_{z=-i} &= g_4(\tau, \sigma), \end{aligned} \quad (4.15)$$

where $g_k \in \text{SU}(2, 2|4)$ ($k = 1, 3$), $g_k \in \text{SU}(2, 2|4)^{\mathbb{C}}$ ($k = 2, 4$), such that

$$\tilde{\tau} g_2 = g_4. \quad (4.16)$$

The associated left-invariant currents are defined as

$$j_1 \equiv g_1^{-1} dg_1, \quad j_2 \equiv g_2^{-1} dg_2, \quad j_3 \equiv g_3^{-1} dg_3, \quad j_4 \equiv g_4^{-1} dg_4, \quad (4.17)$$

and the relations between the gauge field A and the Lax pair \mathcal{L} at each pole are written as

$$\begin{aligned} A|_{z=1} &= -dg_1 g_1^{-1} + \text{Ad}_{g_1} \mathcal{L}|_{z=1}, & A|_{z=i} &= -dg_2 g_2^{-1} + \text{Ad}_{g_2} \mathcal{L}|_{z=i}, \\ A|_{z=-1} &= -dg_3 g_3^{-1} + \text{Ad}_{g_3} \mathcal{L}|_{z=-1}, & A|_{z=-i} &= -dg_4 g_4^{-1} + \text{Ad}_{g_4} \mathcal{L}|_{z=-i}. \end{aligned} \quad (4.18)$$

From the zero structure of the twist function (4.2), we suppose the following ansatz for the Lax pair as

$$\begin{aligned} \mathcal{L} &= \left(z^{-1} V_+^{[-1]} + V_+^{[0]} + z V_+^{[1]} + z^2 V_+^{[2]} \right) d\sigma^+ \\ &+ \left(z^{-2} V_-^{[-2]} + z^{-1} V_-^{[-1]} + V_-^{[0]} + z V_-^{[1]} \right) d\sigma^-, \end{aligned} \quad (4.19)$$

where $V_{\pm}^{[n]}$ ($n = -1, 0, 1$), $V_{\pm}^{[\pm 2]} : \mathcal{M} \rightarrow \mathfrak{su}(2, 2|4)$ are smooth functions such that $\tau \mathcal{L} = \mu_t^* \mathcal{L}$. As we will see later, the ansatz (4.19) works well for both solutions to the boundary equations of motion. Note that the above ansatz (4.19) is not the only possible choice. One may consider other ansatz corresponding to the pure spinor formalism by following [3].

i) the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model. Let us reproduce the GS action of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model from the 4D CS action (2.1).

The boundary conditions (4.10) lead to

$$\begin{aligned}
 j_{1,\pm} &= V_{\pm}^{[0]} + V_{\pm}^{[\pm 2]} + V_{\pm}^{[1]} + V_{\pm}^{[-1]}, \\
 j_{2,\pm} &= V_{\pm}^{[0]} - V_{\pm}^{[\pm 2]} + i V_{\pm}^{[1]} - i V_{\pm}^{[-1]}, \\
 j_{3,\pm} &= V_{\pm}^{[0]} + V_{\pm}^{[\pm 2]} - V_{\pm}^{[1]} - V_{\pm}^{[-1]}, \\
 j_{4,\pm} &= V_{\pm}^{[0]} - V_{\pm}^{[\pm 2]} - i V_{\pm}^{[1]} + i V_{\pm}^{[-1]}.
 \end{aligned} \tag{4.20}$$

By solving these equations with respect to $V_{\pm}^{[n]}$, we obtain

$$\begin{aligned}
 V_{\pm}^{[0]} &= \frac{j_{1,\pm} + j_{2,\pm} + j_{3,\pm} + j_{4,\pm}}{4}, & V_{\pm}^{[\pm 2]} &= \frac{j_{1,\pm} - j_{2,\pm} + j_{3,\pm} - j_{4,\pm}}{4}, \\
 V_{\pm}^{[1]} &= \frac{j_{1,\pm} - i j_{2,\pm} - j_{3,\pm} + i j_{4,\pm}}{4}, & V_{\pm}^{[-1]} &= \frac{j_{1,\pm} + i j_{2,\pm} - j_{3,\pm} - i j_{4,\pm}}{4}.
 \end{aligned} \tag{4.21}$$

Then, $\text{res}_p(\varphi_{\text{str}} \mathcal{L})$ ($p \in \mathfrak{p}$) are evaluated as

$$\begin{aligned}
 \text{res}_{\pm 1}(\varphi_{\text{str}} \mathcal{L}) &= \frac{1}{8} \left(j_{1,+} - (1 \pm i) j_{2,+} + j_{3,+} - (1 \mp i) j_{4,+} \right) d\sigma^+ \\
 &\quad + \frac{1}{8} \left(-j_{1,-} + (1 \mp i) j_{2,-} - j_{3,-} + (1 \pm i) j_{4,-} \right) d\sigma^-, \\
 \text{res}_{\pm i}(\varphi_{\text{str}} \mathcal{L}) &= \frac{1}{8} \left(-(1 \mp i) j_{1,+} + j_{2,+} - (1 \pm i) j_{3,+} + j_{4,+} \right) d\sigma^+ \\
 &\quad + \frac{1}{8} \left((1 \pm i) j_{1,-} - j_{2,-} + (1 \mp i) j_{3,-} - j_{4,-} \right) d\sigma^-.
 \end{aligned} \tag{4.22}$$

Note that the set $\{\text{res}_{\pm 1}(\varphi_{\text{str}} \mathcal{L}), \text{res}_{\pm i}(\varphi_{\text{str}} \mathcal{L})\}$ is invariant under a cyclic permutation of g_k ($k = 1, \dots, 4$). This fact indicates that the associated 2D action also has the same symmetry. In fact, by using (4.22), we obtain the 2D action

$$\begin{aligned}
 S[g_k] &= \frac{1}{16} \int_{\mathcal{M}} \text{Str} \left[\sum_{\sigma \in S^4} \left(j_{\sigma(1),+} - (1+i) j_{\sigma(2),+} + j_{\sigma(3),+} - (1-i) j_{\sigma(4),+} \right) j_{\sigma(1),-} \right. \\
 &\quad \left. - \left(-j_{\sigma(1),-} + (1-i) j_{\sigma(2),-} - j_{\sigma(3),-} + (1+i) j_{\sigma(4),-} \right) j_{\sigma(1),+} \right] d\sigma^+ \wedge d\sigma^-,
 \end{aligned} \tag{4.23}$$

where $\sigma \in S^4$ is a cyclic permutation of the set $\{1, 2, 3, 4\}$. The action (4.23) is clearly invariant under the cyclic permutations of j_k .

Furthermore, we impose relations among j_k ($k = 1, \dots, 4$). From the cyclic symmetry of the 2D action (4.23), we can require the relation

$$j_k = f_s^{k-1}(j) \quad (k = 1, \dots, 4), \tag{4.24}$$

where $j \in \mathfrak{su}(2, 2|4)$ is the left-invariant current for $g \in \text{SU}(2, 2|4)$, and the map $f_s : \mathfrak{su}(2, 2|4)^{\mathbb{C}} \rightarrow \mathfrak{su}(2, 2|4)^{\mathbb{C}}$ is an automorphism of $\mathfrak{su}(2, 2|4)$ satisfying the \mathbb{Z}_4 -grading property $f_s^4 = \text{Id}$. As is well known, the superalgebra $\mathfrak{su}(2, 2|4)$ has the following decomposition into vector subspaces with respect to the \mathbb{Z}_4 -grading structure:

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}, \tag{4.25}$$

where $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)}$ and $\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(3)}$ are the bosonic and fermionic parts of $\mathfrak{su}(2, 2|4)$, respectively, and $\mathfrak{g}^{(0)}$ is identified with a bosonic subgroup $\mathfrak{so}(1, 4) \times \mathfrak{so}(5)$. The commutation relations of $\mathfrak{g}^{(m)}$ satisfy

$$[\mathfrak{g}^{(m)}, \mathfrak{g}^{(n)}] \subset \mathfrak{g}^{(k)} \quad (m + n = k \pmod{4}). \quad (4.26)$$

In order to obtain the GS action, let us take the \mathbb{Z}_4 -grading automorphism f_s such that each subspace $\mathfrak{g}^{(k)}$ ($k = 0, 1, 2, 3$) is the eigenspace of f_s satisfying

$$f_s(\mathfrak{g}^{(k)}) = i^k \mathfrak{g}^{(k)}. \quad (4.27)$$

Note that after taking a supermatrix realization of $\mathfrak{su}(2, 2|4)$, we can write down the explicit expression of f_s (For the details, see [23]).

The additional condition (4.24) enables us to express the functions $V^{[n]}$ in terms of the \mathbb{Z}_4 -graded components $j_{\pm}^{(k)} \in \mathfrak{g}^{(k)}$. In fact, by using (4.24) and (4.27), the left-invariant currents in (4.24) are rewritten as

$$\begin{aligned} j_{1,\pm} &= j_{\pm}^{(0)} + j_{\pm}^{(1)} + j_{\pm}^{(2)} + j_{\pm}^{(3)}, & j_{2,\pm} &= j_{\pm}^{(0)} + i j_{\pm}^{(1)} - j_{\pm}^{(2)} - i j_{\pm}^{(3)}, \\ j_{3,\pm} &= j_{\pm}^{(0)} - j_{\pm}^{(1)} + j_{\pm}^{(2)} - j_{\pm}^{(3)}, & j_{4,\pm} &= j_{\pm}^{(0)} - i j_{\pm}^{(1)} - j_{\pm}^{(2)} + i j_{\pm}^{(3)}. \end{aligned} \quad (4.28)$$

Then, by substituting (4.28) into (4.21), the functions $V^{[n]}$ are given by

$$V_{\pm}^{[0]} = j_{\pm}^{(0)}, \quad V_{\pm}^{[\pm 2]} = j_{\pm}^{(2)}, \quad V_{\pm}^{[1]} = j_{\pm}^{(1)}, \quad V_{\pm}^{[-1]} = j_{\pm}^{(3)}. \quad (4.29)$$

From this result, we immediately obtain the Lax pair

$$\begin{aligned} \mathcal{L} &= \left(z^{-1} j_+^{(3)} + j_+^{(0)} + z j_+^{(1)} + z^2 j_+^{(2)} \right) d\sigma^+ \\ &\quad + \left(z^{-2} j_-^{(2)} + z^{-1} j_-^{(3)} + j_-^{(0)} + z j_-^{(1)} \right) d\sigma^-. \end{aligned} \quad (4.30)$$

The expression (4.30) is precisely the same as the Lax pair constructed in [24].

Next, let us evaluate the 2D action (4.23). By using (4.29), we can see that the contribution to the 2D action from each pole is identical, namely,

$$\text{Str}(\text{res}_p(\varphi_{\text{str}} \mathcal{L}) \wedge g_p^{-1} dg_p) = \frac{1}{2} \text{Str}(j_- d_+(j_+)) d\sigma^+ \wedge d\sigma^-, \quad (4.31)$$

where d_{\pm} are the linear combinations of the projection operators $P_{(i)}$ like

$$d_{\pm} = \pm P_{(1)} + 2P_{(2)} \mp P_{(3)}. \quad (4.32)$$

This fact comes from the cyclic symmetry of the 2D action (4.23). As a result, we obtain

$$S[g_i] = \int_{\mathcal{M}} \text{Str}(j_- d_+(j_+)) d\sigma^+ \wedge d\sigma^-. \quad (4.33)$$

This is nothing but the Metsaev-Tseytlin action of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model [20].

ii) homogeneous YB deformations. Let us next discuss homogeneous YB deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model [11].

We consider the boundary condition (4.12). To avoid confusion of notations, we will replace the functions $V_{\pm}^{[n]}$ appeared in the Lax pair (4.19) with $\bar{V}_{\pm}^{[n]} (\in \mathfrak{su}(2, 2|4))$. Then, from the boundary condition (4.12), we obtain the relations

$$\begin{aligned}
 j_{1,\pm} &= \bar{V}_{\pm}^{[0]} + (1 \mp 2\eta R_{g_1}) \bar{V}_{\pm}^{[\pm 2]} + (1 - \eta R_{g_1}) \bar{V}_{\pm}^{[1]} + (1 + \eta R_{g_1}) \bar{V}_{\pm}^{[-1]}, \\
 j_{2,\pm} &= \bar{V}_{\pm}^{[0]} - (1 \mp 2\eta R_{g_2}) \bar{V}_{\pm}^{[\pm 2]} + i(1 - \eta R_{g_2}) V_{\pm}^{[1]} - i(1 + \eta R_{g_2}) \bar{V}_{\pm}^{[-1]}, \\
 j_{3,\pm} &= \bar{V}_{\pm}^{[0]} + (1 \mp 2\eta R_{g_3}) \bar{V}_{\pm}^{[\pm 2]} - (1 - \eta R_{g_3}) \bar{V}_{\pm}^{[1]} - (1 + \eta R_{g_3}) \bar{V}_{\pm}^{[-1]}, \\
 j_{4,\pm} &= \bar{V}_{\pm}^{[0]} - (1 \mp 2\eta R_{g_4}) \bar{V}_{\pm}^{[\pm 2]} - i(1 - \eta R_{g_4}) \bar{V}_{\pm}^{[1]} + i(1 + \eta R_{g_4}) \bar{V}_{\pm}^{[-1]},
 \end{aligned} \tag{4.34}$$

where the dressed R -operator R_{g_k} ($k = 1, \dots, 4$) is defined as

$$R_{g_k} \equiv \text{Ad}_{g_k}^{-1} \circ R_k \circ \text{Ad}_{g_k}. \tag{4.35}$$

By introducing the linear operator

$$\mathbf{R}_{\mathbf{g}}^{(p)} = \frac{1}{4} (R_{g_1} + i^p R_{g_2} + i^{2p} R_{g_3} + i^{3p} R_{g_4}), \tag{4.36}$$

the equations (4.34) are rewritten as

$$\begin{aligned}
 V_{\pm}^{[0]} &= \bar{V}_{\pm}^{[0]} \mp 2\eta \mathbf{R}_{\mathbf{g}}^{(2)} \bar{V}_{\pm}^{[\pm 2]} - \eta \mathbf{R}_{\mathbf{g}}^{(1)} \bar{V}_{\pm}^{[1]} + \eta \mathbf{R}_{\mathbf{g}}^{(3)} \bar{V}_{\pm}^{[-1]}, \\
 V_{\pm}^{[1]} &= \mp 2\eta \mathbf{R}_{\mathbf{g}}^{(1)} \bar{V}_{\pm}^{[\pm 2]} + (1 - \eta \mathbf{R}_{\mathbf{g}}^{(0)}) \bar{V}_{\pm}^{[1]} + \eta \mathbf{R}_{\mathbf{g}}^{(2)} \bar{V}_{\pm}^{[-1]}, \\
 V_{\pm}^{[\pm 2]} &= (1 \mp 2\eta \mathbf{R}_{\mathbf{g}}^{(0)}) \bar{V}_{\pm}^{[\pm 2]} - \eta \mathbf{R}_{\mathbf{g}}^{(3)} \bar{V}_{\pm}^{[1]} + \eta \mathbf{R}_{\mathbf{g}}^{(1)} \bar{V}_{\pm}^{[-1]}, \\
 V_{\pm}^{[-1]} &= \mp 2\eta \mathbf{R}_{\mathbf{g}}^{(3)} \bar{V}_{\pm}^{[\pm 2]} - \eta \mathbf{R}_{\mathbf{g}}^{(2)} \bar{V}_{\pm}^{[1]} + (1 + \eta \mathbf{R}_{\mathbf{g}}^{(0)}) \bar{V}_{\pm}^{[-1]},
 \end{aligned} \tag{4.37}$$

where the functions $V_{\pm}^{[n]}$ take the expressions (4.21). Since the operator $\mathbf{R}_{\mathbf{g}}^{(p)}$ is skew-symmetric, the equations (4.37) for $\bar{V}_{\pm}^{[n]}$ can be uniquely solved and the associated 2D action can also be written down. However, the resulting 2D action has a rather complex form, and so instead of giving its explicit expression, we will only show that the associated 2D action is invariant under the cyclic permutation of g_k ($k = 1, \dots, 4$).

For this purpose, let us define the map

$$\mathbf{P} : g_k \mapsto g_{k+1}, \tag{4.38}$$

Under this transformation, the linear operator $\mathbf{R}_{\mathbf{g}}^{(p)}$ in (4.36) and the functions $V_{\pm}^{[n]}$ in (4.21) are transformed as

$$\mathbf{P} \left(\mathbf{R}_{\mathbf{g}}^{(p)} \right) = i^{3p} \mathbf{R}_{\mathbf{g}}^{(p)}, \tag{4.39}$$

and

$$\begin{aligned}
 \mathbf{P} \left(V_{\pm}^{[0]} \right) &= V_{\pm}^{[0]}, & \mathbf{P} \left(V_{\pm}^{[\pm 2]} \right) &= -V_{\pm}^{[\pm 2]}, \\
 \mathbf{P} \left(V_{\pm}^{[1]} \right) &= i V_{\pm}^{[1]}, & \mathbf{P} \left(V_{\pm}^{[-1]} \right) &= -i V_{\pm}^{[-1]}.
 \end{aligned} \tag{4.40}$$

From the transformation rules in (4.39) and (4.40), and the equations (4.37), the functions $\overline{V}_{\pm}^{[n]}$ follow the same transformation rules as the functions $V_{\pm}^{[n]}$,

$$\begin{aligned} \mathbf{P}\left(\overline{V}_{\pm}^{[0]}\right) &= \overline{V}_{\pm}^{[0]}, & \mathbf{P}\left(\overline{V}_{\pm}^{[\pm 2]}\right) &= -\overline{V}_{\pm}^{[\pm 2]}, \\ \mathbf{P}\left(\overline{V}_{\pm}^{[1]}\right) &= i\overline{V}_{\pm}^{[1]}, & \mathbf{P}\left(\overline{V}_{\pm}^{[-1]}\right) &= -i\overline{V}_{\pm}^{[-1]}. \end{aligned} \quad (4.41)$$

This fact indicates that the residues $\text{res}_p(\varphi_{\text{str}} \mathcal{L})$ ($p \in \mathfrak{p}$) satisfy

$$\mathbf{P}\left(\text{res}_p(\varphi_{\text{str}} \mathcal{L})\right) = \text{res}_{p+1}(\varphi_{\text{str}} \mathcal{L}), \quad (4.42)$$

where $\text{res}_p(\varphi_{\text{str}} \mathcal{L})$ ($p \in \mathfrak{p}$) is given by

$$\text{res}_p(\varphi_{\text{str}} \mathcal{L}) = \frac{1}{4} \left(i^{n_p-1} \overline{V}_+^{[1]} + 2i^{2(n_p-1)} \overline{V}_+^{[2]} - i^{3(n_p-1)} \overline{V}_+^{[-1]} \right) d\sigma^+ \quad (4.43)$$

$$+ \frac{1}{4} \left(i^{n_p-1} \overline{V}_-^{[1]} - 2i^{2(n_p-1)} \overline{V}_-^{[2]} - i^{3(n_p-1)} \overline{V}_-^{[-1]} \right) d\sigma^-. \quad (4.44)$$

Therefore, the associated 2D action (2.21) is invariant under the permutation of g_k ($k = 1, \dots, 4$).

Thanks to the cyclic symmetry of the 2D action, we can require an additional condition

$$g_k = F_s^{k-1}(g) \quad (k = 1, \dots, 4), \quad (4.45)$$

where $g \in \text{SU}(2, 2|4)$, and the map $F_s : \text{SU}(2, 2|4) \rightarrow \text{SU}(2, 2|4)$ is an automorphism of $\text{SU}(2, 2|4)$ satisfying $F_s^4 = 1$. As in the symmetric coset case, let us take F_s so as to be induced by f_s defined in (4.27). More concretely, when a parameterization of an element $g \in \text{SU}(2, 2|4)$ is taken as

$$g = \exp \left(\sum_{k=0}^3 X^{A_k} T_{A_k}^{(k)} \right), \quad T_{A_k}^{(k)} \in \mathfrak{g}^{(k)} \quad (A_k = 1, \dots, \dim \mathfrak{g}^{(k)}), \quad (4.46)$$

the automorphism F_s is defined as

$$F_s(g) \equiv \exp \left(\sum_{k=0}^3 X^{A_k} f_s(T_{A_k}^{(k)}) \right) = \exp \left(\sum_{k=0}^3 i^k X^{A_k} T_{A_k}^{(k)} \right). \quad (4.47)$$

Here, X^{A_k} are functions of τ and σ . By definition, F_s is an automorphism of $\text{SU}(2, 2|4)$ with the \mathbb{Z}_4 -grading property.

Then, as shown in appendix A.2, the dressed R -operator R_{g_k} that act on the generators of $\mathfrak{su}(2, 2|4)$ should satisfy

$$P^{(m)} \circ R_{g_k} \circ P^{(n)} = i^{(m-n)(k-1)} P^{(m)} \circ R_g \circ P^{(n)}. \quad (4.48)$$

This relation indicates

$$P^{(m)} \circ \mathbf{R}_g^{(p)} \circ P^{(n)} = \begin{cases} P^{(m)} \circ R_g \circ P^{(n)} & m - n + p = 0 \pmod{4} \\ 0 & m - n + p \neq 0 \pmod{4} \end{cases}. \quad (4.49)$$

By using the relation (4.49), the equations in (4.37) can be solved as

$$\bar{V}_{\pm}^{[0]} = J_{\pm}^{(0)}, \quad \bar{V}_{\pm}^{[\pm 2]} = J_{\pm}^{(2)}, \quad \bar{V}_{\pm}^{[1]} = J_{\pm}^{(1)}, \quad \bar{V}_{\pm}^{[-1]} = J_{\pm}^{(3)}, \quad (4.50)$$

where the deformed current J_{\pm} is defined as

$$J_{\pm} \equiv \frac{1}{1 \mp \eta R_g \circ d_{\pm}} j_{\pm}. \quad (4.51)$$

Thus the Lax pair is given by

$$\begin{aligned} \mathcal{L} = & \left(z^{-1} J_+^{(3)} + J_+^{(0)} + z J_+^{(1)} + z^2 J_+^{(2)} \right) d\sigma^+ \\ & + \left(z^{-2} J_-^{(2)} + z^{-1} J_-^{(3)} + J_-^{(0)} + z J_-^{(1)} \right) d\sigma^-. \end{aligned} \quad (4.52)$$

This is nothing but the Lax pair of homogeneous YB deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model [11].

Next, let us derive the associated 2D action. By using (4.50), we find that the contribution to the 2D action from each pole is identical, namely,

$$\text{Str} \left(\text{res}_p(\varphi_{\text{str}} \mathcal{L}) \wedge g_p^{-1} dg_p \right) = \frac{1}{2} \text{Str} (j_- d_+(J_+)) d\sigma^+ \wedge d\sigma^-. \quad (4.53)$$

As a result, we obtain

$$S[g] = \int_{\mathcal{M}} \text{Str} (j_- d_+(J_+)) d\sigma^+ \wedge d\sigma^-. \quad (4.54)$$

This action (4.54) is precisely the same as that of homogeneous YB deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model [11].

5 Conclusion and discussion

In this paper, we have generalized the preceding result on the PCM to the case of the symmetric coset sigma model. By employing the same twist function in the rational description, we have specified boundary conditions which lead to the symmetric coset sigma model and the homogeneous YB-deformed relatives. The same analysis is applicable for the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model. As a result, homogeneous YB-deformations of the $\text{AdS}_5 \times \text{S}^5$ supercoset sigma model have been derived from the 4D CS theory as boundary conditions. In order to discuss the $\text{AdS}_5 \times \text{S}^5$ superstring beyond the sigma model, we have to take the Virasoro conditions into account by following the seminal work [17] in the present formulation. This is one of the most significant issues and the result will be reported in another place [25].

There are some open questions. It is well known that homogeneous YB deformations with abelian classical r -matrices can be seen as twisted boundary conditions [26–30] via non-local gauge transformations. It is interesting to consider the interpretation of this fact from the viewpoint of the 4D CS theory. It is also significant to understand how to realize the sine-Gordon model from the 4D CS theory. The sine-Gordon model can be reproduced

from the $O(3)$ NLSM via the Pohlmeyer reduction at the classical level. Hence it would be nice to study how the Pohlmeyer reduction works in the context of the 4D CS theory.

It is also interesting to study the η -deformation based on the modified classical YB equation as well, though we have discussed only the homogeneous YB-deformations. We will report the result in another place [25].

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A Relations for dressed R -operators

Here we shall prove the relations (3.38) and (4.48) that dressed R -operators should satisfy.

A.1 \mathbb{Z}_2 -grading case

Let us first give a proof of the relation (3.38) for a dressed R -operator.

To begin with, we examine how a dressed R -operator R_g acts on the generators. The adjoint operation with a group element g on the generators $\mathbf{P}_{\hat{a}}$ and $\mathbf{J}_{\hat{a}}$ is expressed as

$$\text{Ad}_g(\mathbf{P}_{\hat{a}}) = [\text{Ad}_g]_{\hat{a}}^{\check{b}} \mathbf{P}_{\check{b}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{J}_{\hat{a}}, \quad \text{Ad}_g(\mathbf{J}_{\hat{a}}) = [\text{Ad}_g]_{\hat{a}}^{\check{a}} \mathbf{P}_{\check{a}} + [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}}. \quad (\text{A.1})$$

Then the action of R_g on $\mathbf{P}_{\hat{a}}$ is evaluated as

$$\begin{aligned} R_g(\mathbf{P}_{\hat{a}}) &= \text{Ad}_{g^{-1}} \circ R([\text{Ad}_g]_{\hat{a}}^{\check{b}} \mathbf{P}_{\check{b}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{J}_{\hat{a}}) \\ &= \text{Ad}_{g^{-1}} \left([\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\check{c}} \mathbf{P}_{\check{c}} + [\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\hat{a}} \mathbf{J}_{\hat{a}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\check{b}} \mathbf{P}_{\check{b}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}} \right) \\ &= [\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\check{c}} [\text{Ad}_{g^{-1}}]_{\check{c}}^{\check{d}} \mathbf{P}_{\check{d}} + [\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\check{c}} [\text{Ad}_{g^{-1}}]_{\check{c}}^{\hat{a}} \mathbf{J}_{\hat{a}} \\ &\quad + [\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\hat{a}} [\text{Ad}_{g^{-1}}]_{\hat{a}}^{\check{c}} \mathbf{P}_{\check{c}} + [\text{Ad}_g]_{\hat{a}}^{\check{b}} R_{\check{b}}^{\hat{a}} [\text{Ad}_{g^{-1}}]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}} \\ &\quad + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\check{b}} [\text{Ad}_{g^{-1}}]_{\check{b}}^{\check{c}} \mathbf{P}_{\check{c}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\check{b}} [\text{Ad}_{g^{-1}}]_{\check{b}}^{\hat{b}} \mathbf{J}_{\hat{b}} \\ &\quad + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\check{b}} \mathbf{P}_{\check{b}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\check{c}} \mathbf{J}_{\check{c}}. \end{aligned} \quad (\text{A.2})$$

Next, let us see the adjoint actions of \tilde{g} , which is related to g through the \mathbb{Z}_2 -grading automorphism (3.37). By using the Campbell-Baker-Hausdorff formula and the \mathbb{Z}_2 -grading

property of \mathfrak{g} , we can obtain

$$\begin{aligned}
 \text{Ad}_{\tilde{g}}(\mathbf{P}_{\hat{a}}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}} + \text{ad}_{X^{\hat{b}}\mathbf{J}_{\hat{b}}})^n (\mathbf{P}_{\hat{a}}) \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} \left((\text{even number of } \text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}}) - (\text{odd number of } \text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}}) \right) (\mathbf{P}_{\hat{a}}) \\
 &= [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}}, \tag{A.3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Ad}_{\tilde{g}}(\mathbf{J}_{\hat{a}}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}} + \text{ad}_{X^{\hat{b}}\mathbf{J}_{\hat{b}}})^n (\mathbf{J}_{\hat{a}}) \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-(\text{odd number of } \text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}}) + (\text{even number of } \text{ad}_{X^{\hat{b}}\mathbf{P}_{\hat{b}}}) \right) (\mathbf{J}_{\hat{a}}) \\
 &= -[\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} + [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}}. \tag{A.4}
 \end{aligned}$$

These results indicate that the adjoint action with \tilde{g} is given by

$$\text{Ad}_{\tilde{g}}(\mathbf{P}_{\hat{a}}) = [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{J}_{\hat{a}}, \quad \text{Ad}_{\tilde{g}}(\mathbf{J}_{\hat{a}}) = -[\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{P}_{\hat{a}} + [\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}}. \tag{A.5}$$

Then, the action of $\tilde{R}_{\tilde{g}}$ on $\mathbf{P}_{\hat{a}}$ defined in (3.12) is given by

$$\begin{aligned}
 \tilde{R}_{\tilde{g}}(\mathbf{P}_{\hat{a}}) &= \text{Ad}_{\tilde{g}^{-1}} \circ \tilde{R}([\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{J}_{\hat{a}}) \\
 &= \text{Ad}_{\tilde{g}^{-1}} \circ f^{-1} \circ R(-[\text{Ad}_g]_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} \mathbf{J}_{\hat{a}}) \\
 &= \text{Ad}_{\tilde{g}^{-1}} \left([\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{c}} \mathbf{P}_{\hat{c}} - [\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{a}} \mathbf{J}_{\hat{a}} + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}} \right) \\
 &= [\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{c}} [\text{Ad}_{g^{-1}}]_{\hat{c}}^{\hat{d}} \mathbf{P}_{\hat{d}} - [\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{c}} [\text{Ad}_{g^{-1}}]_{\hat{c}}^{\hat{a}} \mathbf{J}_{\hat{a}} \\
 &\quad + [\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{a}} [\text{Ad}_{g^{-1}}]_{\hat{a}}^{\hat{c}} \mathbf{P}_{\hat{c}} - [\text{Ad}_g]_{\hat{a}}^{\hat{b}} R_{\hat{b}}^{\hat{a}} [\text{Ad}_{g^{-1}}]_{\hat{a}}^{\hat{b}} \mathbf{J}_{\hat{b}} \\
 &\quad + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\hat{c}} \mathbf{P}_{\hat{c}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\hat{b}} \mathbf{J}_{\hat{b}} \\
 &\quad + [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\hat{b}} \mathbf{P}_{\hat{b}} - [\text{Ad}_g]_{\hat{a}}^{\hat{a}} R_{\hat{a}}^{\hat{b}} [\text{Ad}_{g^{-1}}]_{\hat{b}}^{\hat{c}} \mathbf{J}_{\hat{c}}. \tag{A.6}
 \end{aligned}$$

By using (A.2) and (A.6), we can obtain the relation (3.38).

A.2 $\text{SU}(2, 2|4)$ case

Next, let us show that the action of the dressed R -operator R_{g_k} ($k = 1, \dots, 4$) on the $\mathfrak{su}(2, 2|4)$ generators satisfies the relation (4.48).

As in the previous case, we can see that the adjoint action with g_k on the generators of $\mathfrak{su}(2, 2|4)$ is written as

$$\text{Ad}_{g_k} \circ P^{(n)} = \sum_{s=0}^3 i^{(s-n)(k-1)} P^{(s)} \circ \text{Ad}_g \circ P^{(n)}, \tag{A.7}$$

$$P^{(m)} \circ \text{Ad}_{g_k}^{-1} = \sum_{r=0}^3 i^{(m-r)(k-1)} P^{(m)} \circ \text{Ad}_g^{-1} \circ P^{(r)}. \tag{A.8}$$

By using these relations and the definition (4.14) of R_{g_k} , the projected dressed R -operator $P^{(m)} \circ R_{g_k} \circ P^{(n)}$ can be expressed as

$$\begin{aligned}
 P^{(m)} \circ R_{g_k} \circ P^{(n)} &= P^{(m)} \circ \text{Ad}_{g_k}^{-1} \circ f_s^{k-1} \circ R \circ f_s^{-(k-1)} \circ \left(\sum_{s=0}^3 i^{(s-n)(k-1)} P^{(s)} \circ \text{Ad}_g \circ P^{(n)} \right) \\
 &= P^{(m)} \circ \text{Ad}_{g_k}^{-1} \circ f_s^{k-1} \circ \left(\sum_{s=0}^3 i^{(s-n)(k-1)-s(k-1)} R \circ P^{(s)} \circ \text{Ad}_g \circ P^{(n)} \right) \\
 &= P^{(m)} \circ \text{Ad}_{g_k}^{-1} \circ \left(\sum_{r,s=0}^3 i^{-n(k-1)+r(k-1)} P^{(r)} \circ R \circ P^{(s)} \circ \text{Ad}_g \circ P^{(n)} \right) \\
 &= \left(\sum_{r,s=0}^3 i^{(m-n)(k-1)} P^{(m)} \circ \text{Ad}_{g_k}^{-1} \circ P^{(r)} \circ R \circ P^{(s)} \circ \text{Ad}_g \circ P^{(n)} \right) \\
 &= i^{(m-n)(k-1)} P^{(m)} \circ R_g \circ P^{(n)}. \tag{A.9}
 \end{aligned}$$

Thus the relation (4.48) has been shown.

B Homogeneous bi-YB deformed sigma model

In this appendix, let us derive the action of a homogeneous bi-YB deformed principal chiral model, which is a two-parameter generalization of homogeneous YB deformation. In this case, we use the twist function (3.3) which is the same as in the symmetric coset case.

Boundary condition. A solution to the boundary equations of motion (3.5) is given by

$$(A|_{z=1}, \partial_z A|_{z=1}) \in \mathfrak{g}_{R_R}^{\mathbb{C}}, \quad (A|_{z=-1}, \partial_z A|_{z=-1}) \in \mathfrak{g}_{R_L}^{\mathbb{C}}, \tag{B.1}$$

where $\mathfrak{g}_{R_R}^{\mathbb{C}}$ and $\mathfrak{g}_{R_L}^{\mathbb{C}}$ are defined as

$$\mathfrak{g}_{R_R}^{\mathbb{C}} \equiv \left\{ (2\eta_R R_R(x), x) \mid x \in \mathfrak{g}^{\mathbb{C}} \right\}, \quad \mathfrak{g}_{R_L}^{\mathbb{C}} \equiv \left\{ (-2\eta_L R_L(x), x) \mid x \in \mathfrak{g}^{\mathbb{C}} \right\}. \tag{B.2}$$

Here η_R and η_L are the deformation parameters, and R_R and R_L are linear R -operators satisfying the hCYBE (3.11).

Lax form. Next, let us take \hat{g} at each pole of the twist function (3.3) as

$$\hat{g}(\tau, \sigma, z)|_{z=1} = g_R(\tau, \sigma), \quad \hat{g}(\tau, \sigma, z)|_{z=-1} = g_L(\tau, \sigma), \tag{B.3}$$

where $g_R, g_L \in G^{\mathbb{C}}$ (rather than G). Then, the relation between the gauge field A and the Lax pair \mathcal{L} at each pole is written as, respectively,

$$A|_{z=1} = -dg_R g_R^{-1} + \text{Ad}_{g_R} \mathcal{L}|_{z=1}, \quad A|_{z=-1} = -dg_L g_L^{-1} + \text{Ad}_{g_L} \mathcal{L}|_{z=-1}. \tag{B.4}$$

Since we use the same twist function (3.3) with the symmetric coset case, we suppose the same ansatz for the Lax pair:

$$\mathcal{L} = (U_+ + z V_+) d\sigma^+ + (U_- + z^{-1} V_-) d\sigma^-. \tag{B.5}$$

The solution (B.1) leads to

$$A|_{z=1} = 2\eta_R R_R(\partial_z A|_{z=1}), \quad A|_{z=-1} = -2\eta_L R_L(\partial_z A|_{z=-1}). \quad (\text{B.6})$$

By using (B.4), (B.5) and (B.6), we obtain

$$g_R^{-1} \partial_{\pm} g_R = U_{\pm} + (1 \mp 2\eta_R R_{R,g_R})(V_{\pm}), \quad (\text{B.7})$$

$$g_L^{-1} \partial_{\pm} g_L = U_{\pm} - (1 \mp 2\eta_L R_{L,g_L})(V_{\pm}). \quad (\text{B.8})$$

By solving these equations and removing U_{\pm} from the Lax pair, we obtain the following expression:

$$\begin{aligned} \mathcal{L}_{\pm} &= g_R^{-1} \partial_{\pm} g_R - (1 \mp \eta_R R_{R,g_R})(V_{\pm}) + z^{\pm 1} V_{\pm} \\ &= g_L^{-1} \partial_{\pm} g_L + (1 \mp \eta_L R_{L,g_L})(V_{\pm}) + z^{\pm 1} V_{\pm}, \end{aligned} \quad (\text{B.9})$$

where V_{\pm} contains both g_R and g_L like

$$V_{\pm} = \frac{1}{1 \mp \eta_R R_{R,g_R} \mp \eta_L R_{L,g_L}} \left(\frac{g_R^{-1} \partial_{\pm} g_R - g_L^{-1} \partial_{\pm} g_L}{2} \right). \quad (\text{B.10})$$

Deformed action. Now, we can obtain the action of the homogeneous bi-YB deformed sigma model. By using the expression of the Lax pair (B.9), the residues of $\varphi_c \mathcal{L}$ at $z = \pm 1$ are evaluated as

$$\text{res}_{z=1}(\varphi_c \mathcal{L}) = 4K(V_+ d\sigma^+ - V_- d\sigma^-), \quad (\text{B.11})$$

$$\text{res}_{z=-1}(\varphi_c \mathcal{L}) = -4K(V_+ d\sigma^+ - V_- d\sigma^-). \quad (\text{B.12})$$

Then the 2D action becomes

$$S[g_R, g_L] = K \int_{\Sigma} \langle g_R^{-1} \partial_+ g_R - g_L^{-1} \partial_+ g_L, V_- \rangle d\sigma \wedge d\tau. \quad (\text{B.13})$$

This is an unusual form of the action of the homogeneous bi-YB deformed sigma model.

In order to see the standard expression, let us use a complexified 2D gauge invariance $g_x \mapsto g_x h$ ($h \in G^{\mathbb{C}}$). Then, we can realize the following configuration:

$$g_R = g, \quad g_L = 1, \quad (\text{B.14})$$

where $g \in G$. With this gauge, the action (B.13) reduces to

$$S[g] = \frac{K}{2} \int_{\Sigma} \left\langle g^{-1} \partial_+ g, \frac{1}{1 + \eta_R R_{R,g} + \eta_L R_L} g^{-1} \partial_- g \right\rangle d\sigma \wedge d\tau. \quad (\text{B.15})$$

This is the standard expression of the homogeneous bi-YB deformed sigma model action. Then the Lax pair (B.9) is also simplified as

$$\mathcal{L}_{\pm} = \frac{1 + z^{\pm 1} \mp \eta_L R_L}{2} \left(\frac{1}{1 \mp \eta_R R_{R,g} \mp \eta_L R_L} g^{-1} \partial_{\pm} g \right). \quad (\text{B.16})$$

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References

- [1] K. Costello, E. Witten and M. Yamazaki, *Gauge theory and integrability, I*, [arXiv:1709.09993](https://arxiv.org/abs/1709.09993) [[INSPIRE](#)].
- [2] K. Costello, E. Witten and M. Yamazaki, *Gauge theory and integrability, II*, [arXiv:1802.01579](https://arxiv.org/abs/1802.01579) [[INSPIRE](#)].
- [3] K. Costello and M. Yamazaki, *Gauge theory and integrability, III*, [arXiv:1908.02289](https://arxiv.org/abs/1908.02289) [[INSPIRE](#)].
- [4] B. Vicedo, *Holomorphic Chern-Simons theory and affine Gaudin models*, [arXiv:1908.07511](https://arxiv.org/abs/1908.07511) [[INSPIRE](#)].
- [5] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *A unifying 2d action for integrable σ -models from 4d Chern-Simons theory*, [arXiv:1909.13824](https://arxiv.org/abs/1909.13824) [[INSPIRE](#)].
- [6] C. Klimčík, *Yang-Baxter σ -models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [[hep-th/0210095](https://arxiv.org/abs/hep-th/0210095)] [[INSPIRE](#)].
- [7] C. Klimčík, *On integrability of the Yang-Baxter σ -model*, *J. Math. Phys.* **50** (2009) 043508 [[arXiv:0802.3518](https://arxiv.org/abs/0802.3518)] [[INSPIRE](#)].
- [8] F. Delduc, M. Magro and B. Vicedo, *On classical q -deformations of integrable σ -models*, *JHEP* **11** (2013) 192 [[arXiv:1308.3581](https://arxiv.org/abs/1308.3581)] [[INSPIRE](#)].
- [9] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [[arXiv:1309.5850](https://arxiv.org/abs/1309.5850)] [[INSPIRE](#)].
- [10] F. Delduc, M. Magro and B. Vicedo, *Derivation of the action and symmetries of the q -deformed $AdS_5 \times S^5$ superstring*, *JHEP* **10** (2014) 132 [[arXiv:1406.6286](https://arxiv.org/abs/1406.6286)] [[INSPIRE](#)].
- [11] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2014) 153 [[arXiv:1401.4855](https://arxiv.org/abs/1401.4855)] [[INSPIRE](#)].
- [12] T. Matsumoto and K. Yoshida, *Yang-Baxter σ -models based on the CYBE*, *Nucl. Phys. B* **893** (2015) 287 [[arXiv:1501.03665](https://arxiv.org/abs/1501.03665)] [[INSPIRE](#)].
- [13] K. Sfetsos, *Integrable interpolations: from exact CFTs to non-Abelian T-duals*, *Nucl. Phys. B* **880** (2014) 225 [[arXiv:1312.4560](https://arxiv.org/abs/1312.4560)] [[INSPIRE](#)].
- [14] T.J. Hollowood, J.L. Miramontes and D.M. Schmidt, *Integrable deformations of strings on symmetric spaces*, *JHEP* **11** (2014) 009 [[arXiv:1407.2840](https://arxiv.org/abs/1407.2840)] [[INSPIRE](#)].
- [15] D.M. Schmidt, *Holomorphic Chern-Simons theory and lambda models: PCM case*, *JHEP* **04** (2020) 060 [[arXiv:1912.07569](https://arxiv.org/abs/1912.07569)] [[INSPIRE](#)].
- [16] C. Bassi and S. Lacroix, *Integrable deformations of coupled σ -models*, *JHEP* **05** (2020) 059 [[arXiv:1912.06157](https://arxiv.org/abs/1912.06157)] [[INSPIRE](#)].
- [17] K. Costello and B. Stefański, *The Chern-Simons origin of superstring integrability*, [arXiv:2005.03064](https://arxiv.org/abs/2005.03064) [[INSPIRE](#)].

- [18] O. Fukushima, J.-I. Sakamoto and K. Yoshida, *Comments on η -deformed principal chiral model from 4D Chern-Simons theory*, *Nucl. Phys. B* **957** (2020) 115080 [[arXiv:2003.07309](#)] [[INSPIRE](#)].
- [19] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *On the Hamiltonian integrability of the bi-Yang-Baxter σ -model*, *JHEP* **03** (2016) 104 [[arXiv:1512.02462](#)] [[INSPIRE](#)].
- [20] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys. B* **533** (1998) 109 [[hep-th/9805028](#)] [[INSPIRE](#)].
- [21] H. Itoyama and T. Oota, *The $AdS_5 \times S^5$ superstrings in the generalized light-cone gauge*, *Prog. Theor. Phys.* **117** (2007) 957 [[hep-th/0610325](#)] [[INSPIRE](#)].
- [22] B. Vicedo, *Hamiltonian dynamics and the hidden symmetries of the $AdS_5 \times S^5$ superstring*, *JHEP* **01** (2010) 102 [[arXiv:0910.0221](#)] [[INSPIRE](#)].
- [23] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ superstring. Part I*, *J. Phys. A* **42** (2009) 254003 [[arXiv:0901.4937](#)] [[INSPIRE](#)].
- [24] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[INSPIRE](#)].
- [25] O. Fukushima, J. Sakamoto and K. Yoshida, in preparation.
- [26] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [[hep-th/0503201](#)] [[INSPIRE](#)].
- [27] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018 [[hep-th/0512253](#)] [[INSPIRE](#)].
- [28] T. Matsumoto and K. Yoshida, *Lunin-Maldacena backgrounds from the classical Yang-Baxter equation — towards the gravity/CYBE correspondence*, *JHEP* **06** (2014) 135 [[arXiv:1404.1838](#)] [[INSPIRE](#)].
- [29] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, *JHEP* **06** (2014) 163 [[arXiv:1404.3657](#)] [[INSPIRE](#)].
- [30] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys. B* **915** (2017) 184 [[arXiv:1608.08504](#)] [[INSPIRE](#)].