Note on differential operators, CHY integrands, and unifying relations for amplitudes

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ABSTRACT: An elegant unified web for amplitudes of various theories was given by Cachazo, He and Yuan in the CHY framework a few years ago. Recently, similar web has also been constructed by Cheung, Shen and Wen, which relies on a set of differential operators. In this note, by acting these differential operators on CHY-integrands systematically, we have established the relation between these two approaches. Thus, amplitudes for all theories which have CHY representations, include gravity theory, Einstein-Yang-Mills theory, Einstein-Maxwell theory, pure Yang-Mills theory, Yang-Mills-scalar theory, Born-Infeld theory, Dirac-Born-Infeld theory and its extension, bi-adjoint scalar theory, ϕ^4 theory, non-linear sigma model, as well as special Galileon theory, have been included in the unified web rooted from gravity theory.

Contents

1 Introduction

The unification of different theories is always one of interesting problems in theoretical physics. The modern researches on S-matrix have exhibited amazing structures within amplitudes of gauge and gravity theories, such as the Kawai-Lewellen-Tye (KLT) relations [1], Bern-Carrasco-Johansson (BCJ) color-kinematics duality [2–4], which are invisible in the traditional Lagrangian formulism of quantum field theory. These discoveries hint the existence of some long hidden unifying relations for on-shell amplitudes. A strong evidence for the marvelous unity among amplitudes of different theories has been spelled out in [9] by using the CHY formulae [5–9]. More explicitly, different theories are defined by different CHY-integrands, while they found that CHY-integrands for a wide range of theories can be generated from the CHY-integrand for gravity theory,[1] through the so called compactifying, squeezing as well as the generalized dimensional reduction procedures [9].

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[1]Here the gravity theory has to be understood in a generalized version, i.e., Einstein gravity theory couples to a dilaton and two-forms.

Recently, Cheung, Shen and Wen discovered similar unifying relations for on-shell tree-level amplitudes of a variety of theories from a different angle: by acting some Lorentz and gauge invariant differential operators, one can transmute the physical amplitude of a theory into the one of another theory [10]. In their unified web, amplitudes of various theories include Einstein-Yang-Mills theory, Einstein-Maxwell theory, Born-Infeld theory, Dirac-Born-Infeld theory, special Galileon theory, non-linear sigma model, as well as bi-adjoint scalar theory, can be generated by transmuting the amplitudes of gravity theory. The role of these differential operators has been understood and checked from various angels, such as several explicit examples, factorization property, double copy structure, soft behavior, etc.

Since the similar unified webs for amplitudes of various theories have been given both in [9] and [10], it is very natural to investigate the relation among these two different approaches. In this note, we will establish the exact relation through the CHY formulae [5–9]. Tree-level amplitudes in the CHY formulae are represented as integrals over auxiliary variables as

$$\mathcal{A}_n = \int d\mu_n \, \mathcal{I}^{\mathrm{CHY}} \,, \tag{1.1}$$

where the auxiliary variables are localized by constraints from the so-called scattering equations which depend on the external momenta. In this formulae, the measure part $d\mu_n$ is universal for all theories, while different theories are defined by the so called CHY-integrands $\mathcal{I}^{\mathrm{CHY}}$. Based on this fact, the basic idea of the note can be described as following. Since differential operators discussed in this note are defined through Lorentz invariants include polarization vectors of external particles such as $\epsilon_i \cdot \epsilon_j$ and $\epsilon_i \cdot k_j$, they are commutable[2] with the integral $\int d\mu_n$ over auxiliary variables. Therefore, converting an amplitude is equivalent to converting the CHY-integrand. More explicitly, if two amplitudes $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are related by an operator $\mathcal{O}$ as $\mathcal{A}_\alpha = \mathcal{O}\mathcal{A}_\beta$, analogous relation $\mathcal{I}^{\mathrm{CHY}}_\alpha = \mathcal{O}\mathcal{I}^{\mathrm{CHY}}_\beta$ for integrands must hold, and vice versa. Thus, one can derive the unifying relations systematically by acting operators on CHY-integrands.

Applying differential operators on CHY-integrands, we will re-derive all unifying relation in [10]. We will also define new operators which are composed of basic trace operators, to generate amplitudes of theories having not been mentioned in [10]. Then all amplitudes which have CHY representations in [9] can be brought into the picture of unification: they are generated from the amplitudes of gravity theory via several operators. Other relations among amplitudes indicated by these operators will also be discussed.

The remainder of this paper is organized as follows. In section 2, we give a brief introduction of the Pfaffian and the CHY formulae which are crucial for subsequent discussions. In section 3, we study the effects of three basic operators when acting them on the building blocks of CHY-integrands. Then, in section 4 we will consider the effects of operators built by these basic operators. The unified web and other relations for amplitudes will be presented in section 5. Finally, we end with a summary and discussions in section 6.

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[2]We want to remark that in [10], differential operators such as $\partial_{k_i \cdot k_j}$ have also been discussed. However, these operators will interact with scattering equations, thus we will not use them in this note.

2 Review of Pfaffian and CHY formulae

For reader's convenience, we will briefly discuss the definition of Pfaffian, and rapidly review the CHY formulae.

2.1 Definition of Pfaffian

The definition of Pfaffian is essential for the work in this note. For a $2n \times 2n$ skew symmetric matrix S, Pfaffian is defined as

$$\mathbf{Pf}S = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathbf{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}, \tag{2.1}$$

where S_{2n} is the permutation group of $2n$ elements and $\mathbf{sgn}(\sigma)$ is the signature of σ. More explicitly, let Π be the set of all partitions of $\{1, 2, \cdots, 2n\}$ into pairs without regard to the order. An element α in Π can be written as

$$\alpha = \{(i_1, j_1), (i_2, j_2), \cdots, (i_n, j_n)\}, \tag{2.2}$$

with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$. Now let

$$\pi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{pmatrix} \tag{2.3}$$

be the corresponding permutation of the partition α. If we define

$$S_\alpha = \mathbf{sgn}(\pi_\alpha)\, a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}, \tag{2.4}$$

then the Pfaffian of the matrix A is given as

$$\mathbf{Pf}S = \sum_{\alpha \in \Pi} S_\alpha. \tag{2.5}$$

Both representations (2.1) and (2.5) will be used later. From the (2.5) one can observe that in every term S_α of the Pfaffian, each number of $\{1, 2, \cdots, 2n\}$, as the subscript of the matrix element, will appear once and only once. This observation is simple but useful for latter discussions.

2.2 CHY formulae

With the definition of Pfaffian described above, now we can introduce the CHY formulae [5–9]. In the CHY formulae, tree level amplitudes for n massless particles arise from a multi-dimensional contour integral over the moduli space of genus zero Riemann surfaces with n punctures, $\mathcal{M}_{0,n}$. It can be expressed as

$$\mathcal{A}_n = \int d\mu_n\, \mathcal{I}_L(\{k, \epsilon, z\}) \mathcal{I}_R(\{k, \widetilde{\epsilon}, z\}), \tag{2.6}$$

which possesses the Möbius $SL(2, \mathbb{C})$ invariance. Here k_i, ϵ_i and z_i are the momentum, polarization vector, and puncture location for i^{th} particle, respectively. The measure is defined as

$$d\mu_n \equiv \frac{d^n z}{\text{vol}\, SL(2, \mathbb{C})} \prod_i {}' \delta(\mathcal{E}_i). \tag{2.7}$$

Theory	$\mathcal{I}_L(k,\epsilon,z)$	$\mathcal{I}_R(k,\widetilde{\epsilon},z)$
gravity theory	$\mathbf{Pf}'\Psi$	$\mathbf{Pf}'\Psi$
Einstein-Yang-Mills	$\mathcal{C}_{\mathrm{Tr}_1}\cdots\mathcal{C}_{\mathrm{Tr}_m}\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$	$\mathbf{Pf}'\Psi$
pure Yang-Mills	$\mathcal{C}_n(\sigma)$	$\mathbf{Pf}'\Psi$
Einstein-Maxwell	$\mathbf{Pf}'[\Psi]_{n-2m,2m:n-2m}\mathbf{Pf}[X]_{2m}$	$\mathbf{Pf}'\Psi$
Einstein-Maxwell(photon with flavor)	$\mathbf{Pf}'[\Psi]_{n-2m,2m;n-2m}\mathbf{Pf}[\mathcal{X}]_{2m}$	$\mathbf{Pf}'\Psi$
Born-Infeld	$(\mathbf{Pf}'A)^2$	$\mathbf{Pf}'\Psi$
Yang-Mills-scalar	$\mathcal{C}_{\mathrm{Tr}_1}\cdots\mathcal{C}_{\mathrm{Tr}_m}\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$	$\mathcal{C}_n(\sigma)$
Yang-Mills-scalar(special)	$\mathbf{Pf}'[\Psi]_{n-2m,2m;n-2m}\mathbf{Pf}[\mathcal{X}]_{2m}$	$\mathcal{C}_n(\sigma)$
pure bi-adjoint scalar	$\mathcal{C}_n(\widetilde{\sigma})$	$\mathcal{C}_n(\sigma)$
non-linear sigma model	$(\mathbf{Pf}'A)^2$	$\mathcal{C}_n(\sigma)$
ϕ^4	$\mathbf{Pf}'A\,\mathbf{Pf}[X]_n$	$\mathcal{C}_n(\sigma)$
extended Dirac-Born-Infeld	$\mathcal{C}_{\mathrm{Tr}_1}\cdots\mathcal{C}_{\mathrm{Tr}_m}\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$	$(\mathbf{Pf}'A)^2$
Dirac-Born-Infeld	$\mathbf{Pf}'[\Psi]_{n-2m,2m;n-2m}\mathbf{Pf}[\mathcal{X}]_{2m}$	$(\mathbf{Pf}'A)^2$
special Galileon	$(\mathbf{Pf}'A)^2$	$(\mathbf{Pf}'A)^2$

Table 1. Form of the integrands for various theories.

The δ-functions impose the scattering equations

$$\mathcal{E}_i \equiv \sum_{j\in\{1,2,\ldots,n\}\backslash\{i\}} \frac{s_{ij}}{z_{ij}} = 0\,, \tag{2.8}$$

where $s_{ij} \equiv (k_i + k_j)^2$ is the Mandelstam variable, and $z_{ij} \equiv z_i - z_j$. The scattering equations define the map from the space of kinematic variables to $\mathcal{M}_{0,n}$, and fully localize the integral on their solutions.

The integrand in (2.6) depends on the theory under consideration, and carries all kinematical information of external particles. For any theory known to have a CHY representation, the corresponding integrand can be split into two parts $\mathcal{I}_L$ and $\mathcal{I}_R$, as can be seen in (2.6). Either of them are weight-2 for each variable z_i under the Möbius transformation. We list integrands for various theories as in table 1 [9].[3]

We now explain each ingredient appearing in this table in turn. The $n \times n$ matrixes are defined through

$$A_{ij} = \begin{cases} \dfrac{k_i \cdot k_j}{z_{ij}} & i \neq j\,, \\ 0 & i = j\,, \end{cases} \qquad\qquad B_{ij} = \begin{cases} \dfrac{\epsilon_i \cdot \epsilon_j}{z_{ij}} & i \neq j\,, \\ 0 & i = j\,, \end{cases}$$

$$C_{ij} = \begin{cases} \dfrac{k_i \cdot \epsilon_j}{z_{ij}} & i \neq j\,, \\ -\displaystyle\sum_{l=1,\,l\neq j}^{n} \dfrac{k_l \cdot \epsilon_j}{z_{lj}} & i = j\,, \end{cases} \tag{2.9}$$

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[3]For theories contain gauge or flavor groups, we only show the integrands for color-ordered partial amplitudes instead of full ones.

and

$$
X_{ij} = \begin{cases} \dfrac{1}{z_{ij}} & i \neq j\,, \\[2mm] 0 & i = j\,, \end{cases} \qquad\qquad \mathcal{X}_{ij} = \begin{cases} \dfrac{\delta^{I_i, I_j}}{z_{ij}} & i \neq j\,, \\[2mm] 0 & i = j\,. \end{cases} \tag{2.10}
$$

where δ^{I_i, I_j} forbids the interaction between particles with different flavors. To clarify the dimension, we denote the $n \times n$ matrixes X and $\mathcal{X}$ as $[X]_n$, $[\mathcal{X}]_n$. The $2n \times 2n$ antisymmetric matrix Ψ is given by

$$
\Psi = \left(\begin{array}{c|c} A & C \\ \hline -C^{\mathrm{T}} & B \end{array} \right)\,. \tag{2.11}
$$

The reduced Pfaffian of Ψ is defined as $\mathbf{Pf}'\Psi = \frac{(-)^{i+j}}{z_{ij}}\mathbf{Pf}\Psi^{[i,j]}$, where the notation $\Psi^{[i,j]}$ means the rows and columns i, j of the matrix Ψ have been deleted (with $1 \leq i,j \leq n$). It can be proved that this definition is independent of the choice of i and j. Analogous notation holds for $\mathbf{Pf}'A$.

The definition of Ψ can be generalized to the $(2a+b) \times (2a+b)$ case $[\Psi]_{a,b:a}$ as

$$
[\Psi]_{a,b:a} = \left(\begin{array}{c|c} A_{(a+b)\times(a+b)} & C_{(a+b)\times a} \\ \hline -C^{\mathrm{T}}_{a\times(a+b)} & B_{a\times a} \end{array} \right)\,, \tag{2.12}
$$

here A is a $(a+b) \times (a+b)$ matrix, C is a $(a+b) \times a$ matrix, and B is a $a \times a$ matrix. The definitions of elements of A, B and C are the same as before. The reduced Pfaffian $\mathbf{Pf}'[\Psi]_{a,b:a}$ is defined in the same manner. With the definition of the reduced Pfaffian, one can observe that: *each polarization vector ϵ_i appears once and only once in each term of the reduced Pfaffian.*

Furthermore, starting from the $2n \times 2n$ matrix Ψ, the polynomial $\mathcal{P}_{\{i,j\}}(n,l,m)$ is defined by

$$
\begin{aligned}
\mathcal{P}_{\{i,j\}}(n,l,m) &= \mathbf{sgn}(\{i,j\})\, z_{i_1 j_1} \cdots z_{i_m j_m}\, \mathbf{Pf}'[\Psi]_{n-l,i_1,j_1,\ldots,i_m,j_m:n-l} \\
&= -\mathbf{sgn}(\{i,j\}')\, z_{i_1 j_1} \cdots z_{i_{m-1} j_{m-1}}\, \mathbf{Pf}[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l}\,,
\end{aligned} \tag{2.13}
$$

where $i_k < j_k \in \mathrm{Tr}_k$ and Tr_k's are m sets satisfy[4]

$$
\mathrm{Tr}_1 \cup \mathrm{Tr}_2 \cup \cdots \cup \mathrm{Tr}_m = \{n-l+1, n-l+2, \cdots, n\}\,. \tag{2.14}
$$

In the notation $[\Psi]_{n-l,i_1,j_1,\ldots,i_m,j_m:n-l}$, we explicitly write $\{i_1,j_1,\ldots,i_m,j_m\}$ instead of $2m$ to emphasize the locations of $2m$ rows and $2m$ columns in the original matrix Ψ. Two signatures $\mathbf{sgn}(\{i,j\})$ and $\mathbf{sgn}(\{i,j\}')$ correspond to partitions $\{(i_1,j_1), \cdots, (i_m,j_m)\}$ and $\{(i_1,j_1), \cdots, (i_{m-1},j_{m-1})\}$ respectively, and one can verify $\mathbf{sgn}(\{i,j\}) = \mathbf{sgn}(\{i,j\}')$. In the second line of (2.13), the reduced Pfaffian is calculated by removing rows and columns

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[4]Each set has at least two elements, so in general we have $l \geq 2m$.

i_m and j_m, and $(-)^{(n-l+2m-1)+(n-l+2m)} = (-)$ have been used. Under the definition of $\mathcal{P}_{\{i,j\}}(n,l,m)$ in the second line, the summation $\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$ means

$$\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m) \equiv \sum_{\substack{i_1<j_1\in\mathrm{Tr}_1 \\ \cdots \\ i_{m-1}<j_{m-1}\in\mathrm{Tr}_{m-1}}} \mathcal{P}_{\{i,j\}}(n,l,m) \equiv \mathbf{Pf}'\Pi \,. \tag{2.15}$$

where the sum is over all possible choices of pairs in each trace subset. Notice that one can choose to delete rows and columns belong to any Tr_k when computing the reduced Pfaffian, and $\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$ is independent of the choice since it is equal to the reduced Pfaffian of Π, which is constructed using the squeezing procedure [9].

Finally, the Parke-Taylor factor for ordering σ is given as

$$\mathcal{C}_n(\sigma) = \frac{1}{z_{\sigma_1\sigma_2}z_{\sigma_2\sigma_3}\cdots z_{\sigma_{n-1}\sigma_n}z_{\sigma_n\sigma_1}}\,, \tag{2.16}$$

it implies the color order $\{\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n\}$ for the partial amplitude.

3 Basic operators

In this section, we will consider the effects of acting three basic differential operators given in [10] on the elementary building-blocks of CHY-integrands such as $\mathbf{Pf}'\Psi$, $\mathbf{Pf}'[\Psi]_{a,b:a}$, as well as $\sum_{\{i,j\}}{}'\mathcal{P}_{\{i,j\}}(n,l,m)$.

3.1 Trace operator

The trace operator $\mathcal{T}_{ij}$ is defined as [10]

$$\mathcal{T}_{ij} \equiv \partial_{\epsilon_i\epsilon_j}\,. \tag{3.1}$$

Here $\epsilon_i\epsilon_j$ means $\epsilon_i \cdot \epsilon_j$ and the differential operator is to take derivative regarding to the combination $\epsilon_i\cdot\epsilon_j$. Similar understanding holds for all operators in this note. If one applies $\mathcal{T}_{ij}$ on the reduced Pfaffian $\mathbf{Pf}'\Psi$, only terms containing factor $\epsilon_i\epsilon_j$ (i.e.,element $\Psi_{i+n,j+n}$) provide non-vanishing contributions. Thus performing the operator $\mathcal{T}_{ij}$ is equivalent to the replacement

$$\epsilon_i\epsilon_j \to 1\,, \quad \epsilon_i V \to 0\,, \quad \epsilon_j V \to 0\,, \tag{3.2}$$

where V denotes vectors k_l's or $\epsilon_{l\neq i,j}$'s, since ϵ_i and ϵ_j appear once and only once in each term of the reduced Pfaffian respectively. As noted in [10], the effect is nothing but the dimensional reduction (or the "compactifying" procedure in [9]). Thus we arrive at a new matrix $\widetilde{\Psi}$ satisfies

$$\mathcal{T}_{ij}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'\widetilde{\Psi}\,. \tag{3.3}$$

Without lose of generality, one can assume $\{i,j\} = \{n-1,n\}$,[5] then the new matrix $\widetilde{\Psi}$ is given by

$$\widetilde{\Psi} = \left(\begin{array}{c|c|c} A_{n\times n} & C_{n\times(n-2)} & 0 \\ \hline -C^{\mathrm{T}}_{(n-2)\times n} & B_{(n-2)\times(n-2)} & 0 \\ \hline 0 & 0 & X_{2\times2} \end{array}\right) = \left(\begin{array}{c|c} [\Psi]_{n-2,2:n-2} & 0 \\ \hline 0 & [X]_2 \end{array}\right)\,. \tag{3.4}$$

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[5]This assumption can be realized by moving lows and columns. Since $(n+i)^{\mathrm{th}}$ row and column will be moved simultaneously while moving i^{th} ones, the possible $-$ sign will not arise.

The reduced Pfaffian of the matrix $\widetilde{\Psi}$ can be calculated straightforwardly as

$$\mathbf{Pf}'\widetilde{\Psi} = \mathbf{Pf}'[\Psi]_{n-2,2;n-2}\mathbf{Pf}[X]_2 \,. \tag{3.5}$$

Thus, we find

$$\mathcal{T}_{ij}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'[\Psi]_{n-2,2;n-2}\mathbf{Pf}[X]_2 \,. \tag{3.6}$$

Same analysis gives the result of trace operator acting on generalized matrix $[\Psi]_{a,b:a}$

$$\mathcal{T}_{ij}\mathbf{Pf}'[\Psi]_{a,b:a} = \mathbf{Pf}'[\Psi]_{a-2,b+2:a-2}\mathbf{Pf}[X]_2 \,. \tag{3.7}$$

Repeating the manipulations, multiple action of trace operators give following generalization of (3.6) as

$$\mathcal{T}_{i_1 j_1}\mathcal{T}_{i_2 j_2}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'[\Psi]_{n-4,4;n-4}\mathbf{Pf}[X_1]_2\mathbf{Pf}[X_2]_2 \,,$$

$$\cdots$$

$$\mathcal{T}_{i_1 j_1}\mathcal{T}_{i_2 j_2}\cdots\mathcal{T}_{i_m j_m}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'[\Psi]_{n-2m,2m;n-2m}\mathbf{Pf}[X_1]_2\mathbf{Pf}[X_2]_2\cdots\mathbf{Pf}[X_m]_2$$

$$= \frac{(-)^m}{(z_{i_1 j_1}z_{j_1 i_1})(z_{i_2 j_2}z_{j_2 i_2})\cdots(z_{i_m j_m}z_{j_m i_m})}\mathcal{P}_{\{i,j\}}(n,2m,m)\,, \quad (3.8)$$

where $\mathcal{P}_{\{i,j\}}(n,l,m)$ is defined in (2.13), and we have arranged elements as

$$[X_k]_2 = \left(\begin{array}{c|c} 0 & \frac{1}{z_{i_k j_k}} \\ \hline \frac{1}{z_{j_k i_k}} & 0 \end{array}\right) \,. \tag{3.9}$$

We want to emphasize that the multiple action of $\mathcal{T}_{ij}$ on $\mathbf{Pf}'\Psi$ produce the structure $\mathcal{P}_{\{i,j\}}(n,l,m)$, which is crucial for many theories. This is why it is called the trace operator.

3.2 Insertion operator

The insertion operator is defined by [10]

$$\mathcal{T}_{ikj} \equiv \partial_{k_i\epsilon_k} - \partial_{k_j\epsilon_k} \,. \tag{3.10}$$

As pointed out in [10], $\mathcal{T}_{ikj}$ itself is not a gauge invariant operator, but when it acts on objects obtained after acting one trace operators, it is effectively gauge invariant. Thus we consider the effect of acting this operator on the polynomial $\sum_{\{i,j\}}'\mathcal{P}_{\{i,j\}}(n,l,m)$ only. According to discussions in [10], one should assume that $k \in \{1,2,\cdots,n-l\}$ and $i,j \in \mathrm{Tr}_i$ to protect the gauge invariance. For simplicity, we assume $i,j \in \mathrm{Tr}_m$ and taking the expansion (2.13) where Tr_m has been deleted. This gauge choice will greatly simplify our discussion, since with this choice $k_i\epsilon_k$ can appear in (2.13) only through C_{kk}.

Initially, $\mathbf{Pf}[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l}$ is

$$\mathbf{Pf}[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l} = \sum_{\alpha\in\Pi}\mathbf{sgn}(\pi_\alpha)[\Psi]_{a_1 b_1}[\Psi]_{a_2 b_2}\cdots[\Psi]_{a_{(n'+m')}b_{(n'+m')}}\,, \tag{3.11}$$

where the definition in (2.5) has been used. The element $[\Psi]_{a_i b_i}$ is at the a_i^{th} row and b_i^{th} column of the matrix $[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l}$, and we have defined $n' = n - l$,

$m' = m - 1$. Since ϵ_k appears only in C_{kk}, when acting $\partial_{k_i \epsilon_k}$ on (3.11), only terms containing element $[\Psi]_{k,n'+2m'+k}$ (see the expression (2.12)) can survive. Consider such a term, the remaining part after the action corresponds to a partition of the set $\{1, 2, \cdots, 2(n' + m')\} \setminus \{k, n' + 2m' + k\}$, which has the length $2(n' + m' - 1)$. Such a term appears in the $\mathbf{Pf}[\Psi]_{n-l-1,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l-1}$, weighted by a different signature $\mathbf{sgn}(\pi_{\widetilde{\alpha}})$, where the new matrix $[\Psi]_{n-l-1,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l-1}$ is obtained from the original one $[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l}$ by deleting k^{th} and $(n' + 2m' + k)^{\text{th}}$ rows and columns, and $\mathbf{sgn}(\pi_{\widetilde{\alpha}})$ corresponds to the partition of the length-$2(n' + m' - 1)$ set. By comparing these two special partitions, where one belongs to the original matrix and one belongs to the new one,

$$\alpha = \{(a_1, b_1), (a_2, b_2), \cdots, (k, n' + 2m' + k), \cdots, (a_{(n'+m')}, b_{(n'+m')})\},$$
$$\widetilde{\alpha} = \{(a_1, b_1), (a_2, b_2), \cdots, (a_{(n'+m'-1)}, b_{(n'+m'-1)})\}, \tag{3.12}$$

one can get $\mathbf{sgn}(\pi_\alpha) = \mathbf{sgn}(\pi_{\widetilde{\alpha}})$ since $\widetilde{\alpha}$ is obtained from α by deleting the pair $(k, n' + 2m' + k)$. Using above observation, when we sum all contributions together, we will have

$$\partial_{k_i \epsilon_k} \mathbf{Pf}[\Psi]_{n-l,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l} = \frac{-1}{z_{ik}} \mathbf{Pf}[\Psi]_{n-l-1,i_1,j_1,\ldots,i_{m-1},j_{m-1}:n-l-1}. \tag{3.13}$$

Applying this result to (2.13), we get immediately

$$\mathcal{T}_{ikj}\left(\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l, m)\right) = \left(\frac{1}{z_{jk}} - \frac{1}{z_{ik}}\right)\left(\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l+1, m)\right)$$
$$= \frac{-z_{ij}}{z_{ik} z_{kj}}\left(\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l+1, m)\right). \tag{3.14}$$

Let us give a little bit explanation of the result (3.14). There are two parts. The part $\left(\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l+1, m)\right)$ means we have added a new element k into the set Tr_m. Here is no ordering of the set and every element is at the same footing. The ordering information comes from the part $\frac{-z_{ij}}{z_{ik} z_{kj}}$, especially the denominator factor $z_{ik} z_{kj}$ gives a line connecting i to k and then k to j, i.e., one has inserted the element k between i, j.

To really achieve the goal, from table 1, one can see that $\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l, m)$ always appears together with a series of Parke-Taylor factors $\mathcal{C}_{\text{Tr}_1} \cdots \mathcal{C}_{\text{Tr}_m}$. If the original $\mathcal{C}_{\text{Tr}_m}$ contains $1/z_{ij}$, multiplying the factor $z_{ij}/(z_{ik} z_{jk})$ replaces it with $1/z_{ik} z_{kj}$, therefore implies the new color order $\{\ldots ikj \ldots\}$, i.e., the insertion of the element k between i, j. This explanation tells us how to systematically insert elements into a trace one by one with a well defined sequence of insertion operators.

Since the polynomial $\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n, l, m)$ is independent of the choice of the deleted rows and columns, assuming i and j belong to any other Tr_k will lead to the same conclusion, although the calculation will be more complicate.

3.3 Longitudinal operator

The longitudinal operators are defined via [10]

$$\mathcal{L}_i \equiv \sum_{j \neq i} k_i k_j \partial_{k_j \epsilon_i}, \tag{3.15}$$

and

$$\mathcal{L}_{ij} \equiv -k_i k_j \partial_{\epsilon_i \epsilon_j}. \tag{3.16}$$

Among these two, the $\mathcal{L}_{ij}$ is intrinsically gauge invariant, but $\mathcal{L}_i$ is not.[6] We now discuss the effects of acting them on the reduced Pfaffian $\mathbf{Pf}'[\Psi]_{a,b:a}$.

We first consider the operator $\mathcal{L}_{ij}$. It turns $\epsilon_i \epsilon_j$ into $k_i k_j$, and annihilates all other $\epsilon_i V$'s, $\epsilon_j V$'s. Using the observation that ϵ_i and ϵ_j can appear once and only once respectively, one can conclude that $\mathcal{L}_{ij}$ changes the reduced Pfaffian of the matrix $[\Psi]_{a,b:a}$ as

$$\mathcal{L}_{ij}\,\mathbf{Pf}'\left(\begin{array}{c|c} A_{(a+b)\times(a+b)} & C_{(a+b)\times a} \\ \hline -C_{a\times(a+b)}^{\mathrm{T}} & B_{a\times a} \end{array}\right) \Rightarrow \mathbf{Pf}'\left(\begin{array}{c|c|c} A_{(a+b)\times(a+b)} & C_{(a+b)\times(a-2)} & 0 \\ \hline -C_{(a-2)\times(a+b)}^{\mathrm{T}} & B_{(a-2)\times(a-2)} & 0 \\ \hline 0 & 0 & A_{2\times 2} \end{array}\right). \tag{3.17}$$

Next, we turn to the operator $\mathcal{L}_i$, which replaces every $k_j \epsilon_i$ with $k_j k_i$. Under such replacement, the diagonal elements of the matrix C become

$$C_{ii} \to -\sum_{l=1,\,l\neq i}^{n} \frac{k_l \cdot k_i}{z_{li}}, \tag{3.18}$$

which will vanish due to the scattering equation. Thus, the effect of $\mathcal{L}_i$ is given by

$$\mathcal{L}_i\,\mathbf{Pf}'\left(\begin{array}{c|c} A_{(a+b)\times(a+b)} & C_{(a+b)\times a} \\ \hline -C_{a\times(a+b)}^{\mathrm{T}} & B_{a\times a} \end{array}\right) \Rightarrow \mathbf{Pf}'\left(\begin{array}{c|c|c} A_{(a+b)\times(a+b)} & C_{(a+b)\times(a-2)} & A_{(a+b)\times 2} \\ \hline -C_{(a-2)\times(a+b)}^{\mathrm{T}} & B_{(a-2)\times(a-2)} & 0 \\ \hline A_{2\times(a+b)} & 0 & 0 \end{array}\right). \tag{3.19}$$

At this moment, the meaning of (3.17) and (3.19) is not clear. Actually, the longitudinal operators can not be performed individually to generate any object belongs to physical integrands. Instead, they should be used in a special manner, which will be discussed in the next section.

4 Products of basic operators

Using the products of basic operators, more operators will be constructed. In this section, we will discuss these composed operators, especially their action on the reduced Pfaffian $\mathbf{Pf}'\Psi$, which is the fundamental building-block for the integrand of gravity theory.

4.1 Operator $\mathcal{T}[\alpha]$

The operator $\mathcal{T}[\alpha]$ for a length-m set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is defined as[7] [10]

$$\mathcal{T}[\alpha] \equiv \mathcal{T}_{\alpha_1 \alpha_m} \cdot \prod_{i=2}^{m-1} \mathcal{T}_{\alpha_{i-1} \alpha_i \alpha_m}. \tag{4.1}$$

---

[6]It is useful to compare operators $\mathcal{L}_{ij}$ and $\mathcal{T}_{ij}$: they differ by the factor $k_i \cdot k_j$, which turns the interaction into derivatively coupling. Their common part, i.e., $\partial_{\epsilon_i \epsilon_j}$ plays the same role, i.e., "compactify".

[7]We adopt the convention in [10] that the product of two operators $\mathcal{O}_1 \cdot \mathcal{O}_2$ acts on an amplitude as $(\mathcal{O}_1 \cdot \mathcal{O}_2)\mathcal{A} = \mathcal{O}_2 \mathcal{O}_1 \mathcal{A}$, i.e., the operator $\mathcal{O}_1$ is performed at first, and $\mathcal{O}_2$ secondly.

We now act this operator on $\mathbf{Pf}'\Psi$. Firstly, performing $\mathcal{T}_{\alpha_1\alpha_m}$ gives

$$\mathcal{T}_{\alpha_1\alpha_m}\mathbf{Pf}'\Psi = \frac{-1}{z_{\alpha_1\alpha_m}z_{\alpha_m\alpha_1}}z_{\alpha_1\alpha_m}\mathbf{Pf}'[\Psi]_{n-2,\alpha_1,\alpha_m:n-2}$$

$$= \frac{1}{z_{\alpha_1\alpha_m}z_{\alpha_m\alpha_1}}\mathbf{Pf}[\Psi]_{n-2:n-2}\,, \tag{4.2}$$

where (3.6) and $(-)^{(n-1)+n} = (-)$ have been used. Then one can act $\mathcal{T}_{\alpha_1\alpha_2\alpha_m}$ on it, and use (3.14) to get

$$\mathcal{T}_{\alpha_1\alpha_2\alpha_m}\mathcal{T}_{\alpha_1\alpha_m}\mathbf{Pf}'\Psi = \frac{1}{z_{\alpha_1\alpha_m}z_{\alpha_m\alpha_1}}\frac{-z_{\alpha_1\alpha_m}}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_m}}\mathbf{Pf}[\Psi]_{n-3:n-3}$$

$$= \frac{-1}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_m}z_{\alpha_m\alpha_1}}\mathbf{Pf}[\Psi]_{n-3:n-3}\,. \tag{4.3}$$

Similarly, one can obtain

$$\mathcal{T}_{\alpha_2\alpha_3\alpha_m}\mathcal{T}_{\alpha_1\alpha_2\alpha_m}\mathcal{T}_{\alpha_1\alpha_m}\mathbf{Pf}'\Psi = \frac{-1}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_m}z_{\alpha_m\alpha_1}}\frac{-z_{\alpha_2\alpha_m}}{z_{\alpha_2\alpha_3}z_{\alpha_3\alpha_m}}\mathbf{Pf}[\Psi]_{n-4:n-4}$$

$$= \frac{1}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_3}z_{\alpha_3\alpha_m}z_{\alpha_m\alpha_1}}\mathbf{Pf}[\Psi]_{n-4:n-4}\,. \tag{4.4}$$

This procedure can be repeated recursively, and finally one will arrive

$$\mathcal{T}[\alpha]\mathbf{Pf}'\Psi = \frac{(-)^m}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_3}\cdots z_{\alpha_{m-1}\alpha_m}z_{\alpha_m\alpha_1}}\mathbf{Pf}[\Psi]_{n-m:n-m}$$

$$= (-)^{m+1}\mathcal{C}_\alpha\sum_{\{i,j\}}{}' \mathcal{P}_{\{i,j\}}(n,m,1)\,. \tag{4.5}$$

Above calculation is straightforward as long as $m \leq n-1$. The case $m = n$ needs a careful treatment. When $m = n$, the final insertion operator $\mathcal{T}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}$ acts on the Pfaffian of the 2×2 matrix $[\Psi]_{1:1}$ which is given as

$$[\Psi]_{1:1} = \left(\begin{array}{c|c} 0 & C_{\alpha_{n-1},\alpha_{n-1}} \\ \hline -C^{\mathrm{T}}_{\alpha_{n-1},\alpha_{n-1}} & 0 \end{array}\right)\,. \tag{4.6}$$

The Pfaffian of this matrix is

$$\mathbf{Pf}[\Psi]_{1:1} = C_{\alpha_{n-1},\alpha_{n-1}} = -\sum_{l=1,\,l\neq\alpha_{n-1}}^{n}\frac{k_l\epsilon_{\alpha_{n-1}}}{z_{l,\alpha_{n-1}}}\,. \tag{4.7}$$

Applying $\mathcal{T}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}$ on it, we get

$$\mathcal{T}[\alpha_1,\alpha_2,\cdots,\alpha_n]\mathbf{Pf}'\Psi = \frac{(-)^n}{z_{\alpha_1\alpha_2}z_{\alpha_2\alpha_3}\cdots z_{\alpha_{n-1}\alpha_n}z_{\alpha_n\alpha_1}} = (-)^n\mathcal{C}_n\,. \tag{4.8}$$

The above result can be generalized to multi-trace cases $\mathcal{T}[\alpha_1]\cdot\mathcal{T}[\alpha_2]\cdots$, via general relations (3.8) and (3.14), with the constraint $[\alpha_i]\cap[\alpha_j] = \emptyset$. Let us consider, for example,

$$\mathcal{T}[\alpha]\cdot\mathcal{T}[\beta] = \left(\mathcal{T}_{\alpha_1\alpha_m}\cdot\prod_{i=2}^{m-1}\mathcal{T}_{\alpha_{i-1}\alpha_i\alpha_m}\right)\cdot\left(\mathcal{T}_{\beta_1\beta_l}\cdot\prod_{i=2}^{l-1}\mathcal{T}_{\beta_{i-1}\beta_i\beta_l}\right)$$

$$= \mathcal{T}_{\alpha_1\alpha_m}\cdot\mathcal{T}_{\beta_1\beta_l}\cdot\left(\prod_{i=2}^{m-1}\mathcal{T}_{\alpha_{i-1}\alpha_i\alpha_m}\right)\cdot\left(\prod_{i=2}^{l-1}\mathcal{T}_{\beta_{i-1}\beta_i\beta_l}\right)\,. \tag{4.9}$$

The first step is using (3.8) to obtain

$$\mathcal{T}_{\beta_1 \beta_l} \mathcal{T}_{\alpha_1 \alpha_m} \mathbf{Pf}' \Psi = \left(\frac{-1}{z_{\beta_1 \beta_l} z_{\beta_l \beta_1}} \right) \left(\frac{-1}{z_{\alpha_1 \alpha_m} z_{\alpha_m \alpha_1}} \right) \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,4,2), \qquad (4.10)$$

where

$$\sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,4,2) = \mathcal{P}_{\{i,j\}}(n,4,2) = z_{\beta_1 \beta_l} \mathbf{Pf}[\Psi]_{n-4,\beta_1,\beta_l:n-4} = z_{\alpha_1 \alpha_m} \mathbf{Pf}[\Psi]_{n-4,\alpha_1,\alpha_m:n-4}. \tag{4.11}$$

Secondly, one can use (3.14) to get

$$\mathcal{T}_{\alpha_{m-2}\alpha_{m-1}\alpha_m} \cdots \mathcal{T}_{\alpha_2 \alpha_3 \alpha_m} \mathcal{T}_{\alpha_1 \alpha_2 \alpha_m} \left(\frac{-1}{z_{\alpha_1 \alpha_m} z_{\alpha_m \alpha_1}} \right) \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,4,2)$$

$$= \mathcal{T}_{\alpha_{m-2}\alpha_{m-1}\alpha_m} \cdots \mathcal{T}_{\alpha_2 \alpha_3 \alpha_m} \left(\frac{1}{z_{\alpha_1 \alpha_2} z_{\alpha_2 \alpha_m} z_{\alpha_m \alpha_1}} \right) \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,5,2)$$

$$\vdots$$

$$= (-)^{m+1} \mathcal{C}_\alpha \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,2+m,2). \tag{4.12}$$

Thirdly, we use (3.14) again to obtain

$$\mathcal{T}_{\beta_{l-2}\alpha_{l-1}\alpha_l} \cdots \mathcal{T}_{\beta_2 \beta_3 \beta_l} \mathcal{T}_{\beta_1 \beta_2 \beta_l} \left(\frac{-1}{z_{\beta_1 \beta_l} z_{\beta_l \beta_1}} \right) \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,2+m,2)$$

$$= (-)^{l+1} \mathcal{C}_\beta \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,l+m,2). \tag{4.13}$$

Combining them together we get

$$\mathcal{T}[\alpha] \cdot \mathcal{T}[\beta] \, \mathbf{Pf}' \Psi = (-)^{m+l+2} \mathcal{C}_\alpha \mathcal{C}_\beta \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}(n,l+m,2). \tag{4.14}$$

Now one can see the recursive pattern that

$$\mathcal{T}[\alpha_1] \cdot \mathcal{T}[\alpha_2] \cdots \mathcal{T}[\alpha_k] \, \mathbf{Pf}' \Psi = (-)^{k+\sum |\alpha_i|} \left(\prod \mathcal{C}_{\alpha_i} \right) \sideset{}{'}\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}\left(n, \sum |\alpha_i|, k\right), \qquad (4.15)$$

where $|\alpha_k|$ denotes the length of the set α_k.

4.2 Operator $\mathcal{L} \cdot \mathcal{T}_{ab}$

The operator $\mathcal{L}$ is defined through longitudinal operators as [10]

$$\mathcal{L} \equiv \prod_i \mathcal{L}_i = \widetilde{\mathcal{L}} + \cdots, \quad \text{with } \widetilde{\mathcal{L}} \equiv \sum_{\rho \in \text{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}. \tag{4.16}$$

The expression (4.16) means that at the algebraic level, the effect of $\prod_i \mathcal{L}_i$ is different from that of $\sum_{\rho \in \text{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}$. However, if we consider the combination $\mathcal{L} \cdot \mathcal{T}_{ab} \mathbf{Pf}' \Psi$, and let

subscripts of $\mathcal{L}_i$'s and $\mathcal{L}_{ij}$'s run through all nodes in $\{1, 2, \cdots, n\} \setminus \{a, b\}$, the effects of $\prod_i \mathcal{L}_i$ and $\sum_{\rho \in \text{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}$ are same, give a result which has a meaningful explanation.

Let us first study the effect of the operation $\widetilde{\mathcal{L}} \cdot \mathcal{T}_{ab} \mathbf{Pf}' \Psi$. Since $\widetilde{\mathcal{L}}$ and $\mathcal{T}_{ab}$ are commutable, i.e., $\widetilde{\mathcal{L}} \cdot \mathcal{T}_{ab} = \mathcal{T}_{ab} \cdot \widetilde{\mathcal{L}}$, we will apply the operator $\mathcal{T}_{ab}$ on $\mathbf{Pf}' \Psi$ firstly to get (3.6), then act $\widetilde{\mathcal{L}}$ on it. It is straightforward to see $\sum_{\rho \in \text{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}$ changes the matrix (3.4) into

$$\Psi' = \left(\begin{array}{c|c|c} A_{n \times n} & 0 & 0 \\ \hline 0 & -A_{(n-2) \times (n-2)} & 0 \\ \hline 0 & 0 & X_{2 \times 2} \end{array} \right) , \tag{4.17}$$

due to the previous result (3.17). The Pfaffian of the matrix

$$\left(\begin{array}{c|c} -A_{(n-2) \times (n-2)} & 0 \\ \hline 0 & X_{2 \times 2} \end{array} \right) , \tag{4.18}$$

is just $(-)^{a+b} \mathbf{Pf}'(-A) = (-)^{\frac{n}{2}-1+a+b} \mathbf{Pf}' A$, thus

$$\sum_{\rho \in \text{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij} \cdot \mathcal{T}_{ab} \mathbf{Pf}' \Psi = \mathbf{Pf}' \Psi' = (-)^{\frac{n}{2}-1+a+b} \left(\mathbf{Pf}' A \right)^2 . \tag{4.19}$$

Next we consider the effect of acting $\prod_i \mathcal{L}_i$ on $\mathcal{T}_{ab} \mathbf{Pf}' \Psi$. Using (3.19) we know the operator $\prod_i \mathcal{L}_i$ turns the matrix (3.4) into

$$\Psi'' = \left(\begin{array}{c|c|c} A_{n \times n} & A_{n \times (n-2)} & 0 \\ \hline A_{(n-2) \times n} & 0 & 0 \\ \hline 0 & 0 & X_{2 \times 2} \end{array} \right) , \tag{4.20}$$

thus the reduced Pfaffian is

$$\mathbf{Pf}' \Psi'' = \mathbf{Pf}' \widetilde{A} \, \mathbf{Pf}[X]_2 , \tag{4.21}$$

where

$$\widetilde{A} \equiv \left(\begin{array}{c|c} A_{n \times n} & A_{n \times (n-2)} \\ \hline A_{(n-2) \times n} & 0 \end{array} \right) . \tag{4.22}$$

To compute the reduced Pfaffian of $\widetilde{A}$, we choose a^{th} and b^{th} rows and columns of $A_{n \times n}$ to be removed. Furthermore one can use the relation that for the matrix S with the block structure

$$S = \left(\begin{array}{c|c} M & Q \\ \hline -Q^{\text{T}} & N \end{array} \right) , \tag{4.23}$$

when M is invertible, the Pfaffian of S satisfies

$$\mathbf{Pf} S = \mathbf{Pf} M \, \mathbf{Pf}(N + Q^{\text{T}} M^{-1} Q) . \tag{4.24}$$

Using this, the reduced Pfaffian of $\widetilde{A}$ can be calculated as

$$
\begin{aligned}
\mathbf{Pf}'\widetilde{A} &= \frac{(-)^{a+b}}{z_{ab}}\mathbf{Pf}A_{(n-2)\times(n-2)}\,\mathbf{Pf}\Big(0 + A^{\mathrm{T}}_{(n-2)\times(n-2)}A^{-1}_{(n-2)\times(n-2)}A_{(n-2)\times(n-2)}\Big) \\
&= \frac{(-)^{a+b}}{z_{ab}}\mathbf{Pf}A_{(n-2)\times(n-2)}\,\mathbf{Pf}\Big(0 + A^{\mathrm{T}}_{(n-2)\times(n-2)}\Big) \\
&= \frac{(-)^{a+b}}{z_{ab}}\mathbf{Pf}A_{(n-2)\times(n-2)}\,\mathbf{Pf}\Big(-A_{(n-2)\times(n-2)}\Big) \\
&= (-)^{\frac{n}{2}-1+a+b}z_{ab}\Big(\mathbf{Pf}'A\Big)^2 ,
\end{aligned}
\tag{4.25}
$$

Putting it back we obtain

$$
\prod_i \mathcal{L}_i \cdot \mathcal{T}_{ab}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'\Psi'' = (-)^{\frac{n}{2}-1+a+b}\Big(\mathbf{Pf}'A\Big)^2 .
\tag{4.26}
$$

Above calculations show that

$$
\mathcal{L} \cdot \mathcal{T}_{ab}\,\mathbf{Pf}'\Psi = \widetilde{\mathcal{L}} \cdot \mathcal{T}_{ab}\,\mathbf{Pf}'\Psi = (-)^{\frac{n}{2}-1+a+b}\Big(\mathbf{Pf}'A\Big)^2 .
\tag{4.27}
$$

It is worth to notice that this result is independent of the choice of a and b.

4.3 New operators $\mathcal{T}_{X_{2m}}$ and $\mathcal{T}_{\mathcal{X}_{2m}}$

As can be seen in table 1, the CHY-integrands for several theories require the ingredients $\mathbf{Pf}[X]_{2m}$ and $\mathbf{Pf}[\mathcal{X}]_{2m}$. These objects can also be created from the original matrix Ψ via appropriate operators. Now we give the definition of these new operators.

For a given length-$2m$ set I, we define a new operator as

$$
\mathcal{T}_{X_{2m}} \equiv \sum_{\rho\in\mathrm{pair}} \prod_{i_k,j_k\in\rho} \mathcal{T}_{i_k j_k} .
\tag{4.28}
$$

Here the set of pairs $\{(i_1,j_1),(i_2,j_2),\cdots,(i_m,j_m)\}$ is a partition of I with conditions $i_1 < i_2 < \ldots < i_m$ and $i_t < j_t$, $\forall t$. Using the result in (3.8) as well as the (2.5), one can conclude that the operator $\mathcal{T}_{X_{2m}}$ generates a new matrix

$$
\widetilde{\Psi}^* = \left(\begin{array}{c|c|c}
A_{n\times n} & -C^{\mathrm{T}}_{n\times(n-2m)} & 0 \\
\hline
C_{(n-2m)\times n} & B_{(n-2m)\times(n2-m)} & 0 \\
\hline
0 & 0 & X_{2m\times 2m}
\end{array}\right) ,
\tag{4.29}
$$

such that acting $\mathcal{T}_{X_{2m}}$ on the reduced Pfaffian $\mathbf{Pf}'\Psi$ gives

$$
\mathcal{T}_{X_{2m}}\,\mathbf{Pf}'\Psi = \mathbf{Pf}'\widetilde{\Psi}^* = \mathbf{Pf}'[\Psi]_{n-2m,2m:n-2m}\mathbf{Pf}[X]_{2m} ,
\tag{4.30}
$$

which provides the desired building block $\mathbf{Pf}[X]_{2m}$.

By similar argument, we can also define the operator $\mathcal{T}_{\mathcal{X}_{2m}}$ as

$$
\mathcal{T}_{\mathcal{X}_{2m}} \equiv \sum_{\rho\in\mathrm{pair}} \prod_{i,j\in\rho} \delta^{I_{i_k},I_{j_k}}\mathcal{T}_{i_k j_k} ,
\tag{4.31}
$$

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which is the generalization of $\mathcal{T}_{X_{2m}}$. The $\delta^{I_{i_k},I_{j_k}}$'s turn the matrix $[X]_{2m}$ into $[\mathcal{X}]_{2m}$, therefore we get

$$\mathcal{T}_{\mathcal{X}_{2m}} \mathbf{Pf}'\Psi = \mathbf{Pf}'[\Psi]_{n-2m,2m:n-2m}\mathbf{Pf}[\mathcal{X}]_{2m} , \qquad (4.32)$$

which gives the required building block $\mathbf{Pf}[\mathcal{X}]_{2m}$. Before ending this part, we want to emphasize one important point: since $\mathcal{T}_{ij}$ is intrinsically gauge invariant, so are $\mathcal{T}_{X_{2m}}$ and $\mathcal{T}_{\mathcal{X}_{2m}}$.

5   Unifying relations for amplitudes

With preparations in previous sections, we are ready to exhibit relations between amplitudes. As discussed in section 1, the idea is, differential operators are commutable with the integration over complex variables z_i's, thus the effects of acting them on amplitudes can be realized as acting on corresponding CHY-integrands, and vice versa. Our previous calculations have explicitly established the relation between two approaches in [9] and [10]. In this section, we will apply our results in sections section 3 and section 4 to write down relations between different scattering amplitudes, as did in [9] and [10].

5.1   The unified web

Now we act the operators on CHY integrands for various theories to get the unifying relations for amplitudes. The starting point is the formulation for the gravity theory. The reason is, all operators decrease the spins of external particles, thus the unified web must start from the amplitudes for gravitons which carry highest spins. The integrand of gravity theory is shown in the first line of table 1, two parts $\mathcal{I}_L$ and $\mathcal{I}_R$ depend on two independent sets of polarization vectors $\{\epsilon\}$ and $\{\widetilde{\epsilon}\}$, respectively. Since all operators are defined through partial differentials of some Lorentz invariants contain polarization vectors, it is natural to restrict the effect of them on the $\mathcal{I}_L$ part (or equivalently the $\mathcal{I}_R$ part), by defining operators via ϵ (or $\widetilde{\epsilon}$). Performing operators on the $\mathcal{I}_L$ part and using (3.8), (4.8), (4.15), (4.27), (4.30) and (4.32), after comparing with the middle column of table 1, we get following relations:

$$\begin{aligned}
\mathcal{A}^{\mathrm{EYM}} &= \mathcal{T}[\mathrm{Tr}_1]\cdots\mathcal{T}[\mathrm{Tr}_m]\,\mathcal{A}^{\mathrm{G}} , \\
\mathcal{A}^{\mathrm{YM}} &= \mathcal{T}[i_1\cdots i_n]\,\mathcal{A}^{\mathrm{G}} , \\
\mathcal{A}^{\mathrm{EM}} &= \mathcal{T}_{X_{2m}}\,\mathcal{A}^{\mathrm{G}} , \\
\mathcal{A}^{\mathrm{EM}}_{\mathrm{flavor}} &= \mathcal{T}_{\mathcal{X}_{2m}}\,\mathcal{A}^{\mathrm{G}} , \\
\mathcal{A}^{\mathrm{BI}} &= \mathcal{L}\cdot\mathcal{T}[ab]\,\mathcal{A}^{\mathrm{G}} ,
\end{aligned} \qquad (5.1)$$

up to an overall sign. Here $\mathcal{A}^{\mathrm{G}}$, $\mathcal{A}^{\mathrm{EYM}}$, $\mathcal{A}^{\mathrm{YM}}$, $\mathcal{A}^{\mathrm{EM}}$, $\mathcal{A}^{\mathrm{EM}}_{\mathrm{flavor}}$, $\mathcal{A}^{\mathrm{BI}}$ denote amplitudes of gravity theory, Einstein-Yang-Mills theory, pure Yang-Mills theory, Einstein-Maxwell theory, Einstein-Maxwell theory that photons carry flavors, Born-Infeld theory, respectively.

For the pure Yang-Mills integrand, there is only one copy $\mathbf{pf}'\,\Psi$ depends on polarization vectors, thus operators can be performed directly. Starting from the pure Yang-Mills

integrand, we obtain relations:

$$\mathcal{A}^{\text{YMS}} = \mathcal{T}[\text{Tr}_1] \cdots \mathcal{T}[\text{Tr}_m] \, \mathcal{A}^{\text{YM}} \, ,$$
$$\mathcal{A}^{\text{YMS}}_{\text{special}} = \mathcal{T}_{\mathcal{X}_{2m}} \, \mathcal{A}^{\text{YM}} \, ,$$
$$\mathcal{A}^{\text{BS}} = \mathcal{T}[i_1 \cdots i_n] \, \mathcal{A}^{\text{YM}} \, ,$$
$$\mathcal{A}^{\text{NLSM}} = \mathcal{L} \cdot \mathcal{T}[ab] \, \mathcal{A}^{\text{YM}} \, ,$$
$$\mathcal{A}^{\phi^4} = \mathcal{T}_{X_n} \, \mathcal{A}^{\text{YM}} \, , \tag{5.2}$$

up to an overall sign, where $\mathcal{A}^{\text{YMS}}$, $\mathcal{A}^{\text{YMS}}_{\text{special}}$, $\mathcal{A}^{\text{BS}}$, $\mathcal{A}^{\text{NLSM}}$, $\mathcal{A}^{\phi^4}$ denote amplitudes of Yang-Mills-scalar theory, special Yang-Mills-scalar theory, bi-adjoint scalar theory, non-linear sigma model, as well as ϕ^4 theory, respectively. Notice that the amplitude of ϕ^4 theory is generated via a special $\mathcal{T}_{X_{2m}}$ that $2m = n$.

Applying operators on the Born-Infeld integrand, we get relations:

$$\mathcal{A}^{\text{DBI}}_{\text{ex}} = \mathcal{T}[\text{Tr}_1] \cdots \mathcal{T}[\text{Tr}_m] \, \mathcal{A}^{\text{BI}} \, ,$$
$$\mathcal{A}^{\text{DBI}} = \mathcal{T}_{\mathcal{X}_{2m}} \, \mathcal{A}^{\text{BI}} \, ,$$
$$\mathcal{A}^{\text{NLSM}} = \mathcal{T}[i_1 \cdots i_n] \, \mathcal{A}^{\text{BI}} \, ,$$
$$\mathcal{A}^{\text{SG}} = \mathcal{L} \cdot \mathcal{T}[ab] \, \mathcal{A}^{\text{BI}} \, , \tag{5.3}$$

up to an overall sign, where $\mathcal{A}^{\text{DBI}}_{\text{ex}}$, $\mathcal{A}^{\text{DBI}}$, $\mathcal{A}^{\text{NLSM}}$, $\mathcal{A}^{\text{SG}}$ denote amplitudes of extended Dirac-Born-Infeld theory, Dirac-Born-Infeld theory, non-linear sigma model, special Galileon theory, respectively.

Our results (5.1), (5.2) and (5.3), give not only unified relations presented in [10], but also other relations among theories having CHY representations in [9]. We want to remark that a result in this paper is different from the one in [10], i.e., the Einstein-Maxwell theory: their differential operator is just one term of the operator $\mathcal{T}_{X_{2m}}$ defined in (4.28).

Relations presented above can be organized into table 2. In this table the notations $\mathcal{T}^{\epsilon}[\text{Tr}_i]$ and $\mathcal{T}^{\widetilde{\epsilon}}[\text{Tr}_i]$ means two operators are defined through two independent sets of polarization vectors $\{\epsilon\}$ and $\{\widetilde{\epsilon}\}$ respectively, and so do notations of other operators. If one adds the identical operator $\mathbb{I}$ into the set of operators, table 2 can be summarized as

$$\mathcal{A}^{\text{other}} = \mathcal{O}^{\epsilon} \cdot \mathcal{O}^{\widetilde{\epsilon}} \, \mathcal{A}^{\text{G}}(\epsilon, \widetilde{\epsilon}, k) \, , \tag{5.4}$$

where $\mathcal{O}^{\epsilon}$ and $\mathcal{O}^{\widetilde{\epsilon}}$ denote operators which are defined through $\{\epsilon\}$ and $\{\widetilde{\epsilon}\}$ respectively. Since the manifest double copy structure of the CHY integrands, $\mathcal{O}^{\epsilon}$ and $\mathcal{O}^{\widetilde{\epsilon}}$ are applied on two copies independently at the integrand-level.

5.2 Other relations

Differential operators connect not only amplitudes from different theories, but also amplitudes of same type of theory. For example, let us consider the Einstein-Yang-Mills theory. Let us start from a $(m + n)$-point color-ordered amplitude $\mathcal{A}^{\text{EYM}}(i_1^h, \cdots, i_m^h; j_1^g, \cdots, j_n^g)$, where h and g denote gravitons and gluons respectively with the color order of gluons as

Amplitude	Operator acts on $\mathcal{A}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k)$
$\mathcal{A}^{\mathrm{EYM}}(\epsilon, \widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\epsilon}[\mathrm{Tr}_m]$
$\mathcal{A}^{\mathrm{YM}}(\widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]$
$\mathcal{A}^{\mathrm{EM}}(\epsilon, \widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}_{X_{2m}}$
$\mathcal{A}^{\mathrm{EM}}_{\mathrm{flavor}}(\epsilon, \widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}$
$\mathcal{A}^{\mathrm{BI}}(\widetilde{\epsilon}, k)$	$\mathcal{L}^{\epsilon}\cdot\mathcal{T}^{\epsilon}[ab]$
$\mathcal{A}^{\mathrm{YMS}}(\widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\left(\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_m]\right)$
$\mathcal{A}^{\mathrm{YMS}}_{\mathrm{special}}(\widetilde{\epsilon}, k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\mathcal{T}^{\widetilde{\epsilon}}_{\mathcal{X}_{2m}}$
$\mathcal{A}^{\mathrm{BS}}(k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\mathcal{T}^{\widetilde{\epsilon}}[i'_1\cdots i'_n]$
$\mathcal{A}^{\mathrm{NLSM}}(k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\left(\mathcal{L}^{\widetilde{\epsilon}}\cdot\mathcal{T}^{\widetilde{\epsilon}}_{a'b'}\right)$
$\mathcal{A}^{\phi^4}(k)$	$\mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X_n}$
$\mathcal{A}^{\mathrm{DBI}}_{\mathrm{ex}}(\widetilde{\epsilon}, k)$	$\left(\mathcal{L}^{\epsilon}\cdot\mathcal{T}^{\epsilon}[ab]\right)\cdot\left(\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_m]\right)$
$\mathcal{A}^{\mathrm{DBI}}(\widetilde{\epsilon}, k)$	$\left(\mathcal{L}^{\epsilon}\cdot\mathcal{T}^{\epsilon}[ab]\right)\cdot\mathcal{T}^{\widetilde{\epsilon}}_{\mathcal{X}_{2m}}$
$\mathcal{A}^{\mathrm{SG}}(k)$	$\left(\mathcal{L}^{\epsilon}\cdot\mathcal{T}^{\epsilon}[ab]\right)\cdot\left(\mathcal{L}^{\widetilde{\epsilon}}\cdot\mathcal{T}^{\widetilde{\epsilon}}[a'b']\right)$

Table 2. Unifying relations.

$\{j_1, j_2, \cdots, j_n\}$. Using the relation (3.14), one can act insertion operators to turn gravitons into gluons at any desired positions, such as following:

$$\mathcal{A}^{\mathrm{EYM}}(i_3^h, \cdots, i_m^h; j_1^g, i_1^g, i_2^g, j_2^g, \cdots, j_n^g) = \mathcal{T}_{i_1 i_2 j_2}\,\mathcal{T}_{j_1 i_1 j_2}\,\mathcal{A}^{\mathrm{EYM}}(i_1^h, \cdots, i_m^h; j_1^g, \cdots, j_n^g),$$
$$\mathcal{A}^{\mathrm{EYM}}(i_3^h, \cdots, i_m^h; j_1^g, i_1^g, j_2^g, i_2^g, j_3^g, \cdots, j_n^g) = \mathcal{T}_{j_2 i_2 j_3}\,\mathcal{T}_{j_1 i_1 j_2}\,\mathcal{A}^{\mathrm{EYM}}(i_1^h, \cdots, i_m^h; j_1^g, \cdots, j_n^g),$$
$$\tag{5.5}$$

In above expressions, we have turned two gravitons into gluons, with different orderings: the first one with ordering $\{j_1, i_1, i_2, j_2, \cdots, j_n\}$ and the second one, $\{j_1, i_1, j_2, i_2, j_3, \cdots, j_n\}$, respectively. Situations for other theories can be analyzed similarly.

One can also seek amplitudes for other theories beyond these given in table 2, by acting on the amplitude of gravity theory via other combinations of differential operators. The operator $\mathcal{O}^{\epsilon}$ in (5.4) has 6 choices which are $\mathbb{I}$, $\mathcal{T}[\mathrm{Tr}_1]\cdots\mathcal{T}[\mathrm{Tr}_m]$, $\mathcal{T}[i_1\cdots i_n]$, $\mathcal{T}_{X_{2m}}$, $\mathcal{T}_{\mathcal{X}_{2m}}$, $\mathcal{L}\cdot\mathcal{T}_{ab}$, and so does $\mathcal{O}^{\widetilde{\epsilon}}$. Thus, starting from the CHY-integrand of gravity theory, there are 21 kinds of CHY-integrands can be obtained by performing operators. We now list the remaining 8 cases as following:

$$\mathcal{T}^{\epsilon}_{X_{2m}}\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X'_{2m'}}, \qquad\qquad \mathcal{T}^{\epsilon}[i_1\cdots i_n]\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}},$$
$$\left(\mathcal{L}^{\epsilon}\cdot\mathcal{T}^{\epsilon}[ab]\right)\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}}, \qquad\qquad \left(\mathcal{T}^{\epsilon}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\epsilon}[\mathrm{Tr}_m]\right)\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}},$$
$$\left(\mathcal{T}^{\epsilon}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\epsilon}[\mathrm{Tr}_m]\right)\cdot\mathcal{T}^{\widetilde{\epsilon}}_{\mathcal{X}_{2m}}, \qquad \left(\mathcal{T}^{\epsilon}[\mathrm{Tr}_1]\cdots\mathcal{T}^{\epsilon}[\mathrm{Tr}_m]\right)\cdot\left(\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_{1'}]\cdots\mathcal{T}^{\widetilde{\epsilon}}[\mathrm{Tr}_{m'}]\right),$$
$$\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}\cdot\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}}, \qquad\qquad \mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}\cdot\mathcal{T}^{\widetilde{\epsilon}}_{\mathcal{X}_{2m}}.$$
$$\tag{5.6}$$

Using results in section 4, one can get the corresponding integrands generated by them. If some of these integrands correspond to physical amplitudes, then new unifying relations occurs. The complete analyis of various combinations in (5.6) is beyond the scope of this note and we will leave it to future work. Here we just give some brief discussions.

For the first case $\mathcal{T}^{\epsilon}_{X_{2m}} \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X'_{2m'}}$, when $2m = 2m' = n$, it yields the integrand

$$\mathcal{T}^{\epsilon}_{X_n} \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X_n} \, \mathcal{I}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k, z) = \Big(\mathbf{Pf}' A_n \, \mathbf{Pf}[X]_n \Big) \Big(\mathbf{Pf}' A_n \, \mathbf{Pf}[X]_n \Big) , \tag{5.7}$$

where $\mathcal{I}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k, z)$ denotes the integrand for gravity theory. This is the integrand for Einstein-Maxwell-scalar theory, with all external particles are scalars [11]. This result is just a special case with $n = 2m = 2m'$. As we have emphasized, since $\mathcal{T}^{\epsilon}_{X_n}$ is intrinsically gauge invariant, we can take any length for this operator. Furthermore, the role of $\mathcal{T}_{ij}$ is just to do the dimension reduction. With this understanding, one can see that for general m and m' the $\mathcal{T}^{\epsilon}_{X_{2m}} \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X'_{2m'}} \, \mathcal{A}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k)$ will give the theory obtained from gravity theory by dimension reduction, i.e., the general Einstein-Maxwell-scalar amplitudes, whose external particles can be either gravitons, photons, as well as scalars, i.e.,

$$\mathcal{A}^{\mathrm{EMS}}(\epsilon, \widetilde{\epsilon}, k) = \mathcal{T}^{\epsilon}_{X_{2m}} \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X'_{2m'}} \, \mathcal{A}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k) . \tag{5.8}$$

For the second case $\mathcal{T}^{\epsilon}[i_1 \cdots i_n] \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}}$, when $2m = n$, we get

$$\mathcal{T}^{\epsilon}[i_1 \cdots i_n] \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X_n} \, \mathcal{I}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k, z) = \mathcal{C}_n \mathbf{Pf}' A \, \mathbf{Pf}[X]_n , \tag{5.9}$$

which is the ϕ^4 theory. Again, the operator $\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}}$ can be any length. When $2m < n$, we get the theory obtained by doing dimension reduction from Yang-Mills theory, which is the special Yang-Mills-Scalar theory

$$\mathcal{T}^{\epsilon}[i_1 \cdots i_n] \cdot \mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}} \, \mathcal{I}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k, z) = \mathcal{C}_n \mathbf{Pf}'[\Psi]_{n-2m, 2m:n-2m} \mathbf{Pf}[X]_{2m} . \tag{5.10}$$

Here if we replace $\mathcal{T}^{\widetilde{\epsilon}}_{X_{2m}}$ by $\mathcal{T}^{\widetilde{\epsilon}}_{\mathcal{X}_{2m}}$, we will get the special Yang-Mills-Scalar theory with multiple kinds of scalars, as can be seen in table 2.

Other cases in (5.6) can be discussed similarly. One can obtain more possible integrands via products $\mathcal{O}^{\epsilon} = \mathcal{O}^{\epsilon}_1 \cdots \mathcal{O}^{\epsilon}_a$ and $\mathcal{O}^{\widetilde{\epsilon}} = \mathcal{O}^{\widetilde{\epsilon}}_1 \cdots \mathcal{O}^{\widetilde{\epsilon}}_b$. However, which combinations produce sensible physical scattering amplitudes is not clear and it is worth to be carefully studied. For a physical $\mathcal{O}^{\epsilon} \cdot \mathcal{O}^{\widetilde{\epsilon}} \, \mathcal{I}^{\mathrm{G}}(\epsilon, \widetilde{\epsilon}, k, z)$, information of external particles such as spins and gauge structures can be read out directly from the obtained integrand, but pin down the form of interaction is a hard work.

6 Summary and discussion

To summarize, we have provided manifest connection between two approaches, i.e., the differential operator in [10] and various manipulations (such as compactification and squeezing procedures) in [9]. Using this connection, by acting differential operators on the CHY integrand of gravity theory, one can systematically derive unifying relations for amplitudes of various theories, include Einstein gravity, Einstein-Yang-Mills theory, Einstein-Maxwell

theory, pure Yang-Mills theory, Yang-Mills-scalar theory, Born-Infeld theory, Dirac-Born-Infeld theory and its extension, bi-adjoint scalar theory, ϕ^4 theory, non-linear sigma model, as well as special Galileon theory. Along the line, all unifying relations in [10] have been reproduced, and all theories which have CHY representations in [9] have been included in the unified web. We have also discussed other new relations for amplitudes, which are indicated by our method.

The manifest double copy structure of the CHY integrand permits two sets of operators $\mathcal{O}^\epsilon$ and $\mathcal{O}^{\widetilde{\epsilon}}$ to be applied independently. This advantage simplifies the derivation: it is sufficient to consider the effects of acting operators on the reduced Pfaffian $\mathbf{Pf}'\Psi$.

A natural question will be, why these operators? From discussions in [10], one critical condition is the gauge symmetry. The trace operators protect the gauge invariance while others do not. This is why the insertion and longitudinal operators should be performed after the trace operator. There are other operators, such as $\mathcal{T}_{ijkl}$, have not been used in the construction. Thus it will be interesting to consider broader form of differential operators. Furthermore, how to understand these physical conditions from the point of view of CHY formulae is also important.

Our result can also be used to other studies. For example, recent studies [12–17] have shown how to expand the Einstein-Yang-Mills amplitudes by the Yang-Mills ones. If one acts the differential operator at both sides of the expansion, a differential equation connecting amplitudes of two different theories will be obtained. Solving this differential equation (or doing the integration), we should find amplitudes for particles with higher spins from other ones with lower spins. This is opposite to current construction of united web by starting from highest spin state, i.e., gravitons.

Acknowledgments

References

[1] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A relation between tree amplitudes of closed and open strings*, *Nucl. Phys.* **B 269** (1986) 1 [INSPIRE].

[2] Z. Bern, J.J.M. Carrasco and H. Johansson, *New relations for gauge-theory amplitudes*, *Phys. Rev.* **D 78** (2008) 085011 [arXiv:0805.3993] [INSPIRE].

[3] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative quantum gravity as a double copy of gauge theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [INSPIRE].

[4] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the square of gauge theory*, *Phys. Rev.* **D 82** (2010) 065003 [`arXiv:1004.0693`] [ɪɴSPIRE].

[5] F. Cachazo, S. He and E.Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev.* **D 90** (2014) 065001 [`arXiv:1306.6575`] [ɪɴSPIRE].

[6] F. Cachazo, S. He and E.Y. Yuan, *Scattering of massless particles in arbitrary dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [`arXiv:1307.2199`] [ɪɴSPIRE].

[7] F. Cachazo, S. He and E.Y. Yuan, *Scattering of massless particles: scalars, gluons and gravitons*, *JHEP* **07** (2014) 033 [`arXiv:1309.0885`] [ɪɴSPIRE].

[8] F. Cachazo, S. He and E.Y. Yuan, *Einstein-Yang-Mills scattering amplitudes from scattering equations*, *JHEP* **01** (2015) 121 [`arXiv:1409.8256`] [ɪɴSPIRE].

[9] F. Cachazo, S. He and E.Y. Yuan, *Scattering equations and matrices: from Einstein to Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149 [`arXiv:1412.3479`] [ɪɴSPIRE].

[10] C. Cheung, C.-H. Shen and C. Wen, *Unifying relations for scattering amplitudes*, *JHEP* **02** (2018) 095 [`arXiv:1705.03025`] [ɪɴSPIRE].

[11] F. Cachazo, P. Cha and S. Mizera, *Extensions of theories from soft limits*, *JHEP* **06** (2016) 170 [`arXiv:1604.03893`] [ɪɴSPIRE].

[12] S. Stieberger and T.R. Taylor, *New relations for Einstein-Yang-Mills amplitudes*, *Nucl. Phys.* **B 913** (2016) 151 [`arXiv:1606.09616`] [ɪɴSPIRE].

[13] D. Nandan, J. Plefka, O. Schlotterer and C. Wen, *Einstein-Yang-Mills from pure Yang-Mills amplitudes*, *JHEP* **10** (2016) 070 [`arXiv:1607.05701`] [ɪɴSPIRE].

[14] L. de la Cruz, A. Kniss and S. Weinzierl, *Relations for Einstein-Yang-Mills amplitudes from the CHY representation*, *Phys. Lett.* **B 767** (2017) 86 [`arXiv:1607.06036`] [ɪɴSPIRE].

[15] C.-H. Fu, Y.-J. Du, R. Huang and B. Feng, *Expansion of Einstein-Yang-Mills amplitude*, *JHEP* **09** (2017) 021 [`arXiv:1702.08158`] [ɪɴSPIRE].

[16] F. Teng and B. Feng, *Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame*, *JHEP* **05** (2017) 075 [`arXiv:1703.01269`] [ɪɴSPIRE].

[17] Y.-J. Du, B. Feng and F. Teng, *Expansion of all multitrace tree level eym amplitudes*, *JHEP* **12** (2017) 038 [`arXiv:1708.04514`] [ɪɴSPIRE].