$\mathcal{N} = 1$ superfield description of BPS solutions in 6D gauged SUGRA with 3-branes

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ABSTRACT: We provide $\mathcal{N} = 1$ superfield description of BPS backgrounds in six-dimensional supergravity (6D SUGRA) with 3-branes, which is compactified on a two-dimensional space. The brane terms induce the localized fluxes. We find a useful gauge in which the background equations become significantly simple. This is not the Wess-Zumino gauge, and the relation to the usual component-field expression of 6D SUGRA is not straightforward. One of the equations reduces to the Liouville equation. By moving to the Wess-Zumino gauge, we check that our expressions reproduce the known results of the previous works, which are expressed in the component fields. Our results help us develop the systematic derivation of four-dimensional effective theories that keeps the $\mathcal{N} = 1$ SUSY structure.

KEYWORDS: Field Theories in Higher Dimensions, Flux compactifications, Supergravity Models, Superspaces

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1 Introduction

The $\mathcal{N} = 1$ superfield description of higher-dimensional supersymmetric (SUSY) theories [1]–[9] is useful in various aspects.¹ It makes the expression of the action compact. Especially, when we consider a system with lower-dimensional branes whose dimensions are not less than four, we can express the action keeping the common $\mathcal{N} = 1$ SUSY manifest. It describes the bulk-brane interactions in a transparent manner. Such a description also makes it possible to derive four-dimensional (4D) effective action directly from the original higher-dimensional theory. Besides, the $\mathcal{N} = 1$ superfield formalism is familiar to many researchers, and easy to handle.

When the extra dimensions are compactified on some manifold or orbifold, the moduli fields appear in 4D effective theories. In order to deal with such moduli and discuss the moduli stabilization, we have to work in the context of supergravity (SUGRA). In this paper, we consider six-dimensional (6D) SUGRA compactified on two-dimensional compact spaces, such as a sphere or torus. It is known that a tensor multiplet need to be introduced for the Lagrangian description of 6D SUGRA [10, 11]. We have derived the $\mathcal{N} = 1$ superfield description of the couplings between the tensor and vector multiplets in ref. [12], and extended the result to SUGRA by inserting the superfields that contain the fields in the Weyl multiplets in refs. [13, 14]. The $\mathcal{N} = 1$ superfield action is also a good starting point to derive 4D effective theory keeping the $\mathcal{N} = 1$ SUSY structure, just we did in five-dimensional SUGRA [15]–[20].

We can introduce brane terms localized in the extra dimensions. Here we add brane terms to the bulk action, which lead to the localized magnetic fluxes. They also induce the tensions of the branes, and affect the geometry of the compact space [21–23].

From the superfield action, we can derive the superfield equations of motion (EOMs) straightforwardly. These equations become much simpler for the Bogomol’nyi-Prasad-Sommerfield (BPS) background that preserves $\mathcal{N} = 1$ SUSY. By solving those BPS equations, we obtain the BPS background field configurations. Such backgrounds have been investigated in the previous works, mainly for the sphere compactification [21–23]. The backgrounds in these works are described in terms of the component fields in 6D SUGRA. In this paper, we express the BPS backgrounds in the $\mathcal{N} = 1$ superfield language. As we have shown in our previous work [14], the relation between the superfields and the usual component fields of 6D SUGRA is not simple. Thus expressing the known background configurations in terms of the superfields is a nontrivial task. This helps us develop a systematic derivation of 4D effective theories that keeps the $\mathcal{N} = 1$ SUSY manifest, and enables us to treat the compactifications with different topologies on equal footing.

The paper is organized as follows. In the next section, we briefly review our previous work [14], and provide the bulk and the brane actions in terms of $\mathcal{N} = 1$ superfields. Section 3 is the main part of this paper. We derive the BPS background equations from the superfield action. After choosing an appropriate gauge, we solve the equations and check that the known results are reproduced by the solutions in our superfield approach. In sections 4 and 5, we summarize the results in the previous works in our notation for

¹“ $\mathcal{N} = 1$ ” denotes SUSY with four supercharges in this paper.

the sphere and torus compactifications, respectively. Section 6 is devoted to the summary. In appendix A, we explicitly show some of the components of the superfields in terms of the usual component fields of 6D SUGRA. In appendix B, we list some of the EOMs that are not shown in the text because of their lengthy expressions. We provide a comment on the quantization condition of the total flux in appendix C, and the definitions of the Weierstrass elliptic functions in appendix D.

2 $\mathcal{N} = 1$ superfield description of 6D SUGRA action

In this section, we provide a brief review of our previous results in ref. [14]. The 6D spacetime indices $M, N, \dots = 0, 1, 2, \dots, 5$ are divided into the 4D part $\mu, \nu, \dots = 0, 1, 2, 3$ and the extra-dimensional part $m, n, \dots = 4, 5$. The corresponding local Lorentz indices are denoted by the underbarred ones. We assume that the 4D part of the spacetime has the flat background geometry,² and follow the notation of ref. [26] for the 2-component spinors.

2.1 $\mathcal{N} = 1$ decomposition of 6D supermultiplets

The field content of 6D SUGRA consists of the Weyl multiplet $\mathbb{E}$, the hypermultiplets $\mathbb{H}^A$ ($A = 1, 2, \dots, n_H$), the vector multiplets $\mathbb{V}^I$ ($I = 1, 2, \dots, n_V$), and the tensor multiplet $\mathbb{T}$.³

Each 6D supermultiplet can be decomposed into $\mathcal{N} = 1$ superfields as follows.

Weyl multiplet $\mathbb{E}$

$$\begin{aligned} U^\mu, U^4, U^5, V_E &: \text{Real superfields} \\ S_E &: \text{Chiral superfield} \\ \Psi_4^\alpha, \Psi_5^\alpha &: \text{Spinor superfields} \end{aligned} \quad (2.1)$$

The superfields V_E and one of Ψ_4^α and Ψ_5^α are dependent fields, as will be mentioned in section 2.2.

Hypermultiplet $\mathbb{H}^A$

$$H^{2A-1}, H^{2A} : \text{Chiral superfields} \quad (2.2)$$

The hypermultiplets are divided into the compensator multiplets $A = 1, 2, \dots, n_{\text{comp}}$ and the physical ones $A = n_{\text{comp}} + 1, \dots, n_H$.

Vector multiplet $\mathbb{V}^I$

$$\begin{aligned} V^I &: \text{Real superfield} \\ \Sigma^I &: \text{Chiral superfield} \end{aligned} \quad (2.3)$$

²In 6D SUGRA compactified on a 2D compact space, the 4D flat spacetime is a unique maximally symmetric solution [24, 25].

³We focus on the case of a single tensor multiplet because the theory cannot be described by the Lagrangian in the other cases. Besides, the anomaly cancellation conditions also constrain n_H and n_V and the gauge group [27–29]. In this paper, we do not take account of such constraints, and assume that the gauge groups are Abelian, for simplicity.

E					$\mathbb{H}^A$	$\mathbb{V}^I$		T			field strength			
U^μ	U^m	Ψ_m^α	S_E	V_E	$H^{\bar{A}}$	V^I	Σ^I	$\Upsilon_{T\alpha}$	V_{Tm}	Σ_T	$\mathcal{W}_\alpha^I$	$\mathcal{X}_T$	$\mathcal{Y}_{T\alpha}$	$\mathcal{V}_T$
0	0	$-3/2$	0	-2	$3/2$	0	0	$3/2$	0	0	$3/2$	2	$3/2$	0

Table 1. The Weyl weights of the $\mathcal{N} = 1$ superfields. The index $\bar{A}$ runs from 1 to $2n_H$.

Tensor multiplet T

$$\begin{aligned}
 \Upsilon_{T\alpha} &: \text{Chiral spinor superfield} \\
 V_{T4}, V_{T5} &: \text{Real superfields} \\
 \Sigma_T &: \text{Chiral superfield}
 \end{aligned} \tag{2.4}$$

The correspondence between these superfields and the component fields of 6D SUGRA is summarized in appendix A. The Weyl weights of the superfields are listed in table 1.⁴

Among the above superfields, U^μ corresponds to the 4D part of the Weyl multiplet, and will be dropped in the following expressions because they are irrelevant to the background equations.⁵

The superfields U^m contain e_μ^m , and are used to covariantize the spinor derivatives D_α and $\bar{D}_{\dot{\alpha}}$. Define the operator $\mathcal{P}_U$ that shifts x^m by iU^m .

$$\mathcal{P}_U : x^m \rightarrow x^m + iU^m(x, \theta, \bar{\theta}). \tag{2.5}$$

Then, the covariant derivatives are defined as

$$D_\alpha^{\mathcal{P}} \equiv \bar{\mathcal{P}}_U D_\alpha \bar{\mathcal{P}}_U^{-1}, \quad \bar{D}_{\dot{\alpha}}^{\mathcal{P}} \equiv \mathcal{P}_U \bar{D}_{\dot{\alpha}} \mathcal{P}_U^{-1}. \tag{2.6}$$

The superfields Ψ_m^α contain e_m^μ , and are used to covariantize ∂_m as⁶

$$\nabla_m \equiv \partial_m - \left(\frac{1}{4} \bar{D}^2 \Psi_m^\alpha D_\alpha - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} \Psi_m^\alpha \partial_\mu + \frac{w}{12} \bar{D}^2 D^\alpha \Psi_{m\alpha} \right) + \mathcal{O}(\Psi^2, \Psi U^n), \tag{2.7}$$

in the chiral superspace, and

$$\begin{aligned}
 \nabla_m^{\mathcal{P}} \equiv \mathcal{P}_U \partial_m \mathcal{P}_U^{-1} - & \left(\frac{1}{4} \bar{D}^2 \Psi_m^\alpha D_\alpha + \frac{1}{2} \bar{D}^{\dot{\alpha}} \Psi_m^\alpha \bar{D}_{\dot{\alpha}} D_\alpha + \frac{w+n}{24} \bar{D}^2 D^\alpha \Psi_{m\alpha} \right) \\
 & - \left(\frac{1}{4} D^2 \bar{\Psi}_{m\dot{\alpha}} \bar{D}^{\dot{\alpha}} + \frac{1}{2} D^\alpha \bar{\Psi}_{m\dot{\alpha}} D_\alpha \bar{D}_{\dot{\alpha}} + \frac{w-n}{24} D^2 \bar{D}_{\dot{\alpha}} \bar{\Psi}_m^{\dot{\alpha}} \right) + \mathcal{O}(\Psi^2, \Psi U^n),
 \end{aligned} \tag{2.8}$$

in the full superspace. Here, w and n denote the Weyl and chiral weights, respectively.

⁴We should note that V^I in (A.6) (and U^μ, U^m in (A.2), V_{Tm} in (A.7)) are in the Wess-Zumino gauge. This indicates that we need to choose the Wess-Zumino gauge in order to see the correspondence to the component-field expression of 6D SUGRA.

⁵The U^μ -dependence of the action can be easily recovered by using the result of ref. [30], in which the linearized 4D SUGRA is discussed.

⁶We need not discriminate the flat 4D index $\underline{\mu}$ and the curved one μ at the linearized order since the 4D part of the background spacetime is assumed to be flat ($\langle e_\nu^\mu \rangle = \delta_\nu^\mu$).

The (super) gauge transformations are given by

$$\delta_\Lambda V^I = \Lambda^I + \bar{\Lambda}^I, \quad \delta_\Lambda \Sigma^I = \nabla_E \Lambda^I, \quad (2.9)$$

where the transformation parameters Λ^I are chiral superfields, and

$$\nabla_E \equiv \frac{1}{S_E} \nabla_4 - S_E \nabla_5. \quad (2.10)$$

The gauge-invariant field strength superfields are given by

$$\mathcal{W}_\alpha^I \equiv -\frac{1}{4} (\bar{D}^\mathcal{P})^2 D_\alpha^\mathcal{P} V^I + \mathcal{O}(U^m \Sigma). \quad (2.11)$$

The SUSY extension of the tensor gauge transformation: $B_{MN} \rightarrow B_{MN} + \partial_M \lambda_N - \partial_N \lambda_M$ (λ_M : real parameter) is expressed as

$$\begin{aligned} \delta_G V_{T4} &= -\partial_4 V_G + \text{Re} (S_E \Sigma_G), & \delta_G V_{T5} &= -\partial_5 V_G + \text{Re} \left(\frac{\Sigma_G}{S_E} \right), \\ \delta_G \Upsilon_{T\alpha} &= -\frac{1}{4} \bar{D}^2 D_\alpha V_G, \\ \delta_G \Sigma_T &= -\frac{1}{2} \partial_4 \left(\frac{\Sigma_G}{S_E} \right) + \frac{1}{2} \partial_5 (S_E \Sigma_G), \end{aligned} \quad (2.12)$$

up to U^m - or Ψ_m^α -dependent terms. The transformation parameters V_G and Σ_G are real and chiral superfields respectively, which form a 6D vector multiplet $\mathbb{V}_G$. The field strength superfields invariant under this transformation are⁷

$$\begin{aligned} \mathcal{X}_T &\equiv \frac{1}{2} \text{Im} \left(D^{\mathcal{P}\alpha} \hat{\Upsilon}_{T\alpha} \right), \\ \mathcal{Y}_T &\equiv \frac{1}{2S_E} \mathcal{W}_{T4\alpha} + \frac{S_E}{2} \mathcal{W}_{T5\alpha} + \frac{1}{2} S_E \mathcal{O}_E \Upsilon_{T\alpha}, \\ \mathcal{V}_T &\equiv \text{Re} \left(\nabla_4^\mathcal{P} V_{T5} - \nabla_5^\mathcal{P} V_{T4} + 2J_\mathcal{P} \hat{\Sigma}_T \right), \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} \hat{\Upsilon}_{T\alpha} &\equiv \mathcal{P}_U \Upsilon_{T\alpha}, \\ \mathcal{W}_{Tm\alpha} &\equiv -\frac{1}{4} (\bar{D}^\mathcal{P})^2 D_\alpha^\mathcal{P} V_{Tm} + \mathcal{O}(U^n \Sigma_T), \\ \mathcal{O}_E &\equiv \frac{1}{S_E^2} \nabla_4 + \nabla_5, \\ \hat{\Sigma}_T &\equiv \mathcal{P}_U \Sigma_T, \\ J_\mathcal{P} &\equiv 1 + i\partial_m U^m - \partial_4 U^4 \partial_5 U^5 + \partial_4 U^5 \partial_5 U^4. \end{aligned} \quad (2.14)$$

Note that $J_\mathcal{P}$ is the Jacobian for the shift by $\mathcal{P}_U$.

⁷The U^m - and Ψ_m^α -dependences are determined by the (SUSY extension of) diffeomorphism covariance [14].

2.2 Constraints on tensor multiplets

The tensor multiplet $(Y_{T\alpha}, V_{Tm})$ is subject to the following the constraints, which reduce to the self-dual condition in the global SUSY limit.

$$\begin{aligned}\mathcal{X}_T V_E &= \mathcal{V}_T, \\ \frac{1}{S_E} \mathcal{W}_{T4\alpha} - S_E \mathcal{W}_{T5\alpha} + \nabla_E Y_{T\alpha} &= 0.\end{aligned}\quad (2.15)$$

From the first constraint, the “volume modulus” superfield V_E is expressed in terms of the tensor field strengths $\mathcal{X}_T$ and $\mathcal{V}_T$. Since ∇_E depends on Ψ_m^α , the second constraint indicates that either Ψ_4^α or Ψ_5^α are dependent field, and can be expressed in terms of the other superfields.

2.3 Invariant action

We will omit U^μ and Ψ_m^α in the following because they are irrelevant to the discussions in the next sections.

2.3.1 Bulk action

The 6D SUGRA action is expressed in terms of the $\mathcal{N} = 1$ superfields as

$$\begin{aligned}S_{\text{bulk}} &= \int d^6x (\mathcal{L}_H + \mathcal{L}_{VT}), \\ \mathcal{L}_H &= -2 \int d^4\theta |\mathcal{J}_P| \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T} \right)^{1/2} \left(\hat{H}_{\text{odd}}^\dagger \tilde{d} e^V \hat{H}_{\text{odd}} + \hat{H}_{\text{even}}^\dagger \tilde{d} e^{-V} \hat{H}_{\text{even}} \right), \\ &\quad + \left[\int d^2\theta \left\{ H_{\text{odd}}^t \tilde{d} (\partial_E - \Sigma) H_{\text{even}} - H_{\text{even}}^t \tilde{d} (\partial_E + \Sigma) H_{\text{odd}} \right\} + \text{h.c.} \right], \\ \mathcal{L}_{VT} &= \int d^4\theta f_{IJ} \left[\left\{ -2J_P \hat{\Sigma}^I D^{\mathcal{P}\alpha} V^J \hat{Y}_{T\alpha} + \frac{J_P}{2} (\partial_E^{\mathcal{P}} V^I D^{\mathcal{P}\alpha} V^J - \partial_E^{\mathcal{P}} D^{\mathcal{P}\alpha} V^I V^J) \hat{Y}_{T\alpha} + \text{h.c.} \right\} \right. \\ &\quad \left. + \mathcal{V}_T \left(D^{\mathcal{P}\alpha} V^I \hat{\mathcal{W}}_\alpha^J + \frac{1}{2} V^I D^{\mathcal{P}\alpha} \hat{\mathcal{W}}_\alpha^J + \text{h.c.} \right) \right. \\ &\quad \left. + \frac{\mathcal{X}_T}{R_E^-} \left\{ 4 \left(\partial_E^{\mathcal{P}} V^I - \hat{\Sigma}^I \right)^\dagger \left(\partial_E^{\mathcal{P}} V^J - \hat{\Sigma}^J \right) - 2 \left(\partial_E^{\mathcal{P}} V^I \right)^\dagger \partial_E^{\mathcal{P}} V^J \right. \right. \\ &\quad \left. \left. + \left(2J_P R_E^+ \hat{\Sigma}^I \hat{\Sigma}^J + \text{h.c.} \right) \right\} \right],\end{aligned}\quad (2.16)$$

where $H_{\text{odd}} \equiv (H^1, H^3, \dots, H^{2n_H-1})^t$, $H_{\text{even}} \equiv (H^2, H^4, \dots, H^{2n_H})^t$, $\hat{\Phi} \equiv \mathcal{P}_U \Phi$ for a chiral superfield Φ , $\partial_E \equiv S_E^{-1} \partial_4 - S_E \partial_5$, $\partial_E^{\mathcal{P}} \equiv \mathcal{P}_U \partial_E \mathcal{P}_U^{-1}$, and

$$\begin{aligned}R_E^- &\equiv \frac{1}{2i} \left(J_S^{(2)} \frac{\bar{\hat{S}}_E}{\hat{S}_E} - J_S^{(1)} \frac{\hat{S}_E}{\bar{\hat{S}}_E} \right), \\ R_E^+ &\equiv \frac{1}{2} \left(J_S^{(2)} \frac{\bar{\hat{S}}_E}{\hat{S}_E} + J_S^{(1)} \frac{\hat{S}_E}{\bar{\hat{S}}_E} \right), \\ J_S^{(1)} &\equiv 1 + i (\partial_4 U^4 - \partial_5 U^5) - 2i \bar{\hat{S}}_E^2 \partial_5 U^4 + \mathcal{O}(U^2), \\ J_S^{(2)} &\equiv 1 - i (\partial_4 U^4 - \partial_5 U^5) - \frac{2i}{\bar{\hat{S}}_E^2} \partial_4 U^5 + \mathcal{O}(U^2).\end{aligned}\quad (2.17)$$

The $n_H \times n_H$ constant matrix $\tilde{d}$ is the metric of the hyperscalar space that discriminates the compensator multiplets from the physical ones, and can be chosen as $\tilde{d} = \text{diag}(\mathbf{1}_{n_{\text{comp}}}, -\mathbf{1}_{n_H - n_{\text{comp}}})$. The $n_V \times n_V$ constant matrix f_{IJ} is real and symmetric. In the hyper-sector Lagrangian $\mathcal{L}_H$, the vector multiplets are described in the matrix notation,

$$V \equiv V^I t_I, \quad \Sigma \equiv \Sigma^I t_I, \quad (2.18)$$

where t_I are the generators for the Abelian gauge group, i.e., the charge matrices. Their components are denoted as

$$t_I = \begin{pmatrix} 2c_I & & \\ & \ddots & \\ & & \end{pmatrix}, \quad (2.19)$$

where c_I are the compensator charges.

The above action is invariant under the diffeomorphisms and the Lorentz transformations involving the extra dimensions, and the (super) gauge transformations [14].

2.3.2 Brane action

We also introduce brane terms localized at $x^m = x_k^m$ ($k = 1, \dots, N$).⁸ Here we consider the case of single compensator, i.e., $n_{\text{comp}} = 1$. Since the bulk and the branes feel the same gravity, the chiral compensator superfields appearing in the brane action should originate from the bulk compensator multiplet $\mathbb{H}^1 = (H^1, H^2)$. We should note that H^1 and H^2 cannot mix with each other when $c_I \neq 0$ for some I because they have opposite charges. Thus the brane compensators are either H^1 or H^2 . For simplicity, we assume that all the brane compensators come from $H_{\text{even}}^1 = H^2$. Then we can introduce the following brane terms.

$$S_{\text{brane}} = \int d^6 x \mathcal{L}_{\text{brane}},$$

$$\mathcal{L}_{\text{brane}} = - \int d^4 \theta \sum_{k=1}^N C_k \left(\frac{\mathcal{X}_{\text{T}} R_{\text{E}}^-}{\mathcal{V}_{\text{T}}} \right)^{1/4} \left(\hat{H}_{\text{even}}^1 e^{-2c_I V^I} \hat{H}_{\text{even}}^1 \right)^{1/2} \delta^{(2)}(y - y_k), \quad (2.20)$$

where C_k are real constants, $\vec{y} \equiv (x^4, x^5)^t$ are the extra-dimensional coordinates, and $\vec{y}_k \equiv (x_k^4, x_k^5)^t$ are the brane positions. The powers in (2.20) are determined by the Weyl weight and by requiring that the extra-dimensional components of the sechsbein $e_m^{\underline{n}}$ contained in the superfields are cancelled. (See (A.4), (A.5), and (A.8).) The above terms represent the brane-localized Fayet-Iliopoulos (FI) terms, which lead to the brane tensions and the localized fluxes as we will see in the next section.

3 BPS background

3.1 Background equations of motion

By varying the action in the previous section with respect to the superfields, we obtain the superfield EOMs. The background field configuration can be found by solving them. Here,

⁸We do not consider branes whose codimension is one, for simplicity.

we focus on the background that preserves $\mathcal{N} = 1$ SUSY. Namely, the F- and D-terms of the superfields can be put to zero. Besides, since we are interested in the 4D-Lorentz-invariant background, all the fermionic components and the bosonic components with the Lorentz indices are assumed to have vanishing backgrounds. This means that $U^m = 0$, and S_E , V_{Tm} , Σ_T , H_{even} , H_{odd} , V^I , Σ^I and $D^\alpha \Upsilon_{T\alpha}$ are x^μ - and θ -independent.⁹ Thus the tensor field strengths can be expressed as

$$\mathcal{X}_T = \frac{1}{2} \text{Im} (D^\alpha \Upsilon_{T\alpha}), \quad D^\alpha \mathcal{Y}_{T\alpha} = \frac{1}{2} S_E \mathcal{O}_E (D^\alpha \Upsilon_{T\alpha}). \quad (3.1)$$

Then we have the following EOMs for the background.

For S_E

$$\frac{1}{S_E^2} \left(H_{\text{odd}}^t \tilde{d} \partial_4 H_{\text{even}} - H_{\text{even}}^t \tilde{d} \partial_4 H_{\text{odd}} \right) + \left(H_{\text{odd}}^t \tilde{d} \partial_5 H_{\text{even}} - H_{\text{even}}^t \tilde{d} \partial_5 H_{\text{odd}} \right) = 0. \quad (3.2)$$

For V_{T4} and V_{T5}

$$\partial_4 \left\{ \left(\frac{R_E}{\mathcal{X}_T \mathcal{V}_T} \right)^{1/2} L_H \right\} = \partial_5 \left\{ \left(\frac{R_E}{\mathcal{X}_T \mathcal{V}_T} \right)^{1/2} L_H \right\} = 0, \quad (3.3)$$

where

$$L_H \equiv H_{\text{odd}}^\dagger \tilde{d} e^V H_{\text{odd}} + H_{\text{even}}^\dagger \tilde{d} e^{-V} H_{\text{even}}. \quad (3.4)$$

For H_{even}

$$\left(\partial_E - \frac{1}{2} \mathcal{O}_E S_E + \Sigma \right) H_{\text{odd}} = 0. \quad (3.5)$$

For H_{odd}

$$\left(\partial_E - \frac{1}{2} \mathcal{O}_E S_E - \Sigma \right) H_{\text{even}} = 0. \quad (3.6)$$

For V^I

$$\begin{aligned} & -2 \left(\frac{\mathcal{V}_T R_E}{\mathcal{X}_T} \right)^{1/2} \left(H_{\text{odd}}^\dagger \tilde{d} e^V t_I H_{\text{odd}} - H_{\text{even}}^\dagger \tilde{d} e^{-V} t_I H_{\text{even}} \right) + 2 f_{IJ} (\Sigma^J D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.}) \\ & - \partial_4 \left\{ \frac{2 f_{IJ} \mathcal{X}_T}{R_E \bar{S}_E} (\partial_E V^J - 2 \Sigma^J) + \text{h.c.} \right\} + \partial_5 \left\{ \frac{2 f_{IJ} \mathcal{X}_T \bar{S}_E}{R_E} (\partial_E V^J - 2 \Sigma^J) + \text{h.c.} \right\} \\ & + \sum_{k=1}^N c_I C_k \left(\frac{\mathcal{X}_T R_E}{\mathcal{V}_T} \right)^{1/4} |H_{\text{even}}^1| e^{-c_J V^J} \delta^{(2)}(y - y_k) = 0. \end{aligned} \quad (3.7)$$

For Σ^I

$$H_{\text{odd}}^t \tilde{d} t_I H_{\text{even}} = 0. \quad (3.8)$$

The EOMs for U^4 and U^5 are shown in (B.5) and (B.6).

⁹We do not choose the Wess-Zumino gauge for V_{Tm} and V^I . So their lowest components can have non-vanishing backgrounds.

3.2 Coordinate and gauge choices

It is convenient to choose the coordinates of the extra dimensions such that¹⁰

$$\langle S_E \rangle = e^{-\pi i/4} \equiv \eta. \quad (3.9)$$

Then, we have

$$\begin{aligned} R_E^- &= 1, & R_E^+ &= 0, \\ \partial_E &= 2\bar{\eta}\partial_{\bar{z}}, & \mathcal{O}_E &= 2i\partial_z, \end{aligned} \quad (3.10)$$

where $z \equiv x^4 + ix^5$.

We can gauge away the background of Σ^I by using the transformation (2.9). Then the background EOMs (3.2)–(3.8) are rewritten as

$$\begin{aligned} H_{\text{odd}}^t \tilde{d}\partial_z H_{\text{even}} - H_{\text{even}}^t \tilde{d}\partial_z H_{\text{odd}} &= 0, \\ \partial_z \left(\frac{L_H}{\sqrt{\mathcal{X}_T \mathcal{V}_T}} \right) &= \partial_{\bar{z}} \left(\frac{L_H}{\sqrt{\mathcal{X}_T \mathcal{V}_T}} \right) = 0, \\ \partial_{\bar{z}} H_{\text{even}} &= \partial_{\bar{z}} H_{\text{odd}} = 0, \\ -2 \left(\frac{\mathcal{V}_T}{\mathcal{X}_T} \right)^{1/2} \left(H_{\text{odd}}^\dagger \tilde{d}e^V t_I H_{\text{odd}} - H_{\text{even}}^\dagger \tilde{d}e^{-V} t_I H_{\text{even}} \right) &- \{8f_{IJ} \partial_z (\mathcal{X}_T \partial_{\bar{z}} V^J) + \text{h.c.}\} \\ &+ \sum_k 2c_I C_k \left(\frac{\mathcal{X}_T}{\mathcal{V}_T} \right)^{1/4} |H_{\text{even}}^1| e^{-c_J V^J} \delta^{(2)}(z - z_k) = 0, \\ H_{\text{odd}}^t \tilde{d}t_I H_{\text{even}} &= 0, \end{aligned} \quad (3.11)$$

where $z_k \equiv x_k^4 + ix_k^5$. We have used that $\delta^{(2)}(y - y_k) = 2\delta^{(2)}(z - z_k)$. The second equations can be solved as

$$\frac{L_H}{\sqrt{\mathcal{X}_T \mathcal{V}_T}} \equiv b_H = (\text{real constant}). \quad (3.12)$$

The EOMs for U^m are now written as

$$\begin{aligned} 0 &= - \left(\frac{\mathcal{V}_T^{1/2} L_H}{\mathcal{X}_T^{3/2}} + L_V \right) \partial_z \text{Re} (D^\alpha \Upsilon_{T\alpha}) + 2i\partial_z \left\{ \left(\frac{\mathcal{V}_T}{\mathcal{X}_T} \right)^{1/2} L_H + \mathcal{X}_T L_V \right\} \\ &- 8 \left(\frac{\mathcal{V}_T}{\mathcal{X}_T} \right)^{1/2} \text{Im} \left(H_{\text{even}}^\dagger \tilde{d}e^{-V} \partial_z H_{\text{even}} + H_{\text{odd}}^\dagger \tilde{d}e^V \partial_z H_{\text{odd}} \right) \\ &- 2f_{IJ} \{ i\bar{\eta} (\partial_{\bar{z}} V^I \partial_z V^J - \partial_z \partial_{\bar{z}} V^I V^J) D^\alpha \mathcal{Y}_{T\alpha} - i\eta (\partial_z V^I \partial_{\bar{z}} V^J - \partial_z^2 V^I V^J) \bar{D}_{\dot{\alpha}} \bar{\mathcal{Y}}_T^{\dot{\alpha}} \} \\ &- 2f_{IJ} \{ i\bar{\eta} \partial_{\bar{z}} (\partial_z V^I V^J D^\alpha \mathcal{Y}_{T\alpha}) - i\eta \partial_z (\partial_z V^I V^J \bar{D}_{\dot{\alpha}} \bar{\mathcal{Y}}_T^{\dot{\alpha}}) \\ &\quad + 8i\partial_z (\mathcal{X}_T \partial_z V^I \partial_{\bar{z}} V^J) - 8i\partial_{\bar{z}} (\mathcal{X}_T \partial_z V^I \partial_z V^J) \} + (\text{brane terms}), \end{aligned} \quad (3.13)$$

by combining (B.5) and (B.6). We have used (3.12), and L_V defined by (B.4) becomes

$$L_V = 8f_{IJ} \partial_z V^I \partial_{\bar{z}} V^J. \quad (3.14)$$

¹⁰We can always move to this coordinate system by using the (super) diffeomorphism (i.e., the δ_{Ξ} -transformation in ref. [14]).

3.3 Background solution

For simplicity, we consider a case of $n_V = 1$, and omit the indices I and J in the following. Besides, we focus on a case that only $H_{\text{even}}^1 = H^2$ has a non-vanishing background value among $H^{\bar{A}}$. Then, it must be a constant from (3.11).

$$\begin{aligned} H_{\text{even}}^1 &\equiv h_c = (\text{complex constant}), \\ H_{\text{even}}^{a \neq 1} &= H_{\text{odd}}^b = 0. \end{aligned} \quad (3.15)$$

Thus, L_H in (3.4) is expressed as

$$L_H = |h_c|^2 e^{-2cV}. \quad (3.16)$$

Here we denote the bosonic component of $\Upsilon_{T\alpha}$ as

$$D^\alpha \Upsilon_{T\alpha} = B + 2i\sigma, \quad (3.17)$$

where B and σ are real. Then, $\mathcal{X}_T$ and $D^\alpha \mathcal{Y}_{T\alpha}$ are expressed as

$$\mathcal{X}_T = \sigma, \quad D^\alpha \mathcal{Y}_{T\alpha} = \bar{\eta} \partial_z (B + 2i\sigma). \quad (3.18)$$

Using these results, the background EOM for the vector superfield V in (3.11) becomes

$$\frac{4c|h_c|^4}{b_H \sigma} e^{-4cV} - 16f \text{Re} \partial_z (\sigma \partial_{\bar{z}} V) + \sum_k 2c C_k \sqrt{b_H \sigma} \delta^{(2)}(z - z_k) = 0, \quad (3.19)$$

and (3.13) becomes

$$\begin{aligned} &\frac{i|h_c|^4 e^{-4cV}}{2b_H \sigma^2} \partial_z B - 2if \partial_z V \{-2\partial_{\bar{z}} V \partial_z B + (\partial_z V + \partial_{\bar{z}} V) \partial_{\bar{z}} B + V \partial_z \partial_{\bar{z}} B\} \\ &+ \partial_z \left(\frac{|h_c|^4 e^{-4cV}}{b_H \sigma} \right) + 8f \partial_z V \text{Re} (\partial_z \sigma \partial_{\bar{z}} V + 2\sigma \partial_z \partial_{\bar{z}} V) + (\text{brane terms}) = 0. \end{aligned} \quad (3.20)$$

Using (3.19), the latter becomes¹¹

$$\begin{aligned} &i \left(\frac{|h_c|^4 e^{-4cV}}{2b_H \sigma^2} + 4f \partial_z V \partial_{\bar{z}} V \right) \partial_z B - 2if \partial_z V \text{Re} (2\partial_z V \partial_{\bar{z}} B + V \partial_z \partial_{\bar{z}} B) \\ &- 8f \partial_z V \text{Re} (\partial_z \sigma \partial_{\bar{z}} V) + (\text{brane terms}) = 0. \end{aligned} \quad (3.21)$$

We can see that constant σ and B is a trivial solution, and will focus on it in the following. Then, (3.19) is rewritten as

$$\partial_z \partial_{\bar{z}} \ln \psi = -\frac{K}{2} \psi - 2\pi \sum_k \alpha_k \delta^{(2)}(z - z_k), \quad (3.22)$$

¹¹The “brane terms” here contain the terms proportional to $\partial_z V \delta^{(2)}(z - z_k)$. Since $\partial_z V$ has a singularity at $z = z_k$ as we will see, these “brane terms” are regularization-dependent and we do not evaluate them in this paper.

where

$$\psi \equiv \frac{|h_c|^4}{b_H^2 \sigma^2} e^{-4cV}, \quad K \equiv \frac{2b_H c^2}{f}, \quad \alpha_k \equiv \frac{c^2 C_k \sqrt{b_H}}{4\pi f \sqrt{\sigma}}. \quad (3.23)$$

This is the Liouville equation, and its solution can be expressed in the form of [31, 32]

$$\psi = \frac{4|w'|^2}{K(1+|w|^2)^2}, \quad (3.24)$$

where $w(z)$ is a meromorphic function of z , and $w' \equiv dw/dz$. Noting that

$$\partial_z \partial_{\bar{z}} \ln |z|^2 = 2\pi \delta^{(2)}(z), \quad (3.25)$$

ψ should behave near the brane locations as¹²

$$\psi(z, \bar{z}) \sim \begin{cases} |z - z_k|^{-2\alpha_k} & (z \sim z_k) \\ |z|^{2\alpha_\infty - 4} & (|z| \sim \infty) \end{cases}. \quad (3.26)$$

We should note that there is an ambiguity in the expression of $w(z)$ for a given $\psi(z, \bar{z})$. In fact, ψ does not change under the transformation,

$$w(z) \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot w(z) \equiv \frac{a_{11}w(z) + a_{12}}{a_{21}w(z) + a_{22}}, \quad (3.27)$$

where a_{ij} ($i, j = 1, 2$) are complex constants, and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SU}(2). \quad (3.28)$$

The asymptotic behavior (3.26) is obtained when $w(z)$ behaves as

$$w(z) \sim \begin{cases} M_k \cdot (z - z_k)^{1-\alpha_k} & (z \sim z_k) \\ M_\infty \cdot z^{1-\alpha_\infty} & (|z| \sim \infty) \end{cases} \quad (3.29)$$

for $\alpha_k < 1$ and $M_k, M_\infty \in \text{SU}(2)$.

In summary, the background solution is

$$\begin{aligned} \sigma &= (\text{constant}), \quad B = (\text{constant}), \quad H_{\text{even}} = \begin{pmatrix} h_c \\ 0 \\ \vdots \end{pmatrix}, \quad H_{\text{odd}} = \vec{0}, \\ V &= -\frac{1}{4c} \ln \left\{ \frac{2fb_H \sigma^2}{c^2 |h_c|^4} \frac{|w'|^2}{(1+|w|^2)^2} \right\}, \quad \Sigma = 0, \\ \mathcal{V}_T &= \sigma \psi = \frac{|h_c|^4}{b_H^2 \sigma} e^{-4cV} = \frac{4\sigma |w'|^2}{K(1+|w|^2)^2}. \end{aligned} \quad (3.30)$$

¹²When one of the branes is located at the infinity, we should use another coordinate patch, such as $\tilde{z} \equiv -1/z$, in order to describe it by the delta function.

3.4 Expressions in Wess-Zumino gauge

Here we translate the background (3.30) to the component-field expression in 6D SUGRA. As mentioned in the footnote 4, we need to move to the Wess-Zumino gauge for this purpose. This can be achieved by using the (super) gauge transformation for the background given by

$$\begin{aligned}\tilde{V} &= V + \Lambda + \bar{\Lambda}, & \tilde{\Sigma} &= \Sigma + \partial_E \Lambda = \Sigma + 2\bar{\eta} \partial_{\bar{z}} \Lambda, \\ \tilde{H}_{\text{even}} &= e^{2c\Lambda} H_{\text{even}}, & \tilde{H}_{\text{odd}} &= e^{-2c\Lambda} H_{\text{odd}},\end{aligned}\quad (3.31)$$

(and other superfields are neutral) with

$$\Lambda = -\frac{V}{2} = \frac{1}{8c} \ln \left\{ \frac{2fb_H \sigma^2}{c^2 |h_c|^4} \frac{|w'|^2}{(1+|w|^2)^2} \right\}. \quad (3.32)$$

Then we have the background in this gauge as

$$\begin{aligned}\tilde{H}_{\text{even}} &= \left\{ \frac{2fb_H \sigma^2}{c^2 |h_c|^4} \frac{|w'|^2}{(1+|w|^2)^2} \right\}^{1/4} \begin{pmatrix} h_c \\ 0 \\ \vdots \end{pmatrix}, & \tilde{H}_{\text{odd}} &= \vec{0}, \\ \tilde{V} &= 0, & \tilde{\Sigma} &= \frac{\bar{\eta}}{4c} \left(\frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1+|w|^2} \right), \\ \tilde{\mathcal{V}}_T &= \frac{4\sigma |w'|^2}{K (1+|w|^2)^2},\end{aligned}\quad (3.33)$$

where σ and B are unchanged. Recalling that $S_E = \eta$ in our coordinates and comparing the above expressions with those in appendix A, we obtain

$$\begin{aligned}(E_4 E_5)^{1/4} \phi_2^2 &= \left\{ \frac{2fb_H \sigma^2}{c^2 |h_c|^4} \frac{|w'|^2}{(1+|w|^2)^2} \right\}^{1/4} h_c, & (E_4 E_5)^{1/4} \phi_2^{a \neq 2} &= 0, \\ \frac{i}{2} (A_4 + iA_5) &= \frac{1}{4c} \left(\frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1+|w|^2} \right), \\ e^{(2)} = \psi &= \frac{4|w'|^2}{K (1+|w|^2)^2}, & B_{\underline{45}} &= \frac{B}{4}.\end{aligned}\quad (3.34)$$

Notice that

$$E_5 = iE_4, \quad e^{(2)} = \text{Im} (\bar{E}_4 E_5) = |E_4|^2, \quad (3.35)$$

which follow from $S_E = \eta$, and

$$E_4 E_5 (\phi_2^2)^4 = \frac{2fb_H \sigma^2 (h_c)^4}{c^2 |h_c|^4} \frac{|w'|^2}{(1+|w|^2)^2}, \quad (3.36)$$

where we have used (3.23). Thus, the background can be expressed as

$$\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + |E_4 dz|^2, \\
E_4 &= -iE_5 = \frac{\sqrt{2f}|w'|}{\sqrt{b_H c}(1+|w|^2)} \exp \left\{ i \left(2 \arg(h_c) - \frac{\pi}{4} \right) \right\}, \\
A_4 &= \frac{1}{2c} \text{Im} \left(\frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1+|w|^2} \right), \quad A_5 = -\frac{1}{2c} \text{Re} \left(\frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1+|w|^2} \right), \\
\phi_2^2 &= (b_H \sigma)^{1/2}, \quad \phi_2^{a \neq 2} = 0, \\
\sigma &= (\text{real constant}), \quad B_{45} = e^{(2)} B_{\underline{45}} = \frac{B|w'|^2}{K(1+|w|^2)^2},
\end{aligned} \tag{3.37}$$

where the constants f , b_H , c and B are real, and h_c is complex.

Since the background metric for the compact space is

$$|E_4|^2 dz d\bar{z} = e^{(2)} dz d\bar{z} = \psi dz d\bar{z}, \tag{3.38}$$

and ψ behaves as (3.26) near the singularities, the space has the conical singularities at $z = z_k$, and α_k defined in (3.23) can be identified with the deficit angles, which are proportional to the brane tensions [32].¹³ Besides, the volume of the compact space is given by

$$\text{Vol}^{(2)} = \int dx^4 dx^5 e^{(2)} = \frac{1}{2} \int d^2 z e^{(2)} = \frac{1}{2} \int d^2 z \psi. \tag{3.39}$$

In order for this integral to have a finite value, (3.26) indicates that $\alpha_k < 1$ must be satisfied for all k . Using the Gauss-Bonnet formula, this integral is calculated as [32]

$$\text{Vol}^{(2)} = \frac{2}{K} \int d^2 z \frac{|w'|^2}{(1+|w|^2)^2} = \frac{2\pi}{K} \left(2 - 2g - \sum_k \alpha_k \right), \tag{3.40}$$

where g is the genus of the compact space.

3.5 Localized fluxes and total flux

After moving to the Wess-Zumino gauge, there still remains the gauge degree of freedom. We can add an arbitrary imaginary part of Λ to (3.32) maintaining the background $\tilde{V} = 0$. In such gauges, (the extra-dimensional components of) the gauge potential is expressed as $A_{\bar{z}} = -i\eta\tilde{\Sigma} = -2i\partial_{\bar{z}}\Lambda$. Thus, the field strength $F_{z\bar{z}}$ is

$$F_{z\bar{z}} \equiv \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = -2i\partial_z \partial_{\bar{z}} (\Lambda + \bar{\Lambda}), \tag{3.41}$$

¹³In the Planck unit, the tension τ_k is equal to $2\pi\alpha_k$.

which is certainly gauge-invariant under the remaining gauge transformation.¹⁴ Using (3.32) and (3.22), $F_{z\bar{z}}$ is calculated as

$$\begin{aligned} F_{z\bar{z}} &= 2i\partial_z\partial_{\bar{z}}V = \frac{iK}{4c}\psi + \frac{i\pi}{c}\sum_k\alpha_k\delta^{(2)}(z-z_k) \\ &= \frac{i|w'|^2}{c\left(1+|w|^2\right)^2} + \frac{i\pi}{c}\sum_k\alpha_k\delta^{(2)}(z-z_k). \end{aligned} \quad (3.42)$$

Therefore, the brane terms (2.20) induce the brane-localized fluxes. The total flux is calculated using (3.40) as¹⁵

$$\begin{aligned} \mathcal{B} &= \int dx^4 dx^5 cF_{45} = -i \int d^2z cF_{z\bar{z}} = \int d^2z \frac{|w'|^2}{\left(1+|w|^2\right)^2} + \pi \sum_k \alpha_k \\ &= \pi \left(2 - 2g - \sum_k \alpha_k\right) + \pi \sum_k \alpha_k = \pi(2 - 2g). \end{aligned} \quad (3.43)$$

We have used that $\int dx^4 dx^5 = \frac{1}{2} \int d^2z$, $F_{z\bar{z}} = -2iF_{45}$, and (3.40). Thus, the total flux $\mathcal{B}$ is independent of the brane tensions. Eq. (3.43) indicates that the background solution (3.37) automatically satisfies the flux quantization condition in appendix C.

So far, we have not specified the compact space. The form of $w(z)$ in (3.30) or (3.37) depends on it. In the component-field expressions, this issue is discussed in the previous works [32–34]. In the next two sections, we consider specific compactifications and summarize those results in our notations, for the sake of completeness.

4 Sphere compactification

Let us consider the case that the superfields are defined in the entire complex plane including infinity, i.e., the Riemann sphere. In this case, the tensions are constrained as

$$\alpha_k < 1, \quad \sum_k \alpha_k < 2. \quad (4.1)$$

The second one comes from the condition that the volume (3.40) should be positive.

4.1 In the absence of branes

In the absence of the branes, $w(z)$ has no singularities and is holomorphic over the whole complex plane. Thus we can redefine the complex coordinate as $z \rightarrow \tilde{z} \equiv w(z)$, and obtain

$$ds_2^2 = \frac{4d\tilde{z}d\bar{\tilde{z}}}{K\left(1+|\tilde{z}|^2\right)^2}. \quad (4.2)$$

¹⁴The fact that this is not super-gauge invariant reflects the fact that we cannot construct a field-strength superfield that contains $F_{z\bar{z}}$.

¹⁵If we choose $H_{\text{odd}}^1 = H^1$ as the brane compensator in (2.20) and as the only non-vanishing background among H_{odd} and H_{even} , the total flux becomes $\mathcal{B} = -\pi(2 - 2g)$.

This is nothing but the Fubini-Study metric. Hence the compactified space is a sphere with the radius $1/\sqrt{K}$. In this case, the background (3.37) represents the Salam-Sezgin solution [35].¹⁶

4.2 In the presence of branes

In the presence of the branes, the solution of (3.22) is found by using the technology of the fuchsian equations [32]. In this case, $w(z)$ is given by

$$w(z) = \frac{u_1(z)}{u_2(z)}, \quad (4.3)$$

where $u_1(z)$ and $u_2(z)$ are two linearly independent solutions of the fuchsian equation,

$$\frac{d^2 u}{dz^2} + \sum_{k=1}^{N-1} \left\{ \frac{\alpha_k(2 - \alpha_k)}{4(z - z_k)^2} + \frac{\beta_k}{2(z - z_k)} \right\} u = 0. \quad (4.4)$$

The constants β_k are known as the accessory parameters. The condition that $|z| = \infty$ is a regular singular point requires

$$\begin{aligned} \sum_{k=1}^{N-1} \beta_k &= 0, & \sum_{k=1}^{N-1} \{2\beta_k z_k + \alpha_k(2 - \alpha_k)\} &= \alpha_\infty(2 - \alpha_\infty), \\ \sum_{k=1}^{N-1} \{\beta_k z_k^2 + z_k \alpha_k(2 - \alpha_k)\} &= \beta_\infty. \end{aligned} \quad (4.5)$$

Thus only $N - 3$ parameters among β_k are independent. Going around the singularity $z = z_k$, the two solutions transform as

$$\begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} \rightarrow M_k \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}, \quad (4.6)$$

where the monodromy matrix M_k generically belongs to $\text{SL}(2, \mathbb{C})$. Then, $w(z)$ transforms as

$$w(z) \rightarrow M_k \cdot w(z), \quad (4.7)$$

where the operation of the matrix M_k is defined in (3.27). Hence, in order for $\psi(z, \bar{z})$ to be single-valued on the complex plane, we have to choose the two independent solutions $u_1(z)$ and $u_2(z)$ such that $M_k \in \text{SU}(2)$.

As the simplest example, consider the case that the origin $z = 0$ is the only singularity on the complex plane. In this case, (4.4) becomes

$$\frac{d^2 u}{dz^2} + \frac{\alpha_1(2 - \alpha_1)}{4z^2} u = 0. \quad (4.8)$$

If we choose the two independent solutions as

$$u_1(z) = z^{1 - \frac{\alpha_1}{2}}, \quad u_2(z) = z^{\frac{\alpha_1}{2}}, \quad (4.9)$$

¹⁶The constant B is chosen to zero in ref. [35].

the monodromy matrix becomes $M_1 = \text{diag}(e^{-\pi i \alpha_1}, e^{\pi i \alpha_1})$, which belongs to $\text{SU}(2)$. Thus, the desired background is obtained by

$$w(z) = \frac{u_1(z)}{u_2(z)} = z^{1-\alpha_1},$$

$$\psi(z, \bar{z}) = \frac{4|w'|^2}{K(1+|w|^2)^2} = \frac{4(1-\alpha_1)^2|z|^{-2\alpha_1}}{K(1+|z|^{2-2\alpha_1})^2}. \quad (4.10)$$

Recall that $\alpha_1 < 1$ from the requirement that (3.40) is finite. So the asymptotic behavior of $\psi(z, \bar{z})$ for $|z| \gg 1$ is

$$\psi(z, \bar{z}) \sim \frac{4(1-\alpha_1)^2}{K} |z|^{2\alpha_1-4}, \quad (4.11)$$

which indicates that the infinity is also a singular point with $\alpha_\infty = \alpha_1$ from (3.26). Therefore, there are at least two singularities in the case of the sphere compactification in the presence of the branes. This is in contrast to the torus compactification (see section 5.2). The background solution with (4.10) represents the so-called rugby-ball (or football) solution [21–23]. For the case with more branes, see ref. [32].

The author of ref. [32] focuses on the case that $0 < \alpha_k < 1$. However, this condition can be released as $\alpha_k < 1$ once negative-tension branes are accepted, just like in the Randall-Sundrum model [36]. Especially, when all α_k are integers, the rational functions are allowed as $w(z)$ [31].

5 Torus compactification

Now we consider the case that the extra dimensions are compactified on a torus. The points are identified as

$$z \sim \tilde{z}_{m,n} \equiv z + m + n\tau, \quad (m, n \in \mathbb{Z}) \quad (5.1)$$

where τ is a complex constant, and $\text{Im } \tau > 0$. Since ψ is proportional to $e^{(2)}$, it satisfies the periodic boundary conditions,

$$\psi(\tilde{z}_{m,n}, \bar{\tilde{z}}_{m,n}) = \psi(z, \bar{z}). \quad (5.2)$$

Recalling the redundancy of $w(z)$ under (3.27), they are satisfied when $w(z)$ is subject to the boundary conditions,

$$w(z+1) = \gamma_1 \cdot w(z),$$

$$w(z+\tau) = \gamma_\tau \cdot w(z), \quad (5.3)$$

where

$$\gamma_1, \gamma_\tau \in \text{SU}(2). \quad (5.4)$$

The matrices γ_1 and γ_τ either commute or anticommute to each other [33].

5.1 In the absence of branes

In the absence of the brane terms, i.e., $C_k = 0$, there is no solution to the Liouville equation (3.22) that satisfies the boundary conditions in (5.2), unless $c = 0$. When $c = 0$, i.e., 6D SUGRA is not gauged, we have a constant solution,

$$\psi = (\text{constant}). \quad (5.5)$$

Namely, V and $\mathcal{V}_T$ in (3.30) should be replaced with

$$V = (\text{constant}), \quad \mathcal{V}_T = \frac{|h_c|^4}{b_H^2 \sigma}. \quad (5.6)$$

5.2 Olesen solution

Next we consider the case with branes. The constraints on the tensions in this case become

$$\alpha_k < 1, \quad \sum_k \alpha_k < 0. \quad (5.7)$$

Therefore, negative-tension branes are necessary. The solutions in this case are expressed by using the Weierstrass elliptic functions whose definitions are collected in appendix D.

In contrast to the sphere compactification, there is a solution with one brane, which is found in ref. [34]. Since the expression of the solution becomes simple for a square torus, we consider the case of $\tau = i$ in this subsection. The solution is given by

$$w(z) = \frac{\wp(z)}{\wp(1)}, \quad (5.8)$$

where $\wp(z) \equiv \wp_{2,2i}(z)$ is the Weierstrass p-function. This leads to

$$e^{(2)} = \psi = \frac{4 |\wp(1)|^2 |\wp'(z)|^2}{K \left(|\wp(1)|^2 + |\wp(z)|^2 \right)^2}. \quad (5.9)$$

The corresponding compact space is called the Olesen space [37, 38].

From the definition of $\wp(z)$ in (D.1), we obtain the relation,

$$\wp(i) = -\wp(1). \quad (5.10)$$

Thus, we find from (D.3) that

$$\begin{aligned} w(z+1) &= 1 + \frac{2}{w(z)-1} = \frac{w(z)+1}{w(z)-1}, \\ w(z+i) &= -1 + \frac{2}{w(z)+1} = \frac{-w(z)+1}{w(z)+1}. \end{aligned} \quad (5.11)$$

Namely, the matrices γ_1 and γ_τ in (5.3) are read off as

$$\gamma_1 = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \gamma_\tau = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (5.12)$$

Since these belong to $SU(2)$, (5.9) is doubly periodic.

$$\psi(z+1) = \psi(z+i) = \psi(z). \quad (5.13)$$

As mentioned in ref. [34], this compact space has the positive constant curvature K almost everywhere except for the origin $z=0$. From the definition of $\wp(z)$ in (D.1), we can see that

$$w(z) \sim \frac{1}{z^2}, \quad \psi(z, \bar{z}) \sim |z|^2, \quad (5.14)$$

near the origin. This indicates from (3.26) that there is a conical singularity at $z=0$ with the negative deficit angle (or tension) $\alpha_1 = -1$.

5.3 General solution with branes

The solutions with an arbitrary number of the branes can be expressed by the two special solutions $f_{\varphi_1, \varphi_\tau}(z)$ and $g(z)$, which satisfy

$$\begin{aligned} f_{\varphi_1, \varphi_\tau}(z+1) &= e^{i\varphi_1} f_{\varphi_1, \varphi_\tau}(z), & f_{\varphi_1, \varphi_\tau}(z+\tau) &= e^{i\varphi_\tau} f_{\varphi_1, \varphi_\tau}(z), \\ g(z+1) &= -g(z), & g(z+\tau) &= \frac{1}{g(z)}, \end{aligned} \quad (5.15)$$

where φ_1 and φ_τ are real constants [33]. Here, $f_{\varphi_1, \varphi_\tau}(z)$ is called the elliptic function of the second kind with multipliers of unit modulus, and explicitly given by

$$f_{\varphi_1, \varphi_\tau}(z) = \left\{ s_0 + \sum_{k=1}^N s_k \frac{d^k \zeta}{dz^k}(z - z_0) \right\} \frac{\sigma^N(z - z_0)}{\prod_{k=1}^N \sigma(z - z_k)} e^{\lambda z}, \quad (5.16)$$

where $\zeta(z) \equiv \zeta_{1, \tau}(z)$ and $\sigma(z) \equiv \sigma_{1, \tau}(z)$ are the Weierstrass zeta- and sigma-functions, and

$$\begin{aligned} \lambda &\equiv \frac{\varphi_\tau}{\pi} \zeta\left(\frac{1}{2}\right) - \frac{\varphi_1}{\pi} \zeta\left(\frac{\tau}{2}\right), \\ z_0 &\equiv \frac{1}{2\pi N} (\varphi_\tau - \varphi_1 \tau) + \frac{1}{N} \sum_{k=1}^N z_k. \end{aligned} \quad (5.17)$$

The constants $s_0, s_1, \dots, s_N$ and $z_1, \dots, z_N$ are

$$\begin{aligned} s_0, s_1, \dots, s_N &\in \mathbb{C}, \\ z_1, \dots, z_N &\in \{t_1 + t_2 \tau | 0 \leq t_1, t_2 \leq 1\}. \end{aligned} \quad (5.18)$$

The other special solution $g(z)$ is given by

$$g(z) \equiv -\frac{f_{0, \pi}(z) - 1}{f_{0, \pi}(z) + 1} \cdot \frac{\wp(z) + b_0}{c_0 \wp(z) + d_0}, \quad (5.19)$$

where $\wp(z) \equiv \wp_{2, 2\tau}(z)$, and

$$\begin{aligned} b_0 &\equiv \frac{-e_2 + c_0^2(-2e_1 + e_2)}{1 + c_0^2}, & d_0 &\equiv \frac{c_0(-2e_1 + e_2 - c_0^2 e_2)}{1 + c_0^2}, \\ c_0 &\equiv \sqrt{\frac{-3e_1 + 2\sqrt{(e_1 - e_2)(2e_1 + e_2)}}{e_1 + 2e_2}}, \end{aligned} \quad (5.20)$$

with $e_1 \equiv \wp(1)$, $e_2 \equiv \wp(\tau)$, and $e_3 \equiv -e_1 - e_2$.

The matrices γ_1 and γ_τ for $f_{\varphi_1, \varphi_\tau}(z)$ are

$$\gamma_1 = \begin{pmatrix} e^{i\varphi_1/2} & 0 \\ 0 & e^{-i\varphi_1/2} \end{pmatrix}, \quad \gamma_\tau = \begin{pmatrix} e^{i\varphi_\tau/2} & 0 \\ 0 & e^{-i\varphi_\tau/2} \end{pmatrix}, \quad (5.21)$$

which commute, while those for $g(z)$ are

$$\gamma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma_\tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (5.22)$$

which anti-commute.

In terms of the above special solutions, $w(z)$ is expressed as

$$w(z) = \begin{cases} U \cdot f_{\varphi_1, \varphi_\tau}(z) & (\gamma_1 \text{ and } \gamma_\tau \text{ commute}) \\ U \cdot g(z) & (\gamma_1 \text{ and } \gamma_\tau \text{ anti-commute}) \end{cases}, \quad (5.23)$$

where $U \in \text{U}(2)$ is a constant matrix. Since the matrices in (5.12) anti-commute to each other, the Olesen solution (5.8) can be expressed in the form of $U \cdot g(z)$.

6 Summary

We provided $\mathcal{N} = 1$ superfield description of BPS backgrounds that preserve $\mathcal{N} = 1$ SUSY in 6D SUGRA compactified on a sphere or torus, including brane-localized FI terms. It is obtained by solving the background superfield EOMs, which are derived from the superfield action in our previous paper [14]. We found that the gauge in which $\langle S_E \rangle = e^{-\pi i/4}$ and $\langle \Sigma \rangle = 0$ is convenient to solve the background EOMs. We focused on the case that the dilaton σ and the tensor component B_{45} have constant background, which corresponds to the unwarped geometry of the extra dimensions. One of the background equations in this case is reduced to the Liouville equation, whose solutions have been well-investigated. Moving to the Wess-Zumino gauge for the gauge superfield V , we can read off the component-field expression of the background by using the expressions listed in appendix A. The background obtained in this paper reproduces the known results in the previous works. The brane terms in (2.20) induce the localized magnetic flux in the extra dimensions, keeping the total flux unchanged.

We can also consider the warped geometry by looking for non-constant solutions of (3.21). Such a case was discussed in the component-field expressions in ref. [25]. Because of their complicated definition of the coordinates, it is a nontrivial task to express their solution in our superfield language. Our results help us develop a systematic derivation of 4D effective action from the superfield action of 6D SUGRA with the brane terms. These issues will be discussed in our subsequent papers.

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A Correspondence to the component fields

Here we provide some of the components of the $\mathcal{N} = 1$ superfields in section 2.1 in terms of the component fields in 6D SUGRA [13, 14, 30].

A.1 Weyl multiplet

The 6D Weyl multiplet $\mathbb{E}$ consists of the sechsbein e_M^N , the gravitinos $\psi_{M\bar{\alpha}}^i$, and the auxiliary fields, where $\bar{\alpha}$ is a 6D spinor index, and $i, j = 1, 2$ are the $SU(2)_U$ -doublet indices. The gravitino has the 6D chirality $+$, and is the $SU(2)_U$ -Majorana-Weyl fermion, which can be decomposed into the two 4D Dirac fermions as

$$\psi_M^1 = \begin{pmatrix} \psi_{M\alpha}^+ \\ \bar{\psi}_M^{-\dot{\alpha}} \end{pmatrix}, \quad \psi_M^2 = \begin{pmatrix} -\psi_{M\alpha}^- \\ \bar{\psi}_M^{+\dot{\alpha}} \end{pmatrix}, \quad (\text{A.1})$$

where $\alpha, \dot{\alpha} = 1, 2$ are the 2-component spinor indices. Then the components of the superfields in (2.1) are expressed as

$$\begin{aligned} U^\mu &= (\theta\sigma^\mu\bar{\theta})\tilde{e}_\nu{}^\mu + i\bar{\theta}^2 (\theta\sigma^\mu\bar{\sigma}^\mu\psi_\nu^+) - i\theta^2 (\bar{\theta}\bar{\sigma}^\mu\sigma^\mu\bar{\psi}_\nu^+) + \dots, \\ U^m &= (\theta\sigma^\mu\bar{\theta})e_\mu{}^m - \bar{\theta}^2 (\theta\sigma^\mu\bar{\psi}_\mu^-) (e_4{}^m + ie_5{}^m) + \theta^2 (\bar{\theta}\bar{\sigma}^\mu\psi_\mu^-) (e_4{}^m - ie_5{}^m) \\ &\quad + \dots, \\ \Psi_{m\alpha} &= \frac{i}{2} (\sigma^\mu\bar{\theta})_\alpha e_{m\mu} + \dots, \\ S_E &= \sqrt{\frac{E_4}{E_5}} + \dots, \\ V_E &= e^{(2)} + \dots, \end{aligned} \quad (\text{A.2})$$

where the ellipses denote higher order terms in θ or $\bar{\theta}$, and

$$\begin{aligned} \tilde{e}_\nu{}^\mu &\equiv e_\nu{}^\mu - \langle e_\nu{}^\mu \rangle = e_\nu{}^\mu - \delta_\nu{}^\mu, \\ E_m &\equiv e_m{}^4 + ie_m{}^5, \\ e^{(2)} &\equiv \det(e_m{}^n) = e_4{}^4 e_5{}^5 - e_4{}^5 e_5{}^4. \end{aligned} \quad (\text{A.3})$$

Note that we need not discriminate the 4D flat and curved indices for $\tilde{e}_\nu{}^\mu$ at the linearized order since $\langle e_\nu{}^\mu \rangle = \delta_\nu{}^\mu$.

Thus, the lowest component of R_E^- is calculated as

$$R_E^- = \frac{e^{(2)}}{|E_4 E_5|}. \quad (\text{A.4})$$

A.2 Hypermultiplet

The hypermultiplet $\mathbb{H}^A$ consists of the hyperscalar $(\phi_i^{2A-1}, \phi_i^{2A})$, which are subject to the reality condition: $(\phi_1^{2A-1})^* = \phi_2^{2A}$, $(\phi_1^{2A})^* = -\phi_2^{2A-1}$, the hyperino $(\zeta_{\bar{\alpha}}^{2A-1}, \zeta_{\bar{\alpha}}^{2A})$, which are the symplectic Majorana spinors, and the auxiliary fields. The lowest components of $H^{\bar{A}}$ in (2.2) are given by

$$H^{2A-1} = (E_4 E_5)^{1/4} \phi_2^{2A-1} + \dots, \quad H^{2A} = (E_4 E_5)^{1/4} \phi_2^{2A} + \dots, \quad (\text{A.5})$$

where E_m ($m = 4, 5$) are defined in (A.3).

A.3 Vector multiplet

The vector multiplet $\mathbb{V}^I$ consists of the gauge field A_M^I , the gauginos Ω_α^{Ii} and the auxiliary fields. The gauge field is embedded into the superfields as

$$\begin{aligned} V^I &= -(\theta\sigma^\mu\bar{\theta})A_\mu^I + \dots, \\ \Sigma^I &= \frac{i}{2}\left(\frac{1}{S_E}|A_4^I - S_E|A_5^I\right) + \dots, \end{aligned} \quad (\text{A.6})$$

where $S_E| = \sqrt{E_4/E_5}$ is the lowest component of S_E .

A.4 Tensor multiplet

The tensor multiplet $\mathbb{T}$ consists of a real scalar σ , an anti-symmetric tensor field B_{MN} , the fermionic fields and the auxiliary fields. They are embedded into the superfields as

$$\begin{aligned} \Upsilon_{T\alpha} &= -\theta_\alpha(2B_{45} + i\sigma) - 2i(\sigma^{\mu\nu}\theta)_\alpha B_{\underline{\mu}\underline{\nu}} + \dots, \\ V_{Tm} &= -2(\theta\sigma^\mu\bar{\theta})B_{\underline{\mu}m} + \dots, \\ \Sigma_T &= \frac{e^{(2)}}{2}\sigma - iB_{45} + \dots. \end{aligned} \quad (\text{A.7})$$

From these expressions, we can calculate the lowest components of the field-strength superfields in $\mathbb{T}$ as

$$\mathcal{X}_T = \sigma + \dots, \quad \mathcal{V}_T = e^{(2)}\sigma + \dots. \quad (\text{A.8})$$

B Background equations of motion for U^m

Picking up the linear terms in U^m from the Lagrangian terms in section 2.3, we have

$$\begin{aligned} &-2|J_{\mathcal{P}}|\left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T}\right)^{1/2} \left(\hat{H}_{\text{odd}}^\dagger \tilde{d}e^V \hat{H}_{\text{odd}} + \hat{H}_{\text{even}}^\dagger \tilde{d}e^{-V} \hat{H}_{\text{even}}\right) \\ &\rightarrow 2\left(\frac{R_E^-}{\mathcal{V}_T \mathcal{X}_T}\right)^{1/2} L_H (U^m \text{Im} \partial_m \Sigma_T + \partial_m U^m \text{Im} \Sigma_T) \\ &+ \left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-}\right)^{1/2} L_H \left\{ 2U^m \text{Re} \frac{\bar{S}_E \partial_m S_E}{S_E^2} + (\partial_4 U^4 - \partial_5 U^5) R_E^+ + \frac{\partial_4 U^5}{|S_E|^2} - |S_E|^2 \partial_5 U^4 \right\} \\ &- \frac{1}{2} U^m \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T^3}\right)^{1/2} L_H \text{Re} (\partial_m D^\alpha \Upsilon_{T\alpha}) \\ &+ 4U^m \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T}\right)^{1/2} \text{Im} \left(H_{\text{odd}}^\dagger \tilde{d}e^V \partial_m H_{\text{odd}} + H_{\text{even}}^\dagger \tilde{d}e^{-V} \partial_m H_{\text{even}} \right), \end{aligned} \quad (\text{B.1})$$

where L_H is defined in (3.4),

$$\begin{aligned}
 & J_{\mathcal{P}} f_{IJ} \left\{ -2\hat{\Sigma}^I D^{\mathcal{P}\alpha} V^I \hat{\mathcal{Y}}_{T\alpha} + \frac{1}{2} (\partial_E^{\mathcal{P}} V^I D^{\mathcal{P}\alpha} V^J - \partial_E^{\mathcal{P}} D^{\mathcal{P}\alpha} V^I V^J) \hat{\mathcal{Y}}_{T\alpha} \right\} \\
 & \rightarrow f_{IJ} \left\{ 2iU^m \partial_m (\Sigma^I V^J) \right. \\
 & \quad \left. + \frac{1}{2} (-iU^m \partial_E V^I \partial_m V^J + iU^m \partial_E \partial_m V^I V^J + i\partial_E U^m \partial_m V^I V^J) \right\} D^\alpha \mathcal{Y}_{T\alpha} \\
 & \quad + 2iU^m f_{IJ} \Sigma^I V^J \partial_m D^\alpha \mathcal{Y}_{T\alpha} + \dots, \tag{B.2}
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{\mathcal{X}_T}{R_E^-} f_{IJ} \left\{ 4 \left(\partial_E^{\mathcal{P}} V^I - \hat{\Sigma}^I \right)^\dagger \left(\partial_E^{\mathcal{P}} V^J - \hat{\Sigma}^J \right) - 2 \left(\partial_E^{\mathcal{P}} V^I \right)^\dagger \partial_E^{\mathcal{P}} V^J + \left(2J_{\mathcal{P}} R_E^+ \hat{\Sigma}^I \hat{\Sigma}^J + \text{h.c.} \right) \right\} \\
 & \rightarrow \frac{L_V}{2R_E^-} U^m \text{Re} (\partial_m D^\alpha \Upsilon_{T\alpha}) - \frac{\mathcal{X}_T L_V}{R_E^{-2}} \delta R_E^- \\
 & \quad + \frac{\mathcal{X}_T f_{IJ}}{R_E^-} \left[2i \left(\bar{\partial}_E U^m \partial_m V^I + U^m \partial_m \bar{S}_E \bar{\mathcal{O}}_E V^I \right) (\partial_E V^J - 2\Sigma^J) + 4iU^m \partial_m \bar{\Sigma}^I (\partial_E V^J - \Sigma^J) \right. \\
 & \quad \left. + 2iR_E^+ \partial_m (U^m \Sigma^I \Sigma^J) + 2(\delta R_E^+) \Sigma^I \Sigma^J + \text{h.c.} \right], \tag{B.3}
 \end{aligned}$$

where

$$\begin{aligned}
 \delta R_E^- & \equiv -2U^m \text{Re} \frac{\bar{S}_E \partial_m S_E}{S_E^2} - (\partial_4 U^4 - \partial_5 U^5) R_E^+ - \frac{\partial_4 U^5}{|S_E|^2} + |S_E|^2 \partial_5 U^4, \\
 \delta R_E^+ & \equiv 2U^m \text{Im} \frac{\bar{S}_E \partial_m S_E}{S_E^2} + (\partial_4 U^4 - \partial_5 U^5) R_E^+ - \frac{i\partial_4 U^5}{|S_E|^2} - i|S_E|^2 \partial_5 U^4, \\
 L_V & \equiv f_{IJ} \left\{ 4 \left(\partial_E V^I - \Sigma^I \right)^\dagger (\partial_E V^J - \Sigma^J) - 2 \left(\partial_E V^I \right)^\dagger \partial_E V^J \right. \\
 & \quad \left. + 2\text{Re} \frac{\bar{S}_E}{S_E} (\Sigma^I \Sigma^J + \text{h.c.}) \right\}. \tag{B.4}
 \end{aligned}$$

The ellipsis denotes terms involving the spinor derivative of the superfields other than $\Upsilon_{T\alpha}$ and $\mathcal{Y}_{T\alpha}$. The other Lagrangian terms give no contributions to the background EOMs.

Then we can read off the equation of motion for U^4 as

$$\begin{aligned}
 0 & = -2\text{Im} \Sigma_T \partial_4 \left\{ \left(\frac{R_E^-}{\mathcal{V}_T \mathcal{X}_T} \right)^{1/2} L_H \right\} + 2 \left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} L_H \text{Re} \frac{\bar{S}_E \partial_4 S_E}{S_E^2} \\
 & \quad - \frac{1}{2} \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T^3} \right)^{1/2} L_H \text{Re} (\partial_4 D^\alpha \Upsilon_{T\alpha}) \\
 & \quad + 4 \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T} \right)^{1/2} \text{Im} \left(H_{\text{odd}}^\dagger \tilde{d}e^V \partial_4 H_{\text{odd}} + H_{\text{even}}^\dagger \tilde{d}e^{-V} \partial_4 H_{\text{even}} \right) \\
 & \quad + f_{IJ} \left[\left\{ 2i\partial_4 (\Sigma^I V^J) - \frac{i}{2} \partial_E V^I \partial_4 V^J + \frac{i}{2} \partial_E \partial_4 V^I V^J \right\} D^\alpha \mathcal{Y}_{T\alpha} \right.
 \end{aligned}$$

$$\begin{aligned}
 & +2i\Sigma^I V^J \partial_4 D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \Big] \\
 & - \frac{L_V}{2R_E^-} \text{Re} (\partial_4 D^\alpha \Upsilon_{T\alpha}) + \frac{2\mathcal{X}_T L_V}{R_E^{-2}} \text{Re} \frac{\bar{S}_E \partial_4 S_E}{S_E^2} \\
 & + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left[2i\partial_4 \bar{S}_E \bar{\mathcal{O}}_E V^I (\partial_E V^J - 2\Sigma^J) + 4i\partial_4 \bar{\Sigma}^I (\partial_E V^J - \Sigma^J) + 2iR_E^+ \partial_4 (\Sigma^I \Sigma^J) \right. \\
 & \quad \left. + 4 \left(\text{Im} \frac{\bar{S}_E \partial_4 S_E}{S_E^2} \right) \Sigma^I \Sigma^J + \text{h.c.} \right] \\
 & - \partial_4 \left[\left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} L_H R_E^+ + \left(\frac{if_{IJ}}{2S_E} \partial_4 V^I V^J D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \right) + \frac{\mathcal{X}_T L_V}{R_E^{-2}} R_E^+ \right. \\
 & \quad \left. + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left\{ \frac{2i}{\bar{S}_E} \partial_4 V^I (\partial_E V^J - 2\Sigma^J) + 2i \frac{S_E}{\bar{S}_E} \Sigma^I \Sigma^J + \text{h.c.} \right\} \right] \\
 & - \partial_5 \left[- \left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} L_H |S_E|^2 - \left(\frac{if_{IJ} S_E}{2} \partial_4 V^I V^J D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \right) - \frac{\mathcal{X}_T L_V}{R_E^{-2}} |S_E|^2 \right. \\
 & \quad \left. + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left\{ -2i\bar{S}_E \partial_4 V^I (\partial_E V^J - 2\Sigma^J) - 2i |S_E|^2 \Sigma^I \Sigma^J + \text{h.c.} \right\} \right] \\
 & + (\text{brane terms}), \tag{B.5}
 \end{aligned}$$

and the equation of motion for U^5 as

$$\begin{aligned}
 0 = & -2\text{Im} \Sigma_T \partial_5 \left\{ \left(\frac{R_E^-}{\mathcal{V}_T \mathcal{X}_T} \right)^{1/2} L_H \right\} + 2 \left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} L_H \text{Re} \frac{\bar{S}_E \partial_5 S_E}{S_E^2} \\
 & - \frac{1}{2} \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T^3} \right)^{1/2} L_H \text{Re} (\partial_5 D^\alpha \Upsilon_{T\alpha}) \\
 & + 4 \left(\frac{\mathcal{V}_T R_E^-}{\mathcal{X}_T} \right)^{1/2} \text{Im} \left(H_{\text{odd}}^\dagger \tilde{d}e^V \partial_5 H_{\text{odd}} + H_{\text{even}}^\dagger \tilde{d}e^{-V} \partial_5 H_{\text{even}} \right) \\
 & + f_{IJ} \left[\left\{ 2i\partial_5 (\Sigma^I V^J) - \frac{i}{2} \partial_E V^I \partial_5 V^J + \frac{i}{2} \partial_E \partial_5 V^I V^J \right\} D^\alpha \mathcal{Y}_{T\alpha} \right. \\
 & \quad \left. + 2i\Sigma^I V^J \partial_5 D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \right] \\
 & - \frac{L_V}{2R_E^-} \text{Re} (\partial_5 D^\alpha \Upsilon_{T\alpha}) + \frac{2\mathcal{X}_T L_V}{R_E^{-2}} \text{Re} \frac{\bar{S}_E \partial_5 S_E}{S_E^2} \\
 & + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left[2i\partial_5 \bar{S}_E \bar{\mathcal{O}}_E V^I (\partial_E V^J - 2\Sigma^J) + 4i\partial_5 \bar{\Sigma}^I (\partial_E V^J - \Sigma^J) + 2iR_E^+ \partial_5 (\Sigma^I \Sigma^J) \right. \\
 & \quad \left. + 4 \left(\text{Im} \frac{\bar{S}_E \partial_5 S_E}{S_E^2} \right) \Sigma^I \Sigma^J + \text{h.c.} \right] \\
 & - \partial_4 \left[\left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} \frac{L_H}{|S_E|^2} + \left(\frac{if_{IJ}}{2S_E} \partial_5 V^I V^J D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \right) + \frac{\mathcal{X}_T L_V}{R_E^{-2} |S_E|^2} \right. \\
 & \quad \left. + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left\{ \frac{2i}{\bar{S}_E} \partial_5 V^I (\partial_E V^J - 2\Sigma^J) - \frac{2i}{|S_E|^2} \Sigma^I \Sigma^J + \text{h.c.} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\partial_5 \left[-\left(\frac{\mathcal{V}_T}{\mathcal{X}_T R_E^-} \right)^{1/2} L_H R_E^+ - \left(\frac{if_{IJ} S_E}{2} \partial_5 V^I V^J D^\alpha \mathcal{Y}_{T\alpha} + \text{h.c.} \right) - \frac{\mathcal{X}_T L_V}{R_E^{-2}} R_E^+ \right. \\
 & \quad \left. + \frac{f_{IJ} \mathcal{X}_T}{R_E^-} \left\{ -2i \bar{S}_E \partial_5 V^I (\partial_E V^J - 2\Sigma^J) + 2i \frac{\bar{S}_E}{S_E} \Sigma^I \Sigma^J + \text{h.c.} \right\} \right] \\
 & + (\text{brane terms}).
 \end{aligned} \tag{B.6}$$

The ellipses denote the contributions from the brane terms.

C Total flux quantization

Here we give a comment on the quantization of the total flux. For this purpose, we begin with solving the Maxwell equation in the bulk,

$$\partial_P \left(\sqrt{\det G_{MN}} \sigma F^{PQ} \right) = 0. \tag{C.1}$$

Since we assume that the dilaton σ has a constant background, and the background metric G_{MN} is given by

$$G_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + e^{(2)} \{ (dx^4)^2 + (dx^5)^2 \}, \tag{C.2}$$

the above equation becomes

$$\sigma \partial_m \left(\frac{F_{45}}{e^{(2)}} \right) = 0. \tag{C.3}$$

Namely, the background field strength F_{45} is proportional to $e^{(2)}$. By solving the Einstein equation, we find that $e^{(2)}$ in (3.34) is a solution [32]. Thus, we obtain

$$F_{z\bar{z}} = -2iF_{45} = \frac{ic_F |w'|^2}{(1 + |w|^2)^2}, \tag{C.4}$$

where c_F is an integration constant. Since (C.1) does not include the brane contributions, (C.4) does not have the localized flux terms, in contrast to (3.42). Up to the brane-localized terms, (C.4) can be solved as

$$A_z = \frac{ic_F}{4} \partial_z \ln \frac{|w'|^2}{(1 + |w|^2)^2}. \tag{C.5}$$

In fact, this is the solution including the brane contributions. Hence, (C.4) is modified as

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = \frac{ic_F |w'|^2}{(1 + |w|^2)^2} + i\pi c_F \sum_k \alpha_k \delta^{(2)}(z - z_k). \tag{C.6}$$

Thus the total flux is calculated as

$$\mathcal{B} = -i \int d^2z \, c F_{z\bar{z}} = \pi c c_F (2 - 2g), \tag{C.7}$$

where g is the genus of the compact space. We have used (3.40). The coefficient c_F (or the total flux $\mathcal{B}$) will be quantized by requiring the single-valuedness of the fields.

C.1 Sphere compactification ($g = 0$)

The sphere is covered by the following two coordinate patches.

Patch I: z , which covers the whole points except for the infinity.

Patch II: $\tilde{z} \equiv -1/z$, which covers the whole points except for the origin $z = 0$.

The gauge 1-form A is expressed in the patch I as¹⁷

$$A^{(I)} = A_z dz + A_{\bar{z}} d\bar{z} = \frac{i\mathcal{B}}{8\pi c} dz \partial_z \ln \frac{|dw/dz|^2}{(1+|w|^2)^2} + \text{h.c.}, \quad (\text{C.8})$$

where (C.7) has been used, while it is expressed in the patch II as

$$A^{(II)} = \frac{i\mathcal{B}}{8\pi c} d\tilde{z} \partial_{\tilde{z}} \ln \frac{|dw/d\tilde{z}|^2}{(1+|w|^2)^2} + \text{h.c.} \quad (\text{C.9})$$

Since

$$\frac{dw}{dz} = \frac{d\tilde{z}}{dz} \frac{dw}{d\tilde{z}} = \tilde{z}^2 \frac{dw}{d\tilde{z}}, \quad (\text{C.10})$$

$A^{(I)}$ can be rewritten as

$$\begin{aligned} A^{(I)} &= \frac{i\mathcal{B}}{8\pi c} d\tilde{z} \partial_{\tilde{z}} \ln \frac{|\tilde{z}|^4 |dw/d\tilde{z}|^2}{(1+|w|^2)^2} + \text{h.c.} = A^{(II)} + \frac{i\mathcal{B}}{4\pi c} \left(\frac{d\tilde{z}}{\tilde{z}} - \frac{d\bar{\tilde{z}}}{\bar{\tilde{z}}} \right) \\ &= A^{(II)} + \frac{i\mathcal{B}}{4\pi c} d \ln \left(\frac{\tilde{z}}{\bar{\tilde{z}}} \right) = A^{(II)} - \frac{\mathcal{B}}{2\pi c} d\vartheta, \end{aligned} \quad (\text{C.11})$$

where $\vartheta \equiv \arg \tilde{z}$. Thus, from the patch II to the patch I, the gauge transformation:¹⁸

$$A_M \rightarrow A_M + \partial_M \lambda, \quad \phi_2^2 \rightarrow e^{ic\lambda} \phi_2^2, \quad (\text{C.12})$$

with $\lambda = -\mathcal{B}\vartheta/(2\pi c)$ has to be performed. Therefore, from the single-valuedness of ϕ_2^2 , we obtain the flux quantization condition,

$$\mathcal{B} = 2n\pi. \quad (n \in \mathbb{Z}) \quad (\text{C.13})$$

C.2 Torus compactification ($g = 1$)

In this case, the total flux vanishes, and the coefficient c_F is not quantized by the requirement that the fields are single-valued. In fact, (C.5) is invariant under the translations $z \rightarrow z + 1 \rightarrow z + 1 + \tau \rightarrow z + \tau \rightarrow z$, in contrast to the non-BPS background in the absence of the branes [39]:

$$A_z = -\frac{i\mathcal{B}\bar{z}}{2c\text{Im}\tau}. \quad (\text{C.14})$$

In order to fix c_F , we have to take into account the other equations of motion.

¹⁷We omit the 4D components of A since they do not have non-vanishing background.

¹⁸This can be read off by substituting $\Lambda = \frac{i}{2}\lambda$ in (3.31).

D Weierstrass elliptic functions

The Weierstrass p-, zeta- and sigma-functions are defined as

$$\begin{aligned}\wp_{\omega_1, \omega_2}(z) &\equiv \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left\{ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\}, \\ \zeta_{\omega_1, \omega_2}(z) &\equiv \frac{1}{z} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right), \\ \sigma_{\omega_1, \omega_2}(z) &\equiv z \prod_{(m,n) \neq (0,0)} \left(1 - \frac{z}{\Omega_{m,n}} \right) \exp \left(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2} \right),\end{aligned}\tag{D.1}$$

where $\Omega_{m,n} \equiv m\omega_1 + n\omega_2$, and $m, n \in \mathbb{Z}$. Clearly, $\wp(z)$ is doubly periodic.

$$\wp_{\omega_1, \omega_2}(z + \omega_1) = \wp_{\omega_1, \omega_2}(z + \omega_2) = \wp_{\omega_1, \omega_2}(z).\tag{D.2}$$

It also follows that

$$\begin{aligned}\zeta'_{\omega_1, \omega_2}(z) &= -\wp(z), \\ \zeta_{\omega_1, \omega_2}(z) &= \frac{d \ln \sigma_{\omega_1, \omega_2}(z)}{dz} = \frac{\sigma'_{\omega_1, \omega_2}(z)}{\sigma_{\omega_1, \omega_2}(z)}, \\ \wp_{\omega_1, \omega_2}\left(z + \frac{\omega_1}{2}\right) &= e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp_{\omega_1, \omega_2}(z) - e_1}, \\ \wp_{\omega_1, \omega_2}\left(z + \frac{\omega_2}{2}\right) &= e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\wp_{\omega_1, \omega_2}(z) - e_2},\end{aligned}\tag{D.3}$$

where $e_1 \equiv \wp_{\omega_1, \omega_2}(\omega_1/2)$, $e_2 \equiv \wp_{\omega_1, \omega_2}(\omega_2/2)$, and $e_3 = -e_1 - e_2$.

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References

- [1] N. Marcus, A. Sagnotti and W. Siegel, *Ten-dimensional supersymmetric Yang-Mills theory in terms of four-dimensional superfields*, *Nucl. Phys. B* **224** (1983) 159 [[INSPIRE](#)].
- [2] N. Arkani-Hamed, T. Gregoire and J.G. Wacker, *Higher dimensional supersymmetry in 4D superspace*, *JHEP* **03** (2002) 055 [[hep-th/0101233](#)] [[INSPIRE](#)].
- [3] D. Marti and A. Pomarol, *Supersymmetric theories with compact extra dimensions in $N = 1$ superfields*, *Phys. Rev. D* **64** (2001) 105025 [[hep-th/0106256](#)] [[INSPIRE](#)].
- [4] A. Hebecker, *5D super Yang-Mills theory in 4D superspace, superfield brane operators and applications to orbifold GUTs*, *Nucl. Phys. B* **632** (2002) 101 [[hep-ph/0112230](#)] [[INSPIRE](#)].
- [5] H. Abe, T. Kobayashi, H. Ohki and K. Sumita, *Superfield description of 10D SYM theory with magnetized extra dimensions*, *Nucl. Phys. B* **863** (2012) 1 [[arXiv:1204.5327](#)] [[INSPIRE](#)].

- [6] H. Abe, T. Horie and K. Sumita, *Superfield description of $(4 + 2n)$ -dimensional SYM theories and their mixtures on magnetized tori*, *Nucl. Phys. B* **900** (2015) 331 [[arXiv:1507.02425](#)] [[INSPIRE](#)].
- [7] W.D. Linch, III, M.A. Luty and J. Phillips, *Five-dimensional supergravity in $N = 1$ superspace*, *Phys. Rev. D* **68** (2003) 025008 [[hep-th/0209060](#)] [[INSPIRE](#)].
- [8] F. Paccetti Correia, M.G. Schmidt and Z. Tavartkiladze, *Superfield approach to 5D conformal SUGRA and the radion*, *Nucl. Phys. B* **709** (2005) 141 [[hep-th/0408138](#)] [[INSPIRE](#)].
- [9] H. Abe and Y. Sakamura, *Superfield description of 5D supergravity on general warped geometry*, *JHEP* **10** (2004) 013 [[hep-th/0408224](#)] [[INSPIRE](#)].
- [10] E. Bergshoeff, E. Sezgin and A. Van Proeyen, *Superconformal tensor calculus and matter couplings in six-dimensions*, *Nucl. Phys. B* **264** (1986) 653 [Erratum *ibid.* **B 598** (2001) 667] [[INSPIRE](#)].
- [11] F. Coomans and A. Van Proeyen, *Off-shell $N = (1, 0)$, $D = 6$ supergravity from superconformal methods*, *JHEP* **02** (2011) 049 [Erratum *ibid.* **01** (2012) 119] [[arXiv:1101.2403](#)] [[INSPIRE](#)].
- [12] H. Abe, Y. Sakamura and Y. Yamada, *$N = 1$ superfield description of vector-tensor couplings in six dimensions*, *JHEP* **04** (2015) 035 [[arXiv:1501.07642](#)] [[INSPIRE](#)].
- [13] H. Abe, Y. Sakamura and Y. Yamada, *$N = 1$ superfield description of six-dimensional supergravity*, *JHEP* **10** (2015) 181 [[arXiv:1507.08435](#)] [[INSPIRE](#)].
- [14] H. Abe, S. Aoki and Y. Sakamura, *Full diffeomorphism and Lorentz invariance in 4D $N = 1$ superfield description of 6D SUGRA*, *JHEP* **11** (2017) 146 [[arXiv:1708.09106](#)] [[INSPIRE](#)].
- [15] H. Abe and Y. Sakamura, *Roles of Z_2 -odd $N = 1$ multiplets in off-shell dimensional reduction of 5D supergravity*, *Phys. Rev. D* **75** (2007) 025018 [[hep-th/0610234](#)] [[INSPIRE](#)].
- [16] H. Abe and Y. Sakamura, *Flavor structure with multi moduli in 5D supergravity*, *Phys. Rev. D* **79** (2009) 045005 [[arXiv:0807.3725](#)] [[INSPIRE](#)].
- [17] H. Abe, H. Otsuka, Y. Sakamura and Y. Yamada, *SUSY flavor structure of generic 5D supergravity models*, *Eur. Phys. J. C* **72** (2012) 2018 [[arXiv:1111.3721](#)] [[INSPIRE](#)].
- [18] Y. Sakamura, *One-loop Kähler potential in 5D gauged supergravity with generic prepotential*, *Nucl. Phys. B* **873** (2013) 165 [Erratum *ibid.* **B 873** (2013) 728] [[arXiv:1302.7244](#)] [[INSPIRE](#)].
- [19] Y. Sakamura and Y. Yamada, *Impacts of non-geometric moduli on effective theory of 5D supergravity*, *JHEP* **11** (2013) 090 [Erratum *ibid.* **01** (2014) 181] [[arXiv:1307.5585](#)] [[INSPIRE](#)].
- [20] Y. Sakamura and Y. Yamada, *Natural realization of a large extra dimension in 5D supersymmetric theory*, *PTEP* **2014** (2014) 093B02 [[arXiv:1401.1921](#)] [[INSPIRE](#)].
- [21] S.M. Carroll and M.M. Guica, *Sidestepping the cosmological constant with football shaped extra dimensions*, [[hep-th/0302067](#)] [[INSPIRE](#)].
- [22] I. Navarro, *Codimension two compactifications and the cosmological constant problem*, *JCAP* **09** (2003) 004 [[hep-th/0302129](#)] [[INSPIRE](#)].
- [23] I. Navarro, *Spheres, deficit angles and the cosmological constant*, *Class. Quant. Grav.* **20** (2003) 3603 [[hep-th/0305014](#)] [[INSPIRE](#)].

- [24] G.W. Gibbons, R. Güven and C.N. Pope, *3-branes and uniqueness of the Salam-Sezgin vacuum*, *Phys. Lett. B* **595** (2004) 498 [[hep-th/0307238](#)] [[INSPIRE](#)].
- [25] H.M. Lee and C. Lüdeling, *The general warped solution with conical branes in six-dimensional supergravity*, *JHEP* **01** (2006) 062 [[hep-th/0510026](#)] [[INSPIRE](#)].
- [26] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton Univ. Pr., Princeton, U.S.A., (1992) [[INSPIRE](#)].
- [27] S. Randjbar-Daemi, A. Salam, E. Sezgin and J.A. Strathdee, *An anomaly free model in six-dimensions*, *Phys. Lett. B* **151** (1985) 351 [[INSPIRE](#)].
- [28] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly free chiral theories in six-dimensions*, *Nucl. Phys. B* **254** (1985) 327 [[INSPIRE](#)].
- [29] V. Kumar, D.R. Morrison and W. Taylor, *Global aspects of the space of 6D $N = 1$ supergravities*, *JHEP* **11** (2010) 118 [[arXiv:1008.1062](#)] [[INSPIRE](#)].
- [30] Y. Sakamura, *Direct relation of linearized supergravity to superconformal formulation*, *JHEP* **12** (2011) 008 [[arXiv:1107.4247](#)] [[INSPIRE](#)].
- [31] P.A. Horvathy and J.C. Yera, *Vortex solutions of the Liouville equation*, *Lett. Math. Phys.* **46** (1998) 111 [[hep-th/9805161](#)] [[INSPIRE](#)].
- [32] M. Redi, *Footballs, conical singularities and the Liouville equation*, *Phys. Rev. D* **71** (2005) 044006 [[hep-th/0412189](#)] [[INSPIRE](#)].
- [33] N. Akerblom, G. Cornelissen, G. Stavenga and J.-W. van Holten, *Nonrelativistic Chern-Simons vortices on the torus*, *J. Math. Phys.* **52** (2011) 072901 [[arXiv:0912.0718](#)] [[INSPIRE](#)].
- [34] N. Akerblom and G. Cornelissen, *A compact codimension two braneworld with precisely one brane*, *Phys. Rev. D* **81** (2010) 124025 [[arXiv:1004.1807](#)] [[INSPIRE](#)].
- [35] A. Salam and E. Sezgin, *Chiral compactification on $Minkowski \times S^2$ of $N = 2$ Einstein-Maxwell supergravity in six-dimensions*, *Phys. Lett. B* **147** (1984) 47 [[INSPIRE](#)].
- [36] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [[hep-ph/9905221](#)] [[INSPIRE](#)].
- [37] P. Olesen, *Vacuum structure of the electroweak theory in high magnetic fields*, *Phys. Lett. B* **268** (1991) 389 [[INSPIRE](#)].
- [38] P. Olesen, *Soliton condensation in some selfdual Chern-Simons theories*, *Phys. Lett. B* **265** (1991) 361 [*Erratum ibid.* **B 267** (1991) 541] [[INSPIRE](#)].
- [39] D. Cremades, L.E. Ibáñez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [[hep-th/0404229](#)] [[INSPIRE](#)].