Argyres-Douglas theories and Liouville irregular states

Takahiro Nishinaka and Takahiro Uetoko

Department of Physical Sciences, College of Science and Engineering, Ritsumeikan University, Shiga 525-8577, Japan

E-mail: nishinak@fc.ritsumei.ac.jp, rp0019fr@ed.ritsumei.ac.jp

ABSTRACT: We study irregular states of rank-two and three in Liouville theory, based on an ansatz proposed by D. Gaiotto and J. Teschner. Using these irregular states, we evaluate asymptotic expansions of irregular conformal blocks corresponding to the partition functions of (A_1, A_3) and (A_1, D_4) Argyres-Douglas theories for general Ω -background parameters. In the limit of vanishing Liouville charge, our result reproduces strong coupling expansions of the partition functions recently obtained via the Painlevé/gauge correspondence. This suggests that the irregular conformal block for one irregular singularity of rank 3 on sphere is also related to Painlevé II. We also find that our partition functions are invariant under the action of the Weyl group of flavor symmetries once four and two-dimensional parameters are correctly identified. We finally propose a generalization of this parameter identification to general irregular states of integer rank.

KEYWORDS: Conformal Field Theory, Extended Supersymmetry, Supersymmetric Gauge Theory

ARXIV EPRINT: [1905.03795](https://arxiv.org/abs/1905.03795)

Contents

1	Introduction	1
2	AD theories of class $\mathcal{S}$ and AGT correspondence	3
2.1	AD theories of class $\mathcal{S}$	3
2.2	Irregular states and generalized AGT correspondence	5
3	Partition functions from irregular states	7
3.1	(A_1, D_4) theory from rank-2 irregular state	8
3.1.1	Ansatz in terms of generalized descendants	8
3.1.2	Irregular conformal block	10
3.2	(A_1, A_3) theory from rank-3 irregular state	12
3.2.1	Irregular conformal block	13
4	Enhanced flavor symmetries	16
4.1	Weyl group of the flavor symmetry	16
4.1.1	(A_1, D_4) theory	17
4.1.2	(A_1, A_3) theory	18
4.2	Conjectural dictionary between 4d and 2d parameters	18
4.2.1	(A_1, D_4) partition function	20
4.2.2	(A_1, A_3) partition function	20
5	Conclusions and discussions	21
A	Computations of generalized descendants	22
A.1	Rank-two state $ I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$	22
A.2	Rank-three state $ I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$	22
B	More coefficients for the (A_1, D_4) partition function	23

1 Introduction

Argyres-Douglas (AD) theories are an important series of strongly coupled $\mathcal{N}=2$ superconformal field theories (SCFTs) in four dimensions [1–3]. Although they are often obtained as IR SCFTs of RG-flows from $\mathcal{N}=2$ gauge theories, their physics is still to be understood. One reason for this is that, since the $U(1)_R$ symmetry of the AD theories is accidental in the RG-flows, it is hard to read off the IR physics from its UV description.¹ The accidental $U(1)_R$ symmetry particularly makes it difficult to compute the partition functions of the AD theories via supersymmetric localization. Therefore, until recently, the partition functions of AD theories were only partially computed for massless cases via the holomorphic anomaly equations [14].

¹For recent developments in the study of $\mathcal{N}=1$ Lagrangian theories that flow into the AD theories, see [4–13].

However, there has recently been remarkable progress in the study of the partition functions of AD theories. In particular, the authors of [15] have evaluated strong coupling expansions of the Nekrasov partition functions of the (A_1, A_2) , (A_1, A_3) and (A_1, D_4) AD theories via a remarkable relation to the tau-functions of Painlevé equations, inspired by [16] and subsequent works [17–20].² After these works, the author of [24–26] has nicely shown that these strong coupling expansions can be reproduced from irregular conformal blocks of Virasoro algebra.³ The emergence of the two-dimensional conformal field theory is naturally understood via the AGT correspondence [28, 29] and its generalization [30, 31] that relate the Nekrasov partition functions of 4d $\mathcal{N}=2$ theories to conformal blocks of two-dimensional Liouville theory. Indeed, an attempt to compute the partition function of AD theories as Liouville irregular conformal blocks was partially studied in [30, 31].⁴ Finally, nice matrix model descriptions of this relation between AD theories and Painlevé equations have been studied in [37–39].⁵

In this paper, we extend the irregular conformal block approach to various directions, focusing on the (A_1, A_3) and (A_1, D_4) AD theories. First, we compute the irregular conformal blocks for these two AD theories using an ansatz proposed in [31]. This ansatz is based on the construction of an irregular state in a colliding limit of regular vertex operators, and implies that a rank- n irregular state is expanded in terms of “generalized descendants” of a rank- $(n-1)$ irregular state. Here, generalized descendants are descendants in the sense of Virasoro generators and derivatives with respect to parameters characterizing irregular states. Since this construction of irregular conformal blocks is different from the one used in [24–26], it is non-trivial whether it leads to a consistent result. In this paper, we explicitly show that the ansatz mentioned above gives a consistent result, and moreover we demonstrate that the asymptotic expansion of the ansatz proposed in [31] precisely corresponds to the strong coupling expansion of partition functions obtained in [15].

Second, we reproduce the partition function of the (A_1, A_3) theory from the rank-three irregular state of the Liouville theory. This partition function was reproduced in [26] as the irregular conformal block with an irregular singular point of rank $\frac{3}{2}$ and a regular singular point. On the other hand, from the class $\mathcal{S}$ construction of the (A_1, A_3) theory, the same partition function is expected to be obtained as the one-point function of an irregular vertex operator of rank 3. We explicitly show that this is indeed the case.

Finally, we evaluate these irregular conformal blocks with non-vanishing Liouville charge Q turned on.⁶ Since the Painlevé tau-functions are related to the case of $Q=0$, earlier works focused on the vanishing Liouville charge. On the other hand, in relation to the 4d physics, it is important to evaluate the irregular conformal blocks for non-vanishing Q . Indeed, according to the AGT correspondence, the Liouville charge Q corresponds to $(\epsilon_1 + \epsilon_2)/\sqrt{\epsilon_1 \epsilon_2}$ in four dimensions, where ϵ_1 and ϵ_2 are two Ω -background parameters [67].

²Another important progress in the study of the partition functions of AD theories has been made through the E-string theory on T^2 [21] and quantum periods in the Nekrasov-Shatashvili limit [22, 23].

³See also [27] for closely related results related to asymptotically free gauge theories.

⁴For more works on Liouville irregular states, see [29, 32–36].

⁵Matrix model descriptions of the partition functions of 4d $\mathcal{N}=2$ theories in the context of the AGT correspondence were studied in various papers. See for example [40–66].

⁶Our convention for the Liouville charge is such that the Virasoro central charge is given by $c=1+6Q^2$.

Therefore, the non-vanishing Q leads to the partition function of AD theories with the most general Ω -background turned on. We particularly show that the four-dimensional flavor symmetry implies that there exists an $\mathcal{O}(Q)$ correction to the relation between 4d and 2d parameters. We interpret this $\mathcal{O}(Q)$ correction in terms of the colliding limit of regular vertex operators, which leads us to a conjecture for a general irregular state of integer rank.

The organization of the paper is the following. In section 2, we briefly review the class $\mathcal{S}$ construction of AD theories and the generalized AGT correspondence. In section 3, we evaluate the Liouville irregular states of rank-two and three, using the ansatz proposed in [31]. We then use these irregular states to compute the irregular conformal blocks corresponding to the partition functions of (A_1, D_4) and (A_1, A_3) theories for a general Ω -background parameters ϵ_1 and ϵ_2 . In section 4, we show that the partition functions of the (A_1, D_4) and (A_1, A_3) theories evaluated in section 3 are invariant under the action of the Weyl group of the enhanced flavor symmetry, once parameters in four and two dimensions are appropriately identified. We also conjecture a general identification of the Liouville momentum of irregular states with a 4d mass parameter. In section 5, we conclude with a comment on the symmetry of the partition functions under $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$.

2 AD theories of class $\mathcal{S}$ and AGT correspondence

In this section, we briefly review the (A_1, A_3) and (A_1, D_4) theories as well as the generalized AGT correspondence that relates the partition functions of AD theories to irregular conformal blocks of Liouville theory in two dimensions.

2.1 AD theories of class $\mathcal{S}$

The AD theories are a series of strongly coupled 4d $\mathcal{N}=2$ SCFTs with Coulomb branch operators of fractional dimensions. While the simplest AD theories were originally discovered as IR SCFTs at special points on the Coulomb branch of asymptotically free gauge theories [1–3], their infinite generalizations were constructed by compactifying 6d $(2, 0)$ SCFTs with gauge algebra $\mathfrak{g}$ on a punctured Riemann surface $\mathcal{C}$ [30, 68].⁷ In this paper, we focus on $\mathfrak{g} = A_1$. Then the resulting AD theories are (A_1, A_N) and (A_1, D_N) theories, depending on the number and types of the punctures on $\mathcal{C}$. In the rest of this sub-section, we denote by $\mathcal{T}_{\mathcal{C}}$ the 4d $\mathcal{N}=2$ theory obtained by compactifying 6d $(2, 0)$ A_1 theory on $\mathcal{C}$.

As mentioned above, the 4d theory $\mathcal{T}_{\mathcal{C}}$ depends on the number and types of the punctures on $\mathcal{C}$. Possible punctures on $\mathcal{C}$ are classified into two types; regular and irregular punctures. To describe these two types, let us briefly review how the Coulomb branch moduli space of the 4d theory $\mathcal{T}_{\mathcal{C}}$ is related to the Hitchin system on $\mathcal{C}$. Since we focus on the case of $\mathfrak{g} = A_1$, we consider the A_1 Hitchin system on $\mathcal{C}$, which involves an $SU(2)$ -connection A on $\mathcal{C}$ and an $\mathfrak{su}(2)$ -valued 1-form $\varphi dz + \bar{\varphi} d\bar{z}$ on $\mathcal{C}$. The pair (A, φ) is required to satisfy a set of differential equations called “Hitchin’s equations.” Since the Hitchin’s equations are differential equations, we need to specify the boundary conditions of A and

⁷See also [69] for the construction of AD theories via type II string theory on Calabi-Yau singularities. Indeed, the AD theories we discuss in this paper, (A_1, A_3) and (A_1, D_4) , were first constructed in this reference.

φ at each puncture on $\mathcal{C}$. We call a puncture a “regular puncture” if φ has a simple pole there, while a puncture is called an “irregular puncture” if the singularity of φ is not a simple pole. Suppose that an irregular puncture is located at $z = 0$. When φ behaves as $\varphi \sim 1/z^{n+1}$ around it, we call n the “rank” of the irregular puncture. While the rank could be an integer or a half-integer for $\mathfrak{g} = A_1$, we focus on integer ranks in this paper. The Hitchin moduli space is then defined as the space of solutions to Hitchin’s equations (modulo gauge transformations) with the boundary conditions at punctures kept fixed. The spectral curve of the Hitchin system is defined by

$$\det(xdz - \varphi) = 0. \quad (2.1)$$

A beautiful observation of [70–72] is that this spectral curve is identified with the Seiberg-Witten (SW) curve of the corresponding 4d $\mathcal{N}=2$ theory $\mathcal{T}_{\mathcal{C}}$.⁸ Note that this SW curve, and therefore the 4d theory $\mathcal{T}_{\mathcal{C}}$, depend on the number and types of the punctures on $\mathcal{C}$. In particular, the flavor symmetry of $\mathcal{T}_{\mathcal{C}}$ is naturally encoded in the punctures. Indeed, each regular puncture gives rise to a flavor $SU(2)$ symmetry, while each irregular puncture of integer rank gives rise to a flavor $U(1)$ symmetry in $\mathcal{T}_{\mathcal{C}}$.⁹ As discussed in [71, 72], when $\mathcal{C}$ has no irregular puncture, $\mathcal{T}_{\mathcal{C}}$ is always an $\mathcal{N}=2$ SCFT.¹⁰ On the other hand, when $\mathcal{C}$ has an irregular puncture, the $\mathcal{N}=2$ supersymmetric field theory $\mathcal{T}_{\mathcal{C}}$ is conformal if and only if $\mathcal{C}$ is of genus zero, has one or zero regular puncture, and does not have more than one irregular puncture [30, 68].

Let us now describe the (A_1, A_N) and (A_1, D_N) theories. The (A_1, A_{2n-3}) theory is a 4d $\mathcal{N}=2$ SCFT associated with $\mathcal{C}$ being a sphere with an irregular puncture of rank n . On the other hand, the (A_1, D_{2n}) theory is 4d $\mathcal{N}=2$ SCFT associated with $\mathcal{C}$ which is a sphere with a regular puncture and an irregular puncture of rank n . These two theories are AD theories since they have fractional-dimensional Coulomb branch operators. In this paper, we focus on two AD theories in these series; (A_1, A_3) and (A_1, D_4) . These two AD theories are special in the sense of flavor symmetry enhancement. Indeed, according to the relation between the flavor symmetry and the punctures on $\mathcal{C}$, the manifest flavor symmetries of (A_1, A_{2n-3}) and (A_1, D_{2n}) for an integer n are $U(1)$ and $SU(2) \times U(1)$, respectively. However, for (A_1, A_3) and (A_1, D_4) theories, these manifest flavor symmetries get enhanced to $SU(2)$ and $SU(3)$, respectively. This flavor symmetry enhancement will be important in our discussions in section 4.

For later use, we here write down the SW curves of these two theories. Up to a change of variables, the SW curve (2.1) for the (A_1, A_3) theory is written as¹¹

$$x^2 = \frac{u}{z^4} + \frac{m}{z^5} + \frac{c}{z^6} + \frac{1}{z^8}, \quad (2.3)$$

⁸The Seiberg-Witten 1-form is identified with $x dz$.

⁹Irregular punctures of non-integer rank do not give rise to a flavor symmetry. We will not use them in this paper.

¹⁰To be more precise, $\mathcal{C}$ needs at least three regular punctures for $\mathcal{T}_{\mathcal{C}}$ to be a non-trivial $\mathcal{N}=2$ SCFT.

¹¹We here take the holomorphic coordinate z on $\mathcal{C}$ so that the irregular puncture is located at $z = 0$. In the literature, the SW curve of the (A_1, A_3) theory is usually written as

$$x^2 = z^4 + cz^2 + mz + u, \quad (2.2)$$

which is related to (2.3) by the change of coordinates $(x, z) \rightarrow (-z^2x, 1/z)$. Note that this coordinate change preserves the SW 1-form $x dz$.

where c, m and u are coefficients respectively interpreted as a relevant coupling of dimension $2/3$, a mass parameter, and the vacuum expectation value (vev) of a Coulomb branch operator of dimension $4/3$. On the other hand, for the (A_1, D_4) theory, the SW curve is written as¹²

$$x^2 = \frac{M^2}{z^2} + \frac{u}{z^3} + \frac{m}{z^4} + \frac{c}{z^5} + \frac{1}{z^6}, \quad (2.5)$$

where c and u are respectively a relevant coupling of dimension $1/2$ and the vev of a Coulomb branch operator of dimension $3/2$, while m and M are two mass parameters. See [30, 68] for more detail.

2.2 Irregular states and generalized AGT correspondence

We here review the generalized AGT correspondence that relates the partition functions of AD theories to irregular conformal blocks of two-dimensional Liouville theory. Let $\mathcal{T}_{\mathcal{C}}$ be a 4d $\mathcal{N}=2$ SCFT associated with a Riemann surface $\mathcal{C}$ with only regular punctures. Then the AGT correspondence [28] states that the Nekrasov partition functions [67] of $\mathcal{T}_{\mathcal{C}}$ to the conformal block of Liouville theory on $\mathcal{C}$. Here, each regular puncture on $\mathcal{C}$ corresponds to the insertion of a Virasoro primary vertex operator $e^{2\gamma\phi(z)}$ whose Liouville momentum γ is fixed by the mass parameter associated with the regular puncture. Therefore, when $\mathcal{C}$ has ℓ regular punctures, then the partition function of $\mathcal{T}_{\mathcal{C}}$ is identified with the ℓ -point conformal block of Liouville theory, i.e.,

$$Z_{\mathcal{T}_{\mathcal{C}}}^{\epsilon_1, \epsilon_2} = \mathcal{F}_{\gamma_1}^{\gamma_2} \gamma_3 \cdots \gamma_{\ell-1} \gamma_{\ell}, \quad (2.6)$$

where ϵ_1 and ϵ_2 are the Ω -background parameters turned on in four dimensions. In this paper, we fix the scale of the l.h.s. by setting $\epsilon_1\epsilon_2 = 1$ as in [28]. The ratio of the Ω -background parameters is related to the Liouville charge Q by¹³

$$Q = b + \frac{1}{b}, \quad b^2 \equiv \frac{\epsilon_1}{\epsilon_2}. \quad (2.7)$$

It was also proposed in [28] that the SW curve of $\mathcal{T}_{\mathcal{C}}$ is identified as the Liouville correlator with a stress tensor insertion:

$$x^2 = \frac{\langle \Delta_{\gamma_1} | T(z) e^{2\gamma_2\phi} \cdots e^{2\gamma_{\ell-1}\phi} | \Delta_{\gamma_{\ell}} \rangle}{\langle \Delta_{\gamma_1} | e^{2\gamma_2\phi} \cdots e^{2\gamma_{\ell-1}\phi} | \Delta_{\gamma_{\ell}} \rangle}, \quad (2.8)$$

where $|\Delta_{\gamma}\rangle$ is the Virasoro primary state of holomorphic dimension $\Delta_{\gamma} \equiv \gamma(Q - \gamma)$.

A generalization of the AGT correspondence to AD theories has been discussed in [30, 31],¹⁴ in which an irregular puncture of rank n is conjectured to correspond to

¹²Again, in the literature, this curve is usually written as

$$x^2 = z^2 + cz + m + \frac{u}{z} + \frac{M^2}{z^2}, \quad (2.4)$$

which is related to (2.5) by $(x, z) \rightarrow (-z^2x, 1/z)$.

¹³Recall here that our convention for Q is such that the Virasoro central charge is given by $c = 1 + 6Q^2$.

¹⁴See also [29] for the first generalization to class $\mathcal{S}$ theories whose Riemann surface involves irregular punctures.

a particular linear combination, $|I^{(n)}\rangle$, of a Virasoro primary and its descendants. In this paper, we follow the construction of $|I^{(n)}\rangle$ proposed in [31]. Namely, the state $|I^{(n)}\rangle$ is a solution to the following $2n$ equations involving $(n+1)$ parameters $c_1, \dots, c_n$ and α :

$$L_k |I^{(n)}\rangle = \begin{cases} \Lambda_k |I^{(n)}\rangle & \text{for } n \leq k \leq 2n \\ \left(\Lambda_k + \sum_{\ell=k+1}^{n-1} (\ell - k) c_\ell \frac{\partial}{\partial c_{\ell-k}} \right) |I^{(n)}\rangle & \text{for } 1 \leq k \leq n-1 \\ \left(\Delta_\alpha + \sum_{\ell=1}^n \ell c_\ell \frac{\partial}{\partial c_\ell} \right) |I^{(n)}\rangle & \text{for } k = 0 \end{cases}, \quad (2.9)$$

where L_k are the generators of Virasoro algebra at central charge $c = 1 + 6Q^2$, and

$$\Lambda_k \equiv \begin{cases} -\sum_{\ell=k-n}^n c_\ell c_{k-\ell} & \text{for } n+1 \leq k \leq 2n \\ -\sum_{\ell=1}^{n-1} c_\ell c_{n-\ell} + ((k+1)Q - 2\alpha) c_k & \text{for } 1 \leq k \leq n \end{cases}. \quad (2.10)$$

We call $|I^{(n)}\rangle$ satisfying the above equations an “irregular state” of rank n , or simply a rank- n state. Note that the rank-zero state $|I^{(0)}\rangle$ is naturally identified with a regular primary state of Virasoro algebra.

Note here that $|I^{(n)}\rangle$ is a simultaneous eigenstate of $L_n, \dots, L_{2n-1}$ and L_{2n} with eigenvalues fixed by $c_1, \dots, c_n$ and α .¹⁵ This particularly implies that $L_k |I^{(n)}\rangle = 0$ for all $k > 2n$. We particularly call α the “Liouville momentum” of $|I^{(n)}\rangle$, which is natural in the following sense. In [31], the rank- n state $|I^{(n)}\rangle$ was constructed in a colliding limit of n regular vertex operators $e^{2\gamma\phi(z)}$ to a regular primary state $|\Delta_{\gamma_0}\rangle$.¹⁶ Namely, $|I^{(n)}\rangle$ is obtained from

$$e^{2\gamma_n\phi(z_n)} e^{2\gamma_{n-1}\phi(z_{n-1})} \dots e^{2\gamma_1\phi(z_1)} |\Delta_{\gamma_0}\rangle, \quad (2.11)$$

in the limit $z_1, \dots, z_n \rightarrow 0$. For this limit to make sense, we need to take $\gamma_0, \dots, \gamma_n \rightarrow \infty$ simultaneously with $\sum_{i=0}^n \gamma_i$ kept fixed. This sum of γ_i is identified with α in (2.9), i.e., $\alpha \equiv \sum_{i=0}^n \gamma_i$. Therefore, α characterizing $|I^{(n)}\rangle$ is the sum of the Liouville momenta of regular vertex operators involved in the colliding limit. This point will be important in section 4.

On the other hand, $|I^{(n)}\rangle$ is *not* an eigenstate of $L_0, \dots, L_{n-2}$ or L_{n-1} . Indeed, the actions of $L_0, \dots, L_{n-1}$ involve derivatives with respect to c_i . This means that (2.9) are differential equations, and therefore $|I^{(n)}\rangle$ is not uniquely fixed by $c_1, \dots, c_n$ and α . To fix $|I^{(n)}\rangle$ completely, we need to specify its “boundary condition” or “asymptotic behavior.” In this paper, we assume the following asymptotic behavior of $|I^{(n)}\rangle$ proposed in [31]; in the small c_n limit, $|I^{(n)}\rangle$ behaves as

$$|I^{(n)}(\alpha, \mathbf{c})\rangle = f(\alpha, \mathbf{c}, \beta_{n-1}) \left(|I^{(n-1)}(\beta_{n-1}, \tilde{\mathbf{c}})\rangle + \sum_{k=1}^{\infty} (c_n)^k |I_k^{(n-1)}\rangle \right), \quad (2.12)$$

¹⁵To be precise, eigenstates of $L_n, \dots, L_{2n}$ were considered in [29, 31], while [30] studied eigenstates of L_1 and L_{2n} . The relation between these two were discussed in appendix A of [32]. These two descriptions are both useful in understanding these special states.

¹⁶See section 2.2 in particular. See also section 2 of [56] for a review.

where $\mathbf{c} \equiv (c_1, \dots, c_n)$, $\tilde{\mathbf{c}} \equiv (c_1, \dots, c_{n-1})$, and β_{n-1} is a free parameter characterizing the asymptotic behavior.¹⁷ On the r.h.s. of (2.12), the leading term $|I^{(n-1)}(\beta_{n-1}, \tilde{\mathbf{c}})\rangle$ is the rank- $(n-1)$ irregular state satisfying (2.9) with n and α replaced by $(n-1)$ and β_{n-1} , respectively. Note that β_{n-1} is the Liouville momentum of this leading term. On the other hand, the remaining terms $|I_k^{(n-1)}\rangle$ are *generalized descendants* of $|I^{(n-1)}(\beta_{n-1}, \tilde{\mathbf{c}})\rangle$. Here, generalized descendants are descendants in the sense of Virasoro generators $L_{k<0}$ and the derivatives with respect to $c_1, \dots, c_{n-1}$.¹⁸ The prefactor $f(\alpha, \mathbf{c}, \beta_{n-1})$ in (2.12) is a function necessary for the ansatz (2.12) to satisfy the differential equations (2.9) order by order in c_n .¹⁹

Note here that $|I^{(n-1)}(\beta_{n-1}, \tilde{\mathbf{c}})\rangle$ on the r.h.s. of (2.12) can further be expanded in terms of rank- $(n-2)$ state and its generalized descendants. In this expansion, the Liouville momentum of the rank- $(n-2)$ state is again a free parameter, which we denote by β_{n-2} . Continuing this expansion until we reach the rank-zero state (i.e., regular primary state), we see that there are n free parameters $\beta_0, \dots, \beta_{n-1}$ characterizing the original rank- n state $|I^{(n)}\rangle$. This means that the rank- n state $|I^{(n)}\rangle$ depends not only on α and $\mathbf{c} = (c_1, \dots, c_n)$ but also on $\boldsymbol{\beta} \equiv (\beta_0, \dots, \beta_{n-1})$. In particular, $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{n-1})$ are parameters completely fixing the asymptotic behavior of $|I^{(n)}\rangle$. Therefore, in the rest of this paper, we write $|I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ instead of $|I^{(n)}(\alpha, \mathbf{c})\rangle$ to denote the Liouville irregular state of rank n . This asymptotic expansion of the rank- n irregular state is proposed in [31] based on the colliding limit of regular vertex operators explained around (2.11).

Given the above correspondence between irregular punctures and Liouville irregular states, the partition functions of the (A_1, A_{2n-3}) and (A_1, D_{2n}) theories are expected to be identified with Liouville correlators involving an irregular state. In particular, since $\mathcal{C}$ for the (A_1, A_{2n-3}) theory is a sphere with one irregular puncture of rank n , its partition function is expected to be written as

$$\mathcal{Z}_{(A_1, A_{2n-3})}^{\epsilon_1, \epsilon_2} = \langle 0 | I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle, \quad (2.13)$$

where $|0\rangle$ is the vacuum. On the other hand, $\mathcal{C}$ for the (A_1, D_{2n}) theory is a sphere with one regular puncture and one irregular puncture of rank n , and therefore its partition function is expected to be given by

$$\mathcal{Z}_{(A_1, D_{2n})}^{\epsilon_1, \epsilon_2} = \langle \Delta_\gamma | I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle. \quad (2.14)$$

In the next section, we explicitly identify the irregular states of rank 2 and 3, and then evaluate the partition functions for (A_1, A_3) and (A_1, D_4) theories via the relations (2.13) and (2.14).

3 Partition functions from irregular states

In this section, we explicitly compute the Liouville irregular state of rank 2 and 3, and then evaluate the partition functions of the (A_1, A_3) and (A_1, D_4) theories via (2.13) and (2.14).

¹⁷See in particular section 3.2 and appendix B.2 of the reference [31] for the first discussion of this ansatz. We also mention here that $|I_k^{(n-1)}\rangle$ in (2.12) was denoted as $|I_{nk}^{(n-1)}\rangle$ in [31]. We use $|I_k^{(n)}\rangle$ instead of $|I_{nk}^{(n)}\rangle$ to reduce clutter.

¹⁸These derivatives make sense since $|I^{(n-1)}(\beta_{n-1}, \tilde{\mathbf{c}})\rangle$ depends on $c_1, \dots, c_{n-1}$.

¹⁹For concrete expressions for $f(\alpha, \mathbf{c}; \boldsymbol{\beta})$ for $n = 2$ and 3 , see (3.9) and (3.40) in section 3.

3.1 (A_1, D_4) theory from rank-2 irregular state

Let us first focus on the rank-2 irregular state and the (A_1, D_4) theory. As reviewed in the previous section, the partition function of the (A_1, D_4) theory is expected to be given by

$$Z_{(A_1, D_4)}^{\epsilon_1, \epsilon_2} = \langle \Delta_\gamma | I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle, \quad (3.1)$$

where $|\Delta_\gamma\rangle$ is a regular primary state of holomorphic dimension $\Delta_\gamma \equiv \gamma(Q - \gamma)$, $\mathbf{c}$ and $\boldsymbol{\beta}$ are respectively short-hand notations for (c_1, c_2) and (β_0, β_1) , and $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ is the irregular state of rank two satisfying

$$L_4 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -c_2^2 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.2)$$

$$L_3 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -2c_1 c_2 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.3)$$

$$L_2 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -(c_1^2 + c_2(2\alpha - 3Q)) |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.4)$$

$$L_1 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \left(c_2 \frac{\partial}{\partial c_1} - 2c_1(\alpha - Q) \right) |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.5)$$

$$L_0 |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \left(\Delta_\alpha + c_1 \frac{\partial}{\partial c_1} + 2c_2 \frac{\partial}{\partial c_2} \right) |I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle. \quad (3.6)$$

To identify the relation between the 4d and 2d parameters, let us first compute the SW-curve in terms of the irregular state using the identification (2.8). Since the Virasoro stress tensor is expanded as $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$, the above equations and $\langle \Delta_\gamma | L_0 = \langle \Delta_\gamma | \Delta_\gamma$ imply that

$$\frac{\langle \Delta_\gamma | T(z) | I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle}{\langle \Delta_\gamma | I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle} = \frac{\Delta_\gamma}{z^2} + \frac{\sqrt{-ic_2} u}{z^3} - \frac{c_1^2 + c_2(2\alpha - 3Q)}{z^4} - \frac{2c_1 c_2}{z^5} - \frac{c_2^2}{z^6}, \quad (3.7)$$

where $u \equiv -\sqrt{-ic_2} \frac{\partial}{\partial c_1} \log \langle \Delta_\gamma | I^{(2)}(\alpha, \boldsymbol{\beta}, \mathbf{c}) \rangle - \frac{2c_1}{\sqrt{ic_2}}(\alpha - Q)$. To make a connection to the SW curve (2.5) of the (A_1, D_4) theory, let us rescale the coordinate as $z \rightarrow \sqrt{-ic_2} z$. Since the stress tensor T is of dimension two, this also rescales T as $T \rightarrow -(c_2)^{-2} T$.²⁰ As a result, we have

$$\frac{\langle \Delta_\gamma | T(z) | I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle}{\langle \Delta_\gamma | I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle} = \frac{\Delta_\gamma}{z^2} + \frac{u}{z^3} + \frac{\frac{c_1^2}{c_2} + (2\alpha - 3Q)}{z^4} + \sqrt{\frac{ic_1^2}{c_2}} \frac{1}{z^5} + \frac{1}{z^6}. \quad (3.8)$$

The identification (2.8) then gives us exactly the same curve as the SW-curve (2.5) of the (A_1, D_4) theory. From this curve, we can read off how the 2d parameters are related to the four-dimensional mass parameters, relevant coupling, and the vev of the Coulomb branch operator. In particular, we see that $\sqrt{ic_1^2/c_2}$ is identified with the relevant coupling of dimension 1/2.

3.1.1 Ansatz in terms of generalized descendants

Given the identification of parameters, we now compute the rank-2 irregular state $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$. To that end, we use the ansatz (2.12) for the irregular state reviewed in

²⁰In other words, $T(z)dz$ is invariant under this transformation.

section 2. For the rank-2 irregular state, the ansatz is written as

$$|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = c_1^{\nu_1} c_2^{\nu_2} e^{(\alpha-\beta_1)\frac{c_1^2}{c_2}} \left(|I^{(1)}(\beta_1, c_1; \beta_0)\rangle + \sum_{k=1}^{\infty} (c_2)^k |I_k^{(1)}\rangle \right), \quad (3.9)$$

where the prefactor $f(\alpha, \mathbf{c}, \beta_1) = c_1^{\nu_1} c_2^{\nu_2} e^{(\alpha-\beta_1)\frac{c_1^2}{c_2}}$ was identified in [31]. The states $|I_k^{(1)}\rangle$ are generalized descendants of the rank-one irregular state $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$.²¹ As reviewed in section 2, a generalized descendant is a descendant of $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ in the sense of $L_{-\ell}$ and $\partial/\partial c_1$, i.e., a linear combination of states of the form

$$(L_{-\ell_1})^{p_1} (L_{-\ell_2})^{p_2} \dots (L_{-\ell_n})^{p_n} \frac{\partial^m}{\partial c_1^m} |I^{(1)}(\beta_1, c_1; \beta_0)\rangle, \quad (3.10)$$

for integers $m, n \geq 0$, $p_i > 0$, and $\ell_1 > \ell_2 > \dots > \ell_n > 0$. The coefficients of this linear combination turn out to depend on c_1 and β_1 but are independent of β_0 . As pointed out in [31], $|I_k^{(1)}\rangle$ turns out to be a level- $2k$ generalized descendant, where L_{-k} , $\partial^k/\partial c_1^k$ and $1/c_1^k$ are regarded as level k . The exponents ν_1 and ν_2 in (3.9) are fixed as [31]

$$\nu_1 = 2(\alpha - \beta_1)(Q - \beta_1), \quad \nu_2 = \frac{1}{2}(\beta_1 - \alpha)(3Q - 3\beta_1 - \alpha), \quad (3.11)$$

in order for the ansatz (3.9) to satisfy (3.2)–(3.6). Note here that the ansatz (3.9) particularly implies that in the limit $c_2 \rightarrow 0$ the irregular state behaves as $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle \rightarrow c_1^{\nu_1} c_2^{\nu_2} e^{(\alpha-\beta_1)\frac{c_1^2}{c_2}} |I^{(1)}(\beta_1, c_1; \beta_0)\rangle$. Therefore (3.9) fixes the “boundary condition” for $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ at $c_2 = 0$. The authors of [31] conjectured that (3.2)–(3.6) uniquely determine the generalized descendants $|I_k^{(1)}(\alpha, c_1; \boldsymbol{\beta})\rangle$.

We now explicitly compute $|I_k^{(1)}(\alpha, c_1; \boldsymbol{\beta})\rangle$ by solving (3.2)–(3.6) order by order of c_2 . Since the basic strategy is already discussed in [31], we move the detailed description for the order-by-order computation to appendix A.1, and write down the results here:

$$|I_0^{(1)}\rangle = |I^{(1)}(\beta_1, c_1; \beta_0)\rangle, \quad (3.12)$$

$$|I_1^{(1)}\rangle = \left(\frac{1}{2c_1} L_{-1} + \frac{2\alpha - 3\beta_1}{2c_1} \partial_{c_1} + \frac{\nu_3}{c_1^2} \right) |I^{(1)}(\beta_1, c_1; \beta_0)\rangle, \quad (3.13)$$

$$\begin{aligned} |I_2^{(1)}\rangle = & \left(-\frac{1}{4c_1^2} L_{-2} + \frac{1}{8c_1^2} L_{-1}^2 + \frac{2\alpha - 3\beta_1}{4c_1^2} L_{-1} \partial_{c_1} \right. \\ & + \frac{2(2\alpha - 3\beta_1)^2 - 1}{16c_1^2} \partial_{c_1}^2 + \frac{11\beta_1 - 8\alpha + Q + 4\nu_3}{8c_1^3} L_{-1} \\ & \left. + \frac{-4\alpha^2 + 8\alpha(3\beta_1 + \nu_3) - \beta_1(23\beta_1 + 12\nu_3) + Q(-12\alpha + 11\beta_1)}{8c_1^3} \partial_{c_1} + \frac{\nu_4}{c_1^4} \right) |I^{(1)}(\beta_1, c_1; \beta_0)\rangle, \end{aligned} \quad (3.14)$$

where $\partial_{c_1} \equiv \partial/\partial c_1$, and ν_3 and ν_4 are given by

$$\nu_3 \equiv \frac{1}{2}(3Q + \alpha - 3\beta_1)(\alpha - \beta_1)(Q - \beta_1), \quad (3.15)$$

$$\begin{aligned} \nu_4 \equiv & \frac{(Q - \beta_1)(\alpha - \beta_1)}{8} \left((\alpha - \beta_1)^3 (Q - \beta_1) + (\alpha - \beta_1)^2 (4\beta_1^2 + 6Q^2 - 10\beta_1 Q - 2) \right. \\ & \left. + (\alpha - \beta_1)(3Q - 2\beta_1) (2\beta_1^2 + 3Q^2 - 5\beta_1 Q - 5) - 8\beta_1^2 - 21Q^2 + 26\beta_1 Q + 1 \right). \end{aligned} \quad (3.16)$$

²¹As mentioned already, we here use a slightly different convention for $|I_k^{(2)}\rangle$. Namely, $|I_k^{(2)}\rangle$ here is identified with $|I_{2k}^{(2)}\rangle$ in [31].

From the fact that $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ is an irregular state of rank one, we see that (3.9) with the above expressions substituted correctly satisfies the equations (3.2)–(3.6).

3.1.2 Irregular conformal block

Using the above result for the irregular state of rank two, we here evaluate the c_2 -expansion of the inner product $\langle \Delta_\gamma | I^{(2)}(\alpha, \mathbf{c}; \beta) \rangle$. We particularly show that, in the limit of $Q \rightarrow 0$, this c_2 -expansion is precisely identical to the strong coupling expansion of the partition function of the (A_1, D_4) theory obtained in [15].

To that end, we first evaluate the inner product of (3.9) and $|\Delta_\gamma\rangle$, which reduces to computing the inner product $\langle \Delta_\gamma | I_k^{(1)} \rangle$ for all k . Note here that, since $\langle \Delta_\gamma | L_{-n} = 0$ for $n > 1$, we only need to keep track of terms without L_{-n} in (3.12)–(3.14). Moreover, since the rank-one irregular state satisfies $L_0 |I^{(1)}(\beta_1, c_1; \beta_0)\rangle = (\Delta_{\beta_1} + c_1 \partial_{c_1}) |I^{(1)}(\beta_1, c_1; \beta_0)\rangle$, the action of ∂_{c_1} on $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ is translated into the action of L_0 . Therefore,

$$\langle \Delta_\gamma | (c_1 \partial_{c_1})^k |I^{(1)}(\beta_1, c_1; \beta_0)\rangle = (\Delta_\gamma - \Delta_{\beta_1})^k \langle \Delta_\gamma | I^{(1)}(\beta_1, c_1; \beta_0)\rangle. \quad (3.17)$$

Using this identity, we see that the inner product of $|I^{(2)}(\alpha, \beta, \mathbf{c})\rangle$ and $|\Delta_\gamma\rangle$ is written as

$$\langle \Delta_\gamma | I^{(2)}(\alpha, \mathbf{c}; \beta) \rangle = \langle \Delta_\gamma | I^{(1)}(\beta_1, c_1; \beta_0) \rangle c_1^{\nu_1} c_2^{\nu_2} e^{(\alpha - \beta_1) \frac{c_1^2}{c_2}} \sum_{k=0}^{\infty} \left(\frac{c_2}{i c_1^2} \right)^k D_k(\alpha, \beta_1, \gamma), \quad (3.18)$$

with coefficients D_k depending on α, β_1 and γ . We here show the expressions for the first several coefficients:

$$D_0 = 1, \quad (3.19)$$

$$D_1 = \frac{i}{2} \left((2\alpha - 3\beta_1) \Delta_\gamma + (Q - \beta_1) (6\beta_1^2 - 3\beta_1(2\alpha + Q) + \alpha(\alpha + 3Q)) \right), \quad (3.20)$$

$$\begin{aligned} D_2 = \frac{1}{16} \Big\{ & \Delta_\gamma \left((1 - 2(2\alpha - 3\beta_1)^2) \Delta_\gamma - 72\beta_1^4 - 6\beta_1^2(10\alpha^2 + 6Q^2 + 30\alpha Q - 11) \right. \\ & + 4\beta_1(2\alpha^3 + 3\alpha(5Q^2 - 6) + 21\alpha^2 Q - 6Q) - 8\alpha(Q(\alpha^2 + 3\alpha Q - 3) - 2\alpha) \\ & + 12\beta_1^3(10\alpha + 9Q) - 1 \Big) - (Q - \beta_1) \left(3\beta_1(-24\beta_1^4 + 5\beta_1^2(7 - 6Q^2) + \beta_1 Q(6Q^2 - 35) \right. \\ & + 14Q^2 + 48\beta_1^3 Q - 1) + 6\alpha^2(-16\beta_1^3 + 8\beta_1 + 3Q^3 - 17\beta_1 Q^2 + 5(6\beta_1^2 - 1)Q) \\ & - 2\alpha(-72\beta_1^4 + 70\beta_1^2 + 18\beta_1 Q^3 + (21 - 90\beta_1^2)Q^2 + 4(36\beta_1^2 - 17)\beta_1 Q - 1) \\ & \left. \left. + 2\alpha^4(Q - \beta_1) + 4\alpha^3(3(Q - 2\beta_1)(Q - \beta_1) - 1) \right) \right\}. \end{aligned} \quad (3.21)$$

Some more coefficients for higher orders are shown in appendix A.1.

Let us now consider the prefactor $\langle \Delta_\gamma | I^{(1)}(\beta_1, c_1; \beta_0) \rangle$ in (3.18). Here $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ is a rank-one irregular state satisfying (2.9) for $n = 1$. In the case of $\beta_0 = \beta_1$, the c_1 -expansion of $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ was studied in [29], whose generalization to $\beta_0 \neq \beta_1$ is of the form

$$|I^{(1)}(\beta_1, c_1; \beta_0)\rangle = c_1^{\Delta_{\beta_1} - \Delta_{\beta_0}} \left(|\Delta_{\beta_0}\rangle + \sum_{k=1}^{\infty} (c_1)^k |R_k\rangle \right), \quad (3.22)$$

where the $|R_k\rangle$ is a level- k Virasoro descendant of $|\Delta_{\beta_0}\rangle$.²² While the Virasoro descendants $|R_k\rangle$ are fixed by β_0 and β_1 so that $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ satisfies (2.9), we do not need to

²²Note here that the expression (3.22) fixes the “boundary condition” for $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ at $c_1 = 0$.

evaluate them explicitly. Indeed, since $|R_k\rangle$ are all Virasoro descendants, we see that

$$\langle \Delta_\gamma | I^{(1)}(\beta_1, c_1; \beta_0) \rangle = \begin{cases} c_1^{\Delta_{\beta_1} - \Delta_{\beta_0}} & (\Delta_{\beta_0} = \Delta_\gamma) \\ 0 & (\Delta_{\beta_0} \neq \Delta_\gamma) \end{cases}. \quad (3.23)$$

For this inner product to be non-vanishing, we set $\gamma = \beta_0$ in the rest of this sub-section.

Substituting (3.23) in (3.18), we obtain an explicit expression for $\langle \Delta_{\beta_0} | I^{(2)}(\alpha, \mathbf{c}; \beta) \rangle$. This is then identified via (2.14) with the partition function of the (A_1, D_4) theory, i.e.,

$$\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2} = c_1^{2(\alpha - \beta_1)(Q - \beta_1) + \Delta_{\beta_0} - \Delta_\alpha} c_2^{\frac{1}{2}(\beta_1 - \alpha)(3Q - 3\beta_1 - \alpha)} e^{(\alpha - \beta_1) \frac{c_1^2}{c_2}} \sum_{k=0}^{\infty} \left(\frac{c_2}{ic_1^2} \right)^k D_k(\alpha, \beta_1, \beta_0), \quad (3.24)$$

up to a c_i -independent prefactor, where the coefficients D_i are those in (3.19)–(3.21) with $\gamma = \beta_0$. Recall here that $\sqrt{ic_1^2/c_2}$ is identified with the relevant coupling of dimension 1/2 of the (A_1, D_4) theory. Therefore, (3.24) is an expansion of the (A_1, D_4) partition function in the *inverse* powers of the relevant coupling, i.e., the strong coupling expansion of $\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2}$.

Let us now compare (3.24) with the (A_1, D_4) partition function obtained in [15].²³ Since the expression obtained in [15] is for the case $Q = 0$, we will focus on the $Q \rightarrow 0$ limit of (3.24) in the rest of this sub-section. Note that $Q = 0$ corresponds to $\epsilon_1 + \epsilon_2 = 0$ in four dimensions. We first change the variables from $(c_2/ic_1^2, \alpha, \beta_0, \beta_1)$ to $(s, \nu, \theta_s, \theta_t)$ by

$$\frac{c_2}{ic_1^2} \equiv \frac{1}{s}, \quad \alpha \equiv i(\theta_s + \theta_t), \quad \beta_0 \equiv i(\theta_t - \theta_s), \quad \beta_1 \equiv \frac{i(\theta_s + \theta_t - 3\nu)}{3}. \quad (3.25)$$

Note that these relations could receive $\mathcal{O}(Q)$ corrections, as we will discuss in section 4. We omit such $\mathcal{O}(Q)$ corrections for the moment since we here focus on the limit $Q \rightarrow 0$. With the above change of variables, the partition function for $Q = 0$ is written as

$$\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2} \Big|_{\epsilon_1 + \epsilon_2 = 0} = c_1^{-\frac{(3\nu + 2(\theta_s + \theta_t))^2}{9}} c_2^{-\frac{3\nu^2}{2} + \frac{2(\theta_s + \theta_t)^2}{3}} e^{\nu s + \frac{2}{3}(\theta_s + \theta_t)s} \sum_{k=0}^{\infty} \frac{D_k|_{Q=0}}{s^k}. \quad (3.26)$$

We see that this expression is identical to $\mathcal{G}(\nu, s)$ in eq. (3.48) of [15] obtained via the connection to Painlevé IV, up to an s -independent prefactor. In particular, the asymptotic behavior, $e^{\nu s + \frac{2}{3}(\theta_s + \theta_t)s}$, in the limit $s \rightarrow \infty$ is precisely identical. Moreover, our $D_1|_{Q=0}$ and $D_2|_{Q=0}$ are written as

$$D_1|_{Q=0} = 3\nu^3 - 2\nu(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - \frac{2}{9}(2\theta_s - \theta_t)(\theta_s + \theta_t)(\theta_s - 2\theta_t), \quad (3.27)$$

$$\begin{aligned} D_2|_{Q=0} = & \frac{9\nu^6}{2} + \nu^4 \left(\frac{105}{16} - 6(\theta_s^2 - \theta_s\theta_t + \theta_t^2) \right) - \frac{2}{3}\nu^3(\theta_s - 2\theta_t)(2\theta_s - \theta_t)(\theta_s + \theta_t) \\ & + \nu^2 \left(\frac{1}{2}(\theta_s^2 - \theta_s\theta_t + \theta_t^2)(4(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 11) + \frac{3}{16} \right) \\ & + \frac{\nu}{9}(2\theta_s - \theta_t)(\theta_s + \theta_t)(\theta_s - 2\theta_t)(4(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 7) \\ & + \frac{1}{324} \left(32\theta_s^6 - 96\theta_s^5\theta_t + \theta_s^4(36 - 24\theta_t^2) + 8\theta_s^3\theta_t(26\theta_t^2 - 9) - 3\theta_s^2(8\theta_t^4 - 36\theta_t^2 + 9) \right. \\ & \left. - 3\theta_s\theta_t(32\theta_t^4 + 24\theta_t^2 - 9) + \theta_t^2(32\theta_t^4 + 36\theta_t^2 - 27) \right), \end{aligned} \quad (3.28)$$

²³In this reference, the (A_1, D_4) theory is called the H_2 Argyres-Douglas theory.

which are precisely identical to D_1 and D_2 in eq. (3.49) of [15]. Therefore, the $Q \rightarrow 0$ limit of our partition function $\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2}$ is identical to $\mathcal{G}(\nu, s)$ in [15] up to a prefactor. In particular, the strong coupling expansion in powers of $1/s$ corresponds to the c_2 -expansion of the irregular state $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ arising from the ansatz (3.9).

Note here that the above $Q \rightarrow 0$ limits of D_1 and D_2 were also obtained in [24], with a different parameterization of the irregular conformal block. In particular, the expansion parameter $1/s$ is identified in [24] as the coordinate of the vertex operator corresponding to the regular singularity. On the other hand, we here identify the expansion in powers of $1/s$ as the asymptotic c_2 -expansion arising from the ansatz (3.9) proposed in [31]. These two parameter identifications are expected to be related by a change of coordinate on sphere. We have also evaluated the Q -dependent terms as shown in (3.21), which will be important in our discussion in section 4.

3.2 (A_1, A_3) theory from rank-3 irregular state

Let us now turn to the (A_1, A_3) theory. According to the generalized AGT correspondence, the partition function of the (A_1, A_3) theory is expected to be given by

$$Z_{(A_1, A_3)}^{\epsilon_1, \epsilon_2} = \langle 0 | I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle, \quad (3.29)$$

where $|0\rangle$ is the vacuum state, and $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ is an irregular primary state of rank-three. Here $\mathbf{c} \equiv (c_1, c_2, c_3)$ and $\boldsymbol{\beta} \equiv (\beta_0, \beta_1, \beta_2)$. This irregular state satisfies the following equations:

$$L_6 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -c_3^2 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.30)$$

$$L_5 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -2c_2 c_3 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.31)$$

$$L_4 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -(c_2^2 + 2c_3 c_1) |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.32)$$

$$L_3 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = -2(c_1 c_2 + c_3(\alpha - 2Q)) |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.33)$$

$$L_2 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \left(c_3 \frac{\partial}{\partial c_1} - c_2(2\alpha - 3Q) - c_1^2 \right) |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.34)$$

$$L_1 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \left(2c_3 \frac{\partial}{\partial c_2} + c_2 \frac{\partial}{\partial c_1} - 2c_1(\alpha - Q) \right) |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle, \quad (3.35)$$

$$L_0 |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \left(\Delta_\alpha + c_1 \frac{\partial}{\partial c_1} + 2c_2 \frac{\partial}{\partial c_2} + 3c_3 \frac{\partial}{\partial c_3} \right) |I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle. \quad (3.36)$$

The relation between the 4d and 2d parameters is identified by looking at the Seiberg-Witten curve. Indeed, $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$ implies that

$$\frac{\langle 0 | T(z) | I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle}{\langle 0 | I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta}) \rangle} = \frac{u}{z^4} - \frac{2(c_1 c_2 + c_3(\alpha - 2Q))}{z^5} - \frac{c_2^2 + 2c_3 c_1}{z^6} - \frac{2c_2 c_3}{z^7} - \frac{c_3^2}{z^8}, \quad (3.37)$$

where $u \equiv c_3 \frac{\partial}{\partial c_1} \log \langle 0 | I^{(3)}(\alpha, c_1, c_2, c_3) \rangle - c_2(2\alpha - 3Q) - c_1^2$. By changing the variable as $z \rightarrow -ic_3^{1/3} z / (1 + ic_2 z / 2c_3^{2/3})$, the stress tensor changes as $T \rightarrow -c_3^{-2/3} (1 + ic_2 z / 2c_3^{2/3}) T$.²⁴

²⁴Note that this transformation is generated by L_0 and L_{-1} , and therefore the stress tensor transforms as a tensor under it.

Then the above equation is mapped to

$$\frac{\langle 0|T(z)|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle}{\langle 0|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle} = \frac{\tilde{u}}{z^4} + \frac{2i(2Q - \alpha)}{z^5} + \frac{\frac{(c_2)^2}{(c_3)^{4/3}} - \frac{4c_1}{(c_3)^{1/3}}}{2z^6} + \frac{1}{z^8}, \quad (3.38)$$

where $\tilde{u} \equiv -u/(c_3)^{2/3} + (-8c_1(c_2)^2c_3 + (c_2)^4 + 16c_2(c_3)^2(\alpha - 2Q))/16(c_3)^{8/3}$. Using (2.8), we can relate this to the Seiberg-Witten curve of the (A_1, A_3) theory shown in (2.3). To be concrete, $\tilde{u}$ is identified with the vev of a Coulomb branch operator of dimension $4/3$, $2i(2Q - \alpha)$ is the mass parameter, and $(c_2)^2/(c_3)^{4/3} - 4c_1/(c_3)^{1/3}$ is identified with the relevant coupling of dimension $2/3$. Note that while the irregular state $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ depends on c_1, c_2 and c_3 , the Seiberg-Witten curve depends only on the above particular combination of them. This means that there is a redundancy in describing the partition function of the (A_1, A_3) theory in terms of the 2d irregular conformal block. We use this redundancy to set $c_1 = 0$ in the rest of this section, which turns out to simplify the computation. Under the condition $c_1 = 0$, the 4d relevant coupling of dimension $2/3$ is identified with

$$\frac{(c_2)^2}{(c_3)^{4/3}} = \left(\frac{(c_2)^3}{(c_3)^2} \right)^{\frac{2}{3}}. \quad (3.39)$$

3.2.1 Irregular conformal block

Let us now evaluate the rank-three irregular state $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ explicitly. To that end, we first use the ansatz (2.12) for the rank-three state:

$$|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = c_2^{\rho_2} c_3^{\rho_3} e^{(\alpha - \beta_2)S_3(\mathbf{c})} \left(|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle + \sum_{k=1}^{\infty} (c_3)^k |I_k^{(2)}\rangle \right), \quad (3.40)$$

where $S_3(\mathbf{c}) \equiv \frac{2c_1c_2}{c_3} - \frac{c_2^3}{3c_3^2} - \frac{c_1^2}{c_2}$, $\tilde{\mathbf{c}} \equiv (c_1, c_2)$, $\tilde{\boldsymbol{\beta}} \equiv (\beta_0, \beta_1)$, and $|I_k^{(2)}\rangle$ are generalized descendants of the rank-two irregular state $|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle$.²⁵ The generalized descendants are now of the form

$$(L_{-\ell_1})^{p_1} (L_{-\ell_2})^{p_2} \cdots (L_{-\ell_n})^{p_n} \frac{\partial^{m_1}}{\partial c_1^{m_1}} \frac{\partial^{m_2}}{\partial c_2^{m_2}} |I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle, \quad (3.41)$$

for integers $n, m_i \geq 0$, $p_i > 0$ and $\ell_1 > \ell_2 > \cdots > \ell_n > 0$. The exponents, ρ_2 and ρ_3 , are uniquely determined as

$$\rho_2 = \frac{1}{2}(\beta_2 - \alpha)(5Q - 5\alpha + \beta_2), \quad \rho_3 = \frac{2}{3}(\alpha - \beta_2)(2Q - 2\alpha + \beta_2), \quad (3.42)$$

so that the above ansatz solves (3.30)–(3.36). Note that the ansatz (3.40) again fixes the “asymptotic behavior” of $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ in the limit $c_3 \rightarrow 0$. The generalized descendants $|I_k^{(2)}\rangle$ are conjectured in [31] to be uniquely determined. We explicitly computed them for $k = 1, \dots, 6$, whose first several expressions are shown in appendix A.2.

²⁵As mentioned already, our convention for $|I_k^{(2)}\rangle$ is slightly different from that of [31]. Our $|I_k^{(2)}\rangle$ is identical to $|I_{3k}^{(2)}\rangle$ of [31].

Using the expressions for the generalized descendants, we see that the inner product of (3.40) and $|0\rangle$ is evaluated as

$$\langle 0|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = \langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle c_2^{\rho_2} c_3^{\rho_3} e^{(\alpha-\beta_2)S_3(\mathbf{c})} \left(\sum_{k=0}^{\infty} \left(\frac{3(c_3)^2}{(c_2)^3} \right)^k D_k(\alpha, \beta_2) + \mathcal{O}(c_1) \right), \quad (3.43)$$

where

$$D_0 = 1, \quad (3.44)$$

$$D_1 = -\frac{1}{36}(\alpha - 2\beta_2 + Q)(4\alpha^2 + 34\beta_2^2 - 34\beta_2(\alpha + Q) + 35\alpha Q - 1), \quad (3.45)$$

$$\begin{aligned} D_2 = \frac{1}{2592} & (16\alpha^6 + \alpha^4(1801Q^2 - 200) + 2\alpha^3Q(1365Q^2 - 1489) + 50\alpha Q(19 - 77Q^2) \\ & - 35Q^2 + \alpha^2(1225Q^4 - 6244Q^2 + 169) - 4\beta_2(\alpha - \beta_2 + Q)(84\alpha^4 - 763\alpha^2 \\ & + 595\alpha Q^3 - \beta_2(\alpha - \beta_2 + Q)(561\alpha^2 + 289Q^2 - 1156\beta_2(\alpha - \beta_2 + Q) + 2958\alpha Q \\ & - 2318) + (2483\alpha^2 - 971)Q^2 + (1011\alpha^2 - 3902)\alpha Q + 271) + 312\alpha^5 Q), \end{aligned} \quad (3.46)$$

$$\begin{aligned} D_3 = -\frac{1}{279936} & (\alpha - 2\beta_2 + Q) \left(64\alpha^8 - 2352\alpha^6 + 26796\alpha^4 - 48313\alpha^2 + 42875\alpha^3 Q^5 \right. \\ & + 1225(82\alpha^2 - 327)\alpha^2 Q^4 + 35(2113\alpha^4 - 19818\alpha^2 + 33843)\alpha Q^3 + (18124\alpha^6 \\ & - 382215\alpha^4 + 1420026\alpha^2 - 5005)Q^2 - 2\beta_2(\alpha - \beta_2 + Q)(944\alpha^6 - 28008\alpha^4 \\ & + 243975\alpha^2 + 62475\alpha^2 Q^4 + 70(3214\alpha^2 - 5667)\alpha Q^3 + 3(40417\alpha^4 - 382090\alpha^2 \\ & + 196601)Q^2 - 2\beta_2(\alpha - \beta_2 + Q)(5100\alpha^4 - 34\beta_2(\alpha - \beta_2 + Q)(697\alpha^2 + 289Q^2 \\ & - 1156\beta_2(\alpha - \beta_2 + Q) + 4148\alpha Q - 6852) + 9\alpha^2(21012Q^2 - 11551) \\ & + 105\alpha Q(289Q^2 - 6024) - 98175Q^2 + 65841\alpha^3 Q + 419196) + 6(3212\alpha^4 \\ & - 66615\alpha^2 + 256546)\alpha Q - 224894) + (1808\alpha^6 - 55800\alpha^4 + 517557\alpha^2 \\ & \left. - 481430)\alpha Q + 7560 \right). \end{aligned} \quad (3.47)$$

The prefactor $\langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle$ in (3.43) is evaluated as follows. Since $\langle 0|L_1 = \langle 0|L_0 = 0$, (3.5) and (3.6) imply

$$\left(c_2 \frac{\partial}{\partial c_1} - 2c_1(\beta_2 - Q) \right) \langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle = \left(\Delta_{\beta_2} + c_1 \frac{\partial}{\partial c_1} + 2c_2 \frac{\partial}{\partial c_2} \right) \langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle = 0. \quad (3.48)$$

From this set of differential equations, we see that $\langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle$ is given by

$$\langle 0|I^{(2)}(\beta_2, \tilde{\mathbf{c}}; \tilde{\boldsymbol{\beta}})\rangle = e^{-(Q-\beta_2)\frac{c_1^2}{c_2} - \frac{1}{2}\Delta_{\beta_2}}, \quad (3.49)$$

up to a prefactor independent of c_1 and c_2 . Note here that $\tilde{\boldsymbol{\beta}}$ -dependence only appears in the c_i -independent prefactor.²⁶

²⁶For this prefactor to be non-vanishing, we need to set $\Delta_{\beta_0} = 0$. Indeed, $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ is a linear combination of $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ and its generalized descendants, and $|I^{(1)}(\beta_1, c_1; \beta_0)\rangle$ is a linear combination of $|\Delta_{\beta_0}\rangle$ and its Virasoro descendants. Therefore, $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ is in the Verma module of Virasoro algebra whose highest weight state is $|\Delta_{\beta_0}\rangle$. This means that $\langle 0|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle = 0$ unless $\Delta_{\beta_0} = 0$.

According to (2.13), the partition function of the (A_1, A_3) theory is identified with $\langle 0|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$. In this identification, we can set $c_1 = 0$, as discussed at the beginning of this sub-section. Therefore, we have

$$\mathcal{Z}_{(A_1, A_3)}^{\epsilon_1, \epsilon_2} = \langle 0|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle|_{c_1=0} = c_2^{-\frac{1}{2}\Delta_{\beta_2} + \rho_2} c_3^{\rho_3} e^{-(\alpha - \beta_2)\frac{c_2^3}{3c_3^2}} \sum_{k=0}^{\infty} \left(\frac{3c_3^2}{c_2^3}\right)^k D_k(\alpha, \beta_2), \quad (3.50)$$

up to a c_i -independent prefactor, where the first several coefficients D_k are shown in (3.44)–(3.47). Recall here that $((c_2)^3/(c_3)^2)^{\frac{2}{3}}$ is identified with the relevant coupling of dimension $2/3$ in the (A_1, A_3) theory. Therefore (3.50) is an expansion of the (A_1, A_3) partition function in the inverse powers of the coupling, i.e., the strong coupling expansion.

Let us now compare the $Q \rightarrow 0$ limit of (3.50) with the strong coupling expansion of the (A_1, A_3) partition function evaluated in [15].²⁷ To that end, we first change the variables as

$$\frac{3(c_3)^2}{(c_2)^3} \equiv \frac{1}{s}, \quad \alpha \equiv i\theta, \quad \beta_2 \equiv i\left(\nu + \frac{\theta}{2}\right). \quad (3.51)$$

Here, these relations could receive $\mathcal{O}(Q)$ corrections, as we will discuss in section 4. We omit such $\mathcal{O}(Q)$ corrections for a while since we focus on the $Q \rightarrow 0$ limit here. With the above change of variables, the partition function for $Q = 0$ (or equivalently $\epsilon_1 + \epsilon_2 = 0$) is written as

$$\mathcal{Z}_{(A_1, A_3)}^{\epsilon_1, \epsilon_2}|_{\epsilon_1 + \epsilon_2 = 0} = c_2^{-\frac{5}{4}\theta^2 + 2\theta\nu - \nu^2} c_3^{\frac{1}{6}(\theta - 2\nu)(3\theta - 2\nu)} e^{i(\nu - \frac{\theta}{2})s} \sum_{k=0}^{\infty} \frac{D_k|_{Q=0}}{s^k}, \quad (3.52)$$

where

$$D_1|_{Q=0} = -\frac{17}{9}i\nu^3 + \frac{9\theta^2 - 2}{36}i\nu, \quad (3.53)$$

$$D_2|_{Q=0} = -\frac{289}{162}\nu^6 + \frac{153\theta^2 - 1159}{324}\nu^4 - \left(\frac{\theta^4}{32} - \frac{11\theta^2}{18} + \frac{271}{648}\right)\nu^2 - \frac{\theta^2(11\theta^2 - 68)}{1728}, \quad (3.54)$$

$$\begin{aligned} D_3|_{Q=0} = & \frac{4913}{4374}i\nu^9 - \frac{17(153\theta^2 - 2284)}{5832}i\nu^7 + \frac{1377\theta^4 - 47178\theta^2 + 279464}{23328}i\nu^5 \\ & + \frac{-729\theta^6 + 45648\theta^4 - 700884\theta^2 + 899576}{279936}i\nu^3 \\ & + \frac{-99\theta^6 + 4270\theta^4 - 28504\theta^2 + 3360}{62208}i\nu. \end{aligned} \quad (3.55)$$

We now see that the partition function (3.52) for $Q = 0$ coincides, up to a prefactor, with $\mathcal{G}(\nu, s)$ in eq. (3.32) of [15] obtained via the connection to Painlevé II. In particular, our $D_1|_{Q=0}$ and $D_2|_{Q=0}$ are precisely identical to D_1 and D_2 in eq. (3.33) of [15].²⁸ This

²⁷In this reference, the (A_1, A_3) theory is called the H_1 Argyres-Douglas theory.

²⁸While the D_3 is not explicitly written in [15], the authors of [37] evaluated it using a nice matrix model description as shown in eq. (3.23) of [37]. We then see that the $Q \rightarrow 0$ limit of our D_3 is in perfect agreement with the expression.

particularly means that the strong coupling expansion in powers of $1/s$ corresponds to the asymptotic c_3 -expansion of the rank-3 state $|I^{(3)}(\alpha, \mathbf{c}; \beta)\rangle$.

Note here that (3.53) and (3.54) were also reproduced from the irregular conformal block with one irregular singularity of rank $\frac{3}{2}$ and one regular singularity [26]. On the other hand, we obtained the above results from the irregular conformal block with an irregular singularity of rank 3 without any regular singularity. Therefore, the author of [26] and we computed different irregular conformal blocks of Liouville theory. Nevertheless, the two irregular conformal blocks turn out to be related to the same tau-function of Painlevé II. This is natural from the viewpoint of 4d/2d correspondence. Indeed, the Hitchin systems associated with the two different irregular conformal blocks are isomorphic [68], and therefore these two irregular conformal blocks are expected to be related to the same 4d $\mathcal{N} = 2$ SCFT, i.e., the (A_1, A_3) theory. Our discussion above explicitly confirms this expectation.

4 Enhanced flavor symmetries

Having evaluated the irregular conformal blocks for the (A_1, D_4) and (A_1, A_3) theories, we here discuss their flavor symmetries. In particular, we will show that the strong coupling expansions of $\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2}$ and $\mathcal{Z}_{(A_1, A_3)}^{\epsilon_1, \epsilon_2}$ are invariant under the action of the Weyl group of the enhanced flavor symmetry of the AD theories.

In general, turning on generic values of mass parameters breaks the flavor symmetry group to its maximal torus.²⁹ Therefore the complete flavor symmetry is not visible in the partition function with mass parameters turned on. However, the Weyl group of the flavor symmetry is still a symmetry of the mass-deformed theory, since it is a symmetry of the maximal torus. In particular, the action of the Weyl group permutes the mass parameters in such a way that the partition function is invariant. Below, we identify such an action of the Weyl group for the (A_1, D_4) and (A_1, A_3) theories.

Furthermore, by demanding this Weyl group symmetry extend to the case of non-vanishing Liouville charge Q , we propose an identification of parameters between the Liouville side and the Argyres-Douglas side for general values of the Ω -background parameters. Under this parameter identification, we find that the (A_1, D_4) and (A_1, A_3) partition functions are invariant under $Q \rightarrow -Q$, or equivalently $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$. We will also give a natural explanation for our parameter identification from the AGT viewpoint, which leads us to a conjecture on the parameter identification for a general rank- n irregular state.

4.1 Weyl group of the flavor symmetry

We first study how the Weyl group of the enhanced flavor symmetry is visible in the irregular conformal blocks corresponding to the (A_1, D_4) and (A_1, A_3) theories.

²⁹The reason for this is that, in any 4d $\mathcal{N} = 2$ SCFT, every $\mathcal{N} = 2$ preserving mass deformation operator is in the same superconformal multiplet as a flavor current, and therefore in the adjoint representation of the flavor symmetry.

4.1.1 (A_1, D_4) theory

We start with the (A_1, D_4) partition function (3.24) evaluated as the irregular conformal block with one irregular singularity of rank 2 and one regular singularity. The flavor symmetry of the (A_1, D_4) theory is enhanced from the manifest $SU(2) \times U(1)$ to $SU(3)$, as reviewed in section 2. Therefore its Weyl group S_3 is expected to act on the set of mass parameters so that the (A_1, D_4) partition function is invariant.

Such an action of S_3 can easily be found in the limit of $Q \rightarrow 0$. To see this, let us look at the $Q \rightarrow 0$ limit of $\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2}$ shown in (3.26). Here, explicit expressions for $D_k|_{Q=0}$ are shown in (3.27) – (3.28), where θ_s, θ_t and ν are related by (3.25) to the parameters α, β_0 and β_1 of the corresponding irregular conformal block. As shown in [15], θ_s and θ_t are identified with two mass parameters of the (A_1, D_4) theory. We now see that the expansion coefficients $D_1|_{Q=0}$ and $D_2|_{Q=0}$ are invariant under both of the following two transformations

$$\sigma_1 : \theta_s \longleftrightarrow \theta_t, \quad (4.1)$$

$$\sigma_2 : \theta_s \rightarrow \theta_s - \theta_t, \quad \theta_t \rightarrow -\theta_t. \quad (4.2)$$

Note that the above σ_1 and σ_2 satisfy

$$\sigma_1^2 = \sigma_2^2 = (\sigma_1 \sigma_2)^3 = \mathbf{1}, \quad (4.3)$$

and therefore gives a representation of S_3 . This S_3 is naturally identified with the Weyl group of the flavor $SU(3)$ symmetry of (A_1, D_4) .

Let us now consider the case of $Q \neq 0$. Recall that the Liouville charge Q corresponds to the Ω -background parameters by (2.7). Since the Ω -deformation does not break the flavor symmetry, the above S_3 -invariance of the partition function is expected to extend to the case of $Q \neq 0$. Indeed, we find that D_k in (3.24) are invariant under the S_3 generated by (4.1) and (4.2) if the parameter identification (3.25) receives an $\mathcal{O}(Q)$ -correction as

$$\alpha \equiv i(\theta_s + \theta_t) + \frac{3}{2}Q, \quad \beta_0 \equiv i(\theta_s - \theta_t) + \frac{1}{2}Q, \quad \beta_1 = \frac{i(\theta_s + \theta_t - 3\nu)}{3} + \mathcal{O}(Q), \quad (4.4)$$

where the $\mathcal{O}(Q)$ correction to β_1 is arbitrary as far as it is independent of θ_s and θ_t .³⁰ Note here that the S_3 invariance requires non-vanishing $\mathcal{O}(Q)$ corrections to the parameter identification (3.25). With the modified identification (4.4), we have checked that $D_1, \dots, D_4$ in (3.24) are invariant under the S_3 generated by (4.1) and (4.2). See appendix B for more detail.

The $\mathcal{O}(Q)$ correction to β_1 is not fixed by the S_3 invariance. The reason for this is that β_1 is related to the parameter ν which is identified with the (dual) Coulomb branch parameter a_D by $\nu = ia_D$ (See the last line of section 4 of [15]). Since a_D is neutral under the flavor $SU(3)$ symmetry, so is ν . Then the flavor symmetry is not enough to fix the $\mathcal{O}(Q)$ correction to the relation between β_1 and ν . Nevertheless, we will conjecture in the next sub-section that the correct identification is $\beta_1 = i(\theta_s + \theta_t - 3\nu)/3 + Q$.

³⁰The $\mathcal{O}(Q)$ -term is expected to be independent of θ_s and θ_t , since the Ω -deformation is not coupled with a mass deformation.

4.1.2 (A_1, A_3) theory

We now turn to the (A_1, A_3) partition function evaluated as the irregular conformal block with one irregular singularity of rank 3. In the limit $Q \rightarrow 0$, the strong coupling expansion of the partition function is given by (3.52). We see that $D_1|_{Q=0}, \dots, D_3|_{Q=0}$ in the expansion are invariant under the S_2 -action

$$\sigma : \theta \rightarrow -\theta. \quad (4.5)$$

Since θ is identified with the mass parameter associated with the flavor $SU(2)$ symmetry of the (A_1, A_3) theory, it is natural to identify this S_2 as the Weyl group of the flavor symmetry.

This S_2 -invariance of the partition function can be extended to the case of $Q \neq 0$, if the parameter identification (3.51) receives an $\mathcal{O}(Q)$ -correction as

$$\alpha \equiv i\theta + 2Q, \quad \beta_2 = i\left(\nu + \frac{\theta}{2}\right) + \mathcal{O}(Q), \quad (4.6)$$

where the $\mathcal{O}(Q)$ correction to β_2 is arbitrary as far as it is independent of θ . We have checked that D_1, D_2 and D_3 in (3.45)–(3.47) are invariant under the S_2 -action (4.5), under the modified parameter identification (4.6). Note that the S_2 -invariance of the partition function requires a non-vanishing $\mathcal{O}(Q)$ correction to the parameter identification. The $\mathcal{O}(Q)$ correction to the relation between β_2 and ν is not fixed by the flavor symmetry, since ν is related to the (dual) Coulomb branch parameter and therefore neutral under the flavor symmetry. Nevertheless, the correction identification will be conjectured in the next sub-section to be $\beta_2 = i(\nu + \theta/2) + 3Q/2$.

4.2 Conjectural dictionary between 4d and 2d parameters

As seen in the previous sub-section, the strong coupling expansions of the (A_1, D_4) and (A_1, A_3) partition functions are invariant under the action of the Weyl group of the flavor symmetries. This Weyl group symmetry arises if the parameter identifications, (3.25) and (3.51), are modified by $\mathcal{O}(Q)$ corrections as in (4.4) and (4.6), respectively. This means that there is generally an $\mathcal{O}(Q)$ -correction to the relation between the mass parameters of 4d $\mathcal{N} = 2$ theories and parameters in 2d Liouville theory.

To read off a general rule for this $\mathcal{O}(Q)$ correction, note first that α and β_0 in (4.4) are regarded as the Liouville momenta of a rank-2 state and a rank-0 state, respectively. Indeed, α is the Liouville momentum of $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$,³¹ and β_0 is the Liouville momentum of the regular primary state $|\Delta_{\beta_0}\rangle$ that arises in the expansion of $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ in terms of generalized descendants.³² Similarly, α in (4.6) is the Liouville momentum of the rank-3 state $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$. More generally, as reviewed in section 2, the parameters α and β_k of $|I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ are respectively regarded as the Liouville momenta of rank- n and rank- k states, in the expansion in terms of generalized descendants. This follows from the general ansatz (2.12) for $|I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ proposed in [31].

³¹Recall here that we call α in (2.9) the “Liouville momentum” of the irregular state of rank n .

³²Recall that a rank-zero state is identified as a regular primary state.

With this correspondence in mind, we now see that the first two equations of (4.4) and the first equation of (4.6) satisfy the following rule: *when the Liouville momentum α of a rank- n state is associated with a four-dimensional mass parameter (or the vev of a Coulomb branch parameter) m , the precise relation between them is given by*

$$\alpha = m + \frac{n+1}{2}Q. \quad (4.7)$$

Here m is a mass parameter or the vev of a (dual) Coulomb branch operator, depending on whether the corresponding symmetry is gauged.

Note that the above rule (4.7) is a natural generalization of the original AGT correspondence involving only regular primary states [28]. In [28], it was shown that the Liouville momentum γ of a regular primary state is related to a mass parameter m of the four-dimensional theory by $\gamma = m + Q/2$. Since the regular state is regarded as the rank-zero state, this is a special case of (4.7). The factor $\frac{n+1}{2}$ in (4.7) is understood as follows. Recall that the irregular state of rank n is constructed by taking a colliding limit of $(n+1)$ regular primary states. As reviewed in section 2, the Liouville momentum α of the resulting rank- n state $|I^{(n)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$ is the sum of the Liouville momenta of the original $(n+1)$ regular primary states, say $\gamma_0, \dots, \gamma_n$. On the other hand, the original AGT correspondence implies that the i -th momentum γ_i is related to a four-dimensional mass parameter m_i by $\gamma_i = m_i + Q/2$. Combining these two, we see that

$$\alpha = \sum_{i=0}^n \gamma_i = \left(\sum_{i=0}^n m_i \right) + \frac{n+1}{2}Q. \quad (4.8)$$

By identifying $\sum_{i=0}^n m_i$ as the 4d mass parameter m corresponding to the Liouville momentum α , we see that (4.8) is precisely identical to the rule (4.7).

Given the above interpretation of (4.7), we now conjecture that the rule (4.7) generally applies to the Liouville momentum α of any Liouville irregular state of integer rank n and the corresponding 4d mass parameter m .³³ This particularly implies that the $\mathcal{O}(Q)$ correction to β_1 in (4.4) is fixed by

$$\beta_1 = \frac{i(\theta_s + \theta_t - 3\nu)}{3} + Q, \quad (4.9)$$

and that to β_2 in (4.6) is fixed by

$$\beta_2 = i \left(\nu + \frac{\theta}{2} \right) + \frac{3}{2}Q. \quad (4.10)$$

Below, we rewrite the strong coupling expansions of the (A_1, D_4) and (A_1, A_3) partition functions using (4.4) and (4.6) restricted by (4.9) and (4.10).

³³There are also irregular states of half-integer rank. They are not constructed in a colliding limit of regular primary states, and moreover they have no parameter corresponding to a 4d mass parameter.

4.2.1 (A_1, D_4) partition function

We first study the (A_1, D_4) partition function. Using (4.4) restricted by (4.9), we see that the coefficients D_i in the strong coupling expansion (3.24) are rewritten as

$$D_1 = \left(3\nu^3 - 2\nu(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - \frac{2}{9}(2\theta_s - \theta_t)(\theta_s + \theta_t)(\theta_s - 2\theta_t) \right) - \frac{3\nu}{4}Q^2, \quad (4.11)$$

$$\begin{aligned} D_2 = & \frac{9\nu^6}{2} + \nu^4 \left(\frac{105}{16} - 6(\theta_s^2 - \theta_s\theta_t + \theta_t^2) \right) - \frac{2}{3}\nu^3(\theta_s - 2\theta_t)(2\theta_s - \theta_t)(\theta_s + \theta_t) \\ & + \nu^2 \left(\frac{1}{2}(\theta_s^2 - \theta_s\theta_t + \theta_t^2)(4(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 11) + \frac{3}{16} \right) \\ & + \frac{\nu}{9}(2\theta_s - \theta_t)(\theta_s + \theta_t)(\theta_s - 2\theta_t)(4(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 7) \\ & + \frac{1}{324} \left(32\theta_s^6 - 96\theta_s^5\theta_t + \theta_s^4(36 - 24\theta_t^2) + 8\theta_s^3\theta_t(26\theta_t^2 - 9) - 3\theta_s^2(8\theta_t^4 - 36\theta_t^2 + 9) \right. \\ & \quad \left. - 3\theta_s\theta_t(32\theta_t^4 + 24\theta_t^2 - 9) + \theta_t^2(32\theta_t^4 + 36\theta_t^2 - 27) \right) \\ & + \frac{1}{192} \left(-432\nu^4 + 18\nu^2(16(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 37) + 32\nu(2\theta_s - \theta_t)(\theta_s + \theta_t)(\theta_s - 2\theta_t) \right. \\ & \quad \left. + 104(\theta_s^2 - \theta_s\theta_t + \theta_t^2) - 3 \right) Q^2 + \frac{1}{256}(72\nu^2 + 25)Q^4. \end{aligned} \quad (4.12)$$

Note that they are invariant under the action of S_3 generated by (4.1) and (4.2), as expected. Moreover, they are also invariant under $Q \rightarrow -Q$, or equivalently $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$. We have checked this invariance up to D_4 . We will briefly comment on this symmetry in the next section.

4.2.2 (A_1, A_3) partition function

Let us turn to the (A_1, A_3) partition function. Using (4.6) restricted by (4.10), the coefficients D_k in the strong coupling expansion (3.50) are rewritten as

$$D_1 = -\frac{i\nu(68\nu^2 - 9\theta^2 + 2)}{36} + \frac{19}{36}i\nu Q^2, \quad (4.13)$$

$$\begin{aligned} D_2 = & -\frac{289}{162}\nu^6 + \frac{153\theta^2 - 1159}{324}\nu^4 - \left(\frac{\theta^4}{32} - \frac{11\theta^2}{18} + \frac{271}{648} \right) \nu^2 - \frac{\theta^2(11\theta^2 - 68)}{1728} \\ & + \left(\frac{323}{324}\nu^4 - \left(\frac{19\theta^2}{144} - \frac{349}{162} \right) \nu^2 - \frac{71\theta^2}{864} + \frac{17}{432} \right) Q^2 - \left(\frac{361}{2592}\nu^2 + \frac{131}{1728} \right) Q^4, \end{aligned} \quad (4.14)$$

$$\begin{aligned} D_3 = & \frac{4913i}{4374}\nu^9 - \frac{17i(153\theta^2 - 2284)}{5832}\nu^7 + \frac{i(1377\theta^4 - 47178\theta^2 + 279464)}{23328}\nu^5 \\ & - \frac{i(9\theta^2(81\theta^4 - 5072\theta^2 + 77876) - 899576)}{279936}\nu^3 - \frac{i(99\theta^6 - 4270\theta^4 + 28504\theta^2 - 3360)}{62208}\nu \\ & + \left(-\frac{5491i\nu^7}{5832} + \frac{i(2907\theta^2 - 68839)}{11664}\nu^5 - \frac{i(1539\theta^4 - 94152\theta^2 + 1161988)}{93312}\nu^3 \right. \\ & \quad \left. - \frac{i(1487\theta^4 - 61300\theta^2 + 66264)}{62208}\nu \right) Q^2 + \left(\frac{6137i}{23328}\nu^5 - \frac{i(3249\theta^2 - 118736)}{93312}\nu^3 \right. \\ & \quad \left. - \frac{i(3877\theta^2 - 92390)}{62208}\nu \right) Q^4 + \left(-\frac{6859i\nu^3}{279936} - \frac{2489i\nu}{62208} \right) Q^6. \end{aligned} \quad (4.15)$$

These coefficients are invariant under the S_2 -action (4.5), as expected. Moreover, we see that they are also invariant under $Q \rightarrow -Q$, or equivalently $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$. We will briefly comment on this symmetry in the next section.

5 Conclusions and discussions

In this paper, we have evaluated Liouville irregular states of rank-two and three, based on an ansatz proposed in [31]. Using these two irregular states, we have computed Liouville irregular conformal blocks corresponding to the partition functions of the (A_1, A_3) and (A_1, D_4) theories for general Ω -background parameters ϵ_1 and ϵ_2 . In the limit $\epsilon_1 + \epsilon_2 \rightarrow 0$, our result correctly reproduces the strong coupling expansions of the partition functions obtained from the tau-functions of Painlevé II and IV. This confirms that the ansatz proposed in [31] gives consistent results with [24–26]. Moreover, we have shown that the asymptotic expansion of an irregular state proposed in [31] corresponds to the strong coupling expansion of partition functions studied in [15]. In addition, our result on (A_1, A_3) implies that the irregular conformal block with one irregular singularity of rank 3 is also related to the tau-function of Painlevé II, in a similar way as that with one irregular singularity of rank $\frac{3}{2}$ and a regular singularity [26].

We have also shown that our partition functions are invariant under the action of the Weyl group of the flavor symmetry when the Liouville momentum of irregular states is appropriately identified with the linear combination of a mass parameter and the Ω -background parameters. From this observation, we have conjectured a general relation (4.7) between the Liouville momentum of rank- n irregular state and the corresponding mass parameter in four dimensions. We have also given an interpretation of this conjectured relation in terms of the colliding limit of regular singularities.

With the conjectured relation (4.7), we find that the partition functions $\mathcal{Z}_{(A_1, A_3)}^{\epsilon_1, \epsilon_2}$ and $\mathcal{Z}_{(A_1, D_4)}^{\epsilon_1, \epsilon_2}$ are invariant under $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$ with masses and the vev of Coulomb branch operators fixed. This invariance is consistent with quantum periods of Argyres-Douglas theories recently evaluated in [22, 23, 73]. The quantum periods are $a_I \equiv \oint_{A_I} \lambda$ and $a_{DI} \equiv \oint_{B_I} \lambda$ deformed by the Ω -background $(\epsilon_1, \epsilon_2) = (\hbar, 0)$, where A_I and B_I are canonical 1-cycles of the Seiberg-Witten curve, and λ is the Ω -deformed Seiberg-Witten 1-form. These periods are related to the deformed prepotential $\mathcal{F} \equiv \lim_{(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)} (-\epsilon_1 \epsilon_2 \log \mathcal{Z}^{\epsilon_1, \epsilon_2})$ by $a_D = \partial \mathcal{F} / \partial a$. As shown in [22, 23], the $\hbar$ -expansion of these quantum periods for (A_1, A_r) and (A_1, D_r) theories have only *even* powers of $\hbar$.³⁴ This means that, at least in the limit of $\epsilon_2 \rightarrow 0$, the quantity $\epsilon_1 \epsilon_2 \log \mathcal{Z}^{\epsilon_1, \epsilon_2}$ is invariant under $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$. Our result suggests that this invariance extends to the case of $\epsilon_1, \epsilon_2 \neq 0$. We leave a detailed study of this symmetry for future work.

³⁴In [22], the (A_1, A_3) and (A_1, D_4) theories are realized at the most singular point on the Coulomb branch of $SU(2)$ gauge theory with two and three flavors, respectively. Its generalization to the whole (A_1, A_r) and (A_1, D_r) theories was carefully studied in [23]. See [73] for discussions around the monopole point on the Coulomb branch.

Acknowledgments

The authors thank Katsushi Ito, Yusuke Kimura, Kazunobu Maruyoshi, Takafumi Okubo, Makoto Sakaguchi, Sakura Schäfer-Nameki and Yuji Sugawara for various discussions. T.N. particularly thanks Matthew Buican for various illuminating dicussions in many collaborations on related topics. The authors also thank the organizers of the international workshop “KEK Theory workshop 2018,” where they had many useful discussions on this topic. The work of T.N. is partially supported by JSPS Grant-in-Aid for Early-Career Scientists 18K13547. The work of T.U. is supported by Grant-in-Aid for JSPS Research Fellows 19J11212.

A Computations of generalized descendants

In this appendix, we explain a little more about the computations of generalized descendants involved in irregular states of rank-two and three, following [31].

A.1 Rank-two state $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$

Let us start with the rank-two irregular state $|I^{(2)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$, where $\mathbf{c} = (c_1, c_2)$ and $\boldsymbol{\beta} = (\beta_0, \beta_1)$. According to [31], this state is expected to be expanded as (3.9) in terms of generalized descendants $|I_k^{(1)}\rangle$ for $k = 1, 2, \dots$. These states are fixed order by order so that (3.9) satisfies (3.2)–(3.6). Indeed, substituting (3.9) in (3.2)–(3.6) leads to the following equations for each k :

$$L_{n>4}|I_k^{(1)}\rangle = 0, \quad (\text{A.1})$$

$$L_4|I_k^{(1)}\rangle = -|I_{k-2}^{(1)}\rangle, \quad (\text{A.2})$$

$$L_3|I_k^{(1)}\rangle = -2c_1|I_{k-1}^{(1)}\rangle, \quad (\text{A.3})$$

$$(L_2 + c_1^2)|I_k^{(1)}\rangle = -(2\alpha - 3Q)|I_{k-1}^{(1)}\rangle, \quad (\text{A.4})$$

$$(L_1 + 2c_1(\beta - Q))|I_k^{(1)}\rangle = \left(\frac{\partial}{\partial c_1} + \frac{\nu_1}{c_1}\right)|I_{k-1}^{(1)}\rangle, \quad (\text{A.5})$$

$$L_0|I_k^{(1)}\rangle = \left(\Delta_\beta + 2k + c_1\frac{\partial}{\partial c_1}\right)|I_k^{(1)}\rangle. \quad (\text{A.6})$$

Solving these equations successively for $k = 1, 2, \dots$, we obtain $|I_k^{(1)}\rangle$ for all k . To be concrete, in evaluating $|I_k^{(1)}\rangle$, we write down the most general linear combination of level- $2k$ generalized descendants, and then fix their coefficients by demanding that the equations (A.1)–(A.5) are satisfied. Note that L_{-1} , $1/c_1$ and ∂_{c_1} are all regarded as level-one. The results for $k = 1$ and $k = 2$ are shown in (3.13) and (3.14). In appendix B, we show the coefficients D_k appearing in (3.18) for $k = 1, \dots, 4$. For computing D_k for $k = 1, \dots, 4$, we need to evaluate $|I_k^{(1)}\rangle$ for $k = 1, \dots, 5$.

A.2 Rank-three state $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$

Let us turn to the rank-three irregular state $|I^{(3)}(\alpha, \mathbf{c}; \boldsymbol{\beta})\rangle$, where $\mathbf{c} = (c_1, c_2, c_3)$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$. As in section A.1, we fix the generalized descendants $|I_k^{(2)}\rangle$ in the

ansatz (3.40) by demanding that (3.40) satisfies (3.30)–(3.36). To that end, we substitute (3.40) in (3.30)–(3.36) to obtain

$$L_{n>6}|I_k^{(2)}\rangle = 0, \quad (\text{A.7})$$

$$L_6|I_k^{(2)}\rangle = -|I_{k-2}^{(2)}\rangle \quad (\text{A.8})$$

$$L_5|I_k^{(2)}\rangle = -2c_2|I_{k-2}^{(2)}\rangle, \quad (\text{A.9})$$

$$(L_4 + c_2^2)|I_k^{(2)}\rangle = -2c_1|I_{k-1}^{(2)}\rangle, \quad (\text{A.10})$$

$$(L_3 + 2c_1c_2)|I_k^{(2)}\rangle = -2(\alpha - 2Q)|I_{k-1}^{(2)}\rangle, \quad (\text{A.11})$$

$$(L_2 + c_2(2\beta - 3Q) + c_1^2)|I_k^{(2)}\rangle = \left(\partial_{c_1} - \frac{2c_1}{c_2}(\alpha - \beta)\right)|I_{k-1}^{(2)}\rangle, \quad (\text{A.12})$$

$$(L_1 + 2c_1(\beta - Q) - c_2\partial_{c_1})|I_k^{(2)}\rangle = \left(2\partial_{c_2} + \frac{2c_1^2}{c_2^2}(\alpha - \beta) + \frac{2\rho_2}{c_2}\right)|I_{k-1}^{(2)}\rangle, \quad (\text{A.13})$$

$$L_0|I_k^{(2)}\rangle = (\Delta_\beta + 3k + c_1\partial_{c_1} + 2c_2\partial_{c_2})|I_k^{(2)}\rangle. \quad (\text{A.14})$$

We solve these equations successively for $k = 1, 2, \dots$. The strategy is again to write down the most general linear combination of level- $3k$ generalized descendants for $|I_k^{(2)}\rangle$, and fix their coefficients so that the above equations are satisfied. Note here that L_{-1} , $1/c_1$ and ∂_{c_1} are regarded as level-one, and $1/c_2$ and ∂_{c_2} are regarded as level-two.

As discussed in the 3.2, we only need to evaluate the leading contributions in the $c_1 \rightarrow 0$ limit. Below, we write down expressions for the first several $|I_k^{(2)}\rangle$:

$$|I_0^{(2)}\rangle = |I^{(2)}(\beta, c_1, c_2)\rangle, \quad (\text{A.15})$$

$$|I_1^{(2)}\rangle = \left(\frac{1}{3c_2}L_{-1} + \frac{3\alpha - 4\beta}{3c_2}\partial_{c_1} - \frac{2c_1}{3c_2}\partial_{c_2} - \frac{2(\rho_2 - \beta^2 + \beta\alpha + Q(\beta - \alpha))c_1}{c_2^2} + \frac{2(\beta - \alpha)c_1^3}{3c_2^3}\right)|I^{(2)}(\beta, c_1, c_2)\rangle, \quad (\text{A.16})$$

$$|I_2^{(2)}\rangle = \left(\frac{\rho_4}{c_2^3} - \frac{1}{6c_2^2}L_{-2} + \frac{\frac{Q}{6} + \frac{19}{9}\beta - \frac{5\alpha}{3}}{c_2^2}\partial_{c_2} + \frac{1}{18c_2^2}L_{-1}^2 + \frac{(-4\beta + 3\alpha)}{9c_2^2}L_{-1}\partial_{c_1} + \frac{-3 + 64\beta^2 - 96\beta\alpha + 36\alpha^2}{72c_2^2}\partial_{c_1}^2 + \mathcal{O}(c_1)\right)|I^{(2)}(\beta, c_1, c_2)\rangle, \quad (\text{A.17})$$

where ρ_4 is determined as

$$\rho_4 = \frac{1}{12}(\beta_3 - \alpha)(4\alpha^2 - 26\alpha\beta_3 + 34\beta_3^2 + 35Q^2 + 27\alpha Q - 69\beta_3 Q - 1). \quad (\text{A.18})$$

In section 3.2, we write down expressions for $D_1, \dots, D_3$ appearing in the expansion (3.43). To evaluate these three coefficients, we need to evaluate $|I_k^{(2)}\rangle$ for $k = 1, \dots, 6$, up to sub-leading terms in the limit $c_1 \rightarrow 0$.

B More coefficients for the (A_1, D_4) partition function

We here show the coefficients D_i in (3.26) up to $i = 4$ using (4.4) and (4.9). These coefficients are written in terms of symmetric polynomials in the variables X_1, X_2 and X_3 ,

where

$$\begin{aligned} X_1 &= \theta_s - 2\theta_t, \\ X_2 &= \theta_t - 2\theta_s, \\ X_3 &= \theta_s + \theta_t. \end{aligned} \tag{B.1}$$

This reflects the fact that the mass parameters are permuted by the action of the Weyl group of the flavor $SU(3)$ symmetry of the (A_1, D_4) theory, as discussed in section 4. Indeed, these three variables are permuted under the S_3 Weyl transformations (4.1) and (4.2). The elementary symmetric polynomials s_1, s_2 and s_3 of these variables are given by

$$\begin{aligned} s_1 &= X_1 + X_2 + X_3 = 0, \\ s_2 &= X_1X_2 + X_2X_3 + X_3X_1 = -3(\theta_s^2 - \theta_s\theta_t + \theta_t^2), \\ s_3 &= X_1X_2X_3 = (\theta_s - 2\theta_t)(\theta_t - 2\theta_s)(\theta_s + \theta_t). \end{aligned} \tag{B.2}$$

Since the partition function is supposed to be invariant under the action of the Weyl group, the coefficients D_i of its strong coupling expansion are expected to be written in terms of s_2 and s_3 . Indeed, using (B.2), we may rewrite D_1 and D_2 as

$$D_1 = 3\nu^3 + \frac{2\nu s_2}{3} + \frac{2s_3}{9} - \frac{3\nu}{4}Q^2, \tag{B.3}$$

$$\begin{aligned} D_2 &= \frac{9\nu^6}{2} + \nu^4 \left(2s_2 + \frac{105}{16} \right) + \frac{2\nu^3 s_3}{3} + \frac{\nu^2}{144} (8s_2(4s_2 + 33) + 27) + \frac{\nu}{27} (4s_2 + 21)s_3 \\ &\quad + \frac{1}{324} (s_2(4s_2 + 9) + 8s_3^2) \\ &\quad + \left(-\frac{9\nu^4}{4} + \nu^2 \left(-\frac{s_2}{2} - \frac{111}{32} \right) - \frac{\nu s_3}{6} + \left(-\frac{13s_2}{72} - \frac{1}{64} \right) \right) Q^2 \\ &\quad + \left(\frac{9\nu^2}{32} + \frac{25}{256} \right) Q^4. \end{aligned} \tag{B.4}$$

Moreover, D_3 and D_4 are also evaluated as

$$\begin{aligned} D_3 &= -\frac{9\nu^9}{2} + \nu^7 \left(-3s_2 - \frac{315}{16} \right) - \nu^6 s_3 + \nu^5 \left(-\frac{2s_2^2}{3} - \frac{79s_2}{8} - \frac{411}{16} \right) - \frac{\nu^4}{72} (32s_2 + 273)s_3 \\ &\quad + \frac{\nu^3}{648} (-32s_2^3 - 816s_2^2 - 5535s_2 - 48s_3^2 - 1701) - \frac{\nu^2}{648} (32s_2^2 + 600s_2 + 2475)s_3 \\ &\quad + \frac{\nu}{486} (-4s_2^3 - 105s_2^2 - s_2(8s_3^2 + 297) - 84s_3^2) - \frac{1}{4374} s_3 (12s_2^2 + 351s_2 + 8s_3^2 + 972) \\ &\quad + \left(\frac{27\nu^7}{8} + \nu^5 \left(\frac{3s_2}{2} + \frac{981}{64} \right) + \frac{\nu^4 s_3}{2} + \frac{1}{48} \nu^3 (8s_2^2 + 203s_2 + 1134) + \nu^2 \left(\frac{s_2}{9} + \frac{65}{48} \right) \right. \\ &\quad \left. + \frac{1}{864} \nu (112s_2^2 + 2307s_2 + 16s_3^2 + 621) s_3 + \frac{1}{2592} (104s_2 + 1305)s_3 \right) Q^2 \\ &\quad + \left(-\frac{27\nu^5}{32} - \frac{3}{256} \nu^3 (16s_2 + 247) - \frac{\nu^2 s_3}{16} + \frac{\nu}{768} (-154s_2 - 1695) - \frac{25s_3}{1152} \right) Q^4 \\ &\quad + \left(\frac{9\nu^3}{128} + \frac{75\nu}{1024} \right) Q^6, \end{aligned} \tag{B.5}$$

$$\begin{aligned}
 D_4 = & -\frac{27\nu^{12}}{8} + \nu^{10} \left(-3s_2 - \frac{945}{32} \right) - \nu^9 s_3 + \nu^8 \left(-s_2^2 - \frac{171s_2}{8} - \frac{50049}{512} \right) - \frac{\nu^7}{24} (16s_2 + 189) s_3 \\
 & + \nu^6 \left(-\frac{4s_2^3}{27} - \frac{373s_2^2}{72} - \frac{1737s_2}{32} - \frac{s_3^2}{9} - \frac{34335}{256} \right) - \frac{\nu^5}{432} (64s_2^2 + 1620s_2 + 9567) s_3 \\
 & + \nu^4 \left(-\frac{2s_2^4}{243} - \frac{35s_2^3}{81} - \frac{1153s_2^2}{144} + \left(-\frac{4s_3^2}{81} - \frac{3323}{64} \right) s_2 - \frac{49s_3^2}{72} - \frac{15381}{512} \right) \\
 & - \frac{\nu^3}{11664} s_3 (128s_2^3 + 5280s_2^2 + 70740s_2 + 64s_3^2 + 273861) \\
 & + \frac{\nu^2}{93312} \left(-256s_2^4 - 14976s_2^3 - 8(64s_3^2 + 30969) s_2^2 - 18(832s_3^2 + 45063) s_2 \right. \\
 & \quad \left. - 9(11888s_3^2 + 5103) \right) \\
 & - \frac{\nu}{52488} s_3 (96s_2^3 + 5616s_2^2 + 64s_3^2 s_2 + 77274s_2 + 1008s_3^2 + 207765) \\
 & + \frac{1}{629856} \left(-48s_2^4 - 2808s_2^3 - 3(64s_3^2 + 8829) s_2^2 - 27(400s_3^2 + 1701) s_2 \right. \\
 & \quad \left. - 16s_3^2(4s_3^2 + 4617) \right) \\
 & + \left(\frac{27\nu^{10}}{8} + \frac{9\nu^8}{8} (2s_2 + 27) + \frac{3s_3\nu^7}{4} + \frac{\nu^6}{512} (256s_2^2 + 7760s_2 + 57555) + \frac{\nu^5}{96} (32s_2 + 495) s_3 \right. \\
 & \quad + \nu^4 \left(\frac{s_2^3}{27} + \frac{299s_2^2}{144} + \frac{4803s_2}{128} + \frac{s_3^2}{18} + \frac{193455}{1024} \right) + \frac{\nu^3}{864} (32s_2^2 + 1148s_2 + 10611) s_3 \\
 & \quad + \frac{\nu^2}{82944} (3840s_2^3 + 190656s_2^2 + 8(128s_3^2 + 338931) s_2 + 9(1984s_3^2 + 168183)) \\
 & \quad + \frac{\nu}{46656} s_3 (1344s_2^2 + 52380s_2 + 64s_3^2 + 426951) \\
 & \quad \left. + \frac{1}{373248} (832s_2^3 + 85752s_2^2 + 1664s_3^2 s_2 + 387018s_2 + 41616s_3^2 + 15309) \right) Q^2 \\
 & + \left(-\frac{81\nu^8}{64} - \frac{9\nu^6}{256} (16s_2 + 287) - \frac{3\nu^5 s_3}{16} + \frac{\nu^4}{4096} (-256s_2^2 - 11680s_2 - 126603) \right. \\
 & \quad - \frac{\nu^3}{384} (16s_2 + 331) s_3 + \frac{\nu^2}{36864} (-4256s_2^2 - 157176s_2 - 256s_3^2 - 1416717) \\
 & \quad \left. - \frac{\nu}{6912} (308s_2 + 6507) s_3 + \frac{1}{663552} (-11616s_2^2 - 873720s_2 - 1600s_3^2 - 378189) \right) Q^4 \\
 & + \left(\frac{27\nu^6}{128} + \frac{3\nu^4}{128} (2s_2 + 51) + \frac{\nu^3 s_3}{64} + \frac{3\nu^2}{8192} (272s_2 + 5433) \right. \\
 & \quad \left. + \frac{25\nu s_3}{1536} + \frac{1}{147456} (2600s_2 + 98613) \right) Q^6 \\
 & + \left(-\frac{27\nu^4}{2048} - \frac{225\nu^2}{8192} - \frac{625}{131072} \right) Q^8.
 \end{aligned} \tag{B.6}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] P.C. Argyres and M.R. Douglas, *New phenomena in SU(3) supersymmetric gauge theory*, *Nucl. Phys. B* **448** (1995) 93 [[hep-th/9505062](#)] [[INSPIRE](#)].
- [2] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $\mathcal{N} = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **461** (1996) 71 [[hep-th/9511154](#)] [[INSPIRE](#)].
- [3] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $\mathcal{N} = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **471** (1996) 430 [[hep-th/9603002](#)] [[INSPIRE](#)].
- [4] K. Maruyoshi and J. Song, *Enhancement of Supersymmetry via Renormalization Group Flow and the Superconformal Index*, *Phys. Rev. Lett.* **118** (2017) 151602 [[arXiv:1606.05632](#)] [[INSPIRE](#)].
- [5] K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs*, *JHEP* **02** (2017) 075 [[arXiv:1607.04281](#)] [[INSPIRE](#)].
- [6] P. Agarwal, K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ Deformations and RG flows of $\mathcal{N} = 2$ SCFTs, part II: non-principal deformations*, *JHEP* **12** (2016) 103 [[arXiv:1610.05311](#)] [[INSPIRE](#)].
- [7] K. Maruyoshi, E. Nardoni and J. Song, *Landscape of Simple Superconformal Field Theories in 4d*, *Phys. Rev. Lett.* **122** (2019) 121601 [[arXiv:1806.08353](#)] [[INSPIRE](#)].
- [8] P. Agarwal, A. Sciarappa and J. Song, *$\mathcal{N} = 1$ Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 211 [[arXiv:1707.04751](#)] [[INSPIRE](#)].
- [9] S. Benvenuti and S. Giacomelli, *Abelianization and sequential confinement in $2 + 1$ dimensions*, *JHEP* **10** (2017) 173 [[arXiv:1706.04949](#)] [[INSPIRE](#)].
- [10] S. Benvenuti and S. Giacomelli, *Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 106 [[arXiv:1707.05113](#)] [[INSPIRE](#)].
- [11] S. Giacomelli, *RG flows with supersymmetry enhancement and geometric engineering*, *JHEP* **06** (2018) 156 [[arXiv:1710.06469](#)] [[INSPIRE](#)].
- [12] S. Giacomelli, *Infrared enhancement of supersymmetry in four dimensions*, *JHEP* **10** (2018) 041 [[arXiv:1808.00592](#)] [[INSPIRE](#)].
- [13] F. Carta, S. Giacomelli and R. Savelli, *SUSY enhancement from T-branes*, *JHEP* **12** (2018) 127 [[arXiv:1809.04906](#)] [[INSPIRE](#)].
- [14] M.-x. Huang and A. Klemm, *Holomorphicity and Modularity in Seiberg-Witten Theories with Matter*, *JHEP* **07** (2010) 083 [[arXiv:0902.1325](#)] [[INSPIRE](#)].
- [15] G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa and A. Tanzini, *On Painlevé/gauge theory correspondence*, [arXiv:1612.06235](#) [[INSPIRE](#)].
- [16] O. Gamayun, N. Iorgov and O. Lisovyy, *Conformal field theory of Painlevé VI*, *JHEP* **10** (2012) 038 [Erratum *ibid.* **10** (2012) 183] [[arXiv:1207.0787](#)] [[INSPIRE](#)].
- [17] O. Gamayun, N. Iorgov and O. Lisovyy, *How instanton combinatorics solves Painlevé VI, V and IIIs*, *J. Phys. A* **46** (2013) 335203 [[arXiv:1302.1832](#)] [[INSPIRE](#)].

- [18] N. Iorgov, O. Lisovyy and J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*, *Commun. Math. Phys.* **336** (2015) 671 [[arXiv:1401.6104](#)] [[INSPIRE](#)].
- [19] M.A. Bershtein and A.I. Shchepochkin, *Bilinear equations on Painlevé τ functions from CFT*, *Commun. Math. Phys.* **339** (2015) 1021 [[arXiv:1406.3008](#)] [[INSPIRE](#)].
- [20] P. Gavrylenko and O. Lisovyy, *Fredholm Determinant and Nekrasov Sum Representations of Isomonodromic Tau Functions*, *Commun. Math. Phys.* **363** (2018) 1 [[arXiv:1608.00958](#)] [[INSPIRE](#)].
- [21] K. Sakai, *BPS index and 4d $\mathcal{N} = 2$ superconformal field theories*, *JHEP* **07** (2016) 046 [[arXiv:1603.09108](#)] [[INSPIRE](#)].
- [22] K. Ito and T. Okubo, *Quantum periods for $\mathcal{N} = 2$ SU(2) SQCD around the superconformal point*, *Nucl. Phys. B* **934** (2018) 356 [[arXiv:1804.04815](#)] [[INSPIRE](#)].
- [23] K. Ito, S. Koizumi and T. Okubo, *Quantum Seiberg-Witten curve and Universality in Argyres-Douglas theories*, *Phys. Lett. B* **792** (2019) 29 [[arXiv:1903.00168](#)] [[INSPIRE](#)].
- [24] H. Nagoya, *Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations*, *J. Math. Phys.* **56** (2015) 123505 [[arXiv:1505.02398](#)] [[INSPIRE](#)].
- [25] H. Nagoya, *Conformal blocks and Painlevé functions*, [arXiv:1611.08971](#) [[INSPIRE](#)].
- [26] H. Nagoya, *Remarks on irregular conformal blocks and Painlevé III and II tau functions*, [arXiv:1804.04782](#) [[INSPIRE](#)].
- [27] O. Lisovyy, H. Nagoya and J. Roussillon, *Irregular conformal blocks and connection formulae for Painlevé V functions*, *J. Math. Phys.* **59** (2018) 091409 [[arXiv:1806.08344](#)] [[INSPIRE](#)].
- [28] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[INSPIRE](#)].
- [29] D. Gaiotto, *Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks*, *J. Phys. Conf. Ser.* **462** (2013) 012014 [[arXiv:0908.0307](#)] [[INSPIRE](#)].
- [30] G. Bonelli, K. Maruyoshi and A. Tanzini, *Wild Quiver Gauge Theories*, *JHEP* **02** (2012) 031 [[arXiv:1112.1691](#)] [[INSPIRE](#)].
- [31] D. Gaiotto and J. Teschner, *Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I*, *JHEP* **12** (2012) 050 [[arXiv:1203.1052](#)] [[INSPIRE](#)].
- [32] H. Kanno, K. Maruyoshi, S. Shiba and M. Taki, *W_3 irregular states and isolated $\mathcal{N} = 2$ superconformal field theories*, *JHEP* **03** (2013) 147 [[arXiv:1301.0721](#)] [[INSPIRE](#)].
- [33] Y. Matsuo, C. Rim and H. Zhang, *Construction of Gaiotto states with fundamental multiplets through Degenerate DAHA*, *JHEP* **09** (2014) 028 [[arXiv:1405.3141](#)] [[INSPIRE](#)].
- [34] D. Polyakov and C. Rim, *Irregular Vertex Operators for Irregular Conformal Blocks*, *Phys. Rev. D* **93** (2016) 106002 [[arXiv:1601.07756](#)] [[INSPIRE](#)].
- [35] D. Polyakov and C. Rim, *Vertex Operators for Irregular Conformal Blocks: Supersymmetric Case*, *Phys. Rev. D* **94** (2016) 086011 [[arXiv:1604.08741](#)] [[INSPIRE](#)].
- [36] C. Rim and H. Zhang, *Nekrasov and Argyres-Douglas theories in spherical Hecke algebra representation*, *Nucl. Phys. B* **919** (2017) 182 [[arXiv:1608.05027](#)] [[INSPIRE](#)].
- [37] A. Grassi and J. Gu, *Argyres-Douglas theories, Painlevé II and quantum mechanics*, *JHEP* **02** (2019) 060 [[arXiv:1803.02320](#)] [[INSPIRE](#)].

- [38] H. Itoyama, T. Oota and K. Yano, *Discrete Painleve system and the double scaling limit of the matrix model for irregular conformal block and gauge theory*, *Phys. Lett. B* **789** (2019) 605 [[arXiv:1805.05057](#)] [[INSPIRE](#)].
- [39] H. Itoyama, T. Oota and K. Yano, *Discrete Painleve system for the partition function of $N_f = 2$ SU(2) supersymmetric gauge theory and its double scaling limit*, [arXiv:1812.00811](#) [[INSPIRE](#)].
- [40] H. Itoyama, K. Maruyoshi and T. Oota, *The Quiver Matrix Model and 2d-4d Conformal Connection*, *Prog. Theor. Phys.* **123** (2010) 957 [[arXiv:0911.4244](#)] [[INSPIRE](#)].
- [41] T. Eguchi and K. Maruyoshi, *Penner Type Matrix Model and Seiberg-Witten Theory*, *JHEP* **02** (2010) 022 [[arXiv:0911.4797](#)] [[INSPIRE](#)].
- [42] R. Schiappa and N. Wyllard, *An A_R threesome: Matrix models, 2d CFTs and 4d $\mathcal{N} = 2$ gauge theories*, *J. Math. Phys.* **51** (2010) 082304 [[arXiv:0911.5337](#)] [[INSPIRE](#)].
- [43] A. Mironov, A. Morozov and S. Shakirov, *Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions*, *JHEP* **02** (2010) 030 [[arXiv:0911.5721](#)] [[INSPIRE](#)].
- [44] M. Fujita, Y. Hatsuda and T.-S. Tai, *Genus-one correction to asymptotically free Seiberg-Witten prepotential from Dijkgraaf-Vafa matrix model*, *JHEP* **03** (2010) 046 [[arXiv:0912.2988](#)] [[INSPIRE](#)].
- [45] H. Itoyama and T. Oota, *Method of Generating q -Expansion Coefficients for Conformal Block and $\mathcal{N} = 2$ Nekrasov Function by beta-Deformed Matrix Model*, *Nucl. Phys. B* **838** (2010) 298 [[arXiv:1003.2929](#)] [[INSPIRE](#)].
- [46] A. Mironov, A. Morozov and A. Morozov, *Conformal blocks and generalized Selberg integrals*, *Nucl. Phys. B* **843** (2011) 534 [[arXiv:1003.5752](#)] [[INSPIRE](#)].
- [47] A. Morozov and S. Shakirov, *The matrix model version of AGT conjecture and CIV-DV prepotential*, *JHEP* **08** (2010) 066 [[arXiv:1004.2917](#)] [[INSPIRE](#)].
- [48] T. Eguchi and K. Maruyoshi, *Seiberg-Witten theory, matrix model and AGT relation*, *JHEP* **07** (2010) 081 [[arXiv:1006.0828](#)] [[INSPIRE](#)].
- [49] H. Itoyama, T. Oota and N. Yonezawa, *Massive Scaling Limit of beta-Deformed Matrix Model of Selberg Type*, *Phys. Rev. D* **82** (2010) 085031 [[arXiv:1008.1861](#)] [[INSPIRE](#)].
- [50] K. Maruyoshi and F. Yagi, *Seiberg-Witten curve via generalized matrix model*, *JHEP* **01** (2011) 042 [[arXiv:1009.5553](#)] [[INSPIRE](#)].
- [51] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, *Generalized matrix models and AGT correspondence at all genera*, *JHEP* **07** (2011) 055 [[arXiv:1011.5417](#)] [[INSPIRE](#)].
- [52] H. Itoyama and N. Yonezawa, *ϵ -Corrected Seiberg-Witten Prepotential Obtained From Half Genus Expansion in beta-Deformed Matrix Model*, *Int. J. Mod. Phys. A* **26** (2011) 3439 [[arXiv:1104.2738](#)] [[INSPIRE](#)].
- [53] T. Nishinaka and C. Rim, *β -Deformed Matrix Model and Nekrasov Partition Function*, *JHEP* **02** (2012) 114 [[arXiv:1112.3545](#)] [[INSPIRE](#)].
- [54] D. Galakhov, A. Mironov and A. Morozov, *S-duality as a beta-deformed Fourier transform*, *JHEP* **08** (2012) 067 [[arXiv:1205.4998](#)] [[INSPIRE](#)].
- [55] J.-E. Bourgin, *Large N limit of beta-ensembles and deformed Seiberg-Witten relations*, *JHEP* **08** (2012) 046 [[arXiv:1206.1696](#)] [[INSPIRE](#)].

- [56] T. Nishinaka and C. Rim, *Matrix models for irregular conformal blocks and Argyres-Douglas theories*, *JHEP* **10** (2012) 138 [[arXiv:1207.4480](#)] [[INSPIRE](#)].
- [57] C. Rim, *Irregular conformal block and its matrix model*, [arXiv:1210.7925](#) [[INSPIRE](#)].
- [58] S.-K. Choi and C. Rim, *Parametric dependence of irregular conformal block*, *JHEP* **04** (2014) 106 [[arXiv:1312.5535](#)] [[INSPIRE](#)].
- [59] H. Itoyama, T. Oota and R. Yoshioka, *2d-4d Connection between q -Virasoro/ W Block at Root of Unity Limit and Instanton Partition Function on ALE Space*, *Nucl. Phys. B* **877** (2013) 506 [[arXiv:1308.2068](#)] [[INSPIRE](#)].
- [60] H. Itoyama, T. Oota and R. Yoshioka, *q -Virasoro/ W Algebra at Root of Unity and Parafermions*, *Nucl. Phys. B* **889** (2014) 25 [[arXiv:1408.4216](#)] [[INSPIRE](#)].
- [61] S.K. Choi, C. Rim and H. Zhang, *Virasoro irregular conformal block and beta deformed random matrix model*, *Phys. Lett. B* **742** (2015) 50 [[arXiv:1411.4453](#)] [[INSPIRE](#)].
- [62] C. Rim and H. Zhang, *Classical Virasoro irregular conformal block*, *JHEP* **07** (2015) 163 [[arXiv:1504.07910](#)] [[INSPIRE](#)].
- [63] S.K. Choi and C. Rim, *Irregular matrix model with $\mathcal{W}$ symmetry*, *J. Phys. A* **49** (2016) 075201 [[arXiv:1506.02421](#)] [[INSPIRE](#)].
- [64] C. Rim and H. Zhang, *Classical Virasoro irregular conformal block II*, *JHEP* **09** (2015) 097 [[arXiv:1506.03561](#)] [[INSPIRE](#)].
- [65] S.K. Choi, C. Rim and H. Zhang, *Irregular conformal block, spectral curve and flow equations*, *JHEP* **03** (2016) 118 [[arXiv:1510.09060](#)] [[INSPIRE](#)].
- [66] D. Polyakov and C. Rim, *Super-spectral curve of irregular conformal blocks*, *JHEP* **12** (2016) 004 [[arXiv:1608.04921](#)] [[INSPIRE](#)].
- [67] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [[hep-th/0206161](#)] [[INSPIRE](#)].
- [68] D. Xie, *General Argyres-Douglas Theory*, *JHEP* **01** (2013) 100 [[arXiv:1204.2270](#)] [[INSPIRE](#)].
- [69] S. Cecotti, A. Neitzke and C. Vafa, *R-Twisting and 4d/2d Correspondences*, [arXiv:1006.3435](#) [[INSPIRE](#)].
- [70] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys. B* **500** (1997) 3 [[hep-th/9703166](#)] [[INSPIRE](#)].
- [71] D. Gaiotto, *$\mathcal{N} = 2$ dualities*, *JHEP* **08** (2012) 034 [[arXiv:0904.2715](#)] [[INSPIRE](#)].
- [72] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin Systems and the WKB Approximation*, [arXiv:0907.3987](#) [[INSPIRE](#)].
- [73] K. Ito, S. Kanno and T. Okubo, *Quantum periods and prepotential in $\mathcal{N} = 2$ SU(2) SQCD*, *JHEP* **08** (2017) 065 [[arXiv:1705.09120](#)] [[INSPIRE](#)].