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Boundary conditions and localization on AdS. Part I

Justin R. David,^a Edi Gava,^{b,d} Rajesh Kumar Gupta^c and Kumar Narain^d

^aCentre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India

^bINFN, Sezione di Trieste,
Strada Costiera 11, Trieste, Italy

^cDepartment of Mathematics, King's College London,
The Strand, London WC2R 2LS, U.K.

^dICTP,
Strada Costiera 11, 34151 Trieste, Italy

E-mail: justin@iisc.ac.in, gava@ictp.it, rajesh.gupta@kcl.ac.uk,
narain@ictp.it

ABSTRACT: We study the role of boundary conditions on the one loop partition function the $\mathcal{N} = 2$ chiral multiplet of R-charge Δ on $\text{AdS}_2 \times S^1$. The chiral multiplet is coupled to a background vector multiplet which preserves supersymmetry. We implement normalizable boundary conditions in AdS_2 and develop the Green's function method to obtain the one loop determinant. We evaluate the one loop determinant for two different actions: the standard action and the Q -exact deformed positive definite action used for localization. We show that if there exists an integer n in the interval $D : (\frac{\Delta-1}{2L}, \frac{\Delta}{2L})$, where L being the ratio of radius of AdS_2 to that of S^1 , then the one loop determinants obtained for the two actions differ. It is in this situation that fields which obey normalizable boundary conditions do not obey supersymmetric boundary conditions. However if there are no integers in D , then fields which obey normalizable boundary conditions also obey supersymmetric boundary conditions and the one loop determinants of the two actions precisely agree. We also show that it is only in the latter situation that the one loop determinant obtained by evaluating the index of the D_{10} operator associated with the localizing action agrees with the one loop determinant obtained using Green's function method.

KEYWORDS: Extended Supersymmetry, Field Theories in Lower Dimensions, Supersymmetric Gauge Theory

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1 Introduction

Supersymmetric localization methods which was first introduced in [1] and later developed in [2–4] enable the exact evaluation of observables in supersymmetric quantum field theories. See [5] for a recent review and a list of references. Some of the exact computations have provided highly non-trivial checks of the AdS/CFT correspondence [6–8]. Localization relies on identifying a fermionic symmetry Q , upto boundary terms of the Lagrangian and an addition of a localizing term which is Q exact up to boundary terms. Therefore, supersymmetric theories on compact spaces without boundaries serve as the canonical examples in which the method of localization has been applied.

Extension of localization methods to supersymmetric theories defined on non-compact spaces presents a considerable challenge. Naively we expect that we must ensure the following

1. Choose boundary conditions of all the fields involved such that the boundary terms that result from the Q variation of the original action as well as the localizing term can be neglected. The boundary conditions must also be chosen so that the path integral is well defined. The canonical method of ensuring this is to implement normalizable boundary conditions.
2. The boundary conditions of both bosonic and fermionic fields chosen must be consistent with supersymmetry.

It is a-priori not clear that fields which satisfy normalizable boundary conditions also obey supersymmetric boundary conditions. To study these issues in detail it is best to focus on concrete examples of a class of non-compact spaces. Supersymmetric theories on spaces of the form $\text{AdS}_n \times S^m$ are examples which also have good applications. Localization of $\mathcal{N} = 2$ supergravity on $\text{AdS}_2 \times S^2$ is relevant for obtaining the entropy of extremal black holes in these theories [9–14]. Evaluating the supersymmetric partition function of $\mathcal{N} = 8$ supergravity on AdS_4 is useful in the context of the holographic duality of this theory with the ABJM theory [15]. Localization of supergravity on AdS_5 reduces to a Chern-Simons theory defined on an AdS_3 slice which describes a protected chiral algebra in $\mathcal{N} = 4$ SYM theory in large N limit [16]. Topologically twisted index on $\text{AdS}_2 \times S^1$ of ABJM theory is related to the entropy of magnetically charged black holes in AdS_4 [17]. Apart from these applications, the study of the supersymmetric localization on non compact spaces itself is an interesting problem, see for instance [18].

As we have briefly mentioned above, since there are subtle issues involved in the application of localization on non-compact spaces it is important to study situations in which the results obtained by applying localization can be checked against another independent method. In [19] localization of $\mathcal{N} = 2$ $U(N)$ Chern-Simons theory on $\text{AdS}_2 \times S^1$ was studied. It was shown that it is possible to choose the gauge field, the gaugino and the auxillary field to lie in the space of square integrable normalizable wave functions in AdS_2 and consistent with supersymmetry.¹ Moreover the result for the supersymmetric partition function for this theory on $\text{AdS}_2 \times S^1$ agreed with that on S^3 which is expected from conformal symmetry. However since this theory is also topological one might suspect that this agreement is the result of a coincidence and therefore it is important to examine the matter sector.

In this paper we study the localization of the chiral multiplet of R-charge Δ on $\text{AdS}_2 \times S^1$ where the radius of AdS_2 is L and the radius of S^1 is normalized to unity. The chiral multiplet is also coupled to the background vector multiplet which is supersymmetric and solves the saddle point equations of the localising action of the vector multiplet. The advantage of studying the matter sector is that its action is quadratic. Therefore, its

¹The gauge fixing choice was such that ghosts satisfied supersymmetric boundary conditions, however they were not normalizable.

partition function which is one loop exact can be obtained by using conventional methods and compared against that obtained using localization. Note that when the chiral multiplet is not coupled with the vector, the theory is free and the one loop determinants for the boson and the fermion in $\text{AdS}_2 \times S^1$ for special values of Δ have been obtained earlier [20] using the eigen function method.²

We develop the Green's function approach of evaluating the ratio of bosonic and fermionic one loop determinants in the chiral multiplet coupled to the background vector. We first consider the standard action for the chiral multiplet given by [21]. We discuss in detail both the normalizable and supersymmetric boundary conditions for fields of this theory. We show that when there exist no n in the interval

$$D : \left(\frac{\Delta - 1}{2L}, \frac{\Delta}{2L} \right), \quad (1.1)$$

solutions to equations of motions which obey normalizable boundary conditions also obey supersymmetric boundary conditions.³ We show that the Green's function for both the boson and the fermion of the chiral multiplet in the background of the vector multiplet is exactly solvable. The Green's function can then be related to the variation of one loop determinants with respect to the background value of the vector multiplet in the following way. Let the background value of the vector multiplet be parametrised by α . The variation of the one loop bosonic determinant of an operator $\mathcal{D}_B(\alpha)$ together with a fermionic determinant of an operator $\mathcal{D}_F(\alpha)$ is given by

$$\frac{\delta}{\delta\alpha} \ln Z_{1\text{-loop}}(\alpha) = \text{Tr} \left[G_F \frac{\delta}{\delta\alpha} \mathcal{D}_F(\alpha) \right] - \frac{1}{2} \text{Tr} \left[G_B \frac{\delta}{\delta\alpha} \mathcal{D}_B(\alpha) \right]. \quad (1.2)$$

Then integrating with respect to α enables the evaluation of the partition function. The result is presented in (4.45). At $\alpha = 0$ and special values of Δ, L , the one loop determinants can be evaluated using the eigen function methods following [20] since the theory is free. As a check on the Green's function method we show the result for the one loop determinant at $L = 1, 2$ and at $\alpha = 0$ agrees with the eigen function method.

Next we examine the Q -exact and positive definite action which is used in localization and again evaluate its one loop determinant using the Green's function method. We show that the one loop determinant of the standard action agrees with the Q -exact action only when there exist no integer n in the domain D . As mentioned earlier it is in this situation that fields which obey normalizable boundary conditions also satisfy supersymmetric boundary conditions. We also show that when there exists no integer n in D , the one loop determinant entirely arises from boundary terms which include contributions from $r = 0$ and $r \rightarrow \infty$ in AdS_2 and the result is independent of the details of the precise solutions to equations of motion of the action. Now when there exists an integer n which lies in D , then the one loop determinant of the standard action differs from that of the Q -exact

²In [20] thermal boundary conditions were chosen along S^1 for the fermions. The method can be adapted for fermions obeying periodic boundary conditions along S^1 .

³In this paper, we will refer to this situation as normalizable boundary conditions are compatible with supersymmetric boundary conditions.

deformation. This shows that the one loop determinant certainly is not independent of the Q -exact deformation when fields obeying normalizable boundary conditions do not satisfy supersymmetric boundary conditions.

We also calculate the one loop determinant of the Q -exact action using the index of the corresponding D_{10} operator and implementing supersymmetric boundary conditions. We observe that when there exists no integer n in the domain D as well as $\frac{\Delta-1}{2L}$ is not an integer, then the one loop determinant agrees with that obtained for this action as well as the standard action using the Green's function method. This is in accordance with the expectation that when normalizable and supersymmetric boundary conditions are compatible, methods relying on localization should yield the same answer as Green's function approach implemented with normalizable boundary conditions.

The organisation of this paper is as follows. In section 2 we discuss the supersymmetric transformations as well as the construction of the supersymmetric action for chiral multiplet on $\text{AdS}_2 \times S^1$. In section 2.1 we introduce co-homological variables for the fermions and write down the Kaluza-Klein reduced actions for the chiral multiplet in presence of the vector multiplet. We call this the standard action. This is then repeated for the localising action of the chiral multiplet, we call this the Q -exact deformed action in section 2.2. Next in section 3 we discuss the boundary conditions of the solutions to the equations of motion of these Lagrangians. In section 3.3 we provide a simplified discussion of the boundary conditions. In section 4 we construct the Green's function for operators associated with the standard action of the chiral multiplet as well as the Q -exact action. Here we make the main observation of the paper. We show that if there exists an integer n in the interval (1.1) the result for the one loop determinants for the regular action differs from that of the Q -exact action. We also observe here that once the interval (1.1) admits an integer there are solutions which obey normalizable boundary conditions, but do not satisfy supersymmetric boundary condition. This is the reason that the one loop determinants of the Q -exact action does not agree with the regular action. In section 5 we evaluate the index of the D_{10} operator associated with the localising Q -exact action. Here we observe that it coincides with the result obtained using the Green's function method for the standard as well as the Q -exact action only when the interval (1.1) does not admit an integer and $\frac{\Delta-1}{2L}$ is not an integer. In section 6 we present our conclusions and discuss generalisations. Appendix A contains the details regarding our conventions, killing spinors and the classical supersymmetric solution for the vector multiplet. Appendix B contains the evaluation of one loop determinants of the free chiral multiplet on $\text{AdS}_2 \times S^1$ using the eigen function method. Appendix C tabulates a list of integrals involving products of hypergeometric functions which are used in our evaluation of one loop determinants.

2 Supersymmetry and actions on $\text{AdS}_2 \times S^1$

Before we start discussing the actions, the preliminaries that we require are the supersymmetry transformations of the vector multiplet and a chiral multiplet coupled to a vector multiplet on $\text{AdS}_2 \times S^1$. We take the ratio of the radius of AdS_2 to that of S^1 to be L . The metric is given in (A.7). Let us begin with the Euclidean supersymmetry transformations

of the fields in a vector multiplet. This is given by

$$\begin{aligned}
 Q\lambda &= -\frac{i}{4}\epsilon G - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu}\epsilon - i\gamma^\mu\epsilon(i\nabla_\mu\sigma - V_\mu\sigma), \\
 Q\tilde{\lambda} &= \frac{i}{4}\tilde{\epsilon} G - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu}\tilde{\epsilon} + i\gamma^\mu\tilde{\epsilon}(i\nabla_\mu\sigma + V_\mu\sigma), \\
 Qa_\mu &= \frac{1}{2}\left(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda\right), \\
 Q\sigma &= \frac{1}{2}\left(-\epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda\right), \\
 QG &= -2i\left[\nabla_\mu\left(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda\right) - i\left[\sigma, \epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda\right] - iV_\mu\left(\epsilon\gamma^\mu\tilde{\lambda} + \tilde{\epsilon}\gamma^\mu\lambda\right)\right], \quad (2.1)
 \end{aligned}$$

where ∇_μ is the covariant derivative containing the Christoffel and the gauge connection. The Killing spinors ϵ and $\tilde{\epsilon}$ on $\text{AdS}_2 \times S^1$ as well as the background V_μ is defined in (A.9).⁴ The supersymmetry transformations of the fields of the chiral multiplet coupled to an abelian vector multiplet are given by

$$\begin{aligned}
 Q\phi &= \epsilon\psi, & Q\bar{\phi} &= \tilde{\epsilon}\tilde{\psi}, \\
 Q\psi &= F(\tau, r, \theta)\epsilon + \Gamma^\mu\tilde{\epsilon}D_\mu\phi - iq\sigma\phi\tilde{\epsilon}, \\
 Q\tilde{\psi} &= \bar{F}(\tau, r, \theta)\tilde{\epsilon} + \Gamma^\mu\epsilon D_\mu\bar{\phi} - iq\sigma\bar{\phi}\epsilon, \\
 QF &= D_\mu(\tilde{\epsilon}\Gamma^\mu\psi) + iq\sigma\tilde{\epsilon}\psi - iq\phi\tilde{\epsilon}\tilde{\lambda}, \\
 Q\bar{F} &= D_\mu(\epsilon\Gamma^\mu\tilde{\psi}) + iq\sigma\epsilon\tilde{\psi} + iq\bar{\phi}\epsilon\lambda. \quad (2.2)
 \end{aligned}$$

Here q refers to the charge of the chiral multiplet. The fields (ϕ, ψ, F) has R-charges $(\Delta, \Delta - 1, \Delta - 2)$, while the fields of the anti-chiral multiplet $(\bar{\phi}, \tilde{\psi}, \bar{F})$ has R-charges $(-\Delta, -\Delta + 1, -\Delta + 2)$. The action of the derivative D_μ are defined by

$$\begin{aligned}
 D_\mu\phi &= \left(\nabla_\mu - i\Delta A_\mu + i\frac{\Delta}{2}V_\mu\right)\phi, \\
 D_\mu\psi &= \left(\nabla_\mu - i(\Delta - 1)A_\mu + i\frac{\Delta}{2}V_\mu\right)\psi, \\
 D_\mu\bar{\phi} &= \left(\nabla_\mu + i\Delta A_\mu - i\frac{\Delta}{2}V_\mu\right)\bar{\phi}, \\
 D_\mu\tilde{\psi} &= \left(\nabla_\mu + i(\Delta - 1)A_\mu - i\frac{\Delta}{2}V_\mu\right)\tilde{\psi}
 \end{aligned} \quad (2.3)$$

Note that ∇_μ is the covariant derivative along with the gauge connection. It is convenient to define the variation separately with respect ϵ and $\tilde{\epsilon}$ as

$$\begin{aligned}
 \delta_\epsilon\phi &= \epsilon\psi, & \delta_{\tilde{\epsilon}}\phi &= 0, & \delta_\epsilon\bar{\phi} &= 0, & \delta_{\tilde{\epsilon}}\bar{\phi} &= \tilde{\epsilon}\tilde{\psi}, \\
 \delta_\epsilon\psi &= F(\tau, r, \theta)\epsilon, & \delta_{\tilde{\epsilon}}\psi &= \Gamma^\mu\tilde{\epsilon}D_\mu\phi - iq\sigma\phi\tilde{\epsilon}, \\
 \delta_\epsilon\tilde{\psi} &= \Gamma^\mu\epsilon D_\mu\bar{\phi} - iq\sigma\bar{\phi}\epsilon, & \delta_{\tilde{\epsilon}}\tilde{\psi} &= \bar{F}(\tau, r, \theta)\tilde{\epsilon}, \\
 \delta_\epsilon F &= 0, & \delta_{\tilde{\epsilon}}F &= D_\mu(\tilde{\epsilon}\Gamma^\mu\psi) + iq\sigma\tilde{\epsilon}\psi - iq\phi\tilde{\epsilon}\tilde{\lambda}, \\
 \delta_\epsilon\bar{F} &= D_\mu(\epsilon\Gamma^\mu\tilde{\psi}) + iq\sigma\epsilon\tilde{\psi} + iq\bar{\phi}\epsilon\lambda, & \delta_{\tilde{\epsilon}}\bar{F} &= 0. \quad (2.4)
 \end{aligned}$$

⁴Please see appendix A for our notations and conventions.

Then Q is the sum given by $Q = \delta_\epsilon + \delta_{\tilde{\epsilon}}$. The action of Q^2 on all of the fields can be written compactly as

$$Q^2 = \mathcal{L}_K + \delta_\Lambda^{\text{gauge transf}} + \delta_{\frac{1}{2L}}^{R\text{-symm}}. \quad (2.5)$$

Here $\mathcal{L}_K$ refers to the Lie derivative along the direction of the Killing vector

$$K^\mu = \tilde{\epsilon}\gamma^\mu\epsilon, \quad K = \frac{\partial}{\partial\tau} + \frac{1}{L}\frac{\partial}{\partial\theta}, \quad (2.6)$$

and $\Lambda = \tilde{\epsilon}\epsilon\sigma - K^\rho a_\rho$. We now will need the vector multiplet background about which we will evaluate the one loop determinants of the chiral multiplet. We take this background to be given by

$$a_\mu = 0, \quad \sigma = \frac{i\alpha}{\cosh r}, \quad G = \frac{4i\alpha}{L \cosh^2 r}. \quad (2.7)$$

Here α is a real constant which is matrix valued in the Lie algebra. We can chose it to lie in the Cartan of the gauge group. One can easily verify that this background is invariant under the supersymmetric variation given in (2.1). This background is also the classical solution of the equations of motion as well as minima to the following localizing action of the vector multiplet.

$$QV_{\text{loc}\{\text{bosonic}\}} = \int d^3x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2 \cosh^2 r} D_\mu (\cosh r \sigma) D^\mu (\cosh r \sigma) - \frac{1}{32} \left(G - \frac{4\sigma}{L \cosh r} \right)^2 \right]. \quad (2.8)$$

This localizing action was used to obtain the supersymmetric partition function of Chern-Simons theory on $\text{AdS}_2 \times S^1$ in [19].

2.1 The standard action on $\text{AdS}_2 \times S^1$

In this section we consider a chiral multiplet coupled to an abelian vector multiplet with charge q on supersymmetric $\text{AdS}_2 \times S^1$ background. Since in this paper we are restricting ourselves to only chiral multiplet, the abelian vector field, which couples to either gauge current or global current like flavor current, is restricted to the susy background (2.7). The generalization to the case of chiral multiplet coupled to a non abelian gauge field with gauge group G is straight forward. The supersymmetric action is given by

$$S = \int d^3x \sqrt{g} \left[\mathcal{D}_\mu \bar{\phi} \mathcal{D}_\mu \phi + \left(-\frac{1}{4} q G - \frac{\Delta}{4} R + \frac{1}{2} \left(\Delta - \frac{1}{2} \right) V^2 - q^2 \sigma^2 \right) \bar{\phi} \phi - F \bar{F} + \tilde{\psi} \not{D} \psi + i q \sigma \tilde{\psi} \psi + i q \bar{\phi} (\lambda \psi) - i q \phi (\tilde{\psi} \bar{\lambda}) \right], \quad (2.9)$$

where

$$\begin{aligned} \mathcal{D}_\mu \phi &= \partial_\mu \phi - i \Delta \left(A_\mu - \frac{3}{2} V_\mu \right) \phi - i \left(\Delta - \frac{1}{2} \right) V_\mu \phi - i q a_\mu \phi = D_\mu \phi + \frac{i}{2} V_\mu \phi, \\ \mathcal{D}_\mu \psi &= \nabla_\mu \psi - i (\Delta - 1) \left(A_\mu - \frac{3}{2} V_\mu \right) \psi - i \left(\Delta - \frac{1}{2} \right) V_\mu \psi - i q a_\mu \psi = D_\mu \psi - \frac{i}{2} V_\mu \psi. \end{aligned} \quad (2.10)$$

In the above action Δ is the R-charge of the chiral multiplet, A_μ and V_μ are supergravity background fields whose values are $A_\tau = V_\tau = \frac{1}{L}$ and a_μ is U(1) gauge field and R is the Ricci scalar. This action for the chiral multiplet was considered in [21] and we will refer it to as the ‘standard action’. The action (2.9) is supersymmetric as well as Q -exact upto boundary terms

$$S = \int d^3x \sqrt{g} \frac{1}{\cosh r} \left[\delta_{\tilde{\xi}} \delta_{\xi} \left(\tilde{\psi} \psi + \frac{i}{L \cosh r} \bar{\phi} \phi + 2iq\sigma \bar{\phi} \phi \right) \right] + \text{boundary terms}. \quad (2.11)$$

The boundary terms are given by

$$\begin{aligned} \text{boundary terms} = \int d^3x \sqrt{g} \nabla_\mu \left[-\frac{1}{\cosh r} (\tilde{\epsilon} \tilde{\psi}) (\epsilon \gamma^\mu \psi) - \frac{i}{2} V^\mu \bar{\phi} \phi \right. \\ \left. - \frac{i}{\cosh r} \epsilon^{\mu\rho\nu} (\epsilon \gamma_\nu \tilde{\epsilon}) \bar{\phi} D_\rho \phi + (\epsilon \gamma^\mu \tilde{\epsilon}) \frac{iq\sigma}{\cosh r} \bar{\phi} \phi \right]. \end{aligned} \quad (2.12)$$

It will be pointed out in the section 3 that the above boundary terms vanish with both normalizable as well as supersymmetric boundary conditions. Therefore, we can apply the technique of supersymmetric localization to the chiral multiplet on $\text{AdS}_2 \times S^1$ to evaluate one loop determinants of the fields in the chiral multiplet. Note that we will consider the action in (2.9) in the background of the vector multiplet given in (2.7).

Twisted variables. To proceed with the analysis it is convenient to define the following twisted variables

$$B(\tau, r, \theta) = \tilde{\epsilon} \psi, \quad \tilde{B}(\tau, r, \theta) = \epsilon \tilde{\psi}, \quad C(\tau, r, \theta) = \epsilon \psi, \quad \tilde{C}(\tau, r, \theta) = \tilde{\epsilon} \tilde{\psi}. \quad (2.13)$$

The map to the twisted variables is an one to one map and it can be inverted. In term of these variables the fermions ψ and $\tilde{\psi}$ are given as

$$\psi = \frac{1}{\tilde{\epsilon}\epsilon} \left(-\tilde{\epsilon} C(\tau, r, \theta) + \epsilon B(\tau, r, \theta) \right), \quad \tilde{\psi} = \frac{1}{\tilde{\epsilon}\epsilon} \left(\epsilon \tilde{C}(\tau, r, \theta) - \tilde{\epsilon} \tilde{B}(\tau, r, \theta) \right). \quad (2.14)$$

Kaluza-Klein reduction. In order to simplify the action, we decompose the fields into the Fourier modes labelled by (n, p) along the S^1 and the angle direction of AdS_2 as

$$\begin{aligned} \phi(\tau, r, \theta) &= e^{i(n\tau + p\theta)} f_{n,p}(r), & \bar{\phi}(\tau, r, \theta) &= e^{-i(n\tau + p\theta)} \bar{f}_{n,p}(r), \\ C(\tau, r, \theta) &= e^{i(n\tau + p\theta)} c_{n,p}(r), & \tilde{C}(\tau, r, \theta) &= e^{-i(n\tau + p\theta)} \tilde{c}_{n,p}(r), \\ B(\tau, r, \theta) &= e^{i(n\tau + (p-1)\theta)} b_{n,p}(r), & \tilde{B}(\tau, r, \theta) &= e^{-i(n\tau + (p-1)\theta)} \tilde{b}_{n,p}(r), \\ F(\tau, r, \theta) &= e^{i(n\tau + (p-1)\theta)} F_{n,p}(r), & \bar{F}(\tau, r, \theta) &= e^{-i(n\tau + (p-1)\theta)} \bar{F}_{n,p}(r). \end{aligned} \quad (2.15)$$

In this case the bosonic part of the action (2.9) for a given (n, p) becomes

$$\begin{aligned} S_{B, \{n,p\}} = \frac{1}{4} \int dr \sinh r \left[-4L^2 F_{n,p} \bar{F}_{n,p} + \left(4Ln + 4L^2 n^2 - 2\Delta - 4Ln\Delta + \Delta^2 + \frac{4p^2}{\sinh^2 r} \right. \right. \\ \left. \left. + \frac{4Lq\alpha(-i + Lq\alpha)}{\cosh^2 r} \right) f_{n,p}(r) \bar{f}_{n,p}(r) + 4\partial_r f_{n,p}(r) \partial_r \bar{f}_{n,p}(r) \right], \end{aligned} \quad (2.16)$$

and similarly the fermionic part for a given (n, p) is given by

$$\begin{aligned}
 S_{F,\{n,p\}} = & -\frac{L}{4} \int dr \frac{1}{\cosh^2 r} \left[\tilde{c}_{n,p}(r) \left\{ b_{n,p}(r) (-4 + 2Ln + 4p - \Delta + (-2Ln + \Delta) \cosh 2r) \right. \right. \\
 & + 2 \sinh r \left(i(-2 + 2Ln + 2p - \Delta - 2iLq\alpha) c_{n,p}(r) - 2 \cosh r b'_{n,p}(r) \right) \Big\} \\
 & + \tilde{b}_{n,p}(r) \left\{ c_{n,p}(r) (2Ln + 4p - \Delta + (-2Ln + \Delta) \cosh 2r) \right. \\
 & \left. \left. + 2 \sinh r \left(i(2Ln + 2p - \Delta + 2iLq\alpha) b_{n,p}(r) + 2 \cosh r c'_{n,p}(r) \right) \right\} \right]. \quad (2.17)
 \end{aligned}$$

Note that in these Kaluza-Klein reduced actions we have substituted the background vector multiplet in (2.7). Since F is an auxiliary field, its contribution to the one loop determinant is trivial. Hence forth we will drop it from the action.

Change of variables. We next change to the variable z by defining $z = \tanh^2 r$. Now the origin of AdS_2 is at $z = 0$, while the boundary is at $z = 1$. In this variable the bosonic part of the action becomes

$$\begin{aligned}
 S_{B,\{n,p\}} = & \int_0^1 dz \frac{1}{8z(1-z)^{3/2}} \left[f_{n,p}(z) \bar{f}_{n,p}(z) \left\{ 4p^2(1-z) + z \left(\Delta(\Delta-2) + L(4n(1-\Delta) \right. \right. \right. \\
 & \left. \left. + 4iq(z-1)\alpha \right) + 4L^2(n^2 - q^2(z-1)\alpha^2) \right\} + 16z^2(1-z)^2 f'_{n,p}(z) \bar{f}'_{n,p}(z) \right]. \quad (2.18)
 \end{aligned}$$

In the above we have also included the contribution coming from change of integration measure (i.e. change from r to z).

Now we vary the action w.r.t. $\bar{f}_{n,p}$ to obtain the equation of motion for $f_{n,p}(z)$ which is⁵

$$\begin{aligned}
 2z\sqrt{(1-z)} \partial_z^2 f_{n,p}(z) + \frac{(2-3z)}{\sqrt{1-z}} \partial_z f_{n,p}(z) + \frac{1}{8z\sqrt{(1-z)^3}} \left(-4p^2(1-z) + z(2-\Delta)\Delta \right. \\
 \left. + 4Lz(n(-1+\Delta) + iq\alpha(1-z)) - 4L^2z(n^2 + q^2\alpha^2(1-z)) \right) f_{n,p}(z) = 0. \quad (2.19)
 \end{aligned}$$

Similarly the fermionic part of the action written in the coordinate z is given as

$$\begin{aligned}
 S_{F,\{n,p\}} = & \int_0^1 dz \frac{1}{2} \left(\tilde{b}_{n,p}(z) \tilde{c}_{n,p}(z) \right) \left(-2iL\sqrt{z} \sigma_2 \partial_z + \frac{L(1-2p(1-z) + (-1+2Ln-\Delta)z)}{2\sqrt{z}(1-z)} \sigma_1 \right. \\
 & \left. - \frac{iL}{2\sqrt{z}} \sigma_2 - \frac{iL(-1+2Ln+2p-\Delta)}{2\sqrt{1-z}} + \frac{L(-i+2Lq\alpha)}{2\sqrt{1-z}} \sigma_3 \right) \begin{pmatrix} b_{n,p}(z) \\ c_{n,p}(z) \end{pmatrix}. \quad (2.20)
 \end{aligned}$$

In the above $\{\sigma_i\}$ are Pauli matrices. The corresponding equation of motion for the fermionic variables $(b_{n,p}(z), c_{n,p}(z))$ are given as

$$\begin{aligned}
 \left(-2iL\sqrt{z} \sigma_2 \partial_z + \frac{L(1-2p(1-z) + (-1+2Ln-\Delta)z)}{2\sqrt{z}(1-z)} \sigma_1 \right. \\
 \left. - \frac{iL}{2\sqrt{z}} \sigma_2 - \frac{iL(-1+2Ln+2p-\Delta)}{2\sqrt{1-z}} + \frac{L(-i+2Lq\alpha)}{2\sqrt{1-z}} \sigma_3 \right) \begin{pmatrix} b_{n,p}(z) \\ c_{n,p}(z) \end{pmatrix} = 0. \quad (2.21)
 \end{aligned}$$

⁵The differential operator is not hermitian for real α . This is also true for the operator obtained for fermionic field in which case it is not anti-hermitian. We assume that α is imaginary for which it is hermitian in bosonic case and anti hermitian in fermionic case and at the end we analytically continue back to real α .

These equations provides two first order coupled differential equation in the variable $(b_{n,p}(z), c_{n,p}(z))$. Solving for $b_{n,p}(z)$ in terms of $c_{n,p}(z)$ and its derivative, we obtained

$$b_{n,p}(z) = -\frac{i((2p(-1+z) + (2Ln - \Delta)z)c_{n,p}(z) + 4(-1+z)z \partial_z c_{n,p}(z))}{\sqrt{z(1-z)}(2p - \Delta + 2L(n + iq\alpha))}, \quad (2.22)$$

and substituting this back into the first order derivative equation for $b_{n,p}(z)$, we obtain the second order differential equation involving field $c_{n,p}(z)$ only which is

$$2z\sqrt{(1-z)}\partial_z^2 c_{n,p}(z) + 2\frac{(2-3z)}{\sqrt{1-z}}\partial_z c_{n,p}(z) + \frac{1}{8z\sqrt{(1-z)}^3}\left(-4p^2(1-z) + z(2-\Delta)\Delta + 4Lz(n(-1+\Delta) + iq\alpha(1-z)) - 4L^2z(n^2 + q^2\alpha^2(1-z))\right)c_{n,p}(z) = 0. \quad (2.23)$$

Comparing the above equation with the bosonic equation (2.19), we see that two are identical and therefore admit the same solution. This is not surprising, in fact it follows from supersymmetry that ϕ and $Q\phi$ should obey the same equation of motion. However as we will show in section 3 that normalizable boundary condition for ϕ and fields (B, C) imply different behaviour at the boundary $z \rightarrow 1$.

2.2 Q-exact deformed action

Application of the method of localization requires a positive definite Q -exact action. The action given in (2.9) although Q -exact is not positive definite. One can instead consider adding a Q -exact deformations which gives positive definite contribution to the Lagrangian. For this we choose V to be

$$V = \int d^3x \sqrt{g} \frac{1}{2 \cosh r} \left[\psi \cdot (Q\psi)^* + \tilde{\psi} \cdot (Q\tilde{\psi})^* \right]. \quad (2.24)$$

In the above $*$ is an ordinary complex conjugation. The bosonic part of the above Q -exact deformation is

$$QV|_{\text{bosonic}} = \int d^3x \sqrt{g} \left[-\bar{F}F + g^{\mu\nu} D_\mu \bar{\phi} D_\nu \phi + \frac{i}{\cosh r} \varepsilon^{\mu\rho\alpha} K_\alpha D_\mu \phi D_\rho \bar{\phi} - q^2 \sigma^2 \bar{\phi} \phi \right]. \quad (2.25)$$

In writing the above we have used the reality condition

$$F^* = -\bar{F}, \quad \phi^* = \bar{\phi}. \quad (2.26)$$

By construction the bosonic part of the QV deformation is positive definite⁶ (note that σ is purely imaginary). The fermionic part of the Q -exact deformation is

$$QV|_{\text{fermionic}} = \int d^3x \sqrt{g} \left[\tilde{\psi} \not{D} \psi + 2iV^\mu (\tilde{\psi} \gamma_\mu \psi) + iq\sigma (\tilde{\psi} \psi) - \frac{i}{\cosh r} (V \cdot K) (\tilde{\psi} \psi) - \frac{i}{2 \cosh r} \nabla_\mu [\varepsilon^{\mu\nu\rho} K_\nu (\tilde{\psi} \gamma_\rho \psi)] - \frac{1}{2} \nabla_\mu (\tilde{\psi} \gamma^\mu \psi) \right]. \quad (2.27)$$

⁶The terms involving the derivative of the scalar field in $QV|_{\text{bosonic}}$ can be manifestly written as a sum of squares, $g^{\mu\nu} D_\mu \bar{\phi} D_\nu \phi + \frac{i}{\cosh r} \varepsilon^{\mu\rho\alpha} K_\alpha D_\mu \phi D_\rho \bar{\phi} = X + \frac{1}{2 \cosh^2 r} (K^\alpha D_\alpha \phi)(K^\beta D_\beta \bar{\phi})$, where $X = \frac{1}{2} g^{\mu\nu} (D_\mu \phi + \frac{i}{\cosh r} \varepsilon_{\mu\alpha\beta} K^\alpha D^\beta \phi)(D_\nu \bar{\phi} - \frac{i}{\cosh r} \varepsilon_{\nu\gamma\delta} K^\gamma D^\delta \bar{\phi})$.

Now we have two different actions. The standard action given in (2.9) and the Q -exact deformed action in (2.24). Under the principles of localization, if the fields obey supersymmetric boundary conditions, then the one-loop determinants from either of the actions should yield the same result. To study the effect of boundary conditions in detail we will evaluate the one-loop determinant of both the actions using the Green's function method.

Kaluza-Klein decomposition. As in the previous section we express the bosonic part of the QV action in terms of Fourier modes (2.15) to obtain

$$\begin{aligned}
 QV|_{\text{bosonic}} &= \int dz \left[2z\sqrt{1-z} f'_{n,p}(z) \bar{f}'_{n,p}(z) \right. \\
 &\quad - \frac{1}{2\sqrt{1-z}} \left(2p(z-1) + (2Ln - \Delta)z \right) \frac{d}{dz} (f_{n,p}(z) \bar{f}_{n,p}(z)) \\
 &\quad \left. + \frac{1}{8z(1-z)^{3/2}} \left(4p^2(1-z) - 4Ln\Delta z + \Delta^2 z + 4L^2 z(n^2 + q^2(1-z)\alpha^2) \right) f_{n,p}(z) \bar{f}_{n,p}(z) \right].
 \end{aligned} \tag{2.28}$$

From this we obtain the equation of motion for $f_{n,p}(z)$ which is given as

$$\begin{aligned}
 2z\sqrt{1-z} \partial_z^2 f_{n,p}(z) + \frac{(2-3z)}{\sqrt{1-z}} \partial_z f_{n,p}(z) + \frac{1}{8z\sqrt{(1-z)^3}} \left(4p(p+z)(-1+z) - z\Delta(-4+\Delta+2z) \right. \\
 \left. + 4Lnz(-2+\Delta+z) - 4L^2 z(n^2 + q^2\alpha^2(1-z)) \right) f_{n,p}(z) = 0.
 \end{aligned} \tag{2.29}$$

Similarly, we express the fermionic part of the QV action in terms of Fourier modes and after doing some integration by parts we obtain

$$\begin{aligned}
 QV|_{\text{fermionic}} &= \int_0^1 dz \frac{1}{2} \left(\tilde{b}_{n,p}(z) \tilde{c}_{n,p}(z) \right) \left(2iL\sqrt{z} \sigma_2 \partial_z - \frac{L(1-2p(1-z) + (-1+2Ln-\Delta)z)}{2\sqrt{z}(1-z)} \sigma_1 \right. \\
 &\quad \left. + \frac{iL}{2\sqrt{z}} \sigma_2 + \frac{iL(2Ln+2p-\Delta)}{2\sqrt{1-z}} - \frac{L^2 q \alpha}{2\sqrt{1-z}} \sigma_3 \right) \begin{pmatrix} b_{n,p}(z) \\ c_{n,p}(z) \end{pmatrix}.
 \end{aligned} \tag{2.30}$$

Following the analysis in the previous section we obtain the equation of motion of $c_{n,p}(z)$

$$\begin{aligned}
 2z\sqrt{1-z} \partial_z^2 c_{n,p}(z) + \frac{(2-3z)}{\sqrt{1-z}} \partial_z c_{n,p}(z) + \frac{1}{8z\sqrt{(1-z)^3}} \left(4p(p+z)(-1+z) - z\Delta(-4+\Delta+2z) \right. \\
 \left. + 4Lnz(-2+\Delta+z) - 4L^2 z(n^2 + q^2\alpha^2(1-z)) \right) c_{n,p}(z) = 0,
 \end{aligned} \tag{2.31}$$

with the corresponding relation between $b_{n,p}(z)$ and $c_{n,p}(z)$ which is given as

$$b_{n,p}(z) = - \frac{i((2p(-1+z) + (2Ln - \Delta)z)c_{n,p}(z) + 4(-1+z)z \partial_z c_{n,p}(z))}{\sqrt{z(1-z)}(2p - \Delta + 2L(n + iq\alpha))}. \tag{2.32}$$

3 Boundary conditions

In this section we discuss two boundary conditions that can be chosen for the fields in the chiral multiplet. The consistent choice of boundary conditions to perform the path integral is the normalizable boundary conditions. We will show that this choice of boundary conditions is not always consistent with supersymmetric boundary conditions.

3.1 Normalizable boundary conditions

The normalizable boundary conditions on the bosonic fields $f_{n,p}(z)$ and $\bar{f}_{n,p}(z)$ as $z \rightarrow 1$ are

$$(1-z)^{-1/4} f_{n,p}(z) \rightarrow 0, \quad (1-z)^{-1/4} \bar{f}_{n,p}(z) \rightarrow 0. \quad (3.1)$$

Following (2.13), we find that the normalizable boundary condition on ψ and $\tilde{\psi}$ as $z \rightarrow 1$ imposes the following boundary conditions on twisted variables $b_{n,p}$, $\tilde{b}_{n,p}$ and $c_{n,p}$, $\tilde{c}_{n,p}$,⁷

$$b_{n,p}(z) \rightarrow 0, \quad \tilde{b}_{n,p}(z) \rightarrow 0, \quad c_{n,p}(z) \rightarrow 0, \quad \tilde{c}_{n,p}(z) \rightarrow 0. \quad (3.3)$$

These are also the boundary conditions that ensure that boundary terms that occur on integration by parts vanish. Thus the path integral is well defined with these boundary conditions. It can be seen that these boundary conditions together with the smoothness conditions for the fields in the chiral multiplet near the origin of AdS_2 ensure that the boundary term given in (2.12), that arises on writing the standard action as a Q -exact action, vanishes. Furthermore, these boundary conditions also need to be imposed on the Q -exact deformed action given in (2.25), (2.27) to define the path integral.

The one loop determinants for the operators in the standard action as well as the Q -exact action therefore should be evaluated on the space of solutions satisfying the normalizable boundary conditions. Without knowing the explicit form of the solutions of the differential equations (2.23), (2.23), (2.29), (2.31) by studying the asymptotic behaviour of the solutions we can obtain those that satisfy normalizable boundary conditions. Let us discuss this first for the solutions of the equations of motion of the standard action. From studying the roots of the indicial equation of (2.23) at $z = 1$ we see that the two behaviours for $f_{n,p}$ are given by

$$\begin{aligned} f_{n,p}^+ &\sim (1-z)^{\frac{1}{4}(2Ln-\Delta+2)} + \dots, \\ f_{n,p}^- &\sim (1-z)^{\frac{1}{4}(-2Ln+\Delta)} + \dots. \end{aligned} \quad (3.4)$$

Therefore, for the solutions which are normalizable, that is which satisfies the conditions (3.1) is given by

$$f_{n,p}|_{\text{normalizable}} = \begin{cases} f_{n,p}^+, & \text{for } n > \frac{\Delta-1}{2L} \\ f_{n,p}^-, & \text{for } n < \frac{\Delta-1}{2L}. \end{cases} \quad (3.5)$$

The asymptotic behaviour of $c_{n,p}$ is the same as that of $f_{n,p}$ since its equation given in (2.23) is identical to that of $f_{n,p}$.

$$\begin{aligned} c_{n,p}^+ &\sim (1-z)^{\frac{1}{4}(2Ln-\Delta+2)} + \dots, \\ c_{n,p}^- &\sim (1-z)^{\frac{1}{4}(-2Ln+\Delta)} + \dots. \end{aligned} \quad (3.6)$$

⁷Note that these boundary conditions follow from the standard normalizable fall off behaviour near the boundary of AdS_2 of the boson ϕ and the fermion ψ in the chiral multiplet which are given by

$$\lim_{r \rightarrow \infty} e^{r/2} \phi \sim 0, \quad \lim_{r \rightarrow \infty} e^{r/2} \psi \sim 0. \quad (3.2)$$

Therefore, solutions which satisfy the normalizable boundary conditions in (3.3) are given by

$$c_{n,p}|_{\text{normalizable}} = \begin{cases} c_{n,p}^+, & \text{for } n > \frac{\Delta-2}{2L} \\ c_{n,p}^-, & \text{for } n < \frac{\Delta}{2L}. \end{cases} \quad (3.7)$$

Note that both $c_{n,p}^+$ as well as $c_{n,p}^-$ are admissible in the interval $\frac{\Delta-2}{2L} < n < \frac{\Delta}{2L}$. Finally using (3.6) and (2.22) we obtain the asymptotic behaviour

$$\begin{aligned} b_{n,p}^+ &\sim (1-z)^{\frac{1}{4}(2Ln-\Delta)} + \dots, \\ b_{n,p}^- &\sim (1-z)^{\frac{1}{4}(-2Ln+\Delta+2)} + \dots. \end{aligned} \quad (3.8)$$

Therefore, the solutions which satisfies the boundary conditions (3.3) are given by

$$b_{n,p}|_{\text{normalizable}} = \begin{cases} b_{n,p}^+, & \text{for } n > \frac{\Delta}{2L} \\ b_{n,p}^-, & \text{for } n < \frac{\Delta+2}{2L}. \end{cases} \quad (3.9)$$

Again both $b_{n,p}^+$ as well as $b_{n,p}^-$ are admissible in the interval $\frac{\Delta}{2L} < n < \frac{\Delta+2}{2L}$. Combining (3.5), (3.9) and (3.7) we see that the solutions for the system $\{f, (b, c)\}$ which satisfies normalizable boundary conditions are

$$\{f_{n,p}, (b_{n,p}, c_{n,p})\}|_{\text{normalizable}} = \begin{cases} \{f^+, (b_{n,p}^+, c_{n,p}^+)\}, & \text{for } n > \frac{\Delta}{2L} \\ \{f^+, (b_{n,p}^-, c_{n,p}^-)\}, & \text{for } \frac{\Delta-1}{2L} < n < \frac{\Delta}{2L} \\ \{f^-, (b_{n,p}^-, c_{n,p}^-)\}, & \text{for } n < \frac{\Delta-1}{2L}. \end{cases} \quad (3.10)$$

To arrive at this conclusion it is important to use the fact that the (b, c) system must satisfy (2.22) and therefore, both are either the ‘+’ modes or both ‘−’ modes.

3.2 Supersymmetric boundary condtions

The normalizable boundary conditions discussed in the previous section are not consistent with supersymmetry. Following supersymmetry one can also impose boundary conditions which closes under supersymmetry transformations. Using the normalizable boundary condition on the bosonic field (3.1) and the following supersymmetry transformations

$$\begin{aligned} Qf_{n,p}(r) &= c_{n,p}(r), & Q\bar{f}_{n,p}(r) &= \tilde{c}_{n,p}(r), \\ Qb_{n,p} &= \frac{1}{4L \sinh r} [(2Ln+4p-\Delta+(-2Ln+\Delta) \cosh 2r) f_{n,p}(r) \\ &\quad - 2 \sinh 2r (LF_{n,p} - \partial_r f_{n,p}(r))], \\ Q\tilde{b}_{n,p} &= -\frac{1}{4L \sinh r} [(-2Ln-4p+\Delta+(2Ln-\Delta) \cosh 2r) \bar{f}_{n,p}(r) \\ &\quad + 2 \sinh 2r (L\bar{F}_{n,p} - \partial_r \bar{f}_{n,p}(r))], \end{aligned} \quad (3.11)$$

one can impose the following supersymmetric boundary condition on $b, \tilde{b}$ and $c, \tilde{c}$ as $z \rightarrow 1$ is

$$(1-z)^{1/4} b_{n,p}(z) \rightarrow 0, \quad (1-z)^{1/4} \tilde{b}_{n,p}(z) \rightarrow 0, \quad (1-z)^{-1/4} c_{n,p}(z) \rightarrow 0, \quad (1-z)^{-1/4} \tilde{c}_{n,p}(z) \rightarrow 0. \quad (3.12)$$

We see from the above boundary conditions that the supersymmetry transformations allows the modes for $b_{n,p}(z)$ and $\tilde{b}_{n,p}(z)$ to diverge as $z \rightarrow 1$. Note that the boundary term (2.12) that arises on writing the standard action as a Q -exact term also vanishes using supersymmetric boundary conditions. The way to see this is to observe that under the conditions (3.12) we have

$$\lim_{r \rightarrow \infty} e^{\frac{r}{2}} \tilde{\epsilon} \tilde{\psi} = 0, \quad \lim_{r \rightarrow \infty} (\epsilon \gamma^\mu \psi) \sim e^{\frac{r}{2}}. \quad (3.13)$$

These equations can be obtained by using (3.12) along with (2.13) and (2.14). With this behaviour the first term in boundary term in (2.12) vanishes, while the rest of the terms vanish on the account of the normalizable boundary condition on the bosonic field f . Using the asymptotic behaviours in (3.4), (3.8) and (3.6) and going through the same analysis as in the previous section we obtain the following admissible solutions for the fields.

$$\begin{aligned} f_{n,p}|_{\text{susy}} &= \begin{cases} f_{n,p}^+, & \text{for } n > \frac{\Delta-1}{2L} \\ f_{n,p}^-, & \text{for } n < \frac{\Delta-1}{2L} \end{cases} \\ c_{n,p}|_{\text{susy}} &= \begin{cases} c_{n,p}^+, & \text{for } n > \frac{\Delta-1}{2L} \\ c_{n,p}^-, & \text{for } n < \frac{\Delta-1}{2L} \end{cases} \\ b_{n,p}|_{\text{susy}} &= \begin{cases} b_{n,p}^+, & \text{for } n > \frac{\Delta-1}{2L} \\ b_{n,p}^-, & \text{for } n < \frac{\Delta+3}{2L}. \end{cases} \end{aligned} \quad (3.14)$$

Now using these solutions, the combined system satisfies supersymmetric boundary conditions when

$$\{f_{n,p}, (b_{n,p}, c_{n,p})\}|_{\text{susy}} = \begin{cases} \{f^+, (b_{n,p}^+, c_{n,p}^+)\}, & \text{for } n > \frac{\Delta-1}{2L} \\ \{f^-, (b_{n,p}^-, c_{n,p}^-)\}, & \text{for } n < \frac{\Delta-1}{2L}. \end{cases} \quad (3.15)$$

Comparing (3.10) and (3.15) we see that fields which satisfy normalizable boundary conditions also satisfy supersymmetric boundary conditions unless there exists an integer n in the open interval

$$D : \left(\frac{\Delta-1}{2L}, \frac{\Delta}{2L} \right). \quad (3.16)$$

It is only if there is exits integers in the interval D , the Kaluza-Klein modes which are normalizable do not obey supersymmetric boundary conditions. Now by the principle of localization we expect that if there exist no integer n in the interval D , the one loop determinant of the actions (2.9) and (2.25), (2.27) evaluated on solutions obeying normalizable boundary conditions to agree. This is because in this case they will also obey supersymmetric boundary conditions. Now if there exist an integer n in the interval D we expect the final one loop determinants of the standard action and the Q -exact deformed action to no longer agree. We will demonstrate this explicitly in the next section.

Before we proceed, as an aside we observe from (2.9) or (B.4), after Kaluza-Klein reduction of the scalar ϕ , the equation for $f_{n,p}$ is an equation for minimally coupled scalar field in AdS_2 with

$$(\text{mass})^2 = \frac{1}{L^2} \left(-\frac{1}{4} + \left(n - \frac{\Delta-1}{2} \right)^2 \right). \quad (3.17)$$

From the analysis of [22] we see then that for Kaluza-Klein modes in the interval

$$\frac{\Delta - 3}{2L} < n < \frac{\Delta + 1}{2L}, \quad (3.18)$$

we can use the alternative quantisation for the scalars $f_{n,p}$ by introducing appropriate boundary terms. Note that the above interval contains the domain D . It will be interesting to explore this choice of the boundary conditions. In this paper we restrict ourselves to the standard choice of quantisation conditions for the scalar $f_{n,p}$.

A careful reader will note, that so far we have not considered the case if the either $\frac{\Delta-1}{2L}$ or $\frac{\Delta}{2L}$ is an integer. In such situations, the Kaluza Klein mode is on border of normalizability. Our final result for the one loop determinant obtained after the analysis in section 4.1 and given in (4.45) will also consider these situations. We will appeal to continuity of the partition function in Δ to determine how to incorporate the modes at the edge of normalizability.

3.3 Simplified discussion of the boundary conditions

With the hindsight of the detailed calculation, we provide a simplified understanding of the boundary conditions. Consider the equations of motion (2.19) and (2.22) at the boundary $r \rightarrow \infty$ or $z \rightarrow 1$. It is clear that the vector multiplet background, parameterised by α , do not determine the behaviour of the bosonic fields $f_{n,p}$ or the fermions $b_{n,p}, c_{n,p}$ at the boundary. This is also clear from the action (2.9) and the expectation value of the vector multiplet in (2.7). Therefore, to study the boundary conditions it is sufficient to examine the situation when the chiral multiplet is decoupled from the vector multiplet. The action we need is given in (B.3). The boson ϕ is essentially a massive scalar in AdS_2 with a mass given by

$$m_b^2 L^2 = -\frac{1}{4} + \left(nL - \frac{\Delta - 1}{2} \right)^2. \quad (3.19)$$

The shift in the Kaluza-Klein mode occurs because of the presence of the background A_μ, V_μ fields. Now, given the mass of the scalar in AdS_2 , there are 2 possible behaviours possible as $r \rightarrow \infty$. These are given by

$$\phi^\pm \sim e^{r \left(-\frac{1}{2} \mp \frac{1}{2} \sqrt{1 + 4m_b^2 L^2} \right)} = e^{r \left(-\frac{1}{2} \mp \left(nL - \frac{\Delta - 1}{2} \right) \right)}. \quad (3.20)$$

Therefore, demanding ϕ obeys normalizable boundary conditions imply that we choose

$$\phi|_{\text{normalizable}} = \begin{cases} \phi^+, & \text{for } n > \frac{\Delta - 1}{2L} \\ \phi^-, & \text{for } n < \frac{\Delta - 1}{2L}. \end{cases} \quad (3.21)$$

Now, let's examine the fermions, at $r \rightarrow \infty$. As argued for the bosons the asymptotics can be obtained by studying the behaviour of the Dirac fermion whose action is given in (B.3). Squaring the Dirac equation we see that one obtains the equation of scalar in AdS_2 with mass

$$m_f^2 L^2 = -\frac{1}{4} + \left(nL - \frac{\Delta}{2} \right)^2. \quad (3.22)$$

To ensure that the fermion is normalizable, we need to choose one of the components of the fermion to obey

$$\psi|_{\text{normalizable}} = \begin{cases} \psi^+, & \text{for } n > \frac{\Delta}{2L} \\ \psi^-, & \text{for } n < \frac{\Delta}{2L}. \end{cases} \quad (3.23)$$

Comparing (3.21) and (3.23), we see that there is a possible window when n belongs to the domain D in which one needs to choose the behaviour ϕ^+ for the bosons and ψ^- from the fermions. We summarise the normalizable boundary conditions for both fermions and bosons as follows

$$(\phi, \psi)|_{\text{normalizable}} = \begin{cases} (\phi^+, \psi^+), & \text{for } n > \frac{\Delta}{2L} \\ (\phi^+, \psi^-), & \text{for } D : \frac{\Delta-1}{2L} < n < \frac{\Delta}{2L} \\ (\phi^-, \psi^-), & \text{for } n < \frac{\Delta-1}{2L}. \end{cases} \quad (3.24)$$

We can make the following observations:

- The existence of the interval D occurs is due to the mismatch of the $(\text{mass})^2$ of the fermion and that of the boson.
- Supersymmetry relates the boson ϕ^+ with the fermion ψ^+ and ϕ^- with ψ^- . Therefore, when n belongs to the interval D , normalizable boundary conditions is not compatible with supersymmetry.

Let us now discuss two limits:⁸

$L \rightarrow \infty$ limit. Recall that L is the ratio of the radius of AdS_2 to that of S^1 . In the $L \rightarrow \infty$, the space essentially becomes only AdS_2 . Keeping finite Δ for the $n \neq 0$, note that the mass of the fermion becomes equal to that of the boson, we have

$$\lim_{L \rightarrow \infty, n \neq 0} m_b^2 = m_f^2. \quad (3.25)$$

The asymptotics of the boson ϕ and the fermion ψ are identical and therefore, normalizable boundary conditions are always compatible with supersymmetry. However, one usually restricts the theory to the lightest $n = 0$ mode in the strict $L \rightarrow \infty$ limit. This mode is normalisable as well as compatible with supersymmetry when $\Delta > 1$. But for $\Delta < 1$, the normalizable mode is not compatible with supersymmetry as can be seen from (3.24). For $\Delta = 1$, the boson ϕ^+ is at the border of normalizability and in section 4.1, we discuss how to incorporate such modes by assuming the continuity of the partition function in Δ .

$L \rightarrow 0$ limit. In this limit the space opens up into $\text{AdS}_2 \times R$. The Kaluza-Klein modes become a continuum. Lets keep track of this by taking the limits on the Fourier expansions. The bosons on S^1 were expanded as

$$\begin{aligned} \phi_n(r) &= \int_{-\pi}^{\pi} d\tau \phi(\tau, r) e^{-in\tau}, \\ &= \int_{-\pi}^{\pi} d\tau \phi(\tau, r) e^{-inL \frac{\tau}{L}}. \end{aligned} \quad (3.26)$$

⁸We thank the referee for the suggestion to study these limits.

Now, taking the limit $L \rightarrow 0$ and defining $k = nL, \tau' = \frac{\tau}{L}$ we obtain

$$\phi_k(r) = L \int_{-\infty}^{\infty} \phi(\tau', r) e^{-ikr} . \quad (3.27)$$

The inverse transformation is given by

$$\phi(\tau', r) = \frac{1}{2\pi L} \int_{-\infty}^{\infty} dk \phi_k(r) e^{ik\tau'} . \quad (3.28)$$

From this analysis, it is evident that to take the $L \rightarrow 0$ limit we need to perform the following replacements

$$nL \rightarrow k, \quad \sum_n \rightarrow \frac{1}{2\pi L} \int_{-\infty}^{\infty} dk , \quad (3.29)$$

to take the continuum limit. Now performing the Fourier analysis the equation for ϕ is a scalar in AdS_2 with mass

$$m_b^2 L^2 = -\frac{1}{4} + \left(k - \frac{\Delta-1}{2}\right)^2 . \quad (3.30)$$

The fermion mass is given by

$$m_f^2 L^2 = -\frac{1}{4} + \left(k - \frac{\Delta}{2}\right)^2 \quad (3.31)$$

It is clear that in this limit with finite Δ we have $m_b^2 \neq m_f^2$. We can perform a similar asymptotic analysis and for the Fourier modes in the domain

$$\Delta - 1 < 2k < \Delta , \quad (3.32)$$

normalizable boundary conditions are not compatible with supersymmetry. In the $L \rightarrow 0$ limit, the Kaluza-Klein modes become a continuum and it is always possible to find modes in the interval D .

A simple and natural way to state these observations is that it is only when the mass of the boson and the fermion are equal in AdS_2 , normalisable boundary conditions are *always* compatible with supersymmetry. In the $L \rightarrow \infty$ this is the case for $\Delta > 1$. For any finite L , the masses are never equal on the account of the background fields A_μ, V_μ . In the $L \rightarrow 0$ limit due to the continuum, there are always modes which obey normalizable boundary conditions but not compatible with supersymmetry.

4 One loop determinants from the Green's functions

Note that the action in (2.9) is quadratic in the fields of the chiral multiplet. Therefore, given the background vector multiplet in (2.7) we can in principle obtain its one loop determinant. The general rules of quantum field theory dictates that we use normalizable boundary conditions for the fields in the path integral. However the method of localization will require the boundary conditions also to be consistent with supersymmetry. We have shown in the previous section that these two boundary conditions are not consistent if there

exists an integer n in the domain (3.16). The action given in (2.9) is a simple example of a situation in which one can evaluate the one loop determinant using the method of localization as well as directly by using the Green's function approach. The one loop determinant calculation using Green's function approach relies on the boundary conditions being normalizable while the method of localization relies on the boundary conditions being supersymmetric. Thus if these boundary conditions are not consistent the answer for the one loop determinant will in general be different. It is this phenomenon we wish to make explicit. We will do this by carrying out the following steps:

1. Evaluate the one loop determinant of the action (2.9) in the vector multiplet background (2.7) by developing the Green's function approach. This relies on normalizable boundary conditions.
2. Evaluate the one loop determinant of the Q -exact action (2.25), (2.27) in the vector multiplet background (2.7) again using the Green's function approach.
3. Comparing the results we will see that one loop determinants of the standard action and that of the Q -exact action differ only when there exists an integer n in the domain (3.16).
4. Finally in section 5 we obtain the one loop determinant by evaluating the index of the D_{10} operator corresponding to the localizing action given in (2.24). The method relies on using boundary conditions which are supersymmetric. We observe that this one loop determinant differs from that obtained from the Green's function for the Q -exact action when there exists an integer n in the domain (3.16).

4.1 One loop determinant of the standard action

To evaluate the one loop determinant of the action in (2.9) we first obtain the Green's function for the bosonic and fermionic operators present in this action. Note that the action depends on the parameter α (2.7) which parametrises the expectation value of the scalar of the vector multiplet. Therefore, the one loop determinant will depend on α . Now by the standard rules which relate the one loop determinant to the Green's function we have the equation

$$\frac{\delta}{\delta\alpha} \ln Z_{1\text{-loop}}(\alpha) = \text{Tr} \left[G_F \frac{\delta}{\delta\alpha} \mathcal{D}_F(\alpha) \right] - \frac{1}{2} \text{Tr} \left[G_B \frac{\delta}{\delta\alpha} \mathcal{D}_B(\alpha) \right]. \quad (4.1)$$

Here G_B is the Green's function of the bosonic operator $\mathcal{D}_B$, while G_F is the Green's function of the fermionic operator $\mathcal{D}_F$ which occurs in the action (2.9). Once the Green's functions are known, we can use (4.1) and integrate with respect to α and obtain the one loop determinant up to a constant independent of α . In fact for specific values of Δ, L we will show that this constant is trivial by directly evaluating the one loop determinant with $\alpha = 0$ using the eigen function method.

The Green's functions for bosonic and fermionic operators of the action (2.9) are constructed by solving the differential equations satisfied by the corresponding fields after the Kaluza-Klein reduction. For the bosonic operator, this is given in (2.19) and for

the fermionic operator it is given in (2.21). Note that differential equation for the boson $f_{n,p}(z)$, (2.19), and fermion $c_{n,p}(z)$, (2.23), are same. Since both the equations are same, we just need to find the solution for one of the differential equation to obtain the Green's function. However we need to make sure that the corresponding solution for $b_{n,p}(z)$ obtained by using (2.22) is smooth in the interior and satisfy the right boundary conditions. The solution which is smooth near $z = 0$ for $p > 0$ is

$$S_{1+}(z) = (1-z)^{\frac{1}{4}(-2Ln+\Delta)} z^{p/2} \times {}_2F_1 \left[\frac{1}{4}(2-2Ln+2p+\Delta+2iLq\alpha), \frac{1}{4}(2p+\Delta-2L(n+iq\alpha)), 1+p, z \right], \quad (4.2)$$

which we see by noticing that its $z \rightarrow 0$ behaviour goes like

$$S_{1+}(z) \rightarrow z^{p/2} + \mathcal{O}(z^{p/2+1}). \quad (4.3)$$

The solution which is smooth near $z = 0$ for $p < 0$

$$S_{1-}(z) = (1-z)^{\frac{1}{4}(-2Ln+\Delta)} z^{-p/2} \times {}_2F_1 \left[\frac{1}{4}(2-2Ln-2p+\Delta+2iLq\alpha), \frac{1}{4}(-2p+\Delta-2L(n+iq\alpha)), 1-p, z \right], \quad (4.4)$$

which goes like, for $z \rightarrow 0$

$$S_{1-}(z) \rightarrow z^{-p/2} + \mathcal{O}(z^{-p/2+1}). \quad (4.5)$$

For $p = 0$ both the solutions are identical and we take this to be $S_{1-}(z)$. In this case the second independent solution is singular which goes logarithmic near $z = 0$.

Now to get the behaviour of the solution at $z = 1$, we start with the differential equation (2.19) and substitute $z \rightarrow 1 - y$, then we get

$$\begin{aligned} & \left(2\Delta(1-y) - \Delta^2(1-y) - 4p^2y - 4L(1-y)(n-n\Delta-iqy\alpha) - 4L^2(1-y)(n^2+q^2y\alpha^2) \right) f_{n,p}(y) \\ & + 8(1-y)y \left((1-3y)\partial_y f_{n,p}(y) + 2(1-y)y\partial_y^2 f_{n,p}(y) \right) = 0. \end{aligned} \quad (4.6)$$

In this case we again have two solutions. The solution which satisfy normalizable boundary condition for $n > \frac{\Delta-1}{2L}$ is

$$S_{2+}(z) = (1-z)^{\frac{1}{4}(2+2Ln-\Delta)} z^{-p/2} \times {}_2F_1 \left[\frac{1}{4}(2+2Ln-2p-\Delta-2iLq\alpha), \frac{1}{4}(4-2p-\Delta+2L(n+iq\alpha)), \frac{3}{2}+Ln-\frac{\Delta}{2}, 1-z \right], \quad (4.7)$$

and the one which satisfies normalizable boundary condition for $n < \frac{\Delta-1}{2L}$ is

$$S_{2-}(z) = (1-z)^{\frac{1}{4}(-2Ln+\Delta)} z^{-p/2} \times {}_2F_1 \left[\frac{1}{4}(2-2Ln-2p+\Delta+2iLq\alpha), \frac{1}{4}(-2p+\Delta-2L(n+iq\alpha)), \frac{1}{2}(1-2Ln+\Delta), 1-z \right]. \quad (4.8)$$

For $n = \frac{\Delta-1}{2L}$, both the solutions coincide. However in this case

$$S_{2+}(z) = S_{2-}(z) \sim (1-z)^{1/4} + \dots \quad (4.9)$$

So for this value of n , $f_{n,p}(z)$ is at the border of normalizability. The second independent solution involves a logarithmic divergence.

Now we look for the solution for the fermionic equations. Since the equation for $c(z)$ is same as that of $f(z)$, we will, therefore, have the same solutions. As discussed in section 3.1 the normalizable boundary conditions for $b(z)$ and $c(z)$ are given by

$$b(z) \rightarrow 0, \quad c(z) \rightarrow 0, \quad \text{for } z \rightarrow 1. \quad (4.10)$$

Therefore, $S_{2+}(z)$ is valid solution for $c(z)$ for $n > \frac{\Delta-2}{2L}$ and $S_{2-}(z)$ is valid solution for $n < \frac{\Delta}{2L}$. Thus we see that in the range $\frac{\Delta-2}{2L} < n < \frac{\Delta}{2L}$, both solutions $S_{2+}(z)$ and $S_{2-}(z)$ are admissible solutions for $c(z)$. Now we will substitute the expression for $c(z)$ in (2.22) to obtain $b(z)$ and find its asymptotic behaviour. One gets

$$\begin{aligned} c_{n,p}(z) = S_{2+}(z), \quad b_{n,p}(z) &\sim (1-z)^{\frac{1}{4}(2Ln-\Delta)} + \dots \\ c_{n,p}(z) = S_{2-}(z), \quad b_{n,p}(z) &\sim (1-z)^{\frac{1}{4}(-2Ln+\Delta+2)} + \dots \end{aligned} \quad (4.11)$$

So for $b(z)$, $S_{2+}(z)$ is valid solution for $n > \frac{\Delta}{2L}$ and $S_{2-}(z)$ is valid solution for $n < \frac{\Delta+2}{2L}$. So for (b, c) system, we have valid solution $S_{2+}(z)$ for $n > \frac{\Delta}{2L}$ and $S_{2-}(z)$ is valid solution for $n < \frac{\Delta}{2L}$. So we see that for $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$ both $f(z)$ and $(b(z), c(z))$ have the same admissible solutions. However there is a mismatch of the solution in the range $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$. In this range the valid solution for $f(z)$ is $S_{2+}(z)$ whereas for $(b(z), c(z))$ system, the valid solution is $S_{2-}(z)$. This conclusion was reached earlier in section 3.1 just by the analysis of the asymptotic properties of the solutions. Here we have the explicit solutions of the differential equations (2.19) and (2.21) and the necessary choices of functions so as to satisfy the boundary conditions at $z = 0$ as well at $z = 1$. Using these solutions we can construct the Green's functions corresponding to these differential equations.

We still need to analyse the boundary behaviour of the mode at $n = \frac{\Delta}{2L}$ (if it is an integer). At $n = \frac{\Delta}{2L}$ we have a similar feature as noticed in the bosonic case at $n = \frac{\Delta-1}{2L}$. If we consider $c_{\frac{\Delta}{2L},p}(z) = S_{2+}(z)$ which goes like $\sqrt{1-z}$ as $z \rightarrow 1$, the corresponding mode for $b_{\frac{\Delta}{2L},p}(z)$ goes like $\mathcal{O}(1)$ and is at the border of normalizability. On the other hand for $c_{\frac{\Delta}{2L},p}(z) = S_{2-}(z)$ which goes like $\mathcal{O}(1)$ as $z \rightarrow 1$, the corresponding mode for $b_{\frac{\Delta}{2L},p}(z)$ goes like $\sqrt{1-z}$ and is admissible. Thus for the (b, c) system none of the modes at $n = \frac{\Delta}{2L}$ satisfy strict normalizable boundary conditions.

Construction of the Green's function: $n > \frac{\Delta}{2L}$ or $n < \frac{\Delta-1}{2L}$. In this case both $f(z)$ and $(b(z), c(z))$ have same valid solution as $z \rightarrow 1$, let's call it $S_2(z)$. Also let us assume that the smooth solution near $z \rightarrow 0$ is $S_1(z)$, it will be $S_{1\pm}(z)$ depending on whether p is positive or negative or zero in which case both, $S_{1+}(z)$ and $S_{1-}(z)$, are equal. Let us first construct the Green's function for bosonic field $f_{n,p}$. Given the solutions of (2.19), the Green's function can be written as

$$G_b(z, z') = c_b \left[\Theta(z' - z) S_1(z) S_2(z') + \Theta(z - z') S_1(z') S_2(z) \right]. \quad (4.12)$$

c_b is some constant.

This satisfies the continuity condition

$$\lim_{\epsilon \rightarrow 0} G(z' - \epsilon, z') = \lim_{\epsilon \rightarrow 0} G(z' + \epsilon, z'). \quad (4.13)$$

The discontinuity condition for the first derivative of Green's function implies that

$$S_1(z') \partial_z S_2(z)|_{z=z'} - S_2(z') \partial_z S_1(z)|_{z=z'} = \frac{1}{c_b a(z')}, \quad (4.14)$$

where

$$a(z) = -2z\sqrt{1-z}. \quad (4.15)$$

To fix the constant c_b we consider the Wronskian which is defined as

$$W(z) = \partial_z S_2(z) S_1(z) - \partial_z S_1(z) S_2(z). \quad (4.16)$$

The Wronskian satisfies

$$\partial_z W(z) - \frac{3z-2}{2z(z-1)} W(z) = 0. \quad (4.17)$$

The solution is given by

$$W(z) = \frac{c_1}{z\sqrt{1-z}}. \quad (4.18)$$

Thus, comparing with the condition (4.14) the constants c_1 and c_b are related by

$$c_b = \frac{z\sqrt{1-z}}{c_1 a(z)} = -\frac{1}{2c_1}. \quad (4.19)$$

The constant c_1 can be determined by evaluating $W(z)$ at some value of z . Its value depends on integers (n, p) . We now will determine the constant c_1 for various cases. When $S_1(z) = S_{1+}(z)$ and $S_2(z) = S_{2+}(z)$, then evaluating $W(z)$ near $z \rightarrow 0$, we obtain

$$\lim_{z \rightarrow 0} W(z) = -p \frac{\Gamma(p)\Gamma(\frac{3}{2} + Ln - \frac{\Delta}{2})}{z \Gamma\left[\frac{1}{4}(2 + 2p + 2L(n - iq\alpha) - \Delta)\right] \Gamma\left[\frac{1}{4}(4 + 2p + 2L(n + iq\alpha) - \Delta)\right]}, \quad (4.20)$$

and comparing with (4.18) we obtain the constant

$$c_{1++} = -p \frac{\Gamma(p)\Gamma(\frac{3}{2} + Ln - \frac{\Delta}{2})}{\Gamma\left[\frac{1}{4}(2 + 2p + 2L(n - iq\alpha) - \Delta)\right] \Gamma\left[\frac{1}{4}(4 + 2p + 2L(n + iq\alpha) - \Delta)\right]}. \quad (4.21)$$

Similarly, when $S_1(z) = S_{1-}(z)$ and $S_2(z) = S_{2+}(z)$, then by evaluating $W(z)$ near $z \rightarrow 0$, we obtain

$$c_{1-+} = p \frac{\Gamma(-p)\Gamma(\frac{3}{2} + Ln - \frac{\Delta}{2})}{\Gamma\left[\frac{1}{4}(2 - 2p + 2L(n - iq\alpha) - \Delta)\right] \Gamma\left[\frac{1}{4}(4 - 2p + 2L(n + iq\alpha) - \Delta)\right]}. \quad (4.22)$$

When $S_1(z) = S_{1+}(z)$ and $S_2(z) = S_{2-}(z)$, then by evaluating $W(z)$ near $z \rightarrow 0$, we obtain

$$c_{1+-} = -p \frac{\Gamma(p)\Gamma(\frac{1}{2}(1 - 2Ln + \Delta))}{\Gamma\left[\frac{1}{4}(2 + 2p - 2L(n - iq\alpha) + \Delta)\right] \Gamma\left[\frac{1}{4}(2p - 2L(n + iq\alpha) + \Delta)\right]}. \quad (4.23)$$

When $S_1(z) = S_{1-}(z)$ and $S_2(z) = S_{2-}(z)$, then by evaluating $W(z)$ near $z \rightarrow 0$, we obtain

$$c_{1--} = p \frac{\Gamma(-p)\Gamma(\frac{1}{2}(1-2Ln+\Delta))}{\Gamma\left[\frac{1}{4}(2-2p-2L(n-iq\alpha)+\Delta)\right]\Gamma\left[\frac{1}{4}(-2p-2L(n+iq\alpha)+\Delta)\right]}. \quad (4.24)$$

These constants will play important role in calculating the one loop determinant. For instance for $p > 0$ and $n > \frac{\Delta}{2L}$, the Green's function in (4.12) will involve the constant $c_b = -\frac{1}{2c_{1++}}$. Similar statements apply for all the other cases in (4.22), (4.23) and (4.24).

Let us now compute the Green's function for fermions. The Green's function for the (b, c) system satisfies the first order differential equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} G_f(z, z') = \delta(z - z'), \quad (4.25)$$

where the various elements of the matrix are given in (2.21). The Green's function can then be written in terms of the solutions to the b, c system as

$$G_f(z, z') = c_f \left[\Theta(z' - z) \begin{pmatrix} b_1(z)b_2(z') & b_1(z)S_2(z') \\ b_2(z')S_1(z) & S_1(z)S_2(z') \end{pmatrix} + \Theta(z - z') \begin{pmatrix} b_1(z')b_2(z) & b_2(z)S_1(z') \\ b_1(z')S_2(z) & S_1(z')S_2(z) \end{pmatrix} \right]. \quad (4.26)$$

Here $b_1(z)$ and $b_2(z)$ are determined from (2.22) by substituting $c(z) \rightarrow S_1(z)$ and $c(z) \rightarrow S_2(z)$ respectively. What is left in the construction of the fermionic Green's function is to determine the constant c_f . We now relate c_f to the bosonic constant c_b in (4.19). From (4.25) we see that the continuity constraint of the Green's function given in (4.13) needs to be satisfied only by the diagonal elements. Evaluating the discontinuity of the Green's function in (4.26) we obtain

$$\lim_{\epsilon \rightarrow 0} (G_f(z' - \epsilon, z') - G_f(z' + \epsilon, z')) = -\frac{4c_f c_1}{\sqrt{z'}(2Ln + 2p - \Delta + 2iLq\alpha)} \sigma_2. \quad (4.27)$$

To obtain this we have also used the relation obtained from the Wronksian (4.16) and (4.18) which is given by

$$\partial_z S_2(z) = \frac{c_1 + z\sqrt{1-z}S_2(z)\partial_z S_1(z)}{z\sqrt{1-z}S_1(z)}. \quad (4.28)$$

Integrating the differential equation (4.25) from $z' - \epsilon$ to $z' + \epsilon$ and from the fact that the first order derivative in (2.21) comes with the coefficient $-2iL\sqrt{z}\sigma_2$ we obtain

$$\frac{8iLc_f c_1}{2Ln + 2p - \Delta + 2iLq\alpha} = 1 \Rightarrow c_f = -\frac{(2Ln + 2p - \Delta + 2iLq\alpha)c_b}{4iL}. \quad (4.29)$$

where we have used $c_1 = -\frac{1}{2c_b}$. This completes the construction of the fermionic Green's function. It is important to note that for each of the cases in (4.21), (4.22), (4.23), (4.24) we use (4.29) to obtain the corresponding values of c_f .

We are now ready to compute the contribution to the one loop determinant from the modes in the range $n > \Delta/2L$ and $n < (\Delta - 1)/2L$. If $\mathcal{D}_b(\alpha)$ and $\mathcal{D}_f(\alpha)$ are the differential operator for the complex scalar and fermions, respectively whose determinant

we are interested in to calculate and α is the background value of scalar field in the vector multiplet, then⁹

$$\frac{\delta}{\delta\alpha} \ln \tilde{Z}_{1\text{-loop}}(\alpha) = \text{Tr} \left[G_F \frac{\delta}{\delta\alpha} \mathcal{D}_f(\alpha) \right] - \text{Tr} \left[G_b \frac{\delta}{\delta\alpha} \mathcal{D}_b(\alpha) \right]. \quad (4.30)$$

Here $\tilde{Z}_{1\text{-loop}}(\alpha)$ is the contribution to one loop determinant coming from the modes in the range $n > \Delta/2L$ and $n < (\Delta - 1)/2L$. Now for the specific differential operators of our interest we obtain

$$\frac{\delta}{\delta\alpha} \mathcal{D}_f(\alpha) = \frac{L^2 q}{2\sqrt{1-z}} \sigma_3, \quad \frac{\delta}{\delta\alpha} \mathcal{D}_b(\alpha) = \frac{Lq(-i + 2Lq\alpha)}{2\sqrt{1-z}}. \quad (4.31)$$

Evaluating the difference in the Green's function and performing the trace in position space we obtain

$$\begin{aligned} \frac{\delta}{\delta\alpha} \ln \tilde{Z}_{1\text{-loop}}(\alpha) &= \int_0^1 dz \frac{c_b Lq}{4\sqrt{(1-z)}} \left[2(i - 2Lq\alpha) S_1(z) S_2(z) - i(2p - \Delta + 2L(n + iq\alpha)) \right. \\ &\quad \times \left. \left(S_1(z) S_2(z) + \frac{XY}{(1-z)z(-2p + \Delta - 2L(n + iq\alpha))^2} \right) \right]. \end{aligned} \quad (4.32)$$

Here

$$\begin{aligned} X &= (2p(-1+z) + (2Ln - \Delta)z) S_1(z) + 4(-1+z)z \partial_z S_1(z), \\ Y &= (2p(-1+z) + (2Ln - \Delta)z) S_2(z) + 4(-1+z)z \partial_z S_2(z). \end{aligned} \quad (4.33)$$

After integrating by parts¹⁰ and using the fact that $S_1(z)$ satisfies equation of motion, we find that the integrand is a total derivative and therefore the integral is given by

$$\frac{\delta}{\delta\alpha} \ln \tilde{Z}_{1\text{-loop}} = - \frac{iLq S_2(z) ((2p(-1+z) + (2Ln - \Delta)z) S_1(z) + 4(-1+z)z \partial_z S_1(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n + iq\alpha))c_1} \Big|_0^1. \quad (4.34)$$

We now evaluate the right hand side of the above expression for various range of (n, p) . We see that the expression depends on the integration constant c_1 from the denominator. However the numerator also proportional to c_1 . Therefore, the expression is independent of c_1 and in fact we do not need the explicit expressions given in (4.21)–(4.24) to evaluate it. Let us denote the right hand side of the expression in (4.34) by BT.

$n > \frac{\Delta}{2L}$ and $p > 0$: we have $S_1(z) = S_{1+}(z)$ and $S_2(z) = S_{2+}(z)$. In this case the boundary term is given by

$$\text{BT} = \frac{2iLq}{2p - \Delta + 2L(n + iq\alpha)}. \quad (4.35)$$

$n > \frac{\Delta}{2L}$ and $p \leq 0$: we have $S_1(z) = S_{1-}(z)$ and $S_2(z) = S_{2+}(z)$. In this case the boundary term is given by

$$\text{BT} = 0. \quad (4.36)$$

⁹Note that there is no factor of $\frac{1}{2}$ in the last term because we are considering a complex scalar.

¹⁰Note that integrating by parts is allowed since all the fields satisfy normalizable boundary conditions.

$n < \frac{\Delta-1}{2L}$ and $p > 0$: we have $S_1(z) = S_{1+}(z)$ and $S_2(z) = S_{2-}(z)$. In this case the boundary term is given by

$$\text{BT} = 0. \quad (4.37)$$

$n < \frac{\Delta-1}{2L}$ and $p \leq 0$: we have $S_1(z) = S_{1-}(z)$ and $S_2(z) = S_{2-}(z)$. In this case for the boundary term is given by

$$\text{BT} = -\frac{2iLq}{2p - \Delta + 2L(n + iq\alpha)}. \quad (4.38)$$

We can integrate each of these expressions with respect to α to obtain the one loop determinant.

Construction of the Green's function: $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$. We still need to analyse the region when the Kaluza-Klein mode n lies in the interval $D : \frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$. Of course if there is no integer n in the interval $(\frac{\Delta-1}{2L}, \frac{\Delta}{2L})$ then the contributions to the one loop determinant obtained by integrating (4.35), (4.36), (4.37), (4.38) is the complete answer. For example consider the situation with $\Delta = 3/2$ and $L = 1$ then this range is $(1/4, 3/4)$ and there exists no integer n in this range. In general if $\Delta > 1$ and $L > 1/2$ we see that there are no integers in the domain D . However, consider the case with $\Delta = 1/2$. The theory is conformal at this point, the domain D becomes $(-\frac{1}{4L}, +\frac{1}{4L})$ and the integer $n = 0$ is necessarily lies inside this range.

Let us evaluate the contribution of the Kaluza-Klein modes to the one loop determinant when $n \in D$. For these values of n , the admissible solution, near the boundary of AdS_2 , for $f(z)$ is $S_{2+}(z)$, whereas for $(b(z), c(z))$ system, the valid solution is $S_{2-}(z)$. Furthermore, we will see that for $n \in D$, the variation of the logarithm of one loop determinant is no longer total derivative. It contains a bulk term together with a boundary term.

Let us consider the case with $p > 0$. In this case we construct the bosonic Green's function with the mode $S_{1+}(z)$ and $S_{2+}(z)$ and the fermionic Green's function with the mode $S_{1+}(z)$ and $S_{2-}(z)$. Repeating the previous analysis, we find that the contribution to the variation of one loop determinant in the range $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$ and for $p > 0$ is

$$\begin{aligned} \frac{\delta}{\delta\alpha} \ln \hat{Z}_{1\text{-loop}}^+(\alpha) = & -\frac{iLqS_{2-}(z)((2p(-1+z) + (2Ln - \Delta)z)S_{1+}(z) + 4(-1+z)z\partial_z S_{1+}(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n + iq\alpha))c_{1+-}} \Big|_0^1 \\ & + \text{Bulk term}, \end{aligned} \quad (4.39)$$

where the bulk term is given by

$$\begin{aligned} \text{Bulk term} = & -\int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1+}(z)S_{2-}(z)}{4c_{1+-}\sqrt{1-z}} \\ & + \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1+}(z)S_{2+}(z)}{4c_{1++}\sqrt{1-z}}. \end{aligned} \quad (4.40)$$

In the above expression for the bulk term, the first term comes from the trace of the fermionic Green's function while the second term is the trace of the bosonic Green's function. Similarly the contribution to the variation of one loop determinant in the range

$\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$ and for $p \leq 0$ is

$$\frac{\delta}{\delta\alpha} \ln \hat{Z}_{1\text{-loop}}^-(\alpha) = - \frac{iLqS_{2-}(z)((2p(-1+z) + (2Ln - \Delta)z)S_{1-}(z) + 4(-1+z)z\partial_z S_{1-}(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n + iq\alpha))c_{1--}} \Big|_0^1 + \text{Bulk term}, \quad (4.41)$$

where now the bulk term is given as

$$\begin{aligned} \text{Bulk term} = & - \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1-}(z)S_{2-}(z)}{4c_{1--}\sqrt{1-z}} \\ & + \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1-}(z)S_{2+}(z)}{4c_{1-+}\sqrt{1-z}}. \end{aligned} \quad (4.42)$$

The contributions from the boundary terms in (4.39) and (4.41) are given in (4.37) and (4.38), respectively. Note that the contributions from the boundary terms are easy to integrate with respect to α and result in logarithms to the one loop free energy.

Let us proceed to evaluate the bulk terms (4.40) and (4.42). These terms involve an integration of a product of Hypergeometric functions. The relevant integrals are listed in appendix C. The results are linear combinations of digamma functions. After this we need to further integrate these terms with respect to α . This is reasonably simple to do, since the digamma functions are derivatives of the logarithm of the gamma functions and the variable α occurs linearly in their arguments.

One loop determinant. Assimilating all the contributions of all the Kaluza-Klein modes n , keeping track of the sign of p and performing the integration with respect to α we obtain the following answer of the one loop determinant.

$$\begin{aligned} \ln Z(\alpha, \Delta) = & \sum_{p>0, n>\frac{\Delta}{2L}} \ln \left(p + L(n + iq\alpha) - \frac{\Delta}{2} \right) - \sum_{p\leq 0, n<\frac{\Delta-1}{2L}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) \\ & - \sum_{\substack{p\leq 0 \\ \frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) - \sum_{\substack{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L} \\ p \in \mathbb{Z}}} \sum_{p \in \mathbb{Z}} \ln \frac{\Gamma(\frac{1}{2} + \frac{1}{4}\hat{x})\Gamma(\frac{1}{4}\hat{x}^*)}{\Gamma(\frac{1}{2} + \frac{1}{4}\hat{y})\Gamma(1 + \frac{1}{4}\hat{y}^*)}, \end{aligned} \quad (4.43)$$

where

$$\hat{x} = 2|p| + \Delta - 2Ln + 2iLq\alpha, \quad \hat{y} = 2|p| - \Delta + 2Ln - 2iLq\alpha. \quad (4.44)$$

The first line in (4.43) is the total contribution from the boundary terms for the Kaluza-Klein modes in the range $n > \Delta/2L$ and $n < (\Delta - 1)/2L$. The second line contains the contribution of the bulk term as well as the boundary term in (4.39) and (4.41) when $n \in D$. The bulk integral results in product of Gamma functions. We note that (4.43) is not yet final result as we still need to include contributions from KK modes for which $\frac{\Delta}{2L}$ or $\frac{\Delta-1}{2L}$ or both are integer. However, if Δ and L are such that none of these ratios are integers then (4.43) is the final result. As we noted earlier when such ratios are integer the modes are at the border of normalizability and therefore, it is not very clear how to take

into account their contributions to partition function which is computed with normalizable boundary conditions. In this scenario we take a clue from the free theory partition function obtained in appendix B, that is that the partition function is a continuous function of Δ . We assume that this holds true even in the presence of non zero charge q *i.e.* we require that $\ln Z(\alpha, \Delta)$ is continuous across every real value of Δ . With this requirement we find that the partition function is¹¹

$$\begin{aligned} \ln Z(\alpha, \Delta) = & \sum_{p>0, n \geq \lceil \frac{\Delta}{2L} \rceil} \ln \left(p + L(n + iq\alpha) - \frac{\Delta}{2} \right) - \sum_{p \leq 0, n < \frac{\Delta-1}{2L}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) \\ & - \sum_{\substack{p \leq 0 \\ \lceil \frac{\Delta-1}{2L} \rceil \leq n < \frac{\Delta}{2L}}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) \\ & - \sum_{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \sum_{p \in \mathbb{Z}} \ln \frac{\Gamma(\frac{1}{2} + \frac{1}{4}\hat{x})\Gamma(\frac{1}{4}\hat{x}^*)}{\Gamma(\frac{1}{2} + \frac{1}{4}\hat{y})\Gamma(1 + \frac{1}{4}\hat{y}^*)}. \end{aligned} \quad (4.45)$$

The ceiling function $\lceil x \rceil$ in the above sum gives an integer greater than or equal to x . The equation (4.45) is our final result for the one loop determinant for a chiral multiplet coupled to a supersymmetric background vector multiplet on $\text{AdS}_2 \times S^1$. Let us recall that we have used normalizable boundary conditions in AdS_2 and periodic boundary conditions for the fermions on S^1 .¹²

There could be an α independent constant since we have obtained the result in (4.45) by first differentiating the one loop determinant with respect α , evaluate this using Green's function approach and then integrating back with respect to α . This constant could depend on both L and Δ . To study the Δ dependence, we can repeat the above steps but now taking variation with respect to Δ . It turns out that in the range $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$ where there are no bulk terms, all the steps go through as before, bulk terms cancel and one arrives at boundary terms. However, in this case the boundary term receives extra contributions in addition to the first line in (4.43). These are terms which are independent of p and therefore, vanish after summing p from $-\infty$ to ∞ by using zeta function regularization. Note that the variations with respect to α and Δ are integrable and fix the one loop determinant upto an overall L dependent constant. In the range $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$, the integral appearing in $\text{Tr}[G_F \frac{\delta}{\delta \Delta}]$ is divergent near $z = 1$ (*i.e.* near the boundary of AdS_2). It is possible to regularize properly and compute these variations. We find that bulk terms agree as well with the result obtained from the α variation. We will now show by comparison with explicit eigen function calculations for $\alpha = 0$, $L = 1, L = 2$ and $0 < \Delta < 2$ that (4.45) precisely agrees with the one loop determinant obtained by the eigen function method.

¹¹From (4.43) we find that for any Δ_0 , in particular for $\Delta_0 = 1$ and 0 , $\lim_{\epsilon \rightarrow 0} \ln Z(\alpha, \Delta_0 - \epsilon) = \lim_{\epsilon \rightarrow 0} \ln Z(\alpha, \Delta_0 + \epsilon)$. We arrive at (4.45) by requiring that $\lim_{\epsilon \rightarrow 0} \ln Z(\alpha, \Delta_0 - \epsilon) = \lim_{\epsilon \rightarrow 0} \ln Z(\alpha, \Delta_0 + \epsilon) = \ln Z(\alpha, \Delta_0)$.

¹²If one takes the $L \rightarrow \infty$ retaining the $n \neq 0$ modes, then care must be taken to regularize the leading divergences to obtain a finite answer. However, usually in this limit, we restrict ourselves to the lowest Kaluza-Klein mode $n = 0$. It is straight forward to track the contribution of this mode from the various terms in (4.45) for any positive Δ to obtain the one loop result. Next, formally to take the $L \rightarrow 0$ limit, we use the replacement (3.29) as well as scale $L\alpha \rightarrow \tilde{\alpha}$ in (4.45). The sums over n are replaced by integrals over k in the appropriate regimes. For example, the first term involves an integral over k running with $2k \geq \Delta$.

Comparison with eigen function result: $L = 1$, $0 < \Delta < 2$. When $q = 0$, the standard action in (2.9) reduces to that of the free boson and free fermion in $\text{AdS}_2 \times S^1$. The background vector multiplet decouples. The eigen function method has been applied to evaluate partition functions for such actions by [20]. We use this approach and evaluate the partition function for the chiral multiplet with $q = 0$, $L = 1$ and Δ in the range $0 < \Delta < 2$. This has been done in appendix B. The result is given in (B.45) for $0 < \Delta < 1$ and (B.50) for $1 < \Delta < 2$. Note that these expressions are symmetric under the transformation $\Delta \rightarrow 2 - \Delta$. This symmetry is manifest in the integral representation for the free energy given in (B.14).¹³ It can also be verified easily by examining the partition function in the infinite product form given in (B.45) and (B.50).

We will verify the general result for the partition function given in (4.45) agrees with that obtained by the eigen function method for $L = 1$ for $q = 0$. First let us write down the expression in (4.45) for $L = 1$ and $1 < \Delta < 2$. Note that for this situation, there is no contribution from the second line of (4.45). Thus we obtain

$$\ln Z(\alpha)|_{L=1, 1 < \Delta < 2} = \sum_{p=1, n=1}^{\infty} \ln \left(p+n+iq\alpha - \frac{\Delta}{2} \right) - \sum_{p=0, n=0}^{\infty} \ln \left(p+n-iq\alpha + \frac{\Delta}{2} \right). \quad (4.46)$$

Now for $L = 1$ and $0 < \Delta < 1$, the only integer allowed in the domain D is $n = 0$, the second line in (4.45) contributes. We can therefore take

$$\hat{x}|_{n=0, L=1} = 2|p| + \Delta + 2iq\alpha, \quad \hat{y}|_{n=0, L=1} = 2|p| - \Delta - 2iq\alpha. \quad (4.47)$$

We expand the term involving the gamma functions using the identity

$$\Gamma(z) = \frac{e^{\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)^{-1} e^{\frac{z}{n}}. \quad (4.48)$$

This leads to

$$\begin{aligned} \mathcal{I} &= \sum_{p \in \mathbb{Z}} \ln \frac{\Gamma(\frac{1}{2} + \frac{\hat{x}}{4}) \Gamma(\frac{\hat{x}^*}{4})}{\Gamma(\frac{1}{2} + \frac{\hat{y}}{4}) \Gamma(1 + \frac{\hat{y}^*}{4})} \Big|_{n=0, L=1}, \\ &= - \sum_{n=0, p=1}^{\infty} 2 \ln \left(n + p + \frac{\Delta}{2} + iq\alpha \right) - \sum_{n=0}^{\infty} \ln \left(n + \frac{\Delta}{2} + iq\alpha \right) \\ &\quad + \sum_{n=1, p=1}^{\infty} 2 \ln \left(n + p - \frac{\Delta}{2} - iq\alpha \right) + \sum_{n=1}^{\infty} \ln \left(n - \frac{\Delta}{2} - iq\alpha \right) \\ &\quad - \sum_{p \in \mathbb{Z}} \left[\ln \left(\frac{|p| + \frac{\Delta}{2} - iq\alpha}{|p| + \frac{\Delta}{2} + iq\alpha} \right) - \sum_{n=1}^{\infty} \ln \left(\frac{2n + |p| + \frac{\Delta}{2} - iq\alpha}{2n + |p| + \frac{\Delta}{2} + iq\alpha} \right) \right] \\ &\quad - \sum_{p \in \mathbb{Z}} \left[\ln \left(\frac{2 + |p| + \frac{\Delta}{2} - iq\alpha}{2 + |p| + \frac{\Delta}{2} + iq\alpha} \right) - \sum_{n=1}^{\infty} \ln \left(\frac{2(n+1) + |p| + \frac{\Delta}{2} - iq\alpha}{2(n+1) + |p| + \frac{\Delta}{2} + iq\alpha} \right) \right]. \end{aligned} \quad (4.49)$$

¹³In fact from the integral representation of the free energy (B.14) we see that there are the following symmetries at $q = 0$: $\Delta \rightarrow \Delta + 2L$ for arbitrary L . For $L = \frac{1}{N}$, where N is a positive integer, we have the symmetry $\Delta \rightarrow \frac{2}{N} - \Delta$.

To obtain this we have used $\zeta(0) = -\frac{1}{2}$. Combining this with the rest of the terms of (4.45) we obtain

$$\begin{aligned}
 \ln Z(\alpha)|_{L=1, 0 < \Delta < 1} = & \sum_{n=0, p=1}^{\infty} \ln \left(p+n+\frac{\Delta}{2}+iq\alpha \right) \\
 & + \sum_{n=1, p=1}^{\infty} \ln \left(\frac{p+n+\frac{\Delta}{2}+iq\alpha}{p+n+\frac{\Delta}{2}-iq\alpha} \right) + \sum_{n=1, p=-\infty}^{\infty} \ln \left(\frac{2n+|p|+\frac{\Delta}{2}-iq\alpha}{2n+|p|+\frac{\Delta}{2}+iq\alpha} \right) \\
 & - \sum_{n=0, p=1}^{\infty} \ln \left(n+p-\frac{\Delta}{2}-iq\alpha \right) \\
 & + \sum_{n=1, p=1}^{\infty} \ln \left(\frac{p+n-\frac{\Delta}{2}+iq\alpha}{p+n-\frac{\Delta}{2}-iq\alpha} \right) + \sum_{n=1, p=-\infty}^{\infty} \ln \left(\frac{2n+|p|-\frac{\Delta}{2}-iq\alpha}{2n+|p|+\frac{\Delta}{2}+iq\alpha} \right). \quad (4.50)
 \end{aligned}$$

Note that on re-organising the two terms on the second line of (4.50) we see that they combine and cancel each other. The same property holds also for the two terms of the fourth line of (4.50). Therefore, we are left with

$$\ln Z(\alpha)|_{L=1, 0 < \Delta < 1} = \sum_{n=0, p=1}^{\infty} \ln \left(p+n+\frac{\Delta}{2}+iq\alpha \right) - \sum_{n=0, p=1}^{\infty} \ln \left(n+p-\frac{\Delta}{2}-iq\alpha \right). \quad (4.51)$$

We can now see that replacing $\Delta \rightarrow 2 - \Delta$ in (4.46) we obtain (4.51). Thus this symmetry which observed in the partition function at $L = 1, q = 0$ in (B.14) continues to hold when $q \neq 0$. Furthermore note that on substituting $q = 0$ in (4.46) and (4.51) it precisely agrees with the eigen function partition function in (B.45) and (B.50). This agreement and the fact that (4.45) has the $\Delta \rightarrow 2 - \Delta$ symmetry for $L = 1$ serves as a non-trivial check of the Green's function method. This also implies that the L dependent integration constant in (4.45) is zero at least for $L = 1$.

We have also verified that the partition function (4.45) has the symmetry $\Delta \rightarrow 2L - \Delta$ for $L = \frac{1}{2}$ and non zero q . Given these observations for $L = 1$ and $L = \frac{1}{2}$, it seems likely that the partition function is invariant under $\Delta \rightarrow 2L - \Delta$ for $L = \frac{1}{N}$ and for non zero q , although it will be very nice to prove it in general. Moreover, it is easy to see that the partition function (4.45) is also invariant under $\Delta \rightarrow 2L + \Delta$ for non zero q and arbitrary L . Thus this symmetry also continues to hold for non zero q .

Comparison with the eigen function result: $L = 2, 0 < \Delta < 2$. As a further check on the Green's function method we compare (4.45) at $q = 0$ with the eigen function result for $L = 2$. First consider the domain $1 < \Delta < 2$. In this interval there is no contribution from the second line of (4.45). The first line yields

$$\ln Z|_{L=2, 1 < \Delta < 2} = \sum_{p=1, n=1}^{\infty} \ln \left(p+2n-\frac{\Delta}{2} \right) - \sum_{p=0, n=0}^{\infty} \ln \left(p+2n+\frac{\Delta}{2} \right). \quad (4.52)$$

This expression precisely agrees with that obtained by the eigen function method in (B.58). Now lets examine the domain $0 < \Delta < 1$. The integer allowed in the domain D is $n = 0$,

thus now the second line of (4.45) contributes. Note however since only $n = 0$ contributes, the values of $\hat{x}$ and $\hat{y}$ is independent of L and therefore the contribution of the term involving the gamma function is same as that given in (4.49) with $q = 0$. Taking all this into account we find for $0 < \Delta < 1$, and $L = 2$, (4.45) reduces to

$$\begin{aligned} \ln Z|_{L=2, 0 < \Delta < 1} = & \sum_{p=1, n=1}^{\infty} \ln \left(p + 2n - \frac{\Delta}{2} \right) - \sum_{p=0, n=1}^{\infty} \ln \left(p + 2n + \frac{\Delta}{2} \right) \\ & + \sum_{n=0, p=1}^{\infty} 2 \ln \left(n + p + \frac{\Delta}{2} \right) \\ & - \sum_{n=1, p=1}^{\infty} 2 \ln \left(n + p - \frac{\Delta}{2} \right) - \sum_{n=1}^{\infty} \ln \left(n - \frac{\Delta}{2} \right). \end{aligned} \quad (4.53)$$

Comparing the above equation and the result from the eigen function method in (B.62) and after some obvious rearrangement of terms we see that they precisely agree. The agreement of the result (4.45) with the eigen function method for $L = 2, q = 0$ in domain $0 < \Delta < 2$ serves as another check of the green function method developed in this paper. Note again, the fact that we have obtained agreement with the eigen function method for $L = 2$ implies that the putative L dependent integration constant in (4.45) is zero.

4.2 One loop determinant of the Q -exact action

In this section we repeat the above analysis for the Q -exact deformations presented in the section 2.2. In order to compute the Green's function, we need to find the solutions of the differential equations (2.29) and (2.31). These differential equations are certainly different from that of the equations of motion for f and (b, c) obtained from the standard action which are given in (2.19), (2.23). However it can be easily seen that the asymptotic properties of the solutions as $z \rightarrow 1$ to these fields for both the Q exact action as well the standard action are the same. Also the equation relating the field b to c in (2.22) and (2.32). Therefore, the asymptotic behaviour of the fields are given by (3.4), (3.6) and (3.8). As mentioned earlier we need to impose normalizable boundary conditions on all the fields to ensure that the path integral is well defined. Thus the normalizable solution for all the Kaluza-Klein modes are given by (3.10).

Let us now proceed to solve the equations (2.29) and (2.31) and construct the Green's function. We will implement smoothness near $z = 0$ and normalizable boundary conditions at $z = 1$. The solution for this equation which is smooth near $z = 0$ for $p > 0$ is given by

$$\tilde{S}_{1+}(z) = (1 - z)^{\frac{1}{4}(-2Ln + \Delta)} z^{p/2} {}_2F_1[a_1, b_1, 1 + p, z], \quad (4.54)$$

whereas the solution which is smooth near $z = 0$ for $p < 0$

$$\tilde{S}_{1-}(z) = (1 - z)^{\frac{1}{4}(-2Ln + \Delta)} z^{-p/2} {}_2F_1[\tilde{a}_1, \tilde{b}_1, 1 - p, z]. \quad (4.55)$$

Here

$$\begin{aligned}
 a_1 &= \frac{1}{4} \left(1 - 2Ln + 2p + \Delta - \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 b_1 &= \frac{1}{4} \left(1 - 2Ln + 2p + \Delta + \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 \tilde{a}_1 &= \frac{1}{4} \left(1 - 2Ln - 2p + \Delta - \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 \tilde{b}_1 &= \frac{1}{4} \left(1 - 2Ln - 2p + \Delta + \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right). \quad (4.56)
 \end{aligned}$$

For $p = 0$, both the solution coincide. The second solution for $p = 0$ is logarithmic in z and therefore will not consider in the analysis. Next we look for the solution near $z = 1$. The normalizable solution for $n > \frac{\Delta-1}{2L}$ is

$$\tilde{S}_{2+}(z) = (1-z)^{\frac{1}{4}(2+2Ln-\Delta)} z^{p/2} {}_2F_1 \left[a_2, b_2, \frac{3}{2} + Ln - \frac{\Delta}{2}, 1-z \right], \quad (4.57)$$

whereas the admissible solution for $n < \frac{\Delta-1}{2L}$ is

$$\tilde{S}_{2-}(z) = (1-z)^{\frac{1}{4}(-2Ln+\Delta)} z^{p/2} {}_2F_1 \left[\tilde{a}_2, \tilde{b}_2, \frac{1}{2} - Ln + \frac{\Delta}{2}, 1-z \right]. \quad (4.58)$$

Here

$$\begin{aligned}
 a_2 &= \frac{1}{4} \left(3 + 2Ln + 2p - \Delta - \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 b_2 &= \frac{1}{4} \left(3 + 2Ln + 2p - \Delta + \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 \tilde{a}_2 &= \frac{1}{4} \left(1 - 2Ln + 2p + \Delta - \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right), \\
 \tilde{b}_2 &= \frac{1}{4} \left(1 - 2Ln + 2p + \Delta + \sqrt{1 - 4Ln - 4p + 2\Delta - 4L^2q^2\alpha^2} \right). \quad (4.59)
 \end{aligned}$$

For $n = \frac{\Delta-1}{2L}$, both the solutions coincide and goes like $(1-z)^{1/4}$ which is at the border of normalisability. The other solution is logarithmic in $(1-z)$ near $z = 1$ which is not square integrable.

Analysing the (b, c) system we find the similar situation as in the previous case. For the (b, c) system, we have the admissible solution $\tilde{S}_{2+}(z)$ for $n > \frac{\Delta}{2L}$ and $\tilde{S}_{2-}(z)$ is admissible solution for $n < \frac{\Delta}{2L}$. So we see that for $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$ both $f(z)$ and $(b(z), c(z))$ have the same admissible solutions. However there is mismatch of the solution in the range $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$.

At $n = \frac{\Delta}{2L}$ (if it is an integer) we have the similar feature as explained in the previous section. If we begin with $c_{\frac{\Delta}{2L}, p}(z) = \tilde{S}_{2+}(z)$ which goes like $\sqrt{1-z}$ as $z \rightarrow 1$, the corresponding mode for $b_{\frac{\Delta}{2L}, p}(z)$ goes like $\mathcal{O}(1)$ and is at the border of normalizability. On the other hand for $c_{\frac{\Delta}{2L}, p}(z) = \tilde{S}_{2-}(z)$ which goes like $\mathcal{O}(1)$ as $z \rightarrow 1$, the corresponding mode for $b_{\frac{\Delta}{2L}, p}(z)$ goes like $\sqrt{1-z}$ and is admissible. Thus for the (b, c) system none of the modes at $n = \frac{\Delta}{2L}$ satisfy strict normalizable boundary conditions.

Case 2: $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$. Proceeding as before we find that for $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$, the variation of the one loop determinant is again total derivative and is given by

$$\frac{\delta}{\delta\alpha} \ln \tilde{Z}_{1\text{-loop}}(\alpha) = - \frac{iLq\tilde{S}_2(z)((2p(-1+z) + (2Ln - \Delta)z)\tilde{S}_1(z) + 4(-1+z)z\partial_z\tilde{S}_1(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n + iq\alpha))\tilde{c}_1} \Big|_0^1. \quad (4.60)$$

Here $\tilde{c}_1$ is a constant determined by evaluating the Wronskian near $z = 0$. For various range of (n, p) , this constant is given as follows

$$\begin{aligned} \tilde{c}_{1++} &= -p \frac{\Gamma(p)\Gamma(\frac{3}{2} + Ln - \frac{\Delta}{2})}{\Gamma(a_2)\Gamma(b_2)}, & \text{for } p > 0, n > \frac{\Delta}{2L}, \\ \tilde{c}_{1-+} &= p \frac{\Gamma(-p)\Gamma(\frac{3}{2} + Ln - \frac{\Delta}{2})}{\Gamma(a_2 - p)\Gamma(b_2 - p)}, & \text{for } p \leq 0, n > \frac{\Delta}{2L}, \\ \tilde{c}_{1+-} &= -p \frac{\Gamma(p)\Gamma(\frac{1}{2}(1 - 2Ln + \Delta))}{\Gamma(\tilde{a}_2)\Gamma(\tilde{b}_2)}, & \text{for } p > 0, n < \frac{\Delta - 1}{2L}, \\ \tilde{c}_{1--} &= p \frac{\Gamma(-p)\Gamma(\frac{1}{2}(1 - 2Ln + \Delta))}{\Gamma(\tilde{a}_1 - p)\Gamma(\tilde{b}_2 - p)}, & \text{for } p \leq 0, n < \frac{\Delta - 1}{2L}. \end{aligned} \quad (4.61)$$

Note that though there seems to be an explicit dependence on $\tilde{c}_1$ in denominator of the boundary terms (4.60), the numerator also depends linearly on $\tilde{c}_1$ when we substitute the behaviour of the functions at 0 and 1. In fact for the Kaluza-Klein modes $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$ we do not need the precise knowledge of the normalisation constant $\tilde{c}_1$. The final result for the boundary terms is independent of the constant $\tilde{c}_1$. We now evaluate the right hand side of (4.60), which we denote below by $\widetilde{\text{BT}}$, for various range of (n, p) .

$n > \frac{\Delta}{2L}$ and $p > 0$: In this case we have $S_1(z) = \tilde{S}_{1+}(z)$ and $S_2(z) = \tilde{S}_{2+}(z)$. Therefore, the boundary term is

$$\widetilde{\text{BT}} = \frac{2iLq}{2p - \Delta + 2L(n + iq\alpha)}. \quad (4.62)$$

$n > \frac{\Delta}{2L}$ and $p \leq 0$: In this case we have $S_1(z) = \tilde{S}_{1-}(z)$ and $S_2(z) = \tilde{S}_{2+}(z)$. Therefore, the boundary term is

$$\widetilde{\text{BT}} = 0. \quad (4.63)$$

$n < \frac{\Delta-1}{2L}$ and $p > 0$: In this case we have $S_1(z) = \tilde{S}_{1+}(z)$ and $S_2(z) = \tilde{S}_{2-}(z)$. Therefore, the boundary term is

$$\widetilde{\text{BT}} = 0. \quad (4.64)$$

$n < \frac{\Delta-1}{2L}$ and $p \leq 0$: In this case we have $S_1(z) = \tilde{S}_{1-}(z)$ and $S_2(z) = \tilde{S}_{2-}(z)$. Therefore, the boundary term is

$$\widetilde{\text{BT}} = - \frac{2iLq}{2p - \Delta + 2L(n + iq\alpha)}. \quad (4.65)$$

Case 2: $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$. As in the previous case of the standard action, we also need to consider the contribution to the one loop determinant for $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$. We begin with the case for $p > 0$. In this case we construct the bosonic Green's function with the mode $\tilde{S}_{1+}(z)$ and $\tilde{S}_{2+}(z)$ and the fermionic Green's function with the mode $\tilde{S}_{1+}(z)$ and $\tilde{S}_{2-}(z)$.

Repeating the previous analysis, we find that the contribution to the variation of one loop determinant in this range is

$$\begin{aligned} \frac{\delta}{\delta\alpha} \ln \hat{Z}_{1\text{-loop}}^+(\alpha) = & - \frac{iLq\tilde{S}_{2-}(z)((2p(-1+z) + (2Ln - \Delta)z)\tilde{S}_{1+}(z) + 4(-1+z)z\partial_z\tilde{S}_{1+}(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n+iq\alpha))\tilde{c}_{1+-}} \Big|_0^1 \\ & + \text{Bulk term} \end{aligned} \quad (4.66)$$

where the bulk term is given as

$$\text{Bulk term} = - \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1+}(z) \tilde{S}_{2-}(z)}{2\tilde{c}_{1+-}\sqrt{1-z}} + \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1+}(z) \tilde{S}_{2+}(z)}{2\tilde{c}_{1++}\sqrt{1-z}}. \quad (4.67)$$

In the above expression, the first term comes from the trace of the fermionic Green's function and the second term is the trace of the bosonic Green's function. Similarly the contribution to the variation of one loop determinant in the range $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$ and for $p \leq 0$ is

$$\begin{aligned} \frac{\delta}{\delta\alpha} \ln \hat{Z}_{1\text{-loop}}^- = & - \frac{iLq\tilde{S}_{2-}(z)((2p(-1+z) + (2Ln - \Delta)z)\tilde{S}_{1-}(z) + 4(-1+z)z\partial_z\tilde{S}_{1-}(z))}{2\sqrt{1-z}(2p - \Delta + 2L(n+iq\alpha))\tilde{c}_{1--}} \Big|_0^1 \\ & + \text{Bulk term}, \end{aligned} \quad (4.68)$$

where now the bulk term is given as

$$\text{Bulk term} = - \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1-}(z) \tilde{S}_{2-}(z)}{2\tilde{c}_{1--}\sqrt{1-z}} + \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1-}(z) \tilde{S}_{2+}(z)}{2\tilde{c}_{1-+}\sqrt{1-z}}. \quad (4.69)$$

The contributions from the boundary terms in (4.66) and (4.68) are given in (4.64) and (4.65), respectively. Next we would like to evaluate the bulk terms (4.67) and (4.69) which involve an integration of a product of Hypergeometric functions. The result of these integrations again involve linear combination of digamma functions. They are listed in the appendix C. Then further integrating with respect to α and assimilating all the contributions we obtain the following result for the one loop determinant for the Q -exact localising action.

$$\begin{aligned} \ln Z(\alpha, \Delta) = & \sum_{p>0, n>\frac{\Delta}{2L}} \ln \left(p + L(n+iq\alpha) - \frac{\Delta}{2} \right) - \sum_{p\leq 0, n<\frac{\Delta-1}{2L}} \ln \left(-p - L(n+iq\alpha) + \frac{\Delta}{2} \right) \\ & - \sum_{p\leq 0, \frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \ln \left(-p - L(n+iq\alpha) + \frac{\Delta}{2} \right) - \sum_{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \sum_{p>0} \ln \frac{\Gamma(\tilde{a}_2)\Gamma(b_2)}{\Gamma(\tilde{b}_2)\Gamma(a_2)} \\ & - \sum_{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \sum_{p\leq 0} \ln \frac{\Gamma(\tilde{a}_2-p)\Gamma(b_2-p)}{\Gamma(\tilde{b}_2-p)\Gamma(a_2-p)}. \end{aligned} \quad (4.70)$$

As discussed in the previous section, the above expression for the partition function is not yet the final result for the Q exact deformation. We still need to include contributions from KK modes for which $\frac{\Delta}{2L}$ or $\frac{\Delta-1}{2L}$ or both are integer. However, if Δ and L are such that these ratios are not integers then (4.70) is the final result for the Q -exact deformation. As

we noted earlier when such ratios are integer the modes are at the border of normalizability and therefore, it is not very clear how to take into account their contributions to partition function which is computed with normalizable boundary conditions. Following the previous analysis we also assume here that the partition function is a continuous as a function of Δ even in the presence of non zero charge q *i.e.* we require that $\ln Z(\alpha, \Delta)$ is continuous across every real value of Δ . With this requirement we find that the partition function is

$$\begin{aligned} \ln Z(\alpha, \Delta) = & \sum_{p>0, n \geq \lceil \frac{\Delta}{2L} \rceil} \ln \left(p + L(n + iq\alpha) - \frac{\Delta}{2} \right) - \sum_{p \leq 0, n < \frac{\Delta-1}{2L}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) \\ & - \sum_{p \leq 0, \lceil \frac{\Delta-1}{2L} \rceil \leq n < \frac{\Delta}{2L}} \ln \left(-p - L(n + iq\alpha) + \frac{\Delta}{2} \right) - \sum_{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \sum_{p>0} \ln \frac{\Gamma(\tilde{a}_2) \Gamma(b_2)}{\Gamma(\tilde{b}_2) \Gamma(a_2)} \\ & - \sum_{\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}} \sum_{p \leq 0} \ln \frac{\Gamma(\tilde{a}_2 - p) \Gamma(b_2 - p)}{\Gamma(\tilde{b}_2 - p) \Gamma(a_2 - p)}. \end{aligned} \quad (4.71)$$

Let us compare the result for the one loop determinant of the Q -exact action with that from the standard action in (4.45). We see that when $n > \frac{\Delta}{2L}$ and $n < \frac{\Delta-1}{2L}$ the one loop determinant in (4.70) precisely agrees with that from the standard action. From our discussion in section 3, we know that these Kaluza-Klein modes obey both normalizable and supersymmetric boundary conditions. Therefore, we see explicitly that we can either use the standard action or the Q exact action to obtain the partition function. This then is in accordance with the principle of localization that a Q -exact deformation should not change the result for the partition function. However if there are Kaluza-Klein modes n such that $\frac{\Delta-1}{2L} < n < \frac{\Delta}{2L}$ their contribution to the one loop determinant for the standard action differs from that of the Q -exact action. It is for these modes, that normalizable boundary conditions do not agree with supersymmetric boundary conditions. Therefore, there is no reason why the two one loop determinants, (4.45) and (4.70), obtained from the Green's function method using normalizable boundary conditions should agree.

Furtermore, it is easy to see that the partition function (4.71) is invariant under $\Delta \rightarrow 2L + \Delta$ for non zero q and arbitrary L . Thus this symmetry also continues to hold for Q -exact deformations.

5 One loop determinant from the index of D_{10}

As emphasised several times in the text, to define the path integral one should use normalizable boundary conditions for all the fields involved. We also saw in section 3 that if there exists an integer n in the domain D supersymmetric boundary conditions are not compatible with normalizable boundary conditions. For such a situation we expect that one certainly cannot use the localization methods which rely on supersymmetric boundary conditions to obtain one loop determinant. For the case of supersymmetric boundary conditions one method of obtaining the one loop determinant is to evaluate the index of the D_{10} operator associated with the Q -exact action in 2.2. In this section we do the index computation using the explicit kernel and co-kernel analysis for the D_{10} operator. We will

then compare the result with the Green's function answer obtained in (4.70) and show that when there exists n in the domain D the answers do not agree.

We start with the Q -exact deformation (2.24), which is given as

$$V = \int d^3x \sqrt{g} \frac{1}{\cosh r} \left[\bar{F}(\psi \tilde{\xi}) - \psi \gamma^\mu \xi D_\mu \bar{\phi} - i q \sigma \bar{\phi}(\psi \xi) - F(\tilde{\psi} \xi) + \tilde{\psi} \gamma^\mu \tilde{\xi} D_\mu \phi + i q \sigma \phi(\tilde{\psi} \tilde{\xi}) \right]. \quad (5.1)$$

To recover the terms relevant for D_{10} operator, we express the above action as follows

$$V = \text{Tr} (Q X_0 \ X_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ Q X_1 \end{pmatrix}. \quad (5.2)$$

In the above $X_0 = \{\phi, \bar{\phi}\}$ and $X_1 = \{B, \bar{B}\}$. Now the terms relevant for D_{10} operator are

$$- \frac{\sqrt{g}}{\cosh^2 r} [(\xi \gamma^\mu \xi) B D_\mu \bar{\phi} + (\tilde{\xi} \gamma^\mu \tilde{\xi}) \bar{B} D_\mu \phi]. \quad (5.3)$$

It is important to note that we get the same D_{10} operator from (2.11).

Now let us consider first the kernel equation

$$(\xi \gamma^\mu \xi) D_\mu \bar{\phi} = 0. \quad (5.4)$$

This gives the following first order differential equation

$$\sinh r \left(-n + \frac{\Delta}{2L} \right) \bar{f}_{n,p}(r) + \frac{p}{L \sinh r} \bar{f}_{n,p}(r) + \frac{\cosh r}{L} \partial_r \bar{f}_{n,p}(r) = 0. \quad (5.5)$$

In the above we have expanded the field in terms of Fourier modes $\bar{\phi} = \sum_{n,p} e^{-i(n\tau + p\theta)} \bar{f}_{n,p}(r)$. The solution of the above differential equations is given by

$$\bar{f}_{n,p}(r) = C_1 (\cosh r)^{\frac{1}{4}(2Ln + 4p - \Delta)} (\sinh r)^{\frac{1}{4}(-2Ln - 4p + \Delta)} (\sinh 2r)^{-\frac{1}{4}(-2Ln + \Delta)}. \quad (5.6)$$

We see that as $r \rightarrow \infty$, the solution goes like $\sim e^{-\frac{r}{2}(-2Ln + \Delta)}$ and for $r \rightarrow 0$, the solution goes like $\sim r^{-p}$. Clearly the solution with $p > 0$ is not smooth near $r \rightarrow 0$. Therefore, the solutions with $p > 0$ are excluded.

The other kernel equation is

$$(\tilde{\xi} \gamma^\mu \tilde{\xi}) D_\mu \phi = 0, \quad (5.7)$$

which becomes

$$\sinh r \left(-n + \frac{\Delta}{2L} \right) f_{n,p}(r) + \frac{p}{L \sinh r} f_{n,p}(r) + \frac{\cosh r}{L} \partial_r f_{n,p}(r) = 0. \quad (5.8)$$

In the above we have expanded the field in terms of Fourier modes $\phi = \sum_{n,p} e^{i(n\tau + p\theta)} f_{n,p}(r)$. It is again the same equation as above and, therefore it has the same solution.

Now we look for the cokernel equations. In this case we get

$$D_\mu \left(\frac{\xi \gamma^\mu \xi}{\cosh^2 r} B \right) = 0. \quad (5.9)$$

Using the Killing spinor equation we get

$$\sinh r \left(n - \frac{\Delta}{2L} \right) b_{n,p}(r) + \frac{1-p}{L \sinh r} b_{n,p}(r) + \frac{\cosh r}{L} \partial_r b_{n,p}(r) = 0. \quad (5.10)$$

In the above we have expanded the field in terms of Fourier modes $B = \sum_{n,p} e^{i(n\tau + (p-1)\theta)} b_{n,p}(r)$. We see that it is exactly same as the above equations (equation for ϕ). Therefore, the solution is

$$b_{n,p}(r) = C_2 (\cosh r)^{\frac{1}{4}(-2Ln-4p+4+\Delta)} (\sinh r)^{\frac{1}{4}(2Ln+4p-4-\Delta)} (\sinh 2r)^{-\frac{1}{4}(2Ln-\Delta)}. \quad (5.11)$$

The asymptotic of the above solution is

$$b_{n,p} \sim e^{-\frac{r}{2}(2Ln-\Delta)} \text{ for } r \rightarrow \infty, \text{ and } b_{n,p} \sim r^{p-1}, \text{ } r \rightarrow 0. \quad (5.12)$$

Clearly for $p < 1$ the solution is not smooth and therefore excluded. Similarly we have another cokernel equation

$$D_\mu \left(\frac{\tilde{\xi} \gamma^\mu \tilde{\xi}}{\cosh^2 r} \tilde{B} \right) = 0, \quad (5.13)$$

and using the killing spinor equation we get

$$\sinh r \left(n - \frac{\Delta}{2L} \right) \tilde{b}_{n,p}(r) + \frac{1-p}{L \sinh r} \tilde{b}_{n,p}(r) + \frac{\cosh r}{L} \partial_r \tilde{b}_{n,p}(r) = 0. \quad (5.14)$$

In the above we have expanded the field in terms of Fourier modes $\tilde{B} = \sum_{n,p} e^{-i(n\tau + (p-1)\theta)} \tilde{b}_{n,p}(r)$. We see that it is exactly same as the above equation and therefore, the solution remains same.

5.1 Evaluation of the index

We now impose the following susy boundary conditions at asymptotic infinity and evaluate the index of D_{10} operator. The fall off conditions at infinity are

$$e^{r/2} f_{n,p} \rightarrow 0, \quad e^{-r/2} b_{n,p} \rightarrow 0. \quad (5.15)$$

Similar boundary conditions are imposed on the complex conjugate fields. In this case we see from the asymptotic behaviour of the solutions (5.6) and (5.11) that the mode $\bar{f}_{n,p}$ belongs to the kernel if n satisfies $-2Ln + \Delta - 1 > 0$ and the mode $b_{n,p}$ belongs to co-kernel if n satisfies $2Ln - \Delta + 1 > 0$. Thus the contribution to index from fields $(\phi, \bar{\phi})$ and $(B, \bar{B})$ are

$$\phi : \prod_{n < \frac{\Delta-1}{2L}, p=0}^{-\infty} \left[i \left(n + \frac{p}{L} \right) + iq\Lambda - i \frac{\Delta}{2L} \right], \quad (5.16)$$

$$\bar{\phi} : \prod_{n < \frac{\Delta-1}{2L}, p=0}^{-\infty} \left[-i \left(n + \frac{p}{L} \right) - iq\Lambda + i \frac{\Delta}{2L} \right], \quad (5.17)$$

$$B : \prod_{n > \frac{\Delta-1}{2L}, p=1}^{\infty} \left[i \left(n + \frac{p-1}{L} \right) + iq\Lambda - \frac{i}{2L} (\Delta - 2) \right], \quad (5.18)$$

$$\bar{B} : \prod_{n > \frac{\Delta-1}{2L}, p=1}^{\infty} \left[-i \left(n + \frac{p-1}{L} \right) - iq\Lambda + \frac{i}{2L} (\Delta - 2) \right]. \quad (5.19)$$

Here $\Lambda = i\alpha$. This implies the one loop partition function is given by¹⁴

$$Z^{\text{index}} = \frac{\prod_{n > \frac{\Delta-1}{2L}, p=0}^{\infty} \left[i\left(n + \frac{p}{L}\right) + iq\Lambda - \frac{i}{2L}(\Delta - 2) \right]}{\prod_{n > \frac{1-\Delta}{2L}, p=0}^{\infty} \left[i\left(n + \frac{p}{L}\right) - iq\Lambda + i\frac{\Delta}{2L} \right]} . \quad (5.20)$$

We rewrite this in terms the free energy, we will also remove the factors of i and multiply L in the numerator and denominator. This of course involves scaling the one loop determinant with an infinite L dependent constant. We obtain

$$\ln Z^{\text{index}} = \sum_{p=1, n > \frac{\Delta-1}{2L}} \ln \left(p + L\left(n + iq\alpha\right) - \frac{\Delta}{2} \right) - \sum_{p=0, n > \frac{1-\Delta}{2L}} \ln \left(p + L\left(n - iq\alpha\right) + \frac{\Delta}{2} \right) . \quad (5.21)$$

Let us compare the results for the one loop determinant for the standard action in (4.45) and for the Q -exact action in (4.70) using the Green's function obeying normalizable boundary conditions with (5.21). We see that all the three results for the one loop determinant agree when there exists no integer n in the domain D and $\frac{\Delta-1}{2L}$ is not an integer. From our discussion on boundary conditions we see that this is expected since for this situation the fields obeying normalizable boundary conditions also satisfy supersymmetric boundary conditions.

Discontinuity in the index. We have seen that the one loop determinant of the standard action at $L = 1$ has the symmetry $\Delta \rightarrow 2 - \Delta$. This implies that we can define it such that it is continuous at $\Delta = 1$. Let us examine the behaviour of the index in (5.21) for $L = 1$. When $0 \leq \Delta < 1$, $L = 1$, the result for the index is given by

$$Z_{0 \leq \Delta < 1, L=1}^{\text{index}} = \prod_{m=1}^{\infty} \left(\frac{m + q\Lambda - \frac{\Delta}{2}}{m - q\Lambda + \frac{\Delta}{2}} \right)^m . \quad (5.22)$$

Taking the limit $\Delta \rightarrow 1^-$ we obtain

$$Z_{<} = Z^{\text{index}}|_{\Delta \rightarrow 1^-, L=1} = \prod_{m=1}^{\infty} \left(\frac{m + q\Lambda - \frac{1}{2}}{m - q\Lambda + \frac{1}{2}} \right)^m . \quad (5.23)$$

Similarly, for the case $L = 1$ and $1 < \Delta < 2$, the result of the one loop determinant is given by

$$Z_{1 < \Delta < 2, L=1}^{\text{index}} = \prod_{m=1}^{\infty} \left(\frac{m + q\Lambda - \frac{\Delta-2}{2}}{m - q\Lambda + \frac{\Delta-2}{2}} \right)^m . \quad (5.24)$$

Lets now take the limit $\Delta \rightarrow 1^+$ to get

$$Z_{>} = Z_{\Delta \rightarrow 1^+, L=1}^{\text{index}} = \prod_{m=1}^{\infty} \left(\frac{m + q\Lambda + \frac{1}{2}}{m - q\Lambda - \frac{1}{2}} \right)^m . \quad (5.25)$$

¹⁴If we relax the strict normalizable boundary condition on $f_{n,p}$ (5.15) and allow $e^{r/2} f_{n,p} \sim \mathcal{O}(1)$ as $r \rightarrow \infty$, then the corresponding partition function is $Z^{\text{index}} = \frac{\prod_{n > \frac{\Delta-1}{2L}, p=0}^{\infty} \left[i\left(n + \frac{p}{L}\right) + iq\Lambda - \frac{i}{2L}(\Delta - 2) \right]}{\prod_{n \geq \lceil \frac{1-\Delta}{2L} \rceil, p=0}^{\infty} \left[i\left(n + \frac{p}{L}\right) - iq\Lambda + i\frac{\Delta}{2L} \right]}$. This agrees with the Green's function answer if there are no n 's in D .

Taking the ratio of (5.23) and (5.25) we find

$$\frac{Z_{<}}{Z_{>}} = \prod_{n=0}^{\infty} \left[\left(n + \frac{1}{2} \right)^2 - (q\Lambda)^2 \right]. \quad (5.26)$$

Thus there is a jump in the index at $\Delta = 1$. It is important to note that this feature of the supersymmetric index is not consistent with the partition function of the free theory presented in appendix B where it is continuous in Δ .

6 Conclusions

We have developed the Green's function method to obtain the one loop determinant of the chiral multiplet on $\text{AdS}_2 \times S^1$. It is coupled to a background vector multiplet which preserves supersymmetry. We implemented normalizable boundary conditions on the Green's function in AdS_2 . Our study in this example shows that when fields, which satisfy normalizable boundary conditions, do not obey supersymmetric boundary conditions the partition function does depend on the Q -exact deformation. Furthermore, the one loop determinants evaluated using the index of the D_{10} operator associated with the localizing action do not agree with that using normalizable boundary conditions when these conditions are such that they are not consistent with supersymmetric boundary conditions. This is because the action of Q is non normalizable *i.e.* it takes the space of normalizable wave functions to a space which includes non normalizable wave functions. Therefore, care should be taken when one applies the method of localization to evaluate supersymmetric observables for field theories on non-compact spaces. When the normalizable and supersymmetric boundary conditions are compatible, the result using the Green's function method localizes at the fixed point (which is the origin of AdS_2) and the boundary, and agrees with the result obtained from index calculation by evaluating Kernel and CoKernel of D_{10} operator. This feature is certainly a reflection of supersymmetry and we hope to prove this in general. As the result in our approach is localized at the fixed point and the boundary, it can be calculated more easily by analysing Green's functions locally and does not require finding global solutions of D_{10} operator. The latter in more complicated cases, such as vector multiplets and higher dimensional spaces are highly coupled differential equations, and global solutions to them are very difficult to construct.

One surprising property we noted in our study of the one loop determinant of the chiral multiplet on $\text{AdS}_2 \times S^1$ is when the radius of AdS_2 equals that of S^1 there exists a hidden symmetry¹⁵ $\Delta \rightarrow 2 - \Delta$, where Δ is the R-charge of the chiral multiplet. This symmetry is manifest in the eigen function representation of the partitions function when the chiral multiplet is decoupled from the background vector multiplet. We have seen that it persists when the coupling is turned on. One implication of this symmetry is the following: consider the case when $\Delta = \frac{1}{2}$. This is the situation when the theory is also conformal. For this situation there always exists one integer namely $n = 0$ for which the Kaluza-Klein modes which obeys normalizable boundary conditions are not compatible with supersymmetric

¹⁵We noted in the main text that there is also a symmetry $\Delta \rightarrow 2 + \Delta$ when q is non zero.

boundary conditions. However due to the hidden symmetry, the result for the one loop determinant can be obtained by examining the situation at $\Delta = \frac{3}{2}$ for which the boundary conditions are always compatible with supersymmetry. This is surprising and perhaps this feature is useful. Therefore, it is important to understand if this property persists more generally when the vector multiplet is no longer just a classical background. Such properties if true in general will be useful in the application of localization on such backgrounds.

Localization has been used to evaluate partition functions on $\text{AdS}_2 \times S^2$ with supersymmetric backgrounds corresponding to extremal black holes [9–14]. In light of our results on $\text{AdS}_2 \times S^1$, we have re-examined this question for the partition function of the hypermultiplet on $\text{AdS}_2 \times S^2$ [23]. We find that the normalizable boundary conditions for fields in extremal black hole backgrounds on $\text{AdS}_2 \times S^2$ always satisfy supersymmetric boundary conditions. We can also ask the same questions for the evaluation of partition function of the vector multiplets on both $\text{AdS}_2 \times S^1$ as well as that of $\text{AdS}_2 \times S^2$ using localization. We hope to report on these questions in the near future.

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A Supersymmetry on $\text{AdS}_2 \times S^1$

A.1 Conventions

The covariant derivative of a fermion is given by

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{i}{4} \omega_{\mu ab} \varepsilon^{abc} \gamma_c \right) \psi, \quad \varepsilon^{123} = 1. \quad (\text{A.1})$$

Our choice of gamma matrices are

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (\text{A.2})$$

They satisfy gamma matrices algebra

$$\gamma^a \gamma^b = \delta^{ab} + i \varepsilon^{abc} \gamma_c. \quad (\text{A.3})$$

$$\gamma^{aT} = -C \gamma^a C^{-1}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C^T = -C = C^{-1}. \quad (\text{A.4})$$

In Lorentzian space ψ and $\bar{\psi}$ are complex conjugate to each other but in Euclidean space fermions ψ and $\bar{\psi}$ are independent two component complex spinor. The product of two fermions ϵ and ψ is defined through charge conjugation matrix

$$\epsilon \psi = \epsilon^T C \psi. \quad (\text{A.5})$$

A.2 Killing spinors

The Killing spinor equations are given by

$$\begin{aligned}(\nabla_\mu - iA_\mu)\epsilon &= -\frac{1}{2}H\gamma_\mu\epsilon - iV_\mu\epsilon - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\epsilon, \\(\nabla_\mu + iA_\mu)\tilde{\epsilon} &= -\frac{1}{2}H\gamma_\mu\tilde{\epsilon} + iV_\mu\tilde{\epsilon} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\epsilon}.\end{aligned}\tag{A.6}$$

Here $\varepsilon^{\mu\nu\rho} = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\rho}$, $\epsilon^{\tau\eta\theta} = 1$.

The background metric of $\text{AdS}_2 \times S^1$ is given by

$$ds^2 = d\tau^2 + L^2(dr^2 + \sinh^2 r d\theta^2).\tag{A.7}$$

The vielbein are $e^1 = d\tau$, $e^2 = L dr$, $e^3 = L \sinh r d\theta$.

The non vanishing components of Christoffel symbols and spin connections are given by

$$\Gamma_{\theta\theta}^r = -\cosh r \sinh r, \quad \Gamma_{r\theta}^\theta = \coth r, \quad \omega^3{}_2 = \cosh r d\theta.\tag{A.8}$$

The solution of killing spinor equations is given as

$$\begin{aligned}\epsilon &= e^{\frac{i\theta}{2}} \begin{pmatrix} i \cosh(\frac{r}{2}) \\ \sinh(\frac{r}{2}) \end{pmatrix}, \quad \tilde{\epsilon} = e^{-\frac{i\theta}{2}} \begin{pmatrix} \sinh(\frac{r}{2}) \\ i \cosh(\frac{r}{2}) \end{pmatrix}, \\A_\tau = V_\tau &= \frac{1}{L}, \quad A_{r,\theta} = H = 0.\end{aligned}\tag{A.9}$$

A.3 Localization for vector multiplet on $\text{AdS}_2 \times S^1$

$$\Psi = \frac{i}{2}(\tilde{\epsilon}\lambda + \epsilon\tilde{\lambda}), \quad \Psi_\mu = Qa_\mu = \frac{1}{2}(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda).\tag{A.10}$$

The fermion bilinears are convenient for the index computation. The inverse of the above relations express $(\lambda, \tilde{\lambda})$ in terms of Ψ, Ψ_μ are given by

$$\lambda = \frac{1}{\tilde{\epsilon}\epsilon} [\gamma^\mu\epsilon\Psi_\mu - i\epsilon\Psi], \quad \tilde{\lambda} = \frac{1}{\epsilon\tilde{\epsilon}} [\gamma^\mu\tilde{\epsilon}\Psi_\mu - i\tilde{\epsilon}\Psi].\tag{A.11}$$

The supersymmetry transformation of the bilinears are

$$\begin{aligned}Q\Psi &= \frac{1}{4}(\tilde{\epsilon}\epsilon)G - \frac{i}{2}\lambda(\tilde{\epsilon}\gamma^{\mu\nu}\epsilon\rho)F_{\mu\nu} - \frac{1}{L}\sigma, \\Q\Psi_\mu &= \mathcal{L}_K a_\mu + D_\mu\Lambda.\end{aligned}\tag{A.12}$$

Here $\Lambda = \tilde{\epsilon}\epsilon\sigma - K^\rho a_\rho$. Next we deform the action by a Q -exact term, tQV_{loc} . According to the principle of supersymmetric localization, the partition function does not depend on the parameter t and the choice of V_{loc} . Thus one can take t to infinity. In this limit the path integral receives contribution from the field configurations which are minima of QV_{loc} . One convenient choice of V_{loc} is given by

$$V_{\text{loc}} = \int d^3x \sqrt{g} \frac{1}{(\tilde{\epsilon}\epsilon)^2} \text{Tr} \left[\Psi^\mu (Q\Psi_\mu)^\dagger + \Psi (Q\Psi)^\dagger \right].\tag{A.13}$$

The bosonic part of the QV_{loc} action is given by

$$\begin{aligned}
 & QV_{\text{loc}\{\text{bosonic}\}} \\
 &= \int d^3x \sqrt{g} \frac{1}{2(\tilde{\epsilon}\epsilon)^2} \text{Tr} \left[(Q\Psi^\mu)(Q\Psi_\mu)^\dagger + (Q\Psi)(Q\Psi)^\dagger \right] \\
 &= \int d^3x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2 \cosh^2 r} D_\mu (\cosh r \sigma) D^\mu (\cosh r \sigma) - \frac{1}{32} \left(G - \frac{4\sigma}{L \cosh r} \right)^2 \right].
 \end{aligned} \tag{A.14}$$

The minima of $QV_{\text{loc}\{\text{bosonic}\}}$ are the solutions of the following equations

$$F_{\mu\nu} = 0, \quad D_\mu (\cosh r \sigma) = 0, \quad G = \frac{4\sigma}{L \cosh r}. \tag{A.15}$$

Thus the solution of localization equation upto gauge transformations is given by

$$a_\mu = 0, \quad \sigma = \frac{i\alpha}{\cosh r}, \quad G = \frac{4i\alpha}{L \cosh^2 r}. \tag{A.16}$$

Here α is a real constant matrix valued in Lie algebra.

B Eigen function method for the free chiral multiplet

In this section we will compute the partition function of a free chiral multiplet on $\text{AdS}_2 \times S^1$ using the periodic boundary conditions along the S^1 direction for both the scalar and fermion. The free chiral multiplet is one loop exact and therefore, the one loop determinant can be evaluated exactly. In this section we compute the one loop determinant by expanding the fields in terms of harmonics on AdS_2 . We follow [20] where the eigen function method was used to obtain the free energy of a conformal scalar and massless fermions on $\text{AdS}_2 \times S^1$. In [20] the fermions obeyed thermal boundary conditions on the S^1 , here we impose periodic boundary conditions for the fermions. We can then compare the result with the explicit Green's function calculation presented in the section 4.1.

The Lagrangian for free chiral multiplet is given as

$$\mathcal{L} = \mathcal{D}^\mu \bar{\phi} \mathcal{D}_\mu \phi - \bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi - \bar{F} F - \frac{\Delta}{4} R \bar{\phi} \phi + \frac{1}{2} \left(\Delta - \frac{1}{2} \right) V^2 \bar{\phi} \phi. \tag{B.1}$$

Here Δ is the R-charge of the chiral multiplet i.e. the R-charges of fields (ϕ, ψ, F) are $(\Delta, \Delta - 1, \Delta - 2)$ and the covariant derivative

$$\mathcal{D}_\mu = \nabla_\mu - i\Delta_R \left(A_\mu - \frac{3}{2} V_\mu \right) - i(\Delta_R - r_0) V_\mu, \tag{B.2}$$

where Δ_R is the R-charge of the field and $r_0 = 1/2$ for scalar and $r_0 = -1/2$ for fermion.

After integrating out the auxiliary field and substituting $R = 2V^2$, we get

$$\mathcal{L} = \mathcal{D}^\mu \bar{\phi} \mathcal{D}_\mu \phi - \bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi - \frac{1}{4} V^2 \bar{\phi} \phi. \tag{B.3}$$

Now let us consider first the scalar terms in the Lagrangian which are

$$\mathcal{L}^s = \left(\nabla_\mu + \frac{i}{2}(\Delta - 1)V_\mu \right) \bar{\phi} \left(\nabla_\mu - \frac{i}{2}(\Delta - 1)V_\mu \right) \phi - \frac{1}{4} V^2 \bar{\phi} \phi. \quad (\text{B.4})$$

Now we define $\phi = e^{\frac{i}{2L}(\Delta-1)\tau} \eta$. Then the Lagrangian becomes

$$\mathcal{L}^s = \nabla_\mu \bar{\eta} \nabla_\mu \eta - \frac{1}{4L^2} \bar{\eta} \eta. \quad (\text{B.5})$$

Now expanding each KK mode, labelled by an integer n , in terms of scalar harmonics on AdS_2 with eigen value $(\lambda^2 + \frac{1}{4})/L^2$ with the density of states [24]

$$\mathcal{D}(\lambda) d\lambda = -\lambda \tanh(\pi\lambda) d\lambda, \quad (\text{B.6})$$

the one loop Free energy $F = -\ln Z$ is given by

$$F^s = \sum_{n \in \mathbb{Z}} \int_0^\infty d\lambda \mathcal{D}(\lambda) \ln \left(\frac{\lambda^2}{L^2} + \left(n - \frac{\Delta - 1}{2L} \right)^2 \right). \quad (\text{B.7})$$

Now using the regularized sum (see the appendix B of [20])

$$\sum_{n \in \mathbb{Z}} \ln \left(a^2 + \frac{(n + \alpha)^2}{q^2} \right) = \ln \left[2 \cosh(2\pi q|a|) - 2 \cos(2\pi\alpha) \right], \quad (\text{B.8})$$

we obtain the contribution to the free energy from the scalar field

$$F^s = \frac{\text{Vol}(H_2)}{2\pi} \int_0^\infty d\lambda \lambda \tanh(\pi\lambda) \ln \left[2 \cosh \left(\frac{2\pi\lambda}{L} \right) - 2 \cos \left(\frac{\pi}{L} (1 - \Delta) \right) \right]. \quad (\text{B.9})$$

Now we look at the fermion. We have the following Lagrangian

$$\mathcal{L}^f = -\bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi = -\bar{\psi} \gamma^\mu \left(\nabla_\mu - \frac{i\Delta}{2} V_\mu \right) \psi. \quad (\text{B.10})$$

Now we define $\psi = e^{\frac{i\Delta}{2L}\tau} \theta$, then we get the fermionic Lagrangian

$$\mathcal{L}^f = -\bar{\theta} \gamma^\mu \nabla_\mu \theta. \quad (\text{B.11})$$

As in the scalar case we expand each KK mode in terms of harmonics of a Dirac operator on AdS_2 labelled by eigen value $\pm i \frac{\lambda}{L}$ with density of states

$$\tilde{\mathcal{D}}(\lambda) d\lambda = -2\lambda \coth(\pi\lambda) d\lambda. \quad (\text{B.12})$$

Then the free energy of the periodic fermionic field is given by

$$\begin{aligned} F^f &= -\frac{1}{2} \sum_{n \in \mathbb{Z}} \int_0^\infty d\lambda \tilde{\mathcal{D}}(\lambda) \ln \left[\frac{\lambda^2}{L^2} + \left(n - \frac{\Delta}{2L} \right)^2 \right] \\ &= -\frac{1}{2} \int_0^\infty d\lambda \tilde{\mathcal{D}}(\lambda) \ln \left[2 \cosh \left(2\pi \frac{\lambda}{L} \right) - 2 \cos \left(\frac{\pi\Delta}{L} \right) \right]. \end{aligned} \quad (\text{B.13})$$

Thus the complete free energy is

$$F = F^s + F^f = - \int d\lambda \lambda \tanh(\pi\lambda) \ln \left[2 \cosh \left(2\pi \frac{\lambda}{L} \right) - 2 \cos \left(2\pi \frac{\Delta - 1}{2L} \right) \right] \\ + \int d\lambda \lambda \coth(\pi\lambda) \ln \left[2 \cosh \left(2\pi \frac{\lambda}{L} \right) - 2 \cos \left(2\pi \frac{\Delta}{2L} \right) \right]. \quad (\text{B.14})$$

Thus to get the free energy and the partition function we need to perform the above one dimensional integral. For example in the case of $L = 1$ with $\Delta = \frac{1}{2}$,¹⁶ we get

$$F_{\Delta=\frac{1}{2}} = \frac{1}{4\pi} \left(\frac{\pi}{2} \ln 2 + 2 \text{ Catalan} \right), \quad \text{and} \quad \ln Z_{\Delta=\frac{1}{2}} = -\frac{1}{4\pi} \left(\frac{\pi}{2} \ln 2 + 2 \text{ Catalan} \right), \quad (\text{B.15})$$

and with $\Delta = 0$, we get

$$F_{\Delta=0} = 0, \quad \text{and} \quad Z_{\Delta=0} = 1. \quad (\text{B.16})$$

Also note that for $\Delta = 1$ the scalar η has periodic boundary condition and fermion θ has anti periodic boundary condition along S^1 . In this case we get

$$F_{\Delta=1} = \frac{1}{2} \ln 2. \quad (\text{B.17})$$

Below we will present the result for the free energy for general Δ and, $L = 1$ and $L = 2$.

Case 1: result for $L = 1$ and $0 \leq \Delta < 2$. We will now present the result for general Δ lying in the range $[0, 2)$. We begin with the expression

$$F = - \int_0^\infty d\lambda \lambda \left[\tanh(\pi\lambda) \ln [2 \cosh(2\pi\lambda) + 2 \cos(\pi\Delta)] \right. \\ \left. - \coth(\pi\lambda) \ln [2 \cosh(2\pi\lambda) - 2 \cos(\pi\Delta)] \right]. \quad (\text{B.18})$$

We notice that the integrand has the symmetry $\Delta \rightarrow 2 - \Delta$. Now, we calculate its first derivative with respect to Δ

$$\frac{dF}{d\Delta} = \pi \sin(\pi\Delta) \int_0^\infty d\lambda \lambda \left[\tanh(\pi\lambda) \frac{1}{\cosh(2\pi\lambda) + \cos(\pi\Delta)} \right. \\ \left. + \coth(\pi\lambda) \frac{1}{\cosh(2\pi\lambda) - \cos(\pi\Delta)} \right]. \quad (\text{B.19})$$

Now the integrals on the r.h.s. is calculable which are given as

$$\int_0^\infty d\lambda \lambda \tanh(\pi\lambda) \frac{1}{\cosh(2\pi\lambda) + \cos(\pi\Delta)} \\ = \frac{1}{48\pi^2 \sin^2(\frac{\pi\Delta}{2})} \left[\pi^2 + 6Li_2(-e^{i\pi\Delta}) + 6Li_2(-e^{-i\pi\Delta}) \right], \quad (\text{B.20})$$

and

$$\int_0^\infty d\lambda \lambda \coth(\pi\lambda) \frac{1}{\cosh(2\pi\lambda) - \cos(\pi\Delta)} \\ = -\frac{1}{24\pi^2 \sin^2(\frac{\pi\Delta}{2})} \left[-\pi^2 + 3Li_2(e^{i\pi\Delta}) + 3Li_2(e^{-i\pi\Delta}) \right]. \quad (\text{B.21})$$

¹⁶In this case, the scalar theory is conformal on AdS_2 .

Thus we get the following first order differential equation

$$\frac{dF}{d\Delta} = -\frac{\sin(\pi\Delta)}{16\pi \sin^2(\frac{\pi\Delta}{2})} \left[-\pi^2 + 2Li_2(e^{i\pi\Delta}) + 2Li_2(e^{-i\pi\Delta}) - 2Li_2(-e^{i\pi\Delta}) - 2Li_2(-e^{-i\pi\Delta}) \right]. \quad (B.22)$$

Case: $0 \leq \Delta < 1$. Now we use the following identity to simplify the expression of the dilogarithm

$$Li_2(e^{2i\pi x}) + Li_2(e^{-2i\pi x}) = 2\pi^2 \left(x^2 - x + \frac{1}{6} \right) \quad \text{for } 0 \leq \text{Re } x < 1. \quad (B.23)$$

In this case we get

$$\begin{aligned} \frac{dF}{d\Delta} &= \frac{\sin(\pi\Delta)}{8\pi(\cos(\pi\Delta)-1)} \left[-\pi^2 + 2Li_2(e^{i\pi\Delta}) + 2Li_2(e^{-i\pi\Delta}) - 2Li_2(e^{i\pi(\Delta+1)}) - 2Li_2(e^{-i\pi(\Delta+1)}) \right], \\ &= \frac{\sin(\pi\Delta)}{8\pi(\cos(\pi\Delta)-1)} \left[-\pi^2 + 4\pi^2 \left(\frac{\Delta^2}{4} - \frac{\Delta}{2} + \frac{1}{6} \right) - 4\pi^2 \left(\frac{(\Delta+1)^2}{4} - \frac{\Delta+1}{2} + \frac{1}{6} \right) \right], \\ &= \frac{\pi\Delta}{4} \cot\left(\frac{\pi\Delta}{2}\right). \end{aligned} \quad (B.24)$$

Now integrating over Δ we get

$$F = \frac{\Delta}{2} \ln(1 - e^{i\pi\Delta}) - \frac{i}{2\pi} \left(\frac{\pi^2 \Delta^2}{4} + Li_2(e^{i\pi\Delta}) \right) + C_1, \quad (B.25)$$

where C_1 is some integration constant which we determine by requiring that $F|_{\Delta=0} = 0$.

$$-\frac{i}{2\pi} Li_2(1) + C_1 = 0 \Rightarrow C_1 = \frac{i}{2\pi} \zeta(2). \quad (B.26)$$

Thus the free energy in this case is

$$F = \frac{\Delta}{2} \ln(1 - e^{i\pi\Delta}) - \frac{i}{2\pi} \left(\frac{\pi^2 \Delta^2}{4} + Li_2(e^{i\pi\Delta}) - \zeta(2) \right). \quad (B.27)$$

Case: $1 \leq \Delta < 2$. In this case we get

$$\begin{aligned} \frac{dF}{d\Delta} &= \frac{\sin(\pi\Delta)}{8\pi(\cos(\pi\Delta)-1)} \left[-\pi^2 + 2Li_2(e^{i\pi\Delta}) + 2Li_2(e^{-i\pi\Delta}) - 2Li_2(e^{i\pi(\Delta-1)}) - 2Li_2(e^{-i\pi(\Delta-1)}) \right], \\ &= \frac{\sin(\pi\Delta)}{8\pi(\cos(\pi\Delta)-1)} \left[-\pi^2 + 4\pi^2 \left(\frac{\Delta^2}{4} - \frac{\Delta}{2} + \frac{1}{6} \right) - 4\pi^2 \left(\frac{(\Delta-1)^2}{4} - \frac{\Delta-1}{2} + \frac{1}{6} \right) \right], \\ &= \frac{\pi(2-\Delta)}{4} \cot\left(\frac{\pi\Delta}{2}\right). \end{aligned} \quad (B.28)$$

Integrating with respect to Δ we get

$$F = \frac{2-\Delta}{2} \ln(1 - e^{i\pi(2-\Delta)}) - \frac{i}{2\pi} \left(\frac{\pi^2 (2-\Delta)^2}{4} + Li_2(e^{i\pi(2-\Delta)}) - \zeta(2) \right). \quad (B.29)$$

Case 2: result for $L = 2$ and $0 \leq \Delta < 2$. In this case we get

$$\frac{dF^s}{d\Delta} = -\frac{\pi}{2} \sin\left(\frac{\pi}{2}(\Delta - 1)\right) \int_0^\infty d\lambda \lambda \tanh(\pi\lambda) \frac{1}{\cosh \pi\lambda - \cos\left(\frac{\pi}{2}(\Delta - 1)\right)}. \quad (\text{B.30})$$

Integrating r.h.s. we get

$$\frac{dF^s}{d\Delta} = \frac{1}{48\pi} \cot\left(\frac{\pi\Delta}{2}\right) \left[\pi^2 + 24\text{Li}_2(-ie^{\frac{i\pi\Delta}{2}}) + 24\text{Li}_2(ie^{-\frac{i\pi\Delta}{2}}) \right]. \quad (\text{B.31})$$

Now we evaluate the above expression for different ranges of R-charge Δ . For the case when $0 \leq \Delta < 1$, we get

$$\frac{dF^s}{d\Delta} = \frac{\pi}{16} \left[3 - (1 - \Delta)(\Delta + 3) \right] \cot \frac{\pi\Delta}{2}. \quad (\text{B.32})$$

For the case when $1 \leq \Delta < 2$, we get

$$\frac{dF^s}{d\Delta} = \frac{\pi}{16} \left[3 - (\Delta - 1)(5 - \Delta) \right] \cot \frac{\pi\Delta}{2}. \quad (\text{B.33})$$

In the case of fermion we get for $0 \leq \Delta < 2$

$$\frac{dF^f}{d\Delta} = -\frac{\pi}{8 \sin\left(\frac{\pi\Delta}{2}\right)} \left[-1 + \cos\left(\frac{\pi\Delta}{2}\right) + 2 \cos\left(\frac{\pi\Delta}{2}\right) \left(\frac{\Delta^2}{4} - \Delta \right) \right]. \quad (\text{B.34})$$

Adding the above two for the case when $0 \leq \Delta < 1$, total derivative of free energy is

$$\frac{dF}{d\Delta} = \frac{dF^f}{d\Delta} + \frac{dF^s}{d\Delta} = \frac{\pi}{8 \sin\left(\frac{\pi\Delta}{2}\right)} \left[1 + (3\Delta - 1) \cos\left(\frac{\pi\Delta}{2}\right) \right]. \quad (\text{B.35})$$

Integrating the above we get

$$F = \frac{1}{16} \left(-3\pi i \Delta^2 + 12\Delta \ln(1 - e^{i\pi\Delta}) - 4\ln 2 - 8\ln\left(\cos \frac{\pi\Delta}{4}\right) \right) - \frac{3i\pi}{4} \text{Li}_2(e^{i\pi\Delta}) + C_1, \quad (\text{B.36})$$

where C_1 is an integration constant.

Similarly, for the case when $1 \leq \Delta < 2$ we have

$$\frac{dF}{d\Delta} = \frac{dF^f}{d\Delta} + \frac{dF^s}{d\Delta} = \frac{\pi}{8 \sin\left(\frac{\pi\Delta}{2}\right)} \left[1 + (3 - \Delta) \cos\left(\frac{\pi\Delta}{2}\right) \right]. \quad (\text{B.37})$$

Integrating the r.h.s., we obtain

$$F = -\frac{\Delta}{4} \ln(1 - e^{i\pi\Delta}) + \ln\left(\sin \frac{\pi\Delta}{4}\right) + \frac{1}{2} \ln\left(\cos \frac{\pi\Delta}{4}\right) + \frac{3}{4} \ln 2 + \frac{i}{4\pi} \left(\frac{\pi^2 \Delta^2}{4} + \text{Li}_2(e^{i\pi\Delta}) \right) + C_2. \quad (\text{B.38})$$

Here C_2 is an integration constant.

B.1 Product representation

In this section we express the free energy obtained above as an infinite product. This will be useful for comparison with the answers obtained by Green's function as well as index method. We follow the strategy that the free energy satisfies a first order differential equation, such as (B.24) and (B.37), and identify the differential equation with a differential equation satisfies by certain infinite product. We find that these infinite products are combinations of certain double sine functions [25].

$L = 1$.

Case: $0 \leq \Delta < 1$. Let us consider the following function

$$\tilde{S}_2(z) = \frac{\prod_{n,p=0}^{\infty} (n+p+1-z)}{\prod_{n,p=0}^{\infty} (n+p+1+z)}. \quad (\text{B.39})$$

We see that

$$\frac{\tilde{S}_2'(z)}{\tilde{S}_2(z)} = -\lim_{s \rightarrow 1} \left[\tilde{\zeta}(s, 1-z) + \tilde{\zeta}(s, 1+z) \right], \quad (\text{B.40})$$

where

$$\tilde{\zeta}(s, a) = (1-a)\zeta(s, a) + \zeta(s-1, a), \quad (\text{B.41})$$

and $\zeta(s, a)$ is the Hurwitz zeta function. Taking into account the following relations

$$\zeta(s, a) = \frac{1}{s-1} - \psi(a) + \mathcal{O}(s-1), \quad \zeta(0, a) = \frac{1}{2} - a. \quad (\text{B.42})$$

Here $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$ which satisfies following relations

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad \psi(1-x) - \psi(x) = \pi \cot \pi x. \quad (\text{B.43})$$

We get

$$\frac{\tilde{S}_2'(z)}{\tilde{S}_2(z)} = \pi z \cot \pi z. \quad (\text{B.44})$$

Thus comparing with (B.24), we see that $F(\Delta) = \ln \tilde{S}_2(\frac{\Delta}{2}) - \ln A'$, where A' is independent of Δ . Thus the partition function is

$$Z = e^{-F} = A' \frac{\prod_{n,p=0}^{\infty} (n+p+1+\frac{\Delta}{2})}{\prod_{n,p=0}^{\infty} (n+p+1-\frac{\Delta}{2})} = A' \frac{\prod_{r=1}^{\infty} (r+\frac{\Delta}{2})^r}{\prod_{r=1}^{\infty} (r-\frac{\Delta}{2})^r}. \quad (\text{B.45})$$

Case: $1 \leq \Delta < 2$. Let us consider the following function

$$S_2(z) = \frac{\prod_{n,p=0}^{\infty} (n+p+z)}{\prod_{n,p=0}^{\infty} (n+p+2-z)}. \quad (\text{B.46})$$

We see that

$$\frac{S_2'(z)}{S_2(z)} = \lim_{s \rightarrow 1} \left[\tilde{\zeta}(s, z) + \tilde{\zeta}(s, 2-z) \right]. \quad (\text{B.47})$$

Again using the relations (B.41), (B.42) and (B.43), we get

$$\frac{S'_2(z)}{S_2(z)} = (1-z)[\psi(2-z) - \psi(z)] + \frac{1}{2} - z + \frac{1}{2} - (2-z) = (1-z)\pi \cot \pi z. \quad (\text{B.48})$$

Thus comparing with (B.28) the natural answer for the free energy for this range of the R-charge is

$$F(\Delta) = \ln S_2\left(\frac{\Delta}{2}\right) - \ln \tilde{A}, \quad (\text{B.49})$$

and the partition function is

$$Z = e^{-F} = \frac{\tilde{A}}{S_2(\frac{\Delta}{2})} = \tilde{A} \frac{\prod_{n,p=0}^{\infty} (n+p+2-\frac{\Delta}{2})}{\prod_{n,p=0}^{\infty} (n+p+\frac{\Delta}{2})} = \tilde{A} \frac{\prod_{r=1}^{\infty} (r+1-\frac{\Delta}{2})^r}{\prod_{r=1}^{\infty} (r-1+\frac{\Delta}{2})^r}, \quad (\text{B.50})$$

where $\tilde{A}$ is some constant.

L = 2. Let us start with the range $1 \leq \Delta < 2$.

Case: $1 \leq \Delta < 2$. We begin with the expression

$$S_2(x, (1, 2)) = \frac{\Gamma(3-x, (1, 2))}{\Gamma(x, (1, 2))}, \quad (\text{B.51})$$

where

$$\Gamma(x, (w_1, w_2)) = \frac{1}{\prod_{n_1, n_2=0}^{\infty} (n_1 w_1 + n_2 w_2 + x)}. \quad (\text{B.52})$$

Thus in the form of infinite product we have

$$S_2(x, (1, 2)) = \frac{\prod_{n_1, n_2=0}^{\infty} (2n_1 + n_2 + x)}{\prod_{n_1, n_2=0}^{\infty} (2n_1 + n_2 + 3 - x)}. \quad (\text{B.53})$$

We also see that the function $S_2(x, (1, 2))$ can be written as

$$S_2(x, (1, 2)) = S_2\left(\frac{x}{2}\right) S_2\left(\frac{x+1}{2}\right). \quad (\text{B.54})$$

Differentiating $S_2(x, (1, 2))$ with respect to x , we get

$$\begin{aligned} \frac{S'_2(x, (1, 2))}{S_2(x, (1, 2))} &= \frac{1}{2} \frac{S'_2(\frac{x}{2})}{S_2(\frac{x}{2})} + \frac{1}{2} \frac{S'_2(\frac{x+1}{2})}{S_2(\frac{x+1}{2})}, \\ &= \frac{1}{2} \left(1 - \frac{x}{2}\right) \pi \cot\left(\frac{\pi x}{2}\right) + \frac{1}{2} \left(1 - \frac{x+1}{2}\right) \pi \cot\left(\pi \frac{x+1}{2}\right), \\ &= \frac{\pi}{4} \left[(2-x) \cot\left(\frac{\pi x}{2}\right) - (1-x) \tan\left(\frac{\pi x}{2}\right) \right] \\ &= \frac{\pi}{4 \sin \pi x} \left[(3-2x) \cos \pi x + 1 \right]. \end{aligned} \quad (\text{B.55})$$

Comparing with (B.37) we see that in this case we have

$$\frac{dF}{dx} = \frac{S'_2(x, (1, 2))}{S_2(x, (1, 2))}, \quad \text{for } x = \frac{\Delta}{2}, \quad (\text{B.56})$$

which means $F = \ln S_2(x, (1, 2)) + \ln A$, therefore

$$Z = \frac{A}{S_2(x, (1, 2))} \Big|_{x=\frac{\Delta}{2}}, \quad (\text{B.57})$$

where A is some constant. Therefore, using (B.53), upto an Δ independent constant we obtain

$$\ln Z = \sum_{n_1, n_2=0}^{\infty} \left(\ln \left(2n_1 + n_2 + 3 - \frac{\Delta}{2} \right) - \ln \left(2n_1 + n_2 + \frac{\Delta}{2} \right) \right). \quad (\text{B.58})$$

Case: $0 \leq \Delta < 1$. We notice that the differential equation (B.35) can be written as

$$\frac{dF}{dx} = \frac{\pi}{4 \sin \pi x} \left[1 + (3 - 2x) \cot \pi x \right] + \pi(2x - 1) \cot \pi x, \quad (\text{B.59})$$

where again $x = \frac{\Delta}{2}$. We see that the first part of the r.h.s. in the above equation is same as the r.h.s. of (B.55). The last term can be written as the linear combination of (B.44) and (B.48).

$$\frac{dF}{dx} = \frac{S'_2(x, (1, 2))}{S_2(x, (1, 2))} + \frac{\tilde{S}'_2(x)}{\tilde{S}_2(x)} - \frac{S'_2(x)}{S_2(x)}. \quad (\text{B.60})$$

Thus the partition function is given as

$$Z = \frac{\hat{A} S_2(x)}{S_2(x, (1, 2)) \tilde{S}_2(x)} \Big|_{x=\frac{\Delta}{2}}. \quad (\text{B.61})$$

Here $\hat{A}$ is constant. Using the definitions of the functions S and $\tilde{S}$, the free energy up to a constant independent of Δ is given by

$$\begin{aligned} \ln Z = & \sum_{n_1=1, n_2=0}^{\infty} 2 \ln \left(n_1 + n_2 + \frac{\Delta}{2} \right) + \sum_{n_1=0}^{\infty} \ln \left(n_1 + \frac{\Delta}{2} \right) \\ & - \sum_{n_1, n_2=1}^{\infty} 2 \ln \left(n_1 + n_2 - \frac{\Delta}{2} \right) - \sum_{n_1=1}^{\infty} \ln \left(n_1 - \frac{\Delta}{2} \right) \\ & + \sum_{n_1, n_2=0}^{\infty} \left(\ln \left(2n_1 + n_2 + 3 - \frac{\Delta}{2} \right) - \ln \left(2n_1 + n_2 + \frac{\Delta}{2} \right) \right). \end{aligned} \quad (\text{B.62})$$

C Integrals involving product of hypergeometric functions

The integrals necessary for evaluating the bulk contribution to the one loop determinant of the standard action when there exists an integer in D involve products of hypergeometric functions. They are given by

$$\begin{aligned} \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1+}(z)S_{2-}(z)}{4c_{1+-}\sqrt{1-z}} &= \frac{iLq}{2} \left[\psi \left(\frac{1}{2} + \frac{1}{4}x \right) - \psi \left(\frac{1}{4}x^* \right) \right], \\ \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1+}(z)S_{2+}(z)}{4c_{1++}\sqrt{1-z}} &= -\frac{iLq}{2} \left[\psi \left(\frac{1}{2} + \frac{1}{4}y \right) - \psi \left(1 + \frac{1}{4}y^* \right) \right], \\ \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1-}(z)S_{2-}(z)}{4c_{1--}\sqrt{1-z}} &= \frac{iLq}{2} \left[\psi \left(\frac{1}{2} + \frac{1}{4}\tilde{x} \right) - \psi \left(\frac{1}{4}\tilde{x}^* \right) \right], \\ \int_0^1 dz \frac{Lq(-i + 2Lq\alpha)S_{1-}(z)S_{2+}(z)}{4c_{1-+}\sqrt{1-z}} &= -\frac{iLq}{2} \left[\psi \left(\frac{1}{2} + \frac{1}{4}\tilde{y} \right) - \psi \left(1 + \frac{1}{4}\tilde{y}^* \right) \right]. \end{aligned} \quad (\text{C.1})$$

where $\psi(z)$ is a digamma function defined by

$$\psi(z) = \frac{d}{dz} \ln(\Gamma(z)), \quad (\text{C.2})$$

and x and y are given as

$$x = 2p + \Delta - 2Ln + 2iLq\alpha, \quad y = 2p - \Delta + 2Ln - 2iLq\alpha, \quad (\text{C.3})$$

and $\tilde{x}$ and $\tilde{y}$ are obtained from x and y by replacing $p \rightarrow -p$, respectively. Finally, the $*$ on $x, y, \tilde{x}$ and $\tilde{y}$ denotes the complex conjugation.

The integrals necessary for evaluating the bulk contribution to the one loop determinant of Q -exact action when there exists an integer in D involve products of hypergeometric functions. They are given by

$$\begin{aligned} \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1+}(z) \tilde{S}_{2-}(z)}{2\tilde{c}_{1+-}\sqrt{1-z}} &= \frac{L^2 q^2 \alpha (\psi(\tilde{a}_2) - \psi(\tilde{b}_2))}{2(\tilde{b}_2 - \tilde{a}_2)}, \\ \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1+}(z) \tilde{S}_{2+}(z)}{2\tilde{c}_{1++}\sqrt{1-z}} &= \frac{L^2 q^2 \alpha (\psi(a_2) - \psi(b_2))}{2(b_2 - a_2)}, \\ \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1-}(z) \tilde{S}_{2-}(z)}{2\tilde{c}_{1--}\sqrt{1-z}} &= \frac{L^2 q^2 \alpha (\psi(\tilde{a}_2 - p) - \psi(\tilde{b}_2 - p))}{2(\tilde{b}_2 - \tilde{a}_2)}, \\ \int_0^1 dz \frac{L^2 q^2 \alpha \tilde{S}_{1-}(z) \tilde{S}_{2+}(z)}{2\tilde{c}_{1-+}\sqrt{1-z}} &= \frac{L^2 q^2 \alpha (\psi(a_2 - p) - \psi(b_2 - p))}{2(b_2 - a_2)}, \end{aligned} \quad (\text{C.4})$$

where the arguments of the digamma function are given in (4.59).

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