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The correlahedron

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Abstract: We introduce a new geometric object, the correlahedron, which we conjecture to be equivalent to stress-energy correlators in planar $\mathcal{N} = 4$ super Yang-Mills. Re-expressing the Grassmann dependence of correlation functions of n chiral stress-energy multiplets with Grassmann degree $4k$ in terms of $4(n + k)$-linear bosonic variables, the resulting expressions have an interpretation as volume forms on a $\mathrm{Gr}(n{+}k, 4{+}n{+}k)$ Grassmannian, analogous to the expressions for planar amplitudes via the amplituhedron. The resulting volume forms are to be naturally associated with the correlahedron geometry. We construct such expressions in this bosonised space both directly, in general, from Feynman diagrams in twistor space, and then more invariantly from specific known correlator expressions in analytic superspace. We give a geometric interpretation of the action of the consecutive lightlike limit and show that under this the correlahedron reduces to the squared amplituhedron both as a geometric object as well as directly on the corresponding volume forms. We give an explicit easily implementable algorithm via cylindrical decompositions for extracting the squared amplituhedron volume form from the squared amplituhedron geometry with explicit examples and discuss the analogous procedure for the correlators.

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1 Introduction

Both scattering amplitudes and stress-tensor correlators in $\mathcal{N} = 4$ SYM have been the subject of intense research for a number of years, revealing wonderful discoveries of mathematical structures. We will be focusing on the integrand in this paper following much recent work (see for example [1–8] and references therein). One of the most exciting discoveries is that the perturbative integrands of n-point, ℓ-loop scattering amplitudes in planar $\mathcal{N} = 4$ SYM are equivalent to generalised polyhedra in Grassmannians, with faces and vertices determined by the momenta and helicities of the particles being scattered [9]. This geometrical object was named the *amplituhedron* (see also further developments in [10–17]). On the other hand (the square of) all ℓ-loop amplitudes are limits of tree-level correlation functions of the stress-energy multiplet (correlators) [6, 7, 18–21] suggesting the possibility of a larger geometrical object describing correlators and reducing to the amplituhedron in relevant limits. The purpose of this paper is to give a proposal for this *correlahedron*.

The starting point for the amplituhedron was the introduction of momentum supertwistors followed by a "bosonisation" of their fermionic parts. Hodges showed that they lead to a geometric formulation of the determinants arising from the fermionic coordinates as volumes of a polyhedron in a projective space for the NMHV amplitude [22]. To generalize[1] to higher MHV degree introduce a particle-independent fermionic variable ϕ_I^p, $p = 1, \ldots, k$ where $I = 1, \ldots, 4$ is an R-symmetry index, and send the odd variables χ_i^I to even variables $\xi_i^p = \chi_i^I \phi_I^p$ [9]. Here the range of the index p depends on the helicity structure (or Grassmann degree) of the superamplitude; for N^{k}MHV amplitudes $p = 1, \ldots, k$ and thus momentum supertwistor space (Z, χ) becomes the vector space $\mathbb{C}^{4+k}$ with bosonic variables (Z, ξ). This framework has considerable practical advantages — for example nilpotent superconformal invariants are straightforward to find and non-trivial superconformal identities become manifest generalized Schouten identities. Furthermore the resulting expression can be seen to arise from volume forms on the Grassmannian of k-planes in $4+k$ dimensions, $\mathrm{Gr}(k, 4+k)$. The construction essentially reduces superconformal invariants to projective invariants.

We perform an analogous bosonisation of the stress-tensor correlators. It is not immediately clear how to do this bosonisation starting from supercorrelators in analytic superspace directly. However, recently such correlators were considered via Feynman diagrams in supertwistor space [23] and this formulation leads to a "potential" for the correlation functions. This potential is a correlator of certain 'log det d-bar' operators based on lines in twistor space. These operators are not manifestly gauge invariant, but only become so

[1] In fact this generalization is really that of the dual of the original Hodges framework [11].

when differentiated by a fourth order Grassmann odd differential operator at each point mapping the 'log det d-bar' operators to the gauge invariant super-BPS operators $\mathcal{O}_i$. In the diagram formulation based on an axial gauge, the gauge dependence will manifest itself in dependence on the reference twistor Z_*. We will nevertheless suppress this differentiation in the following and indeed provide ample evidence for the conjecture that there is a Z_* independent 'potential' of the sum of diagrams given by the correlahedron. Indeed there is a simple prescription for lifting this Z_*-independent potential directly from analytic superspace, even though it is not obtained by direct bosonisation of the analytic superspace correlator. We thus rewrite all known stress-tensor supercorrelators in an appropriately bosonised form. These expressions are all equivalent to volume forms on the Grassmannian space $\mathrm{Gr}(k{+}n, 4{+}k{+}n)$.

The key aspect of the amplituhedron however is geometric; it is a generalised polyhedron lying in the *real* Grassmannian $\mathrm{Gr}(k, 4{+}k)$. A natural volume form on this polyhedron, one with log divergences on the boundary and no divergences inside, gives the afore mentioned bosonised amplitudes. We generalise this geometric aspect to the correlahedron, now lying in $\mathrm{Gr}(k{+}n, 4{+}k{+}n)$. More precisely it is the "squared amplituhedron", a larger object than the amplituhedron itself which generalises to the correlahedron. This "squared amplituhedron" corresponds to the square of the superamplitude. A key advantage of the squared amplituhedron is that it has a more explicit definition than the amplituhedron itself being simply defined by explicit inequalities, whereas the amplituhedron requires a further topological degree requirement [24].

The lightlike limit, by which the correlators become the square of superamplitudes, has a natural geometrical interpretation for the correlahedron. Under a partial freezing to a boundary of the correlahedron space, together with a projection, the correlahedron geometry becomes the squared amplituhedron geometry. This same procedure projects the corresponding correlator volume form to the squared amplitude volume form.

For the amplituhedron, the link between the integrands and the geometry arises from the requirement that the volume form should have no divergences inside the amplituhedron and log divergences on its boundary. This volume form *is* essentially the bosonised amplitude. Obtaining this form from the geometry is non-trivial for the amplituhedron, but becomes much simpler for the "squared amplituhedron" due to its more explicit definition. A key point is that the requirement that the volume form have simple poles on the boundary is not sufficient to determine it, but the combinatorics of the positive geometry of the polyhedral description does. This is manifested in a by-product of this work in which we introduce a completely algorithmic and easily computerisable way of obtaining this volume form from the geometry of the squared amplituhedron. The algorithm uses cylindrical decomposition, an active area of research in its own right and with a number of physical applications, which unfortunately however can be doubly exponential in the number of variables. This method quite quickly becomes impractical for large particle number or loop order. Nevertheless in a number of non-trivial examples, we show that the squared amplitude geometry gives the square of the superamplitude. We then explore the corresponding relation between the correlahedron and the bosonised correlators.

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The plan of the paper is thus as follows. In section 2 we introduce our conventions, details of the bosonisation procedure, and the definitions of the Grassmannians in which the various 'hedra lie. In section 3 we then define the various hedra — amplituhedron, squared amplituhedron and correlahedron — as geometrical polytopes in the corresponding Grassmannians. In section 4 we discuss how to write known explicit expressions for correlators as volume forms on the appropriate Grassmannian. In section 5 we consider the lightlike limit of correlators in correlahedron space. We show that the same geometric procedure reduces the geometry of the correlahedron to that of the amplituhedron as well as reducing the corresponding volume form expressions to those of the amplituhedron volume form expressions. Finally in section 6 we consider the connection between the hedron geometry and the hedron volume forms. We develop a simple algorithm using cylindrical decomposition for obtaining the volume form from the geometry and apply it to a number of squared amplituhedrons and a correlahedron example. In an appendix we look at the most non-trivial examples of taking the lightlike limit of the correlahedron.

2 Bosonisation, conventions and -hedron forms

A key aspect of both the amplituhedron and correlahedron is the bosonised superspace. As such in this section we review this procedure for amplitudes and give our proposal for the appropriate bosonised space for correlators. This will also set out our notation and conventions for the rest of the paper.

2.1 Bosonisation

Planar superamplitudes in $\mathcal{N}$=4 SYM can be nicely presented in momentum supertwistor space $\mathbb{C}^{4|4}$ [22]. Bosonisation of superspace for $\mathrm{N}^{k'}$MHV superamplitudes maps momentum supertwistors, which lie in (4|4) dimensions, to a purely bosonic vector of dimension $4+k'$:

$$\mathbb{C}^{4|4} \ni \quad (z|\chi) \;\to\; Z = (z,\zeta) = (z,\chi\phi) \quad \in \mathbb{C}^{4+k'}\,. \tag{2.1}$$

Here z is a bosonic four-dimensional row vector (a twistor), χ is a fermionic 4-vector (the Grassmann odd component of the supertwistor) and ϕ is a Grassmann odd $4 \times k'$ matrix. Thus Z is indeed a Grassmann even (bosonic) $4 + k'$-dimensional row vector.

The amplituhedron space itself is a subset of $\mathrm{Gr}(k', k' + 4)$, the space of k'-planes, Y, in $4 + k'$ dimensions.

The (chiral) correlator on the other hand will be written in terms of a potential on chiral superspace in section 2.2. Chiral super-Minkowski space can be equivalently thought of as the space of 2-planes in supertwistor space. Such 2-planes on supertwistor space are specified by taking two independent supertwistors on the plane. We will write them as a $2 \times (4|4)$ supermatrix $(x|\theta)$ where the two rows of the matrix are the two supertwistors in question, and there is a local GL(2) acting on the left, corresponding to the independence of the plane on the choice of the two supertwistors.[2]

[2]This GL(2) symmetry can be used to set the $2\times$ 4 matrix x to the form $(1_2, \hat{x})$ where $\hat{x}$ is a 2×2 matrix, the standard spinor representation of 4d Minkowski space.

We perform a very similar bosonisation of these co-ordinates as perfomed above for the amplitudes, with the main difference now being that the bosonised supertwistor space lives in $4+n+k$ dimensions rather than $k'+4$ dimensions. So explicitly we map the $2\times(4|4)$ supermatrix to a $2\times(4{+}n{+}k)$ matrix

$$\mathbb{C}^{2\times(4|4)} \ni \quad (x|\theta) \;\to\; X=(x,\xi)=(x,\theta\phi) \quad \in \mathbb{C}^{2\times(4+n+k)} \tag{2.2}$$

where x is the 2×4 matrix representing Minkowski space, θ the 2×4 fermionic matrix (the fermionic part of super Minkowski space) and ϕ is a supplementary Grassmann odd $4\times(4{+}n{+}k)$ matrix, which will be independent of the space-time point. Thus X is a Grassmann even (bosonic) $2\times(4{+}n{+}k)$-dimensional matrix. Furthermore this matrix X has a local $Gl(2)$ acting on the left, inherited from that of the supermatrix, and thus has the natural interpretation of a two-plane in $4{+}n{+}k$ dimensions. We call this bosonised super-Minkowski space $\mathbb{M}_b := \mathrm{Gr}(2,k{+}n{+}4)$. The correlahedron space itself is a subset of $\mathrm{Gr}(k{+}n,k{+}n{+}4)$, the space of $k{+}n$-planes, Y, in $k{+}n{+}4$ dimensions.

We will use the following indices on the bosonised twistor space, space-time and -hedron space

$$\begin{aligned}
&Z_{i'}{}^{\mathcal{A}'}=(z_{i'}{}^{A},\zeta_{i'}{}^{p'})\,, && X_{i\alpha}^{\mathcal{A}}=(x_{i\alpha}{}^{A},\xi_{i\alpha}{}^{p})\,,\\
&\quad i'=1\dots n',\;\; p'=1\dots k',\quad \mathcal{A}'=(A,p'), && \quad i=1\dots n,\;\; p=1\dots n{+}k,\quad \mathcal{A}=(A,p),\\
&Y_{p'}^{\mathcal{A}'}\in Gr(k',4+k') && Y_{p}^{\mathcal{A}}\in Gr(k+n,4+k+n)\,,\\
&\quad A=1\dots 4,\qquad \alpha=1,2\,.
\end{aligned} \tag{2.3}$$

Here the primed indices will correspond to the amplituhedron case and the unprimed to the correlahedron. The index A is for the bosonic twistor coordinates, and α for homogeneous coordinates σ^α on the line in twistor space corresponding to the point X. In certain GL(2) gauge fixings it can be identified with a two-component self-dual spinor index.

Symmetries: for the amplituhedron we have a local $\mathrm{GL}(k')$ acting from the left on the p' index (corresponding to a different choice of basis for the k'-plane Y in $Gr(k',k'{+}4)$) and n GL(1)s acting scaling each Z_i. We also have a global $\mathrm{GL}(k'{+}4)$ acting simultaneously on the right on the $\mathcal{A}'$ index that Y,Z carry (and the $k'+4$ space). For the correlahedron we analogously have a local $\mathrm{GL}(n+k)$ acting from the left on Y, and a global $\mathrm{GL}(4+n+k)$ acting simultaneously on the right of Y and X. In addition there is also a local $\mathrm{GL}(2)^n$ with each GL(2) acting on the α index of $X_{i\alpha}$ and corresponding to simply changing the choice of basis for each of the 2-planes X_i.

2.2 Bosonised correlator potentials

We will consider correlators $\langle\mathcal{O}_1\dots\mathcal{O}_n\rangle$ where $\mathcal{O}_i$ is the super-BPS operator whose leading part is $\mathrm{tr}((y_i\cdot\Phi(x_i))^2)$. Here the y_i are skew matrices over the four component R-symmetry indices that have rank two and the x_i are points in Minkowski space. The supersymmetric extension extends this to a function on analytic superspace [25, 26], however in [23] an alternative formulation for the supersymmetric extension of the chiral correlator was found (see [27–29] for the extension to the full non-chiral case). This describes the correlator in terms of a potential $\mathcal{G}_n$, a function of (x,θ) in chiral super Minkowski space related by

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \left(\prod_{i=1}^{n} D_i^4 \right) \mathcal{G}_n(x_i, \theta_i)$$

where

$$D_i^4 := y_i^{IJ} y_i^{KL} \partial_{\theta_i^{\alpha I}} \partial_{\theta_i^{\beta J}} \partial_{\theta_{i\alpha}^K} \partial_{\theta_{i\beta}^L} .$$

The correlator decomposes into irreducible parts of degree $4k$ in the θs, and the corresponding potentials we denote $\mathcal{G}_{n;k}$ (which thus have degree $\theta^{4(n+k)}$).

We bosonise the dependence on the θs as in (2.2), (2.3) to lift $\mathcal{G}_{n;k}$ to a function $G_{n;k}(X_1,\dots,X_n)$ defined on n copies of $\mathbb{M}_b = Gr(2,4+n+k)$ where the k corresponds to the fermionic degree $4(n+k)$.

Now the potential $\mathcal{G}_{n;k}$ need not be gauge invariant, although the correlator will be after the differentiation. In the twistor Feynman diagram formalism arising from the twistor action, this potential is interpreted as a correlator of certain 'log det d-bar' operators based on lines in twistor space. These operators are not gauge invariant, although become so when differentiated by the D_i^4 when they become the gauge invariant super-BPS operators $\mathcal{O}_i$. In the diagram formulation based on an axial gauge, the gauge dependence will manifest itself in dependence on the reference twistor Z_*. We will suppress this differentiation in the following and indeed there appears to be a natural Z_* independent volume form in correlahedron space, $G_{n;k}$ obtained directly from analytic superspace expressions.

In the limit where n' of the x_i lie on a lightlike polygon, when multiplied by $\prod_{i=1}^{n'}(x_{i\,i+1}^2)$ this correlator degenerates into the loop integrand for the supersymmetric light-like Wilson-loop at loop order $n-n'$. Via the amplitude-Wilson-loop duality this provides the aforementioned link to amplitudes in the planar limit. However, the Wilson-loop to which it degenerates is in the adjoint rather than fundamental representation and so gives the square of that in the fundamental that corresponds to the amplitude. We will see that the correlahedron degenerates geometrically to give the squared amplituhedron in this limit.

2.3 Correlahedron and amplituhedron forms

The correlahedron lives in the Grassmannian $\mathrm{Gr}(n{+}k, 4{+}n{+}k)$, a $4(n{+}k)$ dimensional space whose points are represented by the $(4{+}n{+}k) \times (n{+}k)$ matrix $Y_p^{\mathcal{A}}$ defined up to $\mathrm{GL}(n+k)$ acting on the p-index.

The potential $G_{n;k}$ is given by a volume form $\Omega_{n;k}(Y,X_i)$ on this space. This gives rise to $G_{n;k}$ by the formula

$$G_{n;k}(X_i) := \int \Omega_{n;k}(Y,X_i)\delta^{4(n+k)}(Y;Y_0) \tag{2.4}$$

where $\Omega_{n;k}(Y,X_i)$ is a $4(n+k)$-form on $\mathrm{Gr}(n{+}k, 4{+}n{+}k)$ and

$$\delta^{4(n+k)}(Y;Y_0) := \int d^{(k+n)^2}\rho_s^r \det(\rho)^4 \delta(Y_r - \rho_r^s Y_{s0}) , \quad \text{and} \quad Y_0 = \begin{pmatrix} 0_{4\times(n+k)} \\ 1_{(n+k)\times(n+k)} \end{pmatrix}. \tag{2.5}$$

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In this formula, ρ is a dummy variable that picks out the additional $(n+k) \times (n+k)$-components of Y and takes their determinant which will then provide the bosonised form of the fermionic delta functions.

Similarly the (square of the) amplituhedron lives in the Grassmannian $\mathrm{Gr}(k, 4+k)$ at tree-level with analogous formulae to the above with

$$A_{n;k}(Z_1, \dots, Z_n) = \int \Omega_{n;k}(Y, Z_i)\delta^{4)}(Y; Y_0)\,, \tag{2.6}$$

with an analogous description for the loop integrand that we shall detail later.

Thus the key information of the correlator/amplitude is encoded in the volume form Ω. We first remark that there is a natural weighted volume form on $\mathrm{Gr}(k, 4+k)$ of weight $k(4+k)$ that can be written as

$$\langle Y d^4 Y_1 \rangle \dots \langle Y d^4 Y_k \rangle$$

and similarly on $\mathrm{Gr}(n+k, 4+n+k)$. However, the overall expression must have weight zero in both Y and the X_i and Z_i. The remaining factor that must balance the weights is proposed to be characterized by its poles, although we will first find a representation as a sum of Feynman diagrams, albeit in a gauge dependent form when it comes to the correlator potential. This remaining factor essentially *is* the bosonised correlator after putting $Y \to Y_0$ (which can be done using the global $\mathrm{GL}(n+k+4)$ symmetry). So for the amplitude

$$\Omega_{n;k}(Y, Z_i) = \langle Y d^4 Y_1 \rangle \dots \langle Y d^4 Y_k \rangle \times A_{n;k}(Y, Z_i), \qquad A_{n;k}(Z_i) = A_{n;k}(Y_0, Z_i) \tag{2.7}$$

and for the correlator

$$\Omega_{n;k}(Y, X_i) = \langle Y d^4 Y_1 \rangle \dots \langle Y d^4 Y_{n+k} \rangle \times G_{n;k}(Y, X_i), \qquad G_{n;k}(X_i) = G_{n;k}(Y_0, X_i)\,. \tag{2.8}$$

3 Hedron geometry

In the previous section we saw that correlators, amplitudes (possibly squared) and their loop integrands can be encoded in terms of volume forms on respectively $\mathrm{Gr}(n+k, 4+n+k)$ and $\mathrm{Gr}(k, 4+k)$. A key aspect of the amplituhedron programme is that these forms should be uniquely determined by the 'hedron' geometry.

In this section we first review the main features of the amplituhedron as a geometrical object following [9]. We then introduce a larger object in the same space, $\mathrm{Gr}(k, 4+k)$ which we call the "squared amplituhedron" and which was been hinted at in [24]. This corresponds to the square of the amplitude. Finally we propose a new geometric object, the correlahedron, a subspace of the higher dimensional Grassmannian $\mathrm{Gr}(n+k, 4+n+k)$, and which should correspond to the correlator.

3.1 Amplituhedron

The first definition of the amplituhedron is as the image of the positive Grassmannian $Gr^+(k, n)$ of *positive* k-planes in n dimensions, into $\mathrm{Gr}^+(k, 4+k)$. Positive here means that all ordered $k \times k$ minors are non-negative. The map from $Gr^+(k, n)$ to $Gr^+(k, k+4)$ follows from a linear map from n to $k{+}4$ dimensions given by the external kinematic data in the form of the n bosonised momentum twistors $Z_i^{\mathcal{A}'}$ an $n \times (k{+}4)$ matrix. The matrix $Z_i{}^{\mathcal{A}}$ also has to be positive, ie all its ordered maximal minors must be positive. In summary, the amplituhedron is the set

$$\text{amplituhedron}_{n;k}(Z) = \Big\{ Y \subset \mathrm{Gr}(k, 4{+}k) : \; Y_{p'}^{\mathcal{A}'} = C_{p'}^{i} Z_i^{\mathcal{A}'} \text{ for } C \in Gr^+(k,n) \Big\} . \tag{3.1}$$

One way to give an explicit description of this positive geometry is via a BCFW decomposition of the amplitude in the Grassmannian [30, 31]. It is proposed that this geometric image uniquely determines the volume form Ω as the unique holomorphic volume form of $\mathrm{Gr}(k, 4{+}k)$ that has logarithmic singularities on the boundary of the region (and no singularities inside).

The above is the tree-level amplituhedron. At ℓ-loops there is an analogous object in which the Grassmannian $\mathrm{Gr}(k, 4{+}k)$ is supplemented by ℓ 2-planes orthogonal to Y. The superamplitude is then given as the differential form Ω that has logarithmic divergences on the boundary of this amplituhedron. For more details of the amplituhedron see [9].

The above definition is somewhat implicit. In general the map from C to Y, $Y = CZ$ is a projection from a higher dimension, that maps many points to the same point. It is difficult to extract an explicit logarithmic form (and hence the amplitude) directly from the geometry without the original BCFW decomposition in the Grassmannian. The definition (3.1), together with the positivity of the external data, implies however the explicit $\mathrm{Gr}(k, 4{+}k)$ constraints

$$\langle Y Z_{i-1} Z_i Z_{j-1} Z_j \rangle > 0 \, ,$$

where here $\langle \dots \rangle$ is the skew form over $\mathbb{R}^{4+k}$ with $4 + k$ arguments and

$$\langle Y ABCD \rangle := \langle Y_1 \dots Y_k \, ABCD \rangle \, .$$

These constraints do indeed encode the location of the physical singularities but are not sufficient to fully specify the amplituhedron and in [24] a further topological condition is understood to be required in addition.

3.2 Squared amplituhedron

The above discussion leads us to consider the subspace of $\mathrm{Gr}(k, 4{+}k)$ defined simply by the inequalities:

$$\text{squared amplituhedron}_{n;k}(Z) = \Big\{ Y \in \mathrm{Gr}(k, 4{+}k) : \; \langle Y Z_{i-1} Z_i Z_{j-1} Z_j \rangle > 0 \Big\} . \tag{3.2}$$

We call this the *squared amplituhedron* on the basis of the conjecture that this indeed gives the square of the amplitude. It lies in the same space, $\mathrm{Gr}(k, 4{+}k)$, as the amplituhedron and indeed contains the amplituhedron, but it is defined by explicit constraints in $\mathrm{Gr}(k, 4{+}k)$ (without the additional topological condition specifying the amplituhedron itself). This explicit definition makes the squared amplituhedron much easier to use in practice.

Indeed we find in a number of examples that the logarithmic volume form associated with this region gives the square of the (bosonised) superamplitude. The square of the

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superamplitude of Grassmann degree $4k$ is:

$$(A^2)_{n;k} = \sum_{k'=0}^{k} A_{n;k'} A_{n;k-k'} \,, \tag{3.3}$$

(obtained simply by expanding the square as a sum over k' and taking the relevant piece). In section 6 we give a concrete practical method (for small n, ℓ) for obtaining the differential form, and hence the superamplitude, from the squared amplituhedron using a cylindrical decomposition.

The squared amplituhedron also extends to loop level. The ℓ-loop squared amplituhedron is a subspace of the space of k-planes $Y \in \mathrm{Gr}(k, k+4)$ together with ℓ complementary 2-planes in $\mathbb{R}^{4+k}$, $\mathcal{L}_i \in \mathrm{Gr}(2, 4+k), i = 1, \ldots, \ell$, subject to the following constraints

$$\begin{aligned}&\text{squared amplituhedron}^{(\ell)}_{n;k}(Z)\\ &= \Big\{(Y, \mathcal{L}_1, \ldots, \mathcal{L}_\ell) : \; \langle Y Z_{i-1} Z_i Z_{j-1} Z_j \rangle > 0, \; \langle Y Z_{i-1} Z_i \mathcal{L}_j \rangle > 0, \; \langle Y \mathcal{L}_i \mathcal{L}_j \rangle > 0 \Big\} \,.\end{aligned} \tag{3.4}$$

The logarithmic differential form on this region gives the square of the superamplitude at Grassmann degree k and perturbative order ℓ, explicitly it gives the combination:

$$(A^2)^{(\ell)}_{n;k} = \sum_{\ell'=0}^{\ell} \sum_{k'=0}^{k} A^{(\ell')}_{n;k'} A^{(\ell-\ell')}_{n;k-k'} \,. \tag{3.5}$$

In section 6 we illustrate this squared amplituhedron in some highly non-trivial examples.

3.3 Correlahedron

More importantly for this paper, the squared amplituhedron lends itself to a natural generalisation, the *correlahedron*, on the basis of the conjecture that it should yield the stress-tensor correlator. We propose the correlahedron as a geometrical object lying inside the space of $(k+n)$-planes in $\mathbb{R}^{4+n+k}$, $\mathrm{Gr}(n+k, 4+n+k)$, specified by the inequalities

$$\Big\{ Y \in \mathrm{Gr}(n+k, n+k+4) : \; \langle Y X_i X_j \rangle > 0 \Big\} \,. \tag{3.6}$$

Here the external data $X_i, i = 1, \ldots, n$ are themselves 2-planes, $X_i \in \mathrm{Gr}(2, n+k+4)$, and are equivalent to points in chiral superspace.

It is the purpose of the rest of this paper to motivate and give evidence for the correlahedron. We will begin in the next section by motivating the choice of space in which the correlahedron lives, $\mathrm{Gr}(n+k, n+k+4)$, from an algebraic point of view, starting with the formulation of correlators using Feynman rules in twistor space [23].

4 Hedron volume forms

We now describe the correlahedron volume forms (bosonised correlators) in $\mathrm{Gr}(n{+}k, n{+}k{+}4)$ from a purely algebraic and analytic perspective, translating expressions found both from analytic superspace bootstrap techniques as well as from twistor space Feynman rules into the correlahedron space we propose. For the correlator the expressions arising from twistor Feynman rules will not be gauge invariant, however those arising from analytic superspace bootstrap expressions are, and this shows that there is nevertheless a unique expression in $\mathrm{Gr}(n{+}k, n{+}k{+}4)$ which we propose to be uniquely defined by the correlahedron geometry described in the previous section.

4.1 Hedron expressions from twistor space Feynman diagrams

Here we explain how the amplituhedra and correlahedra Grassmannians described above arise from considering Feynman diagrams in twistor space. The key result is that there will be a volume form Ω_Γ for each Feynman diagram Γ in twistor space. The $\delta^{4(n+k)}(Y;Y_0)$ will be seen to arise automatically from the product of propagators in a diagram. Each propagator will provide one physical singularity, but there will be plenty of spurious singularities in each diagram, that must cancel in the sum for the final correlator or Wilson loop.

The twistor space Feynman rules are described for holomorphic Wilson loops in [4, 21, 32] and the most developed version for the correlators can be found in [23]. In this context we will use the amplitude/Wilson-loop duality to give amplituhedron and squared amplituhedron expressions. This is equivalent to using a momentum twistor formulation of the amplitudes. Furthermore, the polygonal lightlike Wilson-loop in space-time or region-momentum space will be understood as a holomorphic Wilson-loop for a polygonal loop in momentum twistor space.

The diagrams contributing to the ℓ-loop integrand of a holomorphic Wilson loop in twistor space depends on n' twistors $Z_{i'}$ forming the vertices of the polygon in twistor space that corresponds to the edges of the light-like polygonal Wilson-loop in space-time, together with ℓ lines in twistor space corresponding to points in region momentum space for the loop integrand.[3] We will take all our diagrams to be planar (firstly in order that the amplitude/Wilson-loop duality should hold, and to avoid more complicated rules associated with the colour structure). At $\mathrm{N}^{k'}$MHV degree there should be $2\ell + k'$ propagators connecting the lines and polygon. Correlators are computed using essentially the same rules except that the propagators simply connect a collection of n lines together. In this case, it is said to have MHV degree k when there are $n + k$ propagators as each line must have at least two propagators ending on it. In the light-like limit, n' of these n lines will form the sides of the polygon and $n - n' = \ell$ the loop integrand points. In this limit, the diagrams correpond to the amplitude^2 when the planar representation of the diagram extends both outside and inside the Wilson-loop, but reduces to the amplitude itself when only diagrams inside the polygon are allowed.

[3]When the loop integrand is obtained in this way, the region loop momenta come with fermionic coordinates also that need to be integrated out as part of the loop integration, but are part of the supersymmetric correlator.

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In the following, the simplest case treated first is that for the correlator, where only lines in twistor space are needed connected by propagators. The log-det operator insertions give rise to 'MHV vertices' on these lines with a Parke-Taylor structure. We can then incorporate a holomorphic Wilson-loop in twistor space essentially by regarding the edges of the Wilson loop to carry MHV vertices connected together without propagators around the polygon.

4.1.1 Super twistor space Feynman rules

Points in chiral superspace correspond to lines in $\mathbb{CP}^{3|4}$ spanned by the pair of twistors $X^{\mathcal{A}}_{i\alpha}$, $\alpha = 1, 2$ where points on the line are parametrized homogeneously by σ^α by $Z_i(\sigma) = \sigma^\alpha X^{\mathcal{A}}_{i\alpha}$. When we reduce to a Wilson loop, we take the lines $X_{i'}$ $i' = 1, \dots, n$ to intersect in a polygon, but then we must integrate out $4n'$ superfluous fermionic coordinates (the n' lines have $8n'$ fermionic coordinates, whereas the n' twistors only $4n'$, so we require the identification of the fermionic parts of Z_i as a point on X_i with those on the point X_{i+1}.

The propagator connecting twistors Z and Z' corresponds to the delta-function

$$\Delta(Z, Z') := \int \frac{1}{\mathrm{vol}\, Gl(1)} \frac{dr}{r} \frac{ds}{s} \frac{dt}{t} \delta^{4|4}(rZ_* + sZ + tZ') \,. \tag{4.1}$$

To divide by vol $Gl(1)$ we can simply set one of the parameters r, s, t equal to some constant, but it will be convenient to keep the scalings in play. In a diagram a propagator will connect a line X_i at the point $Z_i(\sigma_{ij})$ to another X_j at the point $Z_j(\sigma_{ji})$. Each line X_i supports a vertex corresponding to a 'log det d-bar' operator on the line in twistor space that can in practice be thought of as an MHV vertex with as many legs as propagator insertions on the line. If the number of propagator insertions is M_i, then the insertion points are given by $Z_i(\sigma_m)$, $m = 1, \dots, M_i$ cyclically ordered by the planarity of the diagram. The vertex requires an integration over the insertion points σ_r

$$\int \frac{1}{M_i \mathrm{Vol}(\mathrm{GL}(2) \times \mathrm{GL}(1)^{M_i})} \prod_{m=1}^{M_i} \frac{d^2\sigma_m}{(\sigma_m, \sigma_{m+1})}, \tag{4.2}$$

where the integration points are understood projectively, hence the $\mathrm{GL}(1)^{M_i}$ and the $\mathrm{GL}(2)$ acts on the σ_r and the α index on X. The GL(2)s can all be fixed by setting $X^{\mathcal{A}}_{i\beta} = (\delta_\beta{}^\alpha, x^{\dot\alpha}_{i\beta}, \theta^I_{i\beta})$ although in the Wilson loop context other gauge fixings can be more helpful. The GL(1)s in (4.2) reflect the fact that the σ integrals are projective. However, the parameters s and t in (4.1) provide scalings for the σs, otherwise said the GL(1) quotients in (4.2) can be used to fix the s and t parameter integrals in the propagators so that the s in $sZ_i(\sigma_{ij})$ defines the scale of σ_{ij}. There is precisely one such GL(1) for each of the two insertions of each propagator in the vertex and so we set $s = t = 1$. The remaining GL(1) in the propagator definition can be used to fix r to be constant. It nevertheless has nontrivial weight so we will not set it equal to one, but keep it in the formulae.

With this gauge fixing, the propagator becomes $\bar\delta^{4|4}(rZ_* + \sigma_{ij} \cdot X_i + \sigma_{ji} \cdot X_j)$ where the scaling integrals are now absorbed into those for the σs at each vertex and r is an arbitrary nonzero constant that will not affect the final answer. In the case of a correlator,

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the diagram's contribution $\mathcal{G}_\Gamma$ to the potential $\mathcal{G}$ for the correlator is

$$\mathcal{G}_\Gamma = \int \prod_{i=1}^{n} \prod_{m_i=1}^{M_i} \frac{d^2\sigma_{m_i}}{(\sigma_{m_i}, \sigma_{m_i+1})} \prod_{p=1}^{n+k} \bar{\delta}^{4|4}\left(r_p Z_* + \sigma_{i_p j_p} \cdot X_{i_p} + \sigma_{j_p i_p} \cdot X_{j_p}\right). \tag{4.3}$$

Here the σs are indexed in two ways, firstly by their locations m_i on the ith vertex and secondly at the ends i_p and j_p of the pth propagator.[4]

It was shown in [23] that once the formulae have been differentiated by the product of the D_i^4, diagrams with two adjacent propagators connecting X_i to X_j automatically vanish so we lose nothing by ruling out such diagrams ab initio. It was further shown that spurious singularities associated with the $(\sigma_r \sigma_{r+1})$ factors in the Parke-Taylor denominators cancel via a process of three-way cancellation. This latter property is no longer guaranteed without the D_i^4 differentiations.

We now discuss the extension of the Feynman diagrams to the holomorphic Wilson loop and hence amplitude (perhaps squared). When n' lines $X_{i'}$, $i' = 1 \dots n'$ intersect so that $X_{i'} \cap X_{i'-1} = Z_i$ we obtain a polygon in twistor space with vertices $Z_{i'}$. It was shown in [21] that as this limit is approached, when multiplied by $\prod_{i'=1}^{n} X_{i'-1} \cdot X_{i'}$, the Feynman diagrams become those for the adjoint holomorphic Wilson loop in twistor space which is the same as the adjoint super Wilson-loop in chiral super Minkowski space, that can in turn be identified with the square of the amplitude.

In more detail, the Feynman diagrams for the adjoint holomorphic Wilson loop now has two types of vertices, the lines $X_{i'}$ that form part of the polygon, and those that do not (these latter in this context correspond to the loop variables in the amplitude interpretation or Lagrangian/stress-energy insertions in the Born approximation). In taking the lightlike limit, we simply omit the n' propagators that connect the now joined consecutive $X_{i'}$ and $X_{i'-1}$. However, we do keep the vertices at the $X_{i'}$ including the connections between the $X_{i'}$ in the Parke-Taylor factors. These can be gauge fixed using the GL(2) in (4.2) for the sides of the polygon so that $\sigma \cdot X_{i'} = \sigma^0 Z_{i'} + \sigma^1 Z_{i'+1}$ and the σ at $Z_{i'}$ is $\sigma_0 = (1, 0)$ and that at $Z_{i'+1}$ is $\sigma_{I_{i'}+1} = (0, 1)$ when there are $I_{i'}$ propagators attached to the i' edge of the Wilson loop inside the polygon and $O_{i'}$ outside. Thus it gives rise to a factor

$$\int \prod_{m=0}^{I_{i'}+O_{i'}+1} \frac{d^2\sigma_m}{(\sigma_m, \sigma_{m+1})}, \tag{4.4}$$

where we have taken the GL(1) scalings to be fixed against the propagators as above (although note that this is a different gauge fixing for the Feynman diagrams for the Wilson loop to those given in [21, 32] say). The distinction between the diagrams considered here is that here the planar diagrams have propagators and vertices both outside and inside the

[4] The integrations over the σs can then all be done explicitly against the delta functions with solution

$$\sigma_{ij\alpha} = \frac{(X_{i\alpha} Z_* X_{j1} X_{j2})}{X_i \cdot X_j}.$$

Here $(Z_1 Z_2 Z_3 Z_4)$ is the skew form on the bosonic parts of the four twistors. In doing these integrations against the delta functions, we obtain a Jacobian factor of $X_i \cdot X_j$ in the denominator for each propagator.

Wilson loop, whereas for the Wilson-loop in the fundamental representation and hence the amplitude, they are purely inside.

In order to obtain the loop integrand itself, we must eventually integrate out all the fermionic θ variables at the X_i when X_i is a region loop variable (if we only do D_i^4 we are essentially obtaining the Born level correlator of a Wilson loop with $\operatorname{tr}\Phi^2$ rather than a Lagrangian insertion corresponding to a loop integrand point).

4.1.2 Bosonisation of Feynman diagrams in the correlahedron space

The Hodges bosonisation of the fermionic variables yields

$$\delta^{0|4}(\chi^I) = \int (\chi \cdot \phi)^4 d^4\phi\,,$$

and this motivates the introduction of new such variables $\phi_I{}^p$, four for each propagator $p = 1, \ldots, n{+}k$. This gives the new bosonic variables $\zeta^p = \chi \cdot \phi^p$ In the amplitude formula above we will then replace the ath delta function by

$$\delta^{4|4}(\mathcal{Z}) \to (\zeta^p)^4 \prod_{A=1}^{4} \delta(Z^A)$$

We retrieve the original $\delta^{4|4}(\mathcal{Z})$ by substituting in and integrating out the $\phi_I{}^p$. With this (4.3) becomes

$$G_\Gamma = \int \prod_{i=1}^{n} \prod_{m_i=1}^{M_i} \frac{d^2\sigma_{m_i}}{M_i(\sigma_{m_i}, \sigma_{m_i+1})} \prod_{p=1}^{n+k} (y^p)^4 \delta^4\left(r_p Z_* + \sigma_{i_p j_p} \cdot X_{i_p} + \sigma_{j_p i_p} \cdot X_{j_p}\right), \tag{4.5}$$

where $y^p = \sigma_{i_p j_p} \cdot \xi_{i_p} + \sigma_{j_p i_p} \cdot \xi_{j_p}$.

We can now define the map from the σ parameters to the correlahedron Grassmannian by

$$Y_p^{\mathcal{B}} = r_p Z_*^{\mathcal{B}} + \sigma_{i_p j_p} \cdot X_{i_p}^{\mathcal{B}} + \sigma_{j_p i_p} \cdot X_{j_p}^{\mathcal{B}}\,. \tag{4.6}$$

where here now $\mathcal{B} = (B, q) = (1 \ldots 4, 1 \ldots n + k)$ (we could here include a ζ_*^p part of Z_* in the same way as we could have had a fermionic part of the original reference twistor).

With this, the product over p on the right hand side of (4.5) becomes $\prod_p (y_p^p)^4 \delta^4(Y_p^A)$. However, we obtain the same formula for the super-amplitude if we replace this expression by $\delta^{4(n+k)}(Y; Y_0)$ as defined in (2.5). On performing the ρ-integral in (2.5), this is equivalent to replacing $\prod_p (y_p^p)^4$ by $(\det\{y_p^q\})^4$. This will yield the same super-amplitude up to some numerical factor because, after inserting $y_p^q = \eta^q \phi_p$, the transform back to the supersymmetric correlator/amplitude picks out the coefficient of the top power of ϕs and this will of necessity be the top power of the ηs that provide the arguments of the desired $\delta^{0|4}$s in (4.3). Thus the only required check is that the numerical factor is not zero, which can be done by hand.

Thus, identifying the δ-functions arising from the propagators with $\delta^{4(n+k)}(Y; Y_0)$, we obtain the diagram's contribution to the correlator by the formula

$$G_\Gamma(X_i, Z_*) := \int \Omega_\Gamma(Y, X_i, Z_*) \delta^{4(n+k)}(Y; Y_0)\,. \tag{4.7}$$

where the $4(n+k)$-form Ω_Γ on the correlahedron Grassmannian is the product of the vertices

$$\Omega_\Gamma(Y, X_i, Z_*) := \prod_{i=1}^{n} \prod_{m=1}^{M_i} \frac{d^2\sigma_{ij_m}}{(\sigma_{ij_m}\sigma_{ij_{m+1}})} \,. \tag{4.8}$$

This formula can be expressed in terms of the Ys by using (4.6). We introduce the notation

$$\langle YABCD\rangle := (Y_1Y_2\dots Y_{n+k}ABCD)$$

where on the right hand side $(\dots)$ denotes the natural skew bracket over $4+n+k$ space of objects with an $\mathcal{A}$-index. This expression taken with Y_p as one of A, B, C, D must vanish, so

$$0 = \langle YY_pX_{i1}X_{i_2}X_{j\alpha}\rangle = r_p\langle YZ_*X_{i1}X_{i2}X_{j\alpha}\rangle + \sigma_{j_pi_p\alpha}\langle YX_iX_j\rangle \,,$$

yielding

$$\sigma_{j_pi_p\alpha} = -r_p\frac{\langle YZ_*X_{i_p1}X_{i_p2}X_{j_p\alpha}\rangle}{\langle YX_{i_p}X_{j_p}\rangle} \,. \tag{4.9}$$

Similarly, taking the exterior derivative of (4.6) (regarding X_i and Z_* as constants) and inserting the resulting equation 4 times into $\langle Y\dots\rangle$ we find

$$d^2\sigma_{j_pi_p}d^2\sigma_{i_pj_p} = \frac{\langle Yd^4Y_p\rangle}{\langle YX_{i_p}X_{j_p}\rangle} \,.$$

With this we can write the volume form as

$$\Omega_\Gamma(Y, X_i, Z_*) := \prod_{i=1}^{n} \prod_{m=1}^{M_i} \frac{1}{(\sigma_{ij_m}\sigma_{ij_{m+1}})} \prod_{p=1}^{n+k} \frac{\langle Yd^4Y_p\rangle}{\langle YX_{i_p}X_{j_p}\rangle} \,. \tag{4.10}$$

In this we can see that we have one 'physical' singularity for each propagator namely the $\langle YX_iX_j\rangle$ and four spurious ones, essentially the adjacent Parke-Taylor denominators (shared with the adjacent propagators at the vertex).[5] We remark that the cancellation of these spurious singularities was identified in [23] as being between triples of diagrams that agree everywhere except on a triangle between three vertices X_i, X_j and X_k with each diagram having two out of three propagators around the triangle.

4.2 Invariant correlahedron expressions directly from correlators on analytic superspace

In the previous subection we translated Twistor Feynman expressions directly into correlahedron form expressions. Unfortunately the resulting expressions were not gauge invariant. However we find that one can alternatively lift directly from the analytic superspace expressions in a canonical gauge invariant way to obtain a unique canonical correlahedron volume form for each correlator.

[5] The denominator in (4.9) might be thought to affect the poles in $\langle YX_iX_j\rangle$ for each propagator, but these factors cancel in the final formulae as there are fourth powers of the σs in the $\det\{Y_p^q\}$ in the $\delta^{4(n+k)}(Y; Y_0)$ in (4.7). This factor could for example be incorporated into r_p.

Many correlators have by now been constructed explicitly writing down forms with the correct singularity structure and showing that they satisfy appropriate consistency properties. In particular, for maximal $k = n-4$, they have been constructed up to $n = 14$ (equivalent to 10 loop four-point correlators), and the next to maximal case, $k = 1$, $n = 6$ has also been constructed [8, 33–42]. We will see here that given these expressions, there is a simple procedure to uplift them directly (and uniquely) into correlahedron volume forms on $\mathrm{Gr}(n{+}k, 4{+}n{+}k)$.

We start with the simplest non-trivial correlator, the 5 point $k = 1$ case $G_{5;1}$. The physical singularities are at $\langle Y X_i X_j\rangle$. The correlahedron space for this correlator is $Y \in \mathrm{Gr}(6, 10)$. There is essentially a unique correlahedron form of weight zero in Y and the X_i with simple poles at the physical singularities. It is given by

$$\Omega_{5;1}(Y, X_i) = \frac{\langle Y d^4 Y_1\rangle \dots \langle Y d^4 Y_6\rangle \langle X_1 X_2 X_3 X_4 X_5\rangle^4}{\langle Y X_1 X_2\rangle \langle Y X_1 X_3\rangle \dots \langle Y X_4 X_5\rangle}\,. \tag{4.11}$$

The next simplest cases to consider are the maximally nilpotent ($k = n-4$) correlators. These are described in terms of a single function f [43] which is a conformally covariant, permutation symmetric function of x_{ij}^2

$$f^{(n-4)}(x_{ij}^2)\,. \tag{4.12}$$

These functions are known explicitly for $n \le 14$. The corresponding correlahedron space for these correlators is $Y \in \mathrm{Gr}(2n-4, 2n)$ and the correlahedron forms are given simply as

$$\Omega_{n;n-4}(Y, X_i) = \langle Y d^4 Y_1\rangle \dots \langle Y d^4 Y_{2n-4}\rangle \times \langle X_1 \dots X_n\rangle^4 \times f^{(n-4)}\Big(\langle Y X_i X_j\rangle\Big)\,. \tag{4.13}$$

So for example, for $n = 5$, corresponding to the four-point one-loop correlator, the function f is

$$f^{(1)}(x_{ij}^2) = \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2} \tag{4.14}$$

and we see that making the replacement $x_{ij}^2 \to \langle Y X_i X_j\rangle$, (4.13) correctly reproduces (4.11).

The only non-maximally nilpotent correlator currently known explicitly is the six point $k = 1$ correlator. This was derived in analytic superspace in [42] and also has a straightforward lift to a correlahedron volume form. This correlahedron lives in the space $Y \in \mathrm{Gr}(7, 11)$ so all the angle brackets are 11-brackets in the following expression. Since we are considering 11-brackets but we have 12 points (6-space time points) it is useful to label the 11-brackets by the missing point. So we define

$$\langle \dots \rangle^{i\alpha} := \langle X_{11} X_{12} X_{21} \dots \widehat{X_{i\alpha}} \dots X_{62}\rangle (-1)^{\alpha}\,. \tag{4.15}$$

The correlator $G_{6;1}$ was given in [42] in terms of nilpotent superconformal invariants $\mathcal{I}^{ijkl;\alpha\beta\gamma\delta}$ in analytic superspace. Lifting the correlator to the 11-dimensional correlahedron space, these nilpotent invariants become the following product of 4 11-brackets:

$$\mathcal{I}^{ijkl;\alpha\beta\gamma\delta} = \langle \dots \rangle^{i\alpha} \langle \dots \rangle^{j\beta} \langle \dots \rangle^{k\gamma} \langle \dots \rangle^{l\delta}\,. \tag{4.16}$$

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One interesting consequence of this correlahedron formulation of the correlator as an object in 11-dimensions is that it manifests highly non-trivial identities involving these invariants. It was observed in [42] that the invariants satisfy the non-trivial identity

$$\sum_{i=1}^{6} X_{i\alpha}\mathcal{I}^{ijkl;\alpha\beta\gamma\delta} = 0 \qquad (\text{for all } j,k,l,M,\beta,\gamma,\delta)\,, \tag{4.17}$$

which was found as a non-trivial consequence of superconformal Ward identities. In the 11-dimensional correlahedron space this identity is a straightforward consequence of a generalised Schouten identity in 11 dimensions

$$\sum_{i=1}^{6} X_{i\alpha}\langle\dots\rangle^{i\alpha} = 0\,. \tag{4.18}$$

The correlator itself as a correlahedron volume form has the representation

$$G^{(0)}_{6;1} = \langle Y d^4Y_1\rangle\dots\langle Y d^4 Y_7\rangle \frac{A_2 - 2\,A_1 - 8\,B_2}{\prod_{1\le i<j\le 6}\langle Y X_i X_j\rangle}\,, \tag{4.19}$$

where we introduced the notation

$$\begin{aligned}
A_1 &= \langle Y X_{5\alpha} X_1 X_{6\gamma}\rangle\langle Y X_{5\beta} X_2 X_{6\delta}\rangle\langle Y X_3 X_5\rangle\langle Y X_4 X_6\rangle\mathcal{I}^{5566;\alpha\beta\gamma\delta} + S_6 \text{ permutations}\,,\\
A_2 &= \langle Y X_{5\alpha} X_1 X_{6\gamma}\rangle\langle Y X_{5\beta} X_2 X_{6\delta}\rangle\langle Y X_3 X_4\rangle\langle Y X_5 X_6\rangle\mathcal{I}^{5566;\alpha\beta\gamma\delta} + S_6 \text{ permutations}\,,\\
B_2 &= \langle Y X_{4\alpha} X_3 X_{6\gamma}\rangle\langle Y X_{5\beta} X_2 X_{6\delta}\rangle\langle Y X_1 X_6\rangle\langle Y X_4 X_5\rangle\mathcal{I}^{4566;\alpha\beta\gamma\delta} + S_6 \text{ permutations}\,.
\end{aligned} \tag{4.20}$$

This form was directly lifted from the corresponding formula found in [42] in analytic superspace.

It is clear from this example that the construction of superconformal invariants on analytic superspace has a direct uplift into the correlahedron space more generally. Indeed the invariants $\mathcal{I}^{ijkl;\alpha\beta\gamma\delta}$ are well-defined for $k = n-5$ for *any* n and have the natural uplift to correlahedron space given by (4.16). But furthermore, the $k = 1$ analytic superspace superconformal invariants generalise to lower k. For example, for $k = n-6$ the most general invariants on analytic superspace are of the form $\mathcal{I}^{\{i_1i_2i_3i_4\}\{j_1j_2j_3j_4\};\{\alpha_1\alpha_2\alpha_3\alpha_4\}\{\beta_1\beta_2\beta_3\beta_4\}}$ which is symmetric under simultaneous interchange of $i_a\alpha_a$ with $i_b\alpha_b$ or separately $j_a\beta_a$ with $j_b\beta_b$. These invariants have a natural uplift to correlahedron space:

$$\begin{aligned}
&\mathcal{I}^{\{i_1i_2i_3i_4\}\{j_1j_2j_3j_4\};\{\alpha_1\alpha_2\alpha_3\alpha_4\}\{\beta_1\beta_2\beta_3\beta_4\}}\\
&\quad = \langle\dots\rangle^{i_1\alpha_1j_1\beta_1}\langle\dots\rangle^{i_2\alpha_2j_2\beta_2}\langle\dots\rangle^{i_3\alpha_3j_3\beta_3}\langle\dots\rangle^{i_4\alpha_4j_4\beta_4} + \dots
\end{aligned} \tag{4.21}$$

where $\langle\dots\rangle^{i\alpha j\beta}$ is the $n-2$ bracket with $X_{i\alpha}$ and $X_{j\beta}$ missing and where we sum over the 24 different possible simultaneous permutations of the $j_a\alpha_a$.

Thus we see that, although the direct construction of the correlator potential in correlahedron space arising from the twistor Feynman rules outlined in the previous section yields an expression which explicitly depends on Z_*, nevertheless, there is a canonical Z^*-independent uplift directly from an analytic superspace expression to the correlahedron form. We conjecture that this canonical form is uniquely defined by the correlahedron geometry as we discuss further in section 6.

5 The lightlike limit on the correlahedron

As discussed previously, in the lightlike limit when consecutive space-time points become lightlike separated, the stress-tensor correlator reproduces the light like polygonal Wilson loop in the adjoint representation, and hence the (square of the) amplitude [6, 7, 18–21]. One can take all n points lightlike separated around a polygon, in which case one gets the corresponding tree-level amplitude. Or one can take a non-maximal lightlike limit in which fewer than n points are consecutively lightlike separated. In this limit the resulting object is a square of a loop level amplitude, with the remaining points corresponding to loop variables.

The reduction from the correlator to the amplitude squared is an explicit algebraic process. We first take the limit as $(x_i - x_{i+1})^2 \to 0$ for $i = 1, \dots, n'$ (understood cyclically with $n' + 1 = 1$) of $G_{n;k}(X_i) \prod_{i=1}^{n'}(x_i - x_{i+1})^2$ but we must then also integrate out half the fermionic dependence of the $X_{i'}$ $i' = 1, \dots n'$. We wish to reduce the eight fermionic $\theta_{i'}$s for each $X_{i'}$ to the four fermionic variables for each twistor $Z_{i'}$ that make the corners of the corresponding polygon in twistor space. When $n > n'$ we must furthermore remove all the fermionic dependence of the remaining X_i, $i > n'$ which then have the interpretation of region loop momenta for the loop integrand.

This apparently fairly complicated procedure, however has a very simple and beautiful geometrical interpretation in correlahedron space which we denote "freeze and project". Furthermore the same geometric procedure acts both on the correlahedron *geometry* as well as on the corresponding *algebraic expressions*. This thus gives further confirmation of our conjecture that the correlahedron determines the correlator.

In this section we explain this lightlike limit on the correlahedron showing how it reproduces the corresponding (squared) amplituhedron. We show this both geometrically as well as algebraically.

5.1 The maximal lightlike limit geometrically

Taking the n-point lightlike limit of the correlator $G_{n;k}$ (we will consider lower point lightlike limits shortly) has the following geometric interpretation in correlahedron space. Recall that the correlahedron space is the subspace of the space of $k+n$-planes, Y, in a $4+k+n$-dimensional space bounded as (3.6)

$$Y \in \mathrm{Gr}(n+k, 4+n+k): \qquad \langle Y X_i X_j \rangle > 0 \quad \forall\, i, j = 1 \dots n\,. \tag{5.1}$$

The n-point lightlike limit is obtained by requiring Y to simultaneously lie on multiple boundaries $\langle Y X_i X_{i+1} \rangle = 0, i = 1 \dots n$ of the correlahedron. This can be done by freezing the first n of the Y_p, i.e., Y_i, $i = 1, \dots n$ to lie respectively in the span of the consecutive $X_i \wedge X_{i+1}$.

To further reduce to the amplitude (squared) we need to reduce the fermionic degree by $4n$ and hence the range of the p index inside $\mathcal{A}$ to $p' = 1 \dots k$. This also has a natural geometric interpretation for the correlahedron, namely it corresponds to performing a projection orthogonal to the n frozen intersection points Y_i. Thus the $4+k+n$ dimensional space is projected down to $4+k$ dimensions and the $k+n$-plane in $4+k+n$ dimensions,

Y, is projected to a k-plane in $4{+}k$ dimensions. This k plane gives the (square of the) amplituhedron.

In practical terms we can perform the freezing of Y to the boundary by choosing a $\mathrm{GL}(n+k)$ basis so that $Y = Y_1 \wedge \ldots \wedge Y_{n+k}$ with

$$\begin{aligned} Y_p &= \sigma_i^\alpha X_{i\alpha} - \tau_i^\alpha X_{i+1\,\alpha} \qquad && \text{for } p = i = 1\ldots n\,, \\ Y_p &= \hat{Y}_{p'} && p = n+p', \quad p' = 1\ldots k \end{aligned} \tag{5.2}$$

for some parameters σ_i^α, τ_i^α. We then need to project onto the quotient by $Y_1, \ldots Y_n$. In practice we can pick a basis for the $k{+}n{+}4$ dimensional vector space

$$\text{basis} = \Big\{Y_1, \ldots, Y_n, e_1, \ldots, e_{4+k}\Big\}\,, \tag{5.3}$$

where $e_1, \ldots e_{4+k}$ are any $4+k$ vectors such that this yields an independent basis.[6] We choose $\hat{Y}_{p'}$ to be a linear combination of the $e_{\mathcal{A}'}$ in this basis. The projection takes the form

$$X_{i\alpha} \to \hat{X}_{i\alpha} \quad \text{where} \quad \hat{X}_{i\alpha}^{\mathcal{A}} = \begin{cases} 0 & \mathcal{A} = 1, \ldots, n \\ X_{i\alpha}^{\mathcal{A}} & \mathcal{A} = n{+}1, \ldots, n{+}k{+}4 \end{cases}\,, \tag{5.4}$$

in the basis $\{Y_1 \ldots Y_n, e_1, \ldots e_{4+k}\}$.

We can then define brackets in the obvious way on the hyperplane spanned by $\{e_1, \ldots, e_{4+k}\}$ and it is clear that

$$\langle \hat{\mathcal{X}} \rangle := \langle Y_1 \ldots Y_n \mathcal{X} \rangle\,. \tag{5.5}$$

Here $\mathcal{X}$ represents any collection of $4+k$ independent vectors, and $\hat{\mathcal{X}}$ the same vectors projected onto the hyperplane.

After projecting out the Y_i, (5.2) gives $\sigma_i.\hat{X}_i = \tau_i.\hat{X}_{i+1}$ so we can define

$$Z_i := \sigma_i.X_i = \tau_i.X_{i+1} + Y_i\,. \tag{5.6}$$

Then after the projection

$$\hat{Z}_i := \sigma_i.\hat{X}_i = \tau_i.\hat{X}_{i+1}\,. \tag{5.7}$$

and the projected planes $\hat{X}_i$ intersect each other consecutively at $\hat{Z}_i$ in the projected space.

Thus freezing and projection yields a k-plane $\hat{Y}$ living in the $4{+}k$ dimensional hyperplane spanned by $\{e_1, \ldots e_{4+k}\}$ and we have projected planes $\hat{X}_{i\alpha}$ in the same $4{+}k$ dimensional space. Further we have

$$\langle Y X_i X_j \rangle = \begin{cases} 0 & |i-j| = 1 \quad \text{mod } n \\ \frac{\langle \hat{Y} \hat{Z}_{i-1} \hat{Z}_i \hat{Z}_{j-1} \hat{Z}_j \rangle}{\tau_{i-1}.\sigma_i\, \tau_{j-1}.\sigma_j} & \text{otherwise}\,. \end{cases} \tag{5.8}$$

So the correlahedron space (5.1) reduces to[7]

$$\hat{Y} \in \mathrm{Gr}(k, 4+k): \qquad \langle \hat{Y} Z_{i-1} Z_i Z_{j-1} Z_j \rangle > 0 \tag{5.9}$$

which is the squared amplituhedron (3.2).

[6] Geometrically the span of the e_i give the hyperplane on which we are projecting. However the final result is independent of this choice of hyperplane.

[7] If we assume we have chosen σ_i and τ_i appropriately so that $\tau_{i-1}.\sigma_i > 0$. Indeed different choices of signs here usually but not always yield the same expressions for the correlator. In some cases one has to sum over different choices of signs (see section 6.2.2 for example).

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5.2 Maximal lightlike limit on the hedron volume forms

In the previous section we described the lightlike limit of the correlahedron geometry as a freezing and projection of the space of Ys. Notice that this procedure is not singular as one might expect. It is simply a restriction of the geometry to a partial boundary, followed by a projection.

Here we give a simple algorithm for implementing this exact same procedure directly on the expressions for the correlator as differential forms in correlahedron space. We find that this indeed correctly reduces the correlator expressions to the correct amplitude (squared) expressions.

The fully covariant correlahedron form should have simple poles at $\langle Y X_i X_{i+1}\rangle = 0$ so we can write

$$\Omega(Y, X_i) = \frac{g_{n;k}(Y, X_i)}{\prod_{i=1}^{n}\langle Y X_i X_{i+1}\rangle} \prod_{p=1}^{n+k} \langle Y d^4 Y_p\rangle\,, \tag{5.10}$$

where $g_{n;k}$ has weight two in each X_i and $-(n+k)(k+4)$ in Y. When we freeze Y as in (5.2) we find

$$\langle Y d^4 Y_i\rangle = \langle Y X_i X_{i+1}\rangle d^2\sigma_i d^2\tau_i \qquad i = 1 \dots n\,. \tag{5.11}$$

Thus as $\langle Y X_i X_{i+1}\rangle \to 0$ in the limit, it cancels the corresponding term in the denominator of the correlahedron form to yield a finite result. Inverting (5.6) we obtain

$$X_{i\alpha} = \frac{-\tau_{i-1\,\alpha} Z_i + \sigma_{i\,\alpha}(Z_{i-1} + Y_{i-1})}{\tau_{i-1}.\sigma_i}\,. \tag{5.12}$$

Projecting along Y_i and correspondingly hatting the Zs in this expression sends[8]

$$X_{i\alpha} \to \frac{-\tau_{i-1\,\alpha} \hat{Z}_i + \sigma_{i\,\alpha}(\hat{Z}_{i-1} + Y_{i-1})}{\tau_{i-1}.\sigma_i}\,. \tag{5.13}$$

The correlahedron form is then reduced to the amplituhedron squared form by setting Y to (5.2) and $X_{i\alpha}$ to (5.13) and finally leaving out the σ, τ dependent factors:

$$\begin{aligned}
&\frac{g_{n;k}(Y, X_i)}{\prod_{i=1}^{n}\langle Y X_i X_{i+1}\rangle} \prod_{p=1}^{n+k} \langle Y d^4 Y_p\rangle \to \\
&\quad \left(\prod_{i=1}^{n} d^2\sigma_i d^2\tau_i\right) \left(\prod_{p'=1}^{k} \langle \hat{Y} d^4 \hat{Y}_{p'}\rangle\right) g_{n;k}\left(Y, \frac{-\tau_{i-1\,\alpha} \hat{Z}_i + \sigma_{i\,\alpha}(\hat{Z}_{i-1} + Y_{i-1})}{\tau_{i-1}.\sigma_i}\right) \\
&\quad = \left(\prod_{i=1}^{n} \frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right) \left(\prod_{i=1}^{k} \langle \hat{Y} d^4 \hat{Y}_i\rangle\right) a_{n;k}(\hat{Y}, \hat{Z}_i) \\
&\quad \to \left(\prod_{p'=1}^{k} \langle \hat{Y} d^4 \hat{Y}_i\rangle\right) a_{n;k}(\hat{Y}, \hat{Z}_i)
\end{aligned} \tag{5.14}$$

[8] Note that this is not quite the same as replacing $X_{i\alpha} \to \hat{X}_{i\alpha}$ as the Y_{i-1} remains on the right hand side and is not projected away. We need to do tis in order to make sense of the k+n+4-brackets.

Here to go from the second to the third line, we used the fact that $g_{n;k}$ as defined in (5.10) has homogeneity degree two in each X_i. We have defined $a_{n;k}(\hat{Y}, \hat{Z}_i) = g_{n;k}(Y, -\tau_{i-1\,\alpha}\hat{Z}_i + \sigma_{i\,\alpha}(\hat{Z}_{i-1} + Y_{i-1}))$ by this relation and this should correspond to the square of the amplitude. In particular it should be independent of σ, τ: this is a direct consequence of the amplitude/Wilson loop duality. This σ, τ dependence corresponds to choosing different points on the boundary 4-planes to freeze Y. To go from the third to the fourth line we simply drop the first factor that depends only on σ, τ which we are freezing.

As an explicit example, take the expression for the correlahedron form $G_{5;1}$ (4.11) and perform the above freezing of Y and projection. We have $Y = Y_1 \wedge \cdots \wedge Y_6 \in \mathrm{Gr}(6, 10)$ and we freeze $Y_1, \ldots, Y_5$ as in (5.2), $Y_i = \sigma_i^\alpha X_{i\alpha} - \tau_i^\alpha X_{i+1\,\alpha}$, leaving $Y_6 = \hat{Y}$ orthogonal. Then

$$\begin{aligned}
\prod_{i=1}^{6}\langle Y d^4 Y_i\rangle \frac{\langle X_1X_2X_3X_4X_5\rangle^4}{\langle YX_1X_2\rangle\ldots\langle YX_4X_5\rangle} \xrightarrow[\text{project}X]{\text{freeze }Y} & \left(\prod_{i=1}^{5}\frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right)\frac{\langle Yd^4\hat{Y}\rangle\langle Y_1\ldots Y_5\hat{Z}_1\ldots\hat{Z}_5\rangle^4}{\langle Y\hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\rangle\ldots\langle Y\hat{Z}_5\hat{Z}_1\hat{Z}_2\hat{Z}_3\rangle} \\
= & \left(\prod_{i=1}^{5}\frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right)\frac{\langle\hat{Y}d^4\hat{Y}\rangle\langle\hat{Z}_1\ldots\hat{Z}_5\rangle^4}{\langle\hat{Y}\hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\rangle\ldots\langle\hat{Y}\hat{Z}_5\hat{Z}_1\hat{Z}_2\hat{Z}_3\rangle} \\
& \downarrow \\
& \frac{\langle\hat{Y}d^4\hat{Y}\rangle\langle\hat{Z}_1\ldots\hat{Z}_5\rangle^4}{\langle\hat{Y}\hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\rangle\ldots\langle\hat{Y}\hat{Z}_5\hat{Z}_1\hat{Z}_2\hat{Z}_3\rangle}
\end{aligned} \tag{5.15}$$

Here in the first line we used that under the replacement (5.13)

$$\langle X_1X_2X_3X_4X_5\rangle = \langle Y_1\ldots Y_5\hat{Z}_1\ldots\hat{Z}_5\rangle\prod_{i=1}^{5}(\tau_i.\sigma_{i+1})^{-1}. \tag{5.16}$$

as well as (5.8)

$$\langle YX_iX_j\rangle = \langle Y\hat{Z}_{i-1}\hat{Z}_i\hat{Z}_{j-1}\hat{Z}_j\rangle \times (\tau_{i-1}.\sigma_i\,\tau_{j-1}.\sigma_j)^{-1}. \tag{5.17}$$

Finally in the last step we performed the reduction by simply removing the total derivatives involving the frozen variables $d\sigma_i, d\tau_i$ which appear in an invariant measure.

The final result is precisely the five-point NMHV amplituhedron form.

We note that it is easier to consider the functions without the measures (which are also much closer to the actual correlator/amplitude expressions). Also it is then easier to make particular choices for the σ_i, τ_i for example $\sigma_i^\alpha = (0, 1)$, $\tau_i^\alpha = (1, 0)$. Then the lightlike limit takes the correlahedron expression $g_{n;k}(Y, X_i)$ to the amplitude expression $a_{n;k}(\hat{Y}, \hat{Z}_i)$ via the simple replacements, implementing the action of freezing and projecting (5.14)

$$g_{n;k}(Y, X_i) \xrightarrow[X_{i2}\to-\hat{Z}_i,\,X_{i1}\to-\hat{Z}_{i-1}-Y_{i-1}]{Y_i=X_{i2}-X_{i+1\,1}(i=1\ldots n),\ Y_{n+i}=\hat{Y}_i(i=1\ldots k)} a_{n;k}(\hat{Y}, \hat{Z}_i). \tag{5.18}$$

We give a highly non-trivial example of this reduction procedure in appendix A. There we reduce the correlator $G_{6;1}$ given by the lengthy expression in (4.20) to the corresponding NMHV 6 point amplitude.

5.3 The non-maximal limit geometrically

The maximal, n-point, lightlike limit described above reduces the correlahedron, which lives in $\mathrm{Gr}(n{+}k, 4{+}n{+}k)$ to $\mathrm{Gr}(k, 4{+}k)$ by partial freezing and projecting from Y. Physically it reduces the n-point, Grassmann degree k correlator $G_{n;k}$ to the (square of the) tree level n-point N^kMHV amplitude. However it is also possible to consider lightlike limits of fewer points, $n' < n$. In this limit the correlator reduces to higher loop amplitudes, specifically the $(n{-}n')$-loop, $\mathrm{N}^{k'}$MHV amplitude, $A^{(n-n')}_{n';k'}$ where

$$k' = k - n + n' \,. \tag{5.19}$$

As in section 5.1, the light like limit is taken by setting $\langle Y X_{i'} X_{i'+1}\rangle = 0$ so that we are freezing Y to intersect the n' 4-planes, $X_1 \wedge X_2$, $X_2 \wedge X_3, \dots, X_{n'} \wedge X_1$. We then project through these n' intersection points, but here we *also project through the $n - n'$ additional 2-planes* $X_{n'+1}, \dots X_n$. This extra step corresponds to integrating out the supersymmetric parts of X_i for $i > n'$ leaving a space-time integrand.

The concrete description of this procedure starts as in the maximal case: the imposition of $\langle Y X_i X_{i+1}\rangle = 0$ allows us to gauge fix (freeze) the first n' components of Y to take the form

$$Y_i = \sigma_i^{\alpha} X_{i\alpha} - \tau_i^{\alpha} X_{i+1\,\alpha} \qquad i = 1 \dots n' \text{ (cyclically)}. \tag{5.20}$$

However, in the non-maximal case we further gauge fix the next $2n - 2n'$ components of the Y matrix as follows:

$$\begin{aligned}
Y_{n'+1} &= \mathcal{L}_{1\,1} + \sigma^{\alpha}_{n'+1} X_{n'+1\,\alpha} & \qquad Y_{n'+2} &= \mathcal{L}_{1\,2} + \sigma^{\alpha}_{n'+2} X_{n'+1\,\alpha}\\
Y_{n'+3} &= \mathcal{L}_{2\,1} + \sigma^{\alpha}_{n'+3} X_{n'+2\,\alpha} & \qquad Y_{n'+4} &= \mathcal{L}_{2\,2} + \sigma^{\alpha}_{n'+4} X_{n'+2\,\alpha}\\
&\dots & &\dots\\
Y_{2n-n'-1} &= \mathcal{L}_{n-n'\,1} + \sigma^{\alpha}_{2n-n'-1} X_{n\,\alpha} & \qquad Y_{2n-n'} &= \mathcal{L}_{n-n'\,2} + \sigma^{\alpha}_{2n-n'} X_{n\,\alpha}\,,
\end{aligned} \tag{5.21}$$

where the $\mathcal{L}_{i\alpha}$ are transverse to all the $X_{n'+i\,\alpha}$ and $Y_{i'}, i = 1 \dots n'$. Note that (5.21) is not a restriction on the hyperplane Y but merely on a choice of basis for Y; we can always choose a $\mathrm{GL}(n+k)$ transformation to obtain (5.21), unlike (5.20) which follows from the freezing Y to the boundary of the space.

There will be $n{+}k - (2n{-}n') = k'$ components of Y remaining and we denote these by

$$Y_{2n-n'+p'} = \hat{Y}_{p'} \qquad p' = 1 \dots k' \tag{5.22}$$

and we also insist that they are transverse to both $X_{n'+i\,\alpha}$ and $Y_i, i{=}1 \dots n'$ using $\mathrm{GL}(n{+}k)$.

To make the above statements precise we can choose a basis (but the final answer will be basis independent) for $\mathbb{C}^{k+n+4}$ given by

$$\text{basis} = \left\{ Y_1, \dots, Y_{n'}, X_{n'+1\,1}, X_{n'+1\,2}, \dots, X_{n\,1}, X_{n\,2}, e_1, \dots, e_{k'+4} \right\}, \tag{5.23}$$

where $e_1, \dots e_{k'+4}$ are any $k'+4$ vectors such that this yields an independent basis.[9] The projection then corresponds simply to setting to zero the first $2n-n'$ components of any vector in this basis

$$X_{i\alpha} \to \hat{X}_{i\alpha} \quad \text{where} \quad \hat{X}^{\mathcal{A}}_{i\alpha} = \begin{cases} 0 & \mathcal{A} = 1, \dots, 2n-n' \\ X^{\mathcal{A}}_{i\alpha} & \mathcal{A} = 2n{-}n'{+}1, \dots, n{+}k{+}4 \end{cases} . \tag{5.24}$$

We will have reduced brackets on the projected $k'+4$ dimensional space spanned by $\{e_1, \dots, e_{k'+4}\}$

$$\langle \hat{\mathcal{X}} \rangle := \langle Y_1 \dots Y_n X_{n'+1} \dots X_n \mathcal{X} \rangle . \tag{5.25}$$

Here $\mathcal{X}$ represents any collection of $k'+4$ independent vectors, and $\hat{\mathcal{X}}$ the same vectors projected onto the hyperplane.

As in the maximal case we define

$$Z_i := \sigma_i . X_i = \tau_i . X_{i+1} + Y_i \quad i = 1 \dots n' \tag{5.26}$$

and after projection this implies

$$\hat{Z}_i := \sigma_i . \hat{X}_i = \tau_i . \hat{X}_{i+1} \quad i = 1 \dots n' , \tag{5.27}$$

the projected planes $\hat{X}_i$ intersect each other consecutively at $\hat{Z}_i$ in the projected space.

If we choose coordinates such that $\tau_{i-1}.\sigma_i > 0$ for all $i = 1 \dots n'$ and $\sigma_{n'+2a-1}.\sigma_{n'+2a} > 0$ for all $a = 1 \dots n$ (ie make a choice of orientation for the projection planes) then the correlahedron region becomes

$$\langle Y X_i X_j \rangle > 0 \to \begin{cases} \langle \hat{Y} \hat{Z}_{i-1} \hat{Z}_i \hat{Z}_{j-1} \hat{Z}_j \rangle > 0 & i,j \in \{1, \dots, n'\} \\ \langle \hat{Y} \mathcal{L}_{i-n'} \hat{Z}_{j-1} \hat{Z}_j \rangle > 0 & j \in \{1, \dots, n'\} \qquad i \in \{n'{+}1, \dots, n\} \\ \langle \hat{Y} \mathcal{L}_{j-n'} \hat{Z}_{i-1} \hat{Z}_i \rangle > 0 & i \in \{1, \dots, n'\} \qquad j \in \{n'{+}1, \dots, n\} \\ \langle \hat{Y} \mathcal{L}_{i-n'} \mathcal{L}_{j-n'} \rangle > 0 & i,j \in \{n'{+}1, \dots, n\} \end{cases} \tag{5.28}$$

This region is precisely the loop level squared amplituhedron region (3.4).

5.4 The non-maximal limit on the hedron expressions

As in the maximal case, the "freeze and project" procedure can be applied directly on the correlahedron form also for the non-maximal limit. The procedure in the maximal case was given in section 5.2 and the non-maximal case is very similar. When we freeze Y as in (5.21) we get

$$\langle Y d^4 Y_i \rangle = \langle Y X_i X_{i+1} \rangle d^2 \sigma_i d^2 \tau_i \qquad i = 1 \dots n' . \tag{5.29}$$

We perform the projection on the differential form by the map

$$X_{i\alpha} \to \frac{-\tau_{i-1\,\alpha} \hat{Z}_i + \sigma_{i\,\alpha} (\hat{Z}_{i-1} + Y_{i-1})}{\tau_{i-1} . \sigma_i} \quad i = 1 \dots n' . \tag{5.30}$$

[9] As in the maximal case, geometrically the span of the e_i gives a hyperplane onto which we are projecting the quotient.

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The correlahedron form is then reduced to the amplituhedron form by setting Y as in (5.21) and $X_{i\alpha}$ to (5.30) and finally leaving out the σ,τ dependent pieces:

$$
\begin{aligned}
&\prod_{i=1}^{n+k}\langle Y d^4 Y_i\rangle \times \frac{g_{n;k}(Y,X_i)}{\prod_{i=1}^{n'}\langle Y X_i X_{i+1}\rangle}\\
&\to \left(\prod_{i=1}^{n'} d^2\sigma_i d^2\tau_i\right)\left(\prod_{i=1}^{n-n'}\prod_{\alpha=1}^{2} d^2\sigma_{n'+2i-2+\alpha}\langle Y X_{n'+i} d^2\mathcal{L}_{i\alpha}\rangle\right)\left(\prod_{i=1}^{k'}\langle \hat{Y} d^4\hat{Y}_i\rangle\right) g_{n;k}\Big(Y,X_{i\alpha}\Big)\\
&\to \left(\prod_{i=1}^{n}\frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right)\left(\prod_{i=1}^{n-n'}\frac{d^2\sigma_{n'+2i-1}d^2\sigma_{n'+2i}}{(\sigma_{n'+2i-1}.\sigma_{n'+2i})^2}\prod_{\alpha=1}^{2}\langle Y_1\dots Y_{n'}X_{n'+1}\dots X_n\hat{Y}\mathcal{L}_i d^2\mathcal{L}_{i\alpha}\rangle\right)\\
&\quad\times\left(\prod_{i=1}^{k'}\langle \hat{Y} d^4\hat{Y}_i\rangle\right) a^{(n-n')}_{n;k}\\
&\to \left(\prod_{i=1}^{n-n'}\prod_{\alpha=1}^{2}\langle \hat{Y}\mathcal{L}_i d^2\mathcal{L}_{i\alpha}\rangle\right)\left(\prod_{i=1}^{k'}\langle \hat{Y} d^4\hat{Y}_i\rangle\right) a^{(n-n')}_{n;k}(\hat{Y},\hat{Z}_i,\mathcal{L}_i)
\end{aligned}
\tag{5.31}
$$

We proceeded in three stages. To get the second line we replaced Y with (5.21) and (5.22) to get the third line we replaced $X_{i\alpha}$ with (5.30) and defined $a^{(n-n')}_{n;k}(\hat{Y},\hat{Z}_i,\mathcal{L}_i)$ which should correspond to the square of the amplitude. We also used that

$$
\langle Y X_{n'+i} d^2\mathcal{L}_{i\alpha}\rangle = \langle Y_1\dots Y_{n'}X_{n'+1}\dots X_n\hat{Y}\mathcal{L}_i d^2\mathcal{L}_{i\alpha}\rangle\,. \tag{5.32}
$$

We claim that the precise dependence on σ,τ always has the factorised form of the third line ie

$$
g_{n;k}\Big(Y,X_{i\alpha}\Big)\to\left(\prod_{i=1}^{n'}\frac{1}{(\tau_{i-1}.\sigma_i)^2}\right)\left(\prod_{i=1}^{n-n'}\frac{1}{(\sigma_{n'+2i-1}.\sigma_{n'+2i})^2}\right) a^{(n-n')}_{n;k}(\hat{Y},\hat{Z}_i,\mathcal{L}_i)\,, \tag{5.33}
$$

which can be seen as a consequence of the duality.

Consider for example the *four*-point light-like limit of the five-point correlahedron $G_{5;1}$. We have $Y=Y_1\wedge\cdots\wedge Y_6\ \in\ \mathrm{Gr}(6,10)$ and we freeze $Y_1,\dots,Y_4$ to $Y_i=\sigma^\alpha X_{i\alpha}-\tau^\alpha X_{i+1\,\alpha}$, as in (5.20), leaving Y_5,Y_6 free, which we gauge fix as $Y_5=\mathcal{L}_{1\,1}+\sigma_5^\alpha X_{5\,\alpha}$, $Y_6=\mathcal{L}_{1\,2}+\sigma_6^\alpha X_{5\,\alpha}$ (5.21). The projection means we replace (5.30) $X_{i\alpha}\to\frac{-\tau_{i-1\,\alpha}\hat{Z}_i+\sigma_{i\,\alpha}(\hat{Z}_{i-1}+Y_{i-1})}{\tau_{i-1}.\sigma_i}$ $i=1\dots4$. Then

$$
\begin{aligned}
&\prod_{i=1}^{6}\langle Y d^4 Y_i\rangle\frac{\langle X_1X_2X_3X_4X_5\rangle^4}{\langle YX_1X_2\rangle\dots\langle YX_4X_5\rangle}\\
&\xrightarrow[\text{project}X]{\text{freeze }Y}\left(\prod_{i=1}^{4}\frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right)\frac{d^2\sigma_5 d^2\sigma_6}{(\sigma_5.\sigma_6)^2}\frac{\langle Y_1\dots Y_4X_5\mathcal{L}_1d^2\mathcal{L}_{1\,1}\rangle\langle Y_1\dots Y_4X_5\mathcal{L}_1d^2\mathcal{L}_{1\,2}\rangle\langle Y_1\dots Y_4X_5\hat{Z}_1\dots\hat{Z}_4\rangle^4}{\langle Y_1\dots Y_4X_5\hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\rangle^2\prod_{i=1}^{4}\langle Y_1\dots Y_4X_5\mathcal{L}_1\hat{Z}_{i-1}\hat{Z}_i\rangle}\\
&=\left(\prod_{i=1}^{4}\frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2}\right)\frac{d^2\sigma_5 d^2\sigma_6}{(\sigma_5.\sigma_6)^2}\frac{\langle\mathcal{L}_1d^2\mathcal{L}_{1\,1}\rangle\langle\mathcal{L}_1d^2\mathcal{L}_{1\,2}\rangle\langle\hat{Z}_1\dots\hat{Z}_4\rangle^2}{\prod_{i=1}^{4}\langle\mathcal{L}_1\hat{Z}_{i-1}\hat{Z}_i\rangle}\\
&\qquad\downarrow\\
&\frac{\langle\mathcal{L}_1d^2\mathcal{L}_{1\,1}\rangle\langle\mathcal{L}_1d^2\mathcal{L}_{1\,2}\rangle\langle\hat{Z}_1\dots\hat{Z}_4\rangle^2}{\langle\mathcal{L}_1\hat{Z}_1\hat{Z}_2\rangle\langle\mathcal{L}_1\hat{Z}_2\hat{Z}_3\rangle\langle\mathcal{L}_1\hat{Z}_3\hat{Z}_4\rangle\langle\mathcal{L}_1\hat{Z}_4\hat{Z}_1\rangle}
\end{aligned}
\tag{5.34}
$$

Here we used

$$
\begin{aligned}
\langle Y d^4 Y_i \rangle &= \langle Y X_i X_{i+1} \rangle d^2\sigma_i d^2\tau_i \qquad i = 1 \dots 4 \text{ (cyclically)} \\
\langle Y d^4 Y_5 \rangle &= d^2\sigma_5 \langle Y_1 \dots Y_4 X_5 \mathcal{L}_1 d^2 \mathcal{L}_{1\,1} \rangle \\
\langle Y d^4 Y_6 \rangle &= d^2\sigma_6 \langle Y_1 \dots Y_4 X_5 \mathcal{L}_1 d^2 \mathcal{L}_{1\,2} \rangle
\end{aligned}
\tag{5.35}
$$

and notice that $\langle Y X_i X_{i+1} \rangle$ cancels four terms of the denominator. Also in the first line we used

$$
\langle X_1 X_2 X_3 X_4 X_5 \rangle \to \langle Y_1 \dots Y_4 X_5 \hat{Z}_1 \dots \hat{Z}_4 \rangle \prod_{i=1}^{4} (\tau_i . \sigma_{i+1})^{-1} \tag{5.36}
$$

after the projection (5.30).

The result (5.34) is precisely the one-loop four-point amplituhedron form.

Just as in the maximal case we again note that it is easier to consider the functions without the measures (which are also much closer to the actual correlator/amplitude expressions). Also we can then make particular choices for the σ_i, τ_i for example $\sigma_i = (1,0)$, $\tau_i = (0,1)$. Then the lightlike limit takes the correlahedron expression $g_{n;k}(Y, X_i)$ to the amplitude expression $g(\hat{Y}, \hat{Z}_i)$ via (5.33)

$$
g_{n;k}(Y, X_i) \xrightarrow[X_{i1} \to \hat{Z}_{i-1} + Y_{i-1},\, X_{i2} \to -\hat{Z}_i, (i=1\dots n')\,, Y_{2n-n'+i} = \hat{Y}_i (i=1\dots k')]{Y_i = X_{i1} - X_{i+1\,2} (i=1\dots n'), Y_{n'+2i-2+\alpha} = \mathcal{L}_{1\,\alpha} + \sigma^{\alpha}_{n'+2i-2+\alpha} X_{n'+i\,\alpha}, (i=1\dots n-n')} a^{(n-n')}_{n;k}(\hat{Y}, \hat{Z}_i, \mathcal{L}_i). \tag{5.37}
$$

We give a highly non-trivial example in the non-maximal lightlike limit case in the appendix where we consider the five-point lightlike limit of the six point correlator $G_{6;1}$ and show that it correctly reproduces the five-point one-loop amplitude.

6 Hedron expressions from hedron geometry

We have introduced the correlahedron as a geometric object in $\mathrm{Gr}(k{+}n, k{+}n{+}4)$. We have also shown how to translate explicit expressions for the correlator in analytic superspace to invariant differential forms on $\mathrm{Gr}(k{+}n, k{+}n{+}4)$. The question we wish to address in this section is the direct relation between the correlahedron geometry and the corresponding differential form. We will only give a tentative answer to this question here, leaving further developments to future work. Working towards this however we first concentrate on the analogous issue for the squared amplituhedron. For the amplituhedron itself a prescription for obtaining the amplitude from the geometry was defined in [9]. To obtain the amplitude from the amplituhedron it was conjectured that one takes the volume form with no divergences inside the amplituhedron and logarithmic divergences on the boundary. This defines a volume form on amplituhedron space which is equivalent to the bosonised amplitude. We take exactly the same prescription here for the squared amplituhedron. Furthermore, due to the more explicit description of the squared amplituhedron, we are able to give a simple computerisable algorithm (via cylindrical decomposition) for obtaining this volume form.

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6.1 Practical algorithm for obtaining the hedron form from the hedron region

The amplitu-/correla-hedron is described geometrically as a subspace of a Grassmannian space. In order to relate this to an amplitude or correlator one has to obtain a differential form from this geometry. For the squared amplituhedron this is the unique form which has logarithmic divergences on the boundary of the amplituhedron space and no divergences inside the space. Here we describe a simple algorithm for obtaining the form from the region.

The first step is to obtain a cylindrical decomposition of the region. A cylindrical decomposition of any subset of R^n describes it as a union of regions with the form

$$\left\{ (x_1,\dots,x_n): \begin{array}{c} a < x_1 < b, \\ a(x_1) < x_2 < b(x_1), \\ a(x_1,x_2) < x_3 < b(x_1,x_2), \\ \dots, \\ a(x_1,\dots,x_{n-1}) < x_n < b(x_1,\dots,x_{n-1}) \end{array} \right\}, \tag{6.1}$$

ie each variable is restricted to an interval which depends on the previous variables.

This is exactly the description of a region one needs to perform an integration over the region as a multiple integral. Here however instead of integrating over this region one assigns a differential form to it by assigning to each inequality a dlog:

$$a(x_1,\dots,x_{i-1}) < x_i < b(x_1,\dots,x_{i-1}) \quad \to \quad d\log\left(\frac{x_i - b(x_1,\dots,x_{i-1})}{x_i - a(x_1,\dots,x_{i-1})}\right) \tag{6.2}$$

thus yielding the n-form

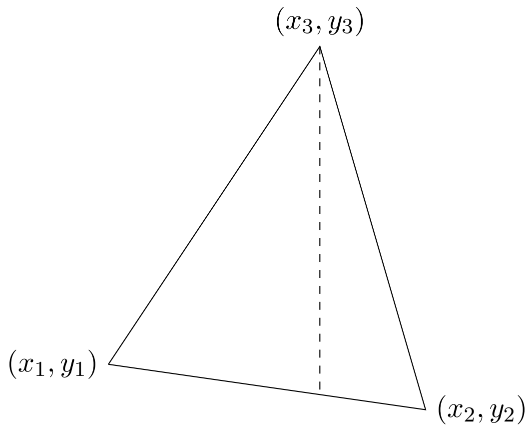
$$\prod_{i=1}^{n} \frac{dx_i\Big(b(x_1,\dots x_{i-1}) - a(x_1,\dots x_{i-1})\Big)}{\Big(x_i - b(x_1,\dots x_{i-1})\Big)\Big(x_i - a(x_1,\dots x_{i-1})\Big)}\,. \tag{6.3}$$

One then simply adds together the contributions from each region. This gives a form with log divergences on each boundary and no divergences inside (as long as the original region is convex).

We here describe this process through the simplest example. We consider the case of a triangle in P^2 with vertices Z_1, Z_2, Z_3. We give them inhomogeneous coordinates $Z_i = (x_i, y_i, 1)$. The region (inside of the triangle) is the space of $Y \in P^2$ such that

$$\langle Y Z_1 Z_2\rangle > 0, \quad \langle Y Z_2 Z_3\rangle > 0, \quad \langle Y Z_3 Z_1\rangle > 0\,. \tag{6.4}$$

Let us also give Y inhomogeneous coordinates $Y = (x, y, 1)$ the region becomes



(6.5)

and can be written as the sum of two regions

$$
\begin{aligned}
&\frac{xy_1 - x_2y_1 - xy_2 + x_1y_2}{x_1 - x_2} < y < \frac{xy_1 - x_3y_1 - xy_3 + x_1y_3}{x_1 - x_3} \text{ and } x_1 < x < x_3 \\
&\frac{xy_1 - x_2y_1 - xy_2 + x_1y_2}{x_1 - x_2} < y < \frac{xy_2 - x_3y_2 - xy_3 + x_2y_3}{x_2 - x_3} \text{ and } x_3 < x < x_2\,.
\end{aligned}
\tag{6.6}
$$

So the differential form corresponding to the above region becomes

$$
\begin{aligned}
&d\log\left(\frac{y-\frac{xy_1-x_3y_1-xy_3+x_1y_3}{x_1-x_3}}{y-\frac{xy_1-x_2y_1-xy_2+x_1y_2}{x_1-x_2}}\right)\wedge d\log\left(\frac{x-x_3}{x-x_1}\right)+d\log\left(\frac{y-\frac{xy_2-x_3y_2-xy_3+x_2y_3}{x_2-x_3}}{y-\frac{xy_1-x_2y_1-xy_2+x_1y_2}{x_1-x_2}}\right)\wedge d\log\left(\frac{x-x_2}{x-x_3}\right)\\
&=\frac{dxdy\,(x_2y_1-x_3y_1-x_1y_2+x_3y_2+x_1y_3-x_2y_3)^2}{(x_1y-x_1y_2-x_2y-xy_1+x_2y_1+xy_2)(x_1y-x_1y_3-x_3y-xy_1+x_3y_1+xy_3)(x_2y-x_2y_3-x_3y-xy_2+x_3y_2+xy_3)}\\
&=\frac{\langle Y d^2Y\rangle\langle Z_1Z_2Z_3\rangle^2}{\langle YZ_1Z_2\rangle\langle YZ_2Z_3\rangle\langle YZ_3Z_1\rangle}\,.
\end{aligned}
\tag{6.7}
$$

To get the second line we simply applied the differential and factorised the result and to obtain the third line we simply rewrote back in homogeneous coordinates. The final result is the 2-form associated with the triangle (see eg [9].)

The above method can be applied more generally and importantly can be simply implemented using a computer algebra programme (for numeric external vertices at least). For example in `mathematica` one can apply the command `CylindricalDecomposition[]` to convert any set of inequalities into the form of a sum of regions upon which we can implement the simple rule (6.2).

In the next two subsections we illustrate this procedure in a number of tree and loop examples.

6.2 Tree level squared amplituhedron examples

6.2.1 Five-point NMHV amplitude

We begin with the simplest physical example, 5 point tree-level. The external data is given by five points, $Z_1, \dots Z_5$, in P^4 and we obtain the geometrical amplituhedron squared region as $Y \in P^4$ subject to

$$\langle Y Z_i Z_{i+1} Z_{i+2} Z_{i+3} \rangle > 0\,. \tag{6.8}$$

This region arises directly from (5.28).

To make this concrete introduce coordinates by $Y = y_1 Z_1 + y_2 Z_2 + y_3 Z_3 + y_4 Z_4 + Z_5$ so the region becomes simply

$$y_1, y_2, y_3, y_4 > 0 \tag{6.9}$$

and the corresponding differential form is then trivially

$$\frac{dy_1 dy_2 dy_3 dy_4}{y_1 y_2 y_3 y_4}\,. \tag{6.10}$$

Finally we can covariantise this differential form to the coordinate independent form

$$\frac{\langle Y d^4 Y \rangle \langle 12345 \rangle^4}{\langle Y1234 \rangle \langle Y2345 \rangle \langle Y3451 \rangle \langle Y4512 \rangle \langle Y5123 \rangle}\,. \tag{6.11}$$

This correctly reproduces the known amplitude (as a form in amplituhedron space). Note that in this case the description is entirely equivalent to the amplituhedron itself (as compared to the squared amplituhedron). We note here that if one instead had a different orientation for one of the $X_i = Z_{i-1} \wedge Z_i$ then although the region (6.8) would be different, the resulting differential form would be the same. For example imagine that instead of $Z_4 \wedge Z_5$ we had the reverse order $Z_5 \wedge Z_4$ with all other edges having the same orientation. Then the corresponding region in P^4 would be defined by $\langle Y Z_2 Z_3 Z_5 Z_4 \rangle > 0$ and $\langle Y Z_5 Z_4 Z_1 Z_2 \rangle > 0$, but with all other inequalities the same. In coordinates we would have $x_2, x_4 > 0$ as before, but this time $x_1 x_3 < 0$. However the resulting form (6.10) is the same.

6.2.2 Six-point NNMHV

The next case, six-point NNMHV is more interesting. We consider the external points $Z_1, \ldots, Z_6 \in P^5$ and the subspace of the Grassmannian of 2-planes $Y = Y_1 \wedge Y_2 \in \mathrm{Gr}(2,6)$ defined by the inequalities

$$\langle Y\, i\, i{+}1\, j\, j{+}1 \rangle > 0\,. \tag{6.12}$$

Note that this is a weaker requirement than that of the amplituhedron which requires all ordered minors in the matrix defining Y to be positive (ie it requires additional constraints such as $\langle Y1235 \rangle > 0$).

Again then to obtain the differential form from this, we first coordinatise the Ys, letting

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b & 0 & c & d \\ 0 & e & f & 1 & g & h \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_6 \end{pmatrix} \tag{6.13}$$

In these coordinates the inequalities (6.12) become

$$e>0, h>0, be-af>0, b>0, -df+bh>0, ce-ag>0, c>0, -dg+ch>0\,. \tag{6.14}$$

Performing a cylindrical decomposition of these inequalities in the order e, h, c, b, g, f, a, d (which seems to give the simplest result — the final answer for the differential form does not depend on this order) gives a description of the region as

$$\begin{aligned}
&e>0\wedge h>0\wedge c>0\wedge b>0\wedge\\
&\left(\left(g<0\wedge\left(\left(f<\frac{bg}{c}\wedge a>\frac{be}{f}\wedge d>\frac{bh}{f}\right)\vee\left(\frac{bg}{c}<f<0\wedge a>\frac{ce}{g}\wedge d>\frac{ch}{g}\right)\right.\right.\right.\\
&\left.\left.\vee\left(f>0\wedge\frac{ce}{g}<a<\frac{be}{f}\wedge\frac{ch}{g}<d<\frac{bh}{f}\right)\right)\right)\\
&\vee\left(g>0\wedge\left(\left(f<0\wedge\frac{be}{f}<a<\frac{ce}{g}\wedge\frac{bh}{f}<d<\frac{ch}{g}\right)\vee\left(0<f<\frac{bg}{c}\wedge a<\frac{ce}{g}\wedge d<\frac{ch}{g}\right)\right.\\
&\left.\left.\left.\vee\left(f>\frac{bg}{c}\wedge a<\frac{be}{f}\wedge d<\frac{bh}{f}\right)\right)\right)\right)
\end{aligned} \tag{6.15}$$

which on performing the replacement (6.2) gives the remarkably simple differential form

$$\frac{2(-abgh-acfh+adfg+3bceh-bdeg-cdef)}{bceh(be-af)(ce-ag)(bh-df)(ch-dg)}da\wedge\cdots\wedge dh\,. \tag{6.16}$$

This lifts into the covariant form

$$\frac{2\langle Yd^4Y_1\rangle\langle Yd^4Y_2\rangle\Big(\langle Y3456\rangle\langle Y2361\rangle\langle Y1245\rangle+\text{ cyclic}\Big)\langle 123456\rangle^4}{\langle Y1245\rangle\langle Y2356\rangle\langle Y3461\rangle\prod_{i=1}^{6}\langle Y\,i\,i{+}1\,i{+}2\,i{+}3\rangle}\,. \tag{6.17}$$

We will discuss the interpretation of this in a moment but first let us check what happens if we switch the orientation of one of the edges. Specifically, we replace the edge $Z_6\wedge Z_1$ with $Z_1\wedge Z_6$. This swaps the inequality of three of the brackets in (6.12): $\langle Y2316\rangle>0$, $\langle Y3416\rangle>0$, $\langle Y4516\rangle>0$. But unlike the NMHV case (where this made no difference to the final differential form) here these swaps of signs make an enormous difference.

Proceeding as in the previous case, with the same coordinates, the inequalities become

$$e>0, h>0, -be+af>0, -ce+ag>0, b>0, -df+bh>0, -c>0, -dg+ch>0 \tag{6.18}$$

and a cylindrical decomposition becomes even simpler:

$$\begin{aligned}
&e>0\wedge h>0\wedge c<0\wedge b>0\wedge\\
&\left(\left(g<0\wedge f<0\wedge a<\frac{be}{f}\wedge d>\frac{ch}{g}\right)\vee\left(g>0\wedge f>0\wedge a>\frac{be}{f}\wedge d<\frac{ch}{g}\right)\right)
\end{aligned} \tag{6.19}$$

yielding the differential form

$$\frac{2da\wedge\cdots\wedge dh}{bceh(be-af)(ch-dg)}\,, \tag{6.20}$$

which in turn covariantises to

$$\frac{2\langle Y d^4 Y_1\rangle\langle Y d^4 Y_2\rangle\langle 123456\rangle^4}{\prod_{i=1}^{6}\langle Y\, i\, i{+}1\, i{+}2\, i{+}3\rangle}\,. \tag{6.21}$$

So in this case we thus obtain two different answers depending on the orientation of the edges. In fact remarkably both answers have a physical meaning. The result arising from the cyclic choice of orientation (6.17) corresponds to the square of the NMHV amplitude $(NMHV_6)^2$ whereas the result from the non-cyclic ordering yields (twice) the NNMHV amplituhedron. The lightlike limit of the correlahedron yields the sum of these two terms. Furthermore we find that all other choices of orientations for the edges yield the same results: an odd number of edge flips yields the amplituhedron, an even number yields $(\mathrm{NMHV})^2$. Given this result it is natural to conjecture that in all cases the correlahedron is the average of all possible orientations of the edges.

6.2.3 Seven-point N^{3}MHV

As a final tree-level example we consider the seven-point N^{3}MHV amplitude, described as a subspace of $\mathrm{Gr}(3,7)$ with the external data Z_i living in P^6. The subspace is defined as the set $Y = Y_1 \wedge Y_2 \wedge Y_3 \in \mathrm{Gr}(3,7)$ such that

$$\langle Y\, i\, i{+}1\, j\, j{+}1\rangle > 0\,. \tag{6.22}$$

Employing the same procedure as previously, we coordinatise $\mathrm{Gr}(3,7)$ as

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & a & b & 0 & c & d & 0 \\ 0 & e & f & 1 & g & h & 0 \\ 0 & i & j & 0 & k & l & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_7 \end{pmatrix} \tag{6.23}$$

and then perform a cylindrical decomposition of the region (6.22) in these variables and then convert the result into a differential form according to (6.2). Remarkably the result is precisely the lightlike limit of the 7 point correlator, or equivalently the square of the amplitude $2N^3MHV_7 + 2NMHV_7 N^2MHV_7$. Explicitly it can be written in correlahedron space as

$$\begin{aligned}&\langle Y d^4 Y_1\rangle\langle Y d^4 Y_2\rangle\langle Y d^4 Y_3\rangle\langle 1234567\rangle^4 \\ &\times\left(\frac{\langle Y7123\rangle}{\langle Y1234\rangle\langle Y1267\rangle\langle Y2345\rangle\langle Y2356\rangle\langle Y2367\rangle\langle Y7134\rangle\langle Y7145\rangle\langle Y7156\rangle}+\dots\right).\end{aligned} \tag{6.24}$$

Here the first (displayed) term is the contribution of the N^{3}MHV amplitude and the dots denote the contributions from the product amplitudes $NMHV_7 N^2MHV_7$. The full expression is most compactly written as the lightlike limit of the correlator, which is the S_7 permutation of a single term. So the bit in brackets in (6.24) can be written

$$\frac{\lim_{x^2_{i\,i+1}\to 0}(x^4_{12}x^2_{34}x^2_{45}x^2_{56}x^2_{67}x^2_{37} \;+\; S_7 \text{ permutations})}{\prod_{i=1}^{7} x^2_{i\,i+2}x^2_{i\,i+3}} \qquad \text{with} \quad x^2_{ij} \to \langle Y\, i{-}1\, i\, j{-}1\, j\rangle\,. \tag{6.25}$$

It is remarkable that this expression arises very simply from the constraints (6.22). Note that unlike the N^{2}MHV case this single choice of edge orientation gives the full answer. Flipping the orientation of one or more of the edges yields exactly the same result in this case.

6.3 Loop level squared amplituhedron examples

As further illustration we now consider some loop level examples where again the cylindrical decomposition procedure correctly reproduces the squared amplitude.

6.3.1 Four-point one-loop

Here we have external twistors $Z_i \in P^3$ and the set of $\mathcal{L} = \mathcal{L}_1 \wedge \mathcal{L}_2 \in \mathrm{Gr}(2,4)$ subject to

$$\langle \mathcal{L}12\rangle > 0,\ \langle \mathcal{L}23\rangle > 0,\ \langle \mathcal{L}34\rangle > 0,\ \langle \mathcal{L}41\rangle > 0\,. \tag{6.26}$$

Putting coordinates for $\mathcal{L}$ as

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_4 \end{pmatrix} \tag{6.27}$$

This yields the differential form

$$\frac{2da \wedge db \wedge dc \wedge dd}{ad(ad-bc)} \tag{6.28}$$

which lifts to

$$\frac{2\langle \mathcal{L}d^2\mathcal{L}_1\rangle\langle \mathcal{L}d^2\mathcal{L}_2\rangle\langle 1234\rangle^2}{\langle \mathcal{L}12\rangle\langle \mathcal{L}23\rangle\langle \mathcal{L}34\rangle\langle \mathcal{L}41\rangle}\,. \tag{6.29}$$

6.3.2 Four-point two-loop

Here we have external twistors $Z_i \in P^3$ and the set of $\mathcal{L} = \mathcal{L}_1 \wedge \mathcal{L}_2 \in \mathrm{Gr}(2,4)$ and $\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \in \mathrm{Gr}(2,4)$ subject to

$$\begin{aligned} &\langle \mathcal{L}12\rangle > 0,\quad \langle \mathcal{L}23\rangle > 0,\quad \langle \mathcal{L}34\rangle > 0,\quad \langle \mathcal{L}41\rangle > 0 \\ &\langle \mathcal{M}12\rangle > 0,\ \langle \mathcal{M}23\rangle > 0,\ \langle \mathcal{M}34\rangle > 0,\ \langle \mathcal{M}41\rangle > 0,\ \langle \mathcal{L}\mathcal{M}\rangle > 0\,. \end{aligned} \tag{6.30}$$

Putting coordinates for $\mathcal{L}$ and $\mathcal{M}$ as

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_4 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & e & f \\ 0 & 1 & g & h \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_4 \end{pmatrix}. \tag{6.31}$$

This yields the differential form (obtained as in the previous cases by writing the inequalities (6.30) in terms of the coordinates (6.31), obtaining a cylindrical decomposition of this region, and then making the replacement (6.2))

$$\frac{2da \wedge \cdots \wedge dh(2ad-2bc+bg+cf+2eh-2fg)}{adeh(ad-bc)(eh-fg)(ad-ah-bc+bg+cf-de+eh-fg)}\,. \tag{6.32}$$

This lifts to

$$\begin{aligned}
&2\langle \mathcal{L}d^2\mathcal{L}_1\rangle\langle \mathcal{L}d^2\mathcal{L}_2\rangle\langle \mathcal{M}d^2\mathcal{M}_1\rangle\langle \mathcal{M}d^2\mathcal{M}_2\rangle\langle 1234\rangle^3 \\
&\times\left(\frac{1}{\langle \mathcal{L}23\rangle\langle \mathcal{L}34\rangle\langle \mathcal{L}41\rangle\langle \mathcal{M}12\rangle\langle \mathcal{M}23\rangle\langle \mathcal{M}41\rangle\langle \mathcal{L}\mathcal{M}\rangle}+\frac{1}{\langle \mathcal{L}12\rangle\langle \mathcal{L}34\rangle\langle \mathcal{L}41\rangle\langle \mathcal{M}12\rangle\langle \mathcal{M}23\rangle\langle \mathcal{M}34\rangle\langle \mathcal{L}\mathcal{M}\rangle}\right.\\
&\quad+\frac{1}{\langle \mathcal{L}12\rangle\langle \mathcal{L}23\rangle\langle \mathcal{L}34\rangle\langle \mathcal{M}12\rangle\langle \mathcal{M}41\rangle\langle \mathcal{M}34\rangle\langle \mathcal{L}\mathcal{M}\rangle}+\frac{1}{\langle \mathcal{L}12\rangle\langle \mathcal{L}23\rangle\langle \mathcal{L}41\rangle\langle \mathcal{M}23\rangle\langle \mathcal{M}41\rangle\langle \mathcal{M}34\rangle\langle \mathcal{L}\mathcal{M}\rangle}\\
&\quad\left.+\frac{\langle 1234\rangle}{\langle \mathcal{L}12\rangle\langle \mathcal{L}23\rangle\langle \mathcal{L}34\rangle\langle \mathcal{L}41\rangle\langle \mathcal{M}12\rangle\langle \mathcal{M}23\rangle\langle \mathcal{M}41\rangle\langle \mathcal{M}34\rangle}\right). \qquad (6.33)
\end{aligned}$$

Here we recognise both the square of the one-loop amplitude (last term) as well as the two loop amplitude (first four terms which are all double boxes). The full expression is precisely the result of taking the lightlike limit of the correlator, ie the square of the four-point amplitude at second order in perturbation theory.

6.3.3 Five-point one-loop

Here we have external twistors $Z_i \in P^4$, the loop 2-plane $\mathcal{L} = \mathcal{L}_1 \wedge \mathcal{L}_2 \in \mathrm{Gr}(2,5)$ as well as $Y \in P^4$. Y and $\mathcal{L}$ satisfy the following inequalities

$$\begin{aligned}
&\langle \mathcal{L}Y12\rangle>0,\ \langle \mathcal{L}Y23\rangle>0,\ \langle \mathcal{L}Y34\rangle>0,\ \langle \mathcal{L}Y45\rangle>0,\ \langle \mathcal{L}Y51\rangle>0\\
&\langle Y1234\rangle>0,\ \langle Y2345\rangle>0,\ \langle Y3451\rangle>0,\ \langle Y4512\rangle>0,\ \langle Y5123\rangle>0\,. \qquad (6.34)
\end{aligned}$$

Putting coordinates for $\mathcal{L}$ and Y as

$$\begin{pmatrix}\mathcal{L}_1\\ \mathcal{L}_2\end{pmatrix}=\begin{pmatrix}1&0&a&b&0\\0&1&c&d&0\end{pmatrix}\begin{pmatrix}Z_1\\ \vdots\\ Z_5\end{pmatrix},\qquad Y=\begin{pmatrix}e&f&1&g&h\end{pmatrix}\begin{pmatrix}Z_1\\ \vdots\\ Z_5\end{pmatrix}, \qquad (6.35)$$

the inequalities (6.34) lead to the differential form

$$-\frac{2adef-2aeg-2bcef+be-cfg+df+2g}{defgh(ad-bc)(ae+cf-1)(adf-ag+b(-c)f+b)}da\wedge db\wedge\cdots\wedge dh\,. \qquad (6.36)$$

This lifts to the co-ordinate independent form

$$\begin{aligned}
&\frac{\langle \mathcal{L}Yd^2\mathcal{L}_1\rangle\langle \mathcal{L}Yd^2\mathcal{L}_2\rangle\langle Yd^4Y\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}\\
&\times\left(\frac{\langle 1234Y\rangle\langle 2345Y\rangle}{\langle \mathcal{L}Y12\rangle\langle \mathcal{L}Y23\rangle\langle \mathcal{L}Y34\rangle\langle \mathcal{L}Y45\rangle}+\frac{\langle 5134Y\rangle\langle 2345Y\rangle}{\langle \mathcal{L}Y23\rangle\langle \mathcal{L}Y34\rangle\langle \mathcal{L}Y45\rangle\langle \mathcal{L}Y51\rangle}\right.\\
&\quad+\frac{\langle 1234Y\rangle\langle 5123Y\rangle}{\langle \mathcal{L}Y12\rangle\langle \mathcal{L}Y23\rangle\langle \mathcal{L}Y34\rangle\langle \mathcal{L}Y51\rangle}+\frac{\langle 1245Y\rangle\langle 5123Y\rangle}{\langle \mathcal{L}Y12\rangle\langle \mathcal{L}Y23\rangle\langle \mathcal{L}Y45\rangle\langle \mathcal{L}Y51\rangle}\\
&\quad\left.+\frac{\langle 1245Y\rangle\langle 5134Y\rangle}{\langle \mathcal{L}Y12\rangle\langle \mathcal{L}Y34\rangle\langle \mathcal{L}Y45\rangle\langle \mathcal{L}Y51\rangle}\right). \qquad (6.37)
\end{aligned}$$

Here we recognise the sum of five box functions (which is the parity even part of the one-loop amplitude) multiplied by the tree-level NMHV amplitude. This is precisely what we expect: the square of the superamplitude at first non-trivial order in both coupling and the Grassmann odd variable expansion is

$$\left(\frac{A^{(0)}_{\rm MHV}+A^{(0)}_{\rm NMHV}+aA^{(1)}_{\rm MHV}+aA^{(1)}_{\rm NMHV}+\dots}{A^{(0)}_{\rm MHV}}\right)^2\Big|_{a^1,\eta^4}=\frac{2A^{(0)}_{\rm MHV}A^{(1)}_{\rm NMHV}+A^{(0)}_{\rm NMHV}A^{(1)}_{\rm MHV}}{\left(A^{(0)}_{\rm MHV}\right)^2}$$
$$=2\frac{A^{(0)}_{\rm NMHV}}{A^{(0)}_{\rm MHV}}\left(M^{(1)}_{\rm NMHV}+M^{(1)}_{\rm MHV}\right), \tag{6.38}$$

where we define $M^{(\ell)}$ to be the loop level amplitude divided by tree-level amplitude of the same helicity structure.

6.4 Obtaining the correlator from the correlahedron

We now arive at the question of how to obtain the correlator from the correlahedron geometry. The obvious method is to attempt the same procedure successfully implemented above for the closely related squared amplituhedron, namely take the unique differential volume form on amplituhedron space with log divergences on the boundary. There are two problems with this. The first problem is purely practical in that the simplest example, the five-point NMHV correlator $G_{5;1}$ is already far too high dimensional for the cylindrical decomposition procedure to give a result (this procedure is doubly exponential in the number of dimensions which is $4(k+n)=24$ in this case). The second problem however is of a more serious nature since it suggests that such a naive implementation of the log divergence criterion does not even apply straightforwardly in this case. The problem is that the correlator apparently can have double poles on the boundary, unlike the amplitude which always has single poles. We have already seen examples of this feature, in for example equation (5.14) where we see a denominator $1/(\tau_{i-1}.\sigma_i)^2$. Such a double pole can not be obtained naively from the cylindrical decomposition procedure.[10]

However since, as we saw in section 5, the correlahedron geometry reduces to the amplituhedron by exactly the same geometric procedure (freeze and project) as the corresponding differential form it would seem puzzling if the procedure for obtaining the differential form from the geometry is very different. The situation can be described by figure 1. (We have displayed multiple arrows from the correlahedron to the amplituhedron to highlight the fact that one can take many different limits to get many different amplitudes from the same correlator.)

A possible resolution of this apparent puzzle arises from a stronger implementation of all the symmetries of the set up before taking the cylindrical decomposition.

[10] Note that log divergence criterion for obtaining the differential form from the geometry is valid also when we go to the boundary of the -hedron space. In other words if we choose Y to saturate one or more of the inequalities $\langle Y\dots\rangle>0$ (so we pin ourselves to the boundary of -hedron space) then implementing the cylindrical decomposition procedure on the remaining inequalities/ variables yields the correct answer for the residue of the expression in this limit.

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$$
\begin{tikzcd}
\textbf{correlahedron} \arrow[rr, "??"] \arrow[d, Rrightarrow, "\textit{freeze/project}"'] & & \textbf{correlator} \arrow[d, Rrightarrow, "\textit{freeze/project}"] \\
\textbf{amplituhedron} \arrow[rr, "\textit{cylindrical decomp}"] & & \textbf{amplitude.}
\end{tikzcd}
$$

Figure 1. Figure schematically illustrating the relationship between the correlator and amplitude (squared) as well as the corresponding relation between the geometric objects, the correlahedron and (squared) amplituhedron.

6.4.1 Toy model reconsidered, implementing the full symmetry

First consider again the simplest toy model case. There we consider points $Y \in \mathrm{Gr}(1,3)$ inside the triangle formed by $Z_1, Z_2, Z_3 \in \mathrm{Gr}(1,3)$. In the standard formulation we let Y (with its two degrees of freedom) vary fully inside the triangle. However in fact the global GL(3) symmetry in this case allows one to completely fix Y, leaving no degrees of freedom at all! To see this, first use GL(3) to fix Z_1, Z_2, Z_3 to the basis elements of $\mathbb{R}^3$ (using a projective rescaling of each if necessary) and set $Y = (y_1, y_2, 1)$. Now consider the residual GL(3) which leaves the external data Z_i invariant. Since the Z_i are projective, the action of the diagonal of GL(3), $\mathrm{diag}(a,b,c)$ can be removed by the projective rescaling. On the other hand this residual GL(3) acts as $Y \to (a/cy_1, b/cy_2, 1)$. Thus by choosing a, b, c appropriately we can use this to set for example $Y = (1,1,1)$. We thus have no degrees of freedom left at all if we implement the GL(3) symmetry!

In fact the triangle form (6.7) can be completely determined (up to an overall numerical constant) by these symmetries alone. Indeed the function of Y, Z_1, Z_2, Z_3 multiplying $\langle Y d^2 Y\rangle$ must be GL(3) covariant, have weight zero in the Z_i and weight 3 in Y. The only possible function with these properties is

$$
\frac{\langle Y d^2 Y\rangle \langle Z_1 Z_2 Z_3\rangle^2}{\langle Y Z_1 Z_2\rangle \langle Y Z_2 Z_3\rangle \langle Y Z_3 Z_1\rangle}\,, \tag{6.39}
$$

the triangular form. So indeed this expression *can* be correctly obtained from no degrees of freedom at all!

6.4.2 Amplituhedron squared reconsidered, implementing the full symmetry

The above example is a bit too trivial, so let us give another example. Indeed one can reconsider the amplituhedron squared examples we looked at in the previous subsections and implement the additional symmetry in a similar way and show that the cylindrical decomposition still gives the right answer which can be covariantised to the full answer in these cases also.

For example, if we reconsider the 6 point $k = 2$ example we looked at in section 6.2.2, we can use the residual GL(6) symmetry to set 5, (a, b, c, d, e), of the 8 variables to unity:

$$
Y = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & f & 1 & g & h \end{pmatrix}. \tag{6.40}
$$

Implementing the cylindrical decomposition procedure exactly as in section 6.2.2 we arrive at the correct answer for the correlahedron form in these reduced variables (assuming the measure reduces in the obvious way to $df \wedge dg \wedge dh$. The full covariant form can then be obtained from this using the full symmetries.

6.4.3 Correlahedron example

Encouraged by the above results we now consider the simplest non-trivial correlation function, the 5 point NMHV correlator.

The correlahedron is the subspace $Y = Y_1 \wedge \ldots \wedge Y_6 \in \mathrm{Gr}(6, 10)$ restricted to the region

$$\langle Y X_i X_j \rangle > 0 \quad i \neq j = 1, \ldots, 5\,. \tag{6.41}$$

We first use GL(10) to choose the 10 external points $X_{i\alpha}$ to be the basis elements. We then note that there is a residual $\mathrm{GL}(2)^5 \subset \mathrm{GL}(10)$ which leaves this external data fixed (up to the GL(2) acting on each $X_{i\alpha}$).

Thus we have a $Y \in \mathrm{Gr}(6, 10)$ with a $\mathrm{GL}(k)$ symmetry acting on the left and a $\mathrm{GL}(2)^5$ on the right. We can put coordinates on this as follows

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e & f \end{pmatrix} \tag{6.42}$$

We claim it is always possible to put Y in this form using the above symmetries. First use the GL(6) on the left to set the matrix consisting of the first six columns of Y to the identity. Then use the residual $\mathrm{GL}(2)^5$ acting on the right of Y, together with compensating $\mathrm{GL}(2)^3 \subset \mathrm{GL}(6)$ acting on the left to restore the form of Y. We can use this residual $\mathrm{GL}(2)^5$ to fix the final four columns of Y to the above form. For example, in the last two stages after fixing all but the bottom right $4{\times}2$ block there is still a residual symmetry, block diagonal $\mathrm{diag}(G, G, G) \subset \mathrm{GL}(6)$ on the left and $\mathrm{diag}(G^{-1}, G^{-1}, G^{-1}, G^{-1}, G^{-1})$ on the right, for G a GL(2) matrix, leaving all but the bottom right $4{\times}2$ block invariant. Using this we can diagonalise the a, b $2{\times}2$ matrix and set one of the off diagonal components of the bottom $2{\times}2$ bocks to 1. The only remaining symmetry is a matrix on the left proportional to the identity and also on the right (with the inverse factor). This GL(1) does not and can never act on Y. Note that the number of variables of Y is the dimension of Y (60) minus the dimension of the residual symmetry $\mathrm{GL}(6) \times \mathrm{GL}(2)^5/\mathrm{GL}(1)$ $(6^2 + 5.2^2 - 1 = 55)$, giving 5, in agreement with our five remaining variables a, b, c, e, f.

Having reduced the variables down to the minimal number consistent with the symmetries, we now perform the cylindrical decomposition procedure. The region (6.41) then corresponds to the restrictions

$$-e + cf > 0, ab > 0, ab - bc - e - af + cf > 0, 1 - c - e - f + cf > 0, (-1 + a)(-1 + b) > 0 \tag{6.43}$$

which upon rewriting as a cylindrical decomposition and converting to a differential form according to (6.2) gives

$$\frac{(a-b)^2 da\, db\, dc\, de\, df}{(a-1)a(b-1)b(e-cf)(c(-f)+c+e+f-1)(ab-af-bc+cf-e)}\,. \tag{6.44}$$

Let us then compare this with the expected answer for the correlator (4.11). With the choice of variables for Y (6.42) this gives

$$\frac{d\mu(a,b,c,e,f)}{(a-1)a(b-1)b(e-cf)(c(-f)+c+e+f-1)(ab-af-bc+cf-e)} \tag{6.45}$$

where $d\mu(a,b,c,e,f)$ is the measure, $\langle Y d^4 Y_1\rangle \dots \langle Y d^4 Y_6\rangle$ reduced to these variables. Remarkably, we get complete agreement on identifying $d\mu(a,b,c,e,f) = (a-b)^2 da\, db\, dc\, de\, df$. Note that the term $(a-b)^2$ is indeed the natural measure factor, the Vandermonde determinant squared, one obtains when writing an integral measure on GL(2) invariant under conjugation in terms of its eigenvalues. Here it was produced directly by the cylindrical decomposition procedure.

So we see that in this case at least, the cylindrical decomposition procedure still works, once all symmetries are correctly taken into account. We leave it to future investigations to firm up this proposal.

7 Conclusions

In this paper we have presented the definition of a new geometric object, the correlahedron, defined as a subspace of $\mathrm{Gr}(n+k, n+k+4)$. We have provided much evidence for its equivalence to the correlator of stress-energy multiplets $G_{n;k}$. We have shown how to obtain the volume form associated with the squared amplituhedron region and its equivalence to squared amplitude expressions in a number of examples. In the process we developed a simple algorithmic procedure for finding the volume form from the region. We have also shown that the correlahedron as a geometric region reduces to the squared amplituhedron as a geometric region (in fact many different squared amplituhedrons in general) via a geometric procedure of freezing the space to a certain boundary and then projecting. The exact same reduction procedure, applied to all known correlator expressions (recast as volume forms in correlahedron space) reduces them correctly to the corresponding squared amplitude expressions.

We believe this gives substantial evidence that the correlahedron geometry is equivalent to the correlator. However the extraction of the relevant volume form is more problematic for the correlator than for the squared amplitude both computationally and conceptually. We overcome both problems in the simplest possible example, by exhausting the full additional symmetries of the problem and only then implementing the cylindrical decomposition procedure. Clearly more work needs to be done however to make the procedure fully concrete, in particular a fuller understanding of the reduced measure.

Our work leaves room for a number of other directions to pursue from here. One of the many remarkable aspects of the -hedron programme, before one even considers the

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geometric one is the bosonisation of nilpotent invariants. This provides an entirely new way to explore nilpotent superconformal invariants in a completely bosonic framework as we saw in section 4.2 and we believe this aspect alone deserves further investigation. One pertinent technical question here is how to extract explicit component correlation functions directly from this bosonised form.

The maximally nilpotent correlator, which in the lightlike limit leads to a sum of products of amplitudes with their conjugates, is a simpler object than the amplitudes themselves and indeed recent high loop four- and five-loop amplitude expressions have been calculated via the correlator [8, 40, 41]. It would be interesting to understand the extent to which one can extract the separate amplitude expressions from the maximally nilpotent correlator at higher than five points.

Another recent development is the computation of higher loop correlators of higher charge BPS operators [44]. It would be interesting to explore how/whether the correlahedron generalises to yield these.

In a different direction, it is important to find a systematic proof of the equivalence between 'hedra and amplitudes and correlators. One approach is to directly work at the level of twistor Feynman graphs. These can individually be mapped to regions in hedron space that together provide a tessellation of the hedron. This seems to be problematic for the amplituhedron itself where sign ambiguities seem to lead to an obstruction to the programme for even k. There is some hope that the more explicit definition provided by the squared amplituhedron might remove these obstructions. Nevertheless, unlike the tesselations provided by the BCFW terms represented in the positive grassmannian, individual 'tiles' seem to have to lie both inside and outside the hedron. As it stands, however, the BCFW description does not apply to correlators, so at this stage, there doesn't seem to be an alternative to the twistor space Feynman diagrams.

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A Lightlike limit of NMHV six points $G_{6;1} \to \mathcal{A}_{6;1}$

A.1 Maximal lightlike limit

As a highly non-trivial example of this lightlike limit procedure we here explicitly reduce the six point "NMHV" correlator $G_{6;1}$ found in [42] to the NMHV 6-point amplitude by performing the "freeze and project" procedure outlined for the correlahedron form in

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section 5.2. In particular we implement the reduction in the form (5.18) and we do it in terms of specific coordinates rather than covariantly.

The correlahedron form for this case is given explicitly in (4.19) where $Y \in \mathrm{Gr}(7,11)$. We choose our basis to be $\{Y_1,\ldots,Y_6,X_{12},X_{22},\ldots X_{52}\}$ where $Y_1,\ldots,Y_6$ are frozen to $Y_i = X_{i1} - X_{i+1\,2} (i = 1\ldots 6)$ as in (5.18). The projection then projects out the first 6 coordinates in this basis and projects onto the final 5 coordinates. So with respect to this basis we have

$$\begin{aligned} X_{i2}^{\mathcal{A}} &= -Z_i^{\mathcal{A}} = -\delta_{i+6}^{\mathcal{A}} & X_{i1}^{\mathcal{A}} &= X_{i-1\,2}^{\mathcal{A}} - Y_{i-1}^{\mathcal{A}} = -\delta_{i+5}^{\mathcal{A}} - \delta_{i-1}^{\mathcal{A}} \quad i=1\ldots5 \\ X_{62}^{\mathcal{A}} &= -Z_6^{\mathcal{A}} = (A,B,\ldots,J,1) & X_{11}^{\mathcal{A}} &= X_{62}^{\mathcal{A}} - Y_6^{\mathcal{A}} = (A,\ldots,E,F-1,G,\ldots J,1) \\ Y_i^{\mathcal{A}} &= \delta_i^{\mathcal{A}} & Y_7^{\mathcal{A}} &= (0\ldots0,1,a,b,c,d)\,. \end{aligned} \tag{A.1}$$

The projection operation then corresponds to projecting onto the last 5 coordinates. In particular we set the variables $A,\ldots,F \to 0$. The projected points have five dimensional coordinates

$$\begin{aligned} \hat{Z}_i &= -\delta_i^{\mathcal{A}'} \quad i=1\ldots5 \\ \hat{Z}_6^{\mathcal{A}'} &= -(G,\ldots,J,1) \\ \hat{Y}^{\mathcal{A}'} &= (1,a,b,c,d) \end{aligned} \tag{A.2}$$

It is straightforward (on a computer) to plug these values into the expression for the correlahedron (4.19) (see (5.18)). We arrive at a rational function of a,b,c,d,G,H,I,J. This rational function is precisely

$$[12345] + [34561] + [56123] \tag{A.3}$$

where

$$[ijklm] = \frac{\langle \hat{Z}_i\hat{Z}_j\hat{Z}_k\hat{Z}_l\hat{Z}_m\rangle^4}{\langle \hat{Y}\hat{Z}_i\hat{Z}_j\hat{Z}_k\hat{Z}_l\rangle\langle \hat{Y}\hat{Z}_j\hat{Z}_k\hat{Z}_l\hat{Z}_m\rangle\langle \hat{Y}\hat{Z}_k\hat{Z}_l\hat{Z}_m\hat{Z}_i\rangle\langle \hat{Y}\hat{Z}_l\hat{Z}_m\hat{Z}_i\hat{Z}_j\rangle\langle \hat{Y}\hat{Z}_m\hat{Z}_i\hat{Z}_j\hat{Z}_k\rangle}\,, \tag{A.4}$$

which we recognise as the NMHV six-point amplituhedron form.

A.2 Non-maximal lightlike limit

In section 5.4 we performed the maximal lightlike limit explicitly on the six point "NMHV" correlahedron expression $G_{6;1}$. We now consider the non-maximal five-point lightlike limit which reduces it to the five-point one-loop amplitude by implementing the non-maximal freeze and project procedure of section 5.4.

We start with the correlahedron form given explicitly in (4.19) where $Y \in \mathrm{Gr}(7,11)$. We choose our basis to be $\{Y_1,\ldots,Y_5,X_{61},X_{62},X_{12},X_{22},X_{32},X_{42}\}$, where $Y_1,\ldots,Y_5$ are frozen to $Y_i = X_{i1} - X_{i+1\,2} (i = 1\ldots 5)$ as in (5.18). The projection then projects out the first 7 coordinates (the five Ys as well as X_6) in this basis and projects onto the final 4 coordinates.

Then with respect to this basis we have

$$
\begin{aligned}
&X_{i2}^{\mathcal{A}} = -Z_i^{\mathcal{A}} = -\delta_{i+7}^{\mathcal{A}} && X_{i1}^{\mathcal{A}} = X_{i-1\,2}^{\mathcal{A}} - Y_{i-1}^{\mathcal{A}} = -\delta_{i+6}^{\mathcal{A}} - \delta_{i-1}^{\mathcal{A}} \quad i = 1\dots 4\\
&X_{52}^{\mathcal{A}} = -Z_5^{\mathcal{A}} = (A, B, \dots, J, 1) && X_{11}^{\mathcal{A}} = X_{52}^{\mathcal{A}} - Y_5^{\mathcal{A}} = (A, \dots, E{-}1, F, G, \dots J, 1)\\
&X_{61}^{\mathcal{A}} = \delta_6^{\mathcal{A}} && X_{62}^{\mathcal{A}} = \delta_7^{\mathcal{A}}\\
&Y_i^{\mathcal{A}} = \delta_i^{\mathcal{A}} \quad i = 1\dots 5 &&\\
&Y_6^{\mathcal{A}} = (0\dots 0, 1, 0, 1, 0, a, b) && Y_7^{\mathcal{A}} = (0\dots 0, 1, 0, 1, c, d)
\end{aligned} \tag{A.5}
$$

The projection operation then corresponds to projecting onto the last 4 coordinates. In particular we set the variables $A, \dots, G \to 0$. The projected points have four dimensional coordinates

$$
\begin{aligned}
\hat{Z}_i^{A'} &= \delta_i^{A'} \quad i = 1\dots 4\\
\hat{Z}_5^{A'} &= -(H, I, J, 1)\\
\mathcal{L}_\alpha^{A'} &= \begin{pmatrix} 1 & 0 & a & b\\ 0 & 1 & c & d \end{pmatrix}
\end{aligned} \tag{A.6}
$$

It is straightforward (on a computer) to plug these values into the expression for the correlahedron (4.19) (see the l.h.s. of (5.37)). We arrive at a rational function of a, b, c, d, H, I, J. This rational function is

$$
\frac{\langle 5123\rangle\langle 1245\rangle}{\langle\mathcal{L}12\rangle\langle\mathcal{L}23\rangle\langle\mathcal{L}51\rangle\langle\mathcal{L}45\rangle} + \frac{\langle 1234\rangle\langle 2345\rangle}{\langle\mathcal{L}12\rangle\langle\mathcal{L}23\rangle\langle\mathcal{L}34\rangle\langle\mathcal{L}45\rangle} + \frac{\big(-\langle\mathcal{L}12\rangle\langle 2345\rangle + \langle\mathcal{L}25\rangle\langle 1234\rangle\big)\langle 1345\rangle}{\langle\mathcal{L}51\rangle\langle\mathcal{L}12\rangle\langle\mathcal{L}23\rangle\langle\mathcal{L}34\rangle\langle\mathcal{L}45\rangle} \tag{A.7}
$$

where

$$
\langle\mathcal{L}12\rangle := \langle\mathcal{L}\hat{Z}_1\hat{Z}_2\rangle\,. \tag{A.8}
$$

This is precisely (up to a numerical factor) the one-loop five-point amplitude given in [45] (eq 6.4 with X chosen to be $X = 45$).

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