

An F-theory realization of the chiral MSSM with \mathbb{Z}_2 -parity

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ABSTRACT: Using F-theory we construct 4D $\mathcal{N} = 1$ SUGRA theories with the Standard Model gauge group, three chiral generations, and matter parity in order to forbid all dimension four baryon and lepton number violating operators. The underlying geometries are derived by constructing smooth genus-one fibered Calabi-Yau fourfolds using toric tops that have a Jacobian fibration with rank one Mordell-Weil group and $SU(3) \times SU(2)$ singularities. The necessary gauge backgrounds on the smooth fourfolds are shown to be fully compatible with the quantization condition, including positive integer D3-tadpoles. This construction realizes for the first time a consistent UV completion of an MSSM-like model with matter parity in F-theory. Moreover our construction is general enough to also exhibit other relevant \mathbb{Z}_2 charge extensions of the MSSM such as lepton and baryon parity. Such models however are rendered inconsistent by non-integer fluxes, which are necessary for producing the exact MSSM chiral spectrum. These inconsistencies turn out to be intimately related to field theory considerations regarding a UV-embedding of the \mathbb{Z}_2 into a $U(1)$ and the resulting discrete anomalies.

KEYWORDS: Discrete Symmetries, F-Theory, Flux compactifications, Supersymmetric Standard Model

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1 Introduction

One of the major goals of string theory is to provide a possible framework to UV complete the Standard Model of particle physics together with gravity. The web of string theories, connected by various dualities, exhibits a rich landscape of possibilities, and each corner provides an interesting starting point towards this goal, coming with its own benefits and challenges. In particular, F-theory [1] provides a very flexible tool to cover the to date largest set of consistent vacua [2–5] from string theory by using a non-perturbative extension of Type IIB strings (see [6–9] for some reviews).

This flexibility lies in the systematic engineering of gauge theories coupled to gravity using powerful tools of algebraic geometry, that combines the strength of various perturbative string theories. In its early days, F-theory has been employed to engineer the whole Standard Model through a unified gauge group $SU(5)$ on a single divisor [10–12], which then has to be broken to the Minimal Supersymmetric Standard Model (MSSM) by a flux in the hypercharge Cartan subgroup [13]. The localization of the $SU(5)$ made a simple local treatment of the compactification space possible, although it was required to enhance these

models by additional Abelian symmetries to control proton decay and Yukawa textures [14]. The global realization of Abelian gauge symmetries and its connection to the Mordell-Weil group of the fibration, although already pointed out earlier [15], was only further explored later on in [16–18]. These developments kicked off the construction of globally consistent realizations of GUTs together with U(1) symmetries [19–25]. Nevertheless, these constructions were still relying on a GUT breaking mechanism via hypercharge flux [26] which is often technically hard to implement or might lead to vector-like exotics [27] when global Wilson lines are used.

However, the newly gained insights into Abelian symmetries made the direct engineering of the MSSM another valid option [28–31]. But similar as in the GUT picture, the MSSM gauge group per se is not enough to forbid various dangerous proton decay inducing operators and must be considered incomplete. One possibility is to extend the symmetries by another gauged U(1) [29, 31], however, one is then faced with the issue of how to lift the additional massless photon from the spectrum. Alternatively, one can add a discrete symmetry, which from the effective field theory perspective is rather minimally invasive and unproblematic, due to the lack of strong anomalies. However this is not true anymore when coupled to (quantum) gravity, as here black hole arguments [32, 33] suggest an inconsistency of every symmetry that does not have a gauged origin. Hence, if we couple gravity to the MSSM, we can only add very specific discrete symmetries that allow a gauging. One of the mildest additions to the MSSM that forbids baryon and lepton number violating operators at the renormalizable level which is also known to be (discrete) anomaly free [34] is matter parity. Therefore, a valid quest is to explore F-theory constructions of such models.

Fortunately, engineering consistent discrete symmetries in F-theory has advanced significantly [35–40] by considering genus-one fibrations. In such a setup, the fibration lacks any globally defined section, but instead has only multi-sections, which nevertheless give rise to a well-defined F-theory vacuum. Indeed, we will demonstrate that in this framework, the MSSM with an additional \mathbb{Z}_2 symmetry can be engineered straightforwardly with toric methods [20, 37, 39, 41, 42]. Moreover, using recent techniques for flux computations [31, 43, 44], we are able to find multiple consistent vacua that exhibit the exact chiral spectrum of the MSSM. It is worth noting that these techniques are only available because the genus-one fibrations we study have crepant resolutions. If one limits oneself to only studying F-theory using Weierstrass models, one would need to first develop methods of computing fluxes on singular manifolds, since the Weierstrass models associated with our genus-one fibrations has terminal singularities [38, 45–47].

This article is structured as follows: in section 2 we review the need for additional discrete symmetries in the MSSM and their field theory constraints from anomaly considerations. In section 3 we present an F-theory model with matter parity, together with the relevant computational details regarding the geometry and G_4 -fluxes. We then demonstrate that these rather formal techniques, when applied to the most simplistic fibrations over the base $B = \mathbb{P}^3$, can produce a number of different, consistent configurations that has the exact chiral MSSM spectrum. Along the way, we highlight the subtle interplay between discrete gauge anomalies in the field theory and intersection number arithmetic of the ge-

ometry. In section 4 we construct another fibration in which we will look for other parity assignments, and confront these with discrete anomaly cancellation conditions. Section 5 summarizes the results and gives an outlook onto possible future directions.

2 Prelude: R-parity violation in the MSSM

In this section we provide a short summary of R-parity violating operators in the MSSM and their tension with experimental bounds. These operators have also appeared in earlier F-theory constructions that directly engineer the MSSM [28–31]. We review possible \mathbb{Z}_2 symmetries that can forbid these operators.

At the renormalizable level, the superpotential of the MSSM can be written as

$$\begin{aligned} \mathcal{W}_{\text{MSSM}} = & Y_{i,j}^u Q \bar{u} H_u + Y_{i,j}^d Q \bar{d} H_d + Y_{i,j}^e \bar{e} L H_d + \mu H_u H_d \\ & + \beta_i L_i H_u + \lambda_{i,j,k} \bar{e} L L + \lambda'_{i,j,k} Q \bar{d} L + \lambda''_{i,j,k} \bar{u} d \bar{d}. \end{aligned} \quad (2.1)$$

The couplings in the second row violate baryon and lepton number conservation and are severely constrained by experimental bounds. These usually come from proton or lepton decays mediated by massive superpartners. The strongest bounds are related to products of the above couplings and less strong for the single ones. Roughly, they are as follows:

$$\begin{aligned} \lambda'' &< 10^{-(2-6)} \left(\frac{\tilde{m}}{100\text{GeV}} \right), & |\lambda \lambda''| &< 10^{-(3-19)} \left(\frac{\tilde{m}}{100\text{GeV}} \right)^2, \\ \lambda' &< 0.01 \left(\frac{\tilde{m}}{100\text{GeV}} \right), & |\lambda' \lambda''| &< 10^{-27} \left(\frac{\tilde{m}}{100\text{GeV}} \right)^2, \\ \lambda &< 0.01 \left(\frac{\tilde{m}}{100\text{GeV}} \right), \end{aligned} \quad (2.2)$$

where \tilde{m} are the masses of the decay mediating superpartners generated upon SUSY breakdown (see [48, 49] and references therein). These couplings can effectively be forbidden by imposing a discrete symmetry on the MSSM which in the simplest case is a \mathbb{Z}_2 . Differing by the discrete charge assignments of MSSM superfields, the phenomenologically most relevant cases are matter, lepton and baryon parity, as summarized in table 1. For each charge assignment we summarize the field theoretically forbidden tree level couplings in table 2. Both matter parities can forbid all unwanted baryon and lepton number violating couplings while lepton and baryon parity can only forbid their respective ones.

Two comments concerning the discrete charge assignments are in order. First we note that the two matter parities are field theoretically equivalent by mixing the $U(1)_Y$ charge with the \mathbb{Z}_2 . Explicitly, we have

$$\mathbb{Z}_2^{M_1} = \mathbb{Z}_2^{M_2} + 6 U(1)_Y \pmod{2}. \quad (2.3)$$

As all $SU(3)$ triplets have hypercharges that are multiples of $1/3$, the above redefinition is trivial for them. Meanwhile, $SU(2)$ doublets as well as the bifundamental states have hypercharge $1/2$ or $1/6$, which leads to a sign flip upon performing the rotation (2.3). Second we note that matter parity is clearly superior to the other \mathbb{Z}_2 charge assignments when it

G_{SM} Rep	Matter
$(\mathbf{3}, \mathbf{2})_{\frac{1}{6}}$	Q
$(\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}}$	\bar{u}
$(\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}$	\bar{d}
$(\mathbf{1}, \mathbf{2})_{\pm\frac{1}{2}}$	L, H_d, H_u
$(\mathbf{1}, \mathbf{1})_1$	e

	Q	\bar{u}	\bar{d}	L	\bar{e}	H_d	H_u
$\mathbb{Z}_2^{M_1}$	+	-	-	+	-	-	-
$\mathbb{Z}_2^{M_2}$	-	-	-	-	-	+	+
\mathbb{Z}_2^L	+	+	+	-	-	+	+
\mathbb{Z}_2^B	+	-	-	-	+	-	-

Table 1. Summary of gauge quantum numbers of chiral MSSM superfields. The left table shows the gauged quantum numbers, while the table on the right shows the \mathbb{Z}_2 charge assignments for matter parities $\mathbb{Z}_2^{M_1}$, $\mathbb{Z}_2^{M_2}$, lepton parity \mathbb{Z}_2^L and bayron parity \mathbb{Z}_2^B .

	Coupling	$\mathbb{Z}_2^{M_1}$	$\mathbb{Z}_2^{M_2}$	\mathbb{Z}_2^L	\mathbb{Z}_2^B
Yukawa-Couplings	$Q\bar{u}H_u$	✓	✓	✓	✓
	$Q\bar{d}H_d$	✓	✓	✓	✓
	$\bar{e}LH_d$	✓	✓	✓	✓
μ -term	H_uH_d	✓	✓	✓	✓
B-& L-Violation	LH_u	X	X	X	✓
	$Q\bar{d}L$	X	X	X	✓
	$\bar{e}LL$	X	X	X	✓
	$\bar{u}\bar{d}\bar{d}$	X	X	✓	X

Table 2. Summary of allowed and forbidden tree level couplings under different \mathbb{Z}_2 symmetry charge assignments. While the first four terms are required to be present the later four should better be forbidden.

comes to the problematic tree level couplings. However, in SUSY breaking schemes where the sfermion masses are large, the other charge assignments might still be phenomenologically interesting. In such cases certain individual couplings might still be within their mild experimental bounds and thus acceptable. Only the products in (2.2) with the λ'' coupling are strongly constrained and those are still forbidden for both parities.

Discrete anomalies. From a phenomenological perspective, a UV completion of the MSSM with gravity might appear to be a purely ideological endeavor. However, if we include an effective global discrete symmetry, the subtle interplay with gravity could affect even low energy physics. Indeed, it has been argued that global symmetries can be broken by quantum gravity effects [32, 33, 50], thus re-introducing the proton decay inducing operators. The only exception are discrete gauge symmetries, i.e., remnants of broken gauge symmetries. This on the other hand puts constraints on the spectrum and charges of \mathbb{Z}_n symmetries as they must be embeddable into an anomaly free Abelian gauge group. In the case $n = 2$, the non-trivial conditions for an anomaly free discrete symmetry that

are inherited from the unbroken phase are [34]:

$$\mathbb{Z}_2 - G^2 : \sum_{\mathbf{R}_G} Q_{\mathbb{Z}_2}(\mathbf{R}) C_G^{(2)}(\mathbf{R}) = m, \quad m \in \mathbb{Z}, \quad (2.4)$$

where G is the non-Abelian gauge group.¹ Here $C_G^{(2)}(\mathbf{R})$ is the quadratic Casimir invariant of the representation \mathbf{R} , normalized such that is $1/2$ for the fundamental representation. As found in [34, 51] and can be easily checked with the spectrum in table 1, out of the above \mathbb{Z}_2 symmetries, only the two matter parities are anomaly free with just the MSSM spectrum, and hence have a corresponding gauge origin in the UV without chiral exotics. We will see in the next section that in global F-theory models, where gravity is automatically coupled to the gauge sector, these subtle field theoretic constraints reappear as geometric consistency conditions.

3 F-theory construction of 4d MSSM vacua with matter parity

In this section, we will present the details of an F-theory compactification which realizes a three-family MSSM vacuum with an additional \mathbb{Z}_2 symmetry that is identified with the matter parity $\mathbb{Z}_2^{M_2}$. The necessary ingredients, which we will now discuss in the same order, are

1. the generic fiber structure that realizes the Abelian part of the gauge symmetry,
2. codimension one singularities (with resolution) corresponding to the non-Abelian gauge algebra,
3. matter representations associated with codimension two fiber components and G_4 -fluxes,
4. specification of the base and fibration and consistent three-family configurations.

By keeping the base generic for the first three points, we will have the capabilities to analyze a large number of concrete models for the last point.

3.1 Toric hypersurface with two bisection classes

With our phenomenological motivations, we seek to realize an F-theory model whose Abelian gauge sector is $U(1) \times \mathbb{Z}_2$. As studied in [52], a straightforward fiber type that does the job is described in terms of one of the 16 reflexive 2D polygons. Labelled as F_2 in [52], the generic fiber \mathfrak{f} of this geometry is a bi-quadric curve, given as the vanishing of the polynomial

$$p_{F_2} = (b_1 y^2 + b_2 s y + b_3 s^2) x^2 + (b_5 y^2 + b_6 s y + b_7 s^2) x t + (b_8 y^2 + b_9 s y + b_{10} s^2) t^2 \quad (3.1)$$

inside the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $([x : t], [y : s])$. By promoting the coefficients b_i to functions over a complex threefold base B , we obtain a genus-one fibered

¹There are also mixed Abelian discrete symmetries, that are less conclusive due to ambiguities in the charge normalization [34].

fourfold $Y \subset \mathcal{A}$. Here, \mathcal{A} is the ambient space obtained by fibering $\mathbb{P}^1 \times \mathbb{P}^1$ over the same base B . The full configuration is summarized via the commutative diagram:

$$\begin{array}{ccccc}
 \mathfrak{f} & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathcal{A} \\
 & \searrow & & \nearrow & \downarrow \\
 & & Y & \xrightarrow{\pi} & B
 \end{array} . \tag{3.2}$$

The ambient space fibration is specified by two line bundles with divisor classes S_7 and S_9 over the base. They determine the relative “twisting” of the fiber coordinates over the base via the linear equivalence relations

$$[x] = [t] - \overline{K}_B + S_9, \quad [y] = [s] - \overline{K}_B + S_7, \tag{3.3}$$

where \overline{K}_B is the anti-canonical class of the base. For Y to be Calabi-Yau, the coefficients b_i of the polynomial (3.1) have to be sections with the following divisor classes:

$$\begin{aligned}
 [b_1] &= 3\overline{K}_B - S_7 - S_9, & [b_2] &= 2\overline{K}_B - S_9, & [b_3] &= \overline{K}_B + S_7 - S_9, \\
 [b_5] &= 2\overline{K}_B - S_7, & [b_6] &= \overline{K}_B, & [b_7] &= S_7, \\
 [b_8] &= \overline{K}_B + S_9 - S_7, & [b_9] &= S_9, & [b_{10}] &= S_7 + S_9 - \overline{K}_B.
 \end{aligned} \tag{3.4}$$

The genus-one fibration Y has no rational section, but two independent bisection classes (3.3), corresponding to the two hyperplanes of the fiber ambient space $\mathbb{P}^1 \times \mathbb{P}^1$ (modulo vertical divisors). They have been shown to give rise to a $U(1) \times \mathbb{Z}_2$ symmetry in F-theory [52, 53]. A multisection of a genus-one fibration is just as good as a section of an elliptic fibration when it comes to identifying the Kaluza-Klein (KK) $U(1)$ in the dual M-theory compactification [45]. However, in the absence of a section, new subtleties arise in the additional possibilities that an n -section can intersect codimension two fiber components. As has been extensively studied in [36–40], one can understand these intersection numbers as the mod n charge of the matter states under a discrete \mathbb{Z}_n symmetry, which in the M-theory phase is mixed with the KK- $U(1)$ to give rise to the massless Abelian vector field dual to the n -section class.

For the model (3.1), we pick the divisor class

$$D_{\mathbb{Z}_2} := [x] \tag{3.5}$$

to be the one dual to the massless vector of the KK/ \mathbb{Z}_2 combination. Then the other bisection class, $[y] = [s] \bmod D_B$, gives rise to another massless vector that uplifts to a genuine massless $U(1)$ gauge field in F-theory. To be precise, this $U(1)$ is dual to a divisor class that is “orthogonal” to $D_{\mathbb{Z}_2}$, which is

$$D_{U(1)} = [y] - [x] + \frac{1}{2}(\overline{K}_B + S_7 - S_9). \tag{3.6}$$

This modified divisor can be understood as a generalized Shioda map for genus-one fibrations with more than one independent multisection class [53].

As extensively studied in [52], this bisection geometry can be obtained through a complex structure deformation of an elliptic fibration with Mordell-Weil rank two, which in F-theory gives rise to a $U(1)^2$ gauge group. The associated conifold transition corresponds in field theory to the Higgsing of one of these $U(1)$ factors to a \mathbb{Z}_2 by giving vev to a singlet of charge $(0, 2)$. Therefore, we see explicitly that the \mathbb{Z}_2 symmetry we construct via F-theory has a gauged origin, and hence should have a consistent quantum gravity embedding.

3.2 Non-Abelian symmetries with matter parity via tops

We include non-Abelian $SU(3) \times SU(2)$ gauge symmetries using the methods of tops [41, 42]. There are four possibilities for each of the $SU(3)$ and $SU(2)$ tops, cf. appendix C. In the following, we will focus on a combination of $SU(3)$ top 1 and $SU(2)$ top 1. The toric description modifies the ambient space \mathcal{A} to include additional toric divisors $\{f_i\}_{i=0,1,2}$ and $\{e_j\}_{j=0,1}$, which themselves are fibered over codimension one loci $\{w_3\}$ resp. $\{w_2\}$ inside the base B . Their restriction, or intersection, with the likewise modified hypersurface $\hat{Y} \equiv Y_{31}$ are now the exceptional, or ‘‘Cartan’’ divisors that resolve the $SU(3)$ resp. $SU(2)$ singularities over respective codimension one loci on B . The \mathbb{P}^1 fibers of these divisors intersect in the affine Dynkin diagrams of the corresponding Lie algebra.

Explicitly, the modified hypersurface equation is given by the vanishing of a polynomial p_{31} , which is a specialization of the polynomial (3.1) with coefficients

$$\begin{aligned} b_1 &= d_1 e_0 f_1, & b_2 &= d_2 e_0 f_0 f_1, & b_3 &= d_3 e_0 f_0^2 f_1, & b_5 &= d_5 f_1 f_2, \\ b_6 &= d_6, & b_7 &= d_7 f_0, & b_8 &= d_8 e_1 f_1 f_2^2, & b_9 &= d_9 e_1 f_2, & b_{10} &= d_{10} e_1 f_0 f_2. \end{aligned} \tag{3.7}$$

The functions d_i are again sections of line bundles over the base, whose divisor classes are related to those without the top (3.4) via

$$\begin{aligned} [d_1] &= [b_1] - W_2, & [d_2] &= [b_2] - W_2 - W_3, & [d_3] &= [b_3] - W_2 - 2W_3, \\ [d_5] &= [b_5], & [d_6] &= [b_6], & [d_7] &= [b_7] - W_3, \\ [d_8] &= [b_8], & [d_9] &= [b_9], & [d_{10}] &= [b_{10}] - W_3, \end{aligned} \tag{3.8}$$

where we have denoted the classes of $\{w_{2/3}\}$ by $W_{2/3}$. Furthermore, we shall denote the classes of the exceptional divisors by F_i resp. E_j . Though these are strictly speaking classes on the ambient space \mathcal{A} , we will abusively use the same notation for their restrictions to Y_{31} .

Through the toric construction, we can straightforwardly determine the linear equivalence relations (LIN) between the divisors and the Stanley-Reisner ideal (SRI), that is, the set of divisors whose intersection product is trivial in the Chow ring. For Y_{31} , these are

$$\begin{aligned} \text{LIN} &= \{[x] = [t] + E_1 + F_2 - \bar{K}_B + S_9, W_2 = E_0 + E_1, \\ & [s] = [y] + F_1 + F_2 + \bar{K}_B - S_7, W_3 = f_0 + f_1 + f_2\}, \\ \text{SRI} &= \{xt, xe_1, xf_2, ys, yf_0, te_0, e_0f_2, tf_1, sf_1, sf_2\}. \end{aligned} \tag{3.9}$$

In the presence of codimension one reducible fibers, the divisors dual to the KK/\mathbb{Z}_2 and the $U(1)$ vector field needs to be refined, in order for these to be ‘‘orthogonal’’ to each other and to the Cartan $U(1)$ s of the non-Abelian symmetries. Geometrically, this is necessary

because the bisections will intersect the fibers of the exceptional divisors non-trivially. For example, while the bisection $[x]$ intersects only the affine node (the fiber component of E_0) of the $SU(2)$, it intersects the $SU(3)$ divisors non-trivially, namely each of the fibers of F_0 and F_1 once. Physically, it would mean that the W-bosons of $SU(3)$ were charged non-trivially under the \mathbb{Z}_2 , which is of course unacceptable. However, much like in the case of $U(1)$ s, we can add a linear combination of the Cartan divisors to the (bi-)section to correct the intersection numbers [37, 39, 43, 54]. For the case at hand, it can be checked that the correct \mathbb{Z}_2 is given by the divisor

$$D_{\mathbb{Z}_2} = [x] + \frac{1}{3}(2F_1 + F_2). \quad (3.10)$$

Similarly, the modified $U(1)$ generator that is orthogonal to the Cartans as well as the \mathbb{Z}_2 is

$$D_{U(1)} = [x] - [y] - \frac{1}{2}E_1 - \left(\frac{1}{3}F_1 + \frac{2}{3}F_2\right) - \frac{1}{3}W_3 + \frac{1}{2}\overline{K}_B + \frac{1}{2}S_7 - \frac{1}{2}S_9. \quad (3.11)$$

Note that we have also flipped the sign of the bisections compared to (3.6), so that it matches the hypercharge $U(1)_Y$ of the MSSM.

3.2.1 The global gauge group structure

Let us briefly discuss the global group structure of this model. Though our fourfold Y_{31} is not elliptically fibered, we can apply the same intersection number argument employed in [55] to determine the constraints on the charges and non-Abelian representations of matter states arising from M2-branes wrapping fibral curves Γ . Essentially, one employs the fact that multisections as integer divisor classes have integer intersection number with any fiber component. This in turn means that the Abelian charges — given by intersection numbers of Γ with the divisors (3.10) and (3.11)—differ by an integer from the specific fractional linear combinations of the non-Abelian weights — given by the fraction linear combination of exceptional divisors in (3.10) and (3.11). For the $U(1)$, the fractional contributions from the exceptional divisors of both $SU(3)$ and $SU(2)$ are exactly those compatible with the \mathbb{Z}_6 quotient of the continuous part of the gauge algebra [55], namely charge $1/2 \bmod \mathbb{Z}$ for doublets, $2/3 \bmod \mathbb{Z}$ for triplets, and $1/6 \bmod \mathbb{Z}$ for bifundamentals.

For the discrete symmetry, note that triplets will generically have charges quantized in $1/3$ with respect to the divisor in (3.10). Because the generic fiber still has intersection 2 with $D_{\mathbb{Z}_2}$, we can still only interpret the charge under $D_{\mathbb{Z}_2}$ modulo 2. Thus, naively, we would expect that the discrete symmetry is enhanced to a \mathbb{Z}_6 by the presence of the non-Abelian symmetries. But not all charges of the \mathbb{Z}_6 can appear! First, it is clear that $SU(2)$ matter will only be charged under a \mathbb{Z}_2 subgroup, because their intersection numbers with $D_{\mathbb{Z}_2}$ are integer. Furthermore, by the analogous argument as in [55], one can construct an order three central element of $SU(3) \times \mathbb{Z}_6$ which acts trivially on any matter states. Essentially, it follows because $(D_{\mathbb{Z}_2} - 1/3(2F_1 + F_2)) \cdot \Gamma = [x] \cdot \Gamma \in \mathbb{Z}$. Since $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$ has a unique order three subgroup, we conclude that the only non-trivially acting part is the \mathbb{Z}_2 . To infer the charges under it, we can simply multiply all intersection numbers with $D_{\mathbb{Z}_2}$ with three and then take the result modulo 2. In order to differentiate it more easily

from the U(1) charges, we will denote even/odd charges by $+/-$. To summarize, the global gauge group of the F-theory model on Y_{31} is

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_6} \times \mathbb{Z}_2. \quad (3.12)$$

Note that it was already anticipated before in [36] that, by engineering a non-Abelian symmetry algebra \mathfrak{g} inside an n -section fibration, the discrete symmetry can be potentially enhanced to $\mathbb{Z}_{n \times r}$, where r is the order of the center $Z(\mathfrak{g})$. In general, $\mathbb{Z}_{n \times r} \neq \mathbb{Z}_n \times \mathbb{Z}_r$ (namely, whenever n and r are *not* coprime), and the enhancement can be physical. However, due to the mechanism that leads to such an enhancement — namely, the divisor associate with the discrete symmetry is shifted by the Cartan divisors — the resulting global gauge group necessarily has to be non-trivial. For example, there is an SU(2) top constructed over the F_2 polygon that has an enhancement, such that the gauge group is $[\mathrm{SU}(2) \times \mathbb{Z}_4]/\mathbb{Z}_2$ (we have omitted the U(1), which itself has a non-trivial gauge group structure associated with the SU(2), see table 13). We will leave a detailed derivation and classification along the lines of [55] for future work.

3.3 Matter surfaces, fluxes and the chiral spectrum

To specify the chiral spectrum of the F-theory compactification, we need two geometric ingredients, namely the surfaces on which the matter states are localized, and the description of the G_4 -flux in terms of their dual four cycle classes. Based on the techniques first developed in [56, 57] and further advanced in [22, 30, 31, 43, 44], we perform a completely base independent analysis of fluxes and chiralities, which then can be straightforwardly applied to specific fibrations.

3.3.1 Matter surfaces and their homology classes

Through the mapping to its Jacobian [52], we can straightforwardly determine the codimension two loci where the fiber singularities of Y_{31} enhance. These are of them form $\{w_i\} \cap \{g_{\mathbf{R}}\}$ for some polynomials $g_{\mathbf{R}}$:

$$\begin{aligned} g_{\mathbf{2}_1} &= d_{10}d_6^2d_8 - d_{10}d_5d_6d_9 - d_6d_7d_8d_9 + d_5d_7d_9^2 + w_3(d_{10}^2d_5^2 - 2d_{10}d_5 + d_7^2d_8^2), \\ g_{\mathbf{2}_2} &= d_1d_3d_6^2 - d_1d_2d_6d_7 + d_1^2d_7^2 + w_3(d_2^2d_5d_7 - d_2d_3d_5d_6 - 2d_1d_3d_5d_7 + d_3^2d_5^2w_3), \\ g_{\mathbf{3}_1} &= d_1, \\ g_{\mathbf{3}_2} &= d_{10}d_6 - d_7d_9, \\ g_{\mathbf{3}_3} &= d_3d_6^2 - d_2d_6d_7 + d_1d_7^2, \\ g_{\mathbf{3}_4} &= d_6^2d_8 - d_5d_6d_9 + d_1d_9^2w_2. \end{aligned} \quad (3.13)$$

Furthermore, there are two charged singlets with U(1) charge 1, but differ in their \mathbb{Z}_2 charge, which are localized over curves given by complicated ideals $I_{+/-}$.

There is also an uncharged singlet with negative \mathbb{Z}_2 parity [52]. These matter states have the same quantum numbers as right-handed neutrinos. Correspondingly, they interact with Higgs and lepton doublets via perturbatively realized Yukawa couplings in the F-theory geometry. The presence of such massless states in the effective field theory not

rep	$U(1) \times \mathbb{Z}_2$	Cartan charges	parent exceptional	base locus	$\mathbb{Z}_2^{M_2}$ Rep.
$\mathbf{2}_1$	$\left(\frac{1}{2}, -\right)$	$(0, 0 1)$	E_0	$\{w_2\} \cap \{g_{\mathbf{2}_1}\}$	L
$\mathbf{2}_2$	$\left(\frac{1}{2}, +\right)$	$(0, 0 -1)$	E_1	$\{w_2\} \cap \{g_{\mathbf{2}_2}\}$	Higgs
$\bar{\mathbf{3}}_1$	$\left(-\frac{2}{3}, -\right)$	$(0, 1 0)$	F_0	$\{w_3\} \cap \{g_{\mathbf{3}_1}\}$	\bar{d}
$\mathbf{3}_2$	$\left(\frac{2}{3}, +\right)$	$(-1, 1 0)$	F_1	$\{w_2\} \cap \{g_{\mathbf{3}_2}\}$	exotic
$\bar{\mathbf{3}}_3$	$\left(\frac{1}{3}, -\right)$	$(1, -1 0)$	F_2	$\{w_3\} \cap \{g_{\mathbf{3}_3}\}$	\bar{u}
$\bar{\mathbf{3}}_4$	$\left(\frac{1}{3}, +\right)$	$(0, 1 0)$	F_0	$\{w_3\} \cap \{g_{\mathbf{3}_4}\}$	exotic
$(\bar{\mathbf{3}}, \mathbf{2})$	$\left(-\frac{1}{6}, -\right)$	$(0, 1 -1)$	E_1, F_0	$\{w_2\} \cap \{w_3\}$	Q
$\mathbf{1}_1$	$(1, -)$	$(0, 0 0)$	n.a.	$V(I_{(1,-)})$	E
$\bar{\mathbf{1}}_2$	$(-1, +)$	$(0, 0 0)$	n.a.	$V(I_{(1,+)})$	exotic
$\mathbf{1}_3$	$(0, -)$	$(0, 0 0)$	n.a.	$V(I_{(0,-)})$	exotic (ν_R)

Table 3. States and charges associated with the matter surfaces. The polynomials g_i defining the matter curves in the base are in equation (3.13). We have included the identification with the MSSM spectrum, where the \mathbb{Z}_2 is identified with matter parity.

only depends on the flux, but also on the complex structure moduli [44, 58]. However, because it is a real representation, there cannot be any net chirality associated with these matter states. Geometrically, this is reflected in the fact that the components of the I_2 fiber associated with this matter are exchanged via monodromy. Consistently, the transversality conditions (3.15) imply that the intersection product between the flux and this singlet’s matter surface is 0. Hence, we will disregard this representation for the rest of this paper, since we are mainly interested in the chiral spectrum.

Over the codimension two loci $C_{\mathbf{R}} = \{w_i\} \cap \{g_{\mathbf{R}}\}$, the reducible fibers contain localized \mathbb{P}^1 components, giving rise to matter states in the representation \mathbf{R} in F-theory. By fibering one such \mathbb{P}^1 over the curve on the base, one obtains a four-cycle $\gamma_{\mathbf{R}}$, a so-called matter surface associated with a weight \mathbf{w} of a representation \mathbf{R} (or its conjugate). To determine the homology classes of these matter surfaces, we use prime ideals techniques and algorithms detailed in [31], utilizing the computer algebra program `Singular` [59]. The specific states, whose matter surfaces we determine this way, are listed in table 3. Their corresponding matter surface classes are collected in the appendix, cf. table 8.

Note that the only bifundamental matter states we have in this model have odd \mathbb{Z}_2 charge. Therefore, this toric model Y_{31} only allows for an identification of the geometrically realized \mathbb{Z}_2 with the second matter parity $\mathbb{Z}_2^{M_2}$ listed in table 1. We will see in the next section other geometries whose corresponding F-theory model may realize the other parities.

3.3.2 Vertical fluxes from matter surfaces

We now turn to the computation of G_4 -fluxes. In practice, these are expressed through their Poincaré-dual four cycle classes (also denoted by G_4), such that the integral giving the chiral index can also be computed via intersection product:

$$\chi(\mathbf{R}) = \int_{\gamma_{\mathbf{R}}} G_4 = G_4 \cdot [\gamma_{\mathbf{R}}] \tag{3.14}$$

Not all four cycles give rise to consistent fluxes. As is well known by now, the fluxes have to satisfy the so-called transversality conditions [60] in order to uplift from M- to F-theory. These conditions have been generalized in [43] to genus-one fibrations:

$$G_4 \cdot D_B^{(1)} \cdot D_B^{(2)} = 0, \quad G_4 \cdot D_B \cdot [x] = 0 \tag{3.15}$$

for some vertical divisors $D_B^{(i)}$. In addition, the flux must not break the non-Abelian symmetries, which requires

$$G_4 \cdot \text{Ex} \cdot D_B = 0, \tag{3.16}$$

where $\text{Ex} \in \{E_1, F_1, F_2\}$ are the Cartan divisors.

For a fibration over a base threefold B , we can use the quotient ring description to determine a basis of vertical fluxes [31]. For generic choices of fibration and base, i.e., such that no further singularity enhancements are induced whose resolution would introduce further divisors, the space of vertical fluxes is spanned by $U(1)$ -fluxes of the form $D_{U(1)} \cdot F$, where $F \in \pi^*(H^{1,1}(B))$, and five non- $U(1)$ -fluxes. In this paper, we follow the method of [44] and express the non- $U(1)$ -fluxes through so-called matter surfaces fluxes. At this point, there is no technical advantage for this procedure, and we could also use the flux basis provided by the algorithm of [31] to compute the chiralities. However, hoping that in future works we will have the computational power to also determine the vector-like spectrum with the methods of [44], we will collect the necessary input in the appendix B. For now, we content ourselves with a basis for vertical fluxes in terms of matter surfaces.

As the name suggests, the matter surface fluxes are constructed using the matter surfaces $[S_{\mathbf{R}}]$. By construction, these surfaces are orthogonal to any curve in the base:

$$[S_{\mathbf{R}}] \cdot D_B^{(1)} \cdot D_B^{(2)} = 0, \tag{3.17}$$

thus automatically satisfying the first of the transversality conditions (3.15). To satisfy the other as well as the gauge symmetry condition (3.16), we can add correction terms of the form $\text{Ex}_i \cdot D_B + D_B^{(a)} \cdot D_B^{(b)}$ to $[S_{\mathbf{R}}]$. Note that these correction terms will not spoil the condition (3.17). Denoting such corrected matter surfaces by $A(\mathbf{R})$, we can choose a basis

of five of them such that together with the U(1)-fluxes, they span the full space of vertical fluxes. Here, we will use the fluxes associated with $\mathbf{2}_2, \mathbf{3}_2, \bar{\mathbf{3}}_4, (\bar{\mathbf{3}}, \mathbf{2}), \bar{\mathbf{1}}_2$, which are:

$$\begin{aligned}
 A(\mathbf{2}_2) &= [S_{\mathbf{2}_2}] + E_1 \cdot (W_2 + W_3 - 3\bar{K}_B + S_9), \\
 A(\mathbf{3}_2) &= [S_{\mathbf{3}_2}] + \frac{1}{3}(-F_1 + F_2) \cdot (S_7 + S_9 - W_3) - \frac{1}{3}W_3 \cdot [g_{\mathbf{3}_2}], \\
 A(\mathbf{3}_4) &= [S_{\bar{\mathbf{3}}_4}] + \frac{1}{3}(F_1 + 2F_2) \cdot (3\bar{K}_B - S_7 + S_9) - \frac{2}{3}W_3 \cdot [g_{\mathbf{3}_4}], \\
 A((\bar{\mathbf{3}}, \mathbf{2})) &= [S_{(\bar{\mathbf{3}}, \mathbf{2})}] + \frac{1}{3}(F_1 + 2F_2) \cdot W_2 - \frac{1}{2}E_1 \cdot W_3 - \frac{1}{6}W_2 \cdot W_3, \\
 A(\bar{\mathbf{1}}_2) &= [S_{\bar{\mathbf{1}}_2}].
 \end{aligned} \tag{3.18}$$

Including the U(1)-flux, we parametrize the most generic vertical G_4 in this model as

$$G_4 = a_1 A(\mathbf{2}_2) + a_2 A(\mathbf{3}_2) + a_3 A(\mathbf{3}_4) + a_4 A((\bar{\mathbf{3}}, \mathbf{2})) + a_5 A(\bar{\mathbf{1}}_2) + D_{U(1)} \wedge F. \tag{3.19}$$

The chiral indices of matter representations can be straightforwardly computed in the quotient ring description of the vertical cohomology ring. Instead, one can also use the more geometric picture laid out in [44, 58] and relate the chiralities to the homology classes of the Yukawa points. Their rather uninspiring explicit expressions are presented in the appendix, Formula (B.3). As a consistency check, it is straightforward to verify that all 4D continuous gauge anomalies induced by the chiral spectrum are indeed canceled. It would be interesting to reproduce this result also in the weakly coupled type IIB limit of this model, along the lines of [61].

3.4 Concrete three family models

We are now in a position to scan for configurations that admit a three family flux solution. Recall that because the bifundamental states in this geometry have odd \mathbb{Z}_2 charge, the only phenomenological parity extension that is compatible is the matter parity $\mathbb{Z}_2^{M_2}$. Then, consistency with the observed spectrum (cf. table 1) requires to have the following chiral indices of the matter representations:

R	$\mathbf{2}_1$	$\mathbf{2}_2$	$\bar{\mathbf{3}}_1$	$\mathbf{3}_2$	$\bar{\mathbf{3}}_3$	$\bar{\mathbf{3}}_4$	$(\bar{\mathbf{3}}, \mathbf{2})$	$\mathbf{1}_{(1,+)}$	$\mathbf{1}_{(1,-)}$
χ	-3	0	3	0	3	0	-3	0	3

(3.20)

In addition to these chiral indices, we have to ensure the vanishing of the flux-induced D-term of the U(1),

$$\xi \sim G_4 \wedge D_{U(1)} \wedge J_B, \tag{3.21}$$

where J_B is the Kähler form of the base. Note that this expression can be easily computed when we express G_4 in terms of the matter surface fluxes (3.18), because the U(1) generator $D_{U(1)}$ is orthogonal to all the correction terms. Hence, the D-term is just a linear combination of the matter curves times the base's Kähler form, where the coefficients are

	(n_2, n_3, s_7, s_9)	flux $\mathcal{F} = (a_1, a_2, a_3, a_4, a_5, \lambda)$	n_{D3}
$\mathbb{Z}_2^{M_2}$	$(1, 3, 7, 3)$	$(0, \frac{7}{4}, -\frac{3}{4}, -2, \frac{3}{4}, -\frac{3}{2})$	33
	$(3, 1, 5, 1)$	$(0, -\frac{5}{12}, -\frac{1}{12}, -\frac{2}{3}, \frac{1}{12}, -\frac{1}{2})$	43
	$(1, 2, 5, 3)$	$(\frac{7}{48}, \frac{1}{6}, -\frac{1}{48}, -\frac{5}{6}, \frac{1}{48}, \frac{5}{24})$	38
	$(2, 1, 5, 2)$	$(\frac{1}{12}, \frac{7}{24}, -\frac{1}{48}, -\frac{5}{6}, \frac{1}{48}, \frac{5}{24})$	44
	$(3, 1, 5, 2)$	$(\frac{7}{96}, \frac{7}{16}, -\frac{1}{32}, -\frac{5}{6}, \frac{1}{32}, \frac{5}{16})$	39

Table 4. The summary of geometric and flux data that lead to three chiral generations in the $\mathbb{Z}_2^{M_2}$ matter parity assignment. The first two configurations each contain a redundant flux parameter; we chose to eliminate this redundancy by setting $a_1 = 0$. For completeness, we have also included the D3-tadpole.

the U(1) charges. For the explicit flux parametrization (3.19), this yields

$$\xi \sim J_B \cdot \left(\frac{a_1}{2} C_{2_2} + \frac{2a_2}{3} C_{3_2} + \frac{a_3}{3} C_{3_2} - \frac{a_4}{6} W_2 W_3 - a_5 C_{1_2} + \underbrace{\left(\frac{1}{2} W_2 + \frac{2}{3} W_3 - 2\bar{K}_B \right)}_{=\pi_*(D_{U(1)} \cdot D_{U(1)})} F \right). \tag{3.22}$$

To find explicit models with this chiral spectrum, we need to specify the base B , the fibration structure in terms of the classes $S_{7/9}$, the choices for the non-Abelian divisors $W_{2/3}$, and the explicit flux which induces the correct chiralities. To make our lives as easy as possible, we will restrict ourselves to the simplest possible base, $B = \mathbb{P}^3$. As we shall see, this choice admits multiple solution and is hence by no means too restrictive.

Realistic chiral models over the base \mathbb{P}^3 . For this simple choice of the base, the only independent divisor class is the hyperplane class H . Parametrizing the divisors in terms of H ,

$$\bar{K}_B = 4H, \quad W_2 = n_2 H, \quad W_3 = n_3 H, \quad S_7 = s_7 H, \quad S_9 = s_9 H. \tag{3.23}$$

The integers n_i, s_j have to be such that the classes (3.8) of the coefficients d_i as well as $W_{2,3}$ are non-negative. On this specific base, the generic vertical flux (3.19) is parametrized by the five coefficients a_i of the matter surfaces fluxes and the U(1)-flux $D_{U(1)} \wedge (\lambda H)$. We collect these numbers in the flux vector $\mathcal{F} = (a_1, a_2, a_3, a_4, a_5, \lambda)$. We can now scan over all possible fibrations over $B = \mathbb{P}^3$ for flux solutions that generate the spectrum (3.20) as well as a vanishing D-term (3.22) (for $B = \mathbb{P}^3$, the Kähler form is simply $J_B \sim H$). In table 4, we list the five configurations satisfying these requirements.

A few comments about these results are in order. First, we emphasize that the search procedure is based on the base *independent* flux (3.19) and chirality computation. This way, we do not have to first construct the full fibration and then construct the fluxes, which, as demonstrated in [31], can be very inefficient (largely due to the technical issues

of triangulating the polytope of the toric ambient space). In particular, as our results in table 4 show, the gauge divisors $W_{2,3}$ are never both toric (i.e., have the divisor class H) in fibrations with three generation flux configurations. Realizing such fibrations explicitly would of course be a good consistency check, but very ineffective for the purpose of scanning a large number of different models. We content ourselves with the verification that these choices of classes generically do not induce any additional non-abelian gauge divisors or any factorization of the generically present matter curves. Second, we point out that first two solutions in table 4 are at the boundary of the allowed region for (n_2, n_3, s_7, s_9) , i.e., at these points, some of the coefficients d_i become constant. Concretely, in the two cases here, it is d_8 . When this happens, also the chosen flux basis (3.19) becomes linearly dependent, meaning that one parameter becomes redundant.² For concreteness, we set $a_1 = 0$ in these cases. Third, note that in all these cases, the D3-tadpole (see below) is positive, which is required for a stable vacuum. Moreover, the fact that all of them are integer is a necessary condition that fluxes are properly quantized. In the following, we will provide further arguments for the correct flux quantization in our realizations.

3.5 Flux quantization and discrete anomalies

The condition for flux quantization reads [62]:

$$G_4 + \frac{1}{2}c_2(Y) \in H^4(\mathbb{Z}, Y), \tag{3.24}$$

where $c_2(Y)$ is the second Chern class of the fourfold. In practice, verifying this condition explicitly is extremely challenging, and we will not attempt it here. However, we will perform several non-trivial sanity checks, which in particular involves the relationship of the quantization condition to the gauge anomalies of the \mathbb{Z}_2 .

3.5.1 D3-tadpole and intersection numbers

One important quantity which has to be integer for a properly quantized flux is the D3-tadpole [63],

$$n_3 = \frac{\chi}{24} - \frac{1}{2} \int_Y G_4 \wedge G_4. \tag{3.25}$$

Here, χ is the Euler characteristic of the fourfold Y , which can be computed as the integral of the fourth Chern class of Y over Y . For toric fibrations, one can compute the Chern classes via adjunction for any base B (see [31, 43, 52] for examples). As we already mentioned above, it turns out that the tadpole is always integer for all three generation flux configurations (cf. table 4).

Furthermore, given the integrality condition (3.24), we necessarily need to have

$$\left(G_4 + \frac{1}{2}c_2(Y) \right) \cdot D_1 \cdot D_2 \in \mathbb{Z} \tag{3.26}$$

²In some cases it can happen, that some constant d_i lead to an enhanced Mordell-Weil group, induced by a multi-section that can become rational. We checked that this jump does not happen in the $[d_8] = 0$ case.

for any two integer divisor classes D_1, D_2 . In practice, we test this condition with D_i being the restrictions of one of the toric divisors from the ambient space. On the hypersurface, these give rise to the Cartan divisors and the four bisections and are thus manifestly integer classes. In the above models with three family spectra, all these intersection numbers are integer, thus further supporting the claim that the flux is properly quantized.

3.5.2 Geometric incarnation of discrete anomaly cancellation

One particularly fascinating aspect of the quantization condition is its relationship to the cancellation of discrete anomalies (2.4). Recall that for the case at hand, we are only interested in the $\mathbb{Z}_2 - G^2$ anomaly. For $G = \text{SU}(2)$, it receives contributions from doublets which have odd \mathbb{Z}_2 charge. Within the spectrum (cf. table 3) of F-theory on Y_{31} , these are the bifundamentals $(\mathbf{3}, \mathbf{2})$ and the doublets $\mathbf{2}_2$. Therefore, the geometric version of the $\mathbb{Z}_2 - \text{SU}(2)^2$ anomaly cancellation condition is

$$\mathcal{A}_{\mathbb{Z}_2 - \text{SU}(2)^2} = 3 \cdot \chi((\mathbf{3}, \mathbf{2})) + \chi(\mathbf{2}_2) = G_4 \cdot (3 [S_{(\mathbf{3}, \mathbf{2})}] + [S_{\mathbf{2}_2}]) \in 2\mathbb{Z}. \quad (3.27)$$

Inserting the matter surface classes and using the transversality (3.15) and gauge symmetry (3.16) constraints of G_4 , this expression simplifies to

$$\mathcal{A}_{\mathbb{Z}_2 - \text{SU}(2)^2} = G_4 \cdot (2 [y] [e_0] - 4 E_1 F_1). \quad (3.28)$$

Thus, the anomaly is canceled if and only if

$$\mathcal{A}_{\mathbb{Z}_2 - \text{SU}(2)^2} \in 2\mathbb{Z} \iff \frac{1}{2} \mathcal{A}_{\mathbb{Z}_2 - \text{SU}(2)^2} = G_4 \cdot ([y] [e_0] - 2 E_1 F_1) \in \mathbb{Z}. \quad (3.29)$$

While this expression depends on the flux and has a priori no reason to be integer, we note that the four cycle class in parenthesis is manifestly integer. Thus, as long as the flux is properly quantized (3.24), one necessarily has to have

$$\left(G_4 + \frac{1}{2} c_2(Y_{31}) \right) \cdot ([y] [e_0] - 2 E_1 F_1) \in \mathbb{Z}. \quad (3.30)$$

Hence, to guarantee (3.29), it suffices to show that

$$\frac{1}{2} c_2(Y_{31}) \cdot ([y] [e_0] - 2 E_1 F_1) \in \mathbb{Z}. \quad (3.31)$$

The same method has been used in [38] to proof that an F-theory model with $\text{SU}(5) \times \mathbb{Z}_2$ has no \mathbb{Z}_2 anomaly. There, to show that the equivalent version of (3.31) held for *any* choice of base and fibration, it was crucial to know how the fluxes and chiralities matched across the conifold transition which uniggsed the \mathbb{Z}_2 into a $\text{U}(1)$. Doing the same for our model here is beyond the scope of this paper. However, we can simply evaluate (3.31) for every explicit choices of base and fibration structure, in particular on which we found three generation configurations. And indeed, it turns out that in all fibrations over \mathbb{P}^3 , the $\mathbb{Z}_2 - \text{SU}(2)^2$ anomaly (3.29) is canceled due to (3.31).

Likewise, the $\mathbb{Z}_2 - \text{SU}(3)^2$ anomaly is

$$\mathcal{A}_{\mathbb{Z}_2 - \text{SU}(3)^2} = \chi(\mathbf{3}_1) + \chi(\mathbf{3}_3) = G_4 \cdot ([S_{\mathbf{3}_1}] + [S_{\mathbf{3}_3}]) = G_4 \cdot (2 F_1 F_2) \stackrel{!}{\in} 2\mathbb{Z}. \quad (3.32)$$

Proceeding analogously as above, this condition is equivalent to

$$\frac{1}{2}c_2(Y_{31}) \cdot F_1 F_2 \in \mathbb{Z}, \quad (3.33)$$

which we can explicitly verify to be true in all cases with $B = \mathbb{P}^3$.

4 Other \mathbb{Z}_2 symmetries assignments

In this section we want to consider other \mathbb{Z}_2 charge assignments that can be of phenomenological relevance. These include the other matter parity assignment $\mathbb{Z}_2^{M_1}$ as well as lepton and baryon parity, as discussed in table 1. In order to realize them, we consider a different top combination, but perform exactly the same steps that were presented in the previous section. Hence, we will be brief about the details in this section.

4.1 Summary of geometric data

The top combination that we are considering is SU(2) top 1 and SU(3) top 2, as given in appendix C. The toric data of the model is summarized in table 5. The matter loci can easily be determined by the information of the two tops given in appendix C including the additional bifundamental representation at $W_2 = W_3 = 0$ as summarized in table 6. The spectrum is very similar as before but includes a \mathbb{Z}_2 -even charged bifundamental. This allows for a straightforward identification of the geometric \mathbb{Z}_2 with the other three parities listed in table 1.

The matter homology classes of the curves in terms of ambient divisors is given in table 9 of appendix A which can be used to obtain the five independent matter surface fluxes:

$$\begin{aligned} A(\mathbf{2}_2) &= [S_{\mathbf{2}_2}] - \frac{1}{2}E_1 \cdot (-2W_2 - W_3 + 6\bar{K}_B - 2S_9), \\ A(\bar{\mathbf{3}}_2) &= [S_{\bar{\mathbf{3}}_2}] - \frac{1}{3}(2F_1 + F_2) \cdot (2\bar{K}_B - W_2 - W_3 + S_7 - S_9) + \frac{1}{3}C_{\mathbf{3}_2}, \\ A(\mathbf{3}_4) &= [S_{\mathbf{3}_4}] + \frac{1}{3}(2F_1 + F_2) \cdot (-W_2 + 5\bar{K}_B - S_7 - S_9) - \frac{1}{3}C_{\mathbf{3}_4}, \\ A((\mathbf{3}, \mathbf{2})) &= [S_{(\mathbf{3}, \mathbf{2})}] + \frac{1}{3}(-F_1 + F_2) \cdot W_2 + \frac{1}{2}E_1 \cdot W_3 - \frac{1}{3}W_2 \cdot W_3, \\ A(\bar{\mathbf{1}}_2) &= [S_{\bar{\mathbf{1}}_2}], \end{aligned} \quad (4.1)$$

where $C_{\mathbf{R}}$ denotes the classes of the associated matter curves. These admit the following algebraic equivalence relations between the above flux basis and other vertical 4-cycles:

$$\begin{aligned} A(\mathbf{2}_2) - A((\mathbf{3}, \mathbf{2})) - 2D_{U(1)} \cdot W_2 + A(\mathbf{2}_1) &= 0, \\ A(\bar{\mathbf{3}}_2) - A(\bar{\mathbf{3}}_1) + D_{U(1)} \cdot W_3 - A(\bar{\mathbf{3}}_2) &= 0, \\ A(\mathbf{3}_4) + A((\mathbf{3}, \mathbf{2})) - D_{U(1)} \cdot W_3 + A(\mathbf{3}_3) &= 0, \\ A((\mathbf{3}, \mathbf{2})) + A(\mathbf{1}_2) + D_{U(1)} \cdot (-6\bar{K}_B + 2W_2 + 3W_3) - A(\mathbf{1}_1) &= 0. \end{aligned} \quad (4.2)$$

The G_4 -flux in the above basis is then given by

$$G_4 = a_1 A(\mathbf{2}_2) + a_2 A(\bar{\mathbf{3}}_2) + a_3 A(\mathbf{3}_4) + a_4 A((\bar{\mathbf{3}}, \mathbf{2})) + a_5 A(\bar{\mathbf{1}}_2) + D_{U(1)} \wedge F. \quad (4.3)$$

top vertices	$SU(2) : e_0 : (0, 0, 1), \quad e_1 : (1, 0, 1),$ $SU(3) : f_0 : (0, 0, -1), \quad f_1 : (0, 1, -1), \quad f_2 : (-1, 1, -1)$
	$b_1 = e_0 f_1 f_2^2 d_1 \quad b_3 = e_0 f_0 f_2 d_3 \quad b_6 = d_6 \quad b_8 = e_1 f_1 d_8 \quad b_{10} = e_1 f_0^2 f_1 d_{10}$ $b_2 = e_0 f_2 d_2 \quad b_5 = f_1 f_2 d_5 \quad b_7 = f_0 d_7 \quad b_9 = e_1 f_0 f_1 d_9$
	SRI: $\{xt, xe_1, xf_1, ys, yf_0, te_0, tf_2, sf_2, e_1 f_2, sf_1\}.$
	$D_{U(1)} = [x] - [y] - \frac{1}{2}E_1 - (\frac{1}{3}F_1 + \frac{2}{3}F_2) - \frac{1}{3}W_3 + \frac{1}{2}\bar{K}_B + \frac{1}{2}S_7 - \frac{1}{2}S_9,$ $D_{\mathbb{Z}_2} = x + \frac{2}{3}(F_1 + 2F_2).$
	$[d_1] = 3\bar{K}_B - S_7 - S_9 - W_2 \quad [d_2] = 2\bar{K}_B - S_9 - W_2 \quad [d_3] = \bar{K}_B + S_7 - S_9 - W_3 - W_2$ $[d_5] = 2\bar{K}_B - S_7 \quad [d_6] = \bar{K}_B \quad [d_7] = S_7 - W_3$ $[d_8] = \bar{K}_B + S_9 - S_7 \quad [d_9] = S_9 - W_3 \quad [d_{10}] = S_9 + S_7 - \bar{K}_B - 2W_3$

Table 5. Geometric data for the hypersurface specialization of the second top combination.

Label	$G_{SM} \times \mathbb{Z}_2$ Rep.	$\mathbb{Z}_2^{M_1}$	\mathbb{Z}_2^L	\mathbb{Z}_2^B
$\mathbf{2}_1$	$(\mathbf{1}, \mathbf{2})_{(-\frac{1}{2}, -)}$	H_d, H_u	L	L
$\mathbf{2}_2$	$(\mathbf{1}, \mathbf{2})_{(-\frac{1}{2}, +)}$	L	H_d, H_u	—
$\bar{\mathbf{3}}_1$	$(\bar{\mathbf{3}}, \mathbf{1})_{(-\frac{2}{3}, -)}$	\bar{u}	—	\bar{u}
$\bar{\mathbf{3}}_2$	$(\bar{\mathbf{3}}, \mathbf{1})_{(-\frac{2}{3}, +)}$	—	\bar{u}	—
$\mathbf{3}_3$	$(\mathbf{3}, \mathbf{1})_{(-\frac{1}{3}, -)}$	\bar{d}	—	\bar{d}
$\mathbf{3}_4$	$(\mathbf{3}, \mathbf{1})_{(-\frac{1}{3}, +)}$	—	\bar{d}	—
$(\bar{\mathbf{3}}, \mathbf{2})$	$(\bar{\mathbf{3}}, \mathbf{2})_{(-\frac{1}{6}, +)}$	Q	Q	Q
$\mathbf{1}_1$	$(\mathbf{1}, \mathbf{1})_{(1, -)}$	\bar{e}	\bar{e}	—
$\mathbf{1}_2$	$(\mathbf{1}, \mathbf{1})_{(-1, +)}$	—	—	\bar{e}

Table 6. Matter curves and their charges as given for the second top combination. MSSM field identifications under various \mathbb{Z}_2 symmetries are given in the last three columns. For each identification we assign the chirality three to MSSM fields whereas states marked with a “—” must be non-chiral.

4.2 Three family searches and discrete anomalies

For concrete three family realizations, we again pick the base to be \mathbb{P}^3 . In this model, we now have the possibility to assign different physical interpretations to the \mathbb{Z}_2 . In each fibration parametrized by (n_2, n_3, s_7, s_9) , we search for flux configurations $(a_1, a_2, a_3, a_4, a_5, \lambda)$ compatible with one of the three possible identifications listed in table 6. In all of them, we again impose the vanishing of the fluxed induced D-term of the U(1). Those solutions that also have a positive integer D3-tadpole are listed in table 7. First we note that the solutions we obtain for the second matter parity assignment have a very similar structure compared

	(n_2, n_3, s_7, s_9)	$(a_1, a_2, a_3, a_4, a_5, \lambda)$	n_{D3}
$\mathbb{Z}_2^{M_1}$	(1,3,7,4)	$(0, -\frac{7}{4}, \frac{3}{4}, 2, \frac{3}{4}, -\frac{3}{2})$	33
	(3,1,5,4)	$(0, -\frac{5}{12}, \frac{1}{12}, \frac{2}{3}, \frac{1}{12}, -\frac{1}{2})$	43
	(1,2,5,4)	$(\frac{7}{48}, -\frac{1}{6}, \frac{1}{48}, -\frac{11}{16}, \frac{1}{48}, -\frac{1}{12})$	38
	(2,1,5,4)	$(\frac{1}{12}, -\frac{7}{24}, \frac{1}{48}, \frac{3}{4}, \frac{1}{48}, -\frac{1}{8})$	44
	(3,1,5,3)	$(\frac{7}{96}, -\frac{7}{16}, \frac{1}{32}, -\frac{73}{96}, \frac{1}{32}, -\frac{1}{8})$	39
\mathbb{Z}_2^L	(1,2,4,4)	$(0, \frac{5}{16}, -\frac{5}{96}, \frac{15}{32}, \frac{1}{6}, -\frac{73}{48})$	43
	(1,2,5,3)	$(0, \frac{1}{12}, -\frac{1}{12}, \frac{1}{2}, 0, 0)$	42
	(1,2,5,5)	$(0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, 0, 0)$	40
	(3,2,5,3)	$(0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, 0, 0)$	32
	(1,2,6,4)	$(\frac{41}{384}, \frac{7}{32}, -\frac{61}{384}, \frac{175}{384}, -\frac{23}{384}, \frac{23}{96})$	39
\mathbb{Z}_2^B	(3,2,5,4)	$(0, 0, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{5}{3})$	34
	(1,2,5,4)	$(-\frac{1}{12}, -\frac{1}{3}, -\frac{1}{12}, \frac{3}{4}, -\frac{1}{12}, \frac{4}{3})$	38

Table 7. The summary of geometric and flux quanta that lead the three chiral generations for the three discrete symmetry assignments of the second top combination.

to the models we obtained in the previous section (cf. table 4), including the same number of D3-branes. This points towards an equivalence between the two fibrations defined via the two different top combinations.³ A more general analysis of this equivalence including the necessary redefinitions of the abelian symmetry generators is left for future research. Secondly it is important to emphasize that only the flux configurations for matter parity fulfill the G_4 -flux integrality condition

$$\left(G_4 + \frac{1}{2}c_2(Y)\right) \cdot D_1 \cdot D_2 \in \mathbb{Z}, \tag{4.4}$$

whereas the lepton and baryon parity assignments do not. Again, we can nicely relate this rather obscure geometric property directly to the cancellation of \mathbb{Z}_2 anomalies.

Namely, these are

$$\begin{aligned} \mathcal{A}_{Z_2-SU(2)^2} &= \chi(\mathbf{2}_1) = G_4 \cdot ([S_{\mathbf{2}_1}]) = -2 [y] E_0 \in 2\mathbb{Z} \\ \mathcal{A}_{Z_2-SU(3)^2} &= \chi(\mathbf{3}_1) + \chi(\mathbf{3}_3) = G_4 \cdot (-[S_{\mathbf{3}_1}] + [S_{\mathbf{3}_3}]) = 2 F_1 F_2 \in 2\mathbb{Z}. \end{aligned} \tag{4.5}$$

Like in the previous section, the quantization condition translates these conditions into a question about integrality of the intersection numbers

$$\frac{1}{2}c_2(Y_{32}) \cdot \begin{cases} [y] E_0 \\ F_1 F_2 \end{cases}, \tag{4.6}$$

which both turn out to be indeed integral for all the fibrations we scanned over. However, we also know that only the matter parity $\mathbb{Z}_2^{M_1}$ assignment of the chiralities is anomaly free, whereas the lepton and baryon parity assignments are not with the spectrum in 6. Since

³The classes of the sections of the two tops can be related by a change $d_1 \leftrightarrow d_8, d_9 \leftrightarrow d_2, d_{10} \leftrightarrow d_3$.

the flux configurations are chosen to reproduce these chiral spectra, we arrive at the same conclusion — but based on field theoretic anomaly considerations — that the flux solutions for these two assignments in table 7 cannot be properly quantized.

5 Summary and conclusion

In this work we have engineered globally consistent four dimensional MSSM-like particle physics models with three chiral generations that admit \mathbb{Z}_2 quantum numbers under the matter parity extension of the Standard Model gauge group. Our compactifications are genus-one fibered fourfolds with G_4 -flux over a simple \mathbb{P}^3 base space that pass all necessary consistency conditions: the G_4 -flux is properly quantized, and the D3-tadpole is canceled with a positive and integral number of D3-branes. For this explicit construction we employed toric geometry to engineer the resolved fourfold which allows the direct computation of all (discrete) gauged quantum numbers. In addition, the fact that the internal space is smooth allows us to easily handle the gauge background, giving us the power to scan systematically for configurations leading to three chiral generations. These constructions are flexible enough to also allow, at least in principle, for other \mathbb{Z}_2 symmetries, such as lepton and baryon parity, by choosing different flux configurations. These models however suffer from non-properly quantized G_4 -fluxes, even though they give three chiral families and a positive integer number of D3-branes. We have shown that this is not just a coincidence, but actually intimately related to the fact that lepton and baryon parities are not free of discrete anomalies with just the MSSM spectrum.

However several interesting questions remain. First it would be exciting to investigate the interplay between fluxes and discrete anomalies further. For example, an analysis similar to [37, 43] of the conifold transition that unhiggses the \mathbb{Z}_2 into a U(1) could allow us to prove the cancellation of discrete anomalies for *generic* fibrations. Moreover we have left out possible Abelian- \mathbb{Z}_2 anomalies as they are hard to investigate in the field theory due to an ambiguous U(1) charge normalization [34]. However, since in F-theory there is a natural charge quantization inherited from the Mordell-Weil lattice [55], one might hope that a more geometric treatment of this issue is possible. Further important steps towards more realistic phenomenology is to understand the full vector-like sector and to decouple possible (vector-like) exotics while keeping one pair of Higgs doublets light. This would also allow us to determine the presence of right-handed neutrinos, whose representation is, at least in principle, realized explicitly in the geometry. Due to recent progress [44, 58] this goal seems to be in reach. However, applying the methods presented there to a complex configuration of matter curves, such as we have in our models, are not feasible with the given algorithms and computing power today. But even without exotics, this model might still suffer from higher dimensional operators in the effective action such as

$$\mathcal{W} \ni \kappa^1 QQQ L + \kappa^2 \overline{uud} E, \quad (5.1)$$

whose coefficients are strongly constrained by proton decay but can not be forbidden with matter parity alone. Hence one might want to construct higher order discrete symmetries, ideally the \mathbb{Z}_6 proton hexality [64] which forbids also other dangerous higher dimensional

operators, and is anomaly free. The classification or construction of higher order (possibly non-Abelian [65–67]) discrete symmetries base-independently beyond \mathbb{Z}_4 [68, 69] are unknown yet (see, however, [70] for some recent examples over specific bases) and, hence, a topic of great interest. Once such a classification is available, we hope that a generalization of our work can realize the chiral MSSM with such a discrete symmetry extension.

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A Homology classes of matter surfaces

In this appendix, we collect the matter surface homology classes of the two top combinations considered in sections 3 and 4, in table 8 and 9, respectively.

R	Matter surface homology classes
2₁	$-(E_1 F_1) - E_1 \bar{K}_B + E_1 S_7 - E_1 S_9 + F_1 W_2 + \bar{K}_B W_2 - S_7 W_2 + S_9 W_2 - W_2[x] - W_2[y] - 2E_1[y] + 2W_2[y]$
2₂	$3E_1 \bar{K}_B - E_1 S_7 - E_1 S_9 + F_2 W_2 - \bar{K}_B W_2 + S_9 W_2 - W_2[x] + 2E_1[y]$
3₁	$F_2^2 - F_2 \bar{K}_B + F_2 S_9 + F_2 W_2 - E_1 W_3 - F_2 W_3 + \bar{K}_B W_3 - S_9 W_3 + W_3[x]$
3₂	$-(E_1 F_1) + F_2^2 + F_1 \bar{K}_B + F_2 \bar{K}_B - F_2 S_7 + W_3[y]$
3₃	$-2F_1 F_2 - F_2^2 + F_2 S_7$
3₄	$E_1 F_1 + 2F_1 F_2 + F_2^2 - 2F_1 \bar{K}_B - F_2 \bar{K}_B - F_2 S_9 + F_1 W_3 + 2\bar{K}_B W_3 - S_7 W_3 + S_9 W_3 - W_3[x] + W_3[y]$
(3, 2)	$E_1 F_0$
1₁	$-2F_1 F_2 - F_2^2 + E_1 \bar{K}_B + 3F_1 \bar{K}_B + F_2 \bar{K}_B + 2\bar{K}_B^2 - E_1 S_7 - F_1 S_7 - 3\bar{K}_B S_7 + S_7^2 + E_1 S_9 - F_1 S_9 + F_2 S_9 + 2\bar{K}_B S_9 - S_7 S_9 - F_1 W_2 - \bar{K}_B W_2 + S_7 W_2 - S_9 W_2 - 4\bar{K}_B[x] + 2S_7[x] + W_2[x] + 2E_1[y] + 2\bar{K}_B[y] + 2S_9[y] - 2W_2[y] - 2W_3[y] - 4[x][y]$
1₂	$-(E_1 F_1) - 2F_1 F_2 - F_2^2 - 2E_1 \bar{K}_B + F_1 \bar{K}_B - 3F_2 \bar{K}_B + 2\bar{K}_B^2 - E_1 S_7 - F_1 S_7 + 2\bar{K}_B S_7 + E_1 S_9 - F_1 S_9 + F_2 S_9 - 3\bar{K}_B S_9 - S_7 S_9 + S_9^2 + E_1 W_2 + F_2 W_2 - \bar{K}_B W_2 + 2E_1 W_3 + F_1 W_3 + 2F_2 W_3 - 3\bar{K}_B W_3 + 2S_9 W_3 + 2\bar{K}_B[x] + 2S_7[x] - W_2[x] - 3W_3[x] + 2E_1[y] - 4\bar{K}_B[y] + 2S_9[y] + W_3[y] - 4[x][y]$

Table 8. Summary of matter homology classes restricted to CY of first top combination.

R	Matter surface homology classes
2₁	$-(E_1\bar{K}_B) - E_1S_7 - E_1S_9 + \bar{K}_BW_2 + S_7W_2 + S_9W_2$ $+2E_1W_3 - 2W_2W_3 + W_2[x] + 2E_1[y] - 2W_2[y]$
2₂	$-(E_1F_1) + 3E_1\bar{K}_B + E_1S_7 - E_1S_9 - 2E_1W_2 + F_2W_2$ $+ \bar{K}_BW_2 - S_9W_2 - E_1W_3 + W_2[x] - 2E_1[y]$
3₁	$F_2^2 + F_2\bar{K}_B - F_2S_9 + W_3[x]$
3₂	$-(E_1F_1) - F_2^2 + F_1\bar{K}_B - F_2\bar{K}_B + F_1S_7 + F_2S_7 - F_1S_9 - F_1W_3 - W_3[y]$
3₃	$2F_1F_2 + F_2^2 + F_2\bar{K}_B + F_2S_9 - 2F_2W_3$
3₄	$E_1F_1 - 2F_1F_2 - F_2^2 - 3F_1\bar{K}_B - 2F_2\bar{K}_B + F_1S_7 + F_2S_7 + F_1S_9$ $- E_1W_3 - F_1W_3 + F_2W_3 + 2\bar{K}_BW_3 - S_9W_3 + W_3[x] - W_3[y]$
(3, 2)	$-(E_1F_1) + F_1W_2$
1₁	$2F_1F_2 + 3F_2^2 + E_1\bar{K}_B + F_1\bar{K}_B + 3F_2\bar{K}_B + 2\bar{K}_B^2 - E_1S_7 - F_1S_7 - 2F_2S_7 - 3\bar{K}_BS_7$ $+ S_7^2 + E_1S_9 + F_1S_9 - F_2S_9 + 2\bar{K}_BS_9 - S_7S_9 - \bar{K}_BW_2 + S_7W_2 - S_9W_2 - 4\bar{K}_B[x]$ $+ 2S_7[x] + W_2[x] + 2E_1[y] + 2\bar{K}_B[y] + 2S_9[y] - 2W_2[y] - 2W_3[y] - 4[x][y]$
1₂	$E_1F_1 + 2F_1F_2 + 3F_2^2 - 2E_1\bar{K}_B - F_1\bar{K}_B + 5F_2\bar{K}_B + 2\bar{K}_B^2 - E_1S_7 - F_1S_7 - 2F_2S_7$ $+ 2\bar{K}_BS_7 + E_1S_9 + F_1S_9 - F_2S_9 - 3\bar{K}_BS_9 - S_7S_9 + S_9^2 + E_1W_2 - F_2W_2 - \bar{K}_BW_2$ $+ S_9W_2 + E_1W_3 + F_1W_3 - F_2W_3 - 2\bar{K}_BW_3 + S_9W_3 + 2\bar{K}_B[x] + 2S_7[x] - W_2[x]$ $- 3W_3[x] + 2E_1[y] - 4\bar{K}_B[y] + 2S_9[y] + W_3[y] - 4[x][y]$

Table 9. Matter surface homology classes of the second top restricted on the fourfold.

B Towards the vector-like spectrum of the first top

In this section we want to give all information that is needed to compute the vector-like spectrum of our first model realizing $\mathbb{Z}_2^{M_2}$ using the methods of [44]. The key point is to assign to each matter curve $C_{\mathbf{R}} \subset B$ a divisor D , i.e., a collection of points on $C_{\mathbf{R}}$, based on the intersection properties between the G_4 -flux and the matter surface $S_{\mathbf{R}}$. By expression the flux in terms of matter surfaces, evaluating this intersection product reduce to properly counting the points, in which various matter surfaces meet. In F-theory geometries, these points are in the fibers over codimension three enhancement loci, i.e., Yukawa points Y_i . Thus, the resulting divisor D is a linear combination $\sum_i \mu_i Y_i$ of these points. From this divisor, one can then extract the left- and right-handed fermions as the sheaf cohomologies

$$h^i(C_{\mathbf{R}}, \mathcal{O}_{C_{\mathbf{R}}}(D) \otimes \sqrt{K_{C_{\mathbf{R}}}}), \quad i = 0, 1, \quad (\text{B.1})$$

where $\sqrt{K_{C_{\mathbf{R}}}}$ is the spin bundle on $C_{\mathbf{R}}$. These cohomologies depend on the complex structure parameters of the fourfold. However, the chiral index $\chi = h^0 - h^1$ is a topological invariant, which is simply the number of points (including signs) that constitutes $D \subset B$. More details and examples can be found in [44].

For the first top realizing the MSSM with matter parity $\mathbb{Z}_2^{M_2}$, we need, in addition to the particular flux basis (3.19) that we have picked, also algebraic equivalence relations

between fluxes and other vertical 4-cycles. Explicitly, these are

$$\begin{aligned}
 A(\mathbf{2}_2) - A((\mathbf{3}, \mathbf{2})) + 2 D_{U(1)} \cdot W_2 + [S_{\mathbf{2}_1}] + \frac{1}{2} E_1 \cdot [\{f_{\mathbf{2}_1}\}] - \frac{1}{2} C_{\mathbf{2}_1} &= 0, \\
 A((\mathbf{3}, \mathbf{2})) - A(\mathbf{3}_2) - D_{U(1)} \cdot W_3 + [S_{\mathbf{3}_1}] + \frac{1}{3} (F_1 + 2 F_2) \cdot [\{f_{\mathbf{3}_1}\}] - \frac{1}{6} C_{\mathbf{3}_1} &= 0, \\
 A(\mathbf{3}_4) + A((\mathbf{3}, \mathbf{2})) + D_{U(1)} \cdot W_3 + [S_{\mathbf{3}_3}] + \frac{1}{3} (F_1 - F_2) \cdot [\{f_{\mathbf{3}_3}\}] - \frac{1}{6} C_{\mathbf{3}_3} &= 0, \\
 A((\mathbf{3}, \mathbf{2})) - A(\mathbf{1}_{(1,+)}) + D_{U(1)} \cdot (6 \bar{K}_B - 2 W_2 - 3 W_3) + [S_{\mathbf{1}_{(1,-)}}] - \frac{1}{2} C_{\mathbf{1}_{(1,-)}} &= 0.
 \end{aligned} \tag{B.2}$$

The Yukawa points and their homology classes are listed in table 10. In terms of the homology classes, we can write the chiral indices of the matter states induced by the flux (3.19) as

$$\begin{aligned}
 \chi(\bar{\mathbf{3}}_1) &= -\frac{2}{3} C_{\bar{\mathbf{3}}_1} F + a_1 [Y_8] + \frac{1}{3} a_2 [Y_{16}] + a_3 \left(\frac{2}{3} [Y_9] - \frac{1}{3} [Y_{13}] \right) - \frac{1}{3} a_4 [Y_8] - a_5 [Y_{12}], \\
 \chi(\mathbf{3}_2) &= \frac{2}{3} C_{\mathbf{3}_2} F + a_2 \left(\frac{1}{3} [Y_{16}] + \frac{1}{3} [Y_6] - \frac{2}{3} C_{\mathbf{3}_2} W_3 \right) - a_3 \left(\frac{4}{3} [Y_{11}] - \frac{1}{3} [Y_{15}] \right) \\
 &\quad + \frac{1}{3} a_4 [Y_6] + a_5 [Y_{15}], \\
 \chi(\bar{\mathbf{3}}_3) &= \frac{1}{3} C_{\bar{\mathbf{3}}_3} F - a_1 [Y_7] + a_2 \left(-\frac{4}{3} [Y_{10}] + \frac{1}{3} [Y_{14}] \right) + a_3 \left(\frac{2}{3} [Y_9] - \frac{1}{3} [Y_{17}] \right) - \frac{1}{3} a_4 [Y_7] + a_5 [Y_{12}], \\
 \chi(\bar{\mathbf{3}}_4) &= \frac{1}{3} C_{\bar{\mathbf{3}}_4} F + a_2 \left(-\frac{4}{3} [Y_{11}] + \frac{1}{3} [Y_{15}] \right) + a_3 \left(\frac{1}{3} [Y_5] - \frac{2}{3} [Y_{18}] - \frac{2}{3} [Y_9] + \frac{1}{3} [Y_{17}] - \frac{1}{3} C_{\bar{\mathbf{3}}_4} W_3 \right) \\
 &\quad + a_4 \left(-\frac{1}{3} [Y_5] + \frac{2}{3} [Y_{18}] \right) + a_5 [Y_{15}], \\
 \chi(\bar{\mathbf{3}}, \mathbf{2}) &= -\frac{1}{6} C_{(\bar{\mathbf{3}}, \mathbf{2})} F + \frac{1}{2} a_1 ([Y_7] - [Y_8]) + \frac{1}{3} a_2 [Y_6] + \frac{1}{6} a_4 C_{(\bar{\mathbf{3}}, \mathbf{2})} (6 \bar{K}_B - 2 W_2 - 3 W_3) \\
 &\quad + a_3 \left(\frac{2}{3} [Y_{18}] - \frac{1}{3} [Y_5] \right), \\
 \chi(\mathbf{2}_2) &= \frac{1}{2} C_{\mathbf{2}_2} F + a_1 \left(-\frac{1}{2} [Y_8] + \frac{1}{2} [Y_7] - C_{\mathbf{2}_2} W_2 + \frac{1}{2} [Y_1] - \frac{1}{2} [Y_3] \right) + a_4 \left(-\frac{1}{2} [Y_8] + \frac{1}{2} [Y_7] \right) \\
 &\quad + 2 a_5 [Y_4], \\
 \chi(\mathbf{2}_1) &= \frac{1}{2} C_{\mathbf{2}_1} F + a_1 \left(\frac{1}{2} [Y_1] - \frac{1}{2} [Y_3] \right) - a_2 [Y_6] - a_3 [Y_5] + a_4 \left(\frac{1}{2} [Y_5] - \frac{1}{2} [Y_6] \right) + 2 a_5 [Y_2], \\
 \chi(\mathbf{1}_{(1,-)}) &= -C_{\mathbf{1}_{(1,-)}} F + 2 a_1 [Y_4] + a_2 [Y_{15}] + a_3 [Y_{15}] + a_5 V, \\
 \chi(\mathbf{1}_{(1,+)}) &= C_{\mathbf{1}_{(1,+)}} F - a_1 [Y_1] - a_2 [Y_{14}] - a_3 [Y_{13}] + a_5 V,
 \end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
 V &= -C_{\mathbf{1}_{(1,+)}} (6 \bar{K}_B - 2 W_2 - 3 W_3) - W_2^2 W_3 - 2 W_2 W_3^2 + 2 W_2^2 \bar{K}_B \\
 &\quad + 13 W_2 W_3 \bar{K}_B + 12 W_3^2 \bar{K}_B - 10 W_2 \bar{K}_B^2 - 24 W_3 \bar{K}_B^2 + 12 \bar{K}_B^3 - 2 W_2 W_3 S_7 \\
 &\quad - 4 W_3^2 S_7 - 4 W_2 \bar{K}_B S_7 - W_3 \bar{K}_B S_7 + 8 \bar{K}_B^2 S_7 + 2 W_2 S_7^2 + 3 W_3 S_7^2 - 4 \bar{K}_B S_7^2 \\
 &\quad + 2 W_2^2 S_9 + 4 W_2 W_3 S_9 - 8 W_2 \bar{K}_B S_9 - 9 W_3 \bar{K}_B S_9 + 8 \bar{K}_B^2 S_9 + 2 W_3 S_7 S_9 \\
 &\quad + 2 W_2 S_9^2 + 3 W_3 S_9^2 - 4 \bar{K}_B S_9^2.
 \end{aligned} \tag{B.4}$$

Coupling	Homology classes
$\mathbf{2}_1 \cdot \mathbf{2}_2 \cdot \overline{\mathbf{1}}_{(1,-)}$	$[Y_1] = W_2 \cdot (6\overline{K}_B^2 - 4\overline{K}_B S_7 + 2S_7^2 + 4\overline{K}_B S_9 - 2S_9^2 - 2\overline{K}_B W_2 - 2S_9 W_2 - \overline{K}_B W_3 - 3S_7 W_3 - S_9 W_3 + W_2 W_3 + 2W_3^2)$
$\mathbf{2}_1 \cdot \mathbf{2}_1 \cdot \overline{\mathbf{1}}_{(1,+)}$	$[Y_2] = W_2 \cdot (2\overline{K}_B S_7 - S_7^2 + \overline{K}_B S_9 + S_9^2 - 2\overline{K}_B W_3 + S_7 W_3 - S_9 W_3)$
$\mathbf{2}_1 \cdot \overline{\mathbf{2}}_2 \cdot \mathbf{1}_{(0,-)}$	$[Y_3] = W_2 \cdot (6\overline{K}_B^2 + 4\overline{K}_B S_7 - 2S_7^2 + 4\overline{K}_B S_9 - 2S_9^2 - 2\overline{K}_B W_2 - 2S_9 W_2 - 9\overline{K}_B W_3 + 3S_7 W_3 - S_9 W_3 + W_2 W_3)$
$\mathbf{2}_2 \cdot \mathbf{2}_2 \cdot \overline{\mathbf{1}}_{(1,+)}$	$[Y_4] = W_2 \cdot (6\overline{K}_B^2 + 2\overline{K}_B S_7 - S_7^2 - 5\overline{K}_B S_9 + S_9^2 - 5\overline{K}_B W_2 + 2S_9 W_2 + W_2^2 - 7\overline{K}_B W_3 + 2S_7 W_3 + 2S_9 W_3 + 2W_2 W_3)$
$\mathbf{2}_1 \cdot \mathbf{3}_4 \cdot (\overline{\mathbf{3}}, \mathbf{2})$	$[Y_5] = W_2 \cdot W_3 \cdot (2\overline{K}_B - S_7 + S_9)$
$\mathbf{2}_1 \cdot \overline{\mathbf{3}}_2 \cdot (\mathbf{3}, \mathbf{2})$	$[Y_6] = W_2 \cdot W_3 \cdot (S_7 + S_9 - W_3)$
$\mathbf{2}_2 \cdot \mathbf{3}_3 \cdot (\overline{\mathbf{3}}, \mathbf{2})$	$[Y_7] = W_2 \cdot W_3 \cdot (3\overline{K}_B + S_7 - S_9 - W_2 - 2W_3)$
$\mathbf{2}_2 \cdot \overline{\mathbf{3}}_1 \cdot (\mathbf{3}, \mathbf{2})$	$[Y_8] = W_2 \cdot W_3 \cdot (3\overline{K}_B - S_7 - S_9 - W_2)$
$\overline{\mathbf{3}}_1 \cdot \overline{\mathbf{3}}_3 \cdot \overline{\mathbf{3}}_4$	$[Y_9] = W_3 \cdot \overline{K}_B (3\overline{K}_B - S_7 - S_9 - W_2)$
$\mathbf{3}_2 \cdot \mathbf{3}_3 \cdot \mathbf{3}_3$	$[Y_{10}] = W_3 \cdot \overline{K}_B (S_7 - W_3)$
$\mathbf{3}_2 \cdot \mathbf{3}_4 \cdot \mathbf{3}_4$	$[Y_{11}] = W_3 \cdot \overline{K}_B S_9$
$\overline{\mathbf{3}}_1 \cdot \mathbf{3}_3 \cdot \mathbf{1}_{(1,+)}$	$[Y_{12}] = W_3 \cdot (3\overline{K}_B - S_7 - S_9 - W_2)(2\overline{K}_B + S_7 - S_9 - W_2 - 2W_3)$
$\overline{\mathbf{3}}_1 \cdot \mathbf{3}_4 \cdot \mathbf{1}_{(1,-)}$	$[Y_{13}] = W_3 \cdot (2\overline{K}_B - S_7 + S_9)(3\overline{K}_B - S_7 - S_9 - W_2)$
$\mathbf{3}_2 \cdot \overline{\mathbf{3}}_3 \cdot \overline{\mathbf{1}}_{(1,-)}$	$[Y_{14}] = W_3 \cdot (\overline{K}_B S_7 + S_7^2 + 3\overline{K}_B S_9 - S_9^2 - S_7 W_2 - S_9 W_2 - \overline{K}_B W_3 - 3S_7 W_3 - S_9 W_3 + W_2 W_3 + 2W_3^2)$
$\mathbf{3}_2 \cdot \overline{\mathbf{3}}_4 \cdot \overline{\mathbf{1}}_{(1,+)}$	$[Y_{15}] = W_3 \cdot (3\overline{K}_B S_7 - S_7^2 + \overline{K}_B S_9 + S_9^2 - 3\overline{K}_B W_3 + S_7 W_3 - S_9 W_3)$
$\overline{\mathbf{3}}_1 \cdot \mathbf{3}_2 \cdot \mathbf{1}_{(0,-)}$	$[Y_{16}] = W_3 \cdot (3\overline{K}_B - S_7 - S_9 - W_2)(S_7 + S_9 - W_3)$
$\overline{\mathbf{3}}_3 \cdot \mathbf{3}_4 \cdot \mathbf{1}_{(0,-)}$	$[Y_{17}] = W_3 \cdot (6\overline{K}_B^2 + \overline{K}_B S_7 - S_7^2 + \overline{K}_B S_9 + 2S_7 S_9 - S_9^2 - 2\overline{K}_B W_2 + S_7 W_2 - S_9 W_2 - 6\overline{K}_B W_3 + 2S_7 W_3 - 2S_9 W_3)$
$(\overline{\mathbf{3}}, \mathbf{2}) \cdot (\overline{\mathbf{3}}, \mathbf{2}) \cdot \overline{\mathbf{3}}_4$	$[Y_{18}] = W_2 \cdot W_3 \cdot \overline{K}_B$

Table 10. Yukawa points of the first top.

C Summary of toric tops

In this section we summarize the toric data of all four SU(2) and SU(3) tops over polygon F_2 , following the prescription of [42]. The factorization of the generic hypersurface is given together with the SR-ideal and the abelian generators. Furthermore we present the matter loci, the class of fiber component we use for the matter surfaces, and the associated representations.

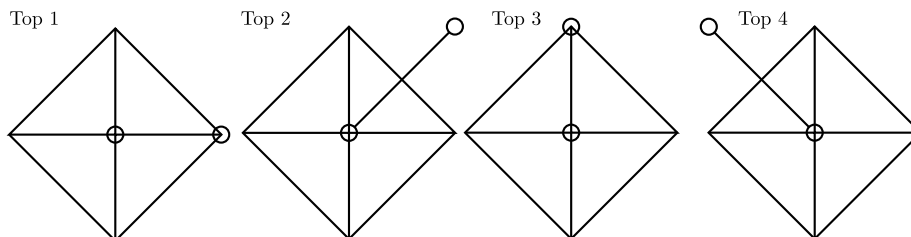


Figure 1. The toric diagram of the four inequivalent SU(2) tops over F_2 .

Factorization:	$b_1 \rightarrow e_0 d_1 \quad b_2 \rightarrow e_0 d_2 \quad b_3 \rightarrow e_0 d_3$ $b_5 \rightarrow d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow d_7$ $b_8 \rightarrow e_1 d_8 \quad b_9 \rightarrow e_1 d_9 \quad b_{10} \rightarrow e_1 d_{10}$			
SRI:	$\{xt, xe_1, ys, te_0\}$			
Charge Generators	$D_{U(1)} = [y] - [x] + 1/2[e_1]$ $D_{Z_2} = [x]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	weight	Rep
$d_{10}^2 d_5^2 + d_{10} d_6^2 d_8 - 2d_{10} d_5 d_7 d_8 + d_7^2 d_8^2$ $-d_6(d_{10} d_5 + d_7 d_8) d_9 + d_5 d_7 d_9^2 = 0$	$(0, 0, 3)$	$[e_0][s]$	$(-1, 1)_{(-\frac{1}{2}, 1)}$	$\mathbf{2}_{(-\frac{1}{2}, -)}$
$d_3^2 d_5^2 - d_2 d_3 d_5 d_6 + d_1 d_3 d_6^2 + d_2^2 d_5 d_7$ $-2d_1 d_3 d_5 d_7 - d_1 d_2 d_6 d_7 + d_1^2 d_7^2 = 0$	$(0, 0, 3)$	$[e_1][s]$	$(1, -1)_{(-\frac{1}{2}, 0)}$	$\mathbf{2}_{(\frac{1}{2}, +)}$
$4d_1 d_{10} - d_6^2 = 0$	$(1, 2, 3)$	–	–	–

Table 11. Summary of SU(2) top 1.

Factorization:	$b_1 \rightarrow d_1 \quad b_2 \rightarrow e_0 d_2 \quad b_3 \rightarrow e_0^2 d_3$ $b_5 \rightarrow e_1 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow e_0 d_7$ $b_8 \rightarrow e_1^2 d_8 \quad b_9 \rightarrow e_1 d_9 \quad b_{10} \rightarrow d_{10}$			
SRI:	$\{xt, ys, xe_0, se_1\}$			
Charge Generators	$D_{U(1)} = [y] - [x] - [e_0]$ $D_{Z_2} = [x]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_1 = 0$	$(0, 0, 3)$	$[e_0][t]$	$(-1, 1)_{(1, 0)}$	$\mathbf{2}_{(1, +)}$
$d_{10} = 0$	$(0, 0, 3)$	$[e_0][y]$	$(-1, 1)_{(-1, 1)}$	$\mathbf{2}_{(-1, -)}$
$-d_{10} d_2^2 + 4d_1 d_{10} d_3$ $-d_3 d_6^2 + d_2 d_6 d_7 - d_1 d_7^2 = 0$	$(0, 0, 3)$	$[e_1][y]$	$(1, -1)_{(0, 0)}$	$\mathbf{2}_{(0, +)}$
$-d_{10} d_5^2 + 4d_1 d_{10} d_8 - d_6^2 d_8$ $+d_5 d_6 d_9 - d_1 d_9^2 = 0$	$(0, 0, 3)$	$[e_0][e_1 + t + y]$	$(1, -1)_{(0, 1)}$	$\mathbf{2}_{(0, -)}$
$4d_1 d_{10} - d_6^2 = 0$	$(1, 2, 3)$	–	–	–

Table 12. Summary of SU(2) top 2.

Factorization:	$b_1 \rightarrow e_1 d_1 \quad b_2 \rightarrow d_2 \quad b_3 \rightarrow e_0 d_3$ $b_5 \rightarrow e_1 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow e_0 d_7$ $b_8 \rightarrow e_1 d_8 \quad b_9 \rightarrow d_9 \quad b_{10} \rightarrow e_0 d_{10}$			
SRI:	$\{xt, ys, ye_0, se_1\}$			
Charge Generators	$D_{U(1)} = [y] - [x] + 1/2[e_1]$ $D_{\mathbb{Z}_2} = [x] + 1/2[e_1]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$-d_2 d_5 d_6 d_8 + d_1 d_6^2 d_8 + d_2^2 d_8^2 + d_2 d_5^2 d_9$ $-d_1 d_5 d_6 d_9 - 2d_1 d_2 d_8 d_9 + d_1^2 d_9^2 = 0$	$(0, 0, 3)$	$[e_1][x + t]$	$(1, -1)_{(\frac{1}{2}, -\frac{1}{2})}$	$\mathbf{2}_{(-\frac{1}{2}, \frac{1}{2})}$
$d_{10} d_3 d_6^2 - d_{10} d_2 d_6 d_7 - 2d_{10} d_2 d_3 d_9$ $+ d_{10}^2 d_2^2 - d_3 d_6 d_7 d_9 + d_2 d_7^2 d_9 + d_3^2 d_9^2 = 0$	$(0, 0, 3)$	$[e_0][x + t]$	$(-1, 1)_{(\frac{1}{2}, \frac{1}{2})}$	$\mathbf{2}_{(\frac{1}{2}, \frac{1}{2})}$
$4d_2 d_9 - d_6^2 = 0$	$(1, 2, 3)$	–	–	–

Table 13. Summary of SU(2) top 3. In this model the discrete symmetry is enhanced to \mathbb{Z}_2 due to the form of $D_{\mathbb{Z}_2}$. Hence, the representations are labeled by \mathbb{Z}_4 charges, which here are multiples of $\frac{1}{2}$ modulo $2\mathbb{Z}$. Note that the charge assignments exhibit the global gauge group structure $[\text{SU}(2) \times \text{U}(1) \times \mathbb{Z}_4]/(\mathbb{Z}_2^{U(1)} \times \mathbb{Z}_2^{\text{bisec}})$. Both quotient factors are embedded in the center of SU(2), however, the first one only affects the U(1) charges while the second one restricts the \mathbb{Z}_4 charges.

Factorization:	$b_1 \rightarrow e_1^2 d_1 \quad b_2 \rightarrow e_1 d_2 \quad b_3 \rightarrow d_3$ $b_5 \rightarrow e_1 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow e_0 d_7$ $b_8 \rightarrow d_8 \quad b_9 \rightarrow e_0 d_9 \quad b_{10} \rightarrow e_0^2 d_{10}$			
SRI:	$\{xt, ys, xe_1, se_1\}$			
Charge Generators	$\sigma(s_1) = [y] - [x]$ $\sigma(s^{(2)}) = [x] + [e_1]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_3 = 0$	$(0, 0, 3)$	$[e_0][y]$	$(-1, 1)_{(-1, 1)}$	$\mathbf{2}_{(-1, -)}$
$d_8 = 0$	$(0, 0, 3)$	$[e_0][x]$	$(-1, 1)_{(1, 0)}$	$\mathbf{2}_{(1, +)}$
$d_3 d_5^2 - d_2 d_5 d_6 + d_1 d_6^2$ $+ d_2^2 d_8 - 4d_1 d_3 d_8 = 0$	$(0, 0, 3)$	$[e_0][x + s]$	$(-1, 1)_{(0, 1)}$	$\mathbf{2}_{(0, -)}$
$-d_{10} d_6^2 + 4d_{10} d_3 d_8 - d_7^2 d_8$ $+ d_6 d_7 d_9 - d_3 d_9^2 = 0$	$(0, 0, 3)$	$[e_1][x + s]$	$(1, -1)_{(0, 0)}$	$\mathbf{2}_{(0, +)}$
$4d_3 d_8 - d_6^2 = 0$	$(1, 2, 3)$	–	–	–

Table 14. Summary of SU(2) top 4.

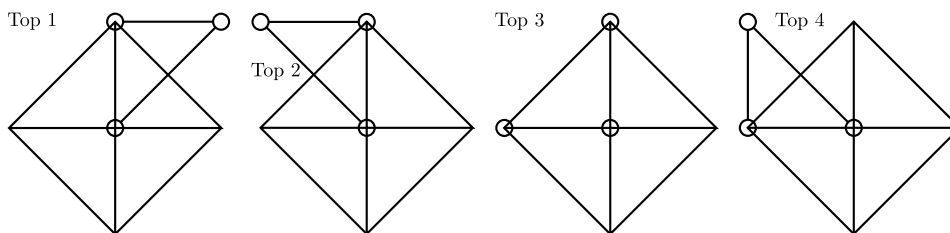


Figure 2. The toric diagram of the four inequivalent SU(3) tops over F_2 .

Factorization:	$b_1 \rightarrow f_1 d_1 \quad b_2 \rightarrow f_0 f_1 d_2 \quad b_3 \rightarrow f_0^2 d_3$ $b_5 \rightarrow f_1 f_2 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow f_0 d_7$ $b_8 \rightarrow f_1 f_2^2 d_8 \quad b_9 \rightarrow f_2 d_9 \quad b_{10} \rightarrow f_0 f_2 d_{10}$			
vertices:	$f_0 : (0, 0, 1), f_1 : (0, 1, 1), f_2 : (1, 1, 1)$			
SRI:	$\{xt, xf_2, ys, yf_0, tf_1, sf_1, sf_2\}$			
Charge Generators	$\sigma(s_1) = [y] - [x] + \frac{1}{3}(2[f_2] + [f_1])$ $\sigma(s^{(2)}) = [x] + \frac{1}{3}(2[f_2] + [f_1])$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_1 = 0$	$(0, 0, 4)$	$[f_0][t]$	$(1, 0)_{(\frac{2}{3}, \frac{1}{3})}$	$\mathbf{3}_{(\frac{2}{3}, -)}$
$d_{10}d_6 - d_7d_9 = 0$	$(0, 0, 4)$	$[f_2][x]$	$(0, -1)_{(\frac{2}{3}, -\frac{2}{3})}$	$\mathbf{3}_{(\frac{2}{3}, +)}$
$d_3d_6^2 - d_2d_6d_7 + d_1d_7^2 = 0$	$(0, 0, 4)$	$[f_1][y]$	$(-1, 1)_{(-\frac{1}{3}, \frac{1}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, -)}$
$d_6^2d_8 - d_5d_6d_9 + d_1d_9^2 = 0$	$(0, 0, 4)$	$[f_0][x]$	$(0, 1)_{(\frac{1}{3}, -\frac{2}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, +)}$
$d_6 = 0$	$(2, 2, 4)$	-	-	-

Table 15. Summary of SU(3) top 1.

Factorization:	$b_1 \rightarrow f_2 d_1 \quad b_2 \rightarrow f_1 d_2 \quad b_3 \rightarrow f_0 f_1 d_3$ $b_5 \rightarrow f_1 f_2 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow f_0 d_7$ $b_8 \rightarrow f_2 d_8 \quad b_9 \rightarrow f_0 f_2 d_9 \quad b_{10} \rightarrow f_0^2 f_2 d_{10}$			
vertices:	$f_0 : (0, 0, 1), f_1 : (-1, 1, 1), f_2 : (0, 1, 1)$			
SRI:	$\{xt, x f_2, ys, y f_0, t f_1, s f_1, s f_2\}$			
Charge Generators	$\sigma(s_1) = [y] - [x] - \frac{1}{3}([f_1] - [f_2])$ $\sigma(s^{(2)}) = [x] + \frac{2}{3}(2[f_1] + [f_2])$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_8 = 0$	$(0, 0, 4)$	$[f_0][x]$	$(1, 0)_{(\frac{2}{3}, \frac{1}{3})}$	$\mathbf{3}_{(\frac{2}{3}, -)}$
$-d_3 d_6 + d_2 d_7 = 0$	$(0, 0, 4)$	$[f_2][t]$	$(0, -1)_{(\frac{2}{3}, -\frac{2}{3})}$	$\mathbf{3}_{(\frac{2}{3}, +)}$
$-d_2 d_5 d_6 + d_1 d_6^2 + d_2^2 d_8 = 0$	$(0, 0, 4)$	$[f_0][t]$	$(0, 1)_{(\frac{1}{3}, \frac{2}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, +)}$
$d_1 0 d_6^2 + d_7^2 d_8 - d_6 d_7 d_9 = 0$	$(0, 0, 4)$	$[f_1][y]$	$(-1, 1)_{(-\frac{1}{3}, \frac{1}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, -)}$
$d_6 = 0$	$(2, 2, 4)$	-	-	-

Table 16. Summary of SU(3) top 2.

Factorization:	$b_1 \rightarrow f_0 f_2 d_1 \quad b_2 \rightarrow f_0 d_2 \quad b_3 \rightarrow f_0^2 f_1 d_3$ $b_5 \rightarrow f_2 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow f_0 f_1 d_7$ $b_8 \rightarrow f_1 f_2^2 d_8 \quad b_9 \rightarrow f_1 f_2 d_9 \quad b_{10} \rightarrow f_1 d_{10}$			
vertices:	$f_0 : (0, 0, 1), f_1 : (-1, 0, 1), f_2 : (0, 1, 1)$			
SRI:	$\{xt, x f_0, x f_2, ys, y f_0, t f_1, s f_2\}$			
Charge Generators	$\sigma(s_1) = [y] - [x] - \frac{2}{3}([f_1] - [f_2])$ $\sigma(s^{(2)}) = [x]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_{10} = 0$	$(0, 0, 4)$	$[f_0][y]$	$(0, 1)_{(-\frac{2}{3}, 1)}$	$\mathbf{3}_{(\frac{2}{3}, -)}$
$d_2 d_5 - d_1 d_6 = 0$	$(0, 0, 4)$	$[f_1][t]$	$(-1, 1)_{(\frac{2}{3}, 0)}$	$\mathbf{3}_{(\frac{2}{3}, +)}$
$d_{10} d_2^2 + d_3 d_6^2 - d_2 d_6 d_7 = 0$	$(0, 0, 4)$	$[f_2][t]$	$(1, -1)_{(\frac{1}{3}, 0)}$	$\mathbf{3}_{(\frac{1}{3}, +)}$
$d_{10} d_5^2 + d_6^2 d_8 - d_5 d_6 d_9 = 0$	$(0, 0, 4)$	$[f_0][s]$	$(1, 0)_{(-\frac{1}{3}, 1)}$	$\mathbf{3}_{(-\frac{1}{3}, -)}$
$d_6 = 0$	$(2, 2, 4)$	-	-	-

Table 17. Summary of SU(3) top 3.

Factorization:	$b_1 \rightarrow f_1^2 f_2 d_1 \quad b_2 \rightarrow f_1 f_2 d_2 \quad b_3 \rightarrow f_2 d_3$ $b_5 \rightarrow f_1 d_5 \quad b_6 \rightarrow d_6 \quad b_7 \rightarrow f_0 f_2 d_7$ $b_8 \rightarrow f_0 f_1 d_8 \quad b_9 \rightarrow f_0 d_9 \quad b_{10} \rightarrow f_0^2 f_2 d_{10}$			
vertices:	$f_0 : (0, 0, 1), f_1 : (-1, 0, 1), f_2 : (-1, 1, 1)$			
SRI:	$\{xt, xf_0, ys, yf_1, tf_2, sf_2, tf_1\}$			
Charge Generators	$\sigma(s_1) = [y] - [x] - \frac{1}{3}([f_1] - [f_2])$ $\sigma(s^{(2)}) = [x] - [f_0]$			
Locus	$V(f, g, \Delta)$	matter \mathbb{P}^1	Weight	Rep
$d_3 = 0$	$(0,0,4)$	$[f_1][t]$	$(-1, 1)_{(\frac{2}{3}, -\frac{1}{3})}$	$\mathbf{3}_{(\frac{2}{3}, -)}$
$-d_6 d_8 + d_5 d_9 = 0$	$(0,0,4)$	$[f_0][x]$	$(1, 0)_{(\frac{2}{3}, \frac{2}{3})}$	$\mathbf{3}_{(\frac{2}{3}, +)}$
$d_3 d_5^2 - d_2 d_5 d_6 + d_1 d_6^2 = 0$	$(0, 0, 4)$	$[f_2][s]$	$(0, -1)_{(-\frac{1}{3}, -\frac{1}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, -)}$
$d_{10} d_6^2 - d_6 d_7 d_9 + d_3 d_9 = 0$	$(0, 0, 4)$	$[f_1][x]$	$(-1, 0)_{(\frac{1}{3}, -\frac{2}{3})}$	$\mathbf{3}_{(-\frac{1}{3}, +)}$
$d_6 = 0$	$(2, 2, 4)$	-	-	-

Table 18. Summary of SU(3) top 4.

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