RECEIVED: June 26, 2019

REVISED: August 30, 2019

ACCEPTED: September 5, 2019

PUBLISHED: September 16, 2019

Classifying solutions to Romans supergravity with a zero B-field

Carolina Matté Gregory

*Instituto de Física, Universidade de Brasília,
70910-900, Brasília, DF, Brasil*

E-mail: carolina.gregory@unb.br

ABSTRACT: Expanding on previous papers, we continue studying Euclidean Romans supergravity in six dimensions with a non-trivial Abelian R-symmetry gauge field. Using a set of differential constraints on a $SU(2)$ structure, we look for further geometric solutions to such equations when we turn off the two-form potential B . We find that the six-dimensional space is described by a two-dimensional fibration over a four-dimensional manifold with a Kähler metric. We then classify these types of solutions.

KEYWORDS: AdS-CFT Correspondence, Supergravity Models, Gauge-gravity correspondence, Supersymmetry and Duality

ARXIV EPRINT: [1906.01660](https://arxiv.org/abs/1906.01660)

Contents

1	Introduction	1
2	Euclidean Romans supergravity	2
3	SU(2) structure	3
4	Classifying solutions with a zero B-field	6
4.1	Taking $B = 0$	6
4.2	Conditions for a supersymmetric M_6	7
4.2.1	Summary	10
4.3	Complex M_6 (setting $d_4\vartheta = 0$)	12
4.3.1	Kähler-Einstein base solutions	12
4.3.2	Product of two Riemann surfaces base solutions	14
5	Conclusion	15

1 Introduction

In previous papers [1, 2], we constructed gravity duals to five-dimensional gauge theories on non-conformally flat backgrounds, specifically, certain families of squashed five-spheres and Sasaki-Einstein manifolds. In [1], we have used the six-dimensional Romans $F(4)$ supergravity [3], which is a consistent truncation of massive IIA supergravity on S^4 [4], and, more recently, it has been shown to be also a consistent truncation of IIB supergravity on a warped product of $S^2 \times \Sigma$, where Σ is a Riemann surface [5, 6]. Having constructed these supergravity solutions, we then computed the holographic free energy $\mathcal{F} = -\log Z$ by holographically renormalizing the on-shell Euclidean action. The perturbative partition function for these theories has been computed in [7] and we have explicitly shown that the large N limit of these partition functions is in precise agreement with the holographic free energies of our supergravity solutions.

In [2], we have shown that real Euclidean supersymmetric solutions to Romans $F(4)$ gauged supergravity, with a non-trivial Abelian R-symmetry gauge field, have a canonical SU(2) structure determined by the Killing spinor. More precisely, we have shown that the supersymmetry conditions together with the equations of motion are equivalent to a set of differential constraints on this SU(2) structure. This geometric formulation then led to a number of interesting applications. We showed that this structure extended into the bulk the conformal boundary SU(2) structure studied in [8], which allowed for the construction of gravity duals to families of five-dimensional gauge theories on rigid backgrounds. As another application we extended several of the results of the previous paper.

In fact, analysing supersymmetric solutions via G-structures has been widely used, first shown in [9], then developed for various string theory settings [10]–[21]. In this paper, we take the same $SU(2)$ structure from [2] and look for further generalizations of the solutions. In particular, we use the approach from [22] to classify the families of solutions arising from a theory with a zero two-form potential B . We find that our six-dimensional space can be seen as a five-dimensional one which is orthogonal to the Killing vector. This five-dimensional space, in turn, can be seen as a product of a one-dimensional space and a four-dimensional space with a Kähler metric.

The plan for the paper is as follows. In section 2, we summarize Euclidean Romans supergravity theory with a non-trivial Abelian R-symmetry gauge field. In section 3, we summarize the differential constraints imposed on the canonical $SU(2)$ structure of this Romans theory to ensure supersymmetric solutions. In section 4 we take the B-field to be equal zero in the differential constraints forementioned, simplifying the problem. These new equations can now be solved in closed form. We end by classifying the families of solutions.

2 Euclidean Romans supergravity

The bosonic fields of the six-dimensional Romans supergravity theory [3] consist of the metric, a scalar field $X = \exp\left(-\frac{\phi}{2\sqrt{2}}\right)$ where ϕ is the dilaton, a two-form potential B , together with an $SO(3)_R \sim SU(2)_R$ R-symmetry gauge field A^i with field strength $F^i = dA^i - \frac{1}{2}\varepsilon_{ijk}A^j \wedge A^k$, where $i = 1, 2, 3$. Here we are working in a gauge in which the Stueckelberg one-form is zero, and we set the gauge coupling constant to 1. The Euclidean signature equations of motion are [1]

$$\begin{aligned} d(X^{-1} * dX) &= -\left(\frac{1}{6}X^{-6} - \frac{2}{3}X^{-2} + \frac{1}{2}X^2\right) * 1 \\ &\quad - \frac{1}{8}X^{-2} \left(\frac{4}{9}B \wedge *B + F^i \wedge *F^i\right) + \frac{1}{4}X^4 H \wedge *H, \\ d(X^4 * H) &= \frac{2i}{9}B \wedge B + \frac{i}{2}F^i \wedge F^i + \frac{4}{9}X^{-2} * B, \\ D(X^{-2} * F^i) &= -iF^i \wedge H. \end{aligned} \quad (2.1)$$

Here $H = dB$ and $D\omega^i = d\omega^i - \varepsilon_{ijk}A^j \wedge \omega^k$ is the $SO(3)$ covariant derivative. Notice that the theory contains Chern-Simons-type couplings, that become purely imaginary in Euclidean signature. The Einstein equation is

$$\begin{aligned} R_{\mu\nu} &= 4X^{-2}\partial_\mu X \partial_\nu X + \left(\frac{1}{18}X^{-6} - \frac{2}{3}X^{-2} - \frac{1}{2}X^2\right)g_{\mu\nu} \\ &\quad + \frac{1}{4}X^4 \left(H_{\mu\nu}^2 - \frac{1}{6}H^2 g_{\mu\nu}\right) + \frac{2}{9}X^{-2} \left(B_{\mu\nu}^2 - \frac{1}{8}B^2 g_{\mu\nu}\right) \\ &\quad + \frac{1}{2}X^{-2} \left((F^i)_{\mu\nu}^2 - \frac{1}{8}(F^i)^2 g_{\mu\nu}\right), \end{aligned} \quad (2.2)$$

where $B_{\mu\nu}^2 = B_{\mu\rho}B_{\nu}^{\rho}$, $H_{\mu\nu}^2 = H_{\mu\rho\sigma}H_{\nu}^{\rho\sigma}$.

A solution is supersymmetric provided there exists a non-trivial $SU(2)_R$ doublet of Dirac spinors ϵ_I , $I = 1, 2$, satisfying the following Killing spinor and dilatino equations

$$D_\mu \epsilon_I = \frac{i}{4\sqrt{2}} \left(X + \frac{1}{3} X^{-3} \right) \Gamma_\mu \Gamma_7 \epsilon_I - \frac{i}{24\sqrt{2}} X^{-1} B_{\nu\rho} (\Gamma_\mu^{\nu\rho} - 6\delta_\mu^\nu \Gamma^\rho) \epsilon_I \\ - \frac{1}{48} X^2 H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \Gamma_7 \epsilon_I + \frac{1}{16\sqrt{2}} X^{-1} F_{\nu\rho}^i (\Gamma_\mu^{\nu\rho} - 6\delta_\mu^\nu \Gamma^\rho) \Gamma_7 (\sigma_i)_I^J \epsilon_J, \quad (2.3)$$

$$0 = -iX^{-1} \partial_\mu X \Gamma^\mu \epsilon_I + \frac{1}{2\sqrt{2}} (X - X^{-3}) \Gamma_7 \epsilon_I + \frac{i}{24} X^2 H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_7 \epsilon_I \\ - \frac{1}{12\sqrt{2}} X^{-1} B_{\mu\nu} \Gamma^{\mu\nu} \epsilon_I - \frac{i}{8\sqrt{2}} X^{-1} F_{\mu\nu}^i \Gamma^{\mu\nu} \Gamma_7 (\sigma_i)_I^J \epsilon_J. \quad (2.4)$$

Here Γ_μ , $\mu = 1, \dots, 6$, are taken to be Hermitian and generate the Clifford algebra $\text{Cliff}(6, 0)$ in an orthonormal frame. We have defined the chirality operator $\Gamma_7 = i\Gamma_{123456}$, which satisfies $(\Gamma_7)^2 = 1$. The covariant derivative acting on the spinor is $D_\mu \epsilon_I = \nabla_\mu \epsilon_I + \frac{i}{2} A_\mu^i (\sigma_i)_I^J \epsilon_J$, where $\nabla_\mu = \partial_\mu + \frac{1}{4} \Omega_\mu^{\nu\rho} \Gamma_{\nu\rho}$ denotes the Levi-Civita spin connection while σ_i , $i = 1, 2, 3$, are the Pauli matrices.

For simplicity we shall consider Abelian solutions in which $A_\mu^1 = A_\mu^2 = 0$, and $A_\mu^3 \equiv \mathcal{A}_\mu$, with field strength $\mathcal{F} \equiv d\mathcal{A}$. Also, as in [1], we consider a “real” class of solutions for which ϵ_I satisfies the symplectic Majorana condition $\varepsilon_I^J \epsilon_J = \mathcal{C} \epsilon_I^* \equiv \epsilon_I^c$, where $\mathcal{C}$ denotes the charge conjugation matrix, satisfying $\Gamma_\mu^T = \mathcal{C}^{-1} \Gamma_\mu \mathcal{C}$. The bosonic fields are all taken to be real, with the exception of the B -field which is purely imaginary and that will later be taken to be zero. With these reality properties one can show that the Killing spinor equation (2.3) and dilatino equation (2.4) for ϵ_2 are simply the charge conjugates of the corresponding equations for ϵ_1 . In this way we effectively reduce to a single Killing spinor $\epsilon \equiv \epsilon_1$, with $SU(2)_R$ doublet $(\epsilon_1, \epsilon_2) = (\epsilon, \epsilon^c)$.

3 $SU(2)$ structure

Consider a Dirac spinor ϵ in six dimensions, such that $(\epsilon_1, \epsilon_2) = (\epsilon, \epsilon^c)$ solves (2.3) and (2.4) above. We may construct the following scalar bilinears

$$S \equiv \epsilon^\dagger \epsilon, \quad \tilde{S} \equiv \epsilon^\dagger \Gamma_7 \epsilon, \quad f \equiv \epsilon^T \epsilon. \quad (3.1)$$

We have chosen a basis for the gamma matrices in which they are purely imaginary and anti-symmetric, with charge conjugation matrix $\mathcal{C} = -i\Gamma_7$. A short computation reveals that

$$d(Xf) = -i(Xf)\mathcal{A}. \quad (3.2)$$

The integrability condition for this equation immediately implies $\mathcal{F} = d\mathcal{A} = 0$ unless $f \equiv 0$ (notice that X is nowhere zero). We will henceforth restrict our analysis to the case $f \equiv 0$, which is necessary for a non-trivial R-symmetry gauge field.¹

¹There are nevertheless interesting solutions for which $f \neq 0$. Indeed, we have constructed a 1/2 BPS solution in [1] lying in this class.

We may then write

$$\epsilon = \epsilon_+ + \epsilon_- , \quad (3.3)$$

where $-\Gamma_7 \epsilon_{\pm} = \pm \epsilon_{\pm}$, and furthermore the condition $f \equiv 0$ allows us to introduce

$$\epsilon_+ = \sqrt{S} \cos \vartheta \eta_1 , \quad \epsilon_- = \sqrt{S} \sin \vartheta \eta_2^* , \quad (3.4)$$

as done in [22]. Here η_1, η_2 are two orthogonal unit norm chiral spinors, so that $\eta_1^\dagger \eta_1 = \eta_2^\dagger \eta_2 = 1$ and $\eta_2^\dagger \eta_1 = 0$. These each define a canonical SU(3) structure, and together determine a canonical SU(2) structure. Concretely, in six dimensions such a structure is specified by two one-forms K_1, K_2 and a triplet of two-forms $J_i, i = 1, 2, 3$, given by

$$\begin{aligned} K_1 - iK_2 &\equiv -\frac{1}{2} \varepsilon^{\alpha\beta} \eta_\alpha^T \Gamma_{(1)} \eta_\beta , \\ J_i &\equiv -\frac{i}{2} \sigma_i^{\alpha\beta} \eta_\alpha^\dagger \Gamma_{(2)} \eta_\beta . \end{aligned} \quad (3.5)$$

Here we have introduced the notation $\Gamma_{(n)} \equiv \frac{1}{n!} \Gamma_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$, where x^μ are local coordinates. We also define

$$\Omega \equiv J_2 + iJ_1 , \quad J \equiv J_3 . \quad (3.6)$$

The canonical SU(2) structure is thus determined by (K_1, K_2, J, Ω) . We note that K_1 and K_2 are orthonormal one-forms, and both are orthogonal to J and Ω , with $J \wedge \Omega = 0$ and $2J \wedge J = \Omega \wedge \bar{\Omega}$.

The SU(2) structure $(S, \vartheta, K_1, K_2, J, \Omega)$ that arises naturally from a supersymmetric solution is thus related to the canonical SU(2) structure by the square norm S and angle ϑ , via (3.4).

It was shown in [2], using the Killing spinor equation (2.3) and dilatino equation (2.4), that the one-form bilinear

$$K \equiv \epsilon^\dagger \Gamma_{(1)} \epsilon = S \sin 2\vartheta K_1 . \quad (3.7)$$

is a Killing one-form, so that the dual vector field $\xi \equiv K^\#$ is a Killing vector. We were able therefore to introduce a local coordinate ψ , so that $\xi = \partial_\psi$ and the metric is independent of ψ . From (3.7) it follows that we may write

$$K_1 = S \sin 2\vartheta (d\psi + \sigma) , \quad (3.8)$$

where $\mathcal{L}_\xi \sigma = 0 = i_\xi \sigma$. In fact, all of the supergravity fields and SU(2) structure are annihilated by $\mathcal{L}_\xi$, with the exception of the complex two-form Ω . The spinor ϵ is a spin^c spinor, charged under the Abelian R-symmetry gauge field $\mathcal{A}$, and provided one makes the gauge choice (3.11) below then also $\mathcal{L}_\xi \Omega = 0$. Thus the vector field $\xi = \partial_\psi$ generates a symmetry of the full solution.

The spinor equations (2.3), (2.4) impose further constraints on the supergravity fields and SU(2) structure. The B -field and R-symmetry gauge field strength $\mathcal{F} = d\mathcal{A}$ may be

written as

$$B = i K_1 \wedge \left[\frac{3}{\sqrt{2} S \sin 2\vartheta} d(XS) + X^{-2} K_2 \right] + B_\perp, \quad (3.9)$$

$$\mathcal{F} = K_1 \wedge \frac{\sqrt{2}}{S \sin 2\vartheta} d(XS \cos 2\vartheta) + \mathcal{F}_\perp, \quad (3.10)$$

where $B_\perp$ and $\mathcal{F}_\perp$ have zero interior contraction with ξ . In particular (3.10) allows us to write

$$\mathcal{A} = -\sqrt{2} X \cot 2\vartheta K_1 + \mathcal{A}_\perp, \quad (3.11)$$

where $i_\xi \mathcal{A}_\perp = 0$ and we have made a partial gauge choice for $\mathcal{A}$. We note that

$$\mathcal{F}_\perp = -\sqrt{2} X S \cos 2\vartheta d\sigma + d\mathcal{A}_\perp. \quad (3.12)$$

We may similarly write the component of $H = dB$ perpendicular to ξ as

$$H_\perp \equiv i \left[\frac{3}{\sqrt{2}} d(XS) + X^{-2} S \sin 2\vartheta K_2 \right] \wedge d\sigma + dB_\perp. \quad (3.13)$$

The differential constraints on the $SU(2)$ structure, also shown in [2], are given by

$$\begin{aligned} X^2 S^2 \sin^2 2\vartheta d\sigma &= -\frac{2\sqrt{2}}{3} X^{-1} S \cos 2\vartheta J - i X^4 S \sin 2\vartheta K_1 \lrcorner * H_\perp \\ &\quad + \sqrt{2} X S \left(\cos 2\vartheta \mathcal{F}_\perp + \frac{2}{3} i B_\perp \right), \\ d(X^{-1} S \cos 2\vartheta J) &= -\frac{3}{2\sqrt{2}} d[(XS)^2 d\sigma] + i X S dB_\perp \\ &\quad + \frac{\sqrt{2}}{3} i X^{-2} S \sin 2\vartheta [K_1 \lrcorner * B_\perp - K_2 \wedge B_\perp], \\ d(X^{-1} S J) &= -\sqrt{2} S \sin 2\vartheta J \wedge K_2 - \frac{3}{2\sqrt{2}} \cos 2\vartheta d[(XS)^2 d\sigma] \\ &\quad + i X S \cos 2\vartheta dB_\perp - \frac{1}{\sqrt{2}} X^{-2} S \sin 2\vartheta [K_1 \lrcorner * \mathcal{F}_\perp - K_2 \wedge \mathcal{F}_\perp], \\ d(S \sin 2\vartheta J \wedge K_2) &= 0, \\ D_\perp(X^{-1} S \sin 2\vartheta \Omega) &= -\sqrt{2} S \Omega \wedge K_2, \\ S^2 J \wedge d\sigma &= -\sqrt{2} S \cos 2\vartheta \left(X + \frac{2}{3} X^{-3} \right) \frac{1}{2} J \wedge J + 2 S K_1 \lrcorner * d\vartheta \\ &\quad + \frac{1}{\sqrt{2}} X^{-1} S J \wedge \left(\cos 2\vartheta d\mathcal{A}_\perp + \frac{2}{3} i B_\perp \right), \\ S^2 \Omega \wedge d\sigma &= -2i S d\vartheta \wedge K_2 \wedge \Omega + \frac{1}{\sqrt{2}} X^{-1} S \Omega \wedge \left(\cos 2\vartheta d\mathcal{A}_\perp + \frac{2}{3} i B_\perp \right), \\ 0 &= X^4 K_2 \lrcorner d(X^{-3} S \sin 2\vartheta) + \sqrt{2} S \left(X^2 - \frac{2}{3} X^{-2} \right) \\ &\quad + \frac{1}{\sqrt{2}} S J \lrcorner \left(\mathcal{F}_\perp + \frac{2}{3} i \cos 2\vartheta B_\perp \right). \end{aligned} \quad (3.14)$$

When supplemented by

$$0 = X^4 S \sin 2\vartheta \, d\sigma \wedge (K_1 \lrcorner * iH_\perp) + d \left[\frac{X^4}{S \sin 2\vartheta} K_1 \lrcorner * d(X^{-2} S \sin 2\vartheta K_2) \right] \\ + \frac{2}{9} B_\perp \wedge B_\perp + \frac{1}{2} \mathcal{F}_\perp \wedge \mathcal{F}_\perp - \frac{4}{9} X^{-2} K_1 \lrcorner * \left[\frac{3}{\sqrt{2} S \sin 2\vartheta} d(XS) + X^{-2} K_2 \right], \quad (3.15)$$

solving these equations is equivalent to finding supergravity solutions to Euclidean Romans theory.

4 Classifying solutions with a zero B-field

Briefly, the idea is to take Euclidean Romans supergravity theory and set the two-form potential B to be zero. This leads us to a simpler set of equations of motion, dilatino and Killing spinor equations.

We analyzed possible solutions to Romans supergravity in previous papers [1, 2], and a particular case where $B = 0$ was discussed in [23]. One can expect, however, to find more possible solutions to this case. By following the procedure adopted in [22], here we classify these families of solutions.

4.1 Taking $B = 0$

Taking the two-form potential $B = 0$ in (3.14) and (3.15) implies

$$X^2 S^2 \sin^2 2\vartheta \, d\sigma = -\frac{2\sqrt{2}}{3} X^{-1} S \cos 2\vartheta \, J + \sqrt{2} X S \cos 2\vartheta \, \mathcal{F}_\perp, \quad (4.1)$$

$$d(X^{-1} S \cos 2\vartheta \, J) = -\frac{3}{2\sqrt{2}} d[(XS)^2 d\sigma], \quad (4.2)$$

$$d(X^{-1} S J) = -\sqrt{2} S \sin 2\vartheta \, J \wedge K_2 - \frac{3}{2\sqrt{2}} \cos 2\vartheta \, d[(XS)^2 d\sigma] \\ - \frac{1}{\sqrt{2}} X^{-2} S \sin 2\vartheta \, [K_1 \lrcorner * \mathcal{F}_\perp - K_2 \wedge \mathcal{F}_\perp], \quad (4.3)$$

$$d(S \sin 2\vartheta \, J \wedge K_2) = 0, \quad (4.4)$$

$$D_\perp(X^{-1} S \sin 2\vartheta \, \Omega) = -\sqrt{2} S \Omega \wedge K_2, \quad (4.5)$$

$$S^2 J \wedge d\sigma = -\sqrt{2} S \cos 2\vartheta \left(X + \frac{2}{3} X^{-3} \right) \frac{1}{2} J \wedge J + 2 S K_1 \lrcorner * d\vartheta \\ + \frac{1}{\sqrt{2}} X^{-1} S J \wedge \cos 2\vartheta \, d\mathcal{A}_\perp, \quad (4.6)$$

$$S^2 \Omega \wedge d\sigma = \frac{1}{\sqrt{2}} X^{-1} S \Omega \wedge \cos 2\vartheta \, d\mathcal{A}_\perp - 2i S d\vartheta \wedge K_2 \wedge \Omega, \quad (4.7)$$

$$0 = X^4 K_2 \lrcorner d(X^{-3} S \sin 2\vartheta) + \sqrt{2} S \left(X^2 - \frac{2}{3} X^{-2} \right) \\ + \frac{1}{\sqrt{2}} S J \lrcorner \mathcal{F}_\perp, \quad (4.8)$$

and

$$0 = d \left[\frac{X^4}{S \sin 2\vartheta} K_1 \lrcorner * d(X^{-2} S \sin 2\vartheta K_2) \right] + \frac{1}{2} \mathcal{F}_\perp \wedge \mathcal{F}_\perp. \quad (4.9)$$

Solving these equations is sufficient to ensure we have a supersymmetric solution to the Euclidean equations of motion of Romans theory in this particular case.

In order to analyse these equations, we will need to break the derivatives (and the remaining of each equation) further into components. We start by defining a radial coordinate in the K_2 direction, this will given by

$$\rho = XS, \quad \text{such that} \quad d\rho = d(XS) . \quad (4.10)$$

The exterior derivative can be written as

$$d = d\psi \wedge \partial_\psi + d\rho \wedge \partial_\rho + d_4 , \quad (4.11)$$

i.e., in the K_1 , K_2 and M_4 directions.

Another term that requires attention is the one-form σ . For instance, one may consider

$$\sigma = \sigma_4 + \sigma_\rho d\rho , \quad (4.12)$$

where σ_4 is the σ component in the direction that is both perpendicular to K_1 and K_2 . Its derivative is then given by

$$d\sigma = d_4\sigma_4 + d\rho \wedge (\partial_\rho\sigma_4) + (d_4\sigma_\rho) \wedge d\rho . \quad (4.13)$$

Notice however that, if we reparametrise ψ (that enters in the definition of the one-form K_1), one can make a gauge choice of shifting it in such a way that

$$\psi \longrightarrow \psi - \int_\rho \sigma_\rho(y, x^i) dy , \quad (4.14)$$

which leads to

$$d\psi \longrightarrow d\psi - \sigma_\rho d\rho - d_4 \int_\rho \sigma_\rho(y, x^i) dy . \quad (4.15)$$

This way, in writing down $d\psi + \sigma$, we have $\sigma_\rho d\rho$ being cancelled, and we can simply write

$$d\psi + \sigma \longrightarrow d\psi + \sigma_4 - d_4 \int_\rho \sigma_\rho(y, x^i) dy , \quad (4.16)$$

and relabel

$$\sigma_4 - d_4 \int_\rho \sigma_\rho(y, x^i) dy \longrightarrow \sigma_4 , \quad (4.17)$$

as both terms are in the d_4 direction. This way, we are free to make a choice where $\sigma \equiv \sigma_4$. Notice however that we still have to consider $d\sigma = d_4\sigma_4 + d\rho \wedge (\partial_\rho\sigma_4)$.

4.2 Conditions for a supersymmetric M_6

Notice that once we take the B -field to be zero in (3.9), we immediately get, from the part in the K_1 direction, that

$$\frac{3}{\sqrt{2}S \sin 2\vartheta} d(XS) + X^{-2} K_2 = 0 . \quad (4.18)$$

As we defined, $d(XS) = d\rho$ (now confirming that K_2 is in fact in the $d\rho$ direction), indeed,

$$K_2 = -\frac{3X^2}{\sqrt{2}S \sin 2\vartheta} d\rho. \quad (4.19)$$

In (4.9), notice that the first term is zero, and it simply reduces to

$$\mathcal{F}_\perp \wedge \mathcal{F}_\perp = 0. \quad (4.20)$$

Equation (4.1) may be used to eliminate the flux $\mathcal{F}_\perp$ in terms of the $SU(2)$ structure (one should consider $\cos 2\vartheta \equiv 0$ as a separate case) such that

$$\mathcal{F}_\perp = \frac{2}{3X^2} J + \frac{\rho \sin^2 2\vartheta}{\sqrt{2} \cos 2\vartheta} d\sigma. \quad (4.21)$$

Let us reparametrize J and Ω by introducing

$$\hat{J} = X^2 J, \quad \text{and} \quad \hat{\Omega} = X^2 \Omega. \quad (4.22)$$

Then from equation (4.4), one reads

$$d(\hat{J} \wedge d\rho) = 0. \quad (4.23)$$

which implies

$$d_4 \hat{J} = 0, \quad (4.24)$$

$$\partial_\psi \hat{J} = 0, \quad (4.25)$$

Next, equation (4.2) reads

$$d\left(\frac{\rho \cos 2\vartheta}{X^4} \hat{J}\right) = -\frac{3}{\sqrt{2}} \rho d\rho \wedge d\sigma. \quad (4.26)$$

This is equivalent to

$$d_4\left(\frac{\cos 2\vartheta}{X^4}\right) = 0, \quad (4.27)$$

so that we can say $\cos 2\vartheta = X^4 \lambda(\rho)$. We then get an equivalent to equation (2.36) in [22], namely

$$\partial_\rho(\rho \lambda(\rho) \hat{J}) = -\frac{3}{\sqrt{2}} \rho d_4 \sigma. \quad (4.28)$$

Notice that from this equation also follows that $d_4 \sigma$ has no components proportional to Ω (but it still could have an anti-self-dual part).

Similarly, equation (4.5) reads

$$d_4\left(\frac{\sin 2\vartheta}{X^4} \hat{\Omega}\right) = -i \mathcal{A}_\perp \wedge \frac{\sin 2\vartheta}{X^4} \hat{\Omega}, \quad (4.29)$$

and

$$\partial_\rho\left(\frac{\rho \sin 2\vartheta}{X^4} \hat{\Omega}\right) = \frac{3}{\sin 2\vartheta} \hat{\Omega}. \quad (4.30)$$

Here again we have used gauge freedom to remove the part of $\mathcal{A}_\perp$ proportional to $d\rho$, so that $\mathcal{A}_\perp = \mathcal{A}_4$. These equations imply that the geometry at constant ρ (and ψ) is (conformally) Kähler, with Kähler metric $\hat{g}_4$ associated to $\hat{J}$ and $\hat{\Omega}$. Moreover, since the derivative of $\hat{\Omega}$ in the ρ direction is proportional to $\hat{\Omega}$, this shows that the associated complex structure $\hat{I}$ is independent of ρ , $\partial_\rho \hat{I} = 0$. Since

$$d_4 \hat{\Omega} = i \hat{P} \wedge \hat{\Omega}, \quad (4.31)$$

where $\hat{P}$ is the canonical Ricci one-form potential, we identify

$$\mathcal{A}_\perp = -\hat{P} - \hat{I} \cdot d_4 \log \left(\frac{\sin 2\vartheta}{X^4} \right). \quad (4.32)$$

Note that this can be rewritten as

$$\mathcal{A}_\perp = -\hat{P} - \hat{I} \cdot d_4 \log \tan 2\vartheta. \quad (4.33)$$

Next, we turn to equation (4.6). Multiplying it by X^2 and substituting from (3.12), this reads

$$X^2 S^2 \sin^2 2\vartheta \Omega \wedge d\sigma = -2i S d\vartheta \wedge K_2 \wedge \Omega + \frac{1}{\sqrt{2}} X S \Omega \wedge \cos 2\vartheta \mathcal{F}_\perp. \quad (4.34)$$

Subtracting $\frac{1}{2}\Omega$ times (4.1) from this previous equation gives

$$\frac{1}{2} \rho^2 \sin^2 2\vartheta \omega \wedge d\sigma = -2i S d\vartheta \wedge K_2 \wedge \Omega. \quad (4.35)$$

And therefore

$$\Omega \wedge d_4 \sigma = 0, \quad (4.36)$$

and

$$\Omega \wedge \left(\partial_\rho \sigma + \frac{6\sqrt{2}X^4}{\rho^2 \sin^3 2\vartheta} i d_4 \vartheta \right) = 0. \quad (4.37)$$

This implies that the one-form in brackets is a $(1, 0)$ -form, and hence

$$\partial_\rho \sigma = -\frac{6\sqrt{2}X^4}{\rho^2 \sin^3 2\vartheta} \hat{I} \cdot d_4 \vartheta. \quad (4.38)$$

Next we turn to equation (4.6). One finds that the component of this equation in the $d\rho$ direction is precisely equivalent to equation (4.38). The remainder of equation (4.6) is equivalent to

$$\frac{1}{2} S^2 \sin^2 2\vartheta J \lrcorner d_4 \sigma = -\sqrt{2} \rho \cos 2\vartheta + \frac{2\sqrt{2} \rho^2 \sin 2\vartheta}{3X^4} \partial_\rho \vartheta. \quad (4.39)$$

Finally we turn to the scalar equation (4.8). After a computation, one remarkably finds precisely equation (4.39) plus a (generically) non-zero function times $\partial_\rho(\rho\lambda(\rho))$. One concludes that

$$\partial_\rho(\rho\lambda(\rho)) = 0 \longrightarrow \lambda(\rho) = \frac{c}{\rho}, \quad (4.40)$$

where c is an integration constant. One can check that equation (4.40) is also true for the four-parameter family of BPS black hole solutions discussed in [23], a highly non-trivial check. We have thus solved these equations for one of the functions in the problem. Notice that now one can write

$$\rho = cX^4 \sec 2\vartheta . \quad (4.41)$$

We thus really have only one free function in the problem, and we can take it to be X .

We have now analysed all the content of all the equations, apart from equations (4.3) and (4.9). After quite a lengthy calculation, and using many of the equations above, one can show that the $d\rho$ component of equation (4.3) is precisely equivalent to (4.39). Notice that the anti-self-dual part of $\mathcal{F}_4$ enters, which is related to $(d_4\sigma)^-$ via (4.21), but this combines with $d_4\sigma$, and in the end only the self-dual part of it remains, and it is proportional to J , as $\Omega \wedge d_4\sigma = 0$. The remainder of equation (4.3) is easier to compute, and one finds an equivalent to (4.38). Thus (4.3) is implied by all the other equations, and hence imposes nothing new.

It thus remains only to impose the equation of motion (4.9). The two components read

$$\partial_\rho \sigma \wedge \mathcal{F}_4 = 0 . \quad (4.42)$$

where from (4.21), we have

$$\mathcal{F}_4 = \frac{2}{3X^2} J + \frac{\rho \sin^2 2\vartheta}{\sqrt{2} \cos 2\vartheta} d_4\sigma , \quad (4.43)$$

together with the scalar equation

$$\|(d_4\sigma)^-\| = \frac{2 \cos 2\vartheta}{\rho \sin^2 2\vartheta} \left[\frac{2\rho \sin 2\vartheta}{3X^2 \cos 2\vartheta} \partial_\rho \vartheta + \frac{1}{X^2} \left(\frac{2}{3} - X^4 \right) \right] . \quad (4.44)$$

We conclude by noting that a few equations are redundant. First (4.38) is precisely the $d\rho$ component of $d(4.33)$. Here we have the second equation

$$d\mathcal{A}_\perp = \frac{2}{3X^4} \hat{J} + \frac{\rho}{\sqrt{2}} (\cos 2\vartheta + \sec 2\vartheta) d\sigma , \quad (4.45)$$

which may be combined with (4.42) to obtain an Einstein-like equation (involving the Ricci form of the Kähler metric). To see this, recall that $\hat{\mathcal{P}} = \frac{1}{2} \hat{I} \dot{d}_2 \log \sqrt{\det \hat{g}}$. But since also $\hat{\Omega} \wedge \hat{\Omega} = 4 \text{vol} = 2\hat{J} \wedge \hat{J}$ is automatically true, it follows by taking ∂_ρ that

$$\hat{J} \lrcorner \partial_\rho \hat{J} = \hat{\Omega} \lrcorner \partial_\rho \hat{\Omega} \quad (4.46)$$

is an identity. Using this, one can check that equations (4.28) and (4.30) in fact imply equation (4.39). The latter is hence implied by the other equations, and it is also redundant.

4.2.1 Summary

We can now put together the necessary and sufficient conditions to have a supersymmetric solution. The metric is given by

$$ds_6^2 = K_1^2 + K_2^2 + g_{\text{SU}(2)} , \quad (4.47)$$

where now $g_{\text{SU}(2)}$ is a conformally Kähler manifold. We can rewrite it as

$$ds_6^2 = X^{-2} \left(\rho^2 \sin^2 2\vartheta (d\psi + \sigma)^2 + \frac{9X^8}{2\rho^2 \sin^2 2\vartheta} d\rho^2 + \hat{g}_{ij} dx^i dx^j \right), \quad (4.48)$$

where $\hat{g}$ is a one-parameter family of Kähler metrics depending only on ρ , for which the complex structure $\hat{I}$ is independent of ρ . The vector ∂_ψ is Killing, and preserves all of the structure. The functions X and ϑ are related by

$$X^4 = \frac{\rho}{c} \cos 2\vartheta, \quad (4.49)$$

where c is a non-zero constant, so that we may substitute

$$\sin^2 2\vartheta = 1 - \frac{c^2 X^8}{\rho^2}. \quad (4.50)$$

The evolution equations for the Kähler structure are

$$\partial_\rho \hat{J} = -\frac{3}{\sqrt{2}c} \rho d_4 \sigma \quad (4.51)$$

and

$$\partial_\rho \left(\tan 2\vartheta \hat{\Omega} \right) = \frac{3}{c \sin 2\vartheta} \hat{\Omega}. \quad (4.52)$$

Notice that $\Omega \wedge d_4 \sigma = 0$ is consistent with equation (4.51) and $\hat{J}$ must remain type (1,1) as the complex structure is independent of ρ .

From (4.33), we have the Einstein-like equation

$$\hat{\mathcal{R}} \equiv d_4 \hat{\mathcal{P}} = -\frac{2}{3X^4} \hat{J} - \frac{\rho}{\sqrt{2}} (\cos 2\vartheta + \sec 2\vartheta) d_4 \sigma - d_4 \cdot \hat{I} \cdot d_4 \log \tan 2\vartheta, \quad (4.53)$$

together with

$$\partial_\rho \sigma = -\frac{6\sqrt{2}X^4}{\rho^2 \sin^3 2\vartheta} \hat{I} \cdot d_4 \vartheta. \quad (4.54)$$

Finally, we must impose the B-field equation of motion components

$$\partial_\rho \sigma \wedge \left[\frac{2}{3X^4} \hat{J} + \frac{\rho \sin^2 2\vartheta}{\sqrt{2} \cos 2\vartheta} d_4 \sigma \right] = 0, \quad (4.55)$$

and

$$\|(d_4 \sigma)^-\| = \frac{2 \cos 2\vartheta}{\rho \sin^2 2\vartheta} \left[\frac{2\rho \sin 2\vartheta}{3X^2 \cos 2\vartheta} \partial_\rho \vartheta + \frac{1}{X^2} \left(\frac{2}{3} - X^4 \right) \right]. \quad (4.56)$$

The norm here is with respect to g_4 (rather than $\hat{g}_4$). Notice that remarkably the supersymmetry equations above are almost exactly the same (essentially up to numerical factors) to the equations in [22].

4.3 Complex M_6 (setting $d_4\vartheta = 0$)

In [22] the equations of this form were solved in closed form, with the additional assumption of $d_4\vartheta = 0$, leading to new solutions. It is then natural, due to the similarity of the system, to make the same assumption here.

In order to have a six-dimensional complex manifold with Hermitian metric, we require the three-form given by $\Omega_{(3)} = \Omega \wedge (K_1 + iK_2)$ to have a derivative in the form

$$d\Omega_{(3)} = A \wedge \Omega_{(3)} + v \wedge \Omega \wedge (K_1 - iK_2), \quad (4.57)$$

with $v = 0$. This restriction will imply that $d\sigma \equiv d_4\sigma$, and implies that $d_4X = d_4\vartheta = d_4S = 0$. From this, one can deduce that

$$\hat{P} = \mathcal{A}_4. \quad (4.58)$$

Next, we may look at (4.53), which reads

$$\hat{\mathcal{R}} = -\frac{2}{3X^4}\hat{J} - \frac{\rho}{\sqrt{2}}(\cos 2\vartheta + \sec 2\vartheta)d_4\sigma. \quad (4.59)$$

The Ricci scalar of the Kähler metric $\hat{g}_4$ is $\hat{R} = \hat{J}^{ij}\hat{\mathcal{R}}_{ij}$, so that using (4.39), we compute

$$\hat{R} = -\frac{4}{3X^4} - \rho(\cos 2\vartheta + \sec 2\vartheta) \left(\frac{2 \cos 2\vartheta}{\rho \sin^2 2\vartheta} - \frac{4}{3X^4 \sin 2\vartheta} \partial_\rho \vartheta \right). \quad (4.60)$$

Since the right hand side is a function only of ρ , we deduce that $d_4\hat{R} = 0$, and $\hat{g}_4$ is a constant scalar curvature Kähler metric (for fixed ρ).

We may similarly compute $\hat{R}_{ij}\hat{R}^{ij} = \hat{\mathcal{R}}_{ij}\hat{\mathcal{R}}^{ij}$ from equation (4.59). Using again (4.39) to compute $\hat{J} \lrcorner d_4\sigma$, and (4.56) to compute $\|(d_4\sigma)^-\|^2$, the right hand side is again a function only of ρ , and we deduce that

$$d_4(\hat{R}_{ij}\hat{R}^{ij}) = 0. \quad (4.61)$$

At fixed ρ , it follows that the Ricci tensor $\hat{R}_{ij}$ has two pairs of constant eigenvalues. This problem is related to the Goldberg conjecture [24], which states that any compact Einstein almost Kähler manifold is Kähler-Einstein, i.e., it has an integrable complex structure. If the Goldberg conjecture is true, then Theorem 2 of [25] implies that a compact Kähler four-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of distinct constant curvatures. If these eigenvalues are the same, then M_4 is a Kähler-Einstein manifold.

We shall consider both cases separately, starting with the latter.

4.3.1 Kähler-Einstein base solutions

For a Kähler -Einstein metric $\hat{\mathcal{R}} \propto \hat{J}$, with the constant of proportionality depending only on ρ . Thus $d_4\sigma$ is also proportional to $\hat{J}$. One checks that there are no solutions with $d_4\sigma = 0$, so, without loss of generality, we set

$$d_4\sigma = \tilde{J}, \quad \hat{J} = F(\rho)\tilde{J}, \quad (4.62)$$

where $\partial_\rho \tilde{J} = 0$. Thus the rescaled Kähler metric $\tilde{J}$ is independent of ρ , and the Kähler-Einstein condition reads

$$\hat{\mathcal{R}} = \tilde{\mathcal{R}} = \kappa \tilde{J}, \quad (4.63)$$

where $\kappa \in \mathbb{R}$ is a constant. Solving first equation (4.51), we find

$$F(\rho) = a - \frac{3\rho^2}{2\sqrt{2}c}, \quad (4.64)$$

where a is an integration constant. Substituting this into (4.59), and X in (4.49), one can find ϑ , given by

$$\cos(2\vartheta) = \left(-1 + \sqrt{1 - \frac{4\sqrt{2}ac}{3\kappa^2}} \right) \frac{\kappa}{\sqrt{2}\rho}. \quad (4.65)$$

Notice that at this point, all the functions have been completely determined. Next, solving (4.52), we can write a and c in terms of κ (where $\hat{\Omega} = F(\rho)\tilde{\Omega}$)

$$a = -\frac{3\kappa}{4}, \quad c = -\frac{\kappa}{\sqrt{2}}. \quad (4.66)$$

It follows that $X \equiv 1$. Finally, notice that $(d_4\sigma)^- = 0$, and one can check that the right hand side of the equation (4.56) is in fact zero. At this point we have solved all the equations.

The final solution is therefore given by

$$F(\rho) = \frac{3\rho^2}{2\kappa} - \frac{3\kappa}{4}, \quad \cos(2\vartheta) = -\frac{\kappa}{\sqrt{2}\rho}, \quad X \equiv 1. \quad (4.67)$$

The six-dimensional metric is

$$ds_6^2 = \frac{9}{2(\rho^2 - \frac{\kappa^2}{2})} d\rho^2 + \left(\rho^2 - \frac{\kappa^2}{2} \right) (d\psi + \sigma)^2 + \frac{3}{2\kappa} \left(\rho^2 - \frac{\kappa^2}{2} \right) \tilde{g}_4, \quad (4.68)$$

where $d\sigma = \tilde{J}$, and $\tilde{g}_4$ is a constant (in ρ) Kähler-Einstein metric with $\tilde{\mathcal{R}} = \kappa \tilde{J}$. The gauge field $\mathcal{A}$ has $d\mathcal{A} = -d\tilde{\mathcal{R}}$, so that $\mathcal{A}$ is a connection on the canonical bundle of M_4 .

The ρ coordinate in the metric (4.68) is somewhat peculiar. A better system of coordinates is set by making the change

$$r^2 \equiv \rho^2 - \frac{\kappa^2}{2}. \quad (4.69)$$

The metric then becomes

$$ds_6^2 = \frac{9}{\kappa^2 + 2r^2} dr^2 + r^2 \left[(d\psi + \sigma)^2 + \frac{3}{2\kappa} \hat{g}_4 \right]. \quad (4.70)$$

This is simply the hyperbolic cone over a regular Sasaki-Einstein manifold. Notice that the full gauge field $\mathcal{F} = 0$. Thus, this solution was known.

4.3.2 Product of two Riemann surfaces base solutions

Analogous to the Kähler-Einstein solutions, we can consider the metric where

$$\hat{J} = F_1(\rho)\tilde{J}_1 + F_2(\rho)\tilde{J}_2, \quad (4.71)$$

with $F_1(\rho)$ and $F_2(\rho)$ depending only on ρ . Then we also have $d_4\sigma$ given by

$$d_4\sigma = c_1\tilde{J}_1 + c_2\tilde{J}_2, \quad (4.72)$$

with the factors c_1 and c_2 being constants. Here the two-forms $\tilde{J}_1$ and $\tilde{J}_2$ are such that $\partial_\rho\tilde{J}_1 = \partial_\rho\tilde{J}_2 = 0$. Again the rescaled Kähler metric is independent of ρ and we can write

$$\hat{\mathcal{R}} = k_1\tilde{J}_1 + k_2\tilde{J}_2, \quad (4.73)$$

where $k_i \in \mathbb{R}$ are constants.

Solving equation (4.51), we find

$$F_1(\rho) = -\frac{3\rho^2 c_1}{2\sqrt{2}c} - a \quad \text{and} \quad F_2(\rho) = -\frac{3\rho^2 c_2}{2\sqrt{2}c} - b, \quad (4.74)$$

where a and b are integration constants. We can now substitute this into (4.59), with X given by (4.49). Notice that there are two equations for ϑ . We find

$$\cos 2\vartheta = \frac{4 a c}{-3k_1 \rho + \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a c c_1)}}, \quad (4.75)$$

and a constraint for b , given by

$$b = 6 \left(\frac{2\sqrt{2}a^2 c c_2 \rho^2}{\left(3k_1 \rho - \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a c c_1)}\right)^2} + \frac{a k_2 \rho}{3k_1 \rho - \sqrt{3\rho^2(3k_1^2 - 4\sqrt{2}a c c_1)}} \right). \quad (4.76)$$

Now solving equation (4.52) order by order, one also finds constraints to c , a and c_2 , namely

$$c = -\frac{k_1}{\sqrt{2}c_1}, \quad a = -\frac{3k_1}{4}, \quad c_2 = c_1 \frac{k_2}{k_1}. \quad (4.77)$$

We then check equation (4.56) and confirm that it holds.

The final solution is a function of the parameters left, i.e., k_1 , k_2 and c_1 , and it is given by

$$F_1(\rho) = \frac{3 c_1^2 \rho^2}{2 k_1} - \frac{3 k_1}{4} \quad \text{and} \quad F_2(\rho) = \frac{3 c_1 c_2 k_2 \rho^2}{2 k_1} - \frac{3 k_1 c_2}{4 c_1}, \quad (4.78)$$

and our starting functions

$$X = 1, \quad \text{and} \quad \cos 2\vartheta = -\frac{k_1}{\sqrt{2} c_1 \rho}. \quad (4.79)$$

The six-dimensional metric is

$$ds_6^2 = \rho^2 \left(1 - \frac{k_1^2}{2 c_1 \rho} \right) (d\psi + \sigma)^2 + \frac{9}{2 \rho^2} \left(1 - \frac{k_1^2}{2 c_1 \rho} \right)^{-1} d\rho^2 \\ + \left(\frac{3 c_1^2 \rho^2}{2 k_1} - \frac{3 k_1}{4} \right) d\tilde{s}^2(C_{k_1}) + \left(\frac{3 c_1 c_2 k_2 \rho^2}{2 k_1} - \frac{3 k_1 c_2}{4 c_1} \right) d\tilde{s}^2(C_{k_2}), \quad (4.80)$$

where $d\tilde{s}^2(C_{k_i})$ are the metrics on a torus ($C_0 \equiv T^2$), or sphere ($C_1 \equiv S^2$) or a hyperbolic space ($C_{-1} \equiv H^2$).

One can reparametrise the metric making $ds_6^2 \rightarrow k_1 ds_6^2$, so that we get

$$ds_6^2 = \rho^2 k_1 \left(1 - \frac{k_1^2}{2 c_1 \rho} \right) (d\psi + \sigma)^2 + \frac{9 k_1}{2 \rho^2} \left(1 - \frac{k_1^2}{2 c_1 \rho} \right)^{-1} d\rho^2 \\ + \left(\frac{3 c_1^2 \rho^2}{2} - \frac{3 k_1^2}{4} \right) d\tilde{s}^2(C_{k_1}) + \left(\frac{3 c_1 c_2 \rho^2}{2} - \frac{3 k_1^2 c_2}{4 c_1} \right) d\tilde{s}^2(C_{k_2}), \quad (4.81)$$

which can be rewritten as

$$ds_6^2 = \rho^2 k_1 \left(1 - \frac{k_1^2}{2 c_1 \rho} \right) (d\psi + \sigma)^2 + \frac{9 k_1}{2 \rho^2} \left(1 - \frac{k_1^2}{2 c_1 \rho} \right)^{-1} d\rho^2 \\ + \left(\frac{3 c_1^2 \rho^2}{2} - \frac{3 k_1^2}{4} \right) d\tilde{s}^2(C_{k_1}) + \left(\frac{3 c_1^2 \rho^2}{2} - \frac{3 k_1^2}{4} \right) \frac{k_2}{k_1} d\tilde{s}^2(C_{k_2}). \quad (4.82)$$

Notice that the factor $\frac{k_1}{k_2}$ changes the curvature of C_{k_2} to C_{k_1} , what we see therefore is a reduction back to the case where the base is simply a Kähler-Einstein manifold. We conclude that a Kähler-Einstein manifold is the most general solution to Romans supergravity with zero B field (and $d_4\vartheta = 0$), completely classifying solutions of this type.

5 Conclusion

We have analysed the differential constraints in the $SU(2)$ structure obtained to ensure a supersymmetric solution to Euclidean Romans gravity theory when we set the two-form potential B to be zero. We were able to solve them analytically and were able to see that the most natural global structure for our six-dimensional space is a (ψ, ρ) -holomorphic complex cone bundle over a Kähler base M_4 , where ψ is the Killing vector direction and ρ an equivalent of a radial direction. We then assumed the Goldberg conjecture to be true, and found that the M_4 manifold has to be either Kähler-Einstein, or a product of two Riemann surfaces.

We found that the only type of product of Riemann surfaces our $SU(2)$ structure allowed is that of two surfaces with the same curvature, returning therefore to the Kähler-Einstein case. This proves that the complex cone holomorphic bundle over a Kähler-Einstein base is the most general type of solution one can get for Romans supergravity theory with a zero B-field.

Acknowledgments

The work of the author was supported by a CNPq junior postdoc scholarship.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] L.F. Alday, M. Fluder, C.M. Gregory, P. Richmond and J. Sparks, *Supersymmetric gauge theories on squashed five-spheres and their gravity duals*, *JHEP* **09** (2014) 067 [[arXiv:1405.7194](#)] [[INSPIRE](#)].
- [2] L.F. Alday, M. Fluder, C.M. Gregory, P. Richmond and J. Sparks, *Supersymmetric solutions to Euclidean Romans supergravity*, *JHEP* **02** (2016) 100 [[arXiv:1505.04641](#)] [[INSPIRE](#)].
- [3] L.J. Romans, *The $F(4)$ Gauged Supergravity in Six-dimensions*, *Nucl. Phys. B* **269** (1986) 691 [[INSPIRE](#)].
- [4] M. Cvetič, H. Lü and C.N. Pope, *Gauged six-dimensional supergravity from massive type IIA*, *Phys. Rev. Lett.* **83** (1999) 5226 [[hep-th/9906221](#)] [[INSPIRE](#)].
- [5] J. Hong, J.T. Liu and D.R. Mayerson, *Gauged Six-Dimensional Supergravity from Warped IIB Reductions*, *JHEP* **09** (2018) 140 [[arXiv:1808.04301](#)] [[INSPIRE](#)].
- [6] E. Malek, H. Samtleben and V. Vall Camell, *Supersymmetric AdS_7 and AdS_6 vacua and their minimal consistent truncations from exceptional field theory*, *Phys. Lett. B* **786** (2018) 171 [[arXiv:1808.05597](#)] [[INSPIRE](#)].
- [7] Y. Imamura, *Supersymmetric theories on squashed five-sphere*, *PTEP* **2013** (2013) 013B04 [[arXiv:1209.0561](#)] [[INSPIRE](#)].
- [8] L.F. Alday, P. Benetti Genolini, M. Fluder, P. Richmond and J. Sparks, *Supersymmetric gauge theories on five-manifolds*, *JHEP* **08** (2015) 007 [[arXiv:1503.09090](#)] [[INSPIRE](#)].
- [9] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [[hep-th/0205050](#)] [[INSPIRE](#)].
- [10] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys. B* **654** (2003) 61 [[hep-th/0211102](#)] [[INSPIRE](#)].
- [11] G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lüst, P. Manousselis and G. Zoupanos, *NonKähler string backgrounds and their five torsion classes*, *Nucl. Phys. B* **652** (2003) 5 [[hep-th/0211118](#)] [[INSPIRE](#)].
- [12] J.P. Gauntlett and S. Pakis, *The Geometry of $D = 11$ Killing spinors*, *JHEP* **04** (2003) 039 [[hep-th/0212008](#)] [[INSPIRE](#)].
- [13] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, *Phys. Rev. D* **69** (2004) 086002 [[hep-th/0302158](#)] [[INSPIRE](#)].
- [14] P. Kaste, R. Minasian and A. Tomasiello, *Supersymmetric M-theory compactifications with fluxes on seven-manifolds and G structures*, *JHEP* **07** (2003) 004 [[hep-th/0303127](#)] [[INSPIRE](#)].
- [15] D. Martelli and J. Sparks, *G structures, fluxes and calibrations in M-theory*, *Phys. Rev. D* **68** (2003) 085014 [[hep-th/0306225](#)] [[INSPIRE](#)].
- [16] K. Behrndt and M. Cvetič, *Supersymmetric intersecting D6-branes and fluxes in massive type IIA string theory*, *Nucl. Phys. B* **676** (2004) 149 [[hep-th/0308045](#)] [[INSPIRE](#)].

- [17] J.P. Gauntlett, J.B. Gutowski and S. Pakis, *The Geometry of $D = 11$ null Killing spinors*, *JHEP* **12** (2003) 049 [[hep-th/0311112](#)] [[INSPIRE](#)].
- [18] K. Behrndt and C. Jeschek, *Fluxes in M-theory on seven manifolds: G structures and superpotential*, *Nucl. Phys. B* **694** (2004) 99 [[hep-th/0311119](#)] [[INSPIRE](#)].
- [19] S. Fidanza, R. Minasian and A. Tomasiello, *Mirror symmetric $SU(3)$ structure manifolds with NS fluxes*, *Commun. Math. Phys.* **254** (2005) 401 [[hep-th/0311122](#)] [[INSPIRE](#)].
- [20] G. Dall'Agata and N. Prezas, *$N = 1$ geometries for M-theory and type IIA strings with fluxes*, *Phys. Rev. D* **69** (2004) 066004 [[hep-th/0311146](#)] [[INSPIRE](#)].
- [21] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five-dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [[hep-th/0209114](#)] [[INSPIRE](#)].
- [22] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of M-theory*, *Class. Quant. Grav.* **21** (2004) 4335 [[hep-th/0402153](#)] [[INSPIRE](#)].
- [23] L.F. Alday, P. Richmond and J. Sparks, *The holographic supersymmetric Renyi entropy in five dimensions*, *JHEP* **02** (2015) 102 [[arXiv:1410.0899](#)] [[INSPIRE](#)].
- [24] S.I. Goldberg, *Integrability of almost Kähler manifolds*, *Proc. Am. Math. Soc.* **21** (1969) 96.
- [25] V. Apostolov, T. Draghici and A. Moroianu, *The holographic supersymmetric Renyi entropy in five dimensions*, *Int. J. Math.* **12** (2001) 769 [[math/0007122](#)].