Constrained BRST-BFV Lagrangian formulations for higher spin fields in Minkowski spaces

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ABSTRACT: BRST-BFV method to construct constrained Lagrangian formulations for (ir)reducible half-integer higher-spin Poincare group representations in Minkowski space is suggested. The procedure is derived by two ways: first, from the unconstrained BRST-BFV method for mixed-symmetry higher-spin fermionic fields subject to an arbitrary Young tableaux with k rows (suggested in *Nucl. Phys. B* **869** (2013) 523, [arXiv:1211.1273](https://arxiv.org/abs/1211.1273)) by extracting the second-class constraints subsystem, $\hat{O}_\alpha = (\hat{O}_\alpha, \hat{O}_\alpha^+)$, from a total superalgebra of constraints, second, in self-consistent way by means of finding BRST-extended initial off-shell algebraic constraints, $\hat{O}_\alpha$. In both cases, the latter constraints supercommute on the constraint surface with constrained BRST operator Q_C and spin operators σ_C^i . The closedness of the superalgebra $\{Q_C, \hat{O}_\alpha, \sigma_C^i\}$ guarantees that the final gauge-invariant Lagrangian formulation is compatible with the off-shell algebraic constraints $\hat{O}_\alpha$ imposed on the field and gauge parameter vectors of the Hilbert space not depending from the ghosts and conversion auxiliary oscillators related to $\hat{O}_\alpha$, in comparison with the vectors for unconstrained BRST-BFV Lagrangian formulation. The suggested constrained BRST-BFV approach is valid for both massive HS fields and integer HS fields in the second-order formulation. It is shown that the respective constrained and unconstrained Lagrangian formulations for (half)-integer HS fields with a given spin are equivalent. The constrained Lagrangians in ghost-independent and component (for initial spin-tensor field) are obtained and shown to coincide with the Fang-Fronsdal formulation for totally-symmetric HS field with respective off-shell gamma-traceless constraints. The triplet and unconstrained quartet Lagrangian formulations for the latter field are derived. The constrained BRST-BFV methods without off-shell constraints describe reducible half-integer HS Poincare group representations with multiple spins as a generalized triplet and provide a starting point for constructing unconstrained Lagrangian formulations by using the generalized quartet

mechanism. A gauge-invariant Lagrangian constrained description for a massive spin-tensor field of spin $n + 1/2$ is obtained using a set of auxiliary Stueckelberg spin-tensors. A concept of BRST-invariant second-class constraints for dynamical systems with mixed-class constraints is suggested, leading to equivalent (w.r.t. the BRST-BFV prescription) results of quantization both at the operator level and in terms of the partition function.

KEYWORDS: Higher Spin Symmetry, Gauge Symmetry, BRST Quantization, Field Theories in Higher Dimensions

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1 Introduction

Higher-spin (HS) field theory, in its various reflections, has been under a long and intense study in order to re-analyse the problems of a unified description for the variety of elementary particles, thereby discovering approaches to a unification of the known and new possible fundamental interactions, and remains part of the LHC experimental program. HS field theory is in close relation to superstring theory due to its tensionless limit [1], which uses a BRST operator to handle an infinite set of HS fields with integer and half-integer generalized spins in d -dimensional ($d \geq 4$) space-time and incorporates HS field theory into superstring theory, providing the consideration of HS theory as an instrument for investigating the structure of superstring theory (for the current status of HS field theory and recent advances in HS theory, see the reviews [2–7]). A powerful systematic

tool used to reconstruct Lagrangian formulations for HS fields described by (spin-)tensor fields as elements of an irreducible representation of the Poincare $\text{ISO}(1, d-1)$ or (anti)-de-Sitter ((A)dS) groups in the respective d -dimensional Minkowski or (A)dS space-time by means of second- or first-order non-Lagrangian equations for free propagating HS integer and half-integer fields, respectively, is based on a BRST construction, developed initially for HS fields with lower spins [8–11] in $\mathbb{R}^{1,d-1}$, extended for higher integer spin fields in $\mathbb{R}^{1,d-1}$ [12–15] and (A)dS_d [16–19], and for the fields with higher half-integer spin [20–22] (for review, see [24]). This approach is based on the BRST-BFV method [25–31], originally developed to solve the problem of Hamiltonian quantization for dynamical systems with first-class constraints in Yang-Mills theories, suggested in [32], and is usually known as the BRST construction. The application of the BRST construction to free HS field theory consists of four steps. First, the conditions that determine representations with a given spin are regarded as a topological (i.e., without Hamiltonian) gauge system of first- and second-class constraints with a spin operator in an auxiliary Fock space. Second, the subsystem of the initial constraints, which contains only the second-class constraints (for massless case), together with the spin operator, is converted, while preserving the initial algebraic structure, into a system of first-class constraints alone, in an enlarged Fock space (for the conversion methods, see [33–35]). Third, with respect to the converted constraints, one constructs a BRST operator, which is a more involved problem for HS fields in (A)dS spaces due to quadratic constraint algebras [36–39]. Fourth, the Lagrangian and the reducible gauge transformations for an HS field are constructed in terms of the BRST operator in such a way that the corresponding equations of motion reproduce the initial constraints. We emphasize that the approach leads automatically to a gauge-invariant Lagrangian description with all the necessary auxiliary and Stueckelberg fields. Applying the BRST-BFV approach to the HS field theory, one usually works within a metric-like formulation due to Fronsdal’s results in totally-symmetric HS fields with integer [40, 41] and half-integer spins [42, 43], whereas in the frame-like formulation [45] the results for the Lagrangian dynamics of HS fields [44] were obtained beyond this construction.

In constructing Lagrangian formulations for free and interacting HS fields, one examines the cases of unconstrained or constrained dynamics, which means, respectively, the absence or presence of consistent usually off-shell holonomic (traceless, γ -traceless, mixed-symmetry) constraints. As a rule, most of the results in the metric- [46–51] and frame-like formulations [52–57] were obtained for off-shell constraints. There exists so-called Maxwell-like formulations for metric-like tensor fields (on flat and to some extent on (A)dS spaces) developed originally with off-shell differential constraints on the gauge parameters and with their resolution by means of a tower of reducible gauge transformations with higher derivatives [58, 59]. We recall that irreducible Poincare or (A)dS group representations in constant curvature space-times is described by mixed-symmetric (MS) HS fields with an arbitrary Young tableaux of k rows, $Y(s_1, \dots, s_k)$ (symmetric basis), determined by more than one spin-like parameters s_i [60–65], and, equivalently, by mixed-antisymmetric (spin-)tensor fields with an arbitrary Young tableaux of l columns, $Y[\hat{s}_1, \dots, \hat{s}_l]$ (antisymmetric basis), the integers or half-integers $\hat{s}_1 \geq \hat{s}_2 \geq \dots \geq \hat{s}_l$ having a spin-like interpretation [66]. BRST-BFV Lagrangian formulations for arbitrary free MS HS fields with integer

and half-integer spins were constructed in an unconstrained form (complete with all the algebraic constraints that follow from the Lagrangian and the tower of the respective gauge transformations after a partial gauge fixing) in our papers [67, 68]. In turn, the notion “unconstrained” has numerous interpretations, and was introduced originally as the geometric formulations of the field equations and Lagrangians for totally-symmetric HS fields both in non-local and local (minimal) representations [69, 70] within the metric-like formulation, and in more symmetric form for Lagrangians with totally-symmetric HS fields in [71, 72] as a “quartet unconstrained formulation” obtained from the triplet formulation with off-shell algebraic constraints [73]. Another usage of this term is due to Lagrangian formulations for MS HS fields in $\mathbb{R}^{1,d-1}$ [74, 75]. Constrained Lagrangians, together with minimal BV actions for totally and mixed-symmetric HS fields with integer spin, were studied within the so-called BRST-BV approach [76–79], which includes the constrained BRST-BFV Lagrangian approach itself. At the same time, no constrained BRST-BFV or BRST-BV constructions for half-integer HS fields in constant curvature spaces have been suggested so far, due to the yet unknown form of consistent (with constrained first-order Lagrangians) holonomic constraints. Moreover, explicit relations between the unconstrained and constrained BRST-BFV approaches to Lagrangian formulations for the same HS field with a given spin have not been established, despite the fact that the respective Lagrangians are to describe dynamics equivalent to the initial relations of an irreducible representation for the HS field. The same problem (which is left out of the paper’s scope) arises as one examines unconstrained (so far undeveloped) and constrained BRST-BV constructions for both integer and half-integer HS fields.

The paper is devoted to solving the following problems:

1. Derivation of constrained BRST-BFV approaches to constrained Lagrangian formulations for MS HS fields in $\mathbb{R}^{1,d-1}$ with given integer and half-integer spins from the respective unconstrained BRST-BFV approaches;
2. Derivation of constrained BRST-BFV approaches to constrained Lagrangian formulations for integer and half-integer MS HS fields in $\mathbb{R}^{1,d-1}$ in a self-consistent way;
3. Study of equivalence between unconstrained and constrained BRST-BFV Lagrangian formulations for the same MS HS field;
4. Derivation of constrained gauge-invariant Lagrangian actions for totally symmetric half-integer massless and massive HS field in the metric formulation with a single spin-tensor, in the triplet and unconstrained quartet forms;
5. Study of a BRST invariant extension of second-class constraints for a general dynamical system with independent sets of first- and second-class constraints and its application to the quantization procedure within both the conventional path integral approach and generalized canonical quantization.

The organization of the paper is as follows. In the section 2 we remind the crucial points of BFV method application to the quantization of dynamical systems subject to the first and second-class constraints in operator and functional integral forms. In section 3, we briefly review the ingredients of the unconstrained BRST-BFV method for gauge-invariant

Lagrangian formulations with free half-integer MS HS fields in Minkowski space, which is the starting point to obtain a constrained BRST-BFV operator, a spin operator and off-shell algebraic constraints in subsection 4.1 of section 4. A self-consistent way to construct the basic elements of the constrained BRST-BFV method is examined in subsection 4.2. Section 5 is devoted to the construction of constrained Lagrangian formulations for free massless and massive half-integer HS fields subject to $Y(s_1, \dots, s_k)$. The case of constrained Lagrangian formulations for integer MS HS fields is examined in subsection 5.2. In section 6, we consider ghost-independent, component spin-tensor and triplet forms of constrained Lagrangians for totally-symmetric fermionic HS fields, in subsection 6.1, and quartet unconstrained Lagrangians for the same fields, together with massive fields, in subsection 6.2. In Conclusion, we present a review of our basic results. Finally, in appendix A, we suggest a new algorithm for quantizing dynamical systems with mixed-class constraints.

We use the conventions of [67, 68], which includes the mostly minus signature for the metric tensor $\eta_{\mu\nu} = \text{diag}(+, -, \dots, -)$, with Lorentz indices $\mu, \nu = 0, 1, \dots, d-1$, the relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ for the Dirac matrices γ^μ , the notation $\varepsilon(A)$, $gh(A)$ for the respective values of Grassmann parity and BFV ghost number of a quantity A , and denote by $[A, B]$ the supercommutator of quantities A, B , which, in case they possess definite values of Grassmann parity, is given by $[A, B] = AB - (-1)^{\varepsilon(A)\varepsilon(B)}BA$.

2 On BRST-BFV method for dynamical systems subject to constraints

Here, we briefly consider some specific points of the BRST-BFV construction (following in part to [25–31], see as well [80]) as applied to the solution of the direct problem of generalized canonical quantization of the dynamical systems subject to the first and second-class constraints in order to calculate average expectation values of the physical quantities on appropriate Hilbert space in gauge-invariant way and to the inverse problem of reconstruction of the Lagrangian formulation for initial non-Lagrangian equations on the (spin)-tensor fields when applying to the HS field theory, firstly on the free and then on the interacting levels.

The constrained dynamical system is described, according to Dirac proposal [81, 82] by the Hamiltonian, $H_0(\Gamma)$, and finite set of the first-class $T_A(\Gamma) = 0$ and second-class constraints, $\Theta_\alpha(\Gamma) = 0$, $A = (A_+, A_-) = 1, \dots, M$; $M = (M_+, M_-)$, $\alpha = (\alpha_+, \alpha_-) = 1, \dots, m$; $m = (m_+, m_-)$ with Grassman gradings $\varepsilon(H_0; T_A; \Theta_\alpha) = (0; \varepsilon_A; \varepsilon_\alpha)$ depending on the phase-space coordinates $\Gamma^p = (q^i, p_i)$, $i = 1, \dots, n$, $n = (n_+, n_-)$ for $(m + 2M) \leq 2n$ in the $2n$ -dimensional phase-space $(\mathcal{M}, \omega)$ with Grassmann-even non-degenerate closed 2-form ω , $d\omega = 0$, and fundamental Poisson superbrackets $\{\Gamma^p, \Gamma^q\} = \omega^{pq}$ at a fixed time instant t for constant $\omega^{pq} = -(-1)^{\varepsilon(\Gamma^p)\varepsilon(\Gamma^q)}\omega^{qp}$ and

$$\{T_A, T_B\} = f_{AB}^C(\Gamma)T_C + f_{AB}^{\alpha\beta}(\Gamma)\Theta_\alpha\Theta_\beta, \quad \{T_A, \Theta_\alpha\} = f_{A\alpha}^C(\Gamma)T_C, \quad (2.1)$$

$$\{\Theta_\alpha, \Theta_\beta\} = \Delta_{\alpha\beta}(\Gamma) + f_{\alpha\beta}^\gamma(\Gamma)\Theta_\gamma, \quad \text{for } \text{sdet} \|\Delta_{\alpha\beta}(\Gamma)\|_{\Theta_\alpha=0} \neq 0, \quad (2.2)$$

$$(f_{AB}^C, f_{AB}^{\alpha\beta}) = -(-1)^{\varepsilon_A\varepsilon_B}(f_{BA}^C, f_{BA}^{\alpha\beta}), \quad (\Delta_{\alpha\beta}, f_{\alpha\beta}^\gamma) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}(\Delta_{\beta\alpha}, f_{\beta\alpha}^\gamma), \quad (2.3)$$

$$\{H_0, \Theta_\alpha\} = V_\alpha^\beta(\Gamma)\Theta_\beta + V_\alpha^A(\Gamma)T_A, \quad \{H_0, T_A\} = V_A^B(\Gamma)T_B + V_A^\alpha(\Gamma)\Theta_\alpha, \quad (2.4)$$

with some functions $f_{AB}^C, f_{AB}^{\alpha\beta}, f_{A\alpha}^C, \Delta_{\beta\alpha}, f_{\beta\alpha}^\gamma, V_\alpha^\beta, V_\alpha^A, V_A^B, V_A^\alpha$ given on $\mathcal{M}$ and for $\varepsilon_A \equiv \varepsilon(T_A), \varepsilon_\alpha \equiv \varepsilon(\Theta_\alpha)$. The dynamics and gauge transformations are determined by the equations,

$$\partial_t \Gamma^p = \{\Gamma^p, H_0\}, \quad T_A = 0, \quad \Theta_\alpha = 0; \quad \delta_\xi \Gamma^p = \{\Gamma^p, T_A(\Gamma)\} \xi^A(t) \quad (2.5)$$

with arbitrary functions $\xi^A(t)$ (functionally independent for linear independent set of $T_A(\Gamma)$) meaning the necessity to introduce gauge conditions $\chi^B(\Gamma) = 0$ such that:

$$\{\chi^A, \chi^B\}|_{T=\Theta=0} = 0, \quad \text{sdet}\|\{T_A, \chi^B\}\}|_{T=\Theta=0} \neq 0 \quad (2.6)$$

and appropriate Cauchy problem: $\Gamma^p(t)|_{t=t_0} = \Gamma_0^p$.

There exists representation for the system of mixed constraints (T_A, Θ_α) in the form $(T_{c|A}, \Phi_\alpha)(\Gamma_c)$ given in the extended phase-space $\mathcal{M}_c$ with coordinates $\Gamma_c^P = (\Gamma^p, \zeta^\alpha)$ for $\varepsilon(\zeta^\alpha) = \varepsilon(\Theta_\alpha)$ being subject to new fundamental Poisson superbrackets,

$$\{\zeta^\alpha, \zeta^\beta\} = \tilde{\omega}^{\alpha\beta} : \{\zeta^\alpha, \Gamma^p\} = \{\tilde{\omega}^{\alpha\beta}, \Gamma^p\} = 0; \quad \tilde{\omega}^{\alpha\beta} = -(-1)^{\varepsilon_\alpha \varepsilon_\beta} \tilde{\omega}^{\beta\alpha}, \quad \text{sdet}\|\tilde{\omega}^{\alpha\beta}\| \neq 0, \quad (2.7)$$

such that the set of $(T_{c|A}, \Phi_\alpha)(\Gamma_c)$ appears by the first-class constraints system, known as the *converted* constraints in $\mathcal{M}_c$ with deformed Hamiltonian $H_{c|0}(\Gamma_c)$. The converted functions satisfy to the involution relations in $\mathcal{M}_c$ for the system of the first-class constraints:

$$\{T_{c|A}, T_{c|B}\} = F_{AB}^C(\Gamma_c) T_{c|C} + F_{AB}^\alpha(\Gamma_c) \Phi_\alpha, \quad \{T_{c|A}, \Phi_\alpha\} = F_{A\alpha}^C(\Gamma_c) T_{c|C}, \quad (2.8)$$

$$\{\Phi_\alpha, \Phi_\beta\} = F_{\alpha\beta}^\gamma(\Gamma_c) \Phi_\gamma, \quad (2.9)$$

$$\{H_{c|0}, \Phi_\alpha\} = V_{c|\alpha}^\beta(\Gamma_c) \Phi_\beta + V_{c|\alpha}^A(\Gamma_c) T_{c|A}, \quad \{H_{c|0}, T_{c|A}\} = V_{c|A}^B(\Gamma_c) T_{c|B} + V_{c|A}^\alpha(\Gamma_c) \Phi_\alpha, \quad (2.10)$$

with, in general, new structure functions, $F_{c|AB}^C, F_{c|AB}^\alpha, F_{c|A\alpha}^C, F_{c|\alpha\beta}^\gamma, V_{c|\alpha}^\beta, V_{c|\alpha}^A, V_{c|A}^B, V_{c|A}^\alpha$ subject to analogous symmetry properties as for unconverted ones in (2.1)–(2.4) and with boundary conditions:

$$(T_{c|A}, \Phi_\alpha, H_{c|0})(\Gamma_c)|_{\zeta=0} = (T_A, \Theta_\alpha, H_0)(\Gamma). \quad (2.11)$$

The dynamics and gauge transformations for converted system are determined by the equations,

$$\partial_t \Gamma_c^P = \{\Gamma_c^P, H_{c|0}\}, \quad \delta_{\xi, \varpi} \Gamma_c^P = \{\Gamma_c^P, T_{c|A}(\Gamma_c)\} \xi^A(t) + \{\Gamma_c^P, \Phi_\alpha(\Gamma_c)\} \varpi^\alpha(t) \quad (2.12)$$

with arbitrary functions $\xi^A(t), \varpi^\alpha(t)$, so that after introduction of the gauge conditions $(\chi_c^B, \Xi^\beta)(\Gamma_c) = 0$ such that: $\{C, D\}|_{T_c=\Phi=0} = 0$, for any $C, D \in (\chi_c^B, \Xi^\beta)$:

$$\text{sdet}\|\{T_{c|A}, \chi_c^B\}\}|_{T_c=\Phi=0} \neq 0, \quad \text{sdet}\|\{\Phi_\alpha, \Xi^\beta\}\}|_{T_c=\Phi=0} \neq 0, \quad (2.13)$$

it leads to solution of appropriate Cauchy problem: $\Gamma_c^P(t_0) = \Gamma_{c|0}^P$ with admissible $\Gamma_{c|0}^P$. The crucial moment in the conversion procedure that the dynamics of initial and converted dynamical systems are equivalent.

The quantization problem for the dynamical system in terms of the functional integrals in both forms: initial and converted should be equivalent when calculating vacuum average expectation values for the quantities given on original phase-space $\mathcal{M}$.

We recall that the total phase space $\mathcal{M}_{\text{tot}}$, ($\mathcal{M} \subset \mathcal{M}_{\text{tot}}$) underlying the BRST-BFV generalized canonical Hamiltonian quantization [25–28] is parameterized (for linearly independent constraints T_A, Θ_α) by the canonical phase-space variables, $\Gamma_T^{\mathbf{P}}, \varepsilon(\Gamma_T^{\mathbf{P}}) = \varepsilon_{\mathbf{P}}$,

$$\Gamma_T^{\mathbf{P}} = (\Gamma^p, \Gamma_{\text{gh}}), \text{ with } \Gamma_{\text{gh}} = (C^A, \bar{\mathcal{P}}_A; \bar{C}_A, \mathcal{P}^A; \pi_A, \lambda^A), \quad (2.14)$$

	C^A	$\bar{\mathcal{P}}_A$	$\bar{C}_A$	$\mathcal{P}^A$	π_A	λ^A
ε	$\varepsilon_A + 1$	$\varepsilon_A + 1$	$\varepsilon_A + 1$	$\varepsilon_A + 1$	ε_A	ε_A
gh	1	-1	-1	1	0	0

(2.15)

with canonical pairs of ghost, $C^A, \bar{\mathcal{P}}_A$, antighost, $\bar{C}_A, \mathcal{P}^A$ and Lagrangian multipliers, π_A, λ^A for χ^A and T_A respectively with non-vanishing fundamental Poisson superbrackets:

$$\{C^A, \bar{\mathcal{P}}_B\} = \delta_B^A, \quad \{\bar{C}_A, \mathcal{P}^B\} = \delta_A^B, \quad \{\pi_A, \lambda^B\} = \delta_A^B. \quad (2.16)$$

The generating functional of Green's functions for a dynamical system in question has the form

$$Z_\Psi(I) = \int d\mu(\Gamma_T) \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{2} \Gamma_T^{\mathbf{P}}(t) \omega_{T|\mathbf{PQ}} \dot{\Gamma}_T^{\mathbf{Q}}(t) - H_\Psi(t) + I(t) \Gamma_T(t) \right] \right\}, \quad (2.17)$$

$$d\mu(\Gamma_T) = d\Gamma_T \delta(\Theta) \text{sdet}^{\frac{1}{2}} \|\{\Theta_\alpha, \Theta_\beta\}\|, \quad d\mu(\Gamma_T) = \prod_t d\mu(\Gamma_T(t)), \quad (2.18)$$

with functional measure $d\mu$ introduced according to [84, 85] and determines the partition function $Z_\Psi = Z_\Psi(0)$ at the vanishing external sources $I_{\mathbf{P}}(t)$ to $\Gamma_T^{\mathbf{P}}$. In (2.17), integration over time is taken over the range $t_{\text{in}} \leq t \leq t_{\text{out}}$; the functions of time $\Gamma_T^{\mathbf{P}}(t) \equiv \Gamma_T^{\mathbf{P}}$ for $t_{\text{in}} \leq t \leq t_{\text{out}}$ are trajectories, $\dot{\Gamma}_T^{\mathbf{P}}(t) \equiv d\Gamma_T^{\mathbf{P}}(t)/dt$; the quantities $\omega_{T|\mathbf{PQ}} = (-1)^{(\varepsilon_{\mathbf{P}}+1)(\varepsilon_{\mathbf{Q}}+1)} \omega_{T|\mathbf{QP}}$ compose an even supermatrix inverse to that with the (constant) elements $\omega_{T|\mathbf{PQ}}^{\mathbf{PQ}} = \{\Gamma_T^{\mathbf{P}}(t), \Gamma_T^{\mathbf{Q}}(t)\}_t$; the unitarizing Hamiltonian $H_\Psi(t) = H_\Psi(\Gamma_T(t))$ is determined by three t -local functions: even-valued $\mathcal{H}(t)$ with $gh(\mathcal{H}) = 0$, odd-valued functions $\Omega(t)$, with $gh(\Omega) = 1$, and $\Psi(t)$, with $gh(\Psi) = -1$, known as the BRST-BFV operator and gauge-fixing Fermion, given by the equations in terms of Dirac superbracket [81, 82] constructed with help of the second-class constraints Θ_α :

$$H_\Psi(t) = \mathcal{H}(t) + \{\Omega(t), \Psi(t)\}_{\mathcal{D}t}, \text{ with } \{A(t), B(t)\}_{\mathcal{D}t} = \left\{ A(\hat{\Gamma}), B(\hat{\Gamma}) \right\}_{\mathcal{D}} \Big|_{\hat{\Gamma}=\hat{\Gamma}(t)} \quad \forall A, B, \quad (2.19)$$

$$\{\Omega, \Omega\}_{\mathcal{D}} \simeq 0, \quad \{\mathcal{H}, \Omega\}_{\mathcal{D}} \simeq 0, \quad (2.20)$$

$$\{A(\Gamma_T), B(\Gamma_T)\}_{\mathcal{D}} = \{A(\Gamma_T), B(\Gamma_T)\} - \{A(\Gamma_T), \Theta_\alpha\} (\Delta^{-1})^{\alpha\beta} \{\Theta_\beta, B(\Gamma_T)\}, \quad (2.21)$$

with the boundary conditions

$$\mathcal{H}|_{\Gamma_{\text{gh}}=0} \simeq H_0(\Gamma), \quad \frac{\delta\Omega}{\delta C^A} \Big|_{\Gamma_{\text{gh}}=0} \simeq T_A(\Gamma), \quad \frac{\delta\Omega}{\delta \mathcal{P}^A} \Big|_{\Gamma_{\text{gh}}=0} = \pi_A(\Gamma), \quad \frac{\delta\Omega}{\delta \bar{C}_A} = 0. \quad (2.22)$$

The sign " $\simeq$ " means weak equality, modulo arbitrary linear combination of the second-class constraints Θ_α [81]. From equations (2.20) and the Jacobi identities for the Dirac superbracket, it follows that

$$\{H_\Psi, \Omega\}_{\mathcal{D}} \simeq 0. \quad (2.23)$$

The solutions for the generating equations (2.20) exist [34, 80, 83] in the form of series in powers of *minimal* ghost coordinates and momenta $C^A, \bar{\mathcal{P}}_A$ with use of $C\bar{\mathcal{P}}$ -ordering up to the second order in Γ_{gh} :

$$\mathcal{H} \simeq H_0 + (-1)^{\varepsilon_C} C^A V_A^C(\Gamma) \bar{\mathcal{P}}_C + O(C^2), \quad (2.24)$$

$$\Omega \simeq C^A \left(T_A + \frac{1}{2} (-1)^{\varepsilon_C + \varepsilon_A} C^B f_{BA}^C(\Gamma) \bar{\mathcal{P}}_C + O(C^2) \right) + \pi_A \mathcal{P}^A \equiv \Omega_{\text{min}} + \pi_A \mathcal{P}^A, \quad (2.25)$$

which encode in $\mathcal{H}$ and $\Omega_{\text{min}} = \Omega_{\text{min}}(\Gamma, \Gamma_{gh|m})$ the structure functions with the terms proportional to only T_A from the algebra of the constraints (2.1)–(2.4). The BRST-BFV operator Ω_{min} depends only on ghost coordinates and momenta from minimal sector $\Gamma_{gh|m} = (C^A, \bar{\mathcal{P}}_A)$ and satisfies to the equation (2.20). In turn, the simple form for the quadratic in powers of $\Gamma_T^{\mathbf{P}}$ gauge fermion to be sufficient for existence of $Z_\Psi(I)$ looks as

$$\Psi = \bar{C}_A \chi^A(\Gamma) + \lambda^A \bar{\mathcal{P}}_A \quad (2.26)$$

with the gauge conditions $\chi^A(\Gamma(t)) = 0$ satisfying to (2.6).

The integrand in (2.17) for $I = 0$ is invariant with respect to the infinitesimal BRST transformations, for odd-valued μ , $\mu^2 = 0$

$$\Gamma_T^{\mathbf{P}} \rightarrow \Gamma_T'^{\mathbf{P}} = \Gamma_T^{\mathbf{P}} (1 + \overleftarrow{s} \mu), \quad \text{with} \quad \overleftarrow{s} = \{\bullet, \Omega\}_{\mathcal{D}}, \quad (2.27)$$

with nilpotent generator $\overleftarrow{s}$: $\overleftarrow{s}^2 = 0$ (by virtue of Jacobi identity for Dirac superbracket and equation (2.20)), realized on the phase-space trajectories $\Gamma_T^{\mathbf{P}}(t)$ (to be solutions of the first equations in (2.5), but with Hamiltonian H_Ψ , with respective Poisson bracket and on the surface $\Theta_\alpha = 0$) as

$$\Gamma_T^{\mathbf{P}}(t) \rightarrow \Gamma_T'^{\mathbf{P}}(t) = \Gamma_T^{\mathbf{P}}(t) (1 + \overleftarrow{s} \mu), \quad \text{with} \quad \Gamma_T^{\mathbf{P}}(t) \overleftarrow{s} = \{\Gamma_T^{\mathbf{P}}(t), \Omega\}_{\mathcal{D}_t}. \quad (2.28)$$

The invariance of the integrand is due to additional (as compared for the Poisson superbracket) property for Dirac superbracket: $\{F, \Theta_\alpha\}_{\mathcal{D}} \simeq 0$ for any function F given on M_{tot} .

For the converted dynamical system with first-class constraints $(T_{c|A}, \Phi_\alpha)(\Gamma_c)$ given in $\mathcal{M}_c$ the total phase space $\mathcal{M}_{c|\text{tot}}$, ($\mathcal{M}_c \subset \mathcal{M}_{\text{tot}}$ and $\mathcal{M}_{\text{tot}} \subset \mathcal{M}_{c|\text{tot}}$) is parameterized by the canonical phase-space variables, $\Gamma_{c|T}^{\mathfrak{P}}$,

$$\Gamma_{c|T}^{\mathfrak{P}} = (\Gamma_c^P, \Gamma_{\text{gh}}, \Gamma_{2|\text{gh}}), \quad \text{with} \quad \Gamma_{2|\text{gh}} = (C^\alpha, \bar{\mathcal{P}}_\alpha; \bar{C}_\alpha, \mathcal{P}^\alpha; \pi_\alpha, \lambda^\alpha), \quad (2.29)$$

	C^α	$\bar{\mathcal{P}}_\alpha$	$\bar{C}_\alpha$	$\mathcal{P}^\alpha$	π_α	λ^α
ε	$\varepsilon_\alpha + 1$	$\varepsilon_\alpha + 1$	$\varepsilon_\alpha + 1$	$\varepsilon_\alpha + 1$	ε_α	ε_α
gh	1	-1	-1	1	0	0

(2.30)

with account for the representation (2.14), (2.15) and properties (2.16) for Γ_{gh} which correspond to only initial first-class constraints T_A subsystem, with canonical pairs of ghost,

C^α , $\overline{\mathcal{P}}_\alpha$, antighost, $\overline{C}_\alpha$, $\mathcal{P}^\alpha$ and Lagrangian multipliers, $\pi_\alpha, \lambda^\alpha$ for Ξ^α and Φ_α respectively with the same non-vanishing fundamental Poisson superbrackets as in (2.16) determined now in $\mathcal{M}_{c|tot}$, with respect to the coordinates $\Gamma_{c|T}^\mathfrak{P}$ so that $\{\Gamma_{gh}, \Gamma_{2|gh}\}_T = 0$.

The generating functional of Green's functions for the dynamical system with converted constraints has the representation

$$Z_{\Psi_c}(I_T) = \int d\Gamma_{c|T} \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{2} \Gamma_{c|T}^\mathfrak{P}(t) \omega_{cT|\mathfrak{P}\Omega} \dot{\Gamma}_{c|T}^\Omega(t) - H_{\Psi_c}(t) + I_T(t) \Gamma_{c|T}(t) \right] \right\}, \quad (2.31)$$

and determines the partition function $Z_{\Psi_c} = Z_{\Psi_c}(0)$ at the vanishing external sources $I_{T|\mathfrak{P}}(t)$ to $\Gamma_{c|T}^\mathfrak{P}$ with the same properties as for (2.17), whereas the quantities $\omega_{cT|\mathfrak{P}\Omega} = (-1)^{(\varepsilon_\mathfrak{P}+1)(\varepsilon_\Omega+1)} \omega_{cT|\Omega\mathfrak{P}}$ compose an even supermatrix inverse to that with (constant) elements $\omega_{c|T}^{\mathfrak{P}\Omega} = \{\Gamma_{c|T}^\mathfrak{P}(t), \Gamma_{c|T}^\Omega(t)\}_T$. Again the unitarizing Hamiltonian $H_{\Psi_c}(t) = H_{\Psi_c}(\Gamma_{c|T}(t))$ is determined by three t -local functions: $\mathcal{H}_c(t)$ with $(\varepsilon, gh)(\mathcal{H}_c) = (0, 0)$, $\Omega_c(t)$, with $(\varepsilon, gh)(\Omega_c) = (1, 1)$, and $\Psi_c(t)$, with $(\varepsilon, gh)(\Psi_c) = (1, -1)$, defined for dynamical system with the converted constraints $T_{c|A}, \Phi_\alpha$:

$$H_{\Psi_c}(t) = \mathcal{H}_c(t) + \{\Omega_c(t), \Psi_c(t)\}_{T|t}, \quad (2.32)$$

$$\{\Omega_c, \Omega_c\}_T = 0, \quad \{\mathcal{H}_c, \Omega_c\}_T = 0, \quad (2.33)$$

subject to the boundary conditions

$$\mathcal{H}_c|_{\Gamma_{gh}=\Gamma_{2|gh}=0} = H_{c|0}(\Gamma_c), \quad \left(\frac{\delta \Omega_c}{\delta C^A}, \frac{\delta \Omega_c}{\delta C^\alpha} \right) \Big|_{\Gamma_{gh}=\Gamma_{2|gh}=0} = (T_{c|A}, \Phi_\alpha), \quad (2.34)$$

$$\left(\frac{\delta \Omega_c}{\delta \mathcal{P}^A}, \frac{\delta \Omega_c}{\delta \mathcal{P}^\alpha} \right) \Big|_{\Gamma_{gh}=\Gamma_{2|gh}=0} = (\pi_A, \pi_\alpha), \quad \frac{\delta \Omega_c}{\delta \overline{C}_A} = \frac{\delta \Omega_c}{\delta \overline{C}_\alpha} = 0. \quad (2.35)$$

From the generating equations (2.33) it follows that total Hamiltonian commutes with BRST charge Ω_c : $\{H_{\Psi_c}, \Omega_c\}_T = 0$. The solutions for the equations (2.33) exist in the form of series in powers of *minimal* ghost coordinates and momenta $C^A, \overline{\mathcal{P}}_A, C^\alpha, \overline{\mathcal{P}}_\alpha$ and up to the second order in Γ_{gh} looks as:

$$\mathcal{H}_c = H_{c|0} + (C^A, C^\alpha) \begin{pmatrix} V_{c|A}^C & V_{c|\alpha}^C \\ V_{c|A}^\beta & V_{c|\alpha}^\beta \end{pmatrix} \begin{pmatrix} (-1)^{\varepsilon_C} \overline{\mathcal{P}}_C \\ (-1)^{\varepsilon_\beta} \overline{\mathcal{P}}_\beta \end{pmatrix} + O(C^2), \quad (2.36)$$

$$\begin{aligned} \Omega_c = (C^A, C^\alpha) & \left[\begin{pmatrix} T_{c|A} \\ \Phi_\alpha \end{pmatrix} + \frac{1}{2} \begin{pmatrix} [C^B F_{BA}^C + C^\beta F_{\beta A}^C] (-1)^{\varepsilon_A} & [C^B F_{BA}^\gamma + C^\beta F_{\beta A}^\gamma] (-1)^{\varepsilon_A} \\ C^B F_{B\alpha}^C (-1)^{\varepsilon_\alpha} & C^\beta F_{\beta\alpha}^\gamma (-1)^{\varepsilon_\alpha} \end{pmatrix} \right. \\ & \left. \times \begin{pmatrix} (-1)^{\varepsilon_C} \overline{\mathcal{P}}_C \\ (-1)^{\varepsilon_\gamma} \overline{\mathcal{P}}_\gamma \end{pmatrix} + O(C^2) \right] + \pi_A \mathcal{P}^A + \pi_\alpha \mathcal{P}^\alpha \equiv \Omega_{c|min} + \pi_A \mathcal{P}^A + \pi_\alpha \mathcal{P}^\alpha, \end{aligned} \quad (2.37)$$

which encode in $\mathcal{H}_c$ and BRST-BFV operator $\Omega_{c|min} \equiv \Omega_{c|min}(\Gamma_c, \Gamma_{gh|m})$ (depending on the ghost coordinates and moments in minimal sector $\Gamma_{gh|m} \equiv (C^A, \overline{\mathcal{P}}_A; C^\alpha, \overline{\mathcal{P}}_\alpha)$) the structure functions from the algebra of the converted constraints (2.8)–(2.10).

The quadratic in powers of $\Gamma_{c|T}^\mathfrak{P}$ gauge fermion Ψ_c to be sufficient for existence of $Z_{\Psi_c}(I_T)$ may be chosen as

$$\Psi_c = (\overline{C}_A, \overline{C}_\alpha) (\chi_c^A(\Gamma_c), \Xi^\alpha(\Gamma_c))^T + (\lambda^A, \lambda^\alpha) (\overline{\mathcal{P}}_A, \overline{\mathcal{P}}_\alpha)^T, \quad (2.38)$$

where the upper sign “ T ” denotes the matrix transposition and the gauge conditions $\chi_c^A(\Gamma_c(t)) = 0$, $\Xi^\alpha(\Gamma_c(t)) = 0$ should satisfy to (2.13).

The integrand in (2.31) for $I_T = 0$ is invariant with respect to the infinitesimal BRST transformations,

$$\Gamma_{c|T}^{\mathfrak{P}} \rightarrow \Gamma_{c|T}^{\mathfrak{P}} = \Gamma_{c|T}^{\mathfrak{P}} (1 + \overleftarrow{s}_c \mu) , \quad \text{with} \quad \overleftarrow{s}_c = \{\bullet, \Omega_c\}_T , \quad (2.39)$$

with nilpotent generator $\overleftarrow{s}_c$: $\overleftarrow{s}_c^2 = 0$, realized on phase-space trajectories $\Gamma_{c|T}^{\mathfrak{P}}(t)$ (to be solutions of the first equations in (2.5), but with Hamiltonian H_{Ψ_c} and respective Poisson bracket) analogously to the rule (2.28) but with Poisson superbracket determined on $\mathcal{M}_{c|\text{tot}}$ instead of Dirac one.

The equivalence of both quantizations for the initial with $(T_A, \Theta_\alpha, H_0)(\Gamma)$ in $\mathcal{M}$ and converted with $(T_{c|A}, \Phi_\alpha, H_{c|0})(\Gamma_c)$ in $\mathcal{M}_c$ dynamical systems means that the vacuum average expectation values for any quantity $A(\Gamma)$ determined on the initial phase-space in $\mathcal{M}$ coincide when calculating with respect to path integrals Z_Ψ (2.17) and Z_{Ψ_c} (2.31):

$$\langle A \rangle_\Psi = \langle \langle A \rangle \rangle_{\Psi_c} \quad \text{where} \quad (2.40)$$

$$\langle A \rangle_\Psi = Z_\Psi^{-1} \int d\mu(\Gamma_T) A(\Gamma) \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{2} \Gamma_T^{\mathbf{P}}(t) \omega_{T|\mathbf{PQ}} \dot{\Gamma}_T^{\mathbf{Q}}(t) - H_\Psi(t) \right] \right\} , \quad (2.41)$$

$$\langle \langle A \rangle \rangle_{\Psi_c} = Z_{\Psi_c}^{-1} \int d\Gamma_{c|T} A(\Gamma) \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{2} \Gamma_{c|T}^{\mathfrak{P}}(t) \omega_{cT|\mathfrak{PQ}} \dot{\Gamma}_{c|T}^{\mathfrak{Q}}(t) - H_{\Psi_c}(t) \right] \right\} . \quad (2.42)$$

On the operator level for the quantization problem let us consider, first, a constrained dynamical system with only second-class constraints Θ_α satisfying to the relations (2.2) and (2.4) for $T_A \equiv 0$. In case of existing the splitting of Θ_α (at least, locally) on two subsystems $\Theta_\alpha(\Gamma) \rightarrow \Theta'_\alpha(\Gamma) = \Lambda_\alpha^\beta(\Gamma) \Theta_\beta(\Gamma) = (\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}})$ with non-degenerate (on the surface $\Theta_\alpha = 0$) supermatrix of rotation of the constraints: $\text{sdet} \|\Lambda_\alpha^\beta\| \neq 0$, for the index α division: $\alpha = (\bar{\alpha}, \underline{\alpha})$ for $\bar{\alpha} = 1, \dots, \frac{1}{2}m$ and $\underline{\alpha} = \frac{1}{2}m + 1, \dots, m$ such that each subsystems $\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}}$ appear by the first-class constraints ones:

$$\{\theta_{\bar{\alpha}}, \theta_{\bar{\beta}}\} = \bar{f}_{\bar{\alpha}\bar{\beta}}^\gamma(\Gamma) \theta_{\bar{\gamma}}, \quad \{\theta_{\underline{\alpha}}, \theta_{\underline{\beta}}\} = \underline{f}_{\underline{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\underline{\gamma}}, \quad (2.43)$$

$$\{\theta_{\bar{\alpha}}, \theta_{\underline{\beta}}\} = \check{\Delta}_{\bar{\alpha}\underline{\beta}}(\Gamma) + \check{f}_{\bar{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\bar{\gamma}} + \check{f}_{\bar{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\underline{\gamma}}, \quad \{H_0, \theta_{\bar{\alpha}}\} = \bar{V}_{\bar{\alpha}}^{\bar{\beta}} \theta_{\bar{\beta}}, \quad \{H_0, \theta_{\underline{\alpha}}\} = \underline{V}_{\underline{\alpha}}^{\underline{\beta}} \theta_{\underline{\beta}}. \quad (2.44)$$

In (2.43), (2.44) the structure functions $\bar{f}_{\bar{\alpha}\bar{\beta}}^\gamma$, $\underline{f}_{\underline{\alpha}\underline{\beta}}^\gamma$, $\check{f}_{\bar{\alpha}\underline{\beta}}^\gamma$, $\check{f}_{\underline{\alpha}\bar{\beta}}^\gamma$, $\bar{V}_{\bar{\alpha}}^{\bar{\beta}}$, $\underline{V}_{\underline{\alpha}}^{\underline{\beta}}$ are related to ones in (2.2), (2.4) for only subsystem of Θ_α with representation for invertible $\|\check{\Delta}_{\bar{\alpha}\underline{\beta}}(\Gamma)\|_{\Theta=0}$ with use of the Leibnitz rule for the Poisson brackets and for $\varepsilon(\Lambda_{\underline{\beta}}^\beta) = \varepsilon_\beta + \varepsilon_{\underline{\beta}}$:

$$\check{\Delta}_{\bar{\alpha}\underline{\beta}}(\Gamma) = \Lambda_{\bar{\alpha}}^\alpha(\Gamma) \Delta_{\alpha\beta}(\Gamma) \Lambda_{\underline{\beta}}^\beta(\Gamma) (-1)^{\varepsilon_\beta(\varepsilon_{\underline{\beta}}+1)} \Rightarrow \text{sdet} \|\check{\Delta}_{\bar{\alpha}\underline{\beta}}(\Gamma)\|_{\Theta'_\alpha=0} \neq 0. \quad (2.45)$$

In the corresponding Hilbert space H_Γ [for the correspondence $\Gamma^p \rightarrow \hat{\Gamma}^p$: $[\hat{\Gamma}^p, \hat{\Gamma}^q] = i\hbar\omega^{pq}$, $\omega^{pq} = \text{const}$, with representation respecting the division for Θ_α , but without exception of the degrees of freedom related to the second-class constraints and with choice of some qp -ordering for $\Theta_\alpha(\hat{\Gamma})$] according to Dirac approach [81, 86] should be realized by means of

only the first-class operator constraints imposing to extract the physical states $|\psi\rangle \in H_\Gamma^{\text{phys}}$ from Hilbert subspace of physical vectors $H_\Gamma^{\text{phys}} \subset H_\Gamma$:

$$\theta_{\bar{\alpha}}(\hat{\Gamma})|\psi\rangle = 0, \quad \forall |\psi\rangle \in H_\Gamma^{\text{phys}}, \quad \text{where } [\theta_{\bar{\alpha}}, \theta_{\bar{\beta}}]|_{\theta_{\bar{\alpha}}=0} = 0. \quad (2.46)$$

The physical states should satisfy to the Schrodinger equation with Hamiltonian not depending on the rest constraints $\theta_{\bar{\alpha}}$, playing the role of the gauge conditions for $\theta_{\bar{\alpha}}$:

$$\left(i\hbar\partial_t - H_0(\hat{\Gamma})|_{\theta_{\bar{\alpha}}=0} \right) |\psi\rangle = 0. \quad (2.47)$$

In turn, for the classically equivalent dynamical system of converted operator first-class constraints $(\Phi_\alpha)(\hat{\Gamma}_c)$ [for the correspondence $\Gamma_c^P \rightarrow \hat{\Gamma}_c^P = (\hat{\Gamma}^P, \hat{\zeta}^\alpha)$: with choice of some ordering for the products in powers of ζ^α additional to qp -ones for $\Phi_\alpha(\hat{\Gamma}_c)$ and without $T_{c|A}$] in $\mathcal{M}_c$ with operator of Hamiltonian $H_{c|0}(\hat{\Gamma}_c)$ (2.9), (2.10) it is valid the

Statement 1. The second-class constraints system $\Theta_\alpha(\hat{\Gamma})$ converted into first-class constraints one $\Phi_\alpha(\hat{\Gamma}, \hat{\zeta})$ with additional to $\hat{\Gamma}^P$ operators $\hat{\zeta}^\alpha$ (2.7), whose number coincides with one of Θ_α , satisfying to the superalgebra: $[\hat{\zeta}^\alpha, \hat{\zeta}^\beta] = i\hbar\tilde{\omega}^{\alpha\beta}$ with constant $\tilde{\omega}^{\alpha\beta}$:

$$[\Phi_\alpha(\hat{\Gamma}, \hat{\zeta}), \Phi_\beta(\hat{\Gamma}, \hat{\zeta})] = F_{\alpha\beta}^\gamma(\hat{\Gamma}, \hat{\zeta})\Phi_\gamma(\hat{\Gamma}, \hat{\zeta}), \quad \text{with } \Phi_\alpha(\hat{\Gamma}, 0) = \Theta_\alpha(\hat{\Gamma}), \quad (2.48)$$

selects the same set of the physical states in H_Γ as the converted constraints $\Phi_\alpha(\hat{\Gamma}, \hat{\zeta})$ select from $H_\Gamma \otimes H_{\hat{\zeta}}$:

$$\begin{aligned} H_\Gamma^{\text{phys}} &= \{|\psi\rangle | \theta_{\bar{\alpha}}(\hat{\Gamma})|\psi\rangle = 0, |\psi\rangle \in H_\Gamma\} \\ &= \{|\chi\rangle | \Phi_\alpha(\hat{\Gamma}, \hat{\zeta})|\chi\rangle = 0, |\chi\rangle \in H_\Gamma \otimes H_{\hat{\zeta}}\}. \end{aligned} \quad (2.49)$$

Note, first, the operator functions $F_{\alpha\beta}^\gamma(\hat{\Gamma}, \hat{\zeta})$ in (2.48) obey to the properties analogous to ones for $f_{\alpha\beta}^\gamma(\hat{\Gamma})$ (2.3) and there are no anomalies in the right-hand side of (2.48) which in opposite case should be proportional to $\hbar^2 D_{\alpha\beta}(\hat{\Gamma}, \hat{\zeta})$. Second, for special (but interesting) cases the additional phase-space operators $\hat{\zeta}^\alpha$ may be chosen, as respecting the division of the second-class constraints: $\Theta'_\alpha = (\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}})$ as follows $\hat{\zeta}^\alpha = (q^\alpha, p^{\bar{\alpha}})$: $[q^\alpha, p^{\bar{\alpha}}] = i\hbar\delta^{\bar{\alpha}\alpha}$, in particular, as for the additional operators (case of additive conversion) $\Phi_\alpha(\hat{\Gamma}_c) = \Theta_\alpha(\hat{\Gamma}) + \vartheta_\alpha(\hat{\zeta})$ for $[\Theta_\alpha, \vartheta_\beta] = 0$, for Fock space $H_{\hat{\zeta}}$ with oscillators

$$(B^a, B^{a+}) \equiv \frac{1}{\sqrt{2}} \left(p^{\bar{\alpha}} - \frac{i}{\hbar} q^\alpha, p^{\bar{\alpha}} + \frac{i}{\hbar} q^\alpha \right) \quad \text{for } a = 1, \dots, \frac{1}{2}m, \quad (2.50)$$

satisfying to $[B^a, B^{b+}] = \delta^{ab}$. Third, the presentation (2.49) permits the boundary conditions for any $|\chi\rangle \in H_\Gamma \otimes H_{\hat{\zeta}}$: $|\chi\rangle|_{\hat{\zeta}=0} = |\psi\rangle \in H_\Gamma$. Fourth, the quantum evolution of the converted constrained system is described by the Schrodinger equation analogous to (2.47)

$$\left(i\hbar\partial_t - H_{c|0}(\hat{\Gamma}) \right) |\chi\rangle = 0. \quad (2.51)$$

Fifth, the Statement 1 guarantees a preservation of explicit Poincare covariance for field-theoretic models, like QED, gravity, models with HS fields when working with converted constraints.

For proper gauge dynamical system with some operator first-class constraints system $T_A(\hat{\Gamma})$ only (without reference to any second-class constraints) it is valid the following

Statement 2. Nilpotent Grassman-odd BRST-BFV operator $Q = \hat{C}^A T_A(\hat{\Gamma}) + \text{“more”}$, $Q \equiv \Omega_{\min}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ from (2.25) with $gh(Q) = 1$ constructed with respect to the system of $T_A(\hat{\Gamma})$ in the Hilbert space $H_Q = H_\Gamma \otimes H_{gh|m}$ admitting Z -grading: $H_Q = \sum_k H_Q^k$, $gh(|\chi^k\rangle) = -k$, for any $|\chi^k\rangle \in H_Q^k$ with Hilbert space $H_{gh|m}$ generated by the operators $\hat{C}^A, \hat{P}_A$ from the minimal sector subject to (2.15) and $[\hat{C}^A, \hat{P}_B] = i\hbar\delta_B^A$ permits to find the physical Hilbert subspace $\mathcal{H}_1^{\text{phys}}$ as follows:

$$\begin{aligned} \mathcal{H}_1^{\text{phys}} &= \{|\psi\rangle | T_A(\hat{\Gamma})|\psi\rangle = 0, |\psi\rangle \in H_\Gamma\} \\ &= \{|\Psi\rangle | gh(|\Psi\rangle) = 0, |\Psi\rangle \in \ker Q / \text{Im } Q\}. \end{aligned} \quad (2.52)$$

for the quotient of the subspace of Q -closed vectors ($\ker Q \in H_Q$) with respect to subspace of Q -exact ones ($\text{Im } Q \in H_Q$). The evolution of the system is described by the Schrodinger equations (2.47) with Hamiltonian $H_0(\hat{\Gamma})$ acting in Hilbert space H_Γ and one with Hamiltonian $\mathcal{H}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ (2.24) but in H_Q^0 .

When presenting operator $Q(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$, $\mathcal{H}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ one should use some ordering for ghost operators in (2.24), (2.25), e.g. $\hat{C}\hat{\mathcal{P}}$ ordering. The representation for $\mathcal{H}_1^{\text{phys}}$ in the second row, equivalently, in terms of BRST Q -complex (equivalent to chain of reducible gauge transformations for state vector: $|\chi^0\rangle$, and gauge parameters: $|\chi^k\rangle$ $k \geq 1$) in H_Q , may be equivalently rewritten as follows

$$\mathcal{H}_1^{\text{phys}} = \{|\chi^0\rangle | \delta|\chi^0\rangle = Q|\chi^1\rangle, \dots, \delta|\chi^{M-1}\rangle = Q|\chi^M\rangle, \delta|\chi^M\rangle = 0, |\chi^k\rangle \in H_Q^k\} \quad (2.53)$$

for $k = 0, \dots, M$ with M being finite for finite number of the constraints $T_A(\hat{\Gamma})$: $[\Phi_A] = M_+ + M_-$, and therefore finite degrees of $\hat{C}^A, \hat{P}_A$ in decomposition of arbitrary vector $|\Psi\rangle \in H(Q)$ in powers of ghosts operators. Thus, $H_Q^{k+M} \equiv 0$ for $k \in \mathbb{N}$, for $M = M_+ + M_-$.

From the Statements 1, 2 it follows the (modulo description the evolution problem)

Corollary 1. The physical states in H_Γ for the dynamical system with second-class constraints, permitting the division on two sets with only first-class constraints, $\Theta_\alpha(\hat{\Gamma}) \rightarrow \Theta'_\alpha(\hat{\Gamma}) = (\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}})$, maybe equivalently presented by nilpotent BRST-BFV operator $Q_{c|2} = \hat{C}^\alpha \Phi_\alpha(\hat{\Gamma}_c) + \text{“more”}$, $Q_c \equiv \Omega_{c|\min}(\hat{\Gamma}_c, \hat{\Gamma}_{gh|m})$ (2.37) for $T_A \equiv 0$, constructed with respect to the system of converted second-class constraints $\Phi_\alpha(\hat{\Gamma}, \hat{\zeta})$ in the Hilbert space $H(Q_{c|2}) = H_\Gamma \otimes H_\zeta \otimes H_{2gh|m}$ in the form:

$$\mathcal{H}_2^{\text{phys}} = \{|\psi\rangle | \theta_{\bar{\alpha}}(\hat{\Gamma})|\psi\rangle = 0, |\psi\rangle \in H_\Gamma\} \quad (2.54)$$

$$= \{|\chi^0\rangle | \delta|\chi^0\rangle = Q_{c|2}|\chi^1\rangle, \dots, \delta|\chi^{M-1}\rangle = Q_{c|2}|\chi^M\rangle, \delta|\chi^M\rangle = 0, |\chi^k\rangle \in H^k(Q_{c|2})\}, \quad (2.55)$$

for $k = 0, \dots, M$, $(\varepsilon, gh)(|\chi^k\rangle) = (k \bmod 2, -k)$.

Finally, for initial dynamical system with mixed-class constraints T_A, Θ_α and Hamiltonian H_0 , satisfying to the operator analog of the superalgebra (2.1)–(2.4), we get to

Corollary 2. The physical states in H_Γ for the dynamical system with first - $T_A(\hat{\Gamma})$ and second-class Θ_α constraints, permitting for the latter the division on two sets with only first-class constraints, $\Theta_\alpha(\hat{\Gamma}) \rightarrow \Theta'_\alpha(\hat{\Gamma}) = (\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}})$ subject to operator analog of the relations (2.43), maybe equivalently presented by nilpotent BRST-BFV operator $Q_c = \hat{C}^\alpha \Phi_\alpha(\hat{\Gamma}_c) + \hat{C}^A T_{c|A}(\hat{\Gamma}_c) + \text{"more"}$, $Q_c \equiv \Omega_{c| \min}(\hat{\Gamma}_c, \hat{\Gamma}_{gh|m})$ (2.37), constructed with respect to the system of converted first $T_{c|A}(\hat{\Gamma}_c)$ and second-class constraints $\Phi_\alpha(\hat{\Gamma}_c)$ in the Hilbert space $H(Q_c) = H_\Gamma \otimes H_\zeta \otimes H_{gh|m} \otimes H_{2|gh|m}$ in the form:

$$\mathcal{H}_{1,2}^{\text{phys}} = \{|\psi\rangle | (T_A(\hat{\Gamma}), \theta_{\bar{\alpha}}(\hat{\Gamma}))|\psi\rangle = (0, 0), |\psi\rangle \in H_\Gamma\} \quad (2.56)$$

$$= \{|\chi^0\rangle | \delta|\chi^0\rangle = Q_c|\chi^1\rangle, \dots, \delta|\chi^{N-1}\rangle = Q_c|\chi^N\rangle, \delta|\chi^N\rangle = 0, |\chi^k\rangle \in H^k(Q_c)\} \quad (2.57)$$

for $k = 0, \dots, N$.

The above Statements and Corollaries are the crucial results in the application of the BRST-BFV method to a canonical quantization of any dynamical system with finite degrees of freedom subject to first- and second-class constraints. In fact, Statement 2 was proved in the excellent textbook [87] (see chapter 14 and Theorem 14.7 therein). The correctness of Statement 1 is based, first of all, on the previously established (see [34, 35]) classical equivalence of a dynamical system with second-class constraints to the same dynamical system with converted first-class constraints. Secondly, the Statement is based on the representation given by the first line of (2.49) for the physical space H_Γ^{phys} of a dynamical system with second-class constraints known from Dirac's work [86]. Therefore, under the absence of anomalies in (2.48) a classically equivalent dynamic al system with converted first-class constraints leads to the same physical space H_Γ^{phys} given by the second line of (2.49), according to the same Dirac quantization concept for a first-class constraints system, also presented in [87] (see chapter 13 and section 13.3 therein).

In case of the first-class constraints subsystem $\{T_A(\hat{\Gamma})\}$ to be closed with respect to Hermitian conjugation: $(T_A)^+ \in \{T_A(\hat{\Gamma})\}$, the results of the Statement 2 and Corollary 2 can be refined. Namely, the property above means that the presentation

$$T_A(\hat{\Gamma}) = (t_{\bar{a}}, t_{\underline{a}}; t_e) \text{ for } (t_{\bar{a}}, t_e)^+ = (t_{\underline{a}}, t_e) \quad (2.58)$$

holds with division of the index A : $A = (\bar{a}, \underline{a}, e)$ for $\bar{a} = 1, \dots, \frac{1}{2}(M-p)$, $\underline{a} = \frac{1}{2}(M-p) + 1, \dots, M-p$ and $e = M-p+1, \dots, M$. Therefore, the only zero-mode constraints t_e and half from the pairs $(t_{\bar{a}}, t_{\underline{a}})$ (e.g. $t_{\bar{a}}$) should be imposed to select the physical state vectors in H_Γ :

$$\mathcal{H}_1^{\text{phys}} = \{|\psi\rangle | (t_{\bar{a}}(\hat{\Gamma}), t_e(\hat{\Gamma}))|\psi\rangle = 0, |\psi\rangle \in H_\Gamma\} \quad (2.59)$$

instead of the whole set of $T_A(\hat{\Gamma})$ in (2.52) and then in (2.56). The argumentation to prove the last condition is analogous to one considered for the case of conditions (from the Virasoro algebra) which have been used to select physical states from the total Hilbert space for bosonic string or for the superstring models [88, 89], but now for the finite set of constraints.

We stress that when analyzing the presentation the structure of the physical states we did not considered the BRST operator, $\Omega(\hat{\Gamma}_T)$, $\Omega_c(\hat{\Gamma}_{c|T})$ and unitarizing Hamiltonian, $H_\Psi(\hat{\Gamma}_T)$, $H_{\Psi_c}(\hat{\Gamma}_{c|T})$ for the dynamical system in question, which act on more wider respective Hilbert spaces (than $H(Q)$, $H(Q_c)$) extended by additional operators $\hat{\bar{C}}$, $\hat{\bar{P}}$, $\hat{\pi}$, $\hat{\lambda}$ from non-minimal sectors. Therefore, it is not necessary to operate with the gauge conditions $\chi^A(\hat{\Gamma})$, $\Xi^\beta(\hat{\Gamma}_c)$ to select physical states from $H(Q)$, $H(Q_c)$. There exists the question does it possible to select any physical vectors for the dynamical system with mixed-class constraints by means of intermediate procedure, with BRST-BFV operator Q for only first-class constraints $T_A(\hat{\Gamma})$ imposed on such state, $Q|\psi\rangle = 0$, with $|\psi\rangle \in H^0(Q)$ together with special extension of the maximal subsystem of the first-class constraints $\theta_{\bar{\alpha}}(\hat{\Gamma})$ from $\Theta_\alpha(\hat{\Gamma})$: $\theta_{\bar{\alpha}}(\hat{\Gamma})|\psi\rangle = 0$? We consider the solution for this problem within application of BRST-BFV approach for the construction of the unconstrained and constrained Lagrangian formulations for HS fields given on $\mathbb{R}^{1,d-1}$.

3 BRST operator for HS symmetry superalgebra for mixed-symmetric fermionic fields

In this section, we shortly repeating a basic results of our research [68] consider a half-integer spin irreducible representation of Poincare group in a Minkowski space $\mathbb{R}^{1,d-1}$ within metric-like formalism which is to be described by a spin-tensor field: $\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \equiv \Psi_{\mu_1^1 \dots \mu_{n_1}^1, \mu_1^2 \dots \mu_{n_2}^2, \dots, \mu_1^k \dots \mu_{n_k}^k} A(x)$, with the Dirac index A (later suppressed) of rank $\sum_{j=1}^k n_j$ and the generalized spin $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2, \dots, n_k + 1/2)$ ($n_1 \geq n_2 \geq \dots \geq n_k > 0$, $k \leq [(d-1)/2]$), which corresponds to a Young tableaux $Y(s_1, s_2, \dots, s_k)$ with k rows of length $n_1, n_2, \dots, n_k$, respectively,

$$Y(s_1, s_2, \dots, s_k) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \mu_1^1 & \mu_2^1 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu_{n_1}^1 \\ \hline \mu_1^2 & \mu_2^2 & \cdot & \cdot & \cdot & \cdot & \mu_{n_2}^2 & \\ \hline \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \hline \mu_1^k & \mu_2^k & \cdot & \cdot & \cdot & \cdot & \mu_{n_k}^k & \\ \hline \end{array}, \quad (3.1)$$

The field is symmetric with respect to permutations of each type of Lorentz indices μ^i and obeys, respectively, to the Dirac, gamma-tracelessness and mixed-symmetry equations [for $i, j = 1, \dots, k$; $l_i, m_i = 1, \dots, n_i$],

$$\epsilon \gamma^\mu \partial_\mu \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0, \quad (3.2)$$

$$\gamma^{\mu_{l_i}^i} \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0, \quad (3.3)$$

$$\Psi_{(\mu^1)_{n_1}, \dots, \underbrace{(\mu^i)_{n_i}, \dots, \mu_1^j \dots \mu_{l_j}^j}_{\dots}, \dots, \mu_{n_j}^j, \dots, (\mu^k)_{n_k}} = 0, \quad i < j, \quad 1 \leq l_j \leq n_j, \quad (3.4)$$

and, in addition for any neighbour rows with equal length: $n_i = n_{i+1} = \dots = n_{i+m}$, $m = 1, \dots, k - i$ the field should be identical with respect to permutation of any two groups

from the total group of the respective indices $(\mu^i)_{n_i}, (\mu^{i+1})_{n_{i+1}}, \dots, (\mu^{i+m})_{n_{i+m}}$. The underlined figure bracket in (3.4) denotes that the indices included in it do not take part in symmetrization, which thus concerns only the indices $(\mu^i)_{n_i}, \mu_{l_j}^j$ in $\{(\mu^i)_{n_i}, \underbrace{\mu_1^j \dots \mu_{l_j}^j}_{\text{bracketed}}\}$.

Equivalently, in terms of the general state (being by a Dirac-like spinor) of the Fock space, $\mathcal{H}$, generated by the k bosonic pairs of creation and annihilation operators $a_i^{\mu_{i_i}^i}, a_i^{+\mu_{i_i}^i}$, $i = 1, \dots, k$:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \dots \sum_{n_k=0}^{n_{k-1}} \frac{\iota^{\sum_i n_i}}{n_1! \times \dots \times n_k!} \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \prod_{i=1}^k \prod_{l_i=1}^{n_i} a_i^{+\mu_{l_i}^i} |0\rangle, \quad (3.5)$$

$$\text{for } [a_{\mu^i}^i, a_{\nu^j}^{j+}] = -\eta_{\mu^i \nu^j} \delta^{ij}, \quad \delta^{ij} = \text{diag}(1, 1, \dots, 1), \quad (3.6)$$

providing the symmetry property of $\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}}$ the set of the relations (3.2)–(3.4) are equivalent for any spin $\mathbf{s}$ to set of $(\frac{1}{2}k(k+1) + 1)$ operator equations:

$$\tilde{t}_0 |\Psi\rangle = \tilde{t}^i |\Psi\rangle = t^{i_1 j_1} |\Psi\rangle = 0, \quad (3.7)$$

$$\text{for } (\tilde{t}_0, \tilde{t}^i, t^{i_1 j_1}) = \left(-i\gamma^\mu \partial_\mu, \gamma^\mu a_\mu^i, a_\mu^{i_1+} a^{j_1 \mu} \right), \quad i_1 < j_1. \quad (3.8)$$

Adding to the relations (3.7), the generalized spin constraints imposed on $|\Psi\rangle$ in terms of number particle operator g_0^i :

$$g_0^i |\Psi\rangle = (n_i + \frac{d}{2}) |\Psi\rangle, \quad \text{with } g_0^i = -\frac{1}{2} \{a_i^{\mu^i}, a_{\mu^i}^+\} \quad (3.9)$$

the irreducible massless of spin $\mathbf{s} = \mathbf{n} + \frac{1}{2}$ Poincare group representation is equivalently to the eqs. (3.2)–(3.4) given by (3.7)–(3.9).

The bosonic character of the primary constraint operators $\tilde{t}_0, \tilde{t}^i$, $\varepsilon(\tilde{t}_0) = \varepsilon(\tilde{t}^i) = 0$ does not permit to obtain the second-order operator $l_0 = \partial^\mu \partial_\mu$ in terms of a commutator constructed from $\tilde{t}_0$, $l_0 \stackrel{?}{=} -\frac{1}{2} [\tilde{t}_0, \tilde{t}_0]$, due to $\varepsilon(\gamma^\mu) = \varepsilon(a_i^\mu) = 0$ (see footnote 5 in [68]). These operators are transformed into fermionic ones (originally proposed in [23], eqs. (16)–(19) for $d = 2N$, and partially following ref. [22]) by means of certain $d + 1$ Grassmann-odd gamma-matrix-like objects $\tilde{\gamma}^\mu, \tilde{\gamma}, \varepsilon(\tilde{\gamma}^\mu) = \varepsilon(\tilde{\gamma}) = 1$, whose explicit realization differs in even, $d = 2N$, and odd, $d = 2N + 1$, $N \in \mathbb{N}$ dimensions. In the former case, we have the following definition [22]:

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}\} = 0, \quad \tilde{\gamma}^2 = -1, \quad \text{so that } \gamma^\mu = \tilde{\gamma}^\mu \tilde{\gamma}, \quad (3.10)$$

with a non-trivial realization for $\tilde{\gamma}$,

$$\tilde{\gamma} = \kappa_d \Pi \gamma_{d+1} \quad \text{for } \gamma_{d+1} = \frac{1}{d!} \left(\prod_{i=1}^d \gamma_{\mu_i} \right) \epsilon^{\mu_1 \dots \mu_d} \quad \text{and } \kappa_d = \begin{cases} 1, & d = 4M, \\ \iota, & d = 4M + 2, \end{cases} \quad (3.11)$$

where the completely antisymmetric Levi-Civita tensor $\epsilon^{\mu_1 \dots \mu_d}$ is normalized by $\epsilon^{01 \dots d-1} = 1$, and the odd unit matrix Π : $\Pi^2 = 1$,¹ changes the Grassmann parity of columns (or

¹ Π is similar to the odd supermatrix $\omega = \|\omega^{\mathbf{AB}}\| = \|(\Gamma^{\mathbf{A}}, \Gamma^{\mathbf{B}})\|$, $\epsilon(\omega) = 1$, resulting from the odd Poisson bracket $(\bullet, \bullet)$ calculated with respect to the field-antifield variables $\Gamma^{\mathbf{A}}$ of the field-antifield formalism [90], which was also used in [103] to construct an $N = 3$ -BRST invariant quantum action for Yang-Mills theories in the minimal configuration space.

rows) alone, in such a way that $\varepsilon(\tilde{\gamma}) = \varepsilon(\gamma_{d+1}) + \varepsilon(\Pi) = 0 + 1 = 1$. Indeed, $\tilde{\gamma}^2 = \kappa_d^2 \Pi^2 (-1)^{d^2/4} \prod_{i=1}^d \gamma_{\mu_i}^2 = \kappa_d^2 (-1)^{d^2/4+d-1} = -1$. In odd space-time dimensions the matrix γ_{d+1} is trivial, but the definition of $\tilde{\gamma}$ in (3.11) remains valid under the following modification of the commutation relations (3.10):

$$[\gamma^\mu, \tilde{\gamma}] = 0, \quad \tilde{\gamma} = \Pi \quad \text{and} \quad \gamma^\mu = \tilde{\gamma}^\mu \tilde{\gamma} \quad \text{so that} \quad \tilde{\gamma}^2 = 1, \quad [\tilde{\gamma}^\mu, \tilde{\gamma}] = 0, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}. \quad (3.12)$$

In both cases, we have a realization of the Clifford algebra for Grassmann-odd gamma-matrix-like objects $\tilde{\gamma}^\mu$ in $\mathbb{R}^{1,d-1}$.² The primary Grassman-odd constraints result in

$$t_0 = -i\tilde{\gamma}^\mu \partial_\mu, \quad t^i = \tilde{\gamma}^\mu a_\mu^i : \quad (t_0, t^i) = \tilde{\gamma}(\tilde{t}_0, \tilde{t}^i). \quad (3.13)$$

The set of primary constraints $\{t_0, t_i, t_{ij}, g_0^i\}$ is closed with respect to the $[\ , \]$ -multiplication if we add to them divergentless, l_i , traceless, l_{ij} , $i \leq j$ and D'alambert operators, l_0 :

$$(l_0, l^i, l^{ij}) = \left(\partial^\mu \partial_\mu, -ia_\mu^i \partial^\mu, \frac{1}{2} a^{i\mu} a_\mu^j \right), \quad (3.14)$$

but for the reality of the Lagrangian we need, in addition, a closedness with respect to the appropriate hermitian conjugation defined by means of odd scalar product in $\mathcal{H}$:

$$\begin{aligned} \langle \tilde{\Phi} | \Psi \rangle &= \int d^d x \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_{k-1}} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{p_1} \cdots \sum_{p_l=0}^{p_{l-1}} \frac{i^{\sum_i n_i} (-i)^{\sum_j p_j}}{n_1! \times \dots \times n_k! p_1! \times \dots \times p_l!} \\ &\quad \times \langle 0 | \prod_{j=1}^l \prod_{m_j=1}^{p_j} a_j^{\nu_{m_j}^j} \Phi_{(\nu^1)_{p_1}, \dots, (\nu^l)_{p_l}}^+ \tilde{\gamma}_0 \Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}} \prod_{i=1}^k \prod_{l_i=1}^{n_i} a_i^{+\mu_{l_i}^i} | 0 \rangle \\ &= \prod_{i=1}^k \frac{(-1)^{s_i}}{s_i!} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_{k-1}} \int d^d x \Phi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}^+ \tilde{\gamma}_0 \Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}, \end{aligned} \quad (3.15)$$

that means the set:

$$o_A = \{t_0, l_0, l_i, l_i^+\} \text{ for } l_i^+ = (l_i)^+ = -ia_{i\mu}^+ \partial^\mu \quad (3.16)$$

composes the first-class constraints subsystem. As a result, the superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$, known as the *half-integer HS symmetry algebra in Minkowski space with a Young tableaux having k rows* [68] and containing the extra operators

$$o_a^+ \equiv (t^{i+}, l^{ij+}, t^{i_1 j_1 +}) = (\tilde{\gamma}^\mu a_\mu^{i+}, \frac{1}{2} a_\mu^{i+} a^{j\mu+}, a_{\mu}^{i_1} a^{j_1 \mu+}), \quad i \leq j; \quad i_1 < j_1, \quad (3.17)$$

in addition to $o_a \equiv (t^i, l^{ij}, t^{i_1 j_1})$, is closed w.r.t. the Hermitian conjugation and the $[\ , \]$ -multiplication. The complete $[\ , \]$ -multiplication table for $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ is given by Multiplication table 1, with commutators present only in the upper subtable, and with the odd-odd part containing only anticommutators in the subtable below. The figure brackets

²The final Lagrangian formulation in terms of the ghost-independent or spin-tensor forms depends only on the standard Grassmann-even matrices γ^μ , and does not depend on $\tilde{\gamma}^\mu, \tilde{\gamma}$, due to the presence of the latter only as even degrees inside the Lagrangian, and due to the homogeneity of the gauge transformations w.r.t. $\tilde{\gamma}$, as shown by the totally-symmetric half-integer case in sections 6.1, 6.2. Therefore, $\tilde{\gamma}^\mu, \tilde{\gamma}$ may be viewed as intermediate objects as compared to γ^μ .

$[\downarrow, \rightarrow]$	$t^{i_1 j_1}$	$t_{i_1 j_1}^+$	l_0	l^i	l^{i+}	$l^{i_1 j_1}$	$l^{i_1 j_1+}$	g_0^i
t_0	0	0	0	0	0	0	0	0
t^{i_2}	$-t^{j_1} \delta^{i_2 i_1}$	$-t_{i_1} \delta^{i_2 j_1}$	0	0	$-t_0 \delta^{i_2 i}$	0	$-\frac{1}{2} t^{\{i_1 + \delta^{j_1\} i_2\}}$	$t^{i_2} \delta^{i_2 i}$
t^{i_2+}	$t^{i_1 + \delta^{i_2 j_1}}$	$t_{j_1}^+ \delta_{i_1}^{i_2}$	0	$t_0 \delta^{i_2 i}$	0	$\frac{1}{2} t^{\{i_1 \delta^{j_1\} i_2\}}$	0	$-t^{i_2+} \delta^{i_2 i}$
$t^{i_2 j_2}$	$A^{i_2 j_2, i_1 j_1}$	$B^{i_2 j_2, i_1 j_1}$	0	$l^{j_2} \delta^{i_2 i}$	$-l^{i_2 + \delta^{j_2 i}}$	$l^{\{j_1 j_2 \delta^{i_1\} i_2\}}$	$-l^{i_2 \{i_1 + \delta^{j_1\} j_2\}}$	$F^{i_2 j_2, i}$
$t_{i_2 j_2}^+$	$-B^{i_1 j_1, i_2 j_2}$	$A_{i_1 j_1, i_2 j_2}^+$	0	$l_{i_2} \delta_{j_2}^i$	$-l_{j_2}^+ \delta_{i_2}^i$	$l_{i_2}^{\{j_1 \delta_{j_2}^{i_1}\}}$	$-l_{j_2}^{\{j_1 + \delta_{i_2}^{i_1}\}}$	$-F_{i_2 j_2}^{i+}$
l_0	0	0	0	0	0	0	0	0
l^j	$-l^{j_1} \delta^{i_1 j}$	$-l_{i_1} \delta_{j_1}^j$	0	0	$l_0 \delta^{j i}$	0	$-\frac{1}{2} l^{\{i_1 + \delta^{j_1\} j\}}$	$l^j \delta^{i j}$
l^{j+}	$l^{i_1 + \delta^{j_1 j}}$	$l_{j_1}^+ \delta_{i_1}^j$	0	$-l_0 \delta^{j i}$	0	$\frac{1}{2} l^{\{i_1 \delta^{j_1\} j\}}$	0	$-l^{j+} \delta^{i j}$
$l^{i_2 j_2}$	$-l^{j_1 \{j_2 \delta^{i_2\} i_1\}}$	$-l_{i_1}^{\{i_2 + \delta_{j_1}^{j_2}\}}$	0	0	$-\frac{1}{2} t^{\{i_2 \delta^{j_2\} i\}}$	0	$L^{i_2 j_2, i_1 j_1}$	$l^{i \{i_2 \delta^{j_2\} i\}}$
$l^{i_2 j_2+}$	$l^{i_1 \{i_2 + \delta^{j_2\} j_1\}}$	$l_{j_1}^{\{j_2 + \delta_{i_1}^{i_2}\}}$	0	$\frac{1}{2} l^{\{i_2 + \delta^{i_2 j_2\} i\}}$	0	$-L^{i_1 j_1, i_2 j_2}$	0	$-l^{i \{i_2 + \delta^{j_2\} i\}}$
g_0^j	$-F^{i_1 j_1, j}$	$F_{i_1 j_1}^{j+}$	0	$-l^i \delta^{i j}$	$l^{i+} \delta^{i j}$	$-l^j \{i_1 \delta^{j_1\} j\}$	$l^j \{i_1 + \delta^{j_1\} j\}$	0

$$\begin{aligned}
 B^{i_2 j_2, i_1 j_1} &= (g_0^{i_2} - g_0^{j_2}) \delta_{i_1}^{i_2} \delta_{j_1}^{j_2} + (t_{j_1}^{j_2} \theta^{j_2 j_1} + t_{j_1}^{j_2+} \theta^{j_1 j_2}) \delta_{i_1}^{i_2} - (t_{i_1}^{+ i_2} \theta^{i_2 i_1} + t_{i_1}^{i_2} \theta^{i_1 i_2}) \delta_{j_1}^{j_2}, \\
 A^{i_2 j_2, i_1 j_1} &= t^{i_1 j_2} \delta^{i_2 j_1} - t^{i_2 j_1} \delta^{i_1 j_2}, \quad F^{i_2 j_2, i} = t^{i_2 j_2} (\delta^{j_2 i} - \delta^{i_2 i}), \\
 L^{i_2 j_2, i_1 j_1} &= \frac{1}{4} \left\{ \delta^{i_2 i_1} \delta^{j_2 j_1} \left[2g_0^{i_2} \delta^{i_2 j_2} + g_0^{i_2} + g_0^{j_2} \right] - \delta^{j_2 i_1} \left[t^{j_1 i_2} \theta^{i_2 j_1} + t^{i_2 j_1} + \theta^{j_1 i_2} \right] \right. \\
 &\quad \left. - \delta^{i_2 i_1} \left[t^{j_1 j_2} \theta^{j_2 j_1} + t^{j_2 j_1} + \theta^{j_1 j_2} \right] \right\};
 \end{aligned}$$

$[\downarrow, \rightarrow]$	t_0	t^i	t^{i+}
t_0	$-2l_0$	$2l^i$	$2l^{i+}$
t^j	$2l^j$	$4l^{j i}$	$2(-g_0^i \delta^{i j} + t^{i j} \theta^{j i} + t^{j i+} \theta^{i j})$
t^{j+}	$2l^{j+}$	$2(-g_0^i \delta^{i j} + t^{j i} \theta^{i j} + t^{i j+} \theta^{j i})$	$4l^{j i+}$

Table 1. Even-even, odd-even and odd-odd parts of HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$.

for the indices i_1, i_2 of table 1 in the quantity $C^{\{i_1 D^{i_2\} i_3 \theta^{i_3 i_2}\}}$ imply symmetrization, $C^{\{i_1 D^{i_2\} i_3 \theta^{i_3 i_2}\}} = C^{i_1 D^{i_2 i_3} \theta^{i_3 i_2}} + D^{i_2} C^{i_1 i_3 \theta^{i_3 i_1}}$; these indices are raised and lowered using the Euclidian metric tensors $\delta^{ij}, \delta_{ij}, \delta_j^i$. The quantity $\theta^{ji} = 1(0)$ for $j > i(j \leq i)$ is the Heaviside θ -symbol θ^{ji} . The products $B_{i_1 j_1}^{i_2 j_2}, A^{i_2 j_2, i_1 j_1}, F^{i_1 j_1, i}, L^{i_2 j_2, i_1 j_1}$ are given explicitly by the relations of table 1 (for details, see [67, 68]). Representing the basis elements of $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$, with allowance for the definitions (3.9), (3.13), (3.14), (3.16) and in agreement with (3.17), as follows:

$$o_I = \{t_0, l_0, l^i, l^{i+}; o_a^+, o_a; g_0^i\} \equiv \{o_A; o_a^+, o_a; g_0^i\}, \quad \text{for } g_0^i = -\frac{1}{2} \{a^{i\mu}, a_{\mu}^{i+}\} \quad \text{and} \quad (3.18)$$

$$\begin{array}{c|c|c|c|c|c|c|c|c|c}
 t_0 & t^i & t^{i+} & l_0 & l^i & l^{i+} & l^{ij} & l^{ij+} & t^{rs} & t^{rs+} \\
 \hline
 -i\tilde{\gamma}^\mu \partial_\mu & \tilde{\gamma}^\mu a_\mu^i & \tilde{\gamma}^\mu a_\mu^{i+} & \partial^\mu \partial_\mu & -i a_\mu^i \partial^\mu & -i a_\mu^{i+} \partial^\mu & \frac{1}{2} a^{i\mu} a_\mu^j & \frac{1}{2} a^{i\mu+} a_\mu^{j+} & a_\mu^{r+} a^{s\mu} & a^{r\mu} a_\mu^{s+}
 \end{array} \quad (3.19)$$

the table of basis $[,]$ -products can be presented as a Lie superalgebra:

$$[o_I, o_J] = f_{IJ}^K o_K, \quad f_{IJ}^K = -(-1)^{\varepsilon(o_I)\varepsilon(o_J)} f_{JI}^K, \quad (3.20)$$

$$[o_a, o_b^+] = \underline{f}_{ab}^c o_c + \bar{f}_{ab}^c o_c^+ + \Delta_{ab}(g_0^i), \quad [o_A, o_B] = f_{AB}^C o_C, \quad [o_a, o_B] = f_{aB}^C o_C, \quad (3.21)$$

$$[o_a, o_b] = f_{ab}^c o_c, \quad [o_a^+, o_b^+] = f_{ab}^{+c} o_c^+, \quad \text{with } (f_{ab}^c)^+ = f_{ab}^{+c}, \quad (3.22)$$

for $\Delta_{ab}(g_0^i) = \sum_i f_{ab}^i g_0^i$. From the Hamiltonian analysis of dynamical systems it follows the operators o_a, o_a^+ are respective operator-valued $2k^2$ bosonic and $2k$ fermionic second-class, as well as o_A are $(2k+1)$ bosonic and 1 fermionic first-class constraint subsystems among $\{o_I\}$ for a topological gauge system (one with vanishing Hamiltonian $H_0 \equiv 0$). The symmetry properties of the real constants $f_{ab}^c, f_{AB}^C, f_{aB}^C$ (being another than general ones in (2.1), (2.2), (2.3) of the section 2) are described in [68]. We only remind the quantities $\Delta_{ab}(g_0^i)$ form a non-degenerate $(k \times k; k^2 \times k^2)$ supermatrix, $\|\Delta_{ab}\|$, in the Fock space $\mathcal{H}$ on the surface $\Sigma \subset \mathcal{H}$: $\|\Delta_{ab}\|_{|\Sigma} \neq 0$, which is determined by the equations $(o_a, t_0, l_0, l^i)|\Psi\rangle = 0$. The only operators g_0^i are not the constraints in $\mathcal{H}$, due to eqs. (3.9), but they select a vector from $|\Psi\rangle$ with fixed value of spin. The second-class constraints o_a, o_a^+ have already splitted on 2 first-class constraints subsystems due to (3.22).

In [68] it is shown the subsuperalgebra of $\{o_a, o_a^+, g_0^i\}$ is isomorphic to orthosymplectic superalgebra $osp(1|2k)$, thus realizing the generalization of Howe duality [91, 92] among whole the set of unitary half-integer HS representations of Lorentz subalgebra $so(1, d-1)$ and $osp(1|2k)$. The rest elements $\{o_A\}$ from $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ forms the subsuperalgebra of Minkowski space $\mathbb{R}^{1,d-1}$ isometries, which has the form of direct sum:

$$\{o_A\} = (T^k \oplus T^{k*} \oplus [T^k, T^{k*}]), \quad [T^k, T^{k*}] \sim l_0 = -t_0^2, \quad (3.23)$$

of k -dimensional commutative algebra $T^k = \{l_i\}$ and its dual $T^{k*} = \{l^{i+}\}$. Finally, HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ represents the semidirect sum of $osp(1|2k)$ [as an algebra of internal derivations of $(T^k \oplus T^{k*})$] with $(T^k \oplus T^{k*} \oplus [T^k, T^{k*}])$,

$$\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1}) = (T^k \oplus T^{k*} \oplus [T^k, T^{k*}]) \oplus osp(1|2k). \quad (3.24)$$

The nilpotent BRST-BFV operator [having a matrix-like $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$ structure] $Q' : (Q')^2 = 0$, for Lie superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ was constructed in [68] with account of (3.20) according to general formula [25–28] (with the $\hat{C}\hat{\mathcal{P}}$ -ordering³):

$$Q' = C^I o_I + \frac{1}{2} C^I C^J f_{JI}^K \bar{\mathcal{P}}_K (-1)^{\varepsilon(o_K) + \varepsilon(o_I)}, \quad [\varepsilon, gh] C^I = [\varepsilon, -gh] \bar{\mathcal{P}}_I = [\varepsilon(o_I) + 1, 1] \quad (3.25)$$

for ghost coordinates, C^I , and momenta, $\bar{\mathcal{P}}_I$ from the minimal sector, subject to $[C^I, \bar{\mathcal{P}}_J] = \delta_J^I$ as for $Q = \Omega_{c|\min}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ in (2.37), but for $\zeta^\alpha \equiv 0$ and with g_0^i considered as the

³Further on, we omit the symbol “ $\hat{}$ ” over an operator for the Hamiltonian ghost coordinates and momenta $\hat{C}^I, \hat{\mathcal{P}}_I$, according to our previous notation [68].

constraints. Explicitly, it has the form,

$$\begin{aligned}
 Q' = & \frac{1}{2}q_0t_0 + q_i^+t^i + \frac{1}{2}\eta_0l_0 + \eta_i^+l^i + \sum_{l \leq m} \eta_{lm}^+l^{lm} + \sum_{l < m} \vartheta_{lm}^+t^{lm} + \frac{1}{2}\eta_g^i g_i \\
 & + \left[\frac{1}{2} \sum_{l,m} (1 + \delta_{lm}) \eta^{lm} q_l^+ - \sum_{l < m} q_l \vartheta^{lm} - \sum_{m < l} q_l \vartheta^{ml+} \right] p_m^+ + \frac{1}{2} \sum_m \eta_g^m (q_m p_m^+ + q_m^+ p_m) \\
 & + \imath \sum_l \left[\frac{1}{2} \eta_l^+ \eta^l \mathcal{P}_0 + \eta_l^+ q^l p_0 - q^l q_l^+ \mathcal{P}_g \right] - \frac{\imath}{2} q_0^2 \mathcal{P}_0 \\
 & - \sum_{i < l < j} \vartheta_{lj}^+ \vartheta_i^{+l} \lambda^{ij} + \frac{\imath}{2} \sum_{l < m} \vartheta_{lm}^+ \vartheta^{lm} (\mathcal{P}_g^m - \mathcal{P}_g^l) - \sum_{l < n < m} \vartheta_{lm}^+ \vartheta^l_n \lambda^{nm} \\
 & + \sum_{n < l < m} \vartheta_{lm}^+ \vartheta_n^m \lambda^{+nl} - \sum_{n, l < m} (1 + \delta_{ln}) \vartheta_{lm}^+ \eta^{l+}_n \mathcal{P}^{mn} + \sum_{n, l < m} (1 + \delta_{mn}) \vartheta_{lm}^+ \eta^m_n \mathcal{P}^{+ln} \\
 & + \frac{\imath}{8} \sum_{l \leq m} (1 + \delta_{lm}) \eta_{lm}^+ \eta^{lm} (\mathcal{P}_g^l + \mathcal{P}_g^m) + \frac{1}{2} \sum_{l \leq m} (1 + \delta_{lm}) \eta_g^l (\eta_{lm}^+ \mathcal{P}^{lm} - \eta_{lm} \mathcal{P}^{lm+}) \\
 & + \frac{1}{2} \sum_{l < m, n \leq m} \eta_{nm}^+ \eta^l_n \lambda^{lm} - 2 \sum_{l < pm} q_l q_m^+ \lambda^{lm} + \frac{1}{2} \sum_{l < m} (\eta_g^m - \eta_g^l) (\vartheta_{lm}^+ \lambda^{lm} - \vartheta_{lm} \lambda^{lm+}) \\
 & - \left[\frac{1}{2} \sum_{l,m} (1 + \delta_{lm}) \eta^m \eta_{lm}^+ + \sum_{l < m} \vartheta_{lm} \eta^{+m} + \sum_{m < l} \vartheta_{ml}^+ \eta^{+m} + 2 \sum_l q_0 q_l^+ \right] \mathcal{P}^l \\
 & - 2 \sum_{l,m} q_l^+ q_m^+ \mathcal{P}^{lm} + \frac{1}{2} \sum_l \eta_g^l (\eta_l^+ \mathcal{P}^l - \eta_l \mathcal{P}^{l+}) + h.c.
 \end{aligned} \tag{3.26}$$

Here, the set of ghost operators $C^I = (q_0, q_i, q_i^+; \eta_0, \eta^i, \eta_i^+, \eta^{ij}, \eta_{ij}^+, \vartheta_{rs}, \vartheta_{rs}^+, \eta_g^i)$ with the Grassmann parity and ghost number given according to (3.25) respectively for the elements $o_I = (t_0, t_i^+, t_i; l_0, l_i^+, l_i, l_{ij}, l_{ij}^+, t_{ij}, t_{ij}^+, g_0^i)$, subject to the properties

$$(\eta^{ij}, \eta_{ij}^+) = (\eta^{ji}, \eta_{ji}^+), \quad (\vartheta_{rs}, \vartheta_{rs}^+) = (\vartheta_{rs}, \vartheta_{rs}^+) \theta^{sr}, \tag{3.27}$$

and their conjugated ghost momenta $\bar{\mathcal{P}}_I$ (composing Wick pairs of ghost operators for the $\{o_I\} \setminus \{t_0, l_0, g_0^i\}$ constraints) with the same properties as those for C^I in (3.27) with the only nonvanishing commutation relations for bosonic

$$[q_i, p_j^+] = [p_i, q_j^+] = \delta_{ij}, \quad [q_0, p_0] = \imath; \tag{3.28}$$

and the anticommutation ones for fermionic ghosts

$$\begin{aligned}
 \{\vartheta_{rs}, \lambda_{tu}^+\} &= \{\lambda_{tu}, \vartheta_{rs}^+\} = \delta_{rt} \delta_{su}, & \{\eta_i, \mathcal{P}_j^+\} &= \{\mathcal{P}_j, \eta_i^+\} = \delta_{ij}, \\
 \{\eta_{lm}, \mathcal{P}_{ij}^+\} &= \{\mathcal{P}_{ij}, \eta_{lm}^+\} = \delta_{li} \delta_{jm}, & \{\eta_0, \mathcal{P}_0\} &= \imath, \quad \{\eta_g^i, \mathcal{P}_g^j\} = \imath \delta^{ij}.
 \end{aligned} \tag{3.29}$$

By construction the property $gh(Q') = 1$ holds, whereas the Hermitian conjugation of zero-mode pairs:

$$(q_0, \eta_0, \eta_g^i, p_0, \mathcal{P}_0, \mathcal{P}_g^i)^+ = (q_0, \eta_0, \eta_g^i, p_0, -\mathcal{P}_0, -\mathcal{P}_g^i) \tag{3.30}$$

provides for Q' to be hermitian $(Q')^+ = Q'$.

Decomposing Q' in powers of zero-mode ghosts $\eta_g^i, \mathcal{P}_g^i$ corresponding to the operators g_0^i not entered into the first-class $\{o_A\}$ and second-class $\{o_a, o_a^+\}$ subsystems of constraints

$$Q' = Q + \eta_g^i \sigma^i(g) + \mathcal{B}^i \mathcal{P}_g^i, \quad (3.31)$$

$$\sigma^i(g) = g_0^i - \eta_i \mathcal{P}_i^+ + \eta_i^+ \mathcal{P}_i + \sum_m (1 + \delta_{im}) (\eta_{im}^+ \mathcal{P}^{im} - \eta_{im} \mathcal{P}_{im}^+) \quad (3.32)$$

$$+ \sum_{l < i} [\vartheta_{li}^+ \lambda^{li} - \vartheta_{li} \lambda_{li}^+] - \sum_{i < l} [\vartheta_{il}^+ \lambda^{il} - \vartheta_{il} \lambda_{il}^+] + q_i p_i^+ + q_i^+ p_i, \quad (3.33)$$

$$Q = \frac{1}{2} q_0 t_0 + q_i^+ t^i + \frac{1}{2} \eta_0 l_0 + \eta_i^+ l^i + \sum_{l \leq m} \eta_{lm}^+ l^{lm} + \sum_{l < m} \vartheta_{lm}^+ t^{lm} + \frac{i}{2} \left(\sum_l \eta_l^+ \eta^l - q_0^2 \right) \mathcal{P}_0$$

$$+ \left[\frac{1}{2} \sum_{l, m} (1 + \delta_{lm}) \eta^{lm} q_l^+ - \sum_{l < m} q_l \vartheta^{lm} - \sum_{m < l} q_l \vartheta^{ml} \right] p_m^+ + i \sum_l \eta_l^+ q^l p_0 - 2 \sum_{l < m} q_l q_m^+ \lambda^{lm}$$

$$- 2 \sum_{l, m} q_l^+ q_m^+ \mathcal{P}^{lm} - \sum_{i < l < j} \vartheta_{lj}^+ \vartheta_i^{+l} \lambda^{ij} - \sum_{l < n < m} \vartheta_{lm}^+ \vartheta_n^l \lambda^{nm} + \sum_{n < l < m} \vartheta_{lm}^+ \vartheta_n^m \lambda^{nl}$$

$$- \sum_{n, l < m} (1 + \delta_{ln}) \vartheta_{lm}^+ \eta^{ln} \mathcal{P}^{mn} + \sum_{n, l < m} (1 + \delta_{mn}) \vartheta_{lm}^+ \eta^m \mathcal{P}^{ln} + \frac{1}{2} \sum_{l < m, n \leq m} \eta_{nm}^+ \eta^n \lambda^{lm}$$

$$- \left[\frac{1}{2} \sum_{l, m} (1 + \delta_{lm}) \eta^m \eta_{lm}^+ + \sum_{l < m} \vartheta_{lm} \eta^{+m} + \sum_{m < l} \vartheta_{ml}^+ \eta^{+m} + 2 \sum_l q_0 q_l^+ \right] \mathcal{P}^l + h.c.,$$

$$\mathcal{B}^i = -2i q^i q_i^+ - i \sum_{l < m} \vartheta_{lm}^+ \vartheta^{lm} (\delta^{mi} - \delta^{li}) + \frac{i}{4} \sum_{l \leq m} (1 + \delta_{lm}) \eta_{lm}^+ \eta^{lm} (\delta^{il} + \delta^{mi}). \quad (3.34)$$

where the operator $\vec{\sigma}(g) = (\sigma^1, \sigma^2, \dots, \sigma^k)(g)$ playing the role of “ancient” of generalized spin operator $\vec{\sigma}$ [68] is Hermitian, $(\sigma^i)^+ = \sigma^i$, as well as $Q: Q^+ = Q$.

From the nilpotency condition for Q' it follows the operator equations in powers of $\eta_g^i, \mathcal{P}_g^i$:

$$(\eta_g^i)^0 (\mathcal{P}_g^j)^0: \quad Q^2 + i \sum_j \mathcal{B}^j \sigma^j(g) = 0, \quad (\eta_g^i)^1 (\mathcal{P}_g^j)^0: \quad \eta_g^i [Q, \sigma^i(g)] = 0, \quad (3.35)$$

$$(\eta_g^i)^0 (\mathcal{P}_g^i)^1: \quad [Q, \mathcal{B}^i] \mathcal{P}_g^i = 0, \quad (\eta_g^i)^1 (\mathcal{P}_g^j)^1: \quad \sum_{i, j} \eta_g^i [\mathcal{B}^j, \sigma^i(g)] \mathcal{P}_g^j = 0, \quad (3.36)$$

where the both the operator nilpotency of the operator Q and its weak nilpotency on some subspaces $\tilde{\mathcal{H}}_{tot|N}^{\sigma(g)}$ of a total Hilbert space $\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}_{gh|m}$ can not be realized, except for the case: $\sigma^i(g) \mathcal{H}_{tot|N}^{\sigma(g)0} = 0$. The latter variant does work for only HS fields with lowest exceptional generalized spin related to the space-time dimension. The second equation in (3.35) and equations (3.36) vanish identically, due to operator identities: $[Q, \sigma^i(g)] = [\mathcal{B}^j, \sigma^i(g)] = 0$ and $[Q, \mathcal{B}^i] = 0$. The non-trivial solution of the first equations in (3.35), (3.36) requires choosing the representation in $\mathcal{H}_{tot}$ and conversion the second-class constraints subsystem $\{o_a, o_a^+\}$ into first-class one.

3.1 Unconstrained Lagrangian formulation

To construct Lagrangian formulation it is impossible to use BRST operator Q' for the initial HS superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$ of the elements $\{o_I\}$. Instead, the additively con-

verted first-class constraints, O_I : $O_I = o_I + o'_I$ and $[o_I, o'_J] = 0$ with unchanged first-class constraints, $O_A = o_A$, i.e. $l'_0 = t'_0 = l'_i = (l'_i)^+ = 0$, are found in [68] for additional parts of the second-class constraints $\{o'_a, o'^+_a\}$ and g_0^i as the polynomials in auxiliary creation and annihilation operators B^{a+}, B^a in new Fock space $\mathcal{H}' \equiv \mathcal{H}_B$: $\mathcal{H} \cap \mathcal{H}' = \emptyset$. The number of new oscillators (with $gh(B^{a+}) = gh(B^a) = 0$) coincides respectively with one for the second-class constraints ($\varepsilon(B_a) = \varepsilon(o_a)$). The supercommutativity requirement, $[o_I, o'_J] = 0$, permits to preserve both for converted O_I and for additional o'_I operators the same superalgebra relations as for $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ given therefore by the eqs. (3.20)–(3.22) and Multiplication table 1, with obvious respective changes: $o_I \rightarrow O_I$ and $o_I \rightarrow o'_I$ with unchanged o_A . The explicit form of $o'_I(B^{a+}, B^a)$ has sufficiently non-trivial form and found from the procedure of generalized Verma module construction for subsuperalgebra of $\{o'_a, o'^+_a, g_0^i\}$ isomorphic to $osp(1|2k)$ presented in the appendix A [68] as well as, so that after oscillator realization over the Heisenberg-Weyl superalgebra, parameterized by $\{B^{a+}; B^a\} = \{f_i, b_{ij}, d_{rs}\theta^{sr}; f_i^+, b_{ij}^+, d_{rs}^+\theta^{sr}\}$ with non-vanishing anticommutators, $\{f_i, f_j^+\} = \delta_{ij}$, for odd oscillators and commutators, $[b_{ij}, b_{lm}^+] = \frac{1}{2}\delta_{i\{l}\delta_{j\}m}$, $[d_{qt}, d_{rs}^+] = \delta_{qr}\delta_{ts}$ for even. Important for further construction additional number particle operators, g_0^i ,

$$g_0^i = h^i + f_i^+ f_i + \sum_{l \leq m} b_{lm}^+ b_{lm} (\delta^{il} + \delta^{im}) + \sum_{r < s} d_{rs}^+ d_{rs} (\delta^{is} - \delta^{ir}), \quad (3.37)$$

contain real-valued quantities h^i , $i = 1, \dots, k$ being by the arbitrary dimensionless constants, introduced in the process of generalized Verma module construction [68], which permits to make the generalized spin equations for proper eigen-vectors in $\mathcal{H}_{c|tot}$: $\mathcal{H}_{c|tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh|m}$: for the spin operator, $\sigma^i(G) \equiv \sigma^i + h^i$, instead of $\sigma^i(g)$:

$$\sigma^i(G)|\chi\rangle = (\sigma^i + h^i)|\chi\rangle = 0 \quad (3.38)$$

with spectra of proper eigen-values for h^i . On the Hilbert subspace $\mathcal{H}_{c|tot|N}^{\sigma(G)}$ from $\mathcal{H}_{c|tot}$ consisting from the proper eigen-vectors satisfying to the equations (3.38) the BRST-BFV operator Q (3.33), but for converted first-class constraints, O_I , without converted number particle operators $G_0^i = g_0^i + g_0^i$ is nilpotent. Namely, the latter operator generates the right BRST complex, whose cohomology in Hilbert subspace $\mathcal{H}_{c|tot|N}^{\sigma(G)k}$ with respective ghost number values spectrum, starting from, $k = 0$ generates correct Lagrangian dynamic for initial spin-tensor with appropriate set of auxiliary spin-tensor fields.

The unconstrained gauge-invariant Lagrangian formulation for HS field $\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}}$ is derived from the sequence of the equations, following from BRST-like equations for $Q'(O)|\chi^0\rangle = 0$, $\delta|\chi^0\rangle = Q'(O)|\chi^1\rangle$, $\dots$, $\delta|\chi^{k-1}\rangle = Q'(O)|\chi^k\rangle$, with $|\chi^k\rangle \in \mathcal{H}_{tot}^k$, due to existence of $\mathbb{Z}$ -grading in $\mathcal{H}_{c|tot}$: $\mathcal{H}_{c|tot} = \bigoplus_k \mathcal{H}_{c|tot}^k$ for $gh(|\chi^k\rangle) = -k$, $k = 0, 1, 2, \dots$. Indeed, decomposing in ghosts η_g^i operator $Q'(O)$ (3.31) and vector $|\chi\rangle$ with choice the standard representation in $\mathcal{H}_{c|tot}$:

$$(q_i, \eta_i, \eta_{ij}, \vartheta_{rs}, p_0, p_i, \mathcal{P}_0, \mathcal{P}_i, \mathcal{P}_{ij}, \lambda_{rs}, \mathcal{P}_g^i)|0\rangle = 0, \quad |0\rangle \in \mathcal{H}_{c|tot}, \quad (3.39)$$

not depending on η_g^i ,

$$\begin{aligned}
 |\chi\rangle = & \sum_n \prod_c (f_c^+)^{n_c^0} \prod_{i \leq j, r < s}^k (b_{ij}^+)^{n_{ij}} (d_{rs}^+)^{p_{rs}} q_0^{n_{b0}} \eta_0^{n_{f0}} \\
 & \times \prod_{e, g, i, j, l \leq m, n \leq o} (q_e^+)^{n_{ae}} (p_g^+)^{n_{bg}} (\eta_i^+)^{n_{fi}} (\mathcal{P}_j^+)^{n_{pj}} (\eta_l^+)^{n_{flm}} (\mathcal{P}_{no}^+)^{n_{pno}} \prod_{r < s, t < u} (\vartheta_{rs}^+)^{n_{frs}} (\lambda_{tu}^+)^{n_{\lambda tu}} \\
 & \times |\Psi(a_i^+)_{(n^0)_c; (n)_{ij}(p)_{rs}}^{n_{b0} n_{f0}; (n)_{ae} (n)_{bg} (n)_{fi} (n)_{pj} (n)_{flm} (n)_{pno} (n)_{frs} (n)_{\lambda tu}} \rangle, \quad (3.40)
 \end{aligned}$$

the following spectral problem is derived

$$Q(O)|\chi\rangle = 0, \quad (\sigma^i + h^i)|\chi\rangle = 0, \quad (\varepsilon, gh_H)(|\chi\rangle) = (1, 0), \quad (3.41)$$

$$\delta|\chi\rangle = Q(O)|\chi^1\rangle, \quad (\sigma^i + h^i)|\chi^1\rangle = 0, \quad (\varepsilon, gh_H)(|\chi^1\rangle) = (0, -1), \quad (3.42)$$

$$\delta|\chi^1\rangle = Q(O)|\chi^2\rangle, \quad (\sigma^i + h^i)|\chi^2\rangle = 0, \quad (\varepsilon, gh_H)(|\chi^2\rangle) = (1, -2), \quad (3.43)$$

$$\delta|\chi^s\rangle = Q(O)|\chi^{s+1}\rangle, \quad (\sigma^i + h^i)|\chi^{s+1}\rangle = 0, \quad (\varepsilon, gh_H)(|\chi^{s+1}\rangle) = (s \bmod 2, -s - 1). \quad (3.44)$$

Note, the brackets $(n^0)_c, (n)_{fi}, (n)_{pj}, (n)_{ij}$ in (3.40) imply, for instance, for $(n^0)_c$ and $(n)_{ij}$, the sets of indices $(n_1^0, \dots, n_k^0)$ and $(n_{11}, \dots, n_{1k}, \dots, n_{k1}, \dots, n_{kk})$. The above sum is taken over $n_{b0}, n_{ae}, n_{bg}, n_{fl}, n_{ij}, p_{rs}$, running from 0 to infinity, and over the remaining n 's from 0 to 1. Thus, the physical state $|\chi^0\rangle$ for the vanishing of all auxiliary operators B^{a+} and ghost variables $q_0, q_i^+, \eta_0, \eta_i^+, p_i^+, \mathcal{P}_i^+, \dots$, contains only the physical string-like vector $|\Psi\rangle = |\Psi(a_i^+)_{(n^0)_c; (n)_{ij}(p)_{rs}}^{0_{b0} 0_{f0}; (0)_{ae} (0)_{bg} (0)_{fi} (0)_{pj} (0)_{flm} (0)_{pno} (0)_{frs} (0)_{\lambda tu}}\rangle$, so that

$$|\chi^0\rangle = |\Psi\rangle + |\Psi_A\rangle, \quad |\Psi_A\rangle \Big|_{(B^{a+}, q_0, q_i^+, \eta_0, \eta_i^+, p_i^+, \mathcal{P}_i^+, \dots)} = 0. \quad (3.45)$$

The middle set of equations (3.41)–(3.44), has the general solution for the set of proper eigenvectors $|\chi^0\rangle_{(n)_k}, |\chi^1\rangle_{(n)_k}, \dots, |\chi^s\rangle_{(n)_k}$, with $(n)_k = (s)_k - \frac{1}{2}$, and the set of the corresponding eigenvalues for values of the parameters h^i ,

$$-h^i = n^i + \frac{d - 4i}{2}, \quad i = 1, \dots, k, \quad (3.46)$$

so that the vector $|\chi^0\rangle_{(n)_k}$ contains the physical field (3.5) and all of its auxiliary fields. Explicitly, the spin and ghost number gh distributions are given by the respective relations being valid for a general case of HS fields subject to $Y(s_1, \dots, s_k)$ and for the subset of “ghost” numbers in (3.40) and (3.5) for fixed values of n_i for $|\chi^l\rangle_{(n)_k}$, $l = 0, \dots, \sum_{o=1}^k n_o + k(k-1)/2$:

$$\begin{aligned}
 n_i = & p_i + n_{ai} + n_i^0 + \sum_{j=1} (1 + \delta_{ij})(n_{ij} + n_{fij} + n_{pij}) + n_{fi} + n_{pi} \\
 & + \sum_{r < i} (p_{ri} + n_{fri} + n_{\lambda ri}) - \sum_{r > i} (p_{ir} + n_{fir} + n_{\lambda ir}), \quad i = 1, \dots, k, \quad (3.47)
 \end{aligned}$$

$$\begin{aligned}
 |\chi^l\rangle_{(n)_k} : \quad & n_{b0} + n_{f0} + \sum_i (n_{fi} - n_{pi} + n_{ai} - n_{bi}) + \sum_{i \leq j} (n_{fij} - n_{pij}) + \sum_{r < s} (n_{frs} - n_{\lambda rs}) = -l. \quad (3.48)
 \end{aligned}$$

For fixed spin $(s)_k = (n)_k + (\frac{1}{2}, \dots, \frac{1}{2})$ the value of h^i is fixed by (3.46) as $h^i(n)$ and $Q(O)|_{[h^i=h^i(n)]} = Q_{(n)_k}$. The second-order equations of motion (3.41) and sequence of the reducible gauge transformations (3.42)–(3.44) on the solutions for middle set of the equations therein has the form:

$$Q_{(n)_k}|\chi^0\rangle_{(n)_k} = 0, \quad \delta|\chi^0\rangle_{(n)_k} = Q_{(n)_k}|\chi^1\rangle_{(n)_k}, \dots, \delta|\chi^{s-1}\rangle_{(n)_k} = Q_{(n)_k}|\chi^s\rangle_{(n)_k}, \quad \delta|\chi^s\rangle_{(n)_k} = 0 \quad (3.49)$$

for $s = \sum_{o=1}^k n_o + k(k-1)/2$ with nilpotent operator $Q_{(n)_k}$ on any $|\chi^m\rangle_{(n)_k}$, $m = 0, \dots, s$: $Q_{(n)_k}^2|\chi^m\rangle_{(n)_k} = 0$. The corresponding formal BRST-like (as for integer HS fields [67]) gauge-invariant action

$$\mathcal{S}_{(n)_k}^{(2)} = \int d\eta_{(n)_k} \langle \tilde{\chi}^0 | K_{(n)_k} Q_{(n)_k} |\chi^0\rangle_{(n)_k}, \quad K = 1 \otimes K' \otimes 1_{gh} : K_{(n)_k} Q_{(n)_k} = Q_{(n)_k}^+ K_{(n)_k} \quad (3.50)$$

first, contains second order derivative l_0 , second the operator K' realizing hermitian conjugation for additional parts to the constraints o'_i .⁴ The dependence on L_0, η_0 from the BRST operator $Q_{(n)_k}$ (3.33) and from the whole set of the vectors $|\chi^l\rangle_{(n)_k}$ maybe removed by means of partial gauge-fixing procedure, based on the extraction of the zero-mode ghosts q_0, η_0 from Q :

$$Q = q_0 \bar{t}_0 + \eta_0 l_0 + \imath(\eta_i^+ q_i - \eta_i q_i^+) p_0 - \imath(q_0^2 - \eta_i^+ \eta_i) \mathcal{P}_0 + \Delta Q, \quad (3.51)$$

where

$$\bar{t}_0 = t_0 - 2q_i^+ \mathcal{P}_i - 2q_i \mathcal{P}_i^+, \quad (3.52)$$

$$\begin{aligned} \Delta Q = & q_i^+ T^i + \eta_i^+ l^i + \sum_{l \leq m} \eta_{lm}^+ L^{lm} + \sum_{l < m} \vartheta_{lm}^+ T^{lm} + \left[\frac{1}{2} \sum_{l,m} (1 + \delta_{lm}) \eta^{lm} q_l^+ \right. \\ & \left. - \sum_{l < m} q_l \vartheta^{lm} - \sum_{m < l} q_l \vartheta^{ml} \right] p_m^+ - 2 \sum_{l < m} q_l q_m^+ \lambda^{lm} - 2 \sum_{l,m} q_l^+ q_m^+ \mathcal{P}^{lm} - \sum_{i < l < j} \vartheta_{lj}^+ \vartheta_i^{+l} \lambda^{ij} \\ & - \sum_{l < n < m} \vartheta_{lm}^+ \vartheta_n^l \lambda^{nm} + \sum_{n < l < m} \vartheta_{lm}^+ \vartheta_n^m \lambda^{+nl} - \sum_{n,l < m} (1 + \delta_{ln}) \vartheta_{lm}^+ \eta^{l+} \mathcal{P}^{mn} \\ & + \sum_{n,l < m} (1 + \delta_{mn}) \vartheta_{lm}^+ \eta^m \mathcal{P}^{+ln} + \frac{1}{2} \sum_{l < m, n \leq m} \eta_{nm}^+ \eta_n^l \lambda^{lm} \\ & \left. - \left[\frac{1}{2} \sum_{l,m} (1 + \delta_{lm}) \eta^m \eta_{lm}^+ + \sum_{l < m} \vartheta_{lm} \eta^{+m} + \sum_{m < l} \vartheta_{ml}^+ \eta^{+m} \right] \mathcal{P}^l + h.c., \quad (3.53) \end{aligned}$$

and for the state vector and gauge parameters, for $s = 0, \dots, \sum_{o=1}^k n_o + k(k-1)/2$:

$$|\chi^s\rangle = \sum_{l=0}^{\sum_{o=1}^k n_o + k(k-1)/2 - 1} q_0^l (|\chi_0^{s(l)}\rangle + \eta_0 |\chi_1^{s(l)}\rangle), \quad gh(|\chi_m^{s(l)}\rangle) = -(s+l+m+l), \quad m = 0, 1. \quad (3.54)$$

⁴ K' , $K' = K'^+$, is determined explicitly by eq. (3.16) of [68] so that the set of additional parts $o'_a(B, B^+)$, $o_a^+(B, B^+)$, g_o^i should be closed with respect to the new Hermitian conjugation: $\langle \tilde{\Psi}_1 | K' o'_a | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' o'_a | \Psi_1 \rangle^*$, $\langle \tilde{\Psi}_1 | K' g_o^i | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' g_o^i | \Psi_1 \rangle^*$.

As the results of gauge-fixing procedure with help of the equations of motion and the set of gauge transformations (3.49) all the components in powers of q_0 , η_0 vector are removed except for two fields for each level, $|\chi_0^{(s(0))}\rangle_{(n)_k}$, $|\chi_0^{(s(1))}\rangle_{(n)_k}$, for $s = 0, \dots, \sum_{o=1}^k n_o + k(k-1)/2$. Therefore, the representation holds:

$$|\chi^s\rangle_{(n)_k} = |\chi_0^{s(0)}\rangle_{(n)_k} + q_0 |\chi_0^{s(1)}\rangle_{(n)_k} - i\eta_0 \bar{t}_0 |\chi_0^{s(1)}\rangle_{(n)_k}, \quad (3.55)$$

which leads to the first-order independent equations of motion for the rest vectors

$$\begin{pmatrix} \bar{t}_0 & \Delta Q \\ \Delta Q & \frac{1}{2}\{\bar{t}_0, \eta_i^+ \eta_i\} \end{pmatrix} \begin{pmatrix} |\chi_0^0\rangle_{(n)_k} \\ |\chi_0^1\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ for } \{A, B\} = AB + BA, \quad (3.56)$$

which follows from the Lagrangian action with help of supermatrix multiplication

$$\mathcal{S}_{(n)_k} = ({}_{(n)_k} \langle \tilde{\chi}_0^0 | {}_{(n)_k} \langle \tilde{\chi}_0^1 |) K_{(n)_k} \begin{pmatrix} \bar{t}_0 & \Delta Q \\ \Delta Q & \frac{1}{2}\{\bar{t}_0, \eta_i^+ \eta_i\} \end{pmatrix} \begin{pmatrix} |\chi_0^0\rangle_{(n)_k} \\ |\chi_0^1\rangle_{(n)_k} \end{pmatrix}, \quad (3.57)$$

where the standard odd scalar product for the creation and annihilation operators is assumed, with the measure $d^d x$ over the Minkowski space. The vectors (Dirac spinors) $|\chi_0^{s(l)}\rangle_{(n)_k}$ (3.54) as the solution of the spin distribution relations (3.47) are the respective vectors $|\chi_0^l\rangle$ in (3.40) for massless ($m = 0$) HS fermionic field $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}(x)$ with the ghost number $gh(|\chi_0^l\rangle_{(n)_k}) = -l$.

The action (3.57) and the equations of motion (3.56) are invariant with respect to reducible gauge transformations:

$$\delta \begin{pmatrix} |\chi_0^{s(0)}\rangle_{(n)_k} \\ |\chi_0^{s(1)}\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} \Delta Q & \frac{1}{2}\{\bar{t}_0, \eta_i^+ \eta_i\} \\ \bar{t}_0 & \Delta Q \end{pmatrix} \begin{pmatrix} |\chi_0^{s+1(0)}\rangle_{(n)_k} \\ |\chi_0^{s+1(1)}\rangle_{(n)_k} \end{pmatrix}, \quad \delta \begin{pmatrix} |\chi_0^{s_{\max}(0)}\rangle_{(n)_k} \\ |\chi_0^{s_{\max}(1)}\rangle_{(n)_k} \end{pmatrix} = 0 \quad (3.58)$$

for $s = 0, 1, \dots, s_{\max} = \sum_{o=1}^k n_o + k(k-1)/2$ with a finite number of reducibility stage to be equal to $(s_{\max} - 1)$.

In [68] it was shown (see appendix C for massive case) that equivalence of the unconstrained BRST-BFV Lagrangian formulation given by (3.57), (3.58) and by (3.56) to the set of the equations (3.2)–(3.4) for spin-tensor field, $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}(x)$, realizing initial irreducible Poincare group representation of spin $\mathbf{s} = (n_1 + \frac{1}{2}, \dots, n_k + \frac{1}{2})$. Note, that in fact, for the massless case the above equivalence follows from the Corollary 2 and comments with (2.58), (2.59) applied for topological dynamical system, where the selection of the solutions $\mathcal{H}_{1,2}^{\text{phys}}$ for the equations (3.2)–(3.4) in $\mathcal{H}$ are written in (2.56), whereas the second-order gauge Lagrangian dynamics are described by the second row (2.57) with $Q_{(n)_k}$ acting on the Fock space subspace, $\mathcal{H}_{c|\text{tot}}^{\sigma_i(G)} \subset \mathcal{H}_{c|\text{tot}}$, playing the role of Q_c in $H(Q_c)$.

Let us consider the HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ realization with use of the initial set of gamma-matrices γ^μ instead of Grassmann-odd ones $\tilde{\gamma}^\mu$ for massless case following to [20, 21]. Doing so, it is natural to assume, that, the Grassmann parity, ε , of γ^μ is not vanishing but composed from two summands, ε_{ISO} , ε_{BFV} induced by the Poincare group $\text{ISO}(1, d-1)$ and from the BRST-BFV method, respectively. It is explicitly given by the rule

$$\varepsilon(\gamma^\mu) = \varepsilon_{\text{ISO}}(\gamma^\mu) + \varepsilon_{\text{BFV}}(\gamma^\mu) = 0 + 1 = 1, \quad (3.59)$$

so that γ^μ have the same properties as for $\tilde{\gamma}^\mu$ of anticommuting with Grassmann-odd ghost oscillators: $\{\gamma^\mu, A\} = 0$ for

$$\varepsilon(A) = \varepsilon_{\text{ISO}}(A) + \varepsilon_{\text{BFV}}(A) = 0 + 1 = 1, \quad A \in \{\eta_i^{(+)}, \eta_{ij}^{(+)}, \vartheta_{rs}^{(+)}, \eta_0, \mathcal{P}_0, \mathcal{P}_i^{(+)}, \mathcal{P}_{ij}^{(+)}, \lambda_{rs}^{(+)}, \mathcal{P}_g^i, \eta_g^i\} \quad (3.60)$$

(for $(\varepsilon_{\text{ISO}}, \varepsilon_{\text{BFV}})\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}} = (1, 0)$) and commuting with rest Grassmann-even ones. Thus, instead of t_0, t_i, t_i^+ the superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$ will contain the constraints $(\tilde{t}_0, \tilde{t}_i, \tilde{t}_i^+) = (-i\gamma^\mu \partial_\mu, \gamma^\mu a_\mu^i, \gamma^\mu a_\mu^{i+})$ (3.8) with unchanged multiplication table 1. Note, first, the Grassmann parities both of BRST-like second order (un)constrained actions (3.50), (5.7) below and for its analogs constructed with use of γ^μ are equal to 1 due to odd-scalar product nature (3.15), where $\varepsilon(\tilde{\gamma}^0) = \varepsilon(\gamma^0) = 1$. Second, when the whole ghost oscillators pairing have calculated, it is natural to factorize ε with respect to ε_{BFV} -parity for γ -matrices passing to factor Grassmann parity: $\varepsilon/\varepsilon_{\text{BFV}}$. Third, the first order (un)constrained actions (3.57), (5.8) appear by bosonic functionals with respect to ε -parity (3.59). The (un)constrained Lagrangian formulations for massless half-integer HS fields in Minkowski space-time with use of γ^μ matrices are equivalent to ones constructed using $\tilde{\gamma}^\mu$ matrices for any dimension d .

Now, we have all the tools to pass to construction of the constrained BRST-BFV approach for the same initial fermionic HS field.

4 Derivation of the constrained BRST operator, spin operator and BRST-extended algebraic constraints

Here, we consider the same main objects as for unconstrained BRST approach but with extracted set of second-class constraints o_a, o_a^+ together with g_0^i (related to $\Delta_{ab}(g_0)$ in (3.21)) from the HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$, which should be imposed on $\mathcal{H}_{\text{tot}}$ in compatible way with the constrained BRST and spin operator for the superalgebra of the rest first-class constraints o_A with space-time derivatives only.

4.1 Reduction from unconstrained BRST operator

Starting from the representation (3.31)–(3.33) for BRST-BFV operator Q' (3.26) written for the converted constraints O_I , let us extract from Q' (hence from Q , $\sigma^i(G)$) the terms corresponding to the Minkowski space $\mathbb{R}^{1, d-1}$ isometries subsuperalgebra (being the ideal in $\mathcal{A}^f(Y(k), \mathbb{R}^{1, d-1})$) of the first-class constraints subsystem $\{o_A\}$ and the summands corresponding to superalgebra isomorphic to $osp(1|2k)$ of the second-class constraints $\{O_a, O_a^+\}$ only:⁵

$$\sigma^i(G) = \sigma_C^i(G) + \sigma^i(O_a, O_a^+), \quad \sigma_C^i(G) = G_0^i - \eta_i \mathcal{P}_i^+ + \eta_i^+ \mathcal{P}_i, \quad (4.1)$$

$$\begin{aligned} \sigma^i(O_a, O_a^+) = & \sum_m (1 + \delta_{im}) (\eta_{im}^+ \mathcal{P}^{im} - \eta_{im} \mathcal{P}_{im}^+) + q_i p_i^+ + q_i^+ p_i + \sum_{l < i} [\vartheta_{li}^+ \lambda^{li} - \vartheta^{li} \lambda_{li}^+] \\ & - \sum_{i < l} [\vartheta_{il}^+ \lambda^{il} - \vartheta^{il} \lambda_{il}^+], \end{aligned} \quad (4.2)$$

⁵In this section and later on, we understand the subscript “C” in $\sigma_C^i(G)$, $Q_C(o_A)$, $|\chi_C^l\rangle$ in the sense that the corresponding objects belong to the constrained BRST-BFV approach unless stated otherwise.

$$Q = Q_C(o_A) + Q(\mathcal{O}_a, \mathcal{O}_a^+), \quad (4.3)$$

$$Q_C(o_A) = q_0 t_0 + \eta_0 l_0 + \eta_i^+ l^i + l^{i+} \eta_i + i \left(\sum_l \eta_l^+ \eta^l - q_0^2 \right) \mathcal{P}_0, \quad (4.4)$$

$$\begin{aligned} Q(\mathcal{O}_a, \mathcal{O}_a^+) = & q_i^+ \mathcal{T}^i + \sum_{l \leq m} \eta_{lm}^+ \mathcal{L}^{lm} + i \sum_l \eta_l^+ q^l p_0 + \sum_{l < m} \vartheta_{lm}^+ \mathcal{T}^{lm} + \left[\frac{1}{2} \sum_{l,m} (1 + \delta_{lm}) \eta^{lm} q_l^+ \right. \\ & - \sum_{l < m} q_l \vartheta^{lm} - \sum_{m < l} q_l \vartheta^{ml+} \left. \right] p_m^+ - 2 \sum_{l < m} q_l q_m^+ \lambda^{lm} - 2 \sum_{l,m} q_l^+ q_m^+ \mathcal{P}^{lm} - \sum_{i < l < j} \vartheta_{lj}^+ \vartheta_i^{+l} \lambda^{ij} \\ & - \sum_{l < n < m} \vartheta_{lm}^+ \vartheta_n^l \lambda^{nm} + \sum_{n < l < m} \vartheta_{lm}^+ \vartheta_n^m \lambda^{+nl} - \sum_{n,l < m} (1 + \delta_{ln}) \vartheta_{lm}^+ \eta^{l+} \mathcal{P}^{mn} \\ & + \sum_{n,l < m} (1 + \delta_{mn}) \vartheta_{lm}^+ \eta_n^m \mathcal{P}^{+ln} + \frac{1}{2} \sum_{l < m, n \leq m} \eta_{nm}^+ \eta_l^n \lambda^{lm} + h.c. \end{aligned} \quad (4.5)$$

with unchanged operator $\mathcal{B}^i$ (3.34). The operator $Q(\mathcal{O}_a, \mathcal{O}_a^+)$ depends on the same, but extended by means of the ghost coordinates and momenta corresponding to the first-class constraints $\{o_A\}$ set of the second-class constraints:

$$\mathcal{T}_i = T_i(B, B^+) - \eta_i p_0 - 2q_0 \mathcal{P}_i, \quad \mathcal{L}_{lm} = L_{lm}(B, B^+) + \frac{1}{2} \eta_{\{m} \mathcal{P}_{l\}}, \quad l \leq m \quad (4.6)$$

$$\mathcal{T}_{rs} = T_{rs}(B, B^+) - \eta_r^+ \mathcal{P}_s - \mathcal{P}_r^+ \eta_s, \quad r < s \quad (4.7)$$

and respective Hermitian conjugated constraints $\mathcal{O}_a^+$. The set of the constraints $\{\mathcal{T}_i, \mathcal{L}_{lm}, \mathcal{T}_{rs}; \mathcal{T}_i^+, \mathcal{L}_{lm}^+, \mathcal{T}_{rs}^+\}$ augmented by the constrained spin operator $\sigma_C^i(G)$ satisfies to the same algebraic relations as the superalgebra of $\{o_a, o_a^+; g_0^i\}$ given by table 1 under replacement $(o_a, o_a^+; g_0^i) \rightarrow (\mathcal{O}_a, \mathcal{O}_a^+; \sigma_C^i(G))$. For vanishing of the additional parts o'_a, o'^+_a (equivalently, quotient of $\mathcal{H}_{c|tot}$ by Fock subspace generated by B^{a+}) the same superalgebra for $([\mathcal{O}_a, \mathcal{O}_a^+]_{(B=B^+=0)}; \sigma_C^i(g))$ acting on $\mathcal{H} \otimes \mathcal{H}_{gh|m}$ holds. We will call the set of the constraints $\{\mathcal{O}_a, \mathcal{O}_a^+\}_{(B=B^+=0)}$ as the *BRST extended constraints*.

From the supercommutator's relations it follows that algebraically independent set for a half of the BRST extended constraints, $\mathcal{O}_a$ is composed from k elements:

$$\{\mathcal{O}_{\bar{a}}\} = \{\mathcal{T}_1, \mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{34}, \dots, \mathcal{T}_{(k-1)k}\}. \quad (4.8)$$

Indeed, the rest (dependent) elements $\{\mathcal{O}_{\mathbf{a}}\}$ from $\{\mathcal{O}_a\} = \{\mathcal{O}_{\bar{a}}, \mathcal{O}_{\mathbf{a}}\}$ are generated as follows:

$$\mathcal{T}_i = \text{ad}_{\mathcal{T}_{(i-1)i}} \dots \text{ad}_{\mathcal{T}_{12}} \mathcal{T}_1 \quad \mathcal{L}_{lm} = \frac{1}{4} \text{ad}_{\mathcal{T}_l} \text{ad}_{\mathcal{T}_{(m-1)m}} \dots \text{ad}_{\mathcal{T}_{12}} \mathcal{T}_1 \quad (4.9)$$

$$\mathcal{T}_{rs} = (-1)^{s-1-r} \text{ad}_{\mathcal{T}_{r(r+1)}} \dots \text{ad}_{\mathcal{T}_{(s-2)(s-1)}} \mathcal{T}_{(s-1)s} \quad (4.10)$$

for $\text{ad}_B A \equiv [B, A]$.

The nilpotency of unconstrained BRST-BFV operator Q' for $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ with following from it equations on unconstrained operators $Q, \sigma^i(g)$ (3.35), but in $\mathcal{H}_{tot}$, leads

to the nontrivial equations for extracted operators (4.1)–(4.5):

$$Q_C^2 = 0, \quad [Q_C, \sigma_C^i(g)] = 0, \quad (4.11)$$

$$[Q_C, Q(\mathcal{O}_a, \mathcal{O}_a^+)] = 0, \quad [Q(\mathcal{O}_a, \mathcal{O}_a^+), \sigma_C^i(G)] = [\sigma^i(\mathcal{O}_a, \mathcal{O}_a^+), Q(\mathcal{O}_a, \mathcal{O}_a^+)], \quad (4.12)$$

$$Q^2(\mathcal{O}_a, \mathcal{O}_a^+) + i \sum_j \mathcal{B}^j \sigma^j(G) = 0. \quad (4.13)$$

Thus, we see from (4.11) that BRST-BFV operator Q_C is nilpotent and supercommute with constrained spin operator $\sigma_C^i(g)$ everywhere in $\mathcal{H}_{\text{tot}}$, whereas the BRST-BFV operator $Q(\mathcal{O}_a, \mathcal{O}_a^+)$ according to (4.13) is nilpotent only on the proper eigen states $|\chi\rangle$ for generalized spin operator $\sigma^i(G)|\chi\rangle = 0$. The set of $\mathcal{O}_a, \mathcal{O}_a^+$ when acting on such $|\chi\rangle$ is classified as the first-class constraint system converted from the second-class constraints o_a, o_a^+ acting on the Hilbert subspace $\mathcal{H} \otimes H_{gh}^{oA}$. From the supercommutativity of both BRST-BFV operators $Q_C, Q(\mathcal{O}_a, \mathcal{O}_a^+)$ in the left equation in (4.12) it follows at the first degree in ghost operator $C^\alpha = (C^{a+}, C^a)$ the supercommutativity of Q_C and $\mathcal{O}_a, \mathcal{O}_a^+$:

$$[Q_C, \mathcal{O}_a|_{(B=B^+=0)}] = 0 \Rightarrow \left([Q_C, \mathcal{O}_a|_{(B=B^+=0)}]\right)^+ = [\mathcal{O}_a^+|_{(B=B^+=0)}, Q_C] = 0. \quad (4.14)$$

Finally, from the left-hand side of the last equations in (4.12) at the first degree in ghost operator C^α we have

$$\begin{aligned} [\mathcal{O}_a|_{(B=B^+=0)}, \sigma_C^i(g)] &= K_a^i \mathcal{O}_a|_{(B=B^+=0)}, \quad K_a^i \in \mathbb{Z}: \\ \text{for } K_a^i &\equiv (K_j^i, K_{lm}^i, K_{rs}^i) = (\delta_j^i, \delta_{lm}(\delta_l^i + \delta_m^i), \delta_{rs}(\delta_s^i - \delta_r^i)) \end{aligned} \quad (4.15)$$

respectively for $\mathcal{O}_a|_{(B=B^+=0)} = (\mathcal{T}_j, \mathcal{L}_{lm}, \mathcal{T}_{rs})$. Therefore, the operators $Q_C, \mathcal{O}_a|_{(B=B^+=0)}, \sigma_C^i(g)$ compose the closed $[\ , \]$ -superalgebra acting in $\mathcal{H} \otimes H_{gh}^{oA}$.

For further research we develop the results known from the general theory of constrained system exposed in the section 2.

Doing so, the application of the Corollary 1 to the BRST-BFV operator, $Q(\mathcal{O}_a, \mathcal{O}_a^+) \equiv Q_{\mathcal{O}}$ (4.5) for the set of converted second-class constraints $(Q_{c|2}, \Phi_\alpha(\hat{\Gamma}_c)$ in (2.54), (2.55)), $\mathcal{O}_\alpha = (\mathcal{O}_a, \mathcal{O}_a^+)$ from the superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ means due to the validity of the nilpotency for $Q_{\mathcal{O}}$ only on the Fock subspace $\mathcal{H}_{\text{tot}}^{\sigma(G)}$ of $\mathcal{H}_{c|\text{tot}}$ which contains only the proper eigen vectors for the spin operator $\sigma^i(G)$, that it is true the

Statement 3. The Fock subspace, $H_{\mathcal{O}_a}^{\sigma_C^i}$: $H_{\mathcal{O}_a}^{\sigma_C^i} \subset \mathcal{H} \otimes H_{gh}^{oA}$, being proper for the constrained spin operator $\sigma_C^i(g) = \sigma^i(G)|_{\mathcal{H} \otimes H_{gh}^{oA}}$ and for the half of the BRST extended second-class constraints not depending on the auxiliary (converted) oscillators: $\mathcal{O}_a|_{B=B^+=0}$ (4.6), (4.7) contains the same (equivalent) set of the states as the Fock subspace in the quotient $\ker Q_{\mathcal{O}} / \text{Im } Q_{\mathcal{O}}$ with zero ghost number and being proper for generalized spin operator $\sigma^i(G)$ (4.1):

$$H_{\mathcal{O}_a}^{\sigma_C^i} = \{|\chi_c\rangle | (\mathcal{O}_a|_{B=B^+=0}, \sigma_C^i(G), gh)|\chi_c\rangle = (0, 0, 0), \quad |\chi_c\rangle \in \mathcal{H} \otimes H_{gh}^{oA}\} \quad (4.16)$$

$$= \{|\chi^0\rangle | \delta|\chi^0\rangle = Q_{\mathcal{O}}|\chi^1\rangle, \dots, \delta|\chi^{s-1}\rangle = Q_{\mathcal{O}}|\chi^s\rangle, \delta|\chi^s\rangle = 0, \quad |\chi^k\rangle \in \mathcal{H}_{\text{tot}}^{\sigma^i(G)|k}\}, \quad (4.17)$$

with $gh(|\chi^k\rangle) = -k$ for $k = 0, \dots, s$, $s = \sum_{o=1}^k n_o + k(k-1)/2$.

From the Statement 3 it follows the important in practice

Corollary 3. A set of the states $H_{\mathcal{O}_a, o_A}^{\sigma_C^i} : H_{\mathcal{O}_a, o_A}^{\sigma_C^i} \subset \mathcal{H} \otimes H_{gh}^{o_A}$ with vanishing ghost number from the Fock subspace: $\ker Q_C / \text{Im } Q_C$, with nilpotent BRST-BFV operator Q_C (4.4) for the subsuperalgebra of constraints o_A acting in $\mathcal{H} \otimes H_{gh}^{o_A}$ being proper eigen-states both for constrained spin operator $\sigma_C^i(g)$ and annihilated by the half of the BRST extended second-class constraints $\mathcal{O}_a|_{B=B^+=0}$ is equivalent to the set of the states from the Fock subspace: $\ker Q / \text{Im } Q$, with BRST-BFV operator Q (4.3) for only system of constraints $\{\mathcal{O}_I \setminus G_0^i\}$ in the HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ to be nilpotent on the proper eigen-states for generalized spin operator $\sigma^i(G)$ (4.1) acting in $\mathcal{H}_{\text{tot}} = \mathcal{H} \otimes H' \otimes H_{gh}$:

$$H_{\mathcal{O}_a, o_A}^{\sigma_C^i} = \{|\psi_c\rangle | (\mathcal{O}_a|_{B=B^+=0}, \sigma_C^i(g), gh)|\psi_c\rangle = (0, 0, 0), |\psi_c\rangle \in \ker Q_C / \text{Im } Q_C\} \quad (4.18)$$

$$= \{|\psi\rangle | (\sigma^i(G), gh)|\psi\rangle = (0, 0), |\psi\rangle \in \ker Q / \text{Im } Q\}. \quad (4.19)$$

Equivalently, in terms of the respective Q - and Q_C -complexes the equivalence above means that the found set of states, $H_{\mathcal{O}_a, o_A}^{\sigma_C^i}$ may be presented according to (4.16), (4.17) as:

$$H_{\mathcal{O}_a, o_A}^{\sigma_C^i} = \{|\chi_c^0\rangle | (Q_C, \mathcal{O}_a|_{B=B^+=0}, \sigma_C^i(g), gh)|\chi_c^l\rangle = (\delta|\chi_c^{l-1}\rangle, 0, 0, -l), |\chi_c^l\rangle \in \mathcal{H} \otimes H_{gh}^{o_A|l}\} \quad (4.20)$$

$$= \{|\chi^0\rangle | (Q, \sigma^i(G), gh)|\chi^m\rangle = (\delta|\chi^{m-1}\rangle, 0, -m) |\chi^m\rangle \in \mathcal{H}_{\text{tot}}^m, m = 0, \dots, s\}, \quad (4.21)$$

for $|\chi_c^{-1}\rangle = |\chi^{-1}\rangle = 0$, $l = 0, 1, \dots, k$ and $s = \sum_{o=1}^k n_o + k(k-1)/2$.

Note, that due to algebraic independence of the set of k constraints $\{\mathcal{O}_{\bar{a}}\} = \{\mathcal{T}_1, \mathcal{T}_{(m-1)m}\}$, $m = 2, \dots, k$ (4.8) among $\mathcal{O}_a$ it is sufficient to impose $\{\mathcal{O}_{\bar{a}}\}$ instead of $\mathcal{O}_a$ in the (4.16), (4.18) and (4.20).

Corollary 3 represents the basis for the equivalent description of the Lagrangian dynamics for free half-integer HS field on $\mathbb{R}^{1,d-1}$ subject to $Y(s_1, \dots, s_k)$ both on a base of unconstrained BRST method and in terms of constrained gauge-invariant Lagrangian formulation with use of constrained BRST approach. Note, earlier the Lagrangian dynamics for free half-integer HS field was developed only within unconstrained BRST method. Because of, HS symmetry algebra $\mathcal{A}(Y(k), \mathbb{R}^{1,d-1})$ [67] for free integer HS field on $\mathbb{R}^{1,d-1}$ subject to $Y(s_1, \dots, s_k)$, in fact, contains in the $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ the Corollary 3 solves the same problem for free integer HS field.

Hence, in order to prove that the set of solutions for the irreducibility representations conditions (3.2)–(3.4) [or in the form (3.7), (3.9)] for half-integer HS fields of fixed spin is equivalent to the Lagrangian equations of motion for more wider set of HS fields to be subject to the reducible gauge transformations, it is enough to consider the constrained gauge-invariant Lagrangian formulation.

4.2 Self-consistent constrained BRST, spin operators, off-shell constraints from HS symmetry superalgebra

The constrained operator quantities $Q_C, \sigma_C^i(g)$ and algebraically independent BRST extended constraints $\{\mathcal{O}_{\bar{a}}\}$ (4.8) may be derived without appealing to the unconstrained

BRST formulation starting explicitly from the equations (3.2)–(3.4), thus revealing the relation with so called tensionless limit of open superstring theory [1], which contains many higher-spin excitations in its spectrum.

Starting from the Poincare group irreducibility conditions (3.2)–(3.4) extracting the massless HS field of spin $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2, \dots, n_k + 1/2)$, realized equivalently for Fock space vector (3.5) by the equations (3.7), (3.9), we consider only the subsuperalgebra of Minkowski space $\mathbb{R}^{1,d-1}$ isometries (3.23) $\{o_A\} = \{t_0, l_0, l_i, l_i^+\}$ as the dynamical constraints being used to construct Lagrangian formulation, whereas the rest (algebraic) primary constraints t_i, t_{rs} (3.13), (3.8) should be imposed as off-shell (i.e. holonomic) constraints on the respective solutions of the Lagrangian equations of motion, but in a *consistent* way. From the algebraic viewpoint, the set of operators t_i, t_{rs} appears by the derivations of the subalgebra $\{o_A\} : [A, o_A] \in \{o_A\}$ for any $A \in \{t_i, t_{rs}\}$. The constraints t_i, t_{rs} itself generates the superalgebra of total set of constraints $\{o_a\} = \{t_i, t_{rs}, l_{lm}\}$ being by the restriction of $\mathcal{O}_a$ (4.9) on the initial Fock space $\mathcal{H}$. The algebraically independent set of $\{o_a\}$ according to (4.8), (4.9) is given by:

$$\{o_a\} = \{t_1, t_{12}, t_{23}, t_{34}, \dots, t_{(k-1)k}\}. \quad (4.22)$$

The constrained BRST operator Q_C for the system of the first-class constraints $\{o_A\}$ in the Hilbert space $\mathcal{H}_C$, $\mathcal{H}_C = \mathcal{H} \otimes H_{gh}^{oA}$ has the form (4.4). The *consistency* condition requires that the set of off-shell constraints t_i, t_{rs} as well as the number particle operator being additively extended in powers of the ghosts variables $C^A, \bar{\mathcal{P}}_A$ in $\mathcal{H}_C$ to $\hat{T}_i, \hat{T}_{rs}$ and $\hat{\sigma}_C^i(g)$:

$$\left(\hat{T}_i, \hat{T}_{rs}, \hat{\sigma}_C^i(g)\right) = \left(t_i + C^A t_{iA}^B \bar{\mathcal{P}}_B, t_{rs} + C^A t_{rsA}^B \bar{\mathcal{P}}_B, g_0^i + C^A g_A^{iB} \bar{\mathcal{P}}_B\right) + o(C\bar{\mathcal{P}}) \quad (4.23)$$

with preservation of the vanishing ghost number: $gh(\hat{T}_i, \hat{T}_{lm}, \hat{\sigma}_C^i(g)) = (0, 0, 0)$ should satisfy to the equations

$$[Q_C, \hat{T}_i] = 0, \quad [Q_C, \hat{T}_{rs}] = 0, \quad [Q_C, \hat{\sigma}_C^i(g)] = 0, \quad (4.24)$$

which we call by the *generating equations* for superalgebra of the constrained BRST, Q_C spin operators $\hat{\sigma}_C^i(g)$ and extended in $\mathcal{H}_C$ off-shell constraints $\hat{T}_i, \hat{T}_{rs}$.

Explicit calculations show, that the solutions for unknown $\hat{\sigma}_C^i(g)$, $\hat{T}_i, \hat{T}_{rs}$ exists for quadratic in powers of ghosts $C^A, \bar{\mathcal{P}}_A$ approximation in the form:

$$C^A \left(t_{iA}^B, t_{rsA}^B, g_A^{iB}\right) \mathcal{P}_B = \left(-\eta_i p_0 - 2q_0 \mathcal{P}_i, -\eta_r^+ \mathcal{P}_s - \mathcal{P}_r^+ \eta_s, \eta_i^+ \mathcal{P}_i - \eta_i \mathcal{P}_i^+\right). \quad (4.25)$$

Comparison with the structure of the BRST extended constraints $\mathcal{T}_i, \mathcal{T}_{rs}$ (4.6) and $\sigma_C^i(G)$ (4.1) restricted to $\mathcal{H}_C$, permits to state on their coincidence (for $h^i = 0$ in case $\sigma_C^i(g) = \sigma_C^i(G)|_{B=0}$):

$$\left(\hat{T}_i, \hat{T}_{rs}, \hat{\sigma}_C^i(g)\right) = \left(\mathcal{T}_i, \mathcal{T}_{rs}, \sigma_C^i(g)|_{h^i=0}\right). \quad (4.26)$$

Therefore, the superalgebra of the constrained operators $\{Q_C, \hat{T}_i, \hat{T}_{rs}, \hat{\sigma}_C^i(g)\}$ derived in self-consistent way coincides with the superalgebra $\{Q_C, \mathcal{T}_i, \mathcal{T}_{rs}, \sigma_C^i(g)|_{h^i=0}\}$ of the operators derived from unconstrained BRST formulation.

As to the difference of the constrained spin operators: *self-consistent*, $\hat{\sigma}_C^i(g)$, and $\sigma_C^i(g)$ then because of the coincidence the set of proper eigen-vectors, $|\chi_c^l\rangle_{(n)_k} \in \mathcal{H}_C$, ($gh(|\chi_c^l\rangle) = -l$) (e.g. due to its supercommutativity: $[\hat{\sigma}_C^i(g), \sigma_C^i(g)] = 0$) the corresponding equations:

$$\hat{\sigma}_C^i(g)|\chi_c^l\rangle_{(n)_k} = \left(n^i + \frac{d-2}{2}\right)|\chi_c^l\rangle_{(n)_k}, \quad \sigma_C^i(g)|\chi_c^l\rangle_{(n)_k} = 0 \quad \text{for } h_C^i = -n^i - \frac{d-2}{2} \quad (4.27)$$

with stabilized $h_C^i \equiv h_C$, $\forall i = 1, \dots, k$, in fact, determine identical spin value distribution:

$$n_i = p_i + n_{fi} + n_{pi}, \quad i = 1, \dots, k, \quad (4.28)$$

for $|\chi_c\rangle$ having the representation

$$|\chi_c\rangle = \sum_n q_0^{n_{b0}} \eta_0^{n_{f0}} \prod_{i,j} (\eta_i^+)^{n_{fi}} (\mathcal{P}_j^+)^{n_{pj}} |\Psi(a_i^+)^{n_{b0}n_{f0};(n)_{fi}(n)_{pj}}\rangle, \quad (4.29)$$

following from (3.40) for $B^+ = 0$, $(C^a, \bar{\mathcal{P}}_a) = 0$ and for

$$|\Psi(a_i^+)^{n_{b0}n_{f0};(n)_{fi}(n)_{pj}}\rangle = |\Psi(a_i^+)^{n_{b0}n_{f0};(0)_{ae}(0)_{bg}(n)_{fi}(n)_{pj}(0)_{flm}(0)_{pno}(0)_{frs}(0)_{\lambda tu}}\rangle.$$

From (4.29) [equivalently from (3.48)] it follows the ghost number distribution for n 's:

$$|\chi_c^l\rangle_{(n)_k} : n_{b0} + n_{f0} + \sum_i (n_{fi} - n_{pi}) = -l. \quad (4.30)$$

Thus, we establish the fact, that the same set of states $H_{(\mathcal{O}_a, o_A)}^{\sigma_C^i}$ (4.20) from Corollary 3 is equivalently reproduced both by the superalgebra of constrained operators: $\{Q_C, \mathcal{T}_i, \mathcal{T}_{rs}, \sigma_C^i(g)\}$ derived from unconstrained superalgebra $\{Q, \sigma^i(G)\}$ and one of $\{Q_C, \hat{\mathcal{T}}_i, \hat{\mathcal{T}}_{rs}, \hat{\sigma}_C^i(g)\}$. Therefore, the two ways of derivation of the constrained BRST Lagrangian formulation (self-consistently and from unconstrained Lagrangian formulation) are equivalent.

5 Constrained gauge-invariant Lagrangian formulations

In this section. we consider the construction of constrained gauge-invariant Lagrangian formulations for mixed-symmetric HS fields in case of massless field with fixed generalized half-integer spin, then in case of one with fixed generalized integer spin, and in case of massive fields.

5.1 Constrained Lagrangian formulation for half-integer HS fields

From the nilpotent BRST operator Q_C , spin operator σ_C^i ($\sigma_C^i \equiv \hat{\sigma}_C^i(g)$), algebraically independent BRST extended constraints $\hat{\mathcal{T}}_i, \hat{\mathcal{T}}_{rs}$, $i, r, s = 1, \dots, k$, $r < s$ we have the

spectral problem, analogous to one for unconstrained case (3.41)–(3.44), but for $|\chi_c^l\rangle \in \mathcal{H}_C^l$:

$$Q_C|\chi_c\rangle=0, \quad \sigma_C^i|\chi_c\rangle=\left(n^i+\frac{d-2}{2}\right)|\chi_c\rangle, \quad (\epsilon, gh_H)(|\chi_c\rangle)=(1,0), \quad (5.1)$$

$$\delta|\chi_c\rangle=Q_C|\chi_c^1\rangle, \quad \sigma_C^i|\chi_c^1\rangle=\left(n^i+\frac{d-2}{2}\right)|\chi_c^1\rangle, \quad (\epsilon, gh_H)(|\chi_c^1\rangle)=(0,-1), \quad (5.2)$$

...

...

...

$$\delta|\chi_c^{s_c-1}\rangle=Q_C|\chi_c^{s_c}\rangle, \quad \sigma_C^i|\chi_c^{s_c}\rangle=\left(n^i+\frac{d-2}{2}\right)|\chi_c^{s_c}\rangle, \quad (\epsilon, gh_H)(|\chi_c^{s_c}\rangle)=(s_c+1, -s_c), \quad (5.3)$$

$$(\widehat{T}_i, \widehat{T}_{rs})|\chi_c^l\rangle=0, \quad l=0,1,\dots,s_c. \quad (5.4)$$

Because of the representations (4.29) and (3.40) the physical state $|\Psi\rangle$ (3.5) contains in $|\chi_c^0\rangle=|\chi_c\rangle$ as it was for $|\chi^0\rangle$ in (3.45), but with $(C^A, \overline{\mathcal{P}}_A)$ -dependent vector $|\Psi_{Ac}\rangle$, $|\Psi_{Ac}\rangle=|\Psi_A\rangle$ when $(B_a^+, \eta_{ij}^+, \mathcal{P}_{ij}^+, \vartheta_{rs}^+, \lambda_{rs}^+)=0$.

The system (5.1)–(5.4) is compatible, due to closedness of the superalgebra $\{Q_C, \sigma_C^i, \widehat{T}_i, \widehat{T}_{rs}\}$. Therefore, its resolution for the joint set of proper eigen-vectors we start from the middle set which determined by (4.27) with distributions (4.28), (4.30) for the proper eigen-vectors $|\chi_c^l\rangle_{(n)_k}$. For fixed spin $(s)_k=(n)_k+(\frac{1}{2}, \dots, \frac{1}{2})$ the solution of the rest equations is written as the second-order equations of motion and sequence of the reducible gauge transformations (5.1)–(5.3) with off-shell constraints:

$$Q_C|\chi_c^0\rangle_{(n)_k}=0, \quad \delta|\chi_c^0\rangle_{(n)_k}=Q_C|\chi_c^1\rangle_{(n)_k}, \dots, \delta|\chi_c^{s_c-1}\rangle_{(n)_k}=Q_C|\chi_c^{s_c}\rangle_{(n)_k}, \delta|\chi_c^{s_c}\rangle_{(n)_k}=0, \quad (5.5)$$

$$(\widehat{T}_i, \widehat{T}_{rs})|\chi_c^l\rangle_{(n)_k}=0, \quad l=0,1,\dots,s_c, \quad \text{for } s_c=k. \quad (5.6)$$

The corresponding BRST-like constrained gauge-invariant action

$$\mathcal{S}_{C|(n)_k}^{(2)}=\int d\eta_{0(n)_k}\langle\tilde{\chi}_c^0|Q_C|\chi_c^0\rangle_{(n)_k}, \quad (5.7)$$

from which follows the equations of motion $Q_C|\chi_c^0\rangle_{(n)_k}=0$, contains second order operator l_0 , but less terms in comparison with its unconstrained analog $\mathcal{S}_{(n)_k}^{(2)}$ (3.50).

Again, repeating the procedure of the removing the dependence on l_0, η_0, q_0 from the BRST operator Q_C (4.4) and from the whole set of the vectors $|\chi_c^l\rangle_{(n)_k}$ as it was done in the section 3.1 by means of partial gauge-fixing we come to the:

Statement 4. The first-order constrained gauge-invariant Lagrangian formulation for half-integer HS field, $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}(x)$ with generalized spin $(s)_k=(n)_k+(\frac{1}{2}, \dots, \frac{1}{2})$, is determined by the action,

$$\mathcal{S}_{C|(n)_k}=\left({}_{(n)_k}\langle\tilde{\chi}_{0|c}^0|{}_{(n)_k}\langle\tilde{\chi}_{0|c}^1|\right)\begin{pmatrix} t_0 & \Delta Q_C \\ \Delta Q_C & t_0\eta_i^+\eta_i \end{pmatrix}\begin{pmatrix} |\chi_{0|c}^0\rangle_{(n)_k} \\ |\chi_{0|c}^1\rangle_{(n)_k} \end{pmatrix} \quad \text{for } \Delta Q_C=\eta_i^+l_i+l_i^+\eta_i, \quad (5.8)$$

invariant with respect to the sequence of the reducible gauge transformations (for $s_c - 1 = (k - 1)$ -being by the stage of reducibility):

$$\delta \begin{pmatrix} |\chi_{0|c}^{l(0)}\rangle_{(n)_k} \\ |\chi_{0|c}^{l(1)}\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} \Delta Q_C & t_0 \eta_i^+ \eta_i \\ t_0 & \Delta Q_C \end{pmatrix} \begin{pmatrix} |\chi_{0|c}^{l+1(0)}\rangle_{(n)_k} \\ |\chi_{0|c}^{l+1(1)}\rangle_{(n)_k} \end{pmatrix}, \quad \delta \begin{pmatrix} |\chi_{0|c}^{k(0)}\rangle_{(n)_k} \\ |\chi_{0|c}^{k(1)}\rangle_{(n)_k} \end{pmatrix} = 0 \quad (5.9)$$

(for $l = -1, 0, \dots, k-1$ and $|\chi_{0|c}^{-1(m)}\rangle = 0$, $m = 0, 1$) with off-shell algebraically independent BRST-extended constraints imposed on the whole set of field and gauge parameters:

$$\widehat{T}_i \left(|\chi_c^{l(0)}\rangle_{(n)_k} + q_0 |\chi_c^{l(1)}\rangle_{(n)_k} \right) = 0, \quad \widehat{T}_{rs} |\chi_c^{l(m)}\rangle_{(n)_k} = 0, \quad l = 0, 1, \dots, k; \quad m = 0, 1. \quad (5.10)$$

The proof is based on the extraction of the zero-mode ghosts q_0, η_0 from Q_C and $|\chi_c^l\rangle_{(n)_k}$:

$$Q_C = q_0 t_0 + \eta_0 l_0 + \imath(\eta_i^+ q_i - \eta_i q_i^+) p_0 - \imath(q_0^2 - \eta_i^+ \eta_i) \mathcal{P}_0 + \Delta Q_C, \quad (5.11)$$

$$|\chi_c^l\rangle = \sum_{m=0}^k q_0^m (|\chi_{0|c}^{l(m)}\rangle + \eta_0 |\chi_{1|c}^{l(m)}\rangle), \quad gh(|\chi_{p|c}^{l(m)}\rangle) = -(p + l + m + l), \quad p = 0, 1 \quad (5.12)$$

(for $|\chi_{1|c}^{l(k)}\rangle \equiv 0$) which lead after gauge-fixing procedure with help of the equations of motion and the set of gauge transformations (5.6) to removing of all the components in powers of q_0, η_0 vector except for two fields for each level, $l = 0, \dots, k$. As the result, the representation

$$|\chi^l\rangle_{(n)_k} = |\chi_0^{l(0)}\rangle_{(n)_k} + q_0 |\chi_0^{l(1)}\rangle_{(n)_k} - i\eta_0 t_0 |\chi_0^{l(1)}\rangle_{(n)_k} \quad (5.13)$$

is true. Inserting (5.13) in (5.4) the off-shell BRST-extended constraints will be presented by the system of the equations (with omitting spin index $(n)_k$):

$$\begin{aligned} \widehat{T}_i |\chi_c^l\rangle &= \widehat{T}_i \left(|\chi_{0|c}^{l(0)}\rangle + q_0 |\chi_{0|c}^{l(1)}\rangle - i\eta_0 t_0 |\chi_{0|c}^{l(1)}\rangle \right) = 0 & \iff & \widehat{T}_i \left(|\chi_{0|c}^{l(0)}\rangle + q_0 |\chi_{0|c}^{l(1)}\rangle \right) = 0 \\ \widehat{T}_{rs} |\chi_c^l\rangle &= \widehat{T}_{rs} \left(|\chi_{0|c}^{l(0)}\rangle + q_0 |\chi_{0|c}^{l(1)}\rangle - i\eta_0 t_0 |\chi_{0|c}^{l(1)}\rangle \right) = 0 & \iff & \widehat{T}_{rs} \left(|\chi_{0|c}^{l(0)}\rangle + q_0 |\chi_{0|c}^{l(1)}\rangle \right) = 0 \end{aligned} \quad (5.14)$$

$$\text{and} \quad t_i t_0 |\chi_{0|c}^{l(1)}\rangle + 2q_0 t_0 \mathcal{P}_i |\chi_{0|c}^{l(1)}\rangle = 0, \quad (5.15)$$

which in terms of q_0 -independent vectors is written as:

$$\begin{aligned} t_i |\chi_{0|c}^{l(0)}\rangle - \eta_i |\chi_{0|c}^{l(1)}\rangle &= 0, & t_i |\chi_{0|c}^{l(1)}\rangle - 2\mathcal{P}_i |\chi_{0|c}^{l(0)}\rangle &= 0, & \mathcal{P}_i |\chi_{0|c}^{l(1)}\rangle &= 0, \\ \widehat{T}_{rs} |\chi_{0|c}^{l(0)}\rangle &= 0, & \widehat{T}_{rs} |\chi_{0|c}^{l(1)}\rangle &= 0. \end{aligned} \quad (5.16)$$

$$\text{and} \quad l_i |\chi_{0|c}^{l(1)}\rangle = -\mathcal{P}_i t_0 |\chi_{0|c}^{l(0)}\rangle. \quad (5.17)$$

To get (5.17) from (5.15) we have used the last equations in the first row of (5.16) and operator identity, $t_i t_0 = l_i - t_0 t_i$.

The solution for the gamma-traceless equations (5.16) leads to η_i^+ -independence of $|\chi_{0|c}^{l(1)}\rangle_{(n)_k}$ and, at most, linear in η_i^+ -dependence of $|\chi_{0|c}^{l(1)}\rangle_{(n)_k}$, due to algebraic consequences from the second and third equations in (5.16): $\mathcal{P}_i \mathcal{P}_j |\chi_{0|c}^{l(0)}\rangle = 0$ and $(\mathcal{P}_i)^2 = 0$.

Consider, now the half of the gauge transformations and equations of motion (for $l = -1$):

$$\delta|\chi_{0|c}^{l(1)}\rangle_{(n)_k} = t_0|\chi_{0|c}^{l+1(0)}\rangle_{(n)_k} + \Delta Q_C|\chi_{0|c}^{l+1(1)}\rangle_{(n)_k}, \quad l = -1, 0, \dots, k-1, \quad (5.18)$$

and apply to them from the left the operator $\mathcal{P}_i$. Because of, η_i^+ -independence of $|\chi_{0|c}^{l(1)}\rangle_{(n)_k}$, the constraints (5.17) coincide for each $l = -1, 0, \dots, k-1$ with respective gauge transformations and equations of motion (for $l = -1$). Therefore, (5.17) do not contain independent [from the constraints (5.16) and sequence of the gauge transformations (5.9)] restrictions on the vectors $|\chi_{0|c}^{l(m)}\rangle_{(n)_k}$. Because of the off-shell constraints (5.16) appear by unfolded decomposition in ghosts of the BRST-extended constraints (5.10) the Statement 4 is completely proved.

From the Statement 4 and Corollary 3 it follows that the unconstrained BRST Lagrangian formulation for the fermionic field, $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}(x)$, of spin $(s)_k = (n)_k + (\frac{1}{2}, \dots, \frac{1}{2})$ given by the action (3.57) and sequence of the gauge transformations (3.58) and constrained BRST Lagrangian formulation given by (5.8)–(5.10) for the same spin-tensor field determine the same dynamic, i.e. equivalent. The respective values of the stage of reducibility are as follows, $s_{\max} - 1 = \sum_{o=1}^k n_o + k(k-1)/2 - 1$ and $s_c - 1 = k - 1$.

Hence, in order to prove that the set of solutions for the irreducibility representations conditions (3.2)–(3.4) [or in the form (3.7), (3.9)] for half-integer HS fields of fixed spin is equivalent to the Lagrangian equations of motion for more wider set of HS fields to be subject to the reducible gauge transformations, it is enough to consider the constrained gauge-invariant Lagrangian formulation. Thus, we come to the basic

Theorem. The set of solutions, $H_{(m, (n)_k)}$, for the equations (3.2)–(3.4) [or in the form (3.7), (3.9)] extracting the Poincare group massless ($m = 0$) irreducible representation of spin $(n)_k + (\frac{1}{2}, \dots, \frac{1}{2})$ in terms of spin-tensor field, $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}$ is equivalent to the solutions of the Lagrangian equations of motion, for $l = -1$ in (5.9) subject to the reducible gauge transformations (5.9) for $l = 0, \dots, k-1$ and off-shell BRST-extended constraints (5.10):

$$\begin{aligned} H_{(0, (n)_k)} &= \{|\Psi\rangle | (t_0, t_i, t_{rs}, g_0^i - [n_i + d/2]) |\Psi\rangle = 0\} \\ &= \left\{ |\chi_{0|c}^e\rangle \left[\begin{pmatrix} t_0 & \Delta Q_C \\ \Delta Q_C & t_0 \eta_i^+ \eta_i \end{pmatrix}, \left\{ \sigma_C^i - n^i - \frac{d-2}{2} \right\} \mathbf{1}_2 \right] \begin{pmatrix} |\chi_{0|c}^0\rangle \\ |\chi_{0|c}^1\rangle \end{pmatrix} = 0, \right. \\ &\quad \delta \begin{pmatrix} |\chi_{0|c}^{l(0)}\rangle \\ |\chi_{0|c}^{l(1)}\rangle \end{pmatrix} = \begin{pmatrix} \Delta Q_C & t_0 \eta_i^+ \eta_i \\ t_0 & \Delta Q_C \end{pmatrix} \begin{pmatrix} |\chi_{0|c}^{l+1(0)}\rangle \\ |\chi_{0|c}^{l+1(1)}\rangle \end{pmatrix}, \delta \begin{pmatrix} |\chi_{0|c}^{k(0)}\rangle \\ |\chi_{0|c}^{k(1)}\rangle \end{pmatrix} = 0 \\ &\quad \left. \left(\hat{T}_i, \hat{T}_{rs}, \left\{ \sigma_C^i - n^i - \frac{d-2}{2} \right\} \right) \left(|\chi_{0|c}^{(l)0}\rangle + q_0 |\chi_{0|c}^{(l)1}\rangle \right) = 0 \right\}, \end{aligned} \quad (5.20)$$

where, $e = 0, 1$; $l = 0, \dots, k-1$; $\mathbf{1}_2$ appears by unit 2×2 matrix.

For massive case this theorem is proved for unconstrained BRST Lagrangian formulation in [68] by explicit resolution of BRST Q -complex within first order Lagrangian formulation. Here we reached the equivalence in question for massless case without above

procedure but with using the results of study the structure of the physical states for topological dynamical system subject to the first and second-class constraints on a base of BRST-BFV approach.

There are some consequences from suggested construction. First, the constrained BRST Lagrangian formulation due to nilpotency of Q_C on $\mathcal{H}_C$ without spin operator imposing may be used to determine Lagrangian formulation for so called fermionic HS fields with *continuous spin* (i.e. with not restricted by the spin constraints set of spin-tensor fields), suggested for totally-symmetric integer HS fields in $d = 4$ in [93]. Second, without off-shell constraints (5.10) imposing, but with generalized spin condition (given by the middle set in (5.1)–(5.3)) we will have so-called *generalized triplet formulation* for half-integer HS fields on Minkovski space-time with many auxiliary spin-tensors with different generalized spin values. Third, including of the part BRST-extended constraints, corresponding to the gamma-trace constraints: $\hat{T}_i |\chi_c^{l(m)}\rangle_{(n)_k} = 0$, (with obvious resolution the mixed-symmetric constraints $\hat{T}_{rs}$ for whole set of field and gauge parameters $|\chi_c^{l(m)}\rangle_{(n)_k}$) into Lagrangian dynamics with help of additional gauge transformations and Lagrangian multipliers permits to construct so-called *generalized quartet formulation* for half-integer HS field with spin $(s)_k = (n)_k + (1/2)_k$, suggested for totally-symmetric integer and half-integer HS fields in [71, 72].

5.2 Constrained Lagrangian formulation for integer HS fields

Another direct consequence from the procedure of constrained Lagrangian formulation for half-integer HS fields concerns, the equivalence among the unconstrained BRST Lagrangian formulation developed in [66, 67] and constrained BRST Lagrangian formulations for the integer HS irreducible representations of the Poincare group in Minkowski space-time with fixed generalized spin $(s)_k = (s_1, s_2, \dots, s_k)$, $s_i \geq s_j, i > j$. and constrained BRST Lagrangian formulation, in fact, suggested here in full details (considered without spin operator for totally-symmetric case in [76, 77] and for mixed-symmetric case in [78]).

Indeed, the HS symmetry algebra $\mathcal{A}(Y(k), \mathbb{R}^{1,d-1})$ for integer HS fields in $\mathbb{R}^{1,d-1}$ [67] (described by tensor fields, $\Phi_{(\mu^1)_{s_1}, \dots, (\mu^k)_{s_k}}(x)$ subject to the d’Alamber, traceless and divergentless equations:

$$\left(\partial^\mu \partial_\mu, \partial_\mu^i, \eta^{\mu_i \mu_j} \right) \Phi_{(\mu^1)_{s_1}, \dots, (\mu^k)_{s_k}}(x) = 0, \quad 1 \leq m_i \leq s_i; \quad i, j = 1, \dots, k, \quad (5.21)$$

instead of (3.2), (3.3) in addition to the mixed-symmetry conditions (3.4)) may be obtained from the superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ by extracting Grassmann-odd generators t_0, t_i, t_i^+ with ignoring the matrix-valued representation for the rest operators o_I^B : $\{o_I^B\} = \{o_I\} \setminus \{o_p^F\}$, for $\{o_p^F\} = \{t_0, t_i, t_i^+\}$, as well as for the respective BRST, Q^B , generalized spin operator σ_B^i not depending on the ghost: $q_0, q_i^{(+)}, p_i^{(+)}, p_0$, and auxiliary (for conversion) oscillators: f_i, f_i^+ , for $\{o_p^F\}$.

The unconstrained Lagrangian formulation, in fact, is given by the second-order Lagrangian action (3.50) adapted for integer spin case:

$$\mathcal{S}_{(s)_k}^B = \int d\eta_{0(s)_k} \langle \chi_B^0 | K_{(s)_k}^B Q_{(s)_k}^B | \chi_B^0 \rangle_{(s)_k}, \quad K^B = 1 \otimes K^{B'} \otimes 1_{gh}: \quad K_{(s)_k}^B Q_{(s)_k}^B = (Q^B)_{(s)_k}^+ K_{(s)_k}, \quad (5.22)$$

reproducing the Lagrangian equations of motion: $Q_{(s)_k}^B |\chi_B^0\rangle_{(s)_k} = 0$, being invariant with respect to reducible gauge transformations

$$\delta |\chi_B^l\rangle_{(s)_k} = Q_{(s)_k}^B |\chi_B^{l+1}\rangle_{(s)_k}, l = 0, 1, \dots, k(k+1) - 1 \dots, \delta |\chi_B^{k(k+1)}\rangle_{(s)_k} = 0 \quad (5.23)$$

and, thus, determining the gauge theory of $[k(k+1) - 1]$ -stage of reducibility. Here, the standard Grassman-even scalar product: $\langle \chi_B | \phi_B \rangle$, for Lorentz- scalar vectors is used, instead of Grassman-odd scalar product and Lorentz-spinor vectors for the fermionic HS fields. The operator K^B is reduced from K and explicitly given in [67] (see for $K^{B'}$ denoted in [67] as K' eq. (3.14), as well as for the BRST operator Q^B eq. (5.2)). The form of the field ($l = 0$) and gauge parameters ($l = 1, \dots, k(k+1)$) $|\chi_B^l\rangle_{(s)_k}$, $gh(|\chi_B^l\rangle) = -l$ is determined accordingly to (3.40) for $n_c^0 = n_{b0} = n_{ae} = n_{bg} = 0$ with component vector $|\Phi(a_i^+)_{(n^0)_c; (n)_{ij}(p)_{rs}}^{0b0f0; (0)_{ae}(0)_{bg}(n)_{fi}(n)_{pj}(n)_{flm}(n)_{pno}(n)_{frs}(n)_{\lambda tu}}\rangle$ (instead of $|\Psi(a_i^+) \dots\rangle$) being by the proper eigen-states for generalized spin operator $\sigma_B^i(G)$, $\sigma_B^i(G) = \sigma^i(G)|_{(q_i^{(+)}=p_i^{(+)}=f_i^{(+)}=0)}$ (3.32), (3.37) for integer spin with value for the h_B^i in

$$\sigma_B^i(G) |\chi_B\rangle_{(s)_k} = (\sigma_B^i + h_B^i) |\chi_B\rangle_{(s)_k} = 0 \implies -h_B^i = s^i + \frac{d-2-4i}{2}, \quad i = 1, \dots, k, \quad (5.24)$$

so that the vector $|\chi_B^0\rangle_{(n)_k}$ contains the physical field $|\Phi\rangle$. In the corresponding spin (3.47) and ghost number (3.48) distributions for $|\chi_B^l\rangle_{(s)_k}$ one should put $n_c^0 = n_{b0} = n_{ae} = n_{bg} = 0$ and $n_i = s_i$.

Note, the Statements 1, 2, 3 are valid for the integer HS field case. The constrained BRST Lagrangian formulation for integer HS field in Minkowski space with generalized spin $(s)_k$ may be derived equivalently both in the self-consistent way and from the unconstrained BRST Lagrangian formulation above as it was done respectively in the subsection 4.2 and subsection 4.1. The final result for constrained BRST Lagrangian formulation for integer HS field under consideration can be determined by the

Statement 5. The constrained gauge-invariant Lagrangian formulation for integer HS field, $\Phi_{(\mu^1)_{s_1}, \dots, (\mu^k)_{s_k}}(x)$ with generalized spin $(s)_k$, is determined by the action and sequence of reducible gauge transformation,

$$\mathcal{S}_{C|(s)_k}^B = \int d\eta_{0(s)_k} \langle \chi_{C|B}^0 | Q_C^B | \chi_{C|B}^0 \rangle_{(s)_k}, \quad \delta |\chi_{C|B}^l\rangle_{(s)_k} = Q_C^B |\chi_{C|B}^{l+1}\rangle_{(s)_k}, \quad \delta |\chi_{C|B}^k\rangle_{(s)_k} = 0, \quad (5.25)$$

$$Q_C^B = Q_C|_{q_0=0} = \eta_0 l_0 + \eta_i^+ l^i + l^{i+} \eta_i + \imath \sum_l \eta_l^+ \eta^l P_0, \quad (5.26)$$

for $l = 0, 1, \dots, k-1$, $|\chi_{C|B}^l\rangle \in \mathcal{H}_{C|tot}^{B|l}$, with $\mathcal{H}_{C|tot}^B = \mathcal{H} \otimes H_{gh}^{oB} = \oplus_l \mathcal{H}_{C|tot}^{B|l}$, $gh(|\chi_{C|B}^l\rangle) = -l$, describing the gauge theory of $(k-1)$ -stage of reducibility with off-shell BRST-extended constraints imposed on the whole set of field and gauge parameters:

$$(\widehat{L}_{ij}, \widehat{T}_{rs}) |\chi_{C|B}^l\rangle_{(s)_k} = 0, \quad l = 0, 1, \dots, k; \quad (\widehat{L}_{ij}, \widehat{T}_{rs}) = \left(l_{ij} + \frac{1}{2} \eta_{\{i} \mathcal{P}_{j\}}, t_{rs} - \eta_r^+ \mathcal{P}_s - \mathcal{P}_r^+ \eta_s \right). \quad (5.27)$$

The set of $|\chi_{c|B}^l\rangle_{(s)_k}$ is determined by the decomposition (4.29) for $n_{b0} \equiv 0$ and appears by the set of proper eigen-states for the constrained spin operator $\sigma_{c|B}^i \equiv \sigma_c^i$ with proper eigen-values:

$$\sigma_C^i |\chi_{c|B}^l\rangle_{(s)_k} = \left(s^i + \frac{d-2}{2}\right) |\chi_{c|B}^l\rangle_{(s)_k}, \quad (5.28)$$

which determine the same spin and ghost number grading (4.28), (4.30) [but for $n_{b0} \equiv 0$] as for the constrained half-integer Lagrangian formulation.

The constrained and unconstrained Lagrangian formulations for integer HS field in Minkowski space-time with generalized spin $(s)_k$ are equivalent, but former one contains less auxiliary HS fields as compared to latter formulation. Concluding, the subsection, note the same comments in the end of the previous subsection, concerning bosonic HS fields with *continuous spin*, generalized triplet formulation and generalized quartet formulation are valid as well.

Let us now shortly consider the Lagrangian formulations for the massive HS fields in Minkowski space-time subject to Young diagram with k rows.

5.3 On constrained Lagrangian formulations for massive fields

The unconstrained BRST Lagrangian formulations for massive half-integer and massive integer HS fields in d -dimensional Minkowski space-time with generalized respective spins, $(n)_k + (\frac{1}{2})_k$, $(s)_k$ were elaborated in [67, 68] (for $k = 1$ and example, see [94, 95]). It was done on a base of the derivation of the HS symmetry superalgebra $\mathcal{A}_m^f(Y(k), \mathbb{R}^{1,d-1})$ for massive half-integer HS fields in $\mathbb{R}^{1,d-1}$ and HS symmetry algebra $\mathcal{A}_m(Y(k), \mathbb{R}^{1,d-1})$ for massive integer HS fields from respective HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d})$ and algebra $\mathcal{A}(Y(k), \mathbb{R}^{1,d})$ for massless HS fields in $\mathbb{R}^{1,d}$ with help of the dimensional reduction (see subsections 3.3 in [67, 68]).

In the case of a massive half-integer HS field $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}(x)$ of spin $(s)_k = (n)_k + (\frac{1}{2})_k$, the Dirac equation

$$(\imath\gamma^\mu\partial_\mu - m)\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0 \iff (\imath\tilde{\gamma}^\mu\partial_\mu - \tilde{\gamma}m)\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0 \quad (5.29)$$

contains a massive term in both even and odd space-time dimensions, but equivalent description in terms of Clifford algebra elements $\tilde{\gamma}^\mu$, $\tilde{\gamma}$ is possible only for $d = 2N$ explicitly given by (3.10), (3.11), with unaltered gamma-traceless and mixed-symmetry equations (3.3), (3.4). The unconstrained Lagrangian formulation in this case is determined by the same relations as those in the massless case with some modifications, first of all, for the initial operators t_0, l_0 ,

$$(t_0, l_0) \rightarrow (\check{t}_0, \check{l}_0) = (t_0 + \tilde{\gamma}m, l_0 + m^2), \quad (5.30)$$

which, along with the remaining unaltered elements from $\mathcal{O}_I$, obey the same HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$, except for the additional commutators

$$[t_i^+, l_j] = \delta_{ij}(\check{t}_0 - \tilde{\gamma}m), \quad [t_i, l_j^+] = -\delta_{ij}(\check{t}_0 - \tilde{\gamma}m), \quad [l_i, l_j^+] = \delta_{ij}(\check{l}_0 - m^2). \quad (5.31)$$

Secondly, the additional parts $o'_I(B, B^+)$ coincide with those of the massless case, whereas the converted set of constraints has the form $O_I^m = \check{o}_I + o'_I$, no longer with a central charge $\tilde{\gamma}m$, for massive HS fields with $\check{o}_I = o_I + \hat{o}_I(b_i, b_i^+)$ (for additional bosonic $2k$ -oscillators $[b_i, b_j^+] = \delta_{ij}$ acting in the Fock space H_m), determined by adding the terms induced by dimensional reduction:

$$\left(\hat{t}_0, \hat{l}_0, \hat{l}_i, \hat{l}_i^+, \hat{g}_0^i\right) = \left(0, 0, mb_i, mb_i^+, b_i^+b_i + \frac{1}{2}\right), \quad (5.32)$$

$$\left(\hat{t}_i, \hat{t}_i^+, \hat{l}_{ij}, \hat{l}_{ij}^+, \hat{t}_{ij}, \hat{t}_{ij}^+\right) = -\left(\tilde{\gamma}b_i, \tilde{\gamma}b_i^+, \frac{1}{2}b_ib_j, \frac{1}{2}b_i^+b_j^+, b_i^+b_j\theta^{ji}, b_ib_j^+\theta^{ji}\right). \quad (5.33)$$

The set of O_I^m satisfies the same HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ as for massless half-integer Poincare group irreducible representations, but for even-valued d . Third, the generalized spin, $\sigma^i(G_m)$, BRST, $Q(O^m)$, operators as well as the arbitrary vector $|\chi_m\rangle \in \mathcal{H}_{c|tot}^m$, $\mathcal{H}_{c|tot}^m = \mathcal{H} \otimes H'_m \otimes H_{gh}$ for $H'_m = H' \otimes H_m$ coincide by the form respectively with ones for massless case (3.32), (3.33) with change $(g, o) \rightarrow (G_m, O^m)$, whereas $|\chi_m\rangle$ has the vector $|\chi\rangle$ (3.40) as the massless limit for $b_i^+ = 0$:

$$|\chi_m\rangle = \sum_{n'_l \geq (0)_l} \prod_{l=1}^k (b_l^+)^{n'_l} |\chi_{(n')_l}(a^+, B^+, q^+, p^+, \eta^+, \mathcal{P}^+, \vartheta^+, \lambda^+)\rangle \text{ for } |\chi_{(n')_l}\rangle \in \mathcal{H}_{tot}. \quad (5.34)$$

Note, the b_l^+ -independent vectors $|\chi_{(n')_l}\rangle$ have the decomposition in powers of oscillators presented by (3.40).

The unconstrained Lagrangian formulation for massive half-integer HS field of spin $(n)_k + (\frac{1}{2})_k$ is determined for $d = 2N$, $N \in \mathbb{N}$ almost the same relations as for massless case (3.57), (3.58) with use of (3.51)–(3.53) for $Q(O^m)$:

$$\mathcal{S}_{(n)_k}^m = \left((n)_k \langle \tilde{\chi}_{m|0}^0 | (n)_k \langle \tilde{\chi}_{m|0}^1 | \right) K_{m|(n)_k} \begin{pmatrix} \bar{t}_0 & \Delta Q \\ \Delta Q & \frac{1}{2} \{ \bar{t}_0, \eta_i^+ \eta_i \} \end{pmatrix} \begin{pmatrix} |\chi_{m|0}^0\rangle_{(n)_k} \\ |\chi_{m|0}^1\rangle_{(n)_k} \end{pmatrix}, \quad (5.35)$$

$$\delta \begin{pmatrix} |\chi_{m|0}^{s(0)}\rangle_{(n)_k} \\ |\chi_{m|0}^{s(1)}\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} \Delta Q & \frac{1}{2} \{ \bar{t}_0, \eta_i^+ \eta_i \} \\ \bar{t}_0 & \Delta Q \end{pmatrix} \begin{pmatrix} |\chi_{m|0}^{s+1(0)}\rangle_{(n)_k} \\ |\chi_{m|0}^{s+1(1)}\rangle_{(n)_k} \end{pmatrix}, \quad \delta \begin{pmatrix} |\chi_{m|0}^{s_{\max}(0)}\rangle_{(n)_k} \\ |\chi_{m|0}^{s_{\max}(1)}\rangle_{(n)_k} \end{pmatrix} = 0, \quad (5.36)$$

where the operator $K_{m|(n)_k}$, which realizes the hermitian conjugation in $\mathcal{H}_{c|tot}^m$: $K_m = 1 \otimes K' \otimes 1_m \otimes 1_{gh}$ [with 1_m being by unit operator in H_m] as well as $\Delta Q = \Delta Q(O^m)$ are obtained after substitution for of the proper eigen-values for real constants h_m^i from the spectrum problem for generalized spin equations, for $e = 0, 1$; $s = 0, \dots, s_{\max}$:

$$\sigma^i(G_m) |\chi_{m|0}^{s(e)}\rangle_{(n)_k} = (\sigma_m^i + h_m^i) |\chi_{m|0}^{s(e)}\rangle_{(n)_k} = 0 \implies h_m^i(n) = - \left(n^i + \frac{d+1-4i}{2} \right), \quad (5.37)$$

as follows, $K_{m|(n)_k} = K_m|_{h_m^i \rightarrow h_m^i(n)}$. From the spin and ghost number distributions (3.47), (3.48) for massless HS field the only first one is modified as:

$$\begin{aligned} n_i &= p_i + n'_i + n_{ai} + n_i^0 + \sum_{j=1}^k (1 + \delta_{ij})(n_{ij} + n_{fij} + n_{pij}) + n_{fi} + n_{pi} \\ &+ \sum_{r < i} (p_{ri} + n_{fri} + n_{\lambda ri}) - \sum_{r > i} (p_{ir} + n_{fir} + n_{\lambda ir}), \quad i = 1, \dots, k. \end{aligned} \quad (5.38)$$

In turn, omitting the details of the derivation of the constrained BRST Lagrangian formulation for massive half-integer HS field, $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}$, of spin $(n)_k + (\frac{1}{2})_k$, the final expressions for the gauge-invariant action, sequence of reducible (of the same stage reducibility as for massless case) gauge transformations and independent off-shell BRST-extended constraints are given by the expressions almost coinciding with (5.8)–(5.10), but in terms of the constrained operators and vectors on $\mathcal{H}_C^m$, $\mathcal{H}_C^m = \mathcal{H}_C \otimes H_m$:

$$\mathcal{S}_{C|(n)_k}^m = \left((n)_k \langle \tilde{\chi}_{m|0|c}^0 | (n)_k \langle \tilde{\chi}_{m|0|c}^1 | \right) \begin{pmatrix} \check{t}_0 & \Delta Q_C \\ \Delta Q_C & \check{t}_0 \eta_i^+ \eta_i \end{pmatrix} \begin{pmatrix} |\chi_{m|0|c}^0\rangle_{(n)_k} \\ |\chi_{m|0|c}^1\rangle_{(n)_k} \end{pmatrix}, \quad (5.39)$$

for $\Delta Q_C = \eta_i^+ \check{l}_i + \check{l}_i^+ \eta_i$,

$$\delta \begin{pmatrix} |\chi_{m|0|c}^{l-1(0)}\rangle_{(n)_k} \\ |\chi_{m|0|c}^{l-1(1)}\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} \Delta Q_C & \check{t}_0 \eta_i^+ \eta_i \\ \check{t}_0 & \Delta Q_C \end{pmatrix} \begin{pmatrix} |\chi_{m|0|c}^{l(0)}\rangle_{(n)_k} \\ |\chi_{m|0|c}^{l(1)}\rangle_{(n)_k} \end{pmatrix}, \quad \delta \begin{pmatrix} |\chi_{m|0|c}^{k(0)}\rangle_{(n)_k} \\ |\chi_{m|0|c}^{k(1)}\rangle_{(n)_k} \end{pmatrix} = 0, \quad (5.40)$$

$$\left(\hat{T}_i^m, \hat{T}_{rs}^m \right) \sum_{e=0}^1 q_0^e |\chi_{m|0|c}^{l(e)}\rangle_{(n)_k} = 0, \quad \text{for } \left(\hat{T}_i^m, \hat{T}_{rs}^m \right) = \left(\hat{T}_i + \hat{t}_i, \hat{T}_{rs} + \hat{t}_{rs}^+ \right), \quad l = 0, 1, \dots, k. \quad (5.41)$$

Here, the constrained field and gauge parameters are by the proper eigen-functions for the constrained massive spin operator, $\sigma_{m|C}^i$ determined as in (4.1):

$$\sigma_{m|C}^i |\chi_{m|0|c}^{l(e)}\rangle_{(n)_k} = \left(n^i + \frac{d-1}{2} \right) |\chi_{m|0|c}^{l(e)}\rangle_{(n)_k}, \quad \text{for } d = 2N, N \in \mathbb{N}, \quad (5.42)$$

which together with constrained BRST operator, Q_C^m : $Q_C^m = Q_C(\check{o}_A)$ (4.4), constraints $\hat{T}_i^m, \hat{T}_{rs}^m$ and algebraically dependent constraints $\hat{L}_{ij}^m$: $\hat{L}_{ij}^m = \hat{L}_{ij} + \hat{l}_{ij}$, forms the same closed superalgebra with respect to $[\ , \]$ -multiplication as their analogs from massless case, namely, nilpotent Q_C^m supercommutes with any from $\{ \hat{T}_i^m, \hat{T}_{rs}^m, \hat{L}_{ij}^m, \sigma_{m|C}^i \}$, which satisfy to the relations in the table 1 for $t_i, l_{ij}, t_{rs}, g_0^i$.

Note, it is easy to establish again that both approaches: first, reduced from unconstrained BRST Lagrangian formulation and, second, obtained in self-consistent way; to derive the constrained BRST Lagrangian formulation for massive HS fields developed in the subsections 4.1, 4.2 are equivalent. The equivalence among the unconstrained and constrained BRST Lagrangian formulations for the same massive half-integer HS field of spin $(n)_k + (\frac{1}{2})_k$ follows, in fact, from the validity of the Statement 4 and Corollary 3 because of the constraints subsystem, $\check{o}_A = \{ \check{t}_0, \check{l}_0, \check{l}_i, \check{l}_i^+ \}$ appears by the first-class ones in $\mathcal{H}_m \equiv \mathcal{H} \otimes H_m$ (not in $\mathcal{H}!$).

The problem of derivation of the (un)constrained Lagrangian formulation for massive half-integer HS fields in odd-valued dimensions, $d = 2N + 1$, may be solved within BRST-BFV approach by means of calculation of whole pairings for ghost $\mathcal{C}, \overline{\mathcal{P}}$ and auxiliary B, B^+ oscillators in (5.35) and (5.39) with use of the partial gauge-fixing procedure up to the representation of the respective Lagrangian formulations where the only initial γ^μ -matrices will survive, without $\tilde{\gamma}$ object due to the property described in the footnote 2. The last Lagrangian formulations obtained for $d = 2N$ can be extrapolated to be by

Lagrangian formulations for odd-valued dimensions, $d = 2N + 1$ if there will not appear another restrictions.

To be complete, we remind the form of unconstrained BRST Lagrangian formulation for massive integer HS field of spin $(s)_k$ in $\mathbb{R}^{1,d-1}$ [67] and present new results for respective constrained BRST Lagrangian formulation. For the former case we ignore, first, spinor-matrix-like $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$ structure for the operators and spinor structure for the states, and extracting from the massive HS symmetry superalgebra $\mathcal{A}_m^f(Y(k), \mathbb{R}^{1,d-1})$ the $(2k + 1)$ Grassman-odd elements: $\check{t}_0, \check{t}_i, \check{t}_i^+$, to get massive HS symmetry algebra $\mathcal{A}_m(Y(k), \mathbb{R}^{1,d-1})$ because of having instead of the wave equation in (5.21) the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2) \Phi_{(\mu^1)_{s_1}, (\mu^2)_{s_2}, \dots, (\mu^k)_{s_k}} = 0. \quad (5.43)$$

Second, we adapt the results of the previous subsection for the massive integer HS field for any dimension d , and representation for the Grassmann-even constraints, $O_I^{B|m}$, to present the unconstrained Lagrangian formulation on a base of one for massless case (5.22), (5.23) with BRST operator $Q_m^B = Q^B(O_m^B)$ for:

$$\mathcal{S}_{m|(s)_k}^B = \int d\eta_{0(s)_k} \langle \chi_{m|B}^0 | K_{m|(s)_k}^B Q_{m|(s)_k}^B | \chi_{m|B}^0 \rangle_{(s)_k}, \quad K_m^B = 1 \otimes K_m^{B'} \otimes 1_{gh}: \quad (5.44)$$

$$K_{m|(s)_k}^B Q_{m|(s)_k}^B = (Q^B)_{m|(s)_k}^+ K_{(s)_k}^B, \quad \delta | \chi_{m|B}^l \rangle_{(s)_k} = Q_{m|(s)_k}^B | \chi_{m|B}^{l+1} \rangle_{(s)_k}, l = 0, 1, \dots, k(k+1) - 1 \dots, \delta | \chi_{m|B}^{k(k+1)} \rangle_{(s)_k} = 0 \quad (5.45)$$

and, thus, determining the gauge theory of $[k(k+1) - 1]$ -stage of reducibility. The form of the field ($l = 0$) and gauge parameters ($l = 1, \dots, k(k+1)$) $| \chi_{m|B}^l \rangle_{(s)_k}$, with the same Grassman and ghost number grading as ones for massless case is determined accordingly to (5.34) for $n_c^0 = n_{b0} = n_{ae} = n_{bg} = 0$ with component vector $|\Phi(a_i^+)^{0_{b0}0_{f0};(0)_{ae}(0)_{bg}(n)_{fi}(n)_{pj}(n)_{flm}(n)_{pno}(n)_{frs}(n)_{\lambda tu}}_{(n')_i(n^0)_c;(n)_{ij}(p)_{rs}} \rangle$ being by the proper eigen-states for massive generalized spin operator $\sigma_{m|B}^i(G)$, $\sigma_{m|B}^i(G) = \sigma_B^i(G_m)$ for integer spin with value for the $h_{m|B}^i$ in

$$\sigma_B^i(G_m) | \chi_{m|B} \rangle_{(s)_k} = (\sigma_{m|B}^i + h_{m|B}^i) | \chi_{m|B} \rangle_{(s)_k} = 0 \implies -h_B^i = s^i + \frac{d-1-4i}{2}, i = 1, \dots, k. \quad (5.46)$$

According to the Statement 5, the constrained gauge-invariant Lagrangian formulation for massive integer HS field, $\Phi_{(\mu^1)_{s_1}, \dots, (\mu^k)_{s_k}}(x)$ with generalized spin $(s)_k$, is determined by the action and sequence of reducible gauge transformation,

$$\mathcal{S}_{m|C|(s)_k}^B = \int d\eta_{0(s)_k} \langle \chi_{m|C|B}^0 | Q_{m|C}^B | \chi_{m|C|B}^0 \rangle_{(s)_k}, \quad (5.47)$$

$$\delta | \chi_{m|C|B}^l \rangle_{(s)_k} = Q_{m|C}^B | \chi_{m|C|B}^{l+1} \rangle_{(s)_k}, l = 0, 1, \dots, k-1 \dots, \delta | \chi_{m|C|B}^k \rangle_{(s)_k} = 0 \quad (5.48)$$

describing the gauge theory of $(k-1)$ -stage of reducibility with off-shell BRST-extended constraints, $\hat{L}_{ij}^m, \hat{T}_{rs}^m$ imposed on the whole set of massive field and gauge parameters:

$$(\hat{L}_{ij}^m, \hat{T}_{rs}^m) | \chi_{m|C|B}^l \rangle_{(s)_k} = 0, \quad l = 0, 1, \dots, k; \quad (\hat{L}_{ij}^m, \hat{T}_{rs}^m) = (\hat{L}_{ij} + \hat{l}_{ij}, \hat{T}_{rs} + \hat{t}_{rs}). \quad (5.49)$$

The set of $|\chi_{m|c|B}^l\rangle_{(s)_k}$ is determined by (4.29) and (5.34) for $n_{b0} \equiv 0$ and appears by the set of proper eigen-states for the massive constrained spin operator $\sigma_{m|C|B}^i \equiv \sigma_{m|C}^i$ with proper eigen-values:

$$\sigma_{m|C}^i |\chi_{m|c|B}^l\rangle_{(s)_k} = \left(s^i + \frac{d-1}{2}\right) |\chi_{m|c|B}^l\rangle_{(s)_k}. \quad (5.50)$$

Again, as well as for the massive half-integer HS fields from the dimensional reduction procedure it follows that both approaches: first, reduced from unconstrained BRST Lagrangian formulation and, second, obtained in self-consistent way; to get the constrained BRST Lagrangian formulation for massive HS fields developed in the subsections 4.1, 4.2 are equivalent. The equivalence among the unconstrained and constrained BRST Lagrangian formulations for the same massive integer HS field of spin $(s)_k$ follows, in fact, from the validity of the Statement 4 and Corollary 3 adapted to integer spin case because of the constraints subsystem, $\delta_A^m = \{\tilde{l}_0, \tilde{l}_i, \tilde{l}_i^+\}$ appears by the first-class ones in $\mathcal{H}_m$.

6 Example: spin $(n + \frac{1}{2})$ totally-symmetric field

Here, we, firstly, realize the general prescriptions of our constrained BRST-BFV Lagrangian formulations in the known case of totally-symmetric fermionic fields in the metric-like formulation.

6.1 Spin $(n + \frac{1}{2})$ field in BRST constrained approach

The equations expressing the belonging of the spin-tensor, $\Psi_{\mu_1 \dots \mu_n}(x)$ to the Poincare group irreducible representation space of spin, $n + \frac{1}{2}$ contains only two equations from (3.2), (3.3)

$$\iota \gamma^\mu \partial_\mu \Psi_{(\mu)_n}(x) = 0, \quad \gamma^{\mu i} \Psi_{\mu_1 \dots \mu_n}(x) = 0 \quad (6.1)$$

which are described by two Grassmann-odd operators $t_0 = -\iota \tilde{\gamma}^\mu \partial_\mu$, $t_1 = \tilde{\gamma}^\mu a_\mu$ ($a_\mu \equiv a_\mu^1$) acting on the basic vector

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \Psi_{(\mu)_n} a^{+\mu_1} \dots a^{+\mu_n} |0\rangle. \quad (6.2)$$

The closed algebra of all constraints: $\{t_0, l_0, t_1, t_1^+, l_1, l_1^+, l_{11}, l_{11}^+; g_0 = -\frac{1}{2}\{a^\mu, a_\mu^+\}\}$ determined by (3.14), (3.17) forms the half-integer HS symmetry algebra in Minkowski space for totally-symmetric fields, $\mathcal{A}^f(Y(1), \mathbb{R}^{1,d-1})$, with the differential first-class $\{t_0, l_0, l_1, l_1^+\}$, holonomic second-class $\{t_1, t_1^+, l_{11}, l_{11}^+\}$ constraints and number particle operator g_0 . Following to the results of the section 4 the nilpotent constrained BRST operator for the first-class system, off-shell BRST extended constraints $\{\hat{T}_1, \hat{L}_{11}\}$ and constrained spin operator $\hat{\sigma}_C(g)$ in the Fock space $\mathcal{H}_C$ have the respective form:

$$Q_C = q_0 t_0 + \eta_0 l_0 + \eta_1^+ l_1 + l_1^+ \eta_1 + \iota(\eta_1^+ \eta_1 - q_0^2) \mathcal{P}_0, \quad (6.3)$$

$$\{\hat{T}_1, \hat{L}_{11}\} = \{t_1 - \eta_1 p_0 - 2q_0 \mathcal{P}_1, l_{11} + \eta_1 \mathcal{P}_1\}, \quad (6.4)$$

$$\hat{\sigma}_C(g) = g_0 + \eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ \quad (6.5)$$

whose closed algebra satisfy to the relations (4.15) for $K_a^i \equiv (K_1^1, K_{11}^1) = (1, 2)$ and to (4.24).

According to the Statement 4 we have the first-order constrained irreducible gauge-invariant Lagrangian formulation for half-integer HS field, $\Psi_{(\mu)_n}(x)$, described by the action

$$\mathcal{S}_{C|(n)} = \left({}_n\langle \tilde{\chi}_{0|c}^0 | {}_n\langle \tilde{\chi}_{0|c}^1 | \right) \begin{pmatrix} t_0 & \eta_1^+ l_1 + l_1^+ \eta_1 \\ \eta_1^+ l_1 + l_1^+ \eta_1 & t_0 \eta_1^+ \eta_1 \end{pmatrix} \begin{pmatrix} |\chi_{0|c}^0\rangle_n \\ |\chi_{0|c}^1\rangle_n \end{pmatrix}, \quad (6.6)$$

to be invariant with respect to the gauge transformations (for $n \geq 1$)

$$\delta \begin{pmatrix} |\chi_{0|c}^0\rangle_n \\ |\chi_{0|c}^1\rangle_n \end{pmatrix} = \begin{pmatrix} \eta_1^+ l_1 + l_1^+ \eta_1 & t_0 \eta_1^+ \eta_1 \\ t_0 & \eta_1^+ l_1 + l_1^+ \eta_1 \end{pmatrix} \begin{pmatrix} |\chi_{0|c}^{1(0)}\rangle_n \\ 0 \end{pmatrix}, \quad \delta |\chi_{0|c}^{1(0)}\rangle_n = 0 \quad (6.7)$$

for $|\chi_{0|c}^{1(1)}\rangle_n \equiv 0$ (because of $gh(|\chi_{0|c}^{1(m)}\rangle_n) = -m - 1$ and $(\mathcal{P}_i^+)^2 = 0$), with off-shell algebraically independent BRST-extended constraint imposed on the fields, $|\chi_{0|c}^m\rangle_n$, $m = 0, 1$ and gauge parameter $|\chi_{0|c}^{1(0)}\rangle_n$:

$$\hat{T}_1 \left(|\chi_{0|c}^0\rangle_n + q_0 |\chi_{0|c}^1\rangle_n \right) = 0, \quad \hat{T}_1 |\chi_{0|c}^{1(0)}\rangle_n = 0. \quad (6.8)$$

The field vectors and gauge parameter being proper for the spin operator $\hat{\sigma}_C(g)$ have the decomposition in ghosts $\eta_1^+, \mathcal{P}_1^+$:

$$|\chi_{0|c}^0\rangle_n = |\Psi\rangle_n + \eta_1^+ \mathcal{P}_1^+ |\chi\rangle_{n-2} = |\Psi\rangle_n + \frac{i^{n-2}}{(n-2)!} \eta_1^+ \mathcal{P}_1^+ \chi^{(\mu)_{n-2}} \prod_{k=1}^{n-2} a_{\mu_k}^+ |0\rangle, \quad (6.9)$$

$$|\chi_{0|c}^1\rangle_n = \mathcal{P}_1^+ \tilde{\gamma} |\chi_1\rangle_{n-1} = \frac{i^{n-1}}{(n-1)!} \mathcal{P}_1^+ \tilde{\gamma} \chi_1^{(\mu)_{n-1}} \prod_{k=1}^{n-1} a_{\mu_k}^+ |0\rangle, \quad (6.10)$$

$$|\chi_{0|c}^{1(0)}\rangle_n = \mathcal{P}_1^+ |\xi\rangle_{n-1} = \frac{i^{n-1}}{(n-1)!} \mathcal{P}_1^+ \xi^{(\mu)_{n-1}} \prod_{k=1}^{n-1} a_{\mu_k}^+ |0\rangle, \quad (6.11)$$

with $|\Psi\rangle$ given by (6.2).

The constraints (6.8) are easily resolved as the gamma-traceless constraint for the gauge parameter spin-tensor $\xi^{(\mu)_{n-1}}$ and triple gamma-traceless constraint for the field spin-tensor $\Psi^{(\mu)}$ and with expressing $\chi_1^{(\mu)_{n-1}}$, $\chi^{(\mu)_{n-2}}$ in terms of $\Psi^{(\mu)}$

$$\hat{T}_1 |\chi_{0|c}^{1(0)}\rangle_n = t_1 |\chi_{0|c}^{1(0)}\rangle_n \iff \gamma^\mu \xi_{(\mu)_{n-1}} = 0, \quad (6.12)$$

$$\hat{T}_1 \left(|\chi_{0|c}^0\rangle_n + q_0 |\chi_{0|c}^1\rangle_n \right) = 0 \iff \left\{ t_1 |\Psi\rangle_n = \tilde{\gamma} |\chi_1\rangle_{n-1}, |\chi\rangle_{n-2} = -\frac{1}{2} t_1 \tilde{\gamma} |\chi_1\rangle_{n-1}, t_1 |\chi\rangle_{n-2} = 0 \right\}. \quad (6.13)$$

Indeed, from (6.13) it follows that

$$(t_1)^3 |\Psi\rangle_n = 0, \quad \tilde{\gamma} |\chi_1\rangle_{n-1} = t_1 |\Psi\rangle_n, \quad |\chi\rangle_{n-2} = -\frac{1}{2} (t_1)^2 |\Psi\rangle_n \quad (6.14)$$

and therefore, $\prod_{i=1}^3 \gamma^{\mu_i} \Psi_{(\mu)_n} = 0$. The resolution of the constraint (6.14) for the fields $|\chi_{0|c}^m\rangle_n$ means the validity of the representation for them denoted as $|\chi_r^m\rangle_n$, $m = 0, 1$:

$$|\chi_r^0\rangle_n = \left(1 - \frac{1}{2} \eta_1^+ \mathcal{P}_1^+ (t_1)^2\right) |\Psi\rangle_n = \frac{i^n}{n!} (a_{\mu_1}^+ a_{\mu_2}^+ + n(n-1) \eta_1^+ \mathcal{P}_1^+ \eta_{\mu_1 \mu_2}) \Psi^{(\mu)_n} \prod_{k=3}^n a_{\mu_k}^+ |0\rangle, \quad (6.15)$$

$$|\chi_r^1\rangle_n = \mathcal{P}_1^+ t_1 |\Psi\rangle_n = -\frac{i^{n-1}}{(n-1)!} \mathcal{P}_1^+ \tilde{\gamma} \gamma_{\mu_1} \Psi^{(\mu)_n} \prod_{k=2}^n a_{\mu_k}^+ |0\rangle. \quad (6.16)$$

Dual (bra-vectors) vectors ${}_n \langle \tilde{\chi}_{0|c}^m |$, $m = 0, 1$ and ${}_n \langle \tilde{\chi}_{0|c}^{1(0)} |$ have the form

$${}_n \langle \tilde{\chi}_r^0 | = {}_n \langle \tilde{\Psi} | \left(1 - \frac{1}{2} (t_1^+)^2 \mathcal{P}_1 \eta_1\right) = -\frac{(-i)^n}{n!} \langle 0 | \prod_{k=2}^n a_{\nu_k} \bar{\Psi}^{(\nu)_n} (a_{\nu_1} a_{\nu_2} + n(n-1) \mathcal{P}_1 \eta_1 \eta_{\nu_1 \nu_2}) \tilde{\gamma} (-1)^d, \quad (6.17)$$

$${}_n \langle \tilde{\chi}_r^1 | = {}_n \langle \tilde{\Psi} | t_1^+ \mathcal{P}_1 = -\frac{(-i)^{n-1}}{(n-1)!} \langle 0 | \prod_{k=2}^n a_{\nu_k} \Psi_{(\nu)_n}^+ \gamma_0 \gamma_{\nu_1} \tilde{\gamma}_0 \mathcal{P}_1 \tilde{\gamma}_0 = \frac{(-i)^{n-1}}{(n-1)!} \langle 0 | \prod_{k=2}^n a_{\nu_k} \bar{\Psi}^{(\nu)_n} \gamma_{\nu_1} \mathcal{P}_1 (-1)^d, \quad (6.18)$$

$${}_n \langle \tilde{\chi}_r^{1(0)} | = {}_n \langle \tilde{\xi} | \mathcal{P}_1 = \frac{(-i)^{n-1}}{(n-1)!} \langle 0 | \prod_{k=1}^{n-1} a_{\nu_k} \xi^{+(\nu)_{n-1}} \mathcal{P}_1 \tilde{\gamma}_0 = -\frac{(-i)^{n-1}}{(n-1)!} \langle 0 | \prod_{k=1}^{n-1} a_{\nu_k} \bar{\xi}^{(\nu)_{n-1}} \mathcal{P}_1 \tilde{\gamma} (-1)^d, \quad (6.19)$$

where use has been made of the definition [20–22, 68] for the Hermitian conjugation of the $\tilde{\gamma}_\nu$ -matrix, adapted to odd and even dimensions d , $(\tilde{\gamma}_\nu)^+ = (-1)^d \tilde{\gamma}_0 \tilde{\gamma}_\nu \tilde{\gamma}_0$; we have also used the properties (3.10), (3.12), the fact that $\{\tilde{\gamma}_0, \mathcal{P}_1\} = \{\tilde{\gamma}, \mathcal{P}_1\} = 0$ and $(\tilde{\gamma}_0)^2 = (-1)^d$, as well as the definition of a Dirac-conjugated spin-tensor, $\bar{\Psi}^{(\nu)_n} = \Psi^{+(\nu)_n} \gamma_0$.

The action (6.6) in terms of independent field vector $|\Psi\rangle_n$ in the ghost-free form (not depending on value of $d \geq 4$) looks as

$$\mathcal{S}_{C|(n)} = {}_n \langle \tilde{\Psi} | \left(t_0 - \frac{1}{4} (t_1^+)^2 t_0 t_1^2 - t_1^+ t_0 t_1 + l_1^+ t_1 + t_1^+ l_1 + \frac{1}{2} t_1^+ l_1^+ t_1^2 + \frac{1}{2} (t_1^+)^2 l_1 t_1 \right) |\Psi\rangle_n, \quad (6.20)$$

to be invariant with respect to the gauge transformations⁶

$$\delta |\Psi\rangle_n = l_1^+ |\xi\rangle_{n-1}. \quad (6.21)$$

The gauge invariance for the action $\mathcal{S}_{C|(n)}$ is easily checked from the Noether identity:

$$\delta \mathcal{S}_{C|(n)} \frac{\overleftarrow{\delta}}{\delta |\xi\rangle} = {}_n \langle \tilde{\Psi} | \left(t_0 - \frac{1}{4} (t_1^+)^2 t_0 t_1^2 - t_1^+ t_0 t_1 + l_1^+ t_1 + t_1^+ l_1 + \frac{1}{2} t_1^+ l_1^+ t_1^2 + \frac{1}{2} (t_1^+)^2 l_1 t_1 \right) l_1^+ = 0, \quad (6.22)$$

modulo the operators $\mathcal{L}(t_1^+, t_0, l_1, l_1^+) t_1$ vanishing when acting on the gamma-traceless vectors like the gauge parameter $|\xi\rangle_{n-1}$ (6.12). Here, the variational derivative of the functional

⁶In terms of only γ^μ -matrices the action (6.20) takes the same form but with operators $(\tilde{t}_1^+, \tilde{t}_0, \tilde{t}_1)$ (3.8).

$\delta\mathcal{S}_{C|(n)} = \langle \tilde{\Psi} | L(t_0, t_1, \dots) | \xi \rangle + \langle \tilde{\xi} | L^+(t_0, t_1, \dots) | \Psi \rangle$ (with the kernel $L(t_0, t_1, \dots)$ written in (6.22)) with respect to the vector $|\xi\rangle$ was introduced.

Indeed, the sum of the first and the fifth terms vanishes, due to $[t_1, l_1^+] = -t_0$. The sum of the fourth term, transformed into $-\frac{1}{2}t_1^+ l_1^+ t_1 t_0 = -t_1^+ l_1^+ l_1$ (due to the half-integer HS symmetry algebra, $\mathcal{A}^f(Y(1), \mathbb{R}^{1,d-1})$) with the third term gives $t_1^+ l_0$. The same expression, but with opposite sign: $-t_1^+ l_0$ follows from the sixth term. Finally, from the second and the last terms we get respectively, $\frac{1}{4}(t_1^+)^2 t_0 t_1 t_0 = \frac{1}{2}(t_1^+)^2 t_0 l_1$ and $-\frac{1}{2}(t_1^+)^2 l_1 t_0$ which justify the validity of (6.22) with accuracy up to the terms $\mathcal{L}(t_1^+, t_0, l_1, l_1^+) t_1$.

In the spin-tensor form the action and the gauge transformations take the familiar form [42, 43] not depending on the values of the dimension $d \geq 4$ and representation for $\tilde{\gamma}_\mu, \tilde{\gamma}$ (3.11), (3.12), with accuracy up to the common coefficient $(n!)^{-1}$, (which may be absorbed into definition of $|\Psi\rangle_n$ with writing $(n!)^{-(1/2)}$ instead of $(n!)^{-1}$ in (6.2)):

$$\begin{aligned} \mathcal{S}_{C|(n)}(\Psi) = & (-1)^n \int d^d x \bar{\Psi}^{(\nu)_n} \left\{ -i\gamma^\mu \partial_\mu \Psi_{(\nu)_n} + \frac{1}{4} n(n-1) \eta_{\nu_{n-1}\nu_n} (i\gamma^\mu \partial_\mu) \eta^{\mu_{n-1}\mu_n} \Psi_{(\nu)_{n-2}\mu_{n-1}\mu_n} \right. \\ & - n\gamma_{\nu_n} (i\gamma^\mu \partial_\mu) \gamma^{\mu_n} \Psi_{(\nu)_{n-1}\mu_n} + n(i\partial_{\nu_n}) \gamma^{\mu_n} \Psi_{(\nu)_{n-1}\mu_n} + n(i\partial^{\mu_n}) \gamma_{\nu_n} \Psi_{(\nu)_{n-1}\mu_n} \\ & \left. - \frac{1}{2} n(n-1) \left(\gamma_{\nu_{n-1}} (i\partial_{\nu_n}) \eta^{\mu_{n-1}\mu_n} \Psi_{(\nu)_{n-2}\mu_{n-1}\mu_n} + \eta_{\nu_{n-1}\nu_n} \gamma^{\mu_{n-1}} (i\partial^{\mu_n}) \Psi_{(\nu)_{n-2}\mu_{n-1}\mu_n} \right) \right\}, \end{aligned} \quad (6.23)$$

$$\begin{aligned} = & (-1)^n \int d^d x \left\{ \bar{\Psi} \not{p} \Psi - \frac{1}{4} n(n-1) \bar{\Psi}'' \not{p} \Psi'' + n \bar{\Psi}' \not{p} \Psi' - n \bar{\Psi} \cdot p \Psi' - n \bar{\Psi}' \cdot p \Psi \right. \\ & \left. + \frac{1}{2} n(n-1) \left(\bar{\Psi}' \cdot p \Psi'' + \bar{\Psi}'' \cdot p \Psi' \right) \right\}, \end{aligned} \quad (6.24)$$

$$\delta\Psi^{(\mu)_n} = - \sum_{i=1}^n \partial^{\mu_i} \xi^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_n}, \quad (6.25)$$

where each term in (6.23) and (6.24) corresponds to the respective summand in (6.20), whereas for the last expression we have used Fang-Fronsdal notations [42, 43] with identifications, $p_\mu = -i\partial_\mu$, $-i\gamma^\mu \partial_\mu = \not{p}$ and $p \cdot \Psi = p_{\mu_1} \Psi^{(\mu)_n}$.

This result was obtained earlier from the unconstrained BRST approach [20, 21] by means of the tedious gauge-fixing procedure eliminating additional reducible set of gauge parameters and additional to $|\Psi\rangle_n$ field vectors from the unconstrained Lagrangian formulation. We stress that the actions $\mathcal{S}_{C|(n)}$ (6.20) and $\mathcal{S}_{C|(n)}(\Psi)$ (6.24) when identifying, $i\Psi^{(\mu)_n} = h^{(\mu)_n}$, coincide with those given by the respective eqs. (122) and (128) of [20, 21].

In turn, the triplet formulation to describe Lagrangian dynamic of massless totally-symmetric field $\Psi_{(\nu)_n}$ with help of the triplet of spin-tensors $\Psi_{(\nu)_n}, \chi_1^{(\mu)_{n-1}}, \chi^{(\mu)_{n-2}}$ and gauge parameter $\xi^{(\mu)_{n-1}}$ subject to the off-shell 3 constraints on the fields vectors, $|\Psi\rangle_n, |\chi_1\rangle_{n-1}, |\chi\rangle_{n-2}$ (6.14) and 1 gamma-traceless constraint on $|\xi\rangle_{n-1}$ (6.12) in the ghost-independent vector form

$$\begin{aligned} \mathcal{S}_{C|(n)}(\Psi, \chi_1, \chi) = & {}_n \langle \tilde{\Psi} | t_0 | \Psi \rangle_n - {}_{n-2} \langle \tilde{\chi} | t_0 | \chi \rangle_{n-2} + {}_{n-1} \langle \tilde{\chi}_1 | \tilde{\gamma} t_0 \tilde{\gamma} | \chi_1 \rangle_{n-1} \\ & - \left({}_{n-1} \langle \tilde{\chi}_1 | \tilde{\gamma} \{ l_1 | \Psi \rangle_n - l_1^+ | \chi \rangle_n \} + h.c. \right), \end{aligned} \quad (6.26)$$

$$\delta \left(|\Psi\rangle_n, |\chi\rangle_{n-2}, |\chi_1\rangle_{n-1} \right) = \left(l_1^+, l_1, \tilde{\gamma} t_0 \right) |\xi\rangle_{n-1} \quad (6.27)$$

coincides with one suggested in [73]. Again, in terms of only γ^μ -matrices the relations (6.26), (6.27) take the same form with operators $(\tilde{t}_1^+, \tilde{t}_0, \tilde{t}_1)$ without $\tilde{\gamma}$ according to footnote 6. With absence of the off-shell constraints the triplet formulation describes the free propagation of couple of massless particles with respective spins $(n + \frac{1}{2}), (n - \frac{1}{2}), \dots, \frac{1}{2}$.

6.2 Unconstrained quartet Lagrangians for spin $(n + \frac{1}{2})$ field and massive case

It was shown in [71, 72] that this formulation maybe described within *unconstrained quartet formulation* with additional to the triplet, compensator field vector $|\varsigma\rangle_{n-2}$, whose gauge transformation is proportional to the constraint on $|\xi\rangle_{n-1}$: $\delta|\varsigma\rangle_{n-2} = \tilde{\gamma}t_1|\xi\rangle_{n-1}$ and the whole off-shell constraints (6.14) are augmented by the terms proportional to $|\varsigma\rangle$ to provide theirs total gauge invariance with respect to (6.27) and above gauge transformations for $|\varsigma\rangle$ as follows

$$\{t_1|\Psi\rangle - \tilde{\gamma}|\chi_1\rangle + l_1^+\tilde{\gamma}|\varsigma\rangle, |\chi\rangle + \frac{1}{2}t_1\tilde{\gamma}|\chi_1\rangle + \frac{1}{2}t_0\tilde{\gamma}|\varsigma\rangle, t_1|\chi\rangle + l_1\tilde{\gamma}|\varsigma\rangle\} = \{0, 0, 0\}. \quad (6.28)$$

Introducing the respective Lagrangian multipliers: fermionic ${}_{n-1}\langle\tilde{\lambda}_1|$, bosonic ${}_{n-2}\langle\tilde{\lambda}_2|$, fermionic ${}_{n-3}\langle\tilde{\lambda}_3|$ with trivial gauge transformations, the equations (6.28) and theirs hermitian conjugated may be derived from the action functional

$$\begin{aligned} \mathcal{S}_{\text{add}|(n)}(\lambda) = & {}_{n-1}\langle\tilde{\lambda}_1|\left(t_1|\Psi\rangle_n - \tilde{\gamma}|\chi_1\rangle_{n-1} + l_1^+\tilde{\gamma}|\varsigma\rangle_{n-2}\right) + {}_{n-2}\langle\tilde{\lambda}_2|\left(|\chi\rangle_{n-2} \right. \\ & \left. + \frac{1}{2}t_1\tilde{\gamma}|\chi_1\rangle_{n-1} + \frac{1}{2}t_0\tilde{\gamma}|\varsigma\rangle_{n-2}\right) + {}_{n-3}\langle\tilde{\lambda}_3|\left(t_1|\chi\rangle_{n-2} + l_1\tilde{\gamma}|\varsigma\rangle_{n-2}\right) + h.c., \end{aligned} \quad (6.29)$$

so that, the gauge-invariant functional,⁷

$$\mathcal{S}_{(n)} = \mathcal{S}_{C|(n)}(\Psi, \chi_1, \chi) + \mathcal{S}_{\text{add}|(n)}(\lambda) \quad (6.30)$$

determines the unconstrained Lagrangian formulation for massless spin-tensor of spin $(n + \frac{1}{2})$ in terms of quartet of the spin-tensor fields $\Psi_{(\nu)_n}, \chi_1^{(\mu)^{n-1}}, \chi^{(\mu)^{n-2}}, \varsigma^{(\mu)^{n-2}}$ with help of three Lagrangian multipliers $\lambda_i^{(\mu)^{n-i}}, i = 1, 2, 3$ with trivial for the latters, as it was shown in [71, 72], dynamics.

The constrained Lagrangian formulation for the massive totally-symmetric spin-tensor may be obtained according to the recipe (5.39)–(5.41) of subsection 5.3. Note that the constrained Lagrangian formulation for such HS fields in constant-curvature spaces was considered using the metric-like formalism in ref. [96] beyond the BRST-BFV approach.

Once again, the constrained first-order irreducible gauge-invariant Lagrangian formulation for a massive half-integer HS field $\Psi_m^{(\mu)^n}(x)$ satisfying, instead of (6.1), the Dirac equation (5.29) for $k = 1$ and any even $d = 2N \geq 4$,

$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi_m^{(\mu)^n} = 0 \iff (i\tilde{\gamma}^\mu\partial_\mu - \tilde{\gamma}m)\Psi_m^{(\mu)^n} = 0, \quad (6.31)$$

⁷As for the Fang-Fronsdal (see footnote 6) and triplet formulations the representation in terms of γ^μ -matrices turn the Lagrangian formulation (6.29) into the same form, but with operators $(\tilde{t}_1^+, \tilde{t}_0, \tilde{t}_1)$, fermionic $\langle\tilde{\lambda}_2| = \langle\tilde{\lambda}_2|\tilde{\gamma}$, property $(t_1, t_0)\tilde{\gamma} = -(\tilde{t}_1, \tilde{t}_0)$ and $\delta|\varsigma\rangle = \tilde{t}_1|\xi\rangle$ without $\tilde{\gamma}$.

is described by the action

$$\mathcal{S}_{C|n}^m = \left({}_n\langle \tilde{\chi}_{m|0|c}^0 | {}_n\langle \tilde{\chi}_{m|0|c}^1 | \right) \begin{pmatrix} \check{t}_0 & \eta_i^+ \check{l}_i + \check{l}_i^+ \eta_i \\ \eta_i^+ \check{l}_i + \check{l}_i^+ \eta_i & \check{t}_0 \eta_1^+ \eta_1 \end{pmatrix} \begin{pmatrix} |\chi_{m|0|c}^0\rangle_n \\ |\chi_{m|0|c}^1\rangle_n \end{pmatrix}, \quad (6.32)$$

$$\delta \begin{pmatrix} |\chi_{m|0|c}^0\rangle_n \\ |\chi_{m|0|c}^1\rangle_n \end{pmatrix} = \begin{pmatrix} \eta_1^+ \check{l}_1 + \check{l}_1^+ \eta_1 & \check{t}_0 \eta_1^+ \eta_1 \\ \check{t}_0 & \eta_1^+ \check{l}_1 + \check{l}_1^+ \eta_1 \end{pmatrix} \begin{pmatrix} |\chi_{m|0|c}^{1(0)}\rangle_n \\ 0 \end{pmatrix}, \quad \delta |\chi_{m|0|c}^{1(0)}\rangle_n = 0 \quad (6.33)$$

with an off-shell algebraically independent BRST-extended constraint imposed on the fields $|\chi_{m|0|c}^e\rangle_n$, $e = 0, 1$, and gauge parameter $|\chi_{m|0|c}^{1(0)}\rangle_n$:

$$\hat{T}_1^m \left(|\chi_{m|0|c}^0\rangle_n + q_0 |\chi_{m|0|c}^1\rangle_n \right) = 0, \quad \hat{T}_1^m |\chi_{m|0|c}^{1(0)}\rangle_n = 0. \quad (6.34)$$

Here, the constrained field and gauge parameters are by the proper eigen-functions for the constrained massive spin operator, $\sigma_{m|C}$ determined as in (4.1) for $G_0 = g_0 + \hat{g}_0$:

$$\sigma_{m|C} |\chi_{m|0|c}^{l(e)}\rangle_n = \left(n + \frac{d-1}{2} \right) |\chi_{m|0|c}^{l(e)}\rangle_n. \quad (6.35)$$

The above operators $\check{o}_I = o_I + \hat{o}_I(b, b^+)$ are determined according to (5.30), (5.32), (5.33):

$$\begin{aligned} (\check{t}_0, \check{l}_0) &= (t_0 + \tilde{\gamma}m, l_0 + m^2), \\ (\hat{l}_1, \hat{l}_1^+, \hat{g}_0, \hat{t}_1, \hat{t}_1^+) &= \left(mb, mb^+, b^+b + \frac{1}{2}, -\tilde{\gamma}b, -\tilde{\gamma}b^+ \right) \end{aligned} \quad (6.36)$$

for 2 additional bosonic oscillators, $[b, b^+] = 1$, acting in the Fock space $\mathcal{H}_m$. The massive field vectors and gauge parameter, being eigenvectors of the spin operator $\sigma_{m|C}$, have the following decompositions in the ghosts η_1^+ , $\mathcal{P}_1^+$ and b^+ :

$$|\chi_{m|0|c}^0\rangle_n = |\Psi_m\rangle_n + \eta_1^+ \mathcal{P}_1^+ |\chi_m\rangle_{n-2} = \sum_{k=0}^n \frac{(b^+)^k}{k!} \Psi_{m|k}\rangle_{n-k} + \eta_1^+ \mathcal{P}_1^+ \sum_{k=0}^{n-2} \frac{(b^+)^k}{k!} |\chi_{m|k}\rangle_{n-k-2}, \quad (6.37)$$

$$|\chi_{m|0|c}^1\rangle_n = \mathcal{P}_1^+ \tilde{\gamma} |\chi_{m|1}\rangle_{n-1} = \mathcal{P}_1^+ \tilde{\gamma} \sum_{k=0}^{n-1} \frac{(b^+)^k}{k!} |\chi_{m|1|k}\rangle_{n-k-1}, \quad (6.38)$$

$$|\chi_{m|0|c}^{1(0)}\rangle_n = \mathcal{P}_1^+ |\xi_m\rangle_{n-1} = \mathcal{P}_1^+ \sum_{k=0}^{n-1} \frac{v^{n-k-1}}{k!(n-k-1)!} (b^+)^k |\xi_{m|k}\rangle_{n-k-1}, \quad (6.39)$$

for $|\Psi_{m|k}\rangle_{n-k}$, $|\chi_{m|k}\rangle_{n-k-2}$, $|\chi_{m|1|k}\rangle_{n-k-1}$, $|\xi_{m|k}\rangle_{n-k-1}$ having the decomposition in a^+ ,

$$|\Psi_{m|k}\rangle_{n-k} = \frac{v^{n-k}}{(n-k)!} \Psi_{m|k}^{(\mu)} \prod_{e=1}^{n-k} a_{\mu_e}^+ |0\rangle, \quad |\chi_{m|k}\rangle_{n-k-2} = \frac{v^{n-k-2}}{(n-k-2)!} \chi_{m|k}^{(\mu)} \prod_{e=1}^{n-k-2} a_{\mu_e}^+ |0\rangle, \quad (6.40)$$

$$\left(|\chi_{m|1|k}\rangle, |\xi_{m|k}\rangle \right)_{n-k-1} = \frac{v^{n-k-1}}{(n-k-1)!} \left(\chi_{m|1|k}^{(\mu)}, \xi_{m|k}^{(\mu)} \right) \prod_{e=1}^{n-k-1} a_{\mu_e}^+ |0\rangle. \quad (6.41)$$

The constraints (6.34) transform the similar massless case (6.12), (6.13) to a ghost independent form,

$$\widehat{T}_1^m |\chi_{m|0|c}^{1(0)}\rangle_n = \check{t}_1 |\chi_{m|0|c}^{1(0)}\rangle_n \iff t_1 |\xi_m\rangle_{n-1} = \tilde{\gamma} b |\xi_m\rangle_{n-1}, \quad (6.42)$$

$$\check{t}_1 |\Psi_m\rangle_n = \tilde{\gamma} |\chi_{m|1}\rangle_{n-1}, \quad |\chi_m\rangle_{n-2} = -\frac{1}{2} \check{t}_1 \tilde{\gamma} |\chi_{m|1}\rangle_{n-1}, \quad \check{t}_1 |\chi_m\rangle_{n-2} = 0. \quad (6.43)$$

The analogue of the constraints (6.14) reads as follows:

$$(\check{t}_1)^3 |\Psi_m\rangle_n = 0, \quad \tilde{\gamma} |\chi_{m|1}\rangle_{n-1} = \check{t}_1 |\Psi_m\rangle_n, \quad |\chi_m\rangle_{n-2} = -\frac{1}{2} (\check{t}_1)^2 |\Psi_m\rangle_n. \quad (6.44)$$

The ghost-independent Lagrangian formulation for any even d in terms of field vector $|\Psi_m(a^+, b^+)\rangle_n$ which contains $(n-1)$ spin-tensor fields $|\Psi_{m|k}(a^+)\rangle_{n-k}$, $k = 1, \dots, n$, in addition to the initial field $|\Psi_{m|0}(a^+)\rangle_n \equiv |\Psi_m(a^+)\rangle_n$ can be obtained from (6.32) and has the form

$$\mathcal{S}_{C|n}^m = {}_n\langle \tilde{\Psi}_m | \left(\check{t}_0 - \frac{1}{4} (\check{t}_1^+)^2 \check{t}_0 \check{t}_1^2 - \check{t}_1^+ \check{t}_0 \check{t}_1 + \check{l}_1^+ \check{t}_1 + \check{t}_1^+ \check{l}_1 + \frac{1}{2} \check{t}_1^+ \check{l}_1^+ \check{t}_1^2 + \frac{1}{2} (\check{t}_1^+)^2 \check{l}_1 \check{t}_1 \right) | \Psi_m \rangle_n, \quad (6.45)$$

$$\delta |\Psi_m(a^+, b^+)\rangle_n = \check{l}_1^+ |\xi_m(a^+, b^+)\rangle_{n-1}, \quad (6.46)$$

with $(n-2)$ spin-tensor gauge parameters (additional to $|\xi_{m|0}(a^+)\rangle_{n-1}$ $|\xi_{m|k}(a^+)\rangle_{n-k-1}$ for $k = 1, \dots, n-1$ and off-shell constraints

$$(\check{t}_1)^3 |\Psi_m\rangle_n = 0, \quad \check{t}_1 |\xi_m\rangle_{n-1} = 0, \quad n > 2. \quad (6.47)$$

Note that the constrained Lagrangian formulation (6.45)–(6.47) is valid for any $d = 2N$ and was suggested in [23] as a product of dimensional reduction (not being the constrained BRST-BFV procedure) for even dimensions only. The triplet and quartet Lagrangian formulations for massive spin-tensor fields have the same form as those of the massless case, albeit with the change

$$\left(t_0, t_i^{(+)}, l_i^{(+)}; |\Psi\rangle, |\chi\rangle, |\chi_1\rangle, |\zeta\rangle, |\lambda_p\rangle, |\xi\rangle \right) \rightarrow \left(\check{t}_0, \check{t}_i^{(+)}, \check{l}_i^{(+)}; |\Psi_m\rangle, |\chi_m\rangle, |\chi_{m|1}\rangle, |\zeta_m\rangle, |\lambda_{m|p}\rangle, |\xi_m\rangle \right), \quad (6.48)$$

(for $p = 1, 2, 3$) in (6.26), (6.27) and (6.28), (6.29), (6.30), respectively, with a representation for the massive vectors as in (6.40), (6.41). Resolving the constraint (6.42) in a b^+ -independent form leads to the representation (for $k = 1, \dots, n-1$):

$$|\xi_{m|k}\rangle_{n-k-1} = (t_1 \tilde{\gamma})^k |\xi_{m|0}\rangle_{n-1} = \tilde{t}_1^k |\xi_{m|0}\rangle_{n-1}. \quad (6.49)$$

with the only independent gauge parameter $|\xi_{m|0}\rangle_{n-1}$. In turn, the $(\tilde{t}_1)^3$ -constraint (6.47) at a fixed degree in $(b^+)^k$ for $k = 0, \dots, n$ is rewritten in an unfolded b^+ -independent form, with $k = 0, \dots, n-3$ and the spin index omitted in $|\Psi_{m|k}\rangle_{n-k}$,

$$t_1^3 |\Psi_{m|k}\rangle - 3\tilde{\gamma} t_1^2 |\Psi_{m|k+1}\rangle - 3t_1 |\Psi_{m|k+2}\rangle + \tilde{\gamma} |\Psi_{m|k+3}\rangle = 0 \quad (6.50)$$

$$\iff \tilde{t}_1^3 |\Psi_{m|k}\rangle - 3\tilde{t}_1^2 |\Psi_{m|k+1}\rangle - 3\tilde{t}_1 |\Psi_{m|k+2}\rangle + |\Psi_{m|k+3}\rangle = 0. \quad (6.51)$$

From the gauge transformations for the field vectors $|\Psi_k\rangle_{n-k}$ at a fixed degree in $(b^+)^k$ for $k = 0, \dots, n$, with allowance for (6.49),

$$(b^+)^k : \quad \delta |\Psi_{m|k}(a^+)\rangle_{n-k} = l_1^+ |\xi_{m|k}(a^+)\rangle_{n-k-1} \theta_{nk} + k m |\xi_{m|k-1}(a^+)\rangle_{n-k} \quad (6.52)$$

$$= \left(l_1^+ t_1 \tilde{\gamma} \theta_{nk} + k m \right) (t_1 \tilde{\gamma})^{k-1} |\xi_{m|0}(a^+)\rangle_{n-1}, \quad (6.53)$$

one can remove the field $|\Psi_n\rangle_0$ by using the gauge parameter $|\xi_{m|n-1}\rangle_0$, so that a constraint appears for the independent gauge parameter $|\xi_{m|0}\rangle_{n-1}$:

$$t_1^{n-1} |\xi_{m|0}\rangle_{n-1} = 0 \iff \prod_{i=1}^{n-1} \gamma_{\mu_i} \xi_{m|0}^{(\mu)^{n-1}} = 0. \quad (6.54)$$

In the spin-tensor notation, the off-shell constraints and gauge transformations read

$$\xi_{m|k}^{(\mu)^{n-k-1}} = (-\imath)^k \prod_{i=n-k}^{n-1} \gamma_{\mu_i} \xi_{m|0}^{(\mu)^{n-1}}, \quad (6.55)$$

$$-\imath \prod_{j=n-k-2}^{n-k} \gamma_{\mu_j} \Psi_{m|k}^{(\mu)^{n-k}} + 3 \prod_{j=n-k-2}^{n-k-1} \gamma_{\mu_j} \Psi_{m|k+1}^{(\mu)^{n-k-1}} + 3\imath \gamma_{\mu_{n-k-2}} \Psi_{m|k+2}^{(\mu)^{n-k-2}} + |\Psi_{m|k+3}^{(\mu)^{n-k-3}} = 0, \quad (6.56)$$

$$\delta \Psi_{m|e}^{(\mu)^{n-e}} = (-\imath)^e \left(- \sum_{i=1}^{n-e} \partial^{\mu_i} \prod_{j=1}^e \gamma_{\nu_j} \xi_{m|0}^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{n-e} (\nu)^e} \theta_{ne} + \imath e m \prod_{j=1}^{e-1} \gamma_{\nu_j} \xi_{m|0}^{(\mu)^{n-e} \nu_1 \dots \nu_{e-1}} \right) \quad (6.57)$$

for $k = 0, \dots, n-3$, $e = 0, \dots, n$. Without removing the spinor $\Psi_{m|n}(x)$, the gauge parameter $\xi_{m|0}^{(\mu)^{n-1}}(x)$ is an unconstrained spin-tensor of rank $(n-1)$. In the opposite case, the constraint (6.54) holds. In terms of the initial constrained massive spin-tensor $\Psi_{m|0}^{(\nu)^n}$ and the auxiliary spin-tensors $\Psi_{m|k}^{(\nu)^{n-k}}$, $k = 1, \dots, n$ with lower ranks and allowance for the representation of the dual vector ${}_n \langle \tilde{\Psi}_m |$

$${}_n \langle \tilde{\Psi}_m | = - \sum_{k=0}^n \frac{(-\imath)^{n-k}}{k!(n-k)!} \langle 0 | b^k \prod_{e=1}^{n-k} a_{\nu_e} \bar{\Psi}_{m|k}^{(\nu)^{n-k}} \tilde{\gamma}, \quad (6.58)$$

the action takes the equivalent forms

$$\begin{aligned}
 \mathcal{S}_{C|(n)}^m(\Psi_{m|k}) = & (-1)^n \int d^d x \sum_{k=0}^n (-1)^k C_n^k \left[-\bar{\Psi}_{m|k}^{(\nu)n-k} (\imath \gamma^\mu \partial_\mu - m) \Psi_{m|k}(\nu)_{n-k} \right. \\
 & - (n-k) \left\{ \bar{\Psi}_{m|k}^{(\nu)n-k} \gamma_{\nu_{n-k}} (\imath \gamma^\mu \partial_\mu + m) \gamma_{\nu_{n-k}} \Psi_{m|k}(\nu)_{n-k-1}^{\nu_{n-k}} \right. \\
 & - \left(\imath \bar{\Psi}_{m|k}^{(\nu)n-k} \gamma_{\nu_{n-k}} (\imath \gamma^\mu \partial_\mu + m) \Psi_{m|k+1}(\nu)_{n-k-1} + h.c. \right) \\
 & \left. + \bar{\Psi}_{m|k+1}^{(\nu)n-k-1} (\imath \gamma^\mu \partial_\mu + m) \Psi_{m|k+1}(\nu)_{n-k-1} \right\} \theta_{nk} \\
 & + \frac{1}{4} \prod_{j=0}^1 (n-k-j) \left\{ \bar{\Psi}_{m|k}^{(\nu)n-k} \eta_{\nu_{n-k-1} \nu_{n-k}} (\imath \gamma^\mu \partial_\mu - m) \eta^{\rho_1 \rho_2} \Psi_{m|k}(\nu)_{n-k-2 \rho_1 \rho_2} \right. \\
 & + \left(\bar{\Psi}_{m|k}^{(\nu)n-k} \eta_{\nu_{n-k-1} \nu_{n-k}} (\imath \gamma^\mu \partial_\mu - m) \Psi_{m|k+2}(\nu)_{n-k-2} + h.c. \right) \\
 & \left. + \bar{\Psi}_{m|k+2}^{(\nu)n-k-2} (\imath \gamma^\mu \partial_\mu - m) \Psi_{m|k+2}(\nu)_{n-k-2} \right\} \theta_{n-1,k} \\
 & - (n-k) \left[\left\{ \bar{\Psi}_{m|k}^{(\nu)n-k} (-\imath \partial_{\nu_{n-k}}) \left(\gamma^\nu \Psi_{m|k}(\nu)_{n-k-1 \nu} - \imath \Psi_{m|k+1}(\nu)_{n-k-1} \right) \right. \right. \\
 & \left. \left. - m \bar{\Psi}_{m|k+1}^{(\nu)n-k-1} \left(\imath \gamma^\nu \Psi_{m|k}(\nu)_{n-k-1 \nu} + \Psi_{m|k+1}(\nu)_{n-k-1} \right) \right\} + h.c. \right] \theta_{nk} \\
 & + \frac{1}{2} \prod_{j=0}^1 (n-k-j) \left[\left\{ \bar{\Psi}_{m|k}^{(\nu)n-k-2 \mu_1 \mu_2} \gamma_{\mu_1} (-\imath \partial_{\mu_2}) \left(\eta^{\rho_1 \rho_2} \Psi_{m|k}(\nu)_{n-k-2 \rho_1 \rho_2} + \Psi_{m|k+2}(\nu)_{n-k-2} \right) \right. \right. \\
 & - \imath m \bar{\Psi}_{m|k+1}^{(\nu)n-k-1} \gamma_{\nu_{n-k-1}} \left(\eta^{\rho_1 \rho_2} \Psi_{m|k}(\nu)_{n-k-2 \rho_1 \rho_2} + \Psi_{m|k+2}(\nu)_{n-k-2} \right) \\
 & + \imath \bar{\Psi}_{m|k+1}^{(\nu)n-k-1} (-\imath \partial_{\nu_{n-k-1}}) \left(\eta^{\rho_1 \rho_2} \Psi_{m|k}(\nu)_{n-k-2 \rho_1 \rho_2} + \Psi_{m|k+2}(\nu)_{n-k-2} \right) \\
 & \left. \left. + m \bar{\Psi}_{m|k+2}^{(\nu)n-k-2} \left(\eta^{\rho_1 \rho_2} \Psi_{m|k}(\nu)_{n-k-2 \rho_1 \rho_2} + \Psi_{m|k+2}(\nu)_{n-k-2} \right) \right\} + h.c. \right] \theta_{n-1,k} \Big] \quad (6.59)
 \end{aligned}$$

$$\begin{aligned}
 = & (-1)^n \int d^d x \sum_{k=0}^n (-1)^k C_n^k \left[\bar{\Psi}_{m|k}(\not{p} + m) \Psi_{m|k} + (n-k) \left\{ \bar{\Psi}'_{m|k}(\not{p} - m) \Psi'_{m|k} \right. \right. \\
 & - \left(\imath \bar{\Psi}'_{m|k}(\not{p} - m) \Psi_{m|k+1} + h.c. \right) + \bar{\Psi}_{m|k+1}(\not{p} - m) \Psi_{m|k+1} \Big\} \theta_{nk} \\
 & - \frac{1}{4} \prod_{j=0}^1 (n-k-j) \left\{ \bar{\Psi}''_{m|k}(\not{p} + m) \Psi''_{m|k} + \left(\bar{\Psi}''_{m|k}(\not{p} + m) \Psi_{m|k+2} + h.c. \right) \right. \\
 & \left. + \bar{\Psi}_{m|k+2}(\not{p} + m) \Psi_{m|k+2} \right\} \theta_{n-1,k} - (n-k) \left[\left\{ \left(\bar{\Psi}_{m|k} \cdot p - \imath m \bar{\Psi}_{m|k+1} \right) \right. \right. \\
 & \times \left(\Psi'_{m|k} - \imath \Psi_{m|k+1} \right) \Big\} + h.c. \Big] \theta_{nk} + \frac{1}{2} \prod_{j=0}^1 (n-k-j) \left[\left\{ \left(\bar{\Psi}'_{m|k} \cdot p - \imath m \bar{\Psi}'_{m|k+1} \right. \right. \right. \\
 & \left. \left. + \imath \bar{\Psi}_{m|k+1} \cdot p + m \bar{\Psi}_{m|k+2} \right) \left(\Psi''_{m|k} + \Psi_{m|k+2} \right) \right\} + h.c. \Big] \theta_{n-1,k} \Big] \quad (6.60)
 \end{aligned}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. Thereby, we have obtained a constrained gauge-invariant Lagrangian formulation for the massive spin-tensor field $\Psi_{m|0}^{(\nu)n} \equiv \Psi_m^{(\nu)n}$ of spin $s = n + 1/2$ in Minkowski space-time of any even dimension. In a similar way, the triplet and quartet Lagrangian

formulations can be derived in spin-tensor forms for the massive spin-tensor field following the recipes for the above massless case with any $d = 2N$. In the massless case determined by the vanishing of all $\Psi_{m|k}^{(\nu)n-k}$, $k \geq 1$ with $m = 0$ in (6.56)–(6.60), we derive a Fang-Fronsdal Lagrangian formulation containing no dependence (!) on either odd or even values of d . Secondly, in view of the spin-tensor representation for the constrained Lagrangian and its gauge symmetries in the massive case not depending on the Grassmann-odd gamma-matrices $\tilde{\gamma}^\mu$, we may use Lagrangian formulation (6.55)–(6.57), (6.59), (6.60) for $d = 2N + 1$ dimensions as well, thus providing its applicability for any $d \geq 4$.⁸ Let us consider a massive field of spin $s = 5/2$, first examined in [97] for $d = 4$. In this case, the off-shell constraints for the independent gauge parameters and the spin-tensor fields $\Psi_{m\mu\nu}$ parameterizing the configuration space,

$$(\Psi_{m|0}^{\mu\nu}, \Psi_{m|1}^\mu, \Psi_{m|2}), \quad (\xi_{m|0}^\mu, \xi_{m|1}) \quad (6.61)$$

look trivial for the fields, which is implied by (6.56), and have the representation (6.55) for ξ , albeit for arbitrary d (for $d = 2N$ by construction and for $d = 2N + 1$ by extrapolation of the Lagrangian formulation (6.55)–(6.57), (6.59) from even to odd dimensions),

$$\xi_{m|1} = (-i)\gamma_\mu \xi_{m|0}^\mu. \quad (6.62)$$

Using the gauge transformations (6.57), we find

$$\delta\Psi_{m|0}^{\mu\nu} = -\partial^\mu \xi_{m|0}^\nu - \partial^\nu \xi_{m|0}^\mu, \quad (6.63)$$

$$\delta\Psi_{m|1}^\mu = i\left(\partial^\mu \gamma_\nu \xi_{m|0}^\nu - i m \xi_{m|0}^\mu\right), \quad (6.64)$$

$$\delta\Psi_{m|2} = 2m\gamma_\mu \xi_{m|0}^\mu, \quad (6.65)$$

where the constraint (6.62) has been resolved, with the gauge parameter $\xi_{m|1}$ expressed in terms of the γ -trace of $\xi_{m|0}^\mu$. Using (6.65), we resolve the Stueckelberg gauge symmetries, thereby removing the field $\Psi_{m|2}$ with a γ -traceless residual gauge parameter $\xi_{m|0}^\mu$, $\gamma_\mu \xi_{m|0}^\mu = 0$. We then decompose the field $\Psi_{m|1}^\mu$ as follows:

$$\Psi_{m|1}^\mu = \Psi_{m|1|\perp}^\mu + \gamma^\mu \Psi_{m|1|L} : \quad \gamma_\mu \Psi_{m|1|\perp}^\mu = 0, \quad (6.66)$$

so that (6.64) implies the gauge transformations for $\Psi_{m|1|\perp}^\mu$ and $\Psi_{m|1|L}$

$$\delta\Psi_{m|1|\perp}^\mu = m\xi_{m|0}^\mu, \quad \delta\Psi_{m|1|L} \equiv \delta(\gamma_\mu \Psi_{m|1}^\mu) = 0 \quad (6.67)$$

whence the spin-tensor $\Psi_{m|1|\perp}^\mu$ is gauged away entirely by using the remaining degrees of freedom in $\xi_{m|0}^\mu$. Thus the theory of a massive field of spin $5/2$ is described, in accordance with [96], by the non-gauge unconstrained fields $\Psi_{m|0}^{\mu\nu}$, $\Psi_{m|1|L}$ whose dynamics is governed

⁸The author highly appreciates the referee's critical remarks on issues concerning an explicit realization of the odd matrix $\tilde{\gamma}$ for different values of d and also concerning a Lagrangian formulation for massive half-integer HS fields. These remarks have proved to be instrumental in solving the mentioned problems.

by an action which follows from (6.60) for $n = 2$ in the Fang-Fronsdal-like notation

$$\begin{aligned}
 \mathcal{S}_{C|2}^m(\Psi_{m|0}, \Psi'_{m|1|L}) = & \int d^d x \left[\bar{\Psi}_{m|0}(\not{p} + m)\Psi_{m|0} + 2\bar{\Psi}'_{m|0}(\not{p} - m)\Psi'_{m|0} \right. \\
 & - \frac{1}{2}\bar{\Psi}''_{m|0}(\not{p} + m)\Psi''_{m|0} - \left\{ \left(2\bar{\Psi}_{m|0} \cdot p\Psi'_{m|0} - \bar{\Psi}'_{m|0} \cdot p\Psi''_{m|0} \right) + h.c. \right\} \\
 & + 2d \left\{ [2 - d]\bar{\Psi}_{m|1|L}\not{p}\Psi_{m|1|L} + dm\bar{\Psi}_{m|1|L}\Psi_{m|1|L} \right\} \\
 & - 2 \left\{ i \left(\bar{\Psi}''_{m|0}(\not{p} - m)\Psi_{m|1|L} - \bar{\Psi}'_{m|0} \cdot p\Psi_{m|1|L} + m\bar{\Psi}_{m|1|L}\Psi''_{m|0} \right. \right. \\
 & \left. \left. - \frac{1}{2}\bar{\Psi}_{m|1|L}(\not{p} - dm)\Psi''_{m|0} \right) + h.c. \right\} \Big]. \tag{6.68}
 \end{aligned}$$

For an arbitrary massive spin-tensor field of spin $n + 1/2$, we can derive from (6.59) or (6.60) a Lagrangian non-gauge description (equivalent to the gauge-invariant one) whose configuration space contains only the constrained fields (due to (6.56))

$$\Psi_{m|0}^{(\mu)_n}, \Psi_{m|1|L}^{(\mu)_{n-2}}, \dots, \Psi_{m|n-2|L}^\mu, \Psi_{m|n-1|L}, \text{ for } \Psi_{m|j|L}^{(\mu)_{n-j-1}} = \gamma_{(\mu)_{n-j}} \Psi_{m|j}^{(\mu)_{n-j}}, j = 1, \dots, n-1. \tag{6.69}$$

The derivations of the constrained Lagrangian formulations for massless and massive spin-tensor fields of spin $s = n + 1/2$ according to general prescriptions of the suggested constrained BRST-BFV approach presents the basic results of the subsection.

7 Conclusion

In this paper, we have developed a constrained BRST-BFV method to construct gauge-invariant Lagrangian formulations for free massless and massive half-integer spin-tensor fields with an arbitrary fixed generalized spin $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2, \dots, n_k + 1/2)$, in Minkowski space-time $\mathbb{R}^{1,d-1}$ of any dimension in the “metric-like” formulation. This result is presented in Statement 4 and explicitly includes, for the field $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}$, a first-order Lagrangian action $S_{C|(n)_k}$ (5.8) invariant with respect to reducible gauge transformations (5.9) (which determine a gauge theory of $(k-1)$ -th stage of reducibility), and off-shell independent BRST-extended constraints $\hat{T}_i, \hat{T}_{rs}$ (4.23), (4.25) imposed on the whole set of field (incorporating the initial spin-tensor $\Psi_{(\mu^1)_{n_1}, \dots, (\mu^k)_{n_k}}$) and gauge parameter vectors (5.10) from the resultant Hilbert space $\mathcal{H} \otimes H_{gh}^{oA}$, whose vectors have the representation (4.29). The crucial point is that the superalgebra formed by a constrained BRST operator (only for the first-class constraint system o_A with a subsuperalgebra of Minkowski space $\mathbb{R}^{1,d-1}$ isometries (3.23) in the HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$), a generalized spin operator, and BRST extended second-class constraints $\{Q_C, \hat{\sigma}_C^i(g), \hat{T}_i, \hat{T}_{rs}, \hat{L}_{ij}\}$ is closed with respect to the $[\cdot, \cdot]$ -multiplication in $\mathcal{H} \otimes H_{gh}^{oA}$ and forms, with the exception of Q_C and with an addition of $\hat{T}_i^+, \hat{T}_{rs}^+, \hat{L}_{ij}^+$, an $osp(1|2k)$ orthosymplectic superalgebra. This fact guarantees a common set of eigenstates in $\mathcal{H} \otimes H_{gh}^{oA}$, which depends on less ghost coordinates and momenta (hence, for a smaller set of auxiliary fields and gauge parameters)

than the ones in the unconstrained Lagrangian formulation (3.57), (3.58) [68] for the same spin-tensor field, and therefore also ensures the consistency of dynamics in the constrained formulation with holonomic off-shell constraints. It is shown in the Theorem (5.19), (5.20), on the basis of general results in operator quantization for dynamical systems with first- and second-class constraints, that the Lagrangian dynamics for the same element of an irreducible half-integer HS representation of Poincare group in $\mathbb{R}^{1,d-1}$ subject to $Y(s_1, \dots, s_k)$ in the constrained and unconstrained BRST-BFV approaches are equivalent, i.e., for both dynamics equivalent to irreducibility conditions with a given spin (3.2)–(3.4). The equivalence of the constrained and unconstrained BRST-BFV methods in question is two-fold. First, it is based on derivation starting from the unconstrained HS symmetry superalgebra $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ and respective BRST operators Q' (3.26) and Q (3.33), as one disregards the additional (due to conversion) oscillators B^a, B^{a+} given by (4.1)–(4.7), and resulting in Statement 3 and Corollary 3 in (4.18), (4.19), or, equivalently, in terms of the Q - and Q_C -complexes in (4.20), (4.21). Second, in a self-consistent form, the nilpotent constrained BRST operator Q_C , the spin operators $\hat{\sigma}_C^i(g)$, and the off-shell BRST-extended constraints $\hat{T}_i, \hat{T}_{rs}$ are derived explicitly in section 4.2 from the irreducibility conditions (3.2)–(3.4), on a basis of solving the generating equations (4.24). The constrained Lagrangian formulation is then obtained from the second-order one (5.5)–(5.7) by partial gauge-fixing, thereby eliminating the zero-mode ghost operators $q_0, \eta_0, p_0, \mathcal{P}_0$ for the Dirac and D'Alembert operators from the whole set of gauge parameters and field vectors, in a way compatible with off-shell BRST-extended constraints having the form (5.16).

As a byproduct, we have derived the constrained BRST-BFV Lagrangian formulation (5.25)–(5.27), for the integer mixed-symmetric HS field $\Phi_{(\mu^1)_{s_1}, \dots, (\mu^k)_{s_k}}$ in $\mathbb{R}^{1,d-1}$ subject to the irreducibility conditions (5.21), from the unconstrained formulation, albeit with BRST-extended traceless off-shell constraints, $\hat{L}_{ij}$, instead of gamma-traceless ones, $\hat{T}_i$. The stages of reducibility for both integer and half-integer massless HS fields coincide and can be used as a starting point for constrained BRST-BFV Lagrangian formulations to accommodate SUSY models of HS fields.

It should be noted that the constrained BRST-BFV approach, as well as the unconstrained one, implies automatically a gauge-invariant Lagrangian description, reflecting the general fact of BV-BFV duality [98–101], which reproduces a Lagrangian action for the initial non-Lagrangian equations (reflecting the fact that the (spin)-tensor belongs to an irreducible representation space of the Poincare group) by means of a Hamiltonian object.

It is shown in section 5.3 that the case of massive half-integer and integer HS fields with a corresponding arbitrary Young tableaux $Y(s_1, \dots, s_k)$ allows one to obtain constrained gauge-invariant Lagrangian formulations, for fermionic (5.39)–(5.41), initially for even dimensions d , and bosonic (5.47)–(5.49) fields for any d , derived from the respective unconstrained formulations (5.35), (5.36) and (5.22), (5.23). This was achieved by dimensional reduction procedure for the respective massless HS symmetry (super)algebra, $\mathcal{A}^{(f)}(Y(k), \mathbb{R}^{1,d})$ in $\mathbb{R}^{1,d}$, to the massive one, $\mathcal{A}_m^{(f)}(Y(k), \mathbb{R}^{1,d-1})$ in $\mathbb{R}^{1,d-1}$, which means conversion for the sets of differential constraints, being this time second-class constraints. In both cases, the constrained BRST operator for differential constraints, the generalized spin operators, and the BRST-extended off-shell constraints are modified by k pairs of

conversion oscillators, b_i, b_i^+ , $i = 1, \dots, k$, thereby preserving their superalgebra, albeit in a larger Hilbert space. Both resulting gauge theories possess the same reducibility stage as the ones for massless fields. We differentiate the cases of odd and even values of space-time dimension d for half-integer HS fields when realizing the explicit Grassmann-odd gamma-matrix-like objects suggested in (3.11) and (3.12) which do not influence either the form of HS symmetry superalgebra or the resulting BRST-BFV Lagrangian formulation. Besides, for massless half-integer HS fields we shown in (3.59), (3.60) the realization of the (un)constrained Lagrangian formulations with only standard γ^μ matrices as Grassmann-odd quantities, following to totally-symmetric HS fields [20, 21].

As an example demonstrating the applicability of the suggested scheme, it is shown that for the particular case of totally-symmetric massless half-integer HS field $\Psi_{(\mu)_n}$ of spin $s = n + \frac{1}{2}$ the constrained BRST-BFV method permits one to immediately reproduce the Fang-Fronsdal [42, 43] gauge-invariant Lagrangian action (6.23)–(6.25) in terms of a triple-gamma-traceless field $\Psi_{(\mu)_n}$ and a gamma-traceless gauge parameter $\xi_{(\mu)_{n-1}}$ for any value of $d \geq 4$. Note that this formulation was reproduced in [20, 21] from an unconstrained Lagrangian with the help of a rather tedious gauge-fixing procedure. The same action $S_{C|(n)}$ in terms of a ghost-independent field vector has the form (6.20), with independent gauge transformations (6.21) and off-shell holonomic constraints $t_1^3|\Psi\rangle_n = 0$, $t_1|\xi\rangle_{n-1} = 0$. The constrained Lagrangian formulation in terms of the triplet of fields $\Psi^{(\mu)_n}$, $\chi_1^{(\mu)_{n-1}}$, $\chi^{(\mu)_{n-2}}$ with the action (6.26), invariant with respect to the gauge transformations (6.27) and subject to the off-shell constraints (6.12), (6.14), coincides with those of [73]. This Lagrangian served as a starting point to construct an unconstrained Lagrangian formulation in a quartet form [71, 72], with the addition of a “fourth” compensating spin-tensor, $\zeta^{(\mu)_{n-2}}$, and 3 Lagrangian multipliers to the rest of the augmented gauge-invariant constraints (6.28), the resulting gauge-invariant action being of the form (6.29), (6.30). A constrained gauge-invariant Lagrangian formulation for a massive spin-tensor for even space-time dimension d has been obtained in the form of ghost-independent Fock space (6.45) and in the Fang-Fronsdal-like spin-tensor form (6.59), (6.60), with a set of $(n-1)$ auxiliary spin-tensors and a single unconstrained gauge parameter $\xi_{m|0}^{(\mu)_{n-1}}$, in accordance with [96, 97], with a different analogue (as for $m = 0$) of the off-shell constraints (6.56) in the total set of fields. Because of the final Lagrangian formulation for massive half-integer field does not depend on the Grassmann-odd matrices $\tilde{\gamma}^\mu, \tilde{\gamma}$, we suggested to use it as an ansatz for the Lagrangian formulation for massive spin-tensor in odd space-time dimension d .

The above construction of the constrained BRST-BFV approach for Lagrangian formulations was considered from the general viewpoint in appendix A for a finite-dimensional dynamical system with Hamiltonian $H_0(\Gamma)$ subject to first-class $T_A(\Gamma) = 0$ and second-class $\Theta_\alpha(\Gamma) = 0$ constraints satisfying special commutation relations only in terms of the Poisson superbrackets (A.1), (A.2). The crucial point here is a construction on the basis of solving the *generating equations* (A.6) for a superalgebra of BRST-extended (in $\mathcal{M}_{\min}$) second-class constraints $\hat{\Theta}_\alpha$, from the requirement of commutation with the BRST charge and Hamiltonian, respecting only the first-class constraints T_A in the minimal sector of ghost coordinates and momenta, and also (trivially) in the total phase-space. The explicit form of $\hat{\Theta}_\alpha$ was found in (A.10), with accuracy up to the second order in C^A , and satisfying

the same Poisson bracket relations (A.22) as in (A.1) for vanishing new structure functions $f_{\alpha\beta A}^{CD}(\Gamma)$ resolving the Jacobi identity for $T_A, \Theta_\alpha, \Theta_\beta$, albeit in $\mathcal{M}_{\min}$ with a BRST-invariant extension of the invertible supermatrix (A.23). The supercommutative algebra of a BRST charge, a unitarizing Hamiltonian, and BRST-invariant second-class constraints lead to a new representation for the generating functional of Green's functions (A.26), (A.27), in terms of $\hat{\Theta}_\alpha$ for the dynamical system in question. At the operator level, the operators of the latter quantities allow one to describe equivalently, with some prescriptions for a choice of ordering in (A.37)–(A.39), a set of physical states by imposing half of the constraints $\hat{\Theta}_\alpha$ on the Hilbert space vectors.

Concluding, we present some ways of extending the results obtained in this paper. First, the development of a Lagrangian construction for tensor and spin-tensor fields with an arbitrary index symmetry in AdS space. Second, the derivation of constrained BRST-BFV Lagrangian formulations for irreducible representations of the SUSY Poincare supergroup along the lines of [102]. Third, the development of the constrained and unconstrained BRST-BV method to construct respective minimal field-antifield BV actions for half-integer HS fields in terms of Fock-space vectors. Fourth, the construction of a quantum action for HS fields within an $N = 1$ BRST approach,⁹ where the space-time variables x^μ are to be considered on equal footing with the total Fock space variables. Fifth, a consistent deformation of the (un)constrained BRST-BFV and BRST-BV approaches applied to bosonic and fermionic mixed-symmetric HS fields will make it possible to construct an interacting theory with mixed-symmetry fermionic HS fields, including the case of curved (AdS) backgrounds. We intend to carry out a study of these problems in our forthcoming works.

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A On quantization of the dynamical system with mixed-class constraints

Let us consider the dynamical system with Hamiltonian, $H_0(\Gamma)$, subject to the finite set of the first-class $T_A(\Gamma) = 0$ and second-class $\Theta_\alpha(\Gamma) = 0$, constraints given on the $2n$ -dimensional phase-space $(\mathcal{M}, \omega)$, $\varepsilon(\omega) = 0$ satisfying to the particular case of the

⁹For a generalization of quantization rules intended to accommodate theories with a gauge group on the basis of an N -parametric BRST symmetry, see [103].

relations (2.1)–(2.4) with respect to the Poisson superbracket, for vanishing functions $V_\alpha^A(\Gamma) = V_\alpha^\beta(\Gamma) = f_{AB}^{\alpha\beta}(\Gamma) = V_\alpha^A(\Gamma) = 0$:

$$\{T_A, T_B\} = f_{AB}^C(\Gamma)T_C, \quad \{T_A, \Theta_\alpha\} = f_{A\alpha}^C(\Gamma)T_C, \quad \{\Theta_\alpha, \Theta_\beta\} = \Delta_{\alpha\beta}(\Gamma) + f_{\alpha\beta}^\gamma(\Gamma)\Theta_\gamma, \quad (\text{A.1})$$

$$\{H_0, \Theta_\alpha\} = 0, \quad \{H_0, T_A\} = V_A^B(\Gamma)T_B, \quad (\text{A.2})$$

where structure functions $\Delta_{\alpha\beta}(\Gamma)$, compose invertible (on the surface, $T_A = \Theta_\alpha = 0$ in $\mathcal{M}$) supermatrix. The dynamics and gauge transformations are described by the equations (2.5).

Because of, the algebra of the functions (T_A, H_0) is closed with respect to $\{, \}$ -multiplication, we may restrict ourselves by choice of the total phase space $\mathcal{M}_{\text{tot}}$, ($\mathcal{M} \subset \mathcal{M}_{\text{tot}}$) underlying the generalized canonical quantization [25–28], which is parameterized (for linearly independent constraints T_A, Θ_α) by the canonical phase-space variables, $\Gamma_T^{\mathbf{P}} = (\Gamma^p, \Gamma_{\text{gh}})$, $\varepsilon(\Gamma_T^{\mathbf{P}}) = \varepsilon_{\mathbf{P}}$, (2.14), with Grassmann and ghost number distributions (2.15). subject to (2.16). In this case the generating functional of Green's functions (and respective partition function for $I_{\mathbf{P}}(t) = 0$) has the form (2.17), (2.18) $I_{\mathbf{P}}(t)$ to $\Gamma_T^{\mathbf{P}}$, but with another unitarizing Hamiltonian $H_{r|\Psi_r}(t) = H_{r|\Psi_r}(\Gamma_T(t))$ to be determined by three t -local functions: $\mathcal{H}_r(t)$ with $(\varepsilon, gh)(\mathcal{H}_r) = (0, 0)$, $\Omega_r(t)$, with $(\varepsilon, gh)(\Omega_r) = (1, 1)$, and $\Psi_r(t)$, with $(\varepsilon, gh)(\Psi) = (1, -1)$ given by the equations in terms of Poisson (instead of Dirac in (2.19)–(2.21)) superbracket:

$$H_{r|\Psi}(t) = \mathcal{H}_r(t) + \{\Omega_r(t), \Psi(t)\}_t, \quad \text{for } \{\Omega_r, \Omega_r\} = 0, \quad \{\mathcal{H}_r, \Omega_r\} = 0, \quad (\text{A.3})$$

with simplest choice for the gauge Fermion Ψ determined by (2.26). The solutions for the generating equations (A.3) are given by the expressions (2.24), (2.25) but with strong equality: in the form of series in powers of *minimal* ghost coordinates and momenta $C^A, \bar{\mathcal{P}}_A$ with use of $C\bar{\mathcal{P}}$ -ordering up to the second order in Γ_{gh} :

$$\mathcal{H}_r = H_0 + (-1)^{\varepsilon_C} C^A V_A^C(\Gamma) \bar{\mathcal{P}}_C + O(C^2), \quad (\text{A.4})$$

$$\Omega_r = \Omega_{r|\text{min}} + \pi_A \mathcal{P}^A = C^A \left(T_A + \frac{1}{2} (-1)^{\varepsilon_C + \varepsilon_A} C^B f_{BA}^C(\Gamma) \bar{\mathcal{P}}_C + O(C^2) \right) + \pi_A \mathcal{P}^A, \quad (\text{A.5})$$

which encode in $\mathcal{H}_r$ and $\Omega_{r|\text{min}} = \Omega_{r|\text{min}}(\Gamma, \Gamma_{gh|m})$ the structure functions of the first-class constraints system T_A .

Let us introduce new (additional) equations in $\mathcal{M}_{\text{tot}}$ without using Dirac superbracket due to the presentation (A.1), (A.2):

$$\{\Omega_{r|\text{min}}, \hat{\Theta}_\alpha\} = 0, \quad \{\mathcal{H}_r, \hat{\Theta}_\alpha\} = 0, \quad (\varepsilon, gh)\hat{\Theta}_\alpha = (\varepsilon_\alpha, 0), \quad (\text{A.6})$$

which we call by the *generating equations* for superalgebra of the BRST extended in $\mathcal{M}_{\text{tot}}$ second-class constraints $\hat{\Theta}_\alpha = \hat{\Theta}_\alpha(\Gamma, \Gamma_{gh|m})$ with boundary condition

$$\hat{\Theta}_\alpha(\Gamma, 0) = \Theta_\alpha(\Gamma). \quad (\text{A.7})$$

The solution of the first equations in (A.6), being analogous as one for the latter equation in (A.3) exists in the form of series in powers of *minimal* ghost coordinates and momenta

$C^A, \bar{\mathcal{P}}_A$ with use of $C\bar{\mathcal{P}}$ -ordering up to the second order in Γ_{gh} because of any p -times applied Poisson bracket from:

$$\{T_{A_1}, \{T_{A_2}, \{, , , , \{T_{A_p}, \Theta_\alpha\} \dots\}\}\} = f_{A_1 \dots A_p \alpha}^A(\Gamma) T_A, \quad (\text{A.8})$$

$$\{T_{A_1}, \{T_{A_2}, \{, , , , \{T_{A_{p-1}}, \{H_0, \Theta_\alpha\}\} \dots\}\}\} = f_{A_1 \dots A_{p-1} \alpha}^A(\Gamma) T_A, \quad (\text{A.9})$$

for $p = 2, \dots$ is proportional to the constraints T_A with some regular f' s. Hence, we have

$$\hat{\Theta}_\alpha(\Gamma, \Gamma_{gh|m}) = \Theta_\alpha(\Gamma) + (-1)^{\varepsilon_\alpha + \varepsilon_C} C^A f_{A\alpha}^C(\Gamma) \bar{\mathcal{P}}_C + O(C^2). \quad (\text{A.10})$$

The validity of the second equations in (A.6) for commutativity of $\hat{\Theta}_\alpha$ with $\mathcal{H}_r$ is considered as the restrictions on the form of $\hat{\Theta}_\alpha$ and will follow up to the second order in $C^A, \bar{\mathcal{P}}_A$, first, from (A.1), (A.2), second, from the Jacobi identity for Poisson brackets of H_0, T_A, Θ_α :

$$\{\{H_0, \Theta_\alpha\}, T_A\} + (-1)^{\varepsilon_A \varepsilon_\alpha} \{\{T_A, H_0\}, \Theta_\alpha\} + \{\{\Theta_\alpha, T_A\}, H_0\} = 0, \quad (\text{A.11})$$

$$\iff (-1)^{\varepsilon_A \varepsilon_\alpha} \left(\{H_0, f_{A\alpha}^C\} + V_A^B f_{B\alpha}^C + f_{A\alpha}^B V_B^C + (-1)^{\varepsilon_C \varepsilon_\alpha} \{V_A^C, \Theta_\alpha\} \right) T_C = 0, \quad (\text{A.12})$$

whose resolution (as it was shown for the first-class constraints in [29–31]) means the presence of new structural functions $f_{A\alpha}^{CD}(\Gamma), f_{A\alpha}^{C\beta}(\Gamma)$ given on $\mathcal{M}$ as follows

$$\{H_0, f_{A\alpha}^C\} + V_A^B f_{B\alpha}^C + f_{A\alpha}^B V_B^C + (-1)^{\varepsilon_C \varepsilon_\alpha} \{V_A^C, \Theta_\alpha\} = -f_{A\alpha}^{CD}(\Gamma) T_D, \quad (\text{A.13})$$

$$\text{with } f_{A\alpha}^{CD}(\Gamma) = -(-1)^{\varepsilon_C \varepsilon_D} f_{A\alpha}^{DC} = 0, \quad (\text{A.14})$$

for the dynamical system in question. The additional terms (proportional to the second order in $(C^A)^2$ in the searched-for BRST extended constraints $\hat{\Theta}_\alpha$ (A.10) should correspond to new structural functions, analogous to one in the right (A.13) which follows from the resolution of the Jacobi identity with Poisson brackets for $p = 2$ in (A.8) of the form

$$(-1)^{\varepsilon_A \varepsilon_B} \{\{T_B, \Theta_\alpha\}, T_A\} + (-1)^{\varepsilon_A \varepsilon_\alpha} \{\{T_A, T_B\}, \Theta_\alpha\} + (-1)^{\varepsilon_B \varepsilon_\alpha} \{\{\Theta_\alpha, T_A\}, T_B\} = 0, \quad (\text{A.15})$$

with two upper indices $f_{AB\alpha}^{CD}$. If $f_{AB\alpha}^{CD} = 0$ then the presentation in (A.10) is final without $O(C^2)$, but with possible $O(C^3)$ terms, whose presence or absence should be described analogously. To establish the algebra of $\hat{\Theta}_\alpha$ with respect to the Poisson superbracket one should to study the Jacobi identity of the form (A.15) but for the quantities $T_A, \Theta_\alpha, \Theta_\beta$

$$(-1)^{\varepsilon_A \varepsilon_\beta} \{\{T_A, \Theta_\alpha\}, \Theta_\beta\} + (-1)^{\varepsilon_\beta \varepsilon_\alpha} \{\{\Theta_\beta, T_A\}, \Theta_\alpha\} + (-1)^{\varepsilon_A \varepsilon_\alpha} \{\{\Theta_\alpha, \Theta_\beta\}, T_A\} = 0 \quad (\text{A.16})$$

$$\Rightarrow (-1)^{\varepsilon_A \varepsilon_\beta} \left(\left[(f_{A\alpha}^D f_{D\beta}^C + (-1)^{\varepsilon_C \varepsilon_\beta} \{f_{A\alpha}^C, \Theta_\beta\}) - (-1)^{\varepsilon_\alpha \varepsilon_\beta} (\alpha \longleftrightarrow \beta) \right] \right. \quad (\text{A.17})$$

$$\left. - (-1)^{\varepsilon_A (\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma)} f_{\alpha\beta}^\gamma f_{A\gamma}^C \right) T_C = (-1)^{\varepsilon_A \varepsilon_\beta} \{T_A, \Delta_{\alpha\beta}\} + (-1)^{\varepsilon_A \varepsilon_\beta} \{T_A, f_{\alpha\beta}^\gamma\} \Theta_\gamma.$$

Due to the linear independence of T_C and Θ_γ on the surface $T_C = \Theta_\gamma = 0$ it follows from the right-hand side of (A.17) it follows that there exist structure functions, $\Delta_{\alpha\beta A}^C(\Gamma), \Delta_{\alpha\beta A}^\gamma(\Gamma)$, regular (i.e. smooth) in $\mathcal{M}$

$$\{T_A, \Delta_{\alpha\beta}\} = -\Delta_{\alpha\beta A}^C(\Gamma) T_C - \Delta_{\alpha\beta A}^\gamma(\Gamma) \Theta_\gamma, \quad (\Delta_{\alpha\beta A}^C, \Delta_{\alpha\beta A}^\gamma) = -(-1)^{\varepsilon_\alpha \varepsilon_\beta} (\Delta_{\beta\alpha A}^C, \Delta_{\beta\alpha A}^\gamma) \quad (\text{A.18})$$

with $\varepsilon(\Delta_{\alpha\beta A}^C, \Delta_{\alpha\beta A}^\gamma) = \varepsilon_\alpha + \varepsilon_\beta + \varepsilon_A + (\varepsilon_C, \varepsilon_\gamma)$ so that the resolution of the equations (A.17) looks as

$$\left(\left[(f_{A\alpha}^D f_{D\beta}^C + (-1)^{\varepsilon_C \varepsilon_\beta} \{f_{A\alpha}^C, \Theta_\beta\}) - (-1)^{\varepsilon_\alpha \varepsilon_\beta} (\alpha \longleftrightarrow \beta) \right] - (-1)^{\varepsilon_A (\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma)} f_{\alpha\beta}^\gamma f_{A\gamma}^C \right) + \Delta_{\alpha\beta A}^C(\Gamma) = f_{\alpha\beta A}^{CD}(\Gamma) T_D, \quad (\text{A.19})$$

$$\{T_A, f_{\alpha\beta}^\gamma\} - \Delta_{\alpha\beta A}^\gamma(\Gamma) = f_{\alpha\beta A}^{\gamma\delta}(\Gamma) \Theta_\delta. \quad (\text{A.20})$$

In (A.19), (A.20) the new structure regular functions, $f_{\alpha\beta A}^{CD}, f_{\alpha\beta A}^{\gamma\delta}$, of the algebra of mixed-class constraints are introduced, with the properties:

$$(f_{\alpha\beta A}^{CD}, f_{\alpha\beta A}^{\gamma\delta}) = -(-1)^{\varepsilon_\alpha \varepsilon_\beta} (f_{\beta\alpha A}^{CD}, f_{\beta\alpha A}^{\gamma\delta}), (f_{\alpha\beta A}^{CD}, f_{\beta\alpha A}^{\gamma\delta}) = -((-1)^{\varepsilon_C \varepsilon_D} f_{\alpha\beta A}^{DC}, (-1)^{\varepsilon_\gamma \varepsilon_\delta} f_{\beta\alpha A}^{\delta\gamma}), \\ \varepsilon(f_{\alpha\beta A}^{CD}, f_{\alpha\beta A}^{\gamma\delta}) = \varepsilon_\alpha + \varepsilon_\beta + \varepsilon_A + (\varepsilon_C + \varepsilon_D, \varepsilon_\gamma + \varepsilon_\delta). \quad (\text{A.21})$$

Therefore, we have from the representation (A.10) with account for (A.1), (2.2), (A.19):

$$\{\hat{\Theta}_\alpha, \hat{\Theta}_\beta\} = \hat{\Delta}_{\alpha\beta}(\Gamma, \Gamma_{gh|m}) + f_{\alpha\beta}^\gamma \hat{\Theta}_\gamma + (-1)^{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_C} C^A f_{\alpha\beta A}^{CD}(\Gamma) T_D \bar{\mathcal{P}}_C + O(C^2), \quad (\text{A.22})$$

$$\text{with } \hat{\Delta}_{\alpha\beta}(\Gamma, \Gamma_{gh|m}) = \Delta_{\alpha\beta}(\Gamma) - (-1)^{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_C} C^A \Delta_{\alpha\beta A}^C \bar{\mathcal{P}}_C + O(C^2), \quad (\text{A.23})$$

that means the BRST invariant extension of the second-class constraints Θ_α satisfies to the same Poisson bracket relations as ones in (A.1) for $f_{\alpha\beta A}^{CD}(\Gamma) = 0$, but in the phase-space $\mathcal{M}_{\min} \subset \mathcal{M}_{\text{tot}}$ with accuracy up to second order in ghost coordinates C^A . The BRST-invariant extension of the invertible supermatrix $\|\Delta_{\alpha\beta}(\Gamma)\|$ (2.2): supermatrix $\|\hat{\Delta}_{\alpha\beta}(\Gamma, \Gamma_{gh|m})\|$, obeys to the same generalized antisymmetry property and is invertible as well on the surface $\hat{\Theta} = 0$ in $\mathcal{M}_{\min}$ due to the representation:

$$\text{sdet} \|\hat{\Delta}_{\alpha\beta}(\Gamma, \Gamma_{gh|m})\| \big|_{\hat{\Theta}_\alpha=0} = \text{sdet} \|\Delta_{\alpha\beta}(\Gamma)\| \times \\ \times \text{sdet} \|(\delta_\beta^\alpha - (-1)^{\varepsilon_\sigma + \varepsilon_\beta + \varepsilon_C} (\Delta^{-1})^{\alpha\sigma} C^A \Delta_{\sigma\beta A}^C \bar{\mathcal{P}}_C + O(C^2))\| \big|_{\hat{\Theta}_\alpha=0} \neq 0. \quad (\text{A.24})$$

If the superalgebra of the constraints T_A, Θ_α, H_0 with respect to the Poisson superbracket appears by Poisson-Lie one, i.e. all the structure functions in (A.1), (A.2) are constants, the form of BRST-invariant constraints $\hat{\Theta}_\alpha$ in (A.10) is exact without terms $O(C^2)$ with unchanged elements, $\hat{\Delta}_{\alpha\beta} = \Delta_{\alpha\beta}$.

In general case, the algebra of the second-class constraints $\hat{\Theta}_\alpha$ is open by first-class constraints term $f_{\alpha\beta A}^{CD}(\Gamma) T_D A$. From the Jacobi identities for the Poisson superbrackets of $\Omega_{r|\min}, \hat{\Theta}_\alpha, \hat{\Theta}_\beta$ and of $\mathcal{H}_r, \hat{\Theta}_\alpha, \hat{\Theta}_\beta$ it follows that $\{\hat{\Theta}_\alpha, \hat{\Theta}_\beta\}$ satisfy the same generating equations (4.24) as ones for $\hat{\Theta}_\alpha$.

Note, that considered in the section 4.2 case of BRST-extended second-class constraints for the constrained BRST-BFV Lagrangian formulation for half-integer HS fields with generating equations (4.24) for the superalgebra of Q_C , spin operators $\hat{\sigma}_C^i(g)$ and extended in $\mathcal{H}_C$ off-shell constraints $\hat{T}_i, \hat{T}_{rs}$ and theirs hermitian conjugated, $\hat{T}_i^+, \hat{T}_{rs}^+$:

$$(\hat{T}_i^+, \hat{T}_{rs}^+) = (t_i^+ + \eta_i^+ p_0 - 2q_0 \mathcal{P}_i^+, t_{rs}^+ - \mathcal{P}_s^+ \eta_r - \eta_s^+ \mathcal{P}_r), \quad (\text{A.25})$$

in terms of supercommutators, in fact for vanishing hamiltonian $H_0 = \mathcal{H}_r = 0$, repeats the construction developed in the appendix. Remind, that $\widehat{\sigma}_C^i(g)$ composes the invertible supermatrix $\|\widehat{\Delta}_{ab}(\widehat{\sigma}_C^i(g))\|$ which may be explicitly constructed with respect to one $\|\Delta_{ab}(g_0^i)\|$ in (3.21) by change of g_0^i (3.9) on $\widehat{\sigma}_C^i(g)$ (4.23), (4.25).

Thus, instead of the functional measure, $d\mu(\Gamma_T)$ (2.18), in the definition of the generating functional $Z_\Psi(I)$ (2.17) one should use measure $d\widehat{\mu}(\Gamma_T)$ in

$$Z_\Psi(I) = \int d\widehat{\mu}(\Gamma_T) \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{2} \Gamma_T^P(t) \omega_{T|PQ} \dot{\Gamma}_T^Q(t) - H_{r|\Psi}(t) + I(t) \Gamma_T(t) \right] \right\}, \quad (\text{A.26})$$

$$d\widehat{\mu}(\Gamma_T) = d\Gamma_T \delta(\widehat{\Theta}) \text{sdet}^{\frac{1}{2}} \|\{\widehat{\Theta}_\alpha, \widehat{\Theta}_\beta\}\|, \quad d\widehat{\mu}(\Gamma_T) = \prod_t d\widehat{\mu}(\Gamma_T(t)), \quad (\text{A.27})$$

being invariant with respect to another [than in (2.27)] BRST transformations with generators $\overleftarrow{s}_r = \{\bullet, \Omega_r\}$.

On the operator level, we suppose, first, that the set of Θ_α admits the splitting (and therefore for BRST-extended constraints $\widehat{\Theta}_\alpha$) as well, at least, locally, on two subsystems, which of them appears by the first-class constraints:

$$\Theta_\alpha(\Gamma) \rightarrow \Theta'_\alpha(\Gamma) = \Lambda_\alpha^\beta(\Gamma) \Theta_\beta(\Gamma) = (\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}}), \quad \text{sdet} \|\Lambda_\alpha^\beta\|_{(\Theta_\alpha=T_A=0)} \neq 0, \quad (\text{A.28})$$

for the index α division: $\alpha = (\bar{\alpha}, \underline{\alpha})$ with $\bar{\alpha} = 1, \dots, \frac{1}{2}m$ and $\underline{\alpha} = \frac{1}{2}m + 1, \dots, m$ such that each subsystems $\theta_{\bar{\alpha}}, \theta_{\underline{\alpha}}$ subject to the relations (2.43), (2.44) between themselves and with specific Poisson brackets with H_0 and T_A following from (A.1), (A.2):

$$\{T_A, \theta_{\bar{\alpha}}\} = \bar{f}_{A\bar{\alpha}}^C(\Gamma) T_C, \quad \{T_A, \theta_{\underline{\alpha}}\} = \underline{f}_{A\underline{\alpha}}^C(\Gamma) T_C, \quad (\text{A.29})$$

$$\{\theta_{\bar{\alpha}}, \theta_{\bar{\beta}}\} = \bar{f}_{\bar{\alpha}\bar{\beta}}^\gamma(\Gamma) \theta_{\bar{\gamma}}, \quad \{\theta_{\underline{\alpha}}, \theta_{\underline{\beta}}\} = \underline{f}_{\underline{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\underline{\gamma}}, \quad (\text{A.30})$$

$$\{\theta_{\bar{\alpha}}, \theta_{\underline{\beta}}\} = \check{\Delta}_{\bar{\alpha}\underline{\beta}}(\Gamma) + \check{f}_{\bar{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\bar{\gamma}} + \check{f}_{\bar{\alpha}\underline{\beta}}^\gamma(\Gamma) \theta_{\underline{\gamma}}, \quad \{H_0, \theta_{\bar{\alpha}}\} = \{H_0, \theta_{\underline{\alpha}}\} = 0, \quad (\text{A.31})$$

with invertible $\|\check{\Delta}_{\alpha\beta}(\Gamma)\|$ satisfying to the relations (2.45).

In the Hilbert space $H(Q_r) = H_\Gamma \otimes H_{gh|m}$ for the minimal sector, [with the correspondence $(\Gamma^p, \hat{C}^A, \hat{P}_A) \rightarrow (\hat{\Gamma}^p, \hat{C}^A, \hat{P}_A)$: $[\hat{\Gamma}^p, \hat{\Gamma}^q] = i\hbar\omega^{pq}$, and $[\hat{C}^A, \hat{P}_B] = i\hbar\delta_B^A$, $\omega^{pq} = \text{const}$] the nilpotency for the operator

$$Q_r(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) = \Omega_{r|\min}(\Gamma, \Gamma_{gh|m})|_{(\Gamma, \Gamma_{gh|m}) \rightarrow (\hat{\Gamma}, \hat{\Gamma}_{gh|m})} : \quad Q_r^2 = (1/2)[Q_r, Q_r] = 0 \quad (\text{A.32})$$

holds, with representation respecting the division for Θ_α , and the choice of some qp -, $\hat{C}\hat{P}$ -orderings for $\Theta_\alpha(\hat{\Gamma})$, Q_r and rest operator functions. The operator form for the *generating equations* (A.6), in addition to commutativity of Q_r with operator of the Hamiltonian, $H_r = \mathcal{H}_r(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$: $[Q_r, H_r] = 0$ looks as:

$$\left[Q_r, \widehat{\Theta}_\alpha(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) \right] = 0, \quad \left[H_r, \widehat{\Theta}_\alpha(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) \right] = 0, \quad (\varepsilon, gh)\widehat{\Theta}_\alpha = (\varepsilon_\alpha, 0), \quad (\text{A.33})$$

for superalgebra of $Q_r, H_r, \widehat{\Theta}_\alpha(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ which determines the BRST extended operator second-class constraints $\widehat{\Theta}_\alpha(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ and, in particular, the first-class subsystem $\widehat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$. The latter, according to (A.28), (being true for $\widehat{\Theta}'_\alpha$) have the form

$$\widehat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) = \theta_{\bar{\alpha}}(\hat{\Gamma}) + (-1)^{\varepsilon_{\bar{\alpha}} + \varepsilon_C} \hat{C}^A \bar{f}_{A\bar{\alpha}}^C(\hat{\Gamma}) \widehat{P}_C + O(\hat{C}^2), \quad (\text{A.34})$$

where for $\Lambda_\alpha^\beta(\Gamma) = \delta_\alpha^\beta$ the equality: $f_{A\bar{\alpha}}^C = \bar{f}_{A\bar{\alpha}}^C$ is true. As it follows from the involution relations for $\theta_{\bar{\alpha}}(\Gamma)$ (A.30) and the $[\ , \]$ -analog of the Poisson brackets for $(\hat{\Theta}_\alpha, \hat{\Theta}_\beta)(\Gamma, \Gamma_{gh|m})$ (A.22) the superalgebra of the operators $Q_r, H_r, \hat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ is closed with respect to $[\ , \]$ -multiplication:

$$\left[Q_r, \hat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) \right] = 0, \quad \left[H_r, \hat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m}) \right] = 0, \quad \left[\hat{\theta}_{\bar{\alpha}}, \hat{\theta}_{\bar{\beta}} \right] = \bar{f}_{\bar{\alpha}\bar{\beta}}^\gamma(\hat{\Gamma}) \hat{\theta}_{\bar{\gamma}}, \quad (\text{A.35})$$

for vanishing the functions $f_{\bar{\alpha}\bar{\beta}A}^{CD}(\hat{\Gamma})$ in the operator analog of (A.19) which follows from the Jacobi identity (A.16), but with respect to supercommutator, $[\ , \]$, and for $T_A(\hat{\Gamma}), \theta_{\bar{\alpha}}(\hat{\Gamma}), \theta_{\bar{\beta}}(\hat{\Gamma})$.

The same superalgebra (A.35) will be obviously true for the set of operators $Q, H_{r|\Psi}(\hat{\Gamma}_T) = H_r + (i\hbar)^{-1}[Q, \Psi]$, $\hat{\theta}_{\bar{\alpha}}(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ which corresponds to the total BRST operator, $Q = \Omega_r(\Gamma_T)|_{\Gamma_T \rightarrow \hat{\Gamma}_T}$, and to unitarizing Hamiltonian $H_{r|\Psi}(\hat{\Gamma}_T)$ with gauge Fermion, $\Psi(\hat{\Gamma}_T)$ in the total, for BFM method [29–31], Hilbert space, $H(Q) = H(Q_r) \otimes H_{gh|nm}$. Here the Hilbert subspace $H_{gh|nm}$ generated by the rest fictitious operator pairs, $\hat{\bar{C}}_A, \hat{P}^A, \hat{\pi}_A, \hat{\lambda}^A$ with non-vanishing supercommutators:

$$[\hat{\bar{C}}_A, \hat{P}^B] = i\hbar \delta_A^B, \quad [\hat{\pi}_A, \hat{\lambda}^B] = i\hbar \delta_A^B. \quad (\text{A.36})$$

Following to the results of the sections 2, 4.1 with Corollaries 2, 3 and in agreement with Dirac approach [81, 86] the physical states $|\psi\rangle \in H_\Gamma^{\text{phys}}$ for constrained dynamical system subject to the operator analogs of the relations (A.1), (A.2) from Hilbert subspace of physical vectors $H_\Gamma^{\text{phys}} \subset H_\Gamma$ may be derived according to

Statement 6. A set of the states H_{Θ_α, T_A} : $H_{\Theta_\alpha, T_A} \subset H(Q_r)$ with vanishing ghost number from the Hilbert subspace: $\ker Q_r / \text{Im } Q_r$, with nilpotent BRST-BFM operator Q_r (A.5), (A.32) for the subsuperalgebra of constraints T_A acting in $H(Q_r)$ both being annihilated by the half of the BRST invariant (extended) second-class constraints $\hat{\theta}_{\bar{\alpha}}$ and satisfying to the Schrodinger equation with Hamiltonian H_r is equivalent to the set of the states from the Hilbert subspace $H_\Gamma^{\text{phys}} \subset H_\Gamma$ annihilated by the constraints $T_A, \theta_{\bar{\alpha}}$ and satisfying the Schrodinger equation with Hamiltonian H_0 :

$$H_{\Theta_\alpha, T_A} = \{ |\psi\rangle | (T_A, \theta_{\bar{\alpha}}, i\hbar \partial_t - H_0(\hat{\Gamma})|_{\theta_{\bar{\alpha}}=0}) |\psi\rangle = (0, 0, 0), |\psi\rangle \in H_\Gamma \} \quad (\text{A.37})$$

$$= \{ |\psi_r\rangle | (\hat{\theta}_{\bar{\alpha}}, i\hbar \partial_t - H_r, gh) |\psi_r\rangle = (0, 0, 0), |\psi_r\rangle \in \ker Q_r / \text{Im } Q_r \}. \quad (\text{A.38})$$

Equivalently, in terms of the respective Q_r -complex the equivalence above means that the found set of states, H_{Θ_α, T_A} may be presented as for the Statement 3 as:

$$H_{\Theta_\alpha, T_A} = \{ |\chi_r^0\rangle | (Q_r, \hat{\theta}_{\bar{\alpha}}, (i\hbar \partial_t - H_r) \delta^{l0}, gh) |\chi_r^l\rangle = (\delta |\chi_r^{l-1}\rangle, 0, 0, -l), |\chi_r^l\rangle \in H^l(Q_r) \} \quad (\text{A.39})$$

for $|\chi_r^{-1}\rangle = |\chi^{-1}\rangle = 0, l = 0, 1, \dots, K_r, K_r \in \mathbb{N}$.

Note, the space H_{Θ_α, T_A} may be equivalently presented in terms of respective Q -complex in the Hilbert space $H(Q) = H(Q_r) \otimes H_{gh|nm}$ with help of the triple of operators $(Q, \hat{\theta}_{\bar{\alpha}}, H_{r|\Psi})$ instead of $(Q_r, \hat{\theta}_{\bar{\alpha}}, H_r)$ in (A.39) but for another natural $K \geq K_r$.

The presentation for the generating functional of Green's functions (A.26), (A.27) in terms of BRST-invariant second-class constraints for the dynamical system in question and Statement 6 present the main results of the appendix. In case of topological system, i.e. for $H_0 = 0$, the last result immediately corresponds to the case of constrained BRST-BFV approach for the Lagrangian formulations for HS fields in the section 5 without extracted from x^μ time coordinate and with special presentation for the invertible supermatrix $\|\hat{\Delta}_{\alpha\beta}\|(\hat{\Gamma}, \hat{\Gamma}_{gh|m})$ (A.23) in terms of constrained generalized spin operator $\sigma_C^i \equiv \hat{\sigma}_C^i(g)$ (4.23), (4.25).

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