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Scalar CFTs and their large N limits

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Abstract: We study scalar conformal field theories whose large N spectrum is fixed by the operator dimensions of either the Ising model or the Lee-Yang edge singularity. Using the numerical bootstrap to study CFTs with $S_N \otimes Z_2$ symmetry, we find a series of kinks whose locations approach $(\Delta_\sigma^{\text{Ising}}, \Delta_\epsilon^{\text{Ising}})$ at $N \to \infty$. Setting $N = 4$, we study the cubic anisotropic fixed point with three spin components. As byproducts of our numerical bootstrap work, we discover another series of kinks whose identification with previous known CFTs remains a mystery. We also show that "minimal models" of $\mathcal{W}_3$ algebra saturate the numerical bootstrap bounds of CFTs with S_3 symmetry.

Keywords: Conformal Field Theory, Renormalization Group

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Contents

1 Introduction

Scalar field theories are useful in studying phase transitions and critical phenomena. A large number of these models have been applied to different condensed matter systems to extract the critical exponents [1, 2]. The simplest example among them, the ϕ^4 theory, can be used to study phase transitions with Z_2 symmetry breaking, which includes the Ising model [3].

The critical exponents calculated in field theories are usually based on certain perturbation methods, such as the ϵ-expansion [3] or the large N expansion (see [71] and references therein). As a non-perturbative method, the conformal bootstrap program [4, 5] has proven to be useful in studying two dimensional conformal field theories. It has played an important role in the classification of two dimensional "minimal models" [6]. In higher dimensions, significant progress was made in the seminal work of [7]. There has been a revival of this program since then. An incomplete list of works on the conformal bootstrap and related topics is [8–70].

The numerical bootstrap is applicable even in regions where neither the ϵ-expansion nor large N works very well. For the three dimensional Ising model, it has provided the most precise critical exponents so far [72–74]. For the perturbative regions, the bootstrap result was also shown to agree with the field theory result. For example, the numerical

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bootstrap for the scaling dimensions of operators in the critical $O(N)$ vector model [75, 76] agrees perfectly with the large N calculation based on the scalar theories [71]. The Borel resummation of ϵ-expansion series for scaling dimensions of operators in the critical Ising model also agrees with the bootstrap result [77].

We will study scalar field theories admitting conformal fixed points whose large N behaviour is controlled by another CFT with central charge of order one. The specific models that we will study are closely related to the continuum limit of the Potts model [78]. We would like to first consider a scalar theory with quartic interaction in $4-\epsilon$ dimensions. The model was referred to as a "restricted Potts model", and was used as an intermediate step to study the continuum limit of the Potts model [79]. This model was also recently revisited in [80]. Its Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{g_1}{8}d_{ijm}d_{klm}\phi^i\phi^j\phi^k\phi^l + \frac{g_2}{8}(\phi^i\phi^i)^2 \tag{1.1}$$

The scalars ϕ^i transform in the $n = N-1$ dimensional representation of the symmetric group S_N. The totally symmetric tensor d_{ijk} is invariant under the action of S_N. The name "restricted Potts model" is due to the fact that besides S_N, it also preserves an extra Z_2 symmetry under which all the scalars change their signs. Its symmetry group is therefore slighter bigger than the S_N symmetry of the original Potts model. Suppose we turn on the a trilinear interaction $\frac{1}{3!}d_{ijk}\phi^i\phi^j\phi^k$, the Z_2 symmetry is broken and one gets the model which describes the continuum limit of the Potts model. The second model that we will consider is a ϕ^3 theory in $6-2\epsilon$ dimensions, given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{g}{6}d_{ijk}\phi^i\phi^j\phi^k \tag{1.2}$$

It can also be used to study the Potts model. In close to six dimensions, quartic interactions of scalars are irrelevant. The ϕ^4 terms in (1.1) can be neglected. The N-state Potts model is known to undergo a first order phase transition for large enough N. In accord with this fact, this ϕ^3 theory is known to have a non-unitary fixed point at imaginary coupling g.

The model (1.1) is known to have two extra fixed points other than the free fixed point and the $O(N)$ invariant fixed point where symmetry is enhanced [79]. In section 2, we look at their operator spectrum to set up the background for the later numerical bootstrap study. Taking the large N limit of the ϵ-expansion series for anomalous dimensions and comparing with the corresponding series in the Ising model, it can be seen that the scaling dimensions of all the operators that we have studied approach a limit fixed by the scaling dimensions of operators in the critical Ising model. The non-unitary fixed point of (1.1), on the other hand, has a large N limit whose operator spectrum is fixed by the Lee-Yang edge singularity.

We then employ the numerical bootstrap method to study CFTs with $S_N \otimes Z_2$ global symmetry. We observe that in three dimensions, there indeed exist a series of kinks, whose locations at large N approach a point given by the scaling dimension of the spin operator σ and the thermal operator ϵ in the critical Ising model. This confirms the large N behaviour predicted by the ϵ-expansion. Setting $N = 4$, we are able to observe the famous cubic anisotropic fixed point [81–84] with three component spins. Interestingly, the scaling dimensions of Δ_ϕ and Δ_S do not agree with the $O(3)$ invariant Heinsberg model,

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as opposed to the prediction in [85]. As a byproduct of our numerical bootstrap study, we also discover a series of new kinks. We are, however, not able to identify them with any CFTs with Lagrangian descriptions. By doing the numerical bootstrap with S_3 symmetry in two dimensions, we also show that the "minimal models" of $\mathcal{W}_3$ algebra saturate the numerical bootstrap bound. These results are presented in section 3.

2 Renormalization of scalar theories

2.1 "Restricted Potts model" → Ising model

For the restricted Potts model (1.1), the invariant tensor d_{ijk} can be constructed explicitly according to [79]. It is possible to define a set of "vielbeins" e_i^α with $\alpha = 1 \ldots N$ and $i = 1 \ldots N-1$ through a recursion relation. These vielbeins tell us how a hypertetrahedron with N vertices can be embedded in $N-1$ dimensional space. From a group theory point of view, the N-dimensional representation is reducible, $N = 1 \oplus n$. Take $N = 3$ as an example, the three vielbeins

$$e^1 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad e^2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad e^3 = (0, 1). \tag{2.1}$$

form an equilateral triangle. The symmetric group S_3 consists of all SO(2) rotations that keep this triangle invariant. Using e_i^α, the totally symmetric tensor is defined as

$$d_{ijk} = \sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha, \tag{2.2}$$

The details of the two loop calculation of (1.1) are summarised in appendix A, which is based on the general formula in [86]. In principle, it is easy to extend the result to three loops using the result of [80]. We will however only focus on the two loop results.

The beta functions of this model have, in total, four fixed points

$$\begin{aligned}
\text{free theory}: &\quad g_1 = g_2 = 0, \\
\text{critical O(n) point}: &\quad g_1 = 0, \quad g_2 \neq 0, \\
P_1: &\quad g_1 \neq 0, \quad g_2 \neq 0, \\
P_2: &\quad g_1 \neq 0, \quad g_2 \neq 0,
\end{aligned} \tag{2.3}$$

since $S_N \otimes Z_2$ is a subgroup of O(n), with $n = N - 1$. The O(n) invariant fixed point is also present. We will focus on the two extra new fixed points P_1 and P_2. The scaling dimensions of the operators we have studied are given in table 1 and table 2. The quadratic operators fall into various irreps of the symmetry group S_N (they are clearly Z_2 even), as

$$\mathrm{n} \otimes \mathrm{n} \to \mathrm{S} \oplus \mathrm{A} \oplus \mathrm{n} \oplus \mathrm{T}'. \tag{2.4}$$

Remember that the product of two vector representations of the O(n) group can be decomposed into three irreducible representations following $\mathrm{n} \otimes \mathrm{n} \to \mathrm{S} \oplus \mathrm{A} \oplus \mathrm{T}$. S denotes the O(n) singlet representation, A denotes the antisymmetric tensor representation, and

Operator	Z_2	Δ	$\Delta_{n\to\infty}$
$\phi \in$ n	$-$	(A.5)	$\Delta_\sigma^{\text{Ising}}$
$\phi^2 \in$ S	$+$	(A.6)	$D - \Delta_\epsilon^{\text{Ising}}$
$\phi^4 \in$ S, 1st	$+$	(A.9)	$2 \times (D - \Delta_\epsilon^{\text{Ising}})$
$\phi^4 \in$ S, 2st	$+$	(A.9)	$\Delta_{\epsilon'}^{\text{Ising}}$
$\phi^2 \in$ n	$+$	(A.7)	$\Delta_\epsilon^{\text{Ising}}$
$\phi^2 \in$ T$'$	$+$	(A.8)	$2 \times \Delta_\sigma^{\text{Ising}}$

Table 1. Scaling dimensions of low-lying operators at the fixed point P_1.

Operator	Z_2	Δ	$\Delta_{n\to\infty}$
$\phi \in$ n	$-$	(A.5)	$\Delta_\sigma^{\text{Ising}}$
$\phi^2 \in$ S	$+$	(A.6)	$\Delta_\epsilon^{\text{Ising}}$
$\phi^4 \in$ S, 1st	$+$	(A.10)	$2 \times \Delta_\epsilon^{\text{Ising}}$
$\phi^4 \in$ S, 2st	$+$	(A.10)	$\Delta_{\epsilon'}^{\text{Ising}}$
$\phi^2 \in$ n	$+$	(A.7)	$\Delta_\epsilon^{\text{Ising}}$
$\phi^2 \in$ T$'$	$+$	(A.8)	$2 \times \Delta_\sigma^{\text{Ising}}$

Table 2. Scaling dimensions of low-lying operators at the fixed point P_2.

T' denotes the symmetric traceless representation. For S_N, because of the existence of the invariant tensor d_{ijk}, the T representation of O(n) branches into n and T'. It is interesting to observe that for both fixed points, the scaling dimensions of low lying operators in the large N limit can be expressed in terms of the Ising model spectrum.

2.2 The spectrum of (de)coupled CFTs

We should mention that the large N behaviour can already be partially inferred from combining the result of [80] and the much earlier work of [87, 88] on cubic anisotropic systems. We will explain this point in the present section, and try to better understand the large N limit.

In [80], another ϕ^4 theory was studied, whereby the model was obtained by replacing the $d_{ijm}d_{klm}$ in (1.1) with

$$Q_{ijkl} = \begin{cases} 1, & \text{if } i = j = k = l, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

This model has a long history of being studied [81–84, 89–91], and certain critical exponents are known up to six loops [92]. It preserves a symmetry group which is the generalized symmetric group $S(2,N) = S_N \otimes Z_2^N$. Like (1.1), it also has four fixed points

$$\begin{aligned} \text{free theory}: &\quad g_1 = g_2 = 0, \\ \text{critical O(N) point}: &\quad g_1 = 0, \quad g_2 \neq 0, \\ \text{cubic anisotropic point}: &\quad g_1 \neq 0, \quad g_2 \neq 0, \\ N\text{-fold product of Ising Models}: &\quad g_1 \neq 0, \quad g_2 = 0. \end{aligned} \tag{2.6}$$

It was shown in [80] that certain numbers that appear in the renormalization calculations of both models have the same large N limit (see section 5.1.2), and therefore the two models approach the same limit at large $N \to \infty$.

It is straightforward to work out the spectrum of the N-fold product of CFTs. Suppose a certain CFT preserves the symmetry group G, then N decoupled copies of this CFT preserve the symmetry group $G \wr S_N = S_N \otimes G^N$. The symbol "$\wr$" stands for wreath product, which can be viewed as a shorthand notation. The group G^N acts independently on each copy of the CFTs, while the S_N group interchanges them. Let's first consider only operators which are invariant under the whole group $G \wr S_N$. Suppose the composite CFT has the following conformal primary operators which are invariant under the action of G,

$$O_1, O_2, O_3, \dots . \tag{2.7}$$

The N-fold product then has the following operators which are also invariant under S_N permutations,

$$\mathcal{O}_1 = \frac{1}{\sqrt{N}} \sum_i O_1^i, \quad \mathcal{O}_2 = \frac{1}{\sqrt{N}} \sum_i O_2^i, \quad \mathcal{O}_3 = \frac{1}{\sqrt{N}} \sum_i O_3^i, \quad \dots, \tag{2.8}$$

where space-time indices are suppressed for simplicity. The index i enumerates the CFT copies. Picking two operators from the same copy, take O_1 and O_2 as an example, one can easily make S_N invariant operators of the following form

$$\frac{1}{\sqrt{N^2 - N}} \sum_{i \neq j} O_1^i O_2^j . \tag{2.9}$$

The coefficient in front of the operators is due to normalization. For operators with non-zero spin, the space-time indices need to be arranged properly for them to have definite spin. The condition $i \neq j$ ensures that the composite operator is made of operators from two different copies of the CFTs, so that it would not be renormalised. The summation over $i \neq j$ pairs ensures S_N invariance. If O_1 and O_2 are scalars, we can also construct the following operators

$$\begin{aligned} [\mathcal{O}_1 \mathcal{O}_2]_{n=0,l=1} &= \frac{1}{\sqrt{N^2 - N}} \sum_{i \neq j} \Delta_2 (\partial_\mu O_1^i) O_2^j - \Delta_1 O_1^j (\partial_\mu O_2^i). \\ [\mathcal{O}_1 \mathcal{O}_2]_{n=1,l=0} &= \frac{1}{\sqrt{N^2 - N}} \sum_{i \neq j} \bigg(\frac{\Delta_1}{2\Delta_1 + 2 - D} (\partial^2 O_1^i) O_2^j \\ &\qquad\qquad - \partial_\mu O_1^i \partial^\mu O_2^j + \frac{\Delta_2}{2\Delta_2 + 2 - D} O_1^i (\partial^2 O_2^j) \bigg). \\ &\dots \end{aligned} \tag{2.10}$$

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We have borrowed the notation $[\mathcal{O}_1\mathcal{O}_2]_{n,l}$ for double trace operators in the AdS/CFT context [93–96]. The scaling dimensions of these operators are simply $\Delta = \Delta_1+\Delta_2+2n+l$. The derivatives acting on the operators are carefully arranged so as to ensure that they are conformal primaries. The procedure of choosing an appropriate derivative structure is exactly the same as constructing conformal primaries for "generalized free fields", as studied in [95]. One can also follow it to construct "double trace" conformal primaries with higher spin and twist. Even though we are not aware of it appearing anywhere in the literature, a similar procedure should exist for constructing double trace operators made of operators with non-zero spins. It is also interesting to look at the 4-pt function consisting of identical scalar operators,

$$\begin{aligned}
&\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \\
&\qquad= \frac{1}{N^2}\sum_{i,j,k,l}\langle O^i(x_1)O^j(x_2)O^k(x_3)O^l(x_4)\rangle \\
&\qquad= \frac{1}{N^2}\sum_{i=j\neq k=l}\frac{1}{x_{12}^{2\Delta_O}x_{34}^{2\Delta_O}}+\frac{1}{N^2}\sum_{i=k\neq j=l}\frac{1}{x_{13}^{2\Delta_O}x_{24}^{2\Delta_O}}+\frac{1}{N^2}\sum_{i=l\neq j=l}\frac{1}{x_{14}^{2\Delta_O}x_{23}^{2\Delta_O}} \\
&\qquad\quad+\frac{1}{N^2}\sum_{i=j=k=l}\langle OOOO\rangle \\
&\qquad= \left(1-\frac{1}{N}\right)\left(\frac{1}{x_{12}^{2\Delta_O}x_{34}^{2\Delta_O}}+\frac{1}{x_{13}^{2\Delta_O}x_{24}^{2\Delta_O}}+\frac{1}{x_{14}^{2\Delta_O}x_{23}^{2\Delta_O}}\right)+\frac{1}{N}\langle OOOO\rangle. \qquad (2.11)
\end{aligned}$$

The condition $i=j\neq k=l$ in the second line makes sure that O^i and O^j come from the same CFT copy, while O^k and O^l comes from a different copy. Its contribution to the four-point function therefore reduces to a product of two-point functions. The leading term in the $\frac{1}{N}$ expansion clearly factorises into disconnected two-point functions. It is equivalent to the boundary four-point function given by a free massive scalar with AdS mass $m^2L^2 = -\Delta_\epsilon^{\text{Ising}}(D-\Delta_\epsilon^{\text{Ising}})$ [93, 95]. The sub-leading behaviour receives contributions from both a disconnected piece and a connected piece, which are given by the four-point function of the composite CFT, as denoted by $\langle OOOO\rangle$.

Specialising to the Ising model, the first three S-channel operators with spin-0 and lowest scaling dimensions are

$$\begin{aligned}
&\tfrac{1}{\sqrt{N}}\textstyle\sum_i \epsilon^i, && \Delta = \Delta_\epsilon^{\text{Ising}} \\
&\tfrac{1}{\sqrt{2N^2-2N}}\textstyle\sum_{i\neq j}\epsilon^i\epsilon^j, && \Delta = 2\times\Delta_\epsilon^{\text{Ising}} \\
&\tfrac{1}{\sqrt{N}}\textstyle\sum_i \epsilon'^i, && \Delta = \Delta_{\epsilon'}^{\text{Ising}}. \qquad (2.12)
\end{aligned}$$

They have the same scaling dimension as the S-channel operators[1] at the fixed point P_2. See table 2. The leading operator in the n-channel is simply ϵ_i (after projecting out the $\sum_i \epsilon_i$ which is S_N invariant). These operators will be important for our later conformal bootstrap study.

[1] "S-channel operators" is short for operators transforming in the singlet representation of S_N.

At the cubic anisotropic fixed point of (2.5), the effective Lagrangian becomes [82],

$$g_1 = g^{\text{Ising}} + \mathcal{O}\left(\frac{1}{N}\right), \quad g_2 = \mathcal{O}\left(\frac{1}{N}\right). \tag{2.13}$$

The action of the model becomes N copies of the Ising model action plus certain $\mathcal{O}(\frac{1}{N})$ corrections. It can be shown that this is also true for both the fixed point P_1 and P_2 of (1.1). At large N, the renormalization is clearly dominated by the Ising model coupling, which explains why the scaling dimensions of the Ising operators appear in the spectrum. Explicit comparison of their operator spectrum shows that P_2 approaches the N-fold product of Ising models, which lives in the UV, while P_1 approaches the IR cubic anisotropic fixed point.

The IR fixed points fit into the class studied by Victor Emery in [88]. Their critical exponents are related to Ising critical exponents by [82, 88, 97]

$$\eta = \eta^{\text{Ising}} + \mathcal{O}\left(\frac{1}{N}\right), \quad \nu = \frac{\nu^{\text{Ising}}}{1-\alpha^{\text{Ising}}} + \mathcal{O}\left(\frac{1}{N}\right), \quad \text{and} \quad \alpha = \frac{\alpha^{\text{Ising}}}{1-\alpha^{\text{Ising}}} + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.14}$$

Translated into operator dimensions, this means

$$\Delta_\phi \to \Delta_\sigma^{\text{Ising}}, \quad \text{and} \quad \Delta_{\phi^2\in S} \to D - \Delta_\epsilon^{\text{Ising}}, \tag{2.15}$$

agreeing exactly with table 1. What's more, operators like

$$\frac{1}{\sqrt{N}}\sum_i \epsilon^i \tag{2.16}$$

self average as in the critical O(N) vector model [71]. Their four-point functions are expected to factorise in the large-N limit as in (2.11). The spectrum of S-channel operators should be exactly the same as the N-fold product of Ising models and also fall into the categories of "single trace operators", "double trace operators" and so on. The only modification that one needs to make is the replacement,

$$\Delta_\epsilon^{\text{Ising}} \to D - \Delta_\epsilon^{\text{Ising}}. \tag{2.17}$$

Notice that in table 1 an operator with $D - \Delta_\epsilon^{\text{Ising}}$ is found to be accompanied by a "double trace" operator with the scaling dimension $2 \times (D - \Delta_\epsilon^{\text{Ising}})$. The IR fixed point can be reached from the UV fixed point of the decoupled Ising model by turning on the double trace deformation $\sum_{i\neq j}\epsilon^i\epsilon^j$. This type of flow at large N was studied in the early days of the AdS/CFT correspondence [98–102]. The replacement (2.17) corresponds to the change of boundary conditions for the scalar field in AdS, and does not change its mass since $M^2_{\text{AdS}}L^2 = -\Delta(D-\Delta)$. The exact same phenomenon happens for O(N) vector models. At the free theory limit, the scaling dimension of the first O(N) singlet operator is given by $\Delta[\sum_i \phi^i\phi^i] = 1$, while at the critical O(N) point, its dimension is given by $\Delta = D-1 = 2$, plus $\frac{1}{N}$ corrections.

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Operator	Δ	$\Delta_{n\to\infty}$
$\phi \in n$	(A.12)	$\Delta_{\phi}^{\text{Lee-Yang}}$
$\phi^2 \in S$	(A.12)	$D - \Delta_{\phi}^{\text{Lee-Yang}}$
$\phi^3 \in S$	(A.12)	$\Delta_{\phi^3}^{\text{Lee-Yang}}$

Table 3. Scaling dimensions for continuum N-state Potts from ϕ^3 theory.

2.3 Potts model $\to$ Lee-Yang singularity

Before closing this section, we briefly mention the large N behaviour of the scalar model (1.2), the continuum limit of N-state Potts models. The theory has a non-unitary fixed point at generic N. It was pointed out in [103] that the $N = 1$ limit of the N-state Potts model gives the percolation model. Based on this fact, people have been using (1.2) to calculate the critical exponents of the percolation problem [104, 105]. The three loop renormalization for operator dimensions is summarised in table 3 (see appendix A.2 for more details). By taking the $N \to \infty$ limit, it is clear that the scaling dimensions of operators are fixed by the spectrum of the Lee-Yang edge singularity CFT. It can also be shown that the coupling constant at large N is given by

$$g = g^{\text{Lee-Yang}} + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.18}$$

By the same argument as in the previous section, operators that are invariant under S_N should fall into the categories of "single trace operators", "double trace operators" and so on. The explicit ϵ-expansion result shows that the single trace spectrum is given by the spectrum of the Lee-Yang edge singularity, with the replacement

$$\Delta_{\phi}^{\text{Lee-Yang}} \to D - \Delta_{\phi}^{\text{Lee-Yang}}. \tag{2.19}$$

3 Bootstraping CFTs with S_N or bigger symmetry

3.1 Bootstraping CFTs with $S_N \otimes Z_2$ symmetry

In this section, we show that the fixed point P_1 studied in the previous section can be observed in numerical bootstrap. Conformal bootstrap is based on crossing symmetry and unitarity. Crossing symmetry means that the following two ways of computing four-point functions should lead to equivalent results

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle. \tag{3.1}$$

The lines connecting the operators denote how the operator product expansion (OPE) is performed. This is true for any conformal field theories. Unitarity on the other hand requires all the OPE coefficients $\lambda_{O_1 O_2 O_3}$ to be real.

By assuming certain conditions on the spectrum of operators that appear in the OPE

$$\phi^i \times \phi^j \sim \sum_O O, \tag{3.2}$$

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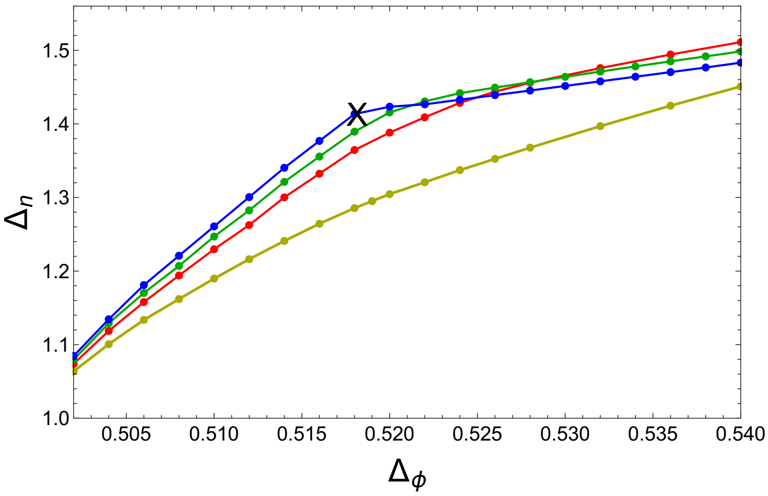

Figure 1. Numerical bootstrap bound on the scaling dimensions of the first n-channel scalar operators in CFTs with $S_N \otimes Z_2$ symmetry (small Δ_ϕ region). Yellow, red, green and blue curves are for $N = 4, 6, 10, 100$ respectively. The black cross denotes the scaling dimension of Ising model operators $(\Delta_\sigma^{\text{Ising}}, \Delta_\epsilon^{\text{Ising}})$. The bounds are obtained at $\Lambda = 19$.

and testing the positivity of $\lambda^2_{\phi\phi O}$, one can then check whether such an assumption is consistent with unitarity and crossing symmetry. We will leave the details of how this method was implemented in appendix B. The conditions that we have assumed for the spectrum are:

- the external operator ϕ^i has scaling dimension Δ_ϕ,
- the first spin-0 operator in the n-channel has scaling dimension greater than or equal to Δ_n,
- all the other operators that appear in $\phi^i \times \phi^j$ have scaling dimensions greater than or equal to the unitarity bound.

We have scanned a certain region of the (Δ_ϕ, Δ_n) plane and the result is presented in figure 1. The result is obtained by setting $\Lambda = 19$, with the range of spins chosen to be $l \in \{1, \dots 25\} \cup \{49, 50\}$. The region above the curves is excluded, which means there are no unitary CFTs with the assumed spectrum.

For large enough N, a clear kink can be observed in the numerical bootstrap curve. The appearance of kinks in numerical bootstrap is a strong indication of the existence of a conformal field theory. More interestingly, as N increases, the location of the kink approaches the point $(\Delta_\sigma^{\text{Ising}}, \Delta_\epsilon^{\text{Ising}})$, as denoted by the black cross in figure 1. This confirms the prediction from the previous section.

From table 1 and 2, it is clear that (Δ_ϕ, Δ_n) should approach $(\Delta_\sigma^{\text{Ising}}, \Delta_\epsilon^{\text{Ising}})$ for both fixed points P_1 and P_2. We can introduce one extra condition on the assumed spectrum:

- The first spin-0 operator in the S-channel has scaling dimension greater than or equal to $\Delta_n + 0.1$.

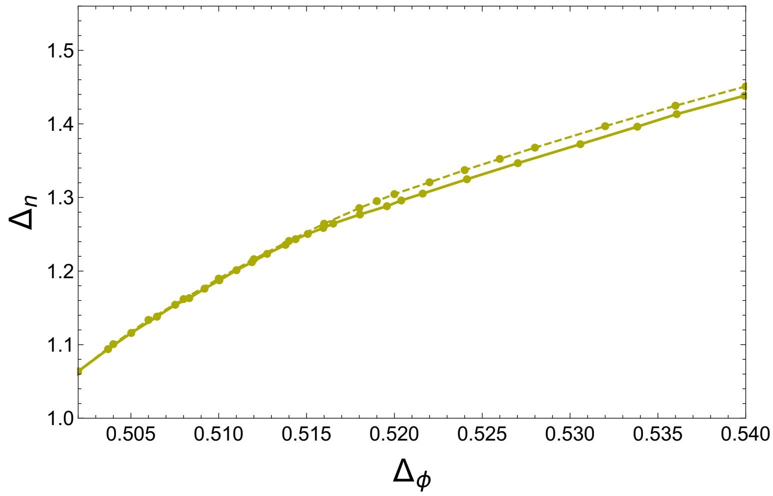

Figure 2. Numerical bootstrap bound on the scaling dimensions of the first n-channel scalar operators in CFTs with $S_4 \otimes Z_2$ symmetry (small Δ_ϕ region). The dashed line is calculated at $\Lambda = 19$, while the solid line is calculated at $\Lambda = 27$. If Λ is further increased, the change of the bounds is no long visible by naked eyes.

At large enough N, this assumption would clearly exclude point P_2, while preserving P_1. Notice that for the fixed point P_1, the leading S-channel operator has scaling dimension $D - \Delta_\epsilon^{\text{Ising}} \approx 1.5874$. While for P_2, the leading S-channel operator has dimension $\Delta_\epsilon^{\text{Ising}} \approx 1.4126$. We have checked that the S_{100} curve has no significant change after introducing this condition. This shows that the fixed point P_1 is located around the kink. At $N = 100$, since the two fixed points are very close to each other in the (Δ_ϕ, Δ_n) plane, it not clear which one of them sits closer to the original bootstrap curve when the above condition is not imposed.

The $N = 4$ case deserves some special attention. The symmetry group $S_4 \otimes Z_2$ is isomorphic to $S_3 \wr Z_2 = S_3 \otimes Z_2^3$ [79]. The two groups clearly have the same order as $4! \times 2 = 3! \times 2^3 = 48$. This means that the "restricted Potts model" (1.1) with $N = 4$ is equivalent to the cubic anisotropic model (2.5) with $N = 3$. From figure 1 itself, it is not clear whether there is a CFT saturating the bootstrap bound or not, since there is no clear kink in the $N = 4$ curve. One can study the corresponding extremal functional [107], as shown in figure 3. The functional is obtained by setting the n-channel gap to saturate the bound on Δ_n computed at $\Lambda = 27$ (which is shown in figure 2).

Clearly the functional is discontinuous at around $\Delta_\phi = 0.5179(2)$. Just like for the 3D Ising model [73], we can treat $(\Delta_\phi, \Delta_S) = (0.5179(2), 1.495(6))$ as our prediction for the scaling dimensions of the corresponding operators. The red error bars are the estimation of the operators' dimensions in the three dimensional O(3) invariant Heisenberg model using the Monte Carlo method [106]. An analysis of the six loop perturbative calculation in the cubic anisotropic model and in the $O(3)$ invariant Heisenberg model shows that their (Δ_ϕ, Δ_S) should agree with each other to high precision [85]. Our non-perturbative result from numerical bootstrap, however, shows this might not be the case.[2] We should emphasis

[2]This difference was noticed recently in [108].

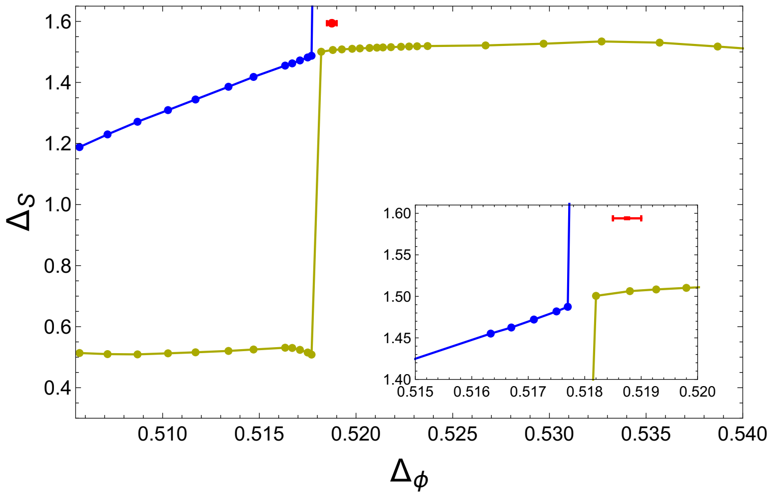

Figure 3. The extremal functional obtained by minimizing the central charge and setting the n-channel gap to saturate the bound on Δ_n computed at $\Lambda = 27$, while the extremal functional itself is calculated at $\Lambda = 23$. The darker yellow and blue dots are the first and second operators respectively appearing in the extremal functional. The red error bars are the scaling dimensions of corresponding operators in the O(3) invariant Heisenberg model obtained using the Monte Carlo Method [106]. We have checked that suppose one increase Λ at which the extremal functional is calculated, the change of dots is no long visible by naked eyes.

here we have assumed that the discontinuity in the extremal functional is caused by the cubic anisotropic fixed point rather than an unknown CFT. This assumption need to be confirmed by measuring the critical exponents of the cubic anisotropic fixed point using either Monte Carlo simulation, especially by measuring the scaling dimension Δ_n explicitly.

3.2 Other bootstrap results: unidentified kinks

The study in the previous section was focused on the region where Δ_ϕ was close to the unitarity bound. It is straight forward to extend the result to the region with much higher Δ_ϕ. This is presented in figure 4. Surprisingly, for large enough N, we can again observe some kinks in the numerical bootstrap curve. Similar kinks can be found on the bootstrap curve obtained by bounding the S-channel operators in O(N) invariant CFTs [109]. Unlike the CFTs in the previous section, we are not able to find Lagrangian descriptions for them. Instead, we will show that these kinks pass some consistency checks for them to actually be CFTs. Any full-fledged conformal field theory necessarily contains the energy momentum tensor in its spectrum. There should be a spin-2 operator saturating the unitarity bound. If the kinks we observe correspond to actual CFTs, they should not survive when a gap is introduced for the spin-2 operators in the S-channel. This fact is tested by adding the following condition on the assumed spectrum

- the first spin-2 operator in the S-channel has scaling dimension greater than or equal to 3.05,

Taking the $N = 10$ curve as an example, the allowed region for (Δ_ϕ, Δ_n) is presented in figure 5. The solid line corresponds to the result without the above condition, while

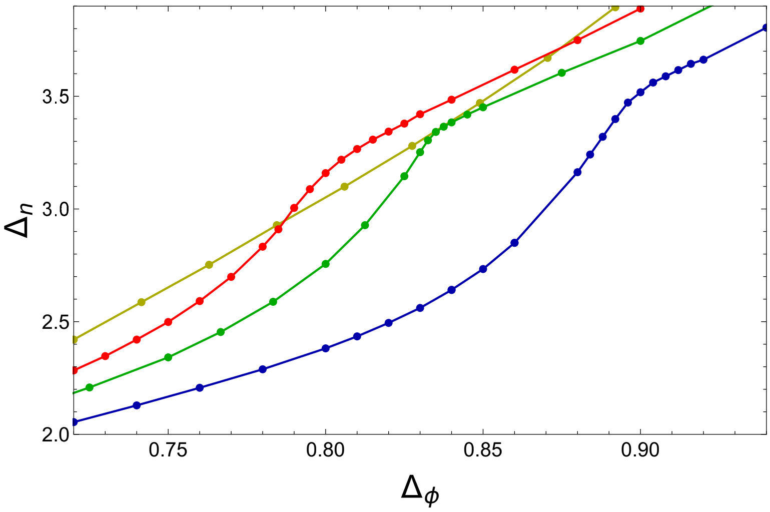

Figure 4. Numerical bootstrap bound on the scaling dimension of the first n-channel scalar operator in CFTs with $S_N \otimes Z_2$ symmetry (large Δ_ϕ region). Yellow, red, green and blue curves are for $N = 4, 6, 10, 100$ respectively. The curves are obtained at $\Lambda = 23$.

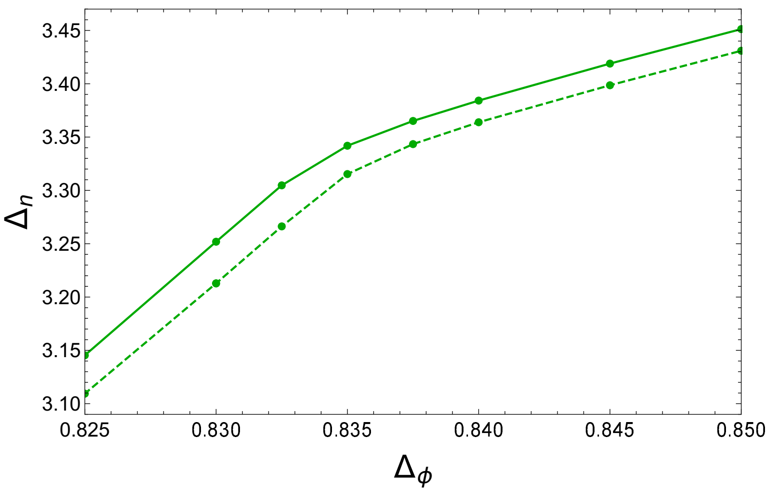

Figure 5. Numerical bootstrap bound on the scaling dimension of the first n-channel scalar operator in CFTs with $S_{10} \otimes Z_2$ symmetry. The solid line corresponds to bounds without a spin-2 gap, while the dashed shows the result when a small gap for the spin-2 operator in the S-channel is introduced.

for the dashed line, the above condition is included. Clearly, when the gap for the spin-2 operator is imposed, the curve moves downward, showing that the energy momentum tensor is present in the spectrum.

3.3 Other bootstrap results: "minimal models" of $\mathcal{W}_3$ algebra

The crossing equations we derived in appendix B apply to CFTs with $S_N \otimes Z_2$ symmetry. The method can also be easily generalized to study CFTs with S_N symmetry, simply by changing the assumptions on the spectrum to be

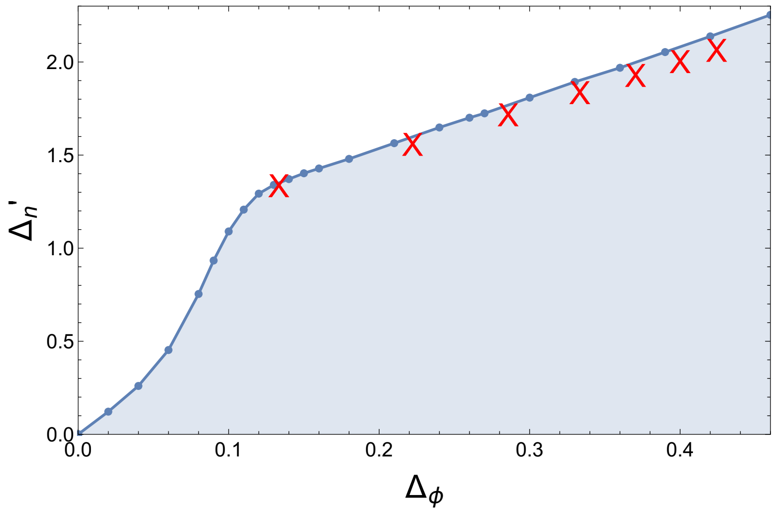

Figure 6. Numerical bootstrap bound on the scaling dimension of the second n-channel scalar operator in CFTs with S_3 symmetry. The crosses correspond to minimal models with $\mathcal{W}_3$ algebra. The first cross to the left is the 3-state Potts model.

- the external operator ϕ^i has scaling dimension Δ_ϕ,
- the first spin-0 operator in the n-channel has scaling dimension Δ_ϕ, while the second spin-0 operator in the n-channel has scaling dimension greater than or equal to Δ_n',
- all other operators that appear in $\phi^i \times \phi^j$ have scaling dimensions greater than or equal to the unitarity bound.

Notice since d_{ijm} is an invariant tensor of the S_N group (which is not invariant under $S_N \otimes Z_2$), the scalar operator ϕ^i would appear in its own OPE, $\phi^i \times \phi^j \sim d_{ijk}\phi^k$.

We have studied the allowed region of (Δ_ϕ, Δ_n') for CFTs with S_3 symmetry in two space-time dimensions. This result is presented in figure 6. We found that "minimal models" of $\mathcal{W}_3$ algebra, as classified in [110], saturate the unitarity bound. The $\mathcal{W}_3$ algebra is an extension of the Virasoro algebra introduced by Zamolodchikov in [111]. It contains the Virasoro algebra as a subalgebra. Besides the usual spin-2 operators L_n, the $\mathcal{W}_3$ algebra contains spin-3 operators W_n which satisfy non-trivial commutation relations with L_n and among themselves. Like for the Virasoro algebra, "minimal models" here means the fusion rules of the models consist of a finite number of irreducible representations of $\mathcal{W}_3$. It was shown in [110] that all these models have a global Z_3 symmetry, therefore, taking into account complex conjugation of complex scalars, one gets the symmetric group $S_3 = Z_3 \otimes Z_2$. The central charges of these models and the scaling dimensions of their $\mathcal{W}_3$

irreducible representations are given by

$$C_p = 2\left(1 - \frac{12}{p(p-1)}\right) \tag{3.3}$$

$$\Delta\left[\Phi\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}\right] = \frac{1}{12p(p+1)}\Big(3((p+1)(n+n') - p(m+m'))^2 + ((p+1)(n-n') - p(m-m'))^2 - 12\Big), \tag{3.4}$$

where m, n, m', n' and p are positive integers whose ranges are $n + n' \leq p - 1$, $m + m' \leq p$ and $p \geq 4$. The horizontal and vertical axis in figure 6 corresponds to operators with

$$\Delta_\phi = 2 \times \Delta\left[\Phi\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right] = \frac{2(p-3)}{3(p+1)}, \quad \text{and} \quad \Delta'_n = 2 \times \Delta\left[\Phi\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\right] = \frac{4(2p-3)}{3(p+1)}, \tag{3.5}$$

respectively. They satisfy

$$\Delta'_n = \frac{5}{2}\Delta_\phi + 1, \tag{3.6}$$

which saturates the numerical bootstrap bound. It was discovered in [112] that minimal models of the Virasoro algebra also saturate the numerical bootstrap bound for CFTs with Z_2 symmetry. It is interesting to observe that minimal models of the $\mathcal{W}_3$ algebra also share the same feature. It would be interesting to extend this result to other $\mathcal{W}$-algebras.

4 Discussion

We have shown that there exist two series of conformal fixed points approaching the (de)coupled Ising model and the Lee-Yang edge singularity respectively in the large N limit. It would be interesting to understand whether it is possible to replace the large N limit by other CFTs such as the XY-model, Heisenberg model, etc. A naive guess is the following. The CFTs that approach the Lee-Yang edge singularity have the symmetry group $S_N \otimes \mathbb{1}$, while the CFTs that approach the Ising model have the symmetry group $S_N \otimes Z_2$. It is therefore natural to consider scalar models with symmetry group $S_N \otimes G$, as a candidate for large N CFTs that approach a CFT with symmetry group G. We leave this for future investigation.

In section 3.1, we have shown that one can observe the fixed point P_1 in the numerical bootstrap curve. It would be interesting to study its spectrum more carefully. The best way to do this is probably by first studying the possibility of isolating this fixed point using mixed correlator bootstrap, along the lines of [74, 76, 113, 114]. For the $N = 4$ special case, a further comparison with experiment or Monte Carlo would also be interesting. It is also desirable to try to extract the $\mathcal{O}(1/N)$ corrections to the operator dimensions and compare them with our numerical bootstrap result. Since the $\mathcal{O}(1/N)$ effect receives contributions from all orders in the ϵ-expansion, a proper resummation is necessary. Finally, it would be interesting to investigate the possibility of performing a proper large-N calculation like in the O(N) vector model (see [71] for a review).

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Before we close, let's think about the large N (de)coupled CFTs in the context of the AdS/CFT correspondence. As explained in section 2.2, the large N spectrum of (de)coupled CFTs naturally breaks down into the categories of "single trace operators", "double trace operators" and so on. It is not yet clear what are the necessary and sufficient conditions for a CFT to have a weakly coupled dual description in AdS [93, 94, 96, 116–118]. As conjectured in [93], besides large N factorization, any CFT with an Einstein-like local bulk dual description must also have a large gap for all single trace operators with spin higher than 2. This is clearly not the case for the large N limit of decoupled CFTs. As shown in section 2.2, the operators that can be interpreted as "single trace" operators are simply the S-channel operators of the component CFT, which clearly contains operators with arbitrary spin. If the dual theory indeed exists, it should be more similar to Vasiliev's higher spin theory [119, 120]. However, since the CFT operators do not saturate the unitarity bound, higher spin symmetry is clearly broken in this case.

Acknowledgments

We would like to thank Youjin Deng, Zhijin Li, Ziyang Meng, Vladislav Vaganov and Yi Pang for helpful discussion and comments. The work of NS is supported by ITP-CAS. The numerical computations in this work are partially supported by HPC Cluster of SKLTP/ITPCAS.

A Renormalization of scalar field theory

A.1 3-loop renormalization of generic ϕ^4 theory in $4-\epsilon$ dimensions

Suppose a group preserves a totally symmetric invariant tensor d_{ijk}, one can define the following constants $\{T_2, T_3, T_5, T_{71}, T_{72}\}$ as [105]

$$\begin{aligned}
d_{i_1 i_3 i_4} d_{i_2 i_3 i_4} &= T_2 \delta_{i_1 i_2} \\
d_{i i_1 i_2} d_{j i_1 i_3} d_{k i_2 i_3} &= T_3 d_{ijk} \\
d_{i i_1 i_2} d_{j i_3 i_4} d_{k i_5 i_6} d_{i_1 i_3 i_5} d_{i_2 i_4 i_6} &= T_5 d_{ijk} \\
d_{i i_1 i_2} d_{j i_3 i_4} d_{k i_5 i_6} d_{i_1 i_3 i_7} d_{i_2 i_5 i_8} d_{i_4 i_6 i_9} d_{i_7 i_8 i_9} &= T_{71} d_{ijk} \\
d_{i i_1 i_2} d_{j i_3 i_4} d_{k i_5 i_6} d_{i_1 i_3 i_7} d_{i_2 i_5 i_8} d_{i_4 i_8 i_9} d_{i_6 i_7 i_9} &= T_{72} d_{ijk} \,.
\end{aligned} \tag{A.1}$$

Using the general formula summarised in [86], we can calculate the scaling dimensions of operators in $4-\epsilon$ up to two loop order. For a scalar field theory given by the Lagrangian (1.1), we get the beta function

$$\begin{aligned}
\beta_1 &= -\epsilon g_1 + \frac{1}{C}\left[A_1 g_1^2 + A_2 g_1 g_2 + A_3 g_2^2 + A_4 g_1^3 + A_5 g_1^2 g_2 + A_6 g_1 g_2^2 + A_7 g_2^3\right] \\
\beta_2 &= -\epsilon g_2 + \frac{1}{C}\left[B_1 g_1^2 + B_2 g_1 g_2 + B_3 g_2^2 + B_4 g_1^3 + B_5 g_1^2 g_2 + B_6 g_1 g_2^2 + B_7 g_2^3\right]
\end{aligned} \tag{A.2}$$

with the coefficient given by table 4.

A_1	$16\pi^2 nT_2^2+96\pi^2 nT_3T_2+256\pi^2 nT_3^2+64\pi^2 nT_5-288\pi^2 T_2^2-192\pi^2 T_3T_2+512\pi^2 T_3^2+128\pi^2 T_5$
A_2	$192\pi^2 nT_2+384\pi^2 nT_3-384\pi^2 T_2+768\pi^2 T_3$
A_3	0
A_4	$nT_2^3+2nT_3T_2^2-20nT_3^2T_2-8nT_5T_2-64nT_3^3-32nT_3T_5-32nT_{71}+30T_2^3+92T_3T_2^2+88T_3^2T_2$ $+16T_5T_2-128T_3^3-64T_3T_5-64T_{71}$
A_5	$-32nT_2^2-112nT_3T_2-192nT_3^2-48nT_5+256T_2^2+64T_3T_2-384T_3^2-96T_5$
A_6	$-5n^2T_2-10n^2T_3-72nT_2-184nT_3+164T_2-328T_3$
A_7	0
B_1	$128\pi^2 T_2^3+320\pi^2 T_3T_2^2-128\pi^2 T_3^2T_2-128\pi^2 T_5T_2$
B_2	$64\pi^2 nT_2^2+128\pi^2 nT_3T_2-128\pi^2 T_2^2+256\pi^2 T_3T_2$
B_3	$16\pi^2 n^2T_2+32\pi^2 n^2T_3+96\pi^2 nT_2+320\pi^2 nT_3-256\pi^2 T_2+512\pi^2 T_3$
B_4	$-16T_2^4-76T_3T_2^3-112T_3^2T_2^2+32T_3T_5T_2+64T_{71}T_2$
B_5	$-5nT_2^3-20nT_3T_2^2-20nT_3^2T_2-86T_2^3-240T_3T_2^2+56T_3^2T_2+96T_5T_2$
B_6	$-44nT_2^2-88nT_3T_2+88T_2^2-176T_3T_2$
B_7	$-9n^2T_2-18n^2T_3-24nT_2-120nT_3+84T_2-168T_3$
C	$256\pi^4\left(nT_2+2nT_3-2T_2+4T_3\right)$

Table 4. Coefficients that appear in β function.

The anomalous dimensions are given by

$$\begin{aligned}
\gamma_\phi &= \frac{g_2^2(n+2)+g_1^2T_2^2+2g_1T_2\left(g_1T_3+2g_2\right)}{1024\pi^4},\\
\gamma_{\phi^2\in S} &= \frac{16\pi^2\left(g_2(n+2)+2g_1T_2\right)-3\left(g_2^2(n+2)+g_1^2T_2^2+2g_1T_2\left(g_1T_3+2g_2\right)\right)}{256\pi^4},\\
\gamma_{\phi^2\in n} &= \frac{\left(g_1T_2+2g_1T_3+2g_2\right)}{16\pi^2}\\
&\quad -\frac{g_2^2(n+6)+g_1^2\left(T_3T_2+6T_3^2+2T_5\right)+8g_2g_1\left(T_2+T_3\right)}{256\pi^4},\\
\gamma_{\phi^2\in T'} &= \frac{1}{256\pi^4(n-2)(n+1)}\Bigg(-g_2^2((n-2))(n+1)(n+6)-4g_1g_2\left(\left(n^2+n-6\right)T_2\right)\\
&\quad +32\pi^2\left(g_1(n-2)T_2-2g_1nT_3+g_2(n-2)(n+1)\right)\\
&\quad +g_1^2\left(4n\left(3T_3^2+\mathrm{T5}\right)+(6-7n)T_2^2-4(n-3)T_3T_2\right)\Bigg)\ldots. \qquad \text{(A.3)}
\end{aligned}$$

Since the symmetric group S_N also preserves a totally symmetric invariant tensor d_{ijk}, they fall into the type of models that can be calculated using the above formulas. Using the explicit construction of d_{ijk} in [79], it is easy to calculate the constants that appear in (A.1), they are

$$\begin{aligned}
T_2 &= \frac{(n-1)(n+1)^2}{n^3},\\
T_3 &= \frac{(n-2)(n+1)^2}{n^3},
\end{aligned}$$

$$
\begin{aligned}
T_5 &= \frac{\left((n-2)^2+1\right)(n+1)^4}{n^6}, \\
T_{71} &= \frac{(n+1)^6\left((n+1)^3-9(n+1)^2+29(n+1)-32\right)}{n^9}, \\
T_{72} &= \frac{(n-2)(n+1)^6\left((n+1)^2-6(n+1)+11\right)}{n^9}.
\end{aligned} \tag{A.4}
$$

Plugging them into (A.2), solving $\beta_1 = \beta_2 = 0$, we find the four fixed points in (2.3). The free fixed point is not renormalised. For other points, we can use (A.3) to get the spectrum. For ϕ, we have

$$
\begin{aligned}
\Delta_\phi^{O(N)} &= 1 - \frac{\epsilon}{2} + \frac{(n+2)\epsilon^2}{4(n+8)^2} + \dots, \\
\Delta_\phi^{P_1} &= 1 - \frac{\epsilon}{2} + \frac{\left(n^2+8n+7\right)\epsilon^2}{108(n+3)^2} + \dots, \\
\Delta_\phi^{P_2} &= 1 - \frac{\epsilon}{2} + \frac{\left(n^4-9n^3+31n^2-45n+22\right)\epsilon^2}{108\left(n^2-5n+8\right)^2} + \dots.
\end{aligned} \tag{A.5}
$$

For the quadratic operator in the S-channel, we have

$$
\begin{aligned}
\Delta_{\phi^2\in S}^{O(N)} &= 2 - \frac{6\epsilon}{n+8} + \frac{(n+2)(13n+44)\epsilon^2}{2(n+8)^3} + \dots \\
\Delta_{\phi^2\in S}^{P_1} &= 2 - \frac{(n+7)\epsilon}{3n+9} - \frac{\left(19n^4-182n^3+672n^2+2614n+1741\right)\epsilon^2}{162(n-5)(n+3)^3} + \dots \\
\Delta_{\phi^2\in S}^{P_2} &= 2 - \frac{2\left(n^2-6n+11\right)\epsilon}{3\left(n^2-5n+8\right)} + \\
&\quad + \frac{\epsilon^2}{162(n-5)\left(n^2-5n+8\right)^3}\Big(19n^7-293n^6+2058n^5-8724n^4+23565n^3 \\
&\qquad -38823n^2+34622n-12424\Big) + \dots
\end{aligned} \tag{A.6}
$$

For the quadratic operator in the n-channel, we have

$$
\begin{aligned}
\Delta_{\phi^2\in n}^{O(N)} &= 2 + \left(\frac{2}{n+8}-1\right)\epsilon + \frac{\left(-n^2+18n+88\right)\epsilon^2}{2(n+8)^3} + \dots \\
\Delta_{\phi^2\in n}^{P_1} &= 2 - \frac{2(n+4)\epsilon}{3(n+3)} + \frac{\left(19n^4+208n^3-318n^2-1616n-1109\right)\epsilon^2}{162(n-5)(n+3)^3} + \dots \\
\Delta_{\phi^2\in n}^{P_2} &= 2 - \frac{2\left(n^2-5n+9\right)\epsilon}{3\left(n^2-5n+8\right)} + \frac{\epsilon^2}{162(n-5)\left(n^2-5n+8\right)^3}\Big(19n^7-379n^6+3200n^5 \\
&\qquad -15310n^4+45287n^3-82423n^2+84478n-37176\Big) + \dots
\end{aligned} \tag{A.7}
$$

For the quadratic operator in the T'-channel, we have

$$
\Delta_{\phi^2\in T'}^{O(N)} = 2 + \left(\frac{2}{n+8}-1\right)\epsilon + \frac{\left(-n^2+18n+88\right)\epsilon^2}{2(n+8)^3} + \dots
$$

$$
\begin{aligned}
\Delta^{P_1}_{\phi^2\in T'} &= 2+\left(\frac{2}{3(n+3)}-1\right)\epsilon-\frac{\left(-3n^4+112n^3+102n^2+864n+1741\right)\epsilon^2}{162(n-5)(n+3)^3}+\dots\\
\Delta^{P_2}_{\phi^2\in T'} &= 2+\left(\frac{2}{3\left(n^2-5n+8\right)}-1\right)\epsilon\\
&\quad+\frac{\epsilon^2}{162(n-5)\left(n^2-5n+8\right)^3}\Big(3n^7-57n^6+548n^5-2938n^4+9239n^3\\
&\qquad\qquad -17917n^2+21242n-12424\Big)+\dots. \qquad \text{(A.8)}
\end{aligned}
$$

The scaling dimension of the quartic operator can be calculated using the eigenvalue of the matrix $\frac{\partial\beta_i}{\partial\lambda_j}$. For P_1, the final result turns out to be

$$
\begin{aligned}
\Delta^{P_1}_{\phi^4\in S,\text{ 1st}} &= 4-\frac{2(n+7)\epsilon}{3(n+3)}\\
&\quad-\frac{\left(19n^5+167n^4-602n^3+3430n^2+13127n+8947\right)\epsilon^2}{81(n-5)(n+3)^3(n+7)}+\dots\\
\Delta^{P_1}_{\phi^4\in S,\text{ 2nd}} &= 4+\frac{\left(17n^3+165n^2+855n+1139\right)\epsilon^2}{27(n+3)^2(n+7)}+\dots \qquad \text{(A.9)}
\end{aligned}
$$

while for the point P_2

$$
\begin{aligned}
\Delta^{P_2}_{\phi^4\in S,\text{1st}} &= 4-\frac{4\left(n^2-6n+11\right)\epsilon}{3\left(n^2-5n+8\right)}\\
&\quad+\frac{(n-1)\epsilon^2}{81(n-5)\left(n^2-6n+11\right)\left(n^2-5n+8\right)^3}\Big(19n^8-388n^7+3583n^6\\
&\quad-19740n^5+71463n^4-174366n^3+278375n^2-262778n+110744\Big)+\dots\\
\Delta^{P_2}_{\phi^4\in S,\text{ 2nd}} &= 4+\frac{\epsilon^2}{27\left(n^2-6n+11\right)\left(n^2-5n+8\right)^2}\Big(-13402+23259n-17709n^2\\
&\qquad +7572n^3-1920n^4+273n^5-17n^6\Big)+\dots \qquad \text{(A.10)}
\end{aligned}
$$

It is useful to record the renormalization of the Ising model here for comparison [2]:

$$
\begin{aligned}
\Delta^{\text{Ising}}_{\sigma} &= 1-\frac{\epsilon}{2}+\frac{\epsilon^2}{108}+\dots,\\
\Delta^{\text{Ising}}_{\epsilon} &= 2-\frac{2}{3}\epsilon+\frac{19}{162}\epsilon^2+\dots,\\
\Delta^{\text{Ising}}_{\epsilon'} &= 4-\frac{17}{27}\epsilon^2+\dots. \qquad \text{(A.11)}
\end{aligned}
$$

A.2 3-loop renormalization of generic ϕ^3 theory in $6-2\epsilon$ dimensions

Three loop renormalization of generic ϕ^3 theory in $D=6-2\epsilon$ was studied by [104, 121]. The four loop result was obtained more recently in [105, 122], where they have also studied the renormalization of the Potts model and the Lee-Yang edge singularity. The authors did not present the result for the N-state Potts model with generic N, but rather focused on the $N \to 1$ limit to study the percolation problem. For the reader's convenience, we will note the generic N result here. Plugging (A.4) into the formulas in [105], one can easily get

$$
\begin{aligned}
\Delta_\phi =& \, 2 - \frac{2(5n-11)\epsilon}{3(3n-7)} - \frac{2(n-1)\left(43n^2-171n+206\right)\epsilon^2}{27(3n-7)^3} \\
&+ \frac{(n-1)\epsilon^3}{243(3n-7)^5}\Big(15552n^4\zeta(3) - 8375n^4 - 129600n^3\zeta(3) + 68025n^3 \\
&\qquad +466560n^2\zeta(3) - 210179n^2 - 829440n\zeta(3) + 300903n \\
&\qquad +580608\zeta(3) - 187238\Big), \\
\Delta_{\phi^2\in S} =& \, 4 - \frac{8(n-4)\epsilon}{3(3n-7)} + \frac{2\left(43n^3-247n^2+857n-653\right)\epsilon^2}{27(3n-7)^3} \\
&+ \frac{1}{243(3n-7)^5}\epsilon^3\Big(-15552n^5\zeta(3) + 8375n^5 + 28512n^4\zeta(3) - 66665n^4 \\
&\qquad -207360n^3\zeta(3) + 163514n^3 + 1467072n^2\zeta(3) \\
&\qquad -224126n^2 - 2887488n\zeta(3) + 450911n + 1614816\zeta(3) - 332009\Big), \\
\Delta_{\phi^3\in S} =& \, 6 + \frac{2\left(-125n^2+544n-671\right)\epsilon^2}{9(3n-7)^2} + \frac{\epsilon^3}{81(3n-7)^4}\Big(38880n^4\zeta(3) + 36755n^4 \\
&\quad -316224n^3\zeta(3) - 319602n^3 + 1187136n^2\zeta(3) + 1123920n^2 - 2265408n\zeta(3) \\
&\quad -1831190n + 1687392\zeta(3) + 1097253\Big).
\end{aligned}
\tag{A.12}
$$

We also note here the renormalization for the Lee-Yang edge singularity for comparison, setting

$$
T_2 = T_3 = T_5 = T_{71} = T_{72} = 1, \tag{A.13}
$$

one gets

$$
\begin{aligned}
\Delta_\phi &= 2 - \frac{10\epsilon}{9} - \frac{86\epsilon^2}{729} + \left(\frac{64\zeta(3)}{243} - \frac{8375}{59049}\right)\epsilon^3 + \mathcal{O}(\epsilon^4), \\
\Delta_{\phi^3} &= 6 - \frac{250\epsilon^2}{81} + \left(\frac{160\zeta(3)}{27} + \frac{36755}{6561}\right)\epsilon^3 + \mathcal{O}(\epsilon^4).
\end{aligned}
\tag{A.14}
$$

It is not necessary to present the dimension of Δ_{ϕ^2}, since it is the conformal descendant of ϕ. As fixed by the equation of motion $\Box\phi \sim \phi^2$, its dimension is $\Delta_{\phi^2} = \Delta_\phi + 2$.

B Bootstrap with S_N symmetry

Using the "vielbeins" e_i^α, besides d_{ijk} defined in (2.2), one can also define the following invariant tensor carrying four indices

$$
Q_{ijkl} = \sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha, \tag{B.1}
$$

They satisfy

$$
d_{ijm}d_{klm} = \frac{n+1}{n}Q_{ijkl} - \frac{(n+1)^2}{n^3}\delta_{ij}\delta_{kl} \tag{B.2}
$$

and

$$d_{ikl}d_{jkl} = \frac{(n-1)(n+1)^2}{n^3}\delta_{ij}. \tag{B.3}$$

The product of two n-dimensional representations can be decomposed as

$$\mathrm{n}\otimes\mathrm{n}\rightarrow \mathrm{S}\oplus\mathrm{A}\oplus\mathrm{n}\oplus\mathrm{T}'.$$

Compared with the product rule for the rotational group $O(n)$, $\mathrm{n}\otimes\mathrm{n}\rightarrow\mathrm{S}\oplus\mathrm{A}\oplus\mathrm{T}$, the T representation of the $O(n)$ group is further decomposed into $\mathrm{n}\oplus$ T', due to the existence of d_{ijk}. One can also define the following linear independent invariant tensors

$$\begin{aligned}
P^{(1)}_{ijkl} &= \frac{1}{n}\delta_{ij}\delta_{kl},\\
P^{(n)}_{ijkl} &= \frac{n^3}{(n-1)(n+1)^2}d_{ijm}d_{klm},\\
P^{(T')}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk}+\frac{1}{2}\delta_{ik}\delta_{jl}-\frac{1}{n}\delta_{ij}\delta_{kl}-\frac{n^3}{(n-1)(n+1)^2}d_{ijm}d_{klm},\\
P^{(A)}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk}-\frac{1}{2}\delta_{ik}\delta_{jl}.
\end{aligned} \tag{B.4}$$

Suppose v_1^i and v_2^i are two vectors carrying indices in the n-dimensional representation of S_N, the tensor

$$P^{(I)}_{ijkl}v_1^k v_2^l \tag{B.5}$$

transforms in the irreducible representation "I" of the S_N group. It can be checked that these projectors satisfy the following relations

$$\begin{aligned}
P^{(I)}_{ijmn}P^{(I)}_{nmkl} &= P^{(I)}_{ijkl},\\
P^{(I)}_{ijkl}\delta_{il}\delta_{jk} &= \mathrm{dim}_I.
\end{aligned} \tag{B.6}$$

where dim_I stands for the dimension of the representation I.

A four-point function in CFTs with S_N global symmetry can be written as

$$\begin{aligned}
&\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_I P^{(I)}_{ijkl}\left(\sum_{\mathcal{O}\in I}\lambda^2_{\mathcal{O}}g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v)\right)\\
&\text{where}\quad I\in\{1^+,n^+,T'^+,A^-\}.
\end{aligned} \tag{B.7}$$

Here $I^\pm$ denotes operators with even(odd) spin and transforms in the irreducible representation "I" of S_N. See [123] for the reason behind the spin choice. Also $g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v)$ is the conformal block which encodes all the kinematics of conformal field theories, which is universal for any CFTs. The dynamical information specific to each CFT, on the other hand, is widely believed to be encoded in the OPE coefficients and the spectrum. An analytical expression for the conformal block in even dimensions was calculated in [124, 125]. Operator product expansions are convergent for conformal field theories, and four-point functions should not depend on how the OPE is preformed, so

$$\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle. \tag{B.8}$$

From this equality we get the following crossing equations

$$\sum_I \sum_{O\in I} \lambda^2_{\phi\phi O} \vec{V}^{(I)}_{\Delta_O, l_O}(u,v) = 0\,, \quad \text{with} \quad I \in \{1^+, n^+, T'^+, A^-\}\,, \tag{B.9}$$

where

$$\vec{V}^{(1^+)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} 0 \\ 0 \\ \frac{F}{n} \\ -\frac{H}{n} \end{pmatrix}, \qquad \vec{V}^{(n^+)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} F \\ 0 \\ \frac{F}{1-n} \\ \frac{H}{n-1} \end{pmatrix}, \tag{B.10}$$

$$\vec{V}^{(T'^+)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} -F \\ \frac{F}{2} \\ \frac{F(n^2-n+2)}{2(n-1)n} \\ \frac{H(n^2-n-2)}{2(n-1)n} \end{pmatrix}, \qquad \vec{V}^{(A^-)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} 0 \\ -\frac{F}{2} \\ \frac{F}{2} \\ \frac{H}{2} \end{pmatrix}. \tag{B.11}$$

Here F and H are short for $F_{\Delta,l}$ and $H_{\Delta,l}$, defined by

$$\begin{aligned} F_{\Delta,l} &= \frac{v^{\Delta_\phi} G_{\Delta,l}(u,v) - u^{\Delta_\phi} G_{\Delta,l}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}, \\ H_{\Delta,l} &= \frac{v^{\Delta_\phi} G_{\Delta,l}(u,v) + u^{\Delta_\phi} G_{\Delta,l}(v,u)}{u^{\Delta_\phi} + v^{\Delta_\phi}}. \end{aligned} \tag{B.12}$$

The logic for numerical bootstrap is to look for a linear functional α such that

$$\begin{aligned} &\alpha(\vec{V}^{(1^+)}_{0,0}) = 1\,, \\ &\alpha(\vec{V}^{(I)}_{\Delta,0}) \geq 0\,, \quad \text{for } \Delta \geq \frac{D-2}{2}\,, \qquad I \in \{1^+, n^+, T'^+\}\,, \\ &\alpha(\vec{V}^{(n^+)}_{\Delta,0}) \geq 0\,, \quad \text{for } \Delta \geq \Delta_n\,, \\ &\alpha(\vec{V}^{(I)}_{\Delta,l}) \geq 0\,, \quad \text{for } \Delta \geq l + D - 2\,, \quad (l = 2,4,6,8,10\dots) \quad \text{and } I \in \{1^+, n^+, T'^+\}, \\ &\alpha(\vec{V}^{(A^-)}_{\Delta,l}) \geq 0\,, \quad \text{for } \Delta \geq l + D - 2\,, \quad (l = 1,3,5,7,9\dots)\,. \end{aligned} \tag{B.13}$$

This realises the conditions imposed on the operator spectrum in section 3.1 to study conformal field theories with $S_N \otimes Z_2$ symmetry. If such a functional can be found, then there is no way for (B.9) to be satisfied with all the $\lambda^2_{\mathcal{O}}$'s being positive. Therefore we conclude that a unitary CFT with $S_N \otimes Z_2$ symmetry and Δ_ϕ must have at least one scalar operator whose dimension is less than Δ_n. For readers interested in the implementation of numerical bootstrap, we refer them to [126] and references therein. The numerical computations in this work were performed using the SDPB package [126]. For the approximation of the conformal blocks, we partially used the code from JuliBoot [127].

Before proceeding, let's recall the dimensions of each representation to be,

$$\dim_S = 1, \quad \dim_n = n, \quad \dim_A = \frac{n(n-1)}{2}, \quad \dim_{T'} = \frac{n(n+1)}{2} - 1 - n. \tag{B.14}$$

For $n = 2$, hence S_3 group, $\dim_{T'} = 0$, one can check that $P^{(T')}_{ijkl} = 0$, and

$$\begin{aligned} P^{(S)}_{ijkl} &= \frac{1}{2}\delta_{ij}\delta_{kl}, \\ P^{(n)}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk} + \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} = \frac{8}{9}d_{ijm}d_{klm}, \\ P^{(A)}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ik}\delta_{jl}. \end{aligned} \tag{B.15}$$

which are the same projectors as for the SO(2) group. Using these projectors, we can derive the following crossing equations

$$\sum_I \sum_{O\in I} \lambda^2_{\phi\phi O} \vec{V}^{(I)}_{\Delta_O, l_O}(u,v) = 0\,, \quad \text{with} \quad I \in \{1^+, n^+, A^-\}\,, \tag{B.16}$$

with

$$\vec{V}^{(1^+)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} 0 \\ F \\ H \end{pmatrix}, \quad \vec{V}^{(n^+)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} F \\ 0 \\ -2H \end{pmatrix}, \quad \vec{V}^{(A^-)}_{\Delta_O,l_O}(u,v) = \begin{pmatrix} -F \\ F \\ -H \end{pmatrix} \tag{B.17}$$

These are exactly the same crossing equations that were used for bootstrapping O(2) invariant CFTs in [75]. However, when studying conformal field theories with S_3 symmetry, since d_{ijm} is an invariant tensor of the S_3 group (which is not invariant under SO(2)), the scalars ϕ^i would appear in its own OPE, $\phi^i \times \phi^j \sim d_{ijk}\phi^k$. We need to search for a linear functional α satisfying (B.13) plus one extra condition

$$\alpha(\vec{V}^{(n^+)}_{\Delta_\phi,0}) \geq 0. \tag{B.18}$$

This is the numerical bootstrap program used in section 3.3.

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