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SU(3) structures on S^2 bundles over four-manifolds

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ABSTRACT: We construct globally-defined SU(3) structures on smooth compact toric varieties (SCTV) in the class of $\mathbb{C}P^1$ bundles over M , where M is an arbitrary SCTV of complex dimension two. The construction can be extended to the case where the base is Kähler-Einstein of positive curvature, but not necessarily toric, and admits a parameter space which includes SU(3) structures of LT type.

KEYWORDS: Differential and Algebraic Geometry, Flux compactifications, Superstring Vacua

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1 Introduction

In many cases the problem of supersymmetric compactification to four-dimensional Minkowski or AdS space can be reformulated as the problem of existence of SU(3) structures with appropriate torsion classes parameterized by the “fluxes”. Although this approach has led to great progress in the construction of string vacua, the search for manifolds with suitable SU(3) structures has been far less systematic than the construction of Calabi-Yau manifolds, for which powerful algebro-geometric tools are available.

In [1] it was proposed to use smooth compact toric varieties (SCTV) as a class of manifolds for which tools from both algebraic and differential geometry can be used, and develop a formalism suitable for the description of $SU(3)$ structures on SCTV. The idea is to use the canonical structure that comes with the symplectic quotient description of the SCTV (metric, complex structure, set of coordinates), and construct on it a different (nonintegrable in general) almost complex structure associated with a globally-defined $SU(3)$ structure.

The construction of $SU(3)$ structures on SCTV proposed in [1] relies on the existence of a one-form K in the parent space of the symplectic quotient, satisfying certain requirements. Thus the problem of constructing $SU(3)$ structures on SCTV is reduced to the problem of constructing one-forms K satisfying the requirements of [1]. Although that reference gave some examples of suitable one-forms, and many more were subsequently constructed in [2], no general formula for K exists satisfying the requirements of [1]. As a result, the search for $SU(3)$ structures on SCTV had up to now proceeded on a case-by-case basis.

In the present paper we extend the formalism of [1] for SCTV to construct globally-defined $SU(3)$ structures on the class $\mathbb{C}P^1$ over M , where M is an arbitrary two-dimensional SCTV. As in [1], our construction is based on the existence of a one-form K which, in our case, is naturally distinguished by the structure of the bundle. This one-form does not have the right $U(1)$ charge (in symplectic-quotient terminology) for the procedure of [1] to go through. A different procedure is used instead, which can be thought of as a generalization of the supersymmetric ansatz of [3, 4] to the toric case, exploiting the local $SU(2)$ structure of the base M of the fibration.

More specifically we give a general formula, eq. (4.11) below, for globally-defined $SU(3)$ structures on all $\mathbb{C}P^1$ bundles whose $U(1)$ charges satisfy eq. (4.1). The latter equation can always be satisfied for any two-dimensional SCTV base, and amounts to choosing a specific twisting of the $\mathbb{C}P^1$ bundle. The $SU(3)$ structures thus constructed admit a space of deformations parameterized by certain functions, described below eq. (4.12). The associated torsion classes depend on these functions, and are nonvanishing in general.

This method can also be used to construct $SU(3)$ structures on S^2 bundles over B_4 , where B_4 is Kähler-Einstein but not necessarily toric, in which case we have an $SU(3)$ -structure ansatz of the general form of [3, 4]. Provided B_4 has positive scalar curvature, i.e. if it is $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, or one of the del Pezzo surfaces dP_3, \dots, dP_8 [5], the total space of the S^2 bundle is complete and the associated metric is regular. We show that the parameter space includes $SU(3)$ structures of LT type, suitable for $\mathcal{N} = 1$ AdS₄ compactifications of massive IIA.

The outline of the remainder of paper is as follows. In section 2 we review the formalism of [1] for SCTV, and introduce the tools that will be used in the rest of the paper. The toric $\mathbb{C}P^1$ bundles are described in section 2.4. In section 3 we work out in detail the example of $\mathbb{C}P^1$ over $\mathbb{C}P^2$. This is the simplest example in the class of toric $\mathbb{C}P^1$ bundles over M , but it already captures the main idea of the construction. The $SU(3)$ structure is constructed in section 3.2. Section 4 discusses the construction of $SU(3)$ structures on toric $\mathbb{C}P^1$ bundles over general two-dimensional SCTV. Section 5 discusses the construction of $SU(3)$ structures on S^2 bundles over four-dimensional Kähler-Einstein bases. We conclude

in section 6. For ease of presentation, many technical details have been moved to the appendices.

2 Review of the formalism

In order to fix the notation and make the paper self-contained, in this section we give a review of the SCTV formalism developed in [1]. Along the way we introduce the tools that will be useful in the rest of the paper. The description of the toric $\mathbb{C}P^1$ bundles is given at the end of the section.

There are various equivalent ways to define a toric variety see e.g. [6], or [7] for an introduction for physicists. In the following we will use the symplectic quotient description, which turns out to be the best suited for the explicit construction of G -structures and the associated differential calculus. The starting point of the symplectic quotient description is a *parent space* \mathbb{C}^k , with coordinates $\{z_i, i = 1, \dots, k\}$, and a set of s linearly-independent integer k -vectors $Q_i^a, \{a = 1, \dots, s\}$ called the *charges*. Let \tilde{M} be the real submanifold defined by the following set of *moment map equations*,

$$Q_i^a |z_i|^2 = \xi^a . \quad (2.1)$$

The real parameters ξ^a are the so-called, *Fayet-Iliopoulos* parameters: they correspond to Kähler moduli, parametrizing the sizes of cycles of the toric variety. On the other hand the topology of the variety is independent of the ξ^a as long as we stay inside the *Kähler cone*, defined by the conditions $\xi^a > 0$. In the following we will always assume this to be the case. The associated toric variety M is given by the quotient $M = \tilde{M}/U(1)^s$ where the *phase vector* $\phi_a \in U(1)^s$ acts on the coordinates $z_i \in \tilde{M}$ through the following *gauge transformations*,

$$z_i \rightarrow \phi \cdot z_i := e^{iQ_i^a \phi_a} z_i . \quad (2.2)$$

Hence M is a manifold of complex dimension $d = k - s$: the equations (2.1) can be thought of as removing s real ‘radial’ directions, whereas the action of (2.2) removes s real ‘angular’ directions. In total the equations (2.1), (2.2) remove s pairs consisting of one radial and one angular variable, which may be thought of as s complex variables.

Since the Q^a are independent as k -vectors, one may choose a set S of s indices such that $Q_b^a, b \in S$, is invertible. The open set $\{z_b \neq 0, b \in S\} \subset \mathbb{C}^k$ then descends to a well-defined open set in M , denoted by U_S . On this patch one can then use the z_b coordinates to compensate the $U(1)^s$ action on the z_α coordinates, where the index α takes values in the complement of S , $\alpha \in \complement S$. One may then define the following gauge-invariant quantities,

$$t_i := z_i \prod_{a \in S} z_a^{-Q_b^a Q_i^b} , \quad (2.3)$$

where we have set,

$$Q_b^a := (Q_b^a)^{-1} . \quad (2.4)$$

Thus, provided $|Q_S| := |\det Q_{b \in S}^a| = 1$, the map,

$$\begin{aligned} \varphi_S : U_S &\rightarrow \mathbb{C}^d \\ [z_i] &\mapsto (t_\alpha)_{\alpha \in \complement S} , \end{aligned} \quad (2.5)$$

where $d := k - s$, is a well-defined homeomorphism, while the transition functions $\varphi_S \circ \varphi_{S'}^{-1}$ are biholomorphic and rational. The charts (U_S, φ_S) form a holomorphic atlas on $M = \bigcup_{|Q_S|=1} U_S$: the t_α , $\alpha \in \mathbb{C}^S$, define d gauge-invariant local holomorphic coordinates on U_S . Note that for $i = c \in S$, we thus find $t_c = 1$. The existence of this covering of M is related to the condition of smoothness for general toric varieties, and can be checked explicitly for the cases we consider here.

To take a simple example, consider the case $s = 1$ and $Q = (1, \dots, 1)$. The corresponding toric variety is the complex projective space $\mathbb{C}P^{k-1}$. Indeed (2.1) gives $\|z\|^2 = \xi$, i.e. $\tilde{M} = S^{2k-1}$. Taking the $U(1)$ quotient, M can be written as $M = (\mathbb{C}^k \setminus \{0\})/\mathbb{C}^*$, the set of complex lines in \mathbb{C}^k . On the patch $U_j = \{z_j \neq 0\}$, the local coordinates take the form,

$$t_i = \frac{z_i}{z_j}, \tag{2.6}$$

which we recognize as the set of canonical coordinates of $\mathbb{C}P^{k-1}$. The z_i on the other hand correspond to homogeneous coordinates of $\mathbb{C}P^{k-1}$.

2.1 Differential forms

We have seen that toric varieties are equipped with systems of complex coordinates which can easily be made explicit. Moreover it is often advantageous to work directly in the parent space \mathbb{C}^k using the homogeneous coordinates z_i . We will be interested in particular in globally-defined differential forms on the manifold M . One way to construct a differential form on M is to start from its local expression on a patch, and make sure a regular global extension exists by checking its compatibility with the transition functions of the cotangent bundle. Working directly in \mathbb{C}^k drastically simplifies this problem: since the topology of the parent space is trivial, a single expression suffices to define differential forms globally. From this point of view the key question is to identify the differential forms of \mathbb{C}^k which descend to well-defined forms on M .

In the following we review how the formalism of [1] can be used to treat this question. Let Φ be a differential form on \mathbb{C}^k . In order for Φ to descend to a well defined form on M , it should be well-defined on \tilde{M} . Hence it should be compatible with the moment map equations (2.1) which imply,

$$Q_i^a \bar{z}_i dz_i + Q_i^a z_i d\bar{z}_i = 0 . \tag{2.7}$$

Consequently Φ should not have any components along the $\Re\eta^a$, where we have defined,

$$\eta^a := Q_i^a \bar{z}_i dz_i . \tag{2.8}$$

In other words, we require,

$$\iota_{\Re(V^a)} \Phi = 0, \tag{2.9}$$

where V^a is the dual of η^a (with respect to the canonical metric of \mathbb{C}^k),

$$V^a := Q_i^a z_i \partial_{z_i} . \tag{2.10}$$

Moreover, Φ should be compatible with the quotient (2.2). On the other hand the $U(1)^s$ action in (2.2) is generated by the vector fields $\Im(V^a)$. Hence the $U(1)^s$ invariance can be stated in terms of the following two conditions:

1. Φ must be constant along $U(1)^s$ orbits, i.e. $\mathcal{L}_{\mathfrak{S}(V^a)}\Phi = 0$.
2. Φ should not have any components along the orbits, i.e. $\iota_{\mathfrak{S}(V^a)}\Phi = 0$.

These conditions have a natural interpretation: first note that a form Φ has charge q^a if it is an eigenvector of the Lie derivative $\mathcal{L}_{\mathfrak{S}(V^a)}$,

$$\mathcal{L}_{\mathfrak{S}(V^a)}\Phi = q^a\Phi. \quad (2.11)$$

We then see that the first of the two conditions above is simply the gauge invariance of Φ , i.e. the condition that the total charge of Φ vanishes. Moreover the second condition combined with (2.9) gives,

$$\iota_{V^a}\Phi = \iota_{\bar{V}^a}\Phi = 0, \quad (2.12)$$

which is equivalent to Φ being *vertical* with respect to V^a .

Thus in order to construct a well-defined form on M descending from a form Φ on \mathbb{C}^k , the gauge invariance of Φ must be imposed from the outset. On the other hand, the verticality condition is purely algebraic and can be imposed by projecting out the components along η^a .

Let us now come to the explicit construction of the vertical projector. We introduce the real symmetric matrix,

$$g^{ab} := \eta^a(V^b) = Q_i^a Q_i^b |z_i|^2. \quad (2.13)$$

The projection P of a $(1,0)$ -form Φ is then given by,

$$P(\Phi) = \Phi - \tilde{g}_{ab}\iota_{V^a}(\Phi)\eta^b, \quad (2.14)$$

where $\tilde{g} = g^{-1}$. This definition of P can be readily extended to all (k,l) -forms [1]. In the following it will be useful to define the vertical projections, $\mathcal{D}z_i$, of the one-forms dz_i ,

$$\mathcal{D}z_i := P(dz_i) = dz_i - \tilde{g}_{ab}Q_j^a Q_i^b \bar{z}_j z_i dz_j = dz_i - h_{ij}z_i \bar{z}_j dz_j, \quad (2.15)$$

(no sum over i) where we have set,

$$h_{ij} := Q_i^a Q_j^b \tilde{g}_{ab}. \quad (2.16)$$

The $\mathcal{D}z_i$ are the building blocks that we will use to construct global forms on M . Note however that since they are not gauge invariant, one must compensate their charge by appropriate (charged) coefficients.

On the other hand the (singular) form $\mathcal{D}z_i/z_i$ is both gauge invariant and vertical and therefore admits an expression in terms of the local coordinates t_i . On the patch U_S we have,

$$\frac{dt_i}{t_i} = \frac{dz_i}{z_i} - \sum_{a \in S} \sum_{b=1}^s Q_b^a Q_i^b \frac{dz_a}{z_a}, \quad (2.17)$$

where we took (2.3) into account. Setting $i = c \in S$ then gives $dt_c = 0$, cf. (2.4). This leaves us with d linearly-independent one-forms dt_α , $\alpha \in \complement S$. We can then compute,

$$\frac{\mathcal{D}z_i}{z_i} = \frac{dt_i}{t_i} - h_{ij}|z_j|^2 \frac{dt_j}{t_j}. \quad (2.18)$$

where we took into account that: $h_{ij}|z_j|^2 Q_j^b = \tilde{g}_{cd} Q_i^c Q_j^d |z_j|^2 Q_j^b = \tilde{g}_{cd} Q_i^c g^{db} = Q_i^b$. As expected, given that the form on the left-hand side of (2.18) is vertical and gauge invariant, the result can be expressed in terms of the local coordinates alone. Note also that the gauge-invariant $|z_j|^2$ can be expressed as a function of t_i using (2.1).

Conversely, (2.17) can be used to express dt_i as a function of $\mathcal{D}z_i$, since dt_i is vertical by definition. We now have all the necessary tools to translate back and forth between the local coordinate system $\{t_i\}$ on M and the global coordinate system $\{z_i\}$ on \mathbb{C}^k .

2.2 The hermitian metric

A useful object on an almost complex manifold M is the *hermitian metric*,

$$\mathfrak{h} = \mathfrak{h}^{i\bar{j}} dz_i \otimes d\bar{z}_{\bar{j}}, \tag{2.19}$$

where $\mathfrak{h}^{i\bar{j}}$ can be thought of as a hermitian positive-definite matrix. The real and imaginary parts of \mathfrak{h} are real bilinear forms, so that,

$$\mathfrak{h} = \mathfrak{g} - iJ, \tag{2.20}$$

where \mathfrak{g} is symmetric, positive definite and can be identified with the Riemannian metric, while J is antisymmetric and can be identified with the almost symplectic form –which is a (1,1)-form with respect to the almost complex structure. In other words the hermitian metric contains both the metric and the almost symplectic form of M .

On toric varieties there is a canonical hermitian metric,

$$\mathfrak{h}(\xi^a) = P(dz_i \otimes d\bar{z}_i) = \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i. \tag{2.21}$$

As we have already noted, the $\mathcal{D}z_i$ above are not linearly independent and do not form a basis of the cotangent bundle of M . Using (2.18) it is not difficult to see that the hermitian metric takes the following form in local coordinates,

$$\mathfrak{h}(\xi^a) = \frac{|z_i|^2}{|t_i|^2} dt_i \otimes d\bar{t}_i - h_{jk} \frac{|z_j|^2}{|t_j|^2} \frac{|z_k|^2}{|t_k|^2} \bar{t}_j dt_j \otimes t_k d\bar{t}_k, \tag{2.22}$$

where we have made use of the identity $h_{ij}h_{ik}|z_i|^2 = h_{jk}$ which can be shown by taking into account the various definitions.

In this case J is in fact the Kähler form of the toric manifold,

$$J = \frac{i}{2} \mathcal{D}z_i \wedge \mathcal{D}\bar{z}_i. \tag{2.23}$$

Although $\mathcal{D}z_i$ are not closed, it can readily be verified that dJ vanishes as it should.

Let us illustrate the above with the example of $\mathbb{C}\mathbb{P}^{k-1}$: on the patch U_k we have $g = \xi$ and $h_{ij} = \frac{1}{\xi}$. Moreover (2.1) gives $|z_k|^2 = \frac{|z_\alpha|^2}{|t_\alpha|^2} = \frac{\xi}{1+t^2}$, where $t^2 := \sum_\alpha |t_\alpha|^2$; $\alpha = 1, \dots, k-1$. Hence,

$$\mathfrak{h}(\xi) = \xi \left(\frac{dt_i \otimes d\bar{t}_i}{1+t^2} - \eta \otimes \bar{\eta} \right), \tag{2.24}$$

where $\eta := \frac{1}{1+t^2} \bar{t}_i dt_i$. We thus recover the Fubini-Study metric and its associated Kähler form.

A hermitian metric also gives rise to a scalar product “.” on forms on M . Since $P^2 = P$, the calculation of the scalar product on vertical forms can be done in the parent space \mathbb{C}^k using the flat metric. Then, using (2.15) we find:

$$\mathcal{D}\bar{z}_i \cdot \mathcal{D}z_j = 2(\delta_{ij} - h_{ij}\bar{z}_i z_j), \tag{2.25}$$

which shows that the $\mathcal{D}z_i$ are not orthogonal.

2.3 SU(d) structures

A Riemannian $2d$ -dimensional manifold M with metric \mathfrak{g} and associated (\mathfrak{g} -compatible) almost complex structure I admits a reduction of its structure group to $U(d)$. At each point over M , the almost complex structure I , which need *not* be integrable, splits the cotangent space of M into a holomorphic and an antiholomorphic subspace, corresponding to the spaces of (1,0)-forms and (0,1)-forms with respect to I . Furthermore a holomorphic top form can be defined, i.e. a $(d,0)$ -form with respect to I , which transforms as a section of the canonical bundle of I . The canonical bundle of I is trivial, and so has vanishing Chern class: $c_1(I) = 0$, precisely when it has a non vanishing global section, i.e. when there is a nowhere-vanishing holomorphic top form. In that case the structure of the manifold is further reduced to $SU(d)$. An equivalent description of an $SU(d)$ structure on M is given by a complex decomposable d -form Ω and a real two-form J such that,

$$\Omega \wedge J = 0 ; \quad \frac{i^{d(d+2)}}{2^d} \Omega \wedge \Omega^* = \frac{1}{d!} J^d . \tag{2.26}$$

In this formulation the Riemannian metric on M is constructed from the pair (J, Ω) .

In six real dimensions ($d = 3$) it is well-known that the topological obstruction for the existence of an $SU(3)$ structure is that the manifold should be spin. We can make contact with the discussion of the previous paragraph by noting that $c_1(I)$ modulo 2 is a topological invariant, and $c_1(I)$ is even in cohomology iff M is spin.¹ Moreover the *torsion classes* characterizing the $SU(3)$ structure are given by the decomposition of $(dJ, d\Omega)$,

$$\begin{aligned} dJ &= \frac{3}{2} \Im(W_1 \Omega^*) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + W_5 \wedge \Omega, \end{aligned} \tag{2.27}$$

where W_1 is a function, W_2 is a (1,1)-form, W_3 is a real $(2,1) \oplus (1,2)$ -form, W_4 is a real one-form and W_5 is a (1,0)-form.

As follows from the previous discussion, it is not always possible to construct an $SU(d)$ structure on an arbitrary toric variety M . Although \mathbb{C}^k has a canonical $SU(k)$ structure

¹Note however that $c_1(I)$ itself is not a topological invariant. A well-known counterexample is $\mathbb{C}\mathbb{P}^3$ which admits both a non-integrable almost complex structure (with $c_1(I) = 0$) and an integrable one (with $c_1(I) \neq 0$). In both cases $c_1(I)$ is even in cohomology, as of course it should, since $\mathbb{C}\mathbb{P}^3$ is spin.

given by,

$$\tilde{J} = \frac{i}{2} dz_i \wedge d\bar{z}_i \tag{2.28}$$

$$\tilde{\Omega} = \bigwedge_i dz_i, \tag{2.29}$$

$\tilde{\Omega}$ does not in general descend to M . One can always define,

$$\hat{J} := P(\tilde{J}), \tag{2.30}$$

which is vertical and gauge invariant: it is the almost symplectic form associated with the hermitian metric (2.22). However $P(\tilde{\Omega})$ vanishes trivially since there are no $(k, 0)$ forms on M . To obtain a $(d := k - s, 0)$ -form on M we must contract $\tilde{\Omega}$ with each of the V^a vectors, so that,

$$\hat{\Omega} := \frac{1}{\sqrt{\det g}} \prod_a \iota_{V^a} \tilde{\Omega}. \tag{2.31}$$

But then $\hat{\Omega}$ has the same charge as $\tilde{\Omega}$, i.e. $q^a = \sum_i Q_i^a$, and so it is not gauge invariant.

On the other hand the pair $(\hat{J}, \hat{\Omega})$ does satisfy the compatibility equations (2.26), thus defining a *local* $SU(d)$ structure on M . Moreover $\hat{\Omega}$ admits a simple expression in terms of local coordinates² on U_S . After some straightforward manipulations we obtain,

$$\hat{\Omega} = (-1)^S Q_S \frac{\prod_i z_i}{\sqrt{\det g}} \bigwedge_{\alpha} \frac{dt_{\alpha}}{t_{\alpha}}, \tag{2.32}$$

where $a \in S$, $\alpha \in {}^{\mathbb{C}}S$ and we have defined,

$$(-1)^S := (-1)^{\sum_{a \in S} a + \frac{(s+1)(s+2)}{2}}. \tag{2.33}$$

In [1] a prescription was given for the construction of *global* $SU(d)$ structures on M .³ It relies on the existence of a one-form K on \mathbb{C}^k with the following properties:

1. It is vertical and $(1,0)$ with respect to the complex structure of \mathbb{C}^k .
2. It has half the charge of $\tilde{\Omega}$.
3. It is nowhere-vanishing.

Given a one-form K on \mathbb{C}^k satisfying the conditions above, [1] showed that a global $SU(3)$ structure on M can be constructed, and provided explicit examples of such a K for certain toric \mathbb{CP}^1 bundles. Many more examples of K were provided for other toric varieties in [2], which also provided explicit computations of the torsion classes of the associated $SU(3)$ structures. However there is no known construction for K that would be applicable in

²Thus defined, $\hat{\Omega}$ is compatible with the transition functions, but the z_i are not strictly functions on U_S since they are not gauge-invariant. A local form could be constructed by substituting z_i with $|z_i|$, at the cost of losing the compatibility with the transition functions.

³Originally presented for $d = 3$, the prescription of [1] is in fact directly generalizable to any dimension.

general, even for a subclass of SCTV, and the search for SU(3) structures on SCTV had so far proceeded in a case by case fashion.

In the following we will present a construction of SU(3) structures valid for toric \mathbb{CP}^1 bundles over any 2d SCTV. As we will see, our method is not equivalent to the prescription of [1], although it also makes use of a certain (1,0)-form on \mathbb{C}^k .

2.4 Toric \mathbb{CP}^1 bundles over SCTV

In [8], the classification of SCTV in three (complex) dimensions was shown to reduce to the classification of certain weighted triangulations of the two-dimensional sphere. In [1] it was shown how to systematically translate the results of [8] into the symplectic quotient language reviewed previously. In the following we will be interested in the subclass of the classification of [8] corresponding to \mathbb{CP}^1 bundles over a two-dimensional SCTV base. However the formalism applies generally to the case of \mathbb{CP}^1 bundles over SCTV, so in this subsection we will keep the dimension of the base arbitrary.

The U(1) charges of these bundles are given by the following set of $(s + 1) \times (k + 2)$ matrices,

$$Q_I^A = \begin{pmatrix} q_i^a & -n^a & 0 \\ 0 & 1 & 1 \end{pmatrix}, \tag{2.34}$$

where $A = 1, \dots, s + 1$, $I = 1, \dots, k + 2$; $n_a \in \mathbb{N}$, $a = 1, \dots, s$, are integers specifying the twisting of the \mathbb{CP}^1 bundle over a SCTV M ; q_i^a , $a = 1, \dots, s$, $i = 1, \dots, k$, are the U(1) charges of the symplectic quotient description of M , which is therefore of complex dimension $d = k - s$. (In subsequent subsections we will specialize to the case $d = 2$.)

The total space of the bundle is constructed by appending two coordinates and one new charge to those of M (given by the q_i^a), as in (2.34), thus obtaining a space of complex dimension $d + 1$. We will use the following notation for the data related to the fiber,

$$u := z_{k+1}; \quad v := z_{k+2}; \quad \xi := \xi^{s+1}. \tag{2.35}$$

The last charge Q_i^{s+1} defines a \mathbb{CP}^1 fiber over M , while the integers n^a determine the twisting of the bundle. Indeed the moment map equations for the total space read,

$$\sum_{i=1}^k q_i^a |z_i|^2 = \xi^a + n^a |u|^2; \quad |u|^2 + |v|^2 = \xi. \tag{2.36}$$

Thus the last two coordinates define a sphere of radius $\sqrt{\xi}$, while the first n coordinates define locally an M_ρ whose ‘‘radii’’ $(\rho^a)^2 := \xi^a + n^a |u|^2$ depend on the fiber. The twisting can be thought of as a consequence of the modified U(1)^{s+1} action.

We would now like to construct a metric that exhibits the bundle structure, i.e. a metric of the form $\mathfrak{h}_{d+1} = \mathfrak{h}_d + \mathfrak{h}_{\mathbb{CP}^1}$, where \mathfrak{h}_d is a metric on M and $\mathfrak{h}_{\mathbb{CP}^1}$ is a metric on the fiber \mathbb{CP}^1 , possibly modified by a connexion on the base. By expanding the canonical metric (2.21) we find,

$$\mathfrak{h}_{d+1}(\xi^A) = \mathfrak{h}_d((\rho^a)^2) + \frac{\hat{g}}{g} |u|^2 |v|^2 \varepsilon \otimes \varepsilon^*, \tag{2.37}$$

where the details of the computation, which are somewhat involved, can be found in appendix A together with the definitions of the various quantities in the second term of the right-hand side above. The one-form ε can be thought of as an analogue of the vertical displacement along the fiber.

3 \mathbb{CP}^1 over \mathbb{CP}^2

Let us now examine in detail the construction of an $SU(3)$ structure on the \mathbb{CP}^1 bundle over \mathbb{CP}^2 . This is the simplest example in the class of 3d SCTV of the form \mathbb{CP}^1 bundle over M , where M is a 2d SCTV, but it already captures the main idea of the construction. We will treat the general case in section 4.

The toric data in this case are: $k = 5$ (the complex dimension of the parent space), $s = 2$ (the number of charges), $d = 3$ (the complex dimension of the toric variety). Explicitly the charges are given by,

$$Q = \begin{pmatrix} 1 & 1 & 1 & -n & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{3.1}$$

where $n \in \mathbb{N}$. The corresponding moment map equations read, using the notation introduced in section 2.4,

$$\begin{aligned} |z_1|^2 + |z_2|^2 + |z_3|^2 &= \xi_1 + n|u|^2 \\ |u|^2 + |v|^2 &= \xi. \end{aligned} \tag{3.2}$$

This is a \mathbb{CP}^1 bundle over \mathbb{CP}^2 , with twisting parameterized by n . We can make this more explicit in local coordinates: on the patch $U_{1,5} := \{z_1, v \neq 0\}$ we define,

$$t_2 := \frac{z_2}{z_1}; \quad t_3 := \frac{z_3}{z_1}; \quad t_4 := \frac{z_1^n u}{v}.$$

Hence t_2, t_3 are local coordinates parameterizing a \mathbb{CP}^2 whereas, for z_1 fixed, t_4 is a local coordinate on a \mathbb{CP}^1 . For $n = 0$, the bundle becomes trivial and we obtain the direct product $\mathbb{CP}^2 \times \mathbb{CP}^1$. We can also see explicitly that the toric variety can be covered with patches of the form U_S , as in (2.3): in the present case S is given by the pair (i, j) where $i = 1, 2, 3$ and $j = 4, 5$, and the moment map equations (3.2) exclude the simultaneous vanishing of z_1, z_2, z_3 or that of u, v . To make contact with our previous discussion of local coordinates, we can check here that $|Q_S| = 1$ for all the S defined above. On the other hand for the patch $U_{4,5}$ we do not get compatible local coordinates in general, since $Q_{S'=\{4,5\}} = -n$, however this patch is not used in the covering of the toric variety by U_S .

Let us now calculate explicitly the various objects introduced in section 2. Since the base is defined by only one charge $q_i = (1, 1, 1)$, the calculations are rather simple. We have,

$$\begin{aligned} \hat{g} &= \rho^2 \\ V_i &= \frac{n}{\rho^2} \\ V &= \frac{n^2}{\rho^2} \\ g &= \xi\rho^2 + n^2|u|^2|v|^2. \end{aligned} \tag{3.3}$$

We thus find,

$$\varepsilon = \frac{dt_4}{t_4} + n\eta,$$

where we have set,

$$\eta := \frac{1}{1+t^2} (\bar{t}_2 dt_2 + \bar{t}_3 dt_3) ; \quad t^2 := |t_2|^2 + |t_3|^2 . \tag{3.4}$$

If we now introduce,

$$\Gamma := \frac{|u|^2|v|^2}{g},$$

the decomposition of the metric (2.37) can be written,

$$\mathfrak{h} = \rho^2 \mathfrak{h}_{\mathbb{CP}^2} + \Gamma \rho^2 \left| \frac{dt_4}{t_4} + n\eta \right|^2, \tag{3.5}$$

where $\mathfrak{h}_{\mathbb{CP}^2}$ is the hermitian Fubini-Study metric of \mathbb{CP}^2 with unit radius, cf. eq. (2.24).

We see the fibration structure appearing naturally in (3.5): the displacement along t_4 is modified by a connection, proportional to η , depending on the variables of the \mathbb{CP}^2 base, t_2, t_3 . Moreover,

$$d\eta = 2i\hat{j}, \tag{3.6}$$

where \hat{j} is the Kähler form of \mathbb{CP}^2 , cf. eq. (2.24). For vanishing n the connection piece drops out from the vertical displacement and the metric becomes that of a direct product as expected.

3.1 Comparison with the literature

Endowed with the hermitian metric (2.24), the base \mathbb{CP}^2 of the \mathbb{CP}^1 fibration is a Kähler-Einstein manifold obeying,

$$d\hat{j} = 0 ; \quad R_{mn} = \lambda g_{mn} . \tag{3.7}$$

i.e. \hat{j} is closed and the Ricci tensor is proportional to the metric. With our conventions, setting $\xi = 1$ gives $\lambda = 6$. Identifying the \mathbb{CP}^1 fiber with S^2 (by forgetting the complex structure), M can be thought of as an S^2 fibration over a Kähler-Einstein base B_4 , denoted by $S^2(B_4)$. These spaces appear naturally in the context of supersymmetric AdS₄ compactifications of M-theory on the so-called $Y^{p,q}(B_4)$ spaces [9, 10], which can be thought of as S^1 fibrations over $S^2(B_4)$. Compactifying M-theory on an appropriately chosen S^1 then leads to $\mathcal{N} = 2$ type IIA solutions of the form AdS₄ × $S^2(B_4)$ [3]. The latter can be deformed to solutions of massive IIA for any Kähler-Einstein base B_4 [4], although regularity requires B_4 to have positive curvature.

In the conventions of [10] the $S^2(B_4)$ metric reads,

$$g = U^{-1} d\tilde{\rho}^2 + \tilde{\rho}^2 g_{\mathbb{CP}^2} + q (d\psi + A)^2, \tag{3.8}$$

where $\tilde{\rho} \in [\tilde{\rho}_1, \tilde{\rho}_2]$ and $\psi \in [0, 2\pi/3]$ are the coordinates of the S^2 fiber (for general λ the period of ψ is $4\pi/\lambda$); U and q are positive functions of $\tilde{\rho}$, vanishing at $\tilde{\rho}_1$ et $\tilde{\rho}_2$. The circle

parameterized by ψ is fibered over the $[\tilde{\rho}_1, \tilde{\rho}_2]$ interval. The connection A is a one-form on the base B_4 obeying,

$$dA = 2\hat{j} . \tag{3.9}$$

At the endpoints of the $\tilde{\rho}$ interval the ψ circle contracts to a point, thus resulting in a total space with the topology of S^2 . The period of ψ is fixed by requiring the metric to be smooth at the endpoints, i.e. that,

$$U^{-1}d\tilde{\rho}^2 + q d\psi^2 \rightarrow du^2 + u^2 d\tilde{\psi}^2 , \quad \text{for } \tilde{\rho} \rightarrow \tilde{\rho}_1, \tilde{\rho}_2 , \tag{3.10}$$

where u is a function of $\tilde{\rho}$ that vanishes at the endpoints $\tilde{\rho}_1, \tilde{\rho}_2$, and we have defined an angular variable $\tilde{\psi} := \lambda\psi/2$ with period 2π .

Moreover the ψ coordinate parameterizes an S^1 fibration in the canonical bundle of B_4 . To see this, note that the connection of the canonical bundle of a Kähler-Einstein space with curvature normalized as in (3.7) obeys,

$$d\mathcal{P} = \lambda\hat{j} , \tag{3.11}$$

cf. appendix B. Comparing with (3.9) we see that $\mathcal{P} = \lambda A/2$, and so the vertical displacement along the S^1 fiber, cf. the last term in (3.8), is proportional to $(d\tilde{\psi} + \mathcal{P})$, as required for the canonical bundle. The fact that λ is positive for \mathbb{CP}^2 guarantees that the total space of the S^1 fibration, written in local coordinates in (3.8), extends globally to a smooth five-dimensional (squashed) Sasaki-Einstein space.

To make contact with the coordinates of (3.5), we must rewrite the \mathbb{CP}^1 fiber coordinate t_4 in terms of a pair of real coordinates. Using the formulas of section C we can rewrite the Riemannian metric \mathfrak{g} and Kähler form J associated with (3.5) for $n \neq 0$. The result reads,

$$\mathfrak{g} = \frac{1}{n^2\Gamma} d\rho^2 + \rho^2 \mathfrak{g}_{\mathbb{CP}^2} + \Gamma \rho^2 (d\varphi + n\Im\eta)^2 , \tag{3.12}$$

and,

$$J = \rho^2 \hat{j} + \frac{\rho}{n} d\rho \wedge (d\varphi + n\Im\eta) , \tag{3.13}$$

where we are using local coordinates on the patch $U_{1,5}$, and $\varphi \in [0, 2\pi]$ denotes the phase of t_4 . The \mathbb{CP}^1 fiber is parameterized by the (ρ, φ) coordinates: φ parameterizes a circle, fibered over the interval $\rho \in [\rho_1, \rho_2] = [\sqrt{\xi_1}, \sqrt{\xi_1 + n\xi}]$, whose radius vanishes at the endpoints. Indeed Γ vanishes for $u = 0$ or $v = 0$ which corresponds respectively to $\rho = \rho_1$ and $\rho = \rho_2$, following from the moment map equations (3.2). Moreover it can be checked that the metric is smooth there.

Furthermore we need to deform the canonical hermitian metric of the toric variety by introducing two warp factors $F(\rho), G(\rho)$ along the base and fiber respectively,

$$\mathfrak{h} = F(\rho) \sum_{i=1}^3 \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i + G(\rho) \sum_{i=4,5} \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i . \tag{3.14}$$

It can then be seen that the functions $F(\rho), G(\rho)$ together with a change of variables $\tilde{\rho} = \tilde{\rho}(\rho)$ may be chosen so that the real and imaginary parts of (3.14) reduce to the metric

in (3.8) and the form J_+ of [10] respectively, provided we set $n = 3$. The details of this exercise can be found in appendix C.

The condition $n = 3$ is also important for the existence of a globally-defined $SU(3)$ structure. We turn to the construction of this structure in section 3.2. Note however that the canonical metric of the SCTV, eq. (2.21), is smooth by construction for all $n \in \mathbb{N}$. This can also be verified explicitly by examination of the local form of the metric in terms of the coordinates (2.3) in each patch U_S .

3.2 The $SU(3)$ structure

In this section we will set $F = G = 1$ for simplicity of presentation: the two warp factors $F(\rho)$, $G(\rho)$ discussed in section 3.1 can be easily reinstated without changing any of the conclusions.

Specializing the formalism of section 2.3 to the present example we obtain a local $SU(3)$ structure $(\hat{J}, \hat{\Omega})$, where \hat{J} is obtained from (3.13) by setting $n = 3$. On the other hand we have,

$$\hat{\Omega} = -\frac{z_5^2}{\sqrt{\det g}} dt_2 \wedge dt_3 \wedge dt_4, \tag{3.15}$$

which is not gauge invariant, so this $SU(3)$ structure is not globally defined. In fact neither of the two local $SU(3)$ structures (J_{\pm}, Ω_{\pm}) of [10] can be globally extended: in the following we will see how to make contact with their results.

Let us first define a local $SU(2)$ structure (\hat{j}, ω) on $\mathbb{C}\mathbb{P}^2$, where,

$$\omega = \frac{1}{(1+t^2)^{3/2}} dt_2 \wedge dt_3, \tag{3.16}$$

and \hat{j} is the Kähler form of $\mathbb{C}\mathbb{P}^2$, cf. eq. (2.24), so that,

$$\begin{aligned} \omega \wedge \hat{j} &= 0 \\ \omega \wedge \omega^* &= 2\hat{j} \wedge \hat{j}. \end{aligned} \tag{3.17}$$

This $SU(2)$ structure is only locally defined since ω has a singularity at $z_1 = 0$, as can be seen by using the transition functions to rewrite ω in a patch where z_1 is allowed to vanish. The $SU(3)$ structures of [10] are then obtained by appending the contribution of the fiber coordinate,

$$J_{\pm} := \rho^2 \hat{j} \pm \frac{i}{2} K \wedge K^*; \quad \Omega_+ := \rho^2 \omega \wedge K; \quad \Omega_- := \rho^2 \omega \wedge K^*, \tag{3.18}$$

where,

$$K := \rho \sqrt{\Gamma} \varepsilon. \tag{3.19}$$

We see that exchanging $K \leftrightarrow K^*$ is equivalent to $(J_+, \Omega_+) \leftrightarrow (J_-, \Omega_-)$.

To better understand the global properties of the Ω_{\pm} , let us start from their local expression on the patch $U_{1,5}$,

$$\begin{aligned} \Omega_+ &= e^{-i\varphi} \frac{|z_5|^2}{\sqrt{\det g}} dt_2 \wedge dt_3 \wedge dt_4 \\ \Omega_- &= e^{i\varphi} \frac{|z_5|^2}{\sqrt{\det g}} dt_2 \wedge dt_3 \wedge (d\bar{t}_4 + 3\bar{t}_4 \bar{\eta}). \end{aligned}$$

We can see that the singularity in ω has been compensated by wedging with K, K^* . On the other hand, we can rewrite Ω_{\pm} in the patch $U_{1,4}$ by using the transition function $t_5 = 1/t_4$,

$$\begin{aligned}\Omega_+ &= e^{i\varphi} \frac{|z_1|^6 |z_4|^2}{\sqrt{\det g}} dt_2 \wedge dt_3 \wedge (-dt_5) \\ \Omega_- &= e^{-i\varphi} \frac{|z_1|^6 |z_4|^2}{\sqrt{\det g}} dt_2 \wedge dt_3 \wedge (-dt_5 + 3t_5 \bar{\eta}) .\end{aligned}$$

We see that Ω_{\pm} has singularities of the form $e^{i\varphi} = t_4/|t_4| = |t_5|/t_5$ at $t_4 = 0$ and $t_5 = 0$: indeed the phase of a complex number z is ambiguous at $z = 0$. It is always possible to soak up one of the two singularities by multiplying or dividing by $e^{i\varphi}$, but never both at the same time. Hence $e^{\pm i\varphi} \Omega_{\pm}$ are well-defined at $t_4 = 0$ but singular at $t_5 = 0$, whereas $e^{\mp i\varphi} \Omega_{\pm}$ are well defined at $t_5 = 0$ but singular at $t_4 = 0$. This problem does not arise for J_{\pm} , since $K \wedge K^*$ does not suffer from any phase ambiguities.

The way out is then to construct an Ω which combines both $e^{\pm i\varphi} \Omega_{\pm}$ and $e^{\mp i\varphi} \Omega_{\pm}$. We can take a hint from the supersymmetric $SU(3)$ structure of [4] which we know is globally well-defined. We use a new coordinate θ instead of ρ , defined by $|u|^2 = \xi \sin^2 \frac{\theta}{2}$. Thus we see that $|v|^2 = \xi \cos^2 \frac{\theta}{2}$ and $\rho^2 = \xi^1 + n \xi \sin^2 \frac{\theta}{2}$, which means that $\theta = 0$ or π for $\rho = \rho_1$ (corresponding to $t_4 = 0$) or $\rho = \rho_2$ (corresponding to $t_5 = 0$), respectively. The idea is then to modify $\omega \rightarrow \hat{\omega}$ by including the problematic phase $e^{i\varphi}$, then define another form $\tilde{\omega}$ with the property that $\tilde{\omega}$ varies from $\hat{\omega}$ to $\hat{\omega}^*$ as θ varies from 0 to π . More specifically we define,

$$\begin{aligned}\hat{\omega} &:= e^{i\varphi} \omega \\ \tilde{j} &:= \sin \theta \Re \hat{\omega} + \cos \theta \hat{j} \\ \tilde{\omega} &:= \cos \theta \Re \hat{\omega} - \sin \theta \hat{j} + i \Im \hat{\omega} ,\end{aligned}\tag{3.20}$$

so that the $SU(3)$ structure is given by,

$$\begin{aligned}J &:= \rho^2 \tilde{j} + \frac{i}{2} K \wedge K^* \\ \Omega &:= \rho^2 \tilde{\omega} \wedge K .\end{aligned}\tag{3.21}$$

The relations (3.17) ensure that (2.26) is satisfied. Moreover at $\theta = 0$ we have $\Omega = e^{i\varphi} \Omega_+$, whereas at $\theta = \pi$ we have $\Omega = - (e^{i\varphi} \Omega_-)^*$. The two singularities have thus been regularized and Ω is globally defined. Thus the pair (J, Ω) is a globally-defined structure $SU(3)$ on the manifold.

Let us make one final comment: the prescription of [1] for constructing global $SU(3)$ structures, reviewed at the end of section 2.3, gives a form Ω which is of type (2,1) with respect to the integrable complex structure of the toric variety. We see that the prescription used here can never coincide with that of [1]: the form Ω defined in eq. (3.21) is of mixed type, varying from (3,0) at $\theta = 0$ to (1,2) at $\theta = \pi$, with respect to the integrable complex structure.

4 \mathbb{CP}^1 over general SCTV

We will now show how to construct a globally-defined $SU(3)$ structure on a canonical (defined in eq. (4.1) below) \mathbb{CP}^1 bundle over a SCTV of complex dimension $d = 2$. This is a generalization, to any SCTV base, of our construction of a globally-defined $SU(3)$ structure on \mathbb{CP}^1 over \mathbb{CP}^2 , discussed in section 3.2.

As we saw explicitly in the special case of \mathbb{CP}^1 over \mathbb{CP}^2 , the canonical metric of the SCTV, eq. (2.21), is smooth for any twisting of the bundle parameterized by $n^a \in \mathbb{N}$. On the other hand the existence of a globally-defined $SU(3)$ structure imposes a topological constraint and hence a constraint on the n^a , as we explain in the following. This constraint is automatically satisfied for the canonical \mathbb{CP}^1 bundle.⁴

We start with a $(d + 1)$ -dimensional toric \mathbb{CP}^1 bundle over a d -dimensional base M , whose charges were given in (2.34). The \mathbb{CP}^1 bundle will be called *canonical* if the charge of z_{k+1} , defining the twisting of the bundle, is taken to compensate exactly for the charges of the base, i.e.,

$$n^a = \sum_{i=1}^k q_i^a. \tag{4.1}$$

As emphasized in [2], the topological condition for the existence of an $SU(3)$ structure on the total space of the SCTV is that its first Chern class should be even. Condition (4.1) guarantees that there is no topological obstruction for the existence of an $SU(3)$ structure. This can be seen as follows: the first Chern class of the SCTV is given by,

$$c_1 = \sum_{I=1}^{k+2} D_I, \tag{4.2}$$

where we have denoted by D_I the divisors corresponding to $\{z_I = 0\}$. On the other hand on a toric variety there are as many linearly-independent divisors as there are $U(1)$ charges [7]. In our case the fact that the local coordinates defined by S in (2.3) are gauge-invariant is equivalent to the linear relations,

$$D_I - \sum_{A \in S} \sum_{B=1}^{s+1} Q_B^A Q_I^B D_A = 0. \tag{4.3}$$

Taking the charges (2.34) into account, and inserting into (4.2) then leads to,

$$c_1 = \sum_{A \in S} \left(\sum_{b=1}^s Q_b^A \left(\sum_{i=1}^k q_i^b - n^b \right) + 2Q_{s+1}^A \right) D_A, \tag{4.4}$$

which, as advertised, is even if the bundle is canonical. More generally, we see that a globally-defined $SU(3)$ structure exists provided $(\sum_{i=1}^k q_i^a - n^a)$ are even for all a [2].

⁴We use the term *canonical metric* for the metric (2.21) of the SCTV, which is defined for all n^a , i.e. for all topologies. On the other hand we use the term *canonical \mathbb{CP}^1 bundle* for the topology defined in eq. (4.1). Hopefully this will not lead to confusion.

We define the usual toric coordinates and a local $SU(d+1)$ structure $(\hat{J}, \hat{\Omega})$ as explained in section 2.3. We recall that $\hat{\Omega}$ is not gauge-invariant: for the canonical $\mathbb{C}\mathbb{P}^1$ bundle it has charge,

$$Q(\hat{\Omega}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2 \end{pmatrix}, \quad (4.5)$$

where we took (4.1) into account.

Following the strategy of section 3.2 we would like to define the analogue of the local $SU(2)$ structure $(\hat{j}, \hat{\omega})$ on the base M , cf. (3.20). As in that case we first note that the $\mathbb{C}\mathbb{P}^1$ fiber distinguishes a one-form K , which we normalize such that $K^* \cdot K = 2$,

$$K := \frac{1}{\sqrt{1 - h_{k+2} \frac{k+2}{k+2}|v|^2}} \mathcal{D}v = \sqrt{\frac{\bar{g}}{\hat{g}}} \frac{\mathcal{D}v}{|u|}. \quad (4.6)$$

Note that K is not globally defined since it is not gauge-invariant. This can be seen explicitly by taking the $u \rightarrow 0$ limit, in which $\mathcal{D}v$ vanishes. Indeed in this limit we have,

$$K \sim \sqrt{\frac{\bar{g}}{\hat{g}}} v \frac{\bar{u}}{|u|} du \sim e^{i(\varphi_v - \varphi_u)} du,$$

where φ_u, φ_v denote the phases of u, v . However $K \wedge K^*$ does not suffer from any phase ambiguity, so that,

$$\hat{j} := \hat{J} - \frac{i}{2} K \wedge K^*, \quad (4.7)$$

is globally well-defined. Furthermore a somewhat tedious calculation which can be found in appendix D shows that $\hat{\Omega}$ can be simplified to,

$$\hat{\Omega} = \frac{(-1)^d}{\sqrt{\hat{g}}} \left(\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \bigwedge_{\alpha \in \mathfrak{L}\hat{S}} \mathcal{D}z_\alpha \right) \wedge e^{i\varphi_u} K. \quad (4.8)$$

Its contraction with K is given by,

$$\frac{1}{2} K^* \cdot \hat{\Omega} = \frac{e^{i\varphi_u}}{\sqrt{\hat{g}}} \left(\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \bigwedge_{\alpha \in \mathfrak{L}\hat{S}} \mathcal{D}z_\alpha \right), \quad (4.9)$$

which is not gauge-invariant. A gauge-invariant local holomorphic form $\hat{\omega}$ on the base can be constructed as follows,

$$\hat{\omega} := \frac{1}{2} e^{-i\varphi_v} K^* \cdot \hat{\Omega}. \quad (4.10)$$

Let us now specialize to $d = 2$. We can apply the procedure of section 3.2 and modify the local $SU(2)$ structure $(\hat{j}, \hat{\omega})$ in order to construct a global $SU(3)$ structure. Since we have $|u|^2 + |v|^2 = \xi$, we can define a parameter $\theta \in [0, \pi]$ such that $|u| = \sqrt{\xi_s} \sin \frac{\theta}{2}$ and

$|v| = \sqrt{\xi_s} \cos \frac{\theta}{2}$. By the same argument as in section 3.2, the SU(3) structure (J, Ω) given by,

$$\begin{aligned} J &:= j + \frac{i}{2} K \wedge K^* \\ \Omega &:= \omega \wedge e^{-i\varphi_v} K, \end{aligned} \tag{4.11}$$

where,

$$\begin{aligned} j &:= \sin \theta \Re \hat{\omega} + \cos \theta \hat{j} \\ \omega &:= \cos \theta \Re \hat{\omega} - \sin \theta \hat{j} + i \Im \hat{\omega}, \end{aligned}$$

can be seen to be globally-defined. Its associated metric is the canonical metric of the SCTV, given in (2.21), (2.37).

The expressions above can be thought of as a natural extension of the ansatz of [3, 4] to the toric case. Note however that, contrary to the simple case discussed in section 3, \hat{j} , $\hat{\omega}$ depend in general on the S^2 -fiber coordinates and should be thought of as forms on the total space. Moreover, except for simple bases such as $\mathbb{C}\mathbb{P}^2$ or $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, there is not a simple relation between the exterior differential of the connection one-form in K and the two-forms on the base, cf. appendix D.1 for more details. As a result, the associated torsion classes are all nonvanishing in general.

This structure could be easily modified by multiplying (j, ω) and K by functions of the coordinates of the S^2 fiber. The associated metric will be modified accordingly to,

$$\mathfrak{h}_3 = |h|^2 \mathfrak{h}_2 + |f|^2 \frac{\hat{g}}{g} |u|^2 |v|^2 K \otimes K^*, \tag{4.12}$$

for some functions of the fiber coordinates, f, h . Indeed modifying the local SU(2) structure via $\omega \rightarrow h^2 \omega$, $j \rightarrow |h|^2 j$, $K \rightarrow fK$ results in the metric (4.12). More generally, an orthogonal transformation can be applied on the triplet $(j, \Re \omega, \Im \omega)$, without changing the metric \mathfrak{h}_2 of the base.

Provided f, h are smooth and nowhere-vanishing, the topology of the total space is that of the SCTV $\mathbb{C}\mathbb{P}^1$ over M . The metric (4.12) is smooth, since it is a smooth deformation of the canonical metric (2.37) of the SCTV. In some cases allowing f, h to have singularities or zeros can lead to a smooth metric on a total space of different topology. We will see an example of this phenomenon in section 5 where an apparently singular metric on S^2 over $\mathbb{C}\mathbb{P}^2$ is in fact the local form of the round metric on S^6 .

5 LT structures on $S^2(B_4)$

We will now show that the sphere bundles of the form $S^2(B_4)$, where B_4 is any four-dimensional Kähler-Einstein space of positive curvature, admit regular globally-defined SU(3) structures of LT type, i.e. such that all torsion classes vanish except for W_1 and W_2 . This is the generic type of SU(3) structure that appears in supersymmetric AdS₄ compactifications of massive IIA supergravity [11].

Let \hat{j} be the Kähler form of B_4 , normalized as in (B.7), (B.8) with $\lambda = 6$, and let $(\hat{j}, \hat{\omega})$ be a local $SU(2)$ structure on B_4 so that,

$$\begin{aligned} \hat{\omega} \wedge \hat{\omega}^* &= 2\hat{j} \wedge \hat{j} ; & \hat{j} \wedge \hat{\omega} &= 0 ; \\ d\mathcal{P} &= 6\hat{j} ; & d\hat{j} &= 0 ; & d\hat{\omega} &= i\mathcal{P} \wedge \hat{\omega} , \end{aligned} \tag{5.1}$$

where \mathcal{P} is the canonical bundle of B_4 , cf. appendix B. We define the following $SU(3)$ structure,

$$\begin{aligned} J &= |h|^2 j + \frac{i}{2} K \wedge K^* \\ \Omega &= h^2 \omega \wedge K , \end{aligned} \tag{5.2}$$

where h is a complex function of θ and,

$$\begin{aligned} j &:= \cos \theta \hat{j} + \sin \theta \Re(e^{i\psi} \hat{\omega}) \\ \omega &:= -\sin \theta \hat{j} + \cos \theta \Re(e^{i\psi} \hat{\omega}) + i \Im(e^{i\psi} \hat{\omega}) \\ K &:= f d\theta + ig(d\psi + \mathcal{P}) , \end{aligned} \tag{5.3}$$

with $\psi \in [0, 2\pi)$ and f, g real functions of θ . The associated metric reads,

$$\mathbf{g} = |h|^2 \mathbf{g}_4 + f^2 d\theta^2 + g^2 (d\psi + \mathcal{P})^2 , \tag{5.4}$$

with \mathbf{g}_4 the Kähler-Einstein metric of B_4 . This ansatz is of the same general form as that of [3, 4]. However here we shall impose different conditions on the torsion classes, leading to an $SU(3)$ structure of LT type instead. As we show below, these conditions can be solved analytically for the functions f, g, h parameterizing the ansatz.

Using eq. (5.1), one can then compute the torsion classes of the $SU(3)$ structure (5.2),

$$\begin{aligned} W_1 &= -\frac{i}{3} \frac{h}{h^*} \left(\frac{1}{f} + \frac{\sin \theta}{g} + 6 \frac{g \sin \theta}{|h|^2} \right) \\ W_2 &= \frac{i}{3} \frac{h}{h^*} \left(\frac{1}{f} + \frac{\sin \theta}{g} - 12 \frac{g \sin \theta}{|h|^2} \right) J^\perp \\ W_3 &= \frac{1}{2} \left(\frac{1}{f} - \frac{\sin \theta}{g} + 6 \frac{g \sin \theta}{|h|^2} \right) \Re \Omega^\perp \\ W_4 &= \left(\frac{|h^2|'}{f|h^2|} - 6 \cos \theta \frac{g}{|h^2|} \right) d\theta \\ W_5 &= \left(\frac{h'}{fh} + \frac{g'}{2fg} - \frac{\cos \theta}{2g} \right) K , \end{aligned} \tag{5.5}$$

where we have introduced the primitive forms,

$$\begin{aligned} J^\perp &= |h|^2 j - iK \wedge K^* \\ \Omega^\perp &= |h|^2 \omega \wedge K^* . \end{aligned} \tag{5.6}$$

Moreover, as we show in appendix E, one can impose $W_3 = W_4 = W_5 = 0$ provided,

$$f = \alpha \left(1 - 6\alpha^2 \frac{\sin^2 \theta}{H} \right)^{-1} ; \quad g = \alpha \sin \theta ; \quad h = \sqrt{H(\theta)} e^{i\beta} , \tag{5.7}$$

with,

$$\begin{aligned}
 H(\theta) &:= \frac{1}{3} \left(\tilde{x} + \frac{\tilde{x}^2}{B} + B \right) \\
 B &:= \left(\frac{27H_0^3}{2} + \tilde{x}^3 + 3\sqrt{3} \sqrt{\frac{27H_0^6}{2} + \tilde{x}^3 H_0^3} \right)^{1/3} \\
 \tilde{x} &:= 9\alpha^2 \sin^2 \theta,
 \end{aligned}
 \tag{5.8}$$

where the real constants α , β and $H_0 \geq 0$ are the parameters of the solution.

For $H_0 > 0$ the functions f , h are nowhere vanishing. Moreover the $\theta \rightarrow 0, \pi$ limit gives a regular metric, provided the period of ψ is 2π . Then by the same argument as in [4, 9], the $SU(3)$ structure (5.2) is globally-defined and the associated metric (5.4) is regular and complete: the (ψ, x^μ) space, where x^μ are the coordinates of B_4 , parametrizes a circle fibration in the canonical bundle \mathcal{L} over B_4 ; it extends to a complete, regular five-dimensional Sasaki-Einstein manifold provided B_4 is Kähler-Einstein of positive curvature [12]. The (ψ, θ) space parameterizes a smooth S^2 , so that the total space has the same topology as $\mathcal{L} \times_{U(1)} \mathbb{C}P^1$, in the notation of [9]. The nonvanishing torsion classes read,

$$\begin{aligned}
 W_1 &= -\frac{2i}{3} \frac{e^{2i\beta}}{\alpha} \\
 W_2 &= \frac{2i}{3} \frac{e^{2i\beta}}{\alpha} \left(1 - \frac{9\alpha^2 \sin^2 \theta}{H} \right) J^\perp.
 \end{aligned}
 \tag{5.9}$$

Therefore the $S^2(B_4)$ bundles admit $SU(3)$ structures of LT type, rendering them suitable as compactification spaces for supersymmetric AdS_4 solutions of massive IIA [11]. Note that unlike the LT $SU(3)$ structures on $S^2(\mathbb{C}P^2)$ discussed in [13] from the point of view of twistor spaces (cf. appendix F) or in [14] from the point of view of cosets, the structure (5.9) does not obey $dW_2 \in (3, 0) \oplus (0, 3)$.⁵ Indeed a direct calculation gives,

$$dW_2 = e^{2i\beta} \left(1 - \frac{9\alpha^2 \sin^2 \theta}{H} \right) \left(\frac{2i}{3\alpha^2} \left(1 - \frac{9\alpha^2 \sin^2 \theta}{H} \right) \Re(e^{-2i\beta} \Omega) - \frac{6i \sin^2 \theta}{H} \Re \Omega^\perp \right).
 \tag{5.10}$$

As a consequence, if these manifolds are to be used as compactification spaces for massive IIA, the Bianchi identity for the RR two-form will require the introduction of (smeared) six-brane sources. Another difference from the LT structures of [13, 14] is that the discussion of this section applies to any $S^2(B_4)$ bundle with Kähler-Einstein base, not only to $B_4 = \mathbb{C}P^2$.

In the case $H_0 = 0$, on the other hand, one obtains the solution,

$$f = 3\alpha ; \quad g = \alpha \sin \theta ; \quad h = 3\alpha \sin \theta e^{i\beta}.
 \tag{5.11}$$

This corresponds to the *nearly Kähler* limit, in which also W_2 vanishes. Moreover the $\theta \rightarrow 0, \pi$ limit results in a conical metric of the form,

$$\mathfrak{g} \sim d\theta^2 + \theta^2 ds_5^2,
 \tag{5.12}$$

⁵It should be possible to make contact with the results of [13, 14] by suitably acting on the vielbein by an orthogonal transformation. There does not seem to exist a simple ansatz for this transformation, which may be rather involved as it could *a priori* depend on all coordinates.

where,

$$ds_5^2 := \mathfrak{g}_4 + \frac{1}{9}(d\psi + \mathcal{P})^2, \tag{5.13}$$

is the canonically normalized metric of a five-dimensional Sasaki-Einstein base written as a circle fibration on the canonical bundle over B_4 ; the normalization is such that the cone metric (5.12) is Ricci-flat. Hence for $H_0 = 0$ the metric presents conical singularities in general, unless B_4 is $\mathbb{C}\mathbb{P}^2$, in which case the associated Sasaki-Einstein metric (5.13) is that of the round sphere, and the associated cone (5.12) is not only Ricci-flat but also flat. Going back to the metric (5.4) we obtain,

$$\mathfrak{g} = 9\alpha^2 (d\theta^2 + \sin^2\theta ds_5^2) . \tag{5.14}$$

We thus see that in the smooth case, $B_4 = \mathbb{C}\mathbb{P}^2$, we obtain a round S^6 of radius 3α . We thus recover the well-known result that the round S^6 admits an associated nearly-Kähler structure.

Let us finally note that we may relax the condition on B_4 , so that B_4 is any four-dimensional Kähler manifold (not necessarily toric, or Einstein). In this case the torsion classes can also be explicitly calculated, cf. appendix G, however we do not expect the structure to admit a global extension to a complete space with a regular metric.

6 Conclusions

The construction of SU(3) structures on SCTV had up to now proceeded on a case-by-case basis. In the present paper we gave a formula for a globally-defined SU(3) structure valid on all canonical $\mathbb{C}\mathbb{P}^1$ bundles over two-dimensional SCTV. This SU(3) structure admits a space of deformations parameterized by certain functions, on which the associated torsion classes depend. The construction is genuinely different from that in [1]: as opposed to the construction in that reference, it produces a holomorphic three-form of varying type (with respect to the integrable complex structure of the SCTV).

Having a general formula for the SU(3) structure opens up the possibility of a systematic (possibly automatized) scan for flux vacua. Such a procedure has been successfully carried out in the case of solvmanifolds [15] and cosets [14], and would be interesting to undertake also in the class $\mathbb{C}\mathbb{P}^1$ over SCTV considered here. It could be extended to $\mathbb{C}\mathbb{P}^1$ fibrations over noncompact toric varieties, as the formalism does not rely on compactness other than in the input of the U(1) charges specifying the toric variety [16].

The construction of the SU(3) structure was also applied to the case of $S^2(B_4)$ bundles. These spaces first appeared as six-dimensional bases of seven-dimensional Sasaki-Einstein spaces in the context of $\mathcal{N} = 2$ AdS₄ vacua of M-theory [9]. It was subsequently realized [10] that reducing along the so-called α -circle produces a (warped) $\mathcal{N} = 2$ AdS₄ \times $S^2(B_4)$ vacuum of IIA. The relevant supersymmetric SU(3) structure, whose existence was implicitly inferred in [10], was first constructed explicitly in [3] for the case $B_4 = \mathbb{C}\mathbb{P}^2$. The generalization to arbitrary B_4 was given in [4].

In the present paper we showed that the $S^2(B_4)$ spaces also admit a different SU(3) structure of LT type, thus making them suitable for $\mathcal{N} = 1$ compactifications of massive

IIA. However, these solutions require (smeared) six-brane sources, complicating their physical interpretation. It is possible that a more general orthogonal rotation of the local $SU(2)$ structure acting on the triplet $(j, \Re\omega, \Im\omega)$ may produce a sourceless LT structure, although the analysis becomes rather cumbersome in this case and we have been unable to obtain a conclusive result.⁶

A Decomposition of the metric

In this section we fill in some of the details leading up to eq. (2.37). We start by defining the vertical one-forms using the formalism introduced in section 2.1, for the total bundle. The various objects are thus given in terms of the charges Q_I^A . Expressing them in terms of the charges of the base q_i^a we have,

$$\begin{aligned}\hat{g}^{ab} &:= q_i^a q_i^b |z_i|^2 \\ \hat{h}_{ij} &:= \hat{g}_{ab} q_i^a q_j^b \\ \hat{\mathcal{D}}z_i &:= \hat{P}(dz_i) = dz_i - \hat{h}_{ij} z_i \bar{z}_j dz_j,\end{aligned}$$

where hatted symbols are used to denote objects relative to the base, in order to distinguish from the objects constructed in (2.1). Note that $\hat{g}, \hat{h}, \hat{\mathcal{D}}z_i$ live on the same space as their non-hatted counterparts, which are the relevant objects for the definition of forms in the symplectic quotient description. This means that they do not have any a priori interpretation as objects on the base. For example, the $|z_i|^2$ do not verify the moment map equations of the base but those of the total bundle, and thus \hat{g}, \hat{h} and $\hat{\mathcal{D}}z_i$ depend on the radii. A quick calculation confirms that the $\hat{\mathcal{D}}z_i$ do obey the expected algebraic relations,

$$\sum_{i=1}^k q_i^a \bar{z}_i \hat{\mathcal{D}}z_i = 0.$$

Recall the form of the canonical metric on a SCTV (the generalization of the Fubiny-Study metric of $\mathbb{C}\mathbb{P}^1$),

$$\mathfrak{h}_{d+1} = \sum_{I=1}^{k+2} \mathcal{D}z_I \otimes \mathcal{D}\bar{z}_I.$$

We will now decompose this metric into base and fiber components. Since the $\mathcal{D}z$ depends on the matrix g^{AB} , the key here will be to decompose it and its inverse along the different bundle directions.

The definition of Q_I^A leads to:

$$g^{AB} = \begin{pmatrix} \hat{g}^{ab} + n^a n^b |u|^2 & -n^a |u|^2 \\ -n^b |u|^2 & \xi \end{pmatrix}.$$

⁶If the existence of a sourceless LT structure could be established within the context of the ansatz of the present paper, it would exist for all $S^2(B_4)$ spaces, not only for $B_4 = \mathbb{C}\mathbb{P}^2$, as our ansatz only relies on the Kähler-Einstein property of the base. As already remarked, a sourceless LT structure does exist on $S^2(\mathbb{C}\mathbb{P}^2)$ [13, 14], but seems to rely on the special properties of $\mathbb{C}\mathbb{P}^2$ as a selfdual Einstein manifold [17]. This is not taken into account by our ansatz, and would not be applicable to the other Kähler-Einstein bases B_4 : $\mathbb{C}\mathbb{P}^2$ is the only Kähler-Einstein four-manifold of positive curvature that is also selfdual.

Moreover we need to express the inverse g_{AB} while keeping track of the inverse, \hat{g}_{ab} , of \hat{g}^{ab} . For this purpose we first need to compute the determinant $g = \det g^{AB}$,

$$g = \begin{vmatrix} \hat{g} + n n^T |u|^2 & n |u|^2 \\ n^T |u|^2 & \xi \end{vmatrix} = \begin{vmatrix} \hat{g} + n n^T |u|^2 \left(1 - \frac{|u|^2}{\xi}\right) & 0 \\ n^T |u|^2 & \xi \end{vmatrix} = \xi \det \left(\hat{g} + \frac{1}{\xi} |u|^2 |v|^2 n n^T \right).$$

We now use the property of multilinearity of the determinant to expand this expression. We then get all different terms of order $s-m$ in g and m in $n n^T$. But since $\text{rank } n n^T = 1$, only the terms of order zero or one remain. The terms of order one are merely the determinant of \hat{g} where the column a has been replaced by the vector $\frac{n^a}{\xi} |u|^2 |v|^2 n$. By expanding along this same column, we exhibit the cofactors of \hat{g} which are independent of this exact column, and are related to the inverse matrix,

$$\det(\hat{g}, g^a \leftrightarrow n) = \sum_a \text{cof}(\hat{g})_{ab} n^b = \hat{g} \hat{g}_{ab} n^b.$$

Thus we have:

$$g = \xi \left(\hat{g} + \frac{n^a}{\xi} |u|^2 |v|^2 \hat{g} \hat{g}_{ab} n^b \right) = \hat{g} (\xi + \hat{g}_{ab} n^a n^b |u|^2 |v|^2).$$

The same trick can be used to compute the inverse matrix:

$$g_{s+1 s+1} = \frac{1}{g} \det(\hat{g} + n n^T |u|^2) = \frac{\hat{g}}{g} (1 + \hat{g}_{ab} n^a n^b |u|^2).$$

Moreover,

$$g_{a s+1} = \frac{1}{g} \det(\hat{g}, g^a \leftrightarrow -|u|^2 n) = \frac{\hat{g}}{g} |u|^2 \hat{g}_{ab} n^b.$$

The last cofactors are somewhat more complicated, since they involve double cofactors. Eventually we get:

$$g_{ab} = \hat{g}_{ab} - \frac{\hat{g}}{g} |u|^2 |v|^2 \hat{g}_{ac} n^c \hat{g}_{bd} n^d.$$

It is now possible to compute the $h_{\mu\nu}$. Let us introduce the objects

$$V := \hat{g}_{ab} n^a n^b, \quad V_i := \hat{g}_{ac} q_i^a n^c, \tag{A.1}$$

in terms of which we obtain,

$$\begin{aligned} h_{ij} &= g_{ab} q_i^a q_j^b = \hat{h}_{ij} - \frac{\hat{g}}{g} |u|^2 |v|^2 V_i V_j \\ h_{ik+1} &= g_{a s+1} q_i^a - g_{ab} q_i^a n^b = -\frac{\hat{g}}{g} V_i |v|^2 \\ h_{ik+2} &= g_{a s+1} q_i^a = \frac{\hat{g}}{g} V_i |u|^2 \\ h_{k+1 k+1} &= g_{s+1 s+1} - 2g_{a s+1} n^a + g_{ab} n^a n^b = \frac{\hat{g}}{g} (1 + V |v|^2) \\ h_{k+1 k+2} &= g_{s+1 s+1} - g_{a s+1} n^a = \frac{\hat{g}}{g} \\ h_{k+2 k+2} &= g_{s+1 s+1} = \frac{\hat{g}}{g} (1 + V |u|^2). \end{aligned} \tag{A.2}$$

We can now compute the $\mathcal{D}z_I$,

$$\frac{\mathcal{D}z_i}{z_i} = \frac{\hat{\mathcal{D}}z_i}{z_i} + \frac{\hat{g}}{g} V_i |u|^2 |v|^2 \varepsilon, \tag{A.3}$$

where,

$$\varepsilon = \frac{du}{u} - \frac{dv}{v} + V_j \bar{z}_j dz_j. \tag{A.4}$$

The last two coordinates correspond to colinear one-forms,

$$\frac{\mathcal{D}u}{u} = \frac{\hat{g}}{g} |v|^2 \varepsilon; \quad \frac{\mathcal{D}v}{v} = -\frac{\hat{g}}{g} |u|^2 \varepsilon.$$

Finally the canonical metric reads,

$$\begin{aligned} \mathfrak{h}_{d+1} &= \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i + \mathcal{D}u \otimes \mathcal{D}\bar{u} + \mathcal{D}v \otimes \mathcal{D}\bar{v} \\ &= \hat{\mathcal{D}}z_i \otimes \hat{\mathcal{D}}\bar{z}_i + \frac{\hat{g}}{g} V_i |u|^2 |v|^2 \hat{\mathcal{D}}z_i \otimes \bar{z}_i K^* + \text{c.c.} \\ &\quad + V_i V_i |z_i|^2 \frac{\hat{g}^2}{g^2} |u|^4 |v|^4 \varepsilon \otimes \varepsilon^* + \frac{\hat{g}^2}{g^2} |u|^2 |v|^2 \xi \varepsilon \otimes \varepsilon^*. \end{aligned} \tag{A.5}$$

On the other hand we have,

$$V_i \bar{z}_i \hat{\mathcal{D}}z_i = \hat{g}_{ab} n^a q_i^b \bar{z}_i \hat{\mathcal{D}}z_i = 0; \quad V_i^2 |z_i|^2 = V,$$

so that the metric simplifies to,

$$\mathfrak{h}_{d+1}(\xi^A) = \mathfrak{h}_d((\rho^a)^2) + \frac{\hat{g}^2}{g^2} |u|^2 |v|^2 (\xi + V |u|^2 |v|^2) \varepsilon \otimes \varepsilon^* \tag{A.6}$$

$$= \mathfrak{h}_d((\rho^a)^2) + \frac{\hat{g}}{g} |u|^2 |v|^2 \varepsilon \otimes \varepsilon^*. \tag{A.7}$$

Note that this decomposition remains valid in the complex local coordinates t_i, t_{k+1} , on the chart U_S defined by $S = \hat{S} \cup \{k+2\}$, in which ε can be written as,

$$\varepsilon = \frac{dt_{k+1}}{t_{k+1}} + \sum_{i=1}^k V_i |z_i|^2 \frac{dt_i}{t_i}.$$

The $\hat{\mathcal{D}}z_i$ happen to be the projections on the space generated by the dt_i , in fact they are related to the dt_i by the relations (2.18) where we take \hat{h}_{ij} instead of h_{IJ} . This justifies that in the decomposition (A.6), the metric on the base is exactly the canonical metric whose radii vary along the fiber.

B Kähler-Einstein manifolds

A Kähler manifold of real dimension $2d$ corresponds to the case of a *local* $SU(d)$ structure where W_5 is the only nonvanishing torsion class,

$$dJ = 0; \quad d\Omega = i\mathcal{P} \wedge \Omega, \tag{B.1}$$

cf. (2.27). The local structure (J, Ω) can also be expressed in terms of bilinears of a locally-defined spinor ζ on M . In terms of this spinor eq. (B.1) can be written equivalently,

$$\nabla_m \zeta = \frac{i}{2} \mathcal{P}_m \zeta, \tag{B.2}$$

where $\mathcal{P} := 2\Im W_5$ is a real one-form. (Note that the existence of the complex structure allows us to reconstruct the torsion W_5 from its imaginary part alone.) Moreover (B.2) can be inverted to obtain \mathcal{P} from the covariant spinor derivative,

$$\mathcal{P}_m = -2i\zeta^\dagger \nabla_m \zeta. \tag{B.3}$$

From (B.3), (B.2), using $\nabla_{[m} \nabla_{n]} \zeta = \frac{1}{8} R_{mnpq} \gamma^{pq} \zeta$ we obtain,

$$d\mathcal{P} = \mathcal{R}, \tag{B.4}$$

where \mathcal{R} is the Ricci form. Hence \mathcal{P} can be identified with the connection of the canonical bundle of M . On the other hand, the Ricci tensor is obtained from the Riemann tensor via,

$$\mathcal{R}_{mn} = \frac{1}{2} R_{mnpq} J^{pq} = R_{mpnq} J^{pq}. \tag{B.5}$$

On a Kähler manifold the Ricci form, the Ricci tensor and the Ricci scalar obey,

$$\mathcal{R}_{mn} = J_m{}^p R_{pn}; \quad \mathcal{R}_{mn} J^{mn} = R. \tag{B.6}$$

Furthermore for a Kähler-Einstein manifold such that,

$$R_{mn} = \lambda g_{mn}, \tag{B.7}$$

eqs. (B.7), (B.6) imply,

$$\mathcal{R} = \lambda J, \tag{B.8}$$

but in general the Ricci form need *not* be proportional to the Kähler form.

The above relations are valid for arbitrary dimension. Specializing to four real dimensions we adopt the notation $(J, \Omega) \rightarrow (\hat{j}, \hat{\omega})$, in accordance with the main text. We may decompose any two-form Φ on the basis of a local $SU(2)$ structure $(\hat{j}, \hat{\omega})$ as follows:

$$\Phi = \varphi \hat{j} + \tilde{\Phi} + \chi \hat{\omega} + \psi \hat{\omega}^*, \tag{B.9}$$

where $\varphi := \frac{1}{4} \hat{j}^{mn} \Phi_{mn}$ is the trace of Φ , and $\tilde{\Phi}$ is (1,1)-traceless: $\hat{j}^{mn} \tilde{\Phi}_{mn} = 0$. Equivalently,

$$\hat{j} \wedge \tilde{\Phi} = 0. \tag{B.10}$$

It is also straightforward to show that $(\hat{j}, \hat{\omega})$ are selfdual forms while (1,1)-traceless forms are anti-selfdual,

$$\star(\hat{j}, \hat{\omega}) = (\hat{j}, \hat{\omega}); \quad \star \tilde{\Phi} = -\tilde{\Phi}. \tag{B.11}$$

In particular for the Ricci form the expansion reads,

$$\mathcal{R} = \frac{1}{4} R \hat{j} + \tilde{\mathcal{R}}. \tag{B.12}$$

Moreover the above properties can be used to calculate,

$$\mathcal{R} \wedge \mathcal{R} = \left(\frac{1}{4} R^2 - \frac{1}{2} R_{mn} R^{mn} \right) \text{vol}_4, \tag{B.13}$$

where the volume is given by,

$$\text{vol}_4 = \frac{1}{2} \hat{j} \wedge \hat{j}. \tag{B.14}$$

C Real coordinates

In this section we explain in detail how to rewrite the hermitian metric (3.5) in terms of real coordinates, and make contact with the metric (3.8). Let us start by rewriting the \mathbb{CP}^1 fiber coordinate t_4 in terms of a pair of real coordinates. It is not necessary to do the same for t_2, t_3 , since the coordinates of the \mathbb{CP}^2 base do not appear explicitly in (3.8). Using eq. (2.36), $|t_4|$ can be written in terms of ρ and the base coordinates,

$$|t_4|^2 = \frac{|z_1|^{2n}|z_4|^2}{|z_5|^2} = \frac{\rho^{2n}}{(1+t^2)^n} \frac{\rho^2 - \rho_1^2}{\rho_2^2 - \rho^2} . \tag{C.1}$$

Let $\varphi \in [0, 2\pi]$ be the phase of t_4 , so that t_4 becomes a function of t_2, t_3, ρ, φ ,

$$\begin{aligned} \frac{dt_4}{t_4} &= \frac{\rho d\rho}{\rho^2 - \rho_1^2} + \frac{\rho d\rho}{\rho^2 - \rho_1^2} + n \frac{d\rho}{\rho} - n \frac{d(t^2)}{2(1+t^2)} + id\varphi \\ &= \frac{d\rho}{n \rho \Gamma} - n \Re \eta + id\varphi . \end{aligned}$$

Moreover we set,

$$\varepsilon := \frac{dt_4}{t_4} + n \eta = \frac{d\rho}{n \rho \Gamma} + i (d\varphi + n \Im \eta) . \tag{C.2}$$

The term $|\varepsilon|^2 := \varepsilon \otimes \bar{\varepsilon}$ appears naturally in (3.5) through the contribution,

$$\varepsilon \otimes \bar{\varepsilon} = \frac{1}{n^2 \rho^2 \Gamma^2} d\rho^2 + (d\varphi + n \Im \eta)^2 - i \frac{1}{n \rho \Gamma} d\rho \wedge (d\varphi + n \Im \eta) . \tag{C.3}$$

The last term on the right-hand side above contributes to the Kähler form, while the rest contributes to the metric. Setting $\psi := \varphi/n$ and $A := \Im \eta$, we recover the terms appearing in (3.8), provided we set $n = 3$. Moreover the relative coefficient between the $d\rho^2$ and the $(d\psi + A)^2$ term is fixed in the expression of $|\varepsilon|^2$, and this determines the change of variables $\rho \rightarrow \tilde{\rho}(\rho)$ by comparing with (3.8). However, performing this change of variables in (3.5) does not directly bring us to the metric of (3.8): there remain two coefficients that still need to be adjusted. This can be achieved by introducing the two warp factors of eq. (3.14) as we now show.

Let us go back to the expression of the metric in terms of $\mathcal{D}z_i$. In local coordinates we have,

$$\begin{aligned} \frac{\mathcal{D}z_1}{z_1} &= n \Gamma \varepsilon - \eta \\ \frac{\mathcal{D}z_2}{z_2} &= \frac{dt_2}{t_2} + n \Gamma \varepsilon - \eta \\ \frac{\mathcal{D}z_3}{z_3} &= \frac{dt_3}{t_3} + n \Gamma \varepsilon - \eta \\ \frac{\mathcal{D}z_4}{z_4} &= \rho^2 \frac{\rho_2^2 - \rho^2}{n \det g} \varepsilon \\ \frac{\mathcal{D}z_5}{z_5} &= \rho^2 \frac{\rho_1^2 - \rho^2}{n \det g} \varepsilon . \end{aligned} \tag{C.4}$$

It follows that the term $\sum_{i=1}^3 \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i$ gives the hermitian metric of $\mathbb{C}\mathbb{P}^2$ plus a $|\varepsilon|^2$ term, whereas $\mathcal{D}z_4, \mathcal{D}z_5$ only contribute to $|\varepsilon|^2$. Let us define,

$$\begin{aligned}
 \mathfrak{h} &= F(\rho) \sum_{i=1}^3 \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i + G(\rho) \sum_{i=4,5} \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i \\
 &= F\rho^2 \mathfrak{h}_{cp2} + \left(F + \left(\frac{1}{n^2\Gamma} - 1 \right) G \right) n^2 \rho^2 \Gamma^2 |\varepsilon|^2 \\
 &= F\rho^2 \mathfrak{h}_{cp2} + \left(F + \left(\frac{1}{n^2\Gamma} - 1 \right) G \right) \left(d\rho^2 + n^4 \rho^2 \Gamma^2 (d\psi + A)^2 - in^2 \rho \Gamma d\rho \wedge (d\psi + A) \right) .
 \end{aligned} \tag{C.5}$$

We can then adjust F, G , and ρ so that,

$$\begin{aligned}
 F\rho^2 &= \tilde{\rho}^2 \\
 \left(F + \left(\frac{1}{n^2\Gamma} - 1 \right) G \right) d\rho^2 &= \frac{1}{U} d\tilde{\rho}^2 \\
 \left(F + \left(\frac{1}{n^2\Gamma} - 1 \right) G \right) n^4 \rho^2 \Gamma^2 &= q .
 \end{aligned} \tag{C.6}$$

These equations can easily be decoupled by first solving for ρ , then for F and finally for G .

D General SCTV base

In the following we give the details of the derivation of eq. (4.8). The first step is writing $\hat{\omega}$ in terms of $\mathcal{D}z$. However this exercise is rather involved, since the $\mathcal{D}z$ are not independent and because of the ambiguity in the decomposition of wedge products. Our starting point is eq. (2.31),

$$\hat{\Omega} = \frac{1}{\sqrt{g}} \bigwedge_A Q_J^A z_J \partial_{z_J} \cdot \bigwedge_I dz_I .$$

In this expression, we notice that the expansion of the contraction with the horizontal vectors amounts to choosing a set S of $s + 1$ integers between 1 and $k + 2$, corresponding to the indices of the contracted coordinates. We compute,

$$\hat{\Omega} = \frac{1}{\sqrt{g}} \sum_S (-1)^S Q_S \prod_{A \in S} z_A \bigwedge_{\alpha \in \mathfrak{L}_S} dz_\alpha ,$$

cf. (2.33), where Q_S is the determinant of the submatrix of Q_I^A whose columns are indexed by S . Notice that if S contains duplicates, or if it does not select independent columns, the determinant vanishes. Thus the sum selects only the sets S for which the matrix Q_A^B is invertible. The sign $(-1)^S$ is the signature of the permutation required to put the $s + 1$ indices of S in the first position, namely:

$$(-1)^S = \sigma(S, \mathfrak{L}_S) = (-1)^{\sum_{a \in S} a + \frac{1}{2}(s+1)(s+2)} . \tag{D.1}$$

We would now like to decompose $\hat{\Omega}$ with respect to the bundle structure. We therefore distinguish four cases:

1. $S \subset [[1, k]]$
2. $S = \hat{S} \cup \{k+1\}$ where $\hat{S} \subset [[1, k]]$, $\#\hat{S} = s-1$
3. $S = \hat{S} \cup \{k+2\}$
4. $S = \check{S} \cup \{k+1, k+2\}$ where $\check{S} \subset [[1, k]]$, $\#\check{S} = s-2$

In the first case we get $Q_S = 0$, since $\text{rank } q_i^a = d < d+1$. In cases 2 and 3 we can easily see that $Q_S = q_{\hat{S}} := \det(q_a^b)_{a \in \hat{S}}$, while $(-1)^S = (-1)^{\hat{S}}(-1)^d$ for case 2, and $(-1)^S = (-1)^{\hat{S}}(-1)^{d+1}$ for case 3. We can now write,

$$\hat{\Omega} = \frac{(-1)^{d+1}}{\sqrt{g}} \sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \bigwedge_{\alpha \in \mathfrak{C}\hat{S}} dz_{\alpha} \wedge (vdu - u dv) + \frac{1}{\sqrt{g}} \Sigma_4,$$

with Σ_4 to be determined. In case 4 we get,

$$\begin{aligned} Q_S &= \det(q_a^b, -n^b)_{a \in \check{S}} \\ &= \det \left(q_a^b, -\sum_{i=1}^k q_i^b \right) \\ &= -\sum_{i=1}^k \det(q_a^b, q_i^b). \end{aligned}$$

In the sum, if $i \in \check{S}$, the determinant cancels out, leaving only a sum over ${}^{\mathfrak{C}}\check{S}$, so that,

$$\Sigma_4 = -\sum_{\check{S}} \sum_{\beta \in {}^{\mathfrak{C}}\check{S}} (-1)^S \det(q_a, q_{\beta})_{a \in \check{S}} \prod_{a \in \check{S}} z_a u v \bigwedge_{\alpha \in {}^{\mathfrak{C}}\check{S}} dz_{\alpha}.$$

We are now ready include this sum in the one over the \hat{S} , which appears in cases 2 and 3: we just need to make the change of variable $\hat{S} = \check{S} \cup \{\beta\}$. However dz_{β} appears in the product, thus we need to shift it to the last position. At the same time we need to move it to its right place inside $\det(q_a, q_{\beta})$ so as to maintain the increasing order of \hat{S} . The number of shifts needed to do so is the number of shifts required to bring β from its place to the end in ${}^{\mathfrak{C}}\check{S}$ plus the number of shifts to bring it from the end to its place in \check{S} ; since ${}^{\mathfrak{C}}\check{S} \cup \check{S} = [[1, k]]$, this is exactly the number of shifts required to bring β from its place to the end in $[[1, k]]$, i.e. $k - \beta$. The last sign we need to compute is,

$$\begin{aligned} (-1)^S &= (-1)^{\sum_{a \in \hat{S}} a + (k+1) + (k+2) - \frac{1}{2}(s+1)(s+2)} \\ &= (-1)^{\sum_{a \in \hat{S}} a - \beta + (k+1) + (k+2) - \frac{1}{2}s(s+1) - (s+1)} \\ &= -(-1)^{\hat{S}} (-1)^{k - \beta + (d+1)}. \end{aligned}$$

Having expressed everything in terms of \hat{S} and β , it is now possible to transform the sum $\sum_{\check{S}} \sum_{\beta \in {}^{\mathfrak{C}}\check{S}}$ in $\sum_{\hat{S}} \sum_{b \in \hat{S}}$,

$$\Sigma_4 = (-1)^{d+1} \sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \bigwedge_{\alpha \in \mathfrak{C}\hat{S}} dz_{\alpha} \wedge \left(u v \sum_{b \in \hat{S}} \frac{dz_b}{z_b} \right).$$

To get a more symmetrical expression we can simply complete the sum $\sum dz_b/z_b$, since the missing terms can be trivially added thanks to the wedge product. The final expression is thus,

$$\hat{\Omega} = \frac{(-1)^{d+1}}{\sqrt{g}} \left(\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \bigwedge_{\alpha \in \mathfrak{L}\hat{S}} \mathcal{D}z_\alpha \right) \wedge \left(v\mathcal{D}u - u\mathcal{D}v + uv \sum_{i=1}^k \frac{\mathcal{D}z_i}{z_i} \right).$$

The dz were ultimately replaced by $\mathcal{D}z$ because $\hat{\Omega}$ is vertical. Now recall that the expression (A.3) decomposes $\mathcal{D}z_i$ into base and fiber parts. Since the metric decomposes correctly into (A.6), the $\mathcal{D}z_i$ are orthogonal to K . Besides, the fiber part can be shown to cancel out in the first factor, so that the first parenthesis is orthogonal to K . Thus we can take the second factor to be proportional to K , and the proportionality factor can be found by computing,

$$\begin{aligned} K^* \cdot \left(v\mathcal{D}u - u\mathcal{D}v + uv \sum_{i=1}^k \frac{\mathcal{D}z_i}{z_i} \right) &= \frac{2}{\sqrt{1 - h_{k+2, k+2} |v|^2}} \left(v(0 - h_{k+1, k+2} \bar{v} u) \right. \\ &\quad \left. - u(1 - h_{k+2, k+2} |v|^2) + uv \sum_{i=1}^k (0 - h_{k+2, i}) \right) \tag{D.2} \\ &= \frac{2u}{\sqrt{1 - h_{k+2, k+2} |v|^2}} \left(-1 + |v|^2 \left(-h_{k+1, k+2} + h_{k+2, k+2} - \sum_{i=1}^k h_{k+2, i} \right) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{i=1}^k h_{k+2, i} &= g_{AB} Q_{k+2}^A \sum_{i=1}^k Q_i^B \\ &= g_{AB} Q_{k+2}^A (Q_{k+2}^B - Q_{k+1}^B) \\ &= h_{k+2, k+2} - h_{k+2, k+1}, \end{aligned}$$

so that,

$$K^* \cdot \left(v\mathcal{D}u - u\mathcal{D}v + uv \sum_{i=1}^k \frac{\mathcal{D}z_i}{z_i} \right) = -2\sqrt{\frac{g}{\hat{g}}} \frac{u}{|u|}.$$

Hence $\hat{\Omega}$ simplifies to the expression in (4.8).

D.1 Torsion classes

For a generic SCTV base all torsion classes are nonvanishing. We will not write them down explicitly in this case, as they are rather cumbersome and not particularly illuminating. The computation boils down to determining the exterior differentials of $\hat{\omega}$ and K . In the following we give some details of the calculation.

In the notation of (2.35), K and $(\hat{j}, \hat{\omega})$ can be written as follows,

$$K = \sqrt{\frac{g}{\hat{g}}} \frac{Dv}{|u|} ; \quad \hat{j} = \hat{J} - \frac{i}{2} K \wedge K^* ; \quad \hat{\omega} = \frac{1}{2} e^{-i\varphi_v} K^* \cdot \hat{\Omega} . \quad (\text{D.3})$$

In terms of the $\hat{D}z_i$, we have,

$$\begin{aligned} \hat{j} &:= \frac{i}{2} \hat{D}z_i \wedge \hat{D}\bar{z}_i \\ \hat{\omega} &:= \frac{e^{i(\varphi_u - \varphi_v)}}{\sqrt{\hat{g}}} \sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \wedge_{\alpha \in \mathfrak{t}_{\hat{S}}} \hat{D}z_\alpha . \end{aligned}$$

Up to a phase (required for gauge invariance) this coincides with the canonical local $SU(2)$ structure of the base. In particular this implies that \hat{j} is Kähler at fixed fiber coordinates. The dependence of \hat{j} on the fiber coordinates is such that \hat{J} is Kähler.

We can also rewrite everything in local complex coordinates on the patch $S = \hat{S} \cup \{k+2\}$:

$$\begin{aligned} K &= -\sqrt{\frac{\hat{g}}{g}} |u|v \left(\frac{dt_{k+1}}{t_{k+1}} + V_i |z_i|^2 \frac{dt_i}{t_i} \right) \\ \hat{\omega} &= \frac{e^{i(\psi + \sum_\alpha \psi_\alpha)}}{\sqrt{\hat{g}}} (-1)^{\hat{S}} q_{\hat{S}} \prod_i |z^i| \bigwedge_\alpha \frac{dt^\alpha}{t^\alpha} = f \bigwedge_\alpha \frac{dt^\alpha}{t^\alpha} , \end{aligned} \quad (\text{D.4})$$

where ψ, ψ_α are the phases of t_{k+1}, t_α . We can now introduce real coordinates θ, ψ on the fiber with $|u|^2 = \xi \sin^2 \frac{\theta}{2}$:

$$K = \frac{1}{2} \left(\gamma d\theta + \frac{\xi}{\gamma} \sin \theta i(d\psi + A) \right) ,$$

where $\gamma = \sqrt{\frac{g}{\hat{g}}} = \sqrt{\xi + \frac{1}{2}\xi^2 V \sin^2 \theta}$ and $A = V_i |z^i|^2 \Im \frac{dt^i}{t^i} = V_i |z^i|^2 d\psi_i$. We also get,

$$dA = \frac{i}{2} V_i \hat{D}z_i \wedge \hat{D}\bar{z}_i + \frac{i}{4} \sin \theta d\theta \wedge V_i^2 (\bar{z}_i \hat{D}z_i - z_i \hat{D}\bar{z}_i) .$$

Differentiating $\hat{\omega}$ leads to another one-form,

$$\begin{aligned} d\hat{\omega} &= \frac{df}{f} \wedge \hat{\omega} \\ &= \left(-\frac{\xi}{2} V_i (1 - \hat{h}_{ii} |z_i|^2) \sin \theta d\theta + id\psi \right. \\ &\quad \left. + \sum_j id\psi_j + \frac{1}{2} (1 - \hat{h}_{jj} |z_j|^2) \left(\frac{\hat{D}\bar{z}^j}{\bar{z}^j} - \frac{\hat{D}z_j}{z_j} \right) \right) \wedge \hat{\omega} . \end{aligned} \quad (\text{D.5})$$

Alternatively, in terms of t^i ,

$$d\hat{\omega} = \left(-\frac{\xi}{2} V_i (1 - \hat{h}_{ii} |z_i|^2) \sin \theta d\theta + i(d\psi + A + d\psi_i |z_i|^2 (h_{ii} - h_{ij} h_{jj} |z_j|^2)) \right) \wedge \hat{\omega} .$$

We can write,

$$A' := A + d\psi_i |z_i|^2 (h_{ii} - h_{ij} h_{jj} |z_j|^2) = A + B,$$

where B comes from the derivatives of \hat{g} and is nonvanishing in general. For simple bases such as \mathbb{CP}^2 or $\mathbb{CP}^1 \times \mathbb{CP}^1$, \hat{g} is constant and thus B vanishes. The $d\theta$ term comes from the deformation of the base metric along the direction θ . Note also that at fixed θ , $dA' \propto \mathcal{R}$ where \mathcal{R} is the Ricci form of the base, *cf.* (B.4).

E LT structures

In this section we fill out the details leading up to eq. (5.8). Plugging the following general ansatz in the decomposition of $dJ, d\Omega$,

$$\begin{aligned} dJ &= \frac{3\alpha_1}{2} \Im\Omega - \frac{3\alpha_2}{2} \Re\Omega + \alpha_3 \Re K + \alpha_4 \Im K + \alpha_5 \Re\Omega^\perp + \alpha_6 \Im\Omega^\perp \\ d\Omega &= a_1 J \wedge J + a_2 K^* \wedge \Omega + a_3 J^\perp \wedge J, \end{aligned} \tag{E.1}$$

for some real and complex parameters $\alpha_1, \dots, \alpha_6$ and a_1, \dots, a_3 respectively, and using eqs. (5.1), (5.2), (5.3), we arrive at the torsion classes given in (5.5). Imposing $W_3 = W_4 = W_5 = 0$ leads to,

$$\begin{aligned} W_3 &: \frac{1}{f} - \frac{\sin\theta}{g} + 6 \frac{g \sin\theta}{|h|^2} = 0 \\ W_4 &: \frac{|h^2|'}{f|h^2|} - 6 \cos\theta \frac{g}{|h^2|} = 0 \\ W_5 &: \frac{h'}{fh} + \frac{g'}{2fg} - \frac{\cos\theta}{2g} = 0. \end{aligned} \tag{E.2}$$

From $W_5 - \bar{W}_5$ we see that the phase of h must be constant but is otherwise unconstrained by the equations, i.e.,

$$h = |h|e^{i\beta}, \tag{E.3}$$

for some real constant $\beta \in [0, 2\pi)$. Moreover we set $H := |h|^2$, for some nonnegative function H . Since $d\psi$ is not defined at $\theta = 0, \pi$, regularity requires that the coefficient of $d\psi + A$ should vanish at the poles. It is therefore convenient to set $g := G \sin\theta$ for some function G . The equations now read,

$$\begin{aligned} \frac{1}{f} - \frac{1}{G} + 6 \frac{G \sin^2\theta}{H} &= 0 \\ H' - 3 \sin 2\theta G f &= 0 \\ \frac{H'}{H} + \frac{G'}{G} + \cot\theta - \frac{f \cot\theta}{G} &= 0, \end{aligned} \tag{E.4}$$

where we have assumed that f, h are nonvanishing. Plugging the first two into the third then implies,

$$G = \alpha, \tag{E.5}$$

for some real constant α . The system is then solved as in eq. (5.8), where H satisfies,

$$H' \left(1 - 6\alpha^2 \frac{\sin^2 \theta}{H} \right) = 3\alpha^2 \sin 2\theta . \tag{E.6}$$

We immediately see that $H(\theta) = 9\alpha^2 \sin^2 \theta$ is a special solution. Moreover the differential equation imposes $H(\pi - \theta) = H(\theta)$. It is thus convenient to introduce a new function $\varphi(x)$, where $x := \sin^2 \theta$ and $H := 9\alpha^2 x \varphi(x)$, in terms of which the equation becomes,

$$\frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2} \varphi' = \frac{1}{x} . \tag{E.7}$$

Integrating over x between X_0 and X we obtain,

$$\int_{X_0}^X \frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2} \varphi' dx = \log \frac{X}{X_0} , \tag{E.8}$$

where $\varphi_0 := \varphi(X_0)$. On the other hand,

$$\frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2} = -\frac{2}{3} \frac{1}{\varphi} - \frac{1}{3} \frac{1}{\varphi - 1} . \tag{E.9}$$

Since $\varphi \geq 0$ and $\varphi - 1, \varphi_0 - 1$ have the same sign, we find,

$$-\frac{2}{3} \log \frac{\varphi}{\varphi_0} - \frac{1}{3} \log \frac{\varphi - 1}{\varphi_0 - 1} = \log \frac{X}{X_0} , \tag{E.10}$$

which leads to,

$$\varphi^2(\varphi - 1) = \frac{X_0^3}{X^3} \varphi_0^2(\varphi_0 - 1) . \tag{E.11}$$

Rewriting the above in terms of H which, contrary to φ , is necessarily everywhere well-defined, we obtain,

$$H^2(H - 9\alpha^2 X) = H_0^2(H_0 - 9\alpha^2 X_0) = \text{constant} . \tag{E.12}$$

We can henceforth assume $X_0 = 0$ without loss of generality, which leads to,

$$H^2(H - 9\alpha^2 X) - H_0^3 = 0 . \tag{E.13}$$

It is easy to see that the above polynomial in H is increasing for negative H , until it attains the value $-H_0^3 \leq 0$ at $H = 0$. It then decreases until $H = 6\alpha^2 X$, from which point on it becomes increasing. Therefore if we impose $H_0 > 0$ the polynomial only vanishes once, for $H > 6\alpha^2 X \geq 0$. For $H_0 = 0$, there are two solutions: $H = 0$ (which must be discarded) and the special solution $H = 9\alpha^2 X$. We conclude that for any $H_0 \geq 0$, there is a unique solution to the differential equation with the boundary conditions $H(0) = H_0 = H(\pi)$; it is given in eq. (5.8) of the main text.

F Twistor spaces

There is an alternative description of the total space of the $\mathbb{C}\mathbb{P}^1$ fibration over $\mathbb{C}\mathbb{P}^2$ in terms of twistor spaces. More generally, for the purposes of the present section we may replace the $\mathbb{C}\mathbb{P}^2$ base by any four-dimensional Kähler space B_4 .

Consider B_4 equipped with its canonical complex structure \hat{I} and a hermitian metric \mathfrak{g} . Let us introduce a complex zweibein z_1, z_2 , so that $\hat{I}z_k = iz_k$, for $k = 1, 2$. These forms are of course only locally defined, since B_4 is not parallelizable in general. We can thus express the metric and the local $SU(2)$ structure on B_4 in terms of the complex zweibein,

$$\begin{aligned} \mathfrak{g} &= z_1 \bar{z}_1 + z_2 \bar{z}_2 \\ \hat{j} &= \frac{i}{2} (z_1 \wedge \bar{z}_1 + z_2 \wedge \bar{z}_2) \\ \hat{\omega} &= z_1 \wedge z_2 . \end{aligned}$$

At any one point $x \in B_4$, $\hat{j}_x, \hat{\omega}_x$ form an $SU(2)$ structure on the tangent space $T_x B_4$. The latter is equipped with a complex structure and a scalar product given by \hat{I}_x and \mathfrak{g}_x respectively. Moreover the relation,

$$\hat{I}_m^k = \mathfrak{g}^{kn} \hat{j}_{mn} ,$$

allows us to identify the complex structure with a real selfdual form. The latter are parameterized as follows, see appendix B,

$$j_x = \alpha \hat{j}_x + \frac{\beta}{2} \hat{\omega}_x + \frac{\beta^*}{2} \hat{\omega}_x^* ,$$

where α is real and $\alpha^2 + |\beta|^2 = 1$. Hence the space of complex structures I_x compatible with the metric \mathfrak{g}_x forms a sphere whose coordinates $\theta \in [0, \pi]$, $\psi \in [0, 2\pi)$ are defined by $\alpha = \cos \theta$, $\beta = \sin \theta e^{i\psi}$, so that I_x is associated with the two-form,

$$j_x = \cos \theta \hat{j}_x + \sin \theta \Re(e^{i\psi} \hat{\omega}_x) .$$

Extending this procedure to each point on B_4 then defines an almost complex structure I over the whole manifold (unlike \hat{I} , I will not be integrable in general). Over each point on B_4 an almost complex structure compatible with the metric of B_4 can be thought of as a point on the sphere S^2 parameterized by (θ, ψ) . Hence the space of almost complex structures on B_4 is a fiber bundle S^2 over B_4 denoted by $\text{Tw}(B_4)$, the *twistor space* of B_4 .

The zweibein z_1, z_2 is no longer compatible with the almost complex structure I associated with the real two-form j given above. Rather we define,

$$\begin{aligned} f_1 &:= \cos \frac{\theta}{2} e^{i\frac{\psi}{2}} z_1 + i \sin \frac{\theta}{2} e^{-i\frac{\psi}{2}} \bar{z}_2 \\ f_2 &:= \cos \frac{\theta}{2} e^{i\frac{\psi}{2}} z_2 - i \sin \frac{\theta}{2} e^{-i\frac{\psi}{2}} \bar{z}_1 , \end{aligned} \tag{F.1}$$

so that $If_k = if_k$. In terms of the new zweibein the local $SU(2)$ structure and the metric read,

$$\begin{aligned} \mathfrak{g} &= f_1 \bar{f}_1 + f_2 \bar{f}_2 \\ j &= \frac{i}{2} (f_1 \wedge \bar{f}_1 + f_2 \wedge \bar{f}_2) \\ \omega &= f_1 \wedge f_2 = \cos \theta \Re(e^{i\psi} \hat{\omega}) - \sin \theta \hat{j} + i \Im(e^{i\psi} \hat{\omega}), \end{aligned}$$

which is precisely of the form of (3.20). Let us also note that the choice of zweibein compatible with I is only determined up to a phase. The latter leaves j and the metric invariant but acts nontrivially on ω , thus changing the $SU(2)$ structure.

We have seen that $I_x(\theta, \psi)$ defines an almost complex structure on the base. Together with the natural complex structure of the sphere (thought of as a $\mathbb{C}P^1$) we can construct an almost complex structure on the total space,

$$I_{\pm} = \begin{pmatrix} I_x(\theta, \psi) & 0_{4 \times 2} \\ 0_{2 \times 4} & \begin{matrix} 0 & \pm \frac{1}{\sin \theta} \\ \mp \sin \theta & 0 \end{matrix} \end{pmatrix},$$

so that f_1, f_2 and $K = d\theta + i \sin \theta (d\psi + A)$ are eigenforms of I_{\pm} with eigenvalue $\pm i$. We can thus take (f_1, f_2, K) as the vielbein on $\text{Tw}(B_4)$. More generally we could modify (f_1, f_2, K) by introducing “warp factors” as in (5.3) below.

G Torsion classes for Kähler base

As mentioned in section 5 we may relax the condition on the base of $S^2(B_4)$, so that B_4 is a generic four-dimensional Kähler manifold. The torsion classes can also be straightforwardly calculated in this case. Note however that this is only a local calculation: without additional constraints, we do not expect there to exist a global extension to a complete space.

Let us postulate a globally-defined $SU(3)$ structure as in (5.3) on a $\mathbb{C}P^1$ bundle with metric,

$$g_6 = |h|^2 g_4 + K K^* ; \quad K = f d\theta + i g (d\psi + A), \tag{G.1}$$

where f, g, h are *a priori* complex functions; θ and ψ parameterize the S^2 fiber; the one-form A satisfies (B.1), (B.4) for $(J, \Omega) \rightarrow (\hat{j}, \hat{\omega})$. We will impose further restrictions on f, g, h ; these functions must be regular and non-vanishing, except for g which must vanish at $\theta = 0$ and $\theta = \pi$. The most general situation we will consider here is that df, dg, dh live on the space spanned by K, K^* (this restricts the dependance on the coordinates). Explicitly we expand,

$$df = f_1 K + f_2 K^*, \tag{G.2}$$

and similarly for g, h . It is also possible restrict the dependance on θ alone.

The calculation of the torsion classes proceeds in the same fashion as in appendix E, with the following result,

$$\begin{aligned}
 W_1 &= -\frac{2i}{3} \frac{h}{h^*} \left(\frac{g + f \sin \theta}{fg^* + f^*g} + \frac{R}{2} g \frac{\sin \theta}{|h|^2} \right) \\
 W_2 &= \frac{2i}{3} \frac{h}{h^*} \left(\frac{g + f \sin \theta}{fg^* + f^*g} - Rg \frac{\sin \theta}{|h|^2} \right) J^\perp \\
 W_3 &= -\frac{1}{2} (fg^* + f^*g) d\theta \wedge \tilde{R} + \Re \left(\frac{g - f \sin \theta}{fg^* + f^*g} + \frac{R}{2} g \frac{\sin \theta}{|h|^2} \right) \Omega^\perp \\
 W_4 &= d(\log |h^2|) - \frac{R}{2|h^2|} (fg^* + f^*g) \cos \theta d\theta \\
 W_5^* &= \frac{1}{fg^* + f^*g} \left(f \cos \theta + f_1 g - f g_1 - (f^* g_2 + f_2 g^*) - 2(fg^* + f^*g) \frac{h_2}{h} \right) K^* .
 \end{aligned} \tag{G.3}$$

Our degrees of freedom in the above are a somewhat redundant: a phase change of K can be absorbed in h so that f or g can be taken real. Let us also note that in general a cross term $(fg^* - f^*g)d\theta(d\psi + A)$ appears in the metric. If we want this to vanish, we must impose f and g to be colinear, so that they can both be taken real.

Furthermore if we want to impose $W_4 = 0$, we must restrict h to depend only on θ , in which case we get,

$$h_1 = \frac{g^* h'}{fg^* + f^*g} ; \quad h_2 = \frac{gh'}{fg^* + f^*g} . \tag{G.4}$$

Therefore f and g must also be restricted so that $R(f^*g + fg^*)$ is a function of θ alone.

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